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Weibull-Linear Exponential Distribution and Its Applications

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Abstract: In this article, a new four-parameter lifetime distribution, namely, the Weibull-Linear exponential distribution is defined and studied. Several of its structural properties such as quartiles, moments, mean waiting time, mean residual lifetime, Renyi entropy, mode, and order statistics are derived. Based on the idea of the Weibull $T-X$ family, the new density function of this model is developed. The model parameters, as well as some of the lifetime parameters (reliability and failure rate functions), are estimated using the maximum likelihood method. Asymptotic confidence intervals estimates of the model parameters are also evaluated by using the Fisher information matrix. Moreover, to construct the asymptotic confidence intervals of the reliability and failure rate functions, we need to find their variance of them, which are approximated by the delta method. A real data set is used to illustrate the application of the Weibull-Linear Exponential distribution.

Keywords: Weibull-Linear exponential model; Weuible $T-X$ family; Properties model; Parameters estimation; Progressive Type-II censoring.

1 Introduction

Continuous univariate distributions have been developed hundreds of times. Recently, researchers from biomedical sciences, and engineering in many areas, reach that data sets which follow the classical distributions are near exceptions rather than reality. As extended distributions have a clear demand, a significant progress action has been taken to generalize some well-known distributions and it is applied successfully in these area's problems.

The linear exponential distribution(LED) has many uses for modeling medical studies and lifetime data in reliability, for example, [1], studied the survival pattern of patients with plasmacytic myeloma. The LED is also known as the Linear Failure Rate distribution, having exponential and Rayleigh distributions as special cases is a very well-known distribution for modeling lifetime data in medical studies and reliability analysis see [2]. It is also modeled phenomenon with an increasing failure rate see [3]. For more details about the LED and its applications, see [4]. [5] studied the generalization of LED. [6] studied the Transmuted LED and a new generalization of LED. [7] studied On transmuted generalized LED. [8] studied Bivariate Exponentiated Generalized LED: Properties, Inference and Applications. [9] studied Progressively censored data from the generalized linear exponential distribution moments and estimation. [10] studied Modified Beta Linear Exponential Distribution with Hydrologic Applications.
The Weibull distribution is a well-known distribution due to its wide use to model various types of data. The Weibull distribution was originally introduced by [11]. Besides the many applications of the Weibull distribution has been widely used in lifetime analysis, quality control, survival and reliability analyses, failure analysis, hydrology, material science, physics, meteorology, and medicine for more details [12]. Many authors developed generalizations of the Weibull distribution. For example, the Generalized Weibull distribution by [13], the Exponentiated-Weibull distribution by [14], the beta-Weibull distribution by [15], Weibull-Pareto Distribution and Its Applications by [16], a new Weibull-like distribution by [17], Lomax Weibull Distribution by [18], On Gamma Inverse Weibull distribution by [19], Weibull Inverse Lomax Distribution by [20], On the Exponentiated Weibull Rayleigh Distribution by [21], Exponentiated

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additive Weibull distribution by [22], Transmuted Weibull distribution and its applications by [23], The Weibull Frechet Distribution and its Applications by [24], A New Bivariate Extended Generalized Inverted Kumaraswamy Weibull Distribution by [25]. The probability density function (PDF), cumulative distribution function (CDF), reliability function and hazard function of the LED are given by, respectively, (for $x > 0$) The probability density function (PDF), cumulative distribution function (CDF), reliability function and hazard function of the LED are given by, respectively, (for $x > 0$)

$$F(x) = 1 - exp\left[-\left(\lambda x + \frac{\theta x^2}{2}\right)\right], \qquad \lambda > 0, \theta > 0, \tag{1}$$

$$f(x) = (\lambda + \theta x) exp\left[-\left(\lambda x + \frac{\theta x^2}{2}\right)\right], \qquad \lambda > 0, \theta > 0, \tag{2}$$

$$S(t) = exp\left[-\left(\lambda t + \frac{\theta t^2}{2}\right)\right], \qquad \lambda > 0, \theta > 0, \tag{3}$$

and

$$h(t) = (\lambda + \theta t), \qquad \lambda > 0, \theta > 0, \tag{4}$$

where θ is scale parameter and λ is the shape parameter

Let $F(x)$ be the CDF of any random variable X and $r(t)$ be PDF of a random variable T defined on $[0, \infty]$. The CDF of the generalized family of distributions defined by [26] is given by

$$G(x) = \int_0^{-\log 1 - F(x)} r(t) dt. \tag{5}$$

This family is called Transformed-Transformer ($T - X$ family) . The PDF of Weibull distribution with parameters c and γ is

$$r(t) = \left(\frac{c}{\gamma}\right)\left(\frac{t}{\gamma}\right)^{(c-1)} \exp\left[-\left(\frac{t}{\gamma}\right)^c\right], \qquad t \geq 0, \gamma > 0, c > 0, \tag{6}$$

where γ and c are scale and shape parameters respectively. From (5) the PDF of $Weibull - X$ family take the following form

$$g(x) = \left(\frac{c}{\gamma}\right)\left(\frac{f(x)}{1 - F(x)}\right)\left(\frac{-\log(1 - F(x))}{\gamma}\right)^{c-1} exp\left[-\left(-\frac{\log(1 - F(x))}{\gamma}\right)^c\right]. \tag{7}$$

The layout of this paper is organized as follows: In Section 2 we define the Weibull-Linear exponential distribution and provide some plots for its PDF, CDF, survival function $S(t)$ and hazard rate function $h(t)$. Some mathematical properties of the new distribution are obtained in Section 3. In Section 4 the researchers will consider the distribution of the order statistics of WLED. The MLEs of the model parameters based on the complete sample and progressive Type-II Censoring are investigated in Section 5. Application to real data is discussed in Section 6. Comparison between WLED and other distributions is discussed in Section 7. Finally, some concluding remarks are given in Section 8.

2 The Weibull-Linear Exponential Distribution

If X is a Linear Exponential random variable with PDF and CDF are given in (1) and (2), then (7) reduces to

$$g(x) = \left(\frac{c}{\gamma^c}\right)\left(\lambda + \theta x\right)\left(\lambda x + \frac{\theta}{2}x^2\right)^{c-1} exp\left[-\left(\frac{\lambda x + \frac{\theta}{2}x^2}{\gamma}\right)^c\right], \theta, \gamma, c, \lambda > 0. \tag{8}$$

A random variable X with the PDF $g(x)$ in (8) is said to follow the Weibull-Linear Exponential distribution (WLED) and will be denoted by $WLED(\theta,\gamma,c,\lambda)$. From (8), the CDF of WLED given by

$$G(x)=1-exp\left[-\left(\frac{\lambda x+\frac{\theta}{2}x^2}{\gamma}\right)^c\right],\theta,\gamma,c,\lambda>0, \tag{9}$$

where c, λ are the shape paramaters and θ, γ are the scale parameters. The WLED distribution is approaches flexibelity when its parameters are changed. It gives the following special models: For $\theta=0$, $\lambda=1$, the $WLED$ reduces to Weibull distribution. For $c=1$, $\gamma=1$, the $WLED$ reduces to LED. For $c=1$, $\gamma=1$ and $\theta=0$, the $WLED$ reduces to Exponential distribution. For $c=1$, $\gamma=1$ and $\lambda=0$, the $WLED$ reduces to Rayleigh distribution where $\theta=\frac{1}{\sigma^2}$.

The plots of the WLED for some parameter values given in Figure 1. The reliability function $S(x)$ of WLED is given by

$$S(x)=exp\left[-\left(\frac{\lambda x+\frac{\theta}{2}x^2}{\gamma}\right)^c\right], \qquad x\geq 0,\gamma>0,\theta>0,c>0,\lambda>0. \tag{10}$$

The limit of the WLED survival function as $x\to 0$ and $x\to\infty$ are given respectively by

$$\lim_{x\to 0}S(x)=\begin{cases}1, & c\geq 1\\ 1, & c=1\\ 1, & c<1\end{cases}, \qquad \lim_{x\to\infty}S(x)=\begin{cases}\infty, & c\geq 1\\ \infty, & c=1\\ \infty, & c<1\end{cases}. \tag{11}$$

The hazard rate function $h(x)$ of WLED is given by

$$h(x)=\left(\frac{c}{\gamma^c}\right)\left(\lambda+\theta x\right)\left(\lambda x+\frac{\theta}{2}x^2\right)^{c-1}, \qquad x\geq 0,\gamma>0,\theta>0,c>0,\lambda>0. \tag{12}$$

The limit of the WLED hazard function as $x\to 0$ and $x\to\infty$ are given respectively by

$$\lim_{x\to 0}h_g(x)=\begin{cases}0, & c\geq 1\\ \frac{\lambda c}{\gamma^c}, & c=1\\ 0, & c<1\end{cases}, \qquad \lim_{x\to\infty}h_g(x)=\begin{cases}\infty, & c\geq 1\\ \infty, & c=1\\ \infty, & c<1\end{cases}. \tag{13}$$

The limiting behaviors of the WLE PDF and the hazard rate function as $x\to 0$ and $x\to\infty$ are given respectively by

$$\lim_{x\to 0}g(x)=\begin{cases}0, & c\geq 1\\ \frac{\lambda c}{\gamma^c}, & c=1\\ 0, & c<1\end{cases}, \qquad \lim_{x\to\infty}g(x)=\begin{cases}0, & c\geq 1\\ \infty, & c=1\\ \infty, & c<1\end{cases}. \tag{14}$$

The derivative $h(x)$ with respect to x of the hazard function rate for WLED is given by

$$h'(x)=\left(\frac{c}{\gamma^c}\right)\left(\lambda x+\frac{\theta}{2}x^2\right)^{c-2}\left[(c-1)(\lambda+\theta x)^2+\theta(\lambda x+\frac{\theta}{2}x^2)\right]. \tag{15}$$

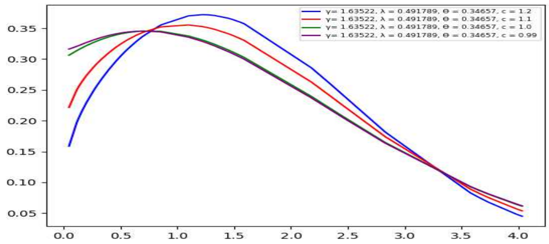
The limit for the derivative $h(x)$ is given by

$$\lim_{x\to 0}h'(x)=\begin{cases}0, & c\geq 2\\ 0, & c=2\end{cases}, \qquad \lim_{x\to\infty}h'(x)=\begin{cases}\infty, & c\geq 2\\ \infty, & c=2\end{cases}. \tag{16}$$

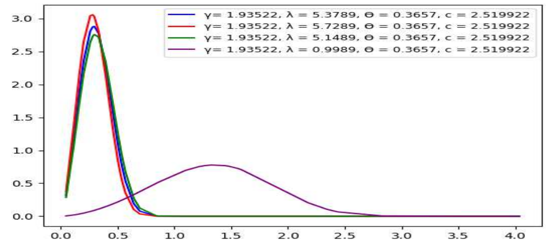
Their exist $x^*=\sqrt{(c^2-\frac{c-1}{\theta})(\frac{\lambda^2}{c+\theta-\frac{3}{2}})-(\frac{c\lambda}{2\theta(c-1)-1})}$ that makes $h'(x)$ called a monotonic function if $c<2$.

3 Some Structural and Statistical Properties

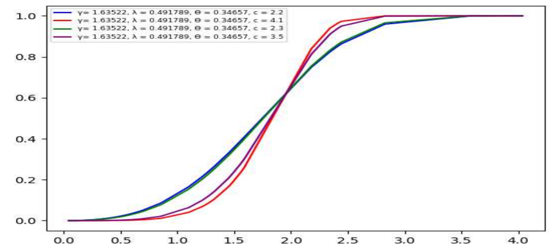
In this section, we investigate some structural and statistical properties of the WLED including quartiles, moments, mean waiting time, Renyi entropy, mean residual lifetime, and mode. To determine some structural properties of the WLED we use algebraic expansions rather than computing them directly using numerical integration of the WLED density function. Programming software such as Python, R, and Mathematica give analytical facilities to use these results in practice.



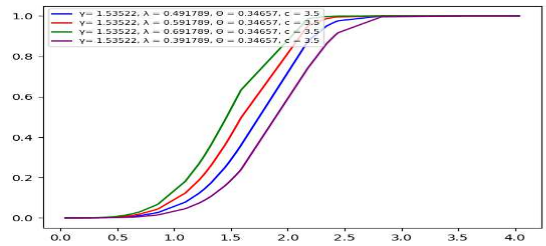
PDF of WLED for various values of c



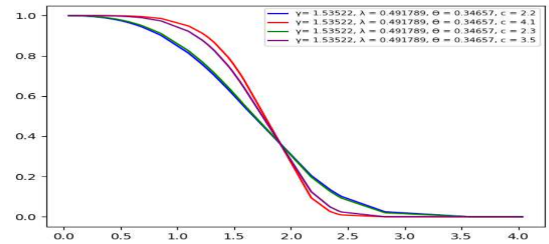
PDF of WLED for various values of λ



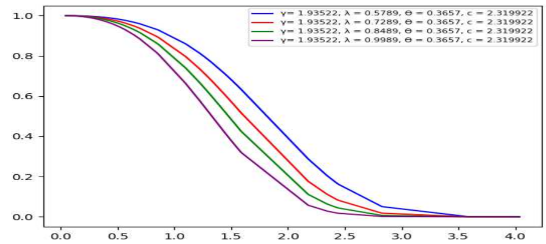
CDF of WLED for various values of c



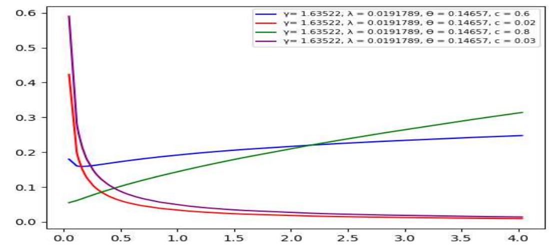
CDF of WLED for various values of λ



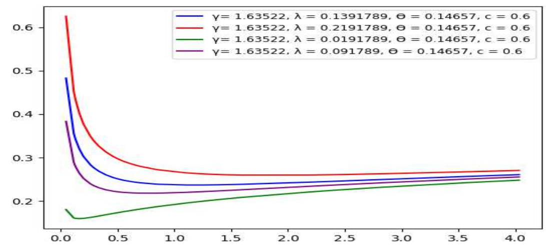
$S(x)$ of WLED for various values of c



$S(x)$ of WLED for various values of λ



$h(x)$ of WLED for various values of c



$h(x)$ of WLED for various values of λ

Figure 1. Plots of the WLED for some parameters values.

3.1 Quartiles

Quantile defines a specific part of a data set. The quartiles are Special quantiles. The quantile determines how many values are above or below a specific limit in a distribution. Thus if we divide a distribution into four equal partitions, we will speak of four quartiles. The first quartile includes all values that are smaller than a quarter of all values. In a graphical representation, it corresponds to 25% of the total area of distribution. The two lower quartiles comprise 50% of all distribution values. The interquartile range between the first and third quartile equals the range in which 50% of all values lie that are distributed around the mean. Furthermore, by inverting (9) the quantile function (qf) of the WLED is obtained as

$$q = F(X_q). \\ = 1 - exp\left[\frac{\lambda x_q + \frac{\theta}{2}{x_q}^2}{\gamma}\right]^c, \tag{17}$$

$$X_q = \sqrt{\frac{\lambda^2}{\theta^2} - \frac{2\gamma}{\theta}\left(log(1-q)\right)^{\frac{1}{c}}} - \frac{\lambda}{\theta}, \qquad 0 < q < 1, \tag{18}$$

where q is a uniform random variable on the unite interval $(0,1)$, i.e $q \in (0,1)$. Thus, the WLED random variable is easly simulated. The median of the WLED is obtained By setting $q = 0.5$ in (9).

3.2 Moments

Moments play important role in any statistical analysis, especially in applications. They are used for finding measures of central tendency, skewness and kurtosis, dispersion, and others. Many authors deal with moments for new lifetime models, see for example Tian et al. [6] derived the transmuted linear exponential distribution: a new generalization of the LED, [24] studied the moment from the Weibull Frechet distribution and its applications and [27] studied the Weibull-Dagum distribution: properties and applications and recently [28] studied the moment generating function for the exponentiated kumaraswamy linear exponential distribution: theory and application. The moment generating function for the WLED can be written as

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} g(x)dx = \int_0^\infty \sum_{i=0}^\infty \frac{t^i x^i}{i!} g(x)dx = 1 + \sum_{i=1}^\infty \frac{t^i}{i!} E(x^i), \tag{19}$$

if X follows the WLED, then

$$E(x^i) = \int_0^\infty x^i g(x)dx, \tag{20}$$

$$E(x^i) = \int_0^\infty x^i \left(\frac{c(\lambda+\theta x)}{\gamma^c}\right)\left(\lambda x + \frac{\theta}{2}x^2\right)^{c-1} exp\left[-\left(\frac{\lambda x + \frac{\theta}{2}x^2}{\gamma}\right)^c\right]dx, \tag{21}$$

$$E(x^i) = \int_0^\infty \frac{x^i}{\gamma} d\left(-exp\left[-\left(\frac{\lambda x + \frac{\theta}{2}x^2}{\gamma}\right)^c\right]\right)dx, \tag{22}$$

let $y = \left(\frac{\lambda x + \frac{\theta}{2}x^2}{\gamma}\right)^c$

$$E(x^i) = \frac{1}{\gamma}\sum_{j=0}^{i}\left(-1\right)^j\left(\frac{\lambda}{\theta}\right)^{i-j}\left(\frac{\lambda^2}{\theta^2}\right)^{\frac{j}{2}}\sum_{m=0}^\infty k^m \Gamma(\frac{m}{c}+1), \tag{23}$$

where $k = \frac{2\gamma\theta}{\lambda^2}$.

$$M_x(t) = 1 + \left[\frac{1}{\gamma}\sum_{i=1}^\infty \frac{t^i}{i!}\left[\frac{1}{\gamma}\sum_{j=0}^{i}\left(-1\right)^j\left(\frac{\lambda}{\theta}\right)^{i-j}\left(\frac{\lambda^2}{\theta^2}\right)^{j/2}\sum_{m=0}^\infty k^m \Gamma\left(\frac{m}{c}+1\right)\right]\right]. \tag{24}$$

The first and the second moment can be determined from (24) respectively, as follows

$$E(X) = \frac{1}{\gamma}\left(\frac{\lambda}{\theta} - \left(\frac{\lambda^2}{\theta^2}\right)^{1/2}\right)\sum_{m=0}^\infty k^m \Gamma\left(\frac{m}{c}+1\right), \tag{25}$$

and

$$E(X^2) = \frac{1}{\gamma}\left[\left(\frac{\lambda}{\theta}\right)^2 - \left(\frac{\lambda}{\theta}\right)\left(\frac{\lambda^2}{\theta^2}\right)^{1/2} + \left(\frac{\lambda^2}{\theta^2}\right)\right]\sum_{m=0}^\infty k^m \Gamma\left(\frac{m}{c}+1\right). \tag{26}$$

Furthermore, the variance for the WLED can be obtained by

$$Var(X) = \left(E(X)\right)^2 - E(X^2). \tag{27}$$

From (25) and (26), we obtain

$$Var(X) = \left(\frac{-\lambda^2}{\theta^2}\right)\left(\frac{1}{\gamma}\right)\sum_{m=0}^{\infty} k^m \Gamma\left(\frac{m}{c}+1\right). \tag{28}$$

3.3 Mean Waiting Time

Considerably the mean waiting time is the criterion used to measure the effectiveness of the service. The pattern which is most effective in decreasing waiting time can be determined by comparing many different services pattern. [29] introduced the formula for the mean waiting time in a G/G/1 queue. [30] introduce mean waiting time as a measure of effectiveness. [31] propose a theoretical analysis of mean waiting time for message delivery in lattice ad hoc networks. [32] studied the difference in mean waiting times between two classes of customers in a single-server FIFO queue. For the WLED the mean waiting time can be written as

$$m(t) = \frac{1}{F(t)}\int_0^t F(x)dx, \tag{29}$$

$$\begin{aligned} m(t) = & \frac{1}{1 - exp\left[-\left(\frac{1}{\gamma^c}\right)\left(\lambda t + \frac{\theta}{2}t^2\right)^c\right]} \\ & \times \left[t - \sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{\gamma})^{ck}}{k!}\sum_{r=0}^{ck}\binom{ck}{r}\lambda^r\left(\frac{\theta}{2}\right)^{ck-r} t^{2ck-r+1}dx\right]. \end{aligned} \tag{30}$$

3.4 Renyi's Entropy

The entropy of a random variable X is a measure of uncertainty variation. It is an important concept in many fields such as ecology, statistics as indices of diversity, reliability, and also important in science, especially theory of communication, physics, probability, and quantum information, which can be used as a measure of entanglement. The Renyi entropy is studied by survival authors, see for example [33], [34], [24]. [35] and [36] introduced the definition of Renyi entropy as

$$I_\alpha(X) = \frac{1}{1-\alpha}log\left(\int_0^\infty g(x;\theta,\gamma,c,\lambda)^\alpha dx\right), \qquad \alpha > 0 \qquad and \qquad \alpha \neq 1, \tag{31}$$

If $X \sim WLED(\theta,\gamma,c,\lambda)$, from (8), and some simplify we get

$$\begin{aligned} I_\alpha(X) = & \frac{1}{1-\alpha}log\left[\frac{\theta}{\lambda}\gamma^{\alpha(c-1)}\left(\frac{\lambda c}{\gamma^c}\right)^\alpha\left(\frac{1}{\alpha}\right)^{\frac{\alpha(c-1)}{c}}\sum_{i=0}^{\infty}\binom{\alpha}{i}\sum_{j=0}^{\infty}\binom{j}{i}\left(\frac{\lambda}{\theta}\right)^{i-j}(-1)^j \right. \\ & \left. \sum_{k=0}^{\infty}\binom{\frac{i}{2}}{k}\left(\frac{2\gamma}{\theta}\right)^k\left(\frac{y}{\alpha}\right)^{\frac{k}{c}}\left(\frac{\lambda^2}{\theta^2}\right)^k\Gamma\left(\frac{\alpha(c+1)+k-1}{c}\right)\right], bx. \end{aligned} \tag{32}$$

3.5 Mean Residual Life Time

It is natural to research surrogates which not depend on the entire right tail of PDF, such as mean residual lifetime and median residual life, and corresponding residual life quantiles. Several studies have addressed this topic. Firstly, [37] introduced the median residual lifetime and characterization theorem with the application. [38], [39] studied the comparison of two life distributions on the basis of their percentile residual life functions. [40] investigated the median residual lifetime and its properties. [41] studied the nonparametric inference on median residual life function. Recently, [42] investigated the estimation of mean residual life based on ranked set sampling. For the WLED the mean residual lifetime can be written as

$$\begin{aligned} m(t) &= \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x)dx \\ &= \frac{1}{\overline{F}(t)} \left(\int_0^{\infty} \overline{F}(x)dx - \int_0^t \overline{F}(x)dx \right) \\ &= 1 - \sum_{k=0}^{\infty} \frac{-1^k \left(\frac{1}{\gamma}\right)^{ck}}{k!} \left[\sum_{r=0}^{ck} \binom{ck}{r} (\lambda)^r (\frac{\theta}{2})^{ck-r} \frac{(t)^{2ck-r+1}}{2ck-r+1} \right]. \end{aligned} \tag{33}$$

3.6 Mode

[43] defined Mode as the value that occurs most frequently in the data. Some data sets do not have a mode because each value occurs only once. However, some data sets can have more than one mode. This happens because the data set has two or more values of equal frequency which is greater than that of any other value. A measure of central tendency is a single value that attempts to describe a set of data by identifying the central position within that set of data. As such, measures of central tendency are sometimes called measures of central location. They are also classed as summary statistics. The mean (often called the average) is most likely the measure of central tendency that you are most familiar with, but there are others, such as the median and the mode.

The mean, median and mode are all valid measures of central tendency, but under different conditions, some measures of central tendency become more appropriate to use than others. The mode of the WLED is given by equating to zero the derivative with respect to x of the PDF $g(x)$ in (8). From (8), the first derivative of PDF is given by

$$g'(x) = \left(\lambda x + \frac{\theta}{2}x^2\right)\left[\left(\lambda x + \frac{\theta}{2}x^2\right)^{c-1} + \frac{\gamma^{c-1}}{c(\lambda+\theta x)^2}\right] + (c-1). \tag{34}$$

The limit of $g'(x)$ as $x \to 0$ and $x \to \infty$ are given respectively by

$$\lim_{x\to 0} g'(x) = \begin{cases} 0, & c \geq 1 \\ 0, & c = 1 \end{cases}, \qquad \lim_{x\to\infty} g'(x) = \begin{cases} \infty, & c \geq 1 \\ \infty, & c = 1 \end{cases}. \tag{35}$$

Their exist x^* that makes $g'(x)$ called a monotonic function if $c < 1$.

4 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from $WLED(\theta, \gamma, c, \lambda)$ with PDF and CDF given by (8) and (9), respectively. Let $X_{1:n} \leqslant X_{2:n} \leqslant ... \leqslant X_{n:n}$ denote the order statistics obtained from this sample. In this section we provide the expressions for the probability density function of $X_{n:n}$, and the moments of $X_{r:n}, r = 1, 2, ..., n$. Also, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ are presented. The PDF of $X_{r:n}$ is given by

$$g_{r:n}(x) = \frac{1}{B(r, n-r+1)} [G(x;\theta,\gamma,c,\lambda)]^{r-1} [1 - G(x;\theta,\gamma,c,\lambda)]^{n-r} g(x;\theta,\gamma,c,\lambda), \tag{36}$$

where $G(x;\theta,\gamma,c,\lambda)$ and $g(x;\theta,\gamma,c,\lambda)$ are the PDF and CDF of the WLED given by (8) and (9), respectively, and $B(.,.)$, is the beta function. For $x > 0$ and $0 < G(x;\theta,\gamma,c,\lambda) < 1$,we can use the binomial series expansion for $[1 - G(x;\theta,\gamma,c,\lambda)]^{n-r}$, hence

$$g_{r:n}(x) = \frac{1}{B(r, n-r+1)} g(x;\theta,\gamma,c,\lambda) \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [G(x;\theta,\gamma,c,\lambda)]^{r+j-1}. \tag{37}$$

From (8), (9) and (37), we get

$$
\begin{aligned}
g_{r:n}(x) = \frac{1}{B(r,n-r+1)}\left(\frac{c}{\gamma^c}\right)\left(\lambda+\theta x\right)\left(\lambda x+\frac{\theta}{2}x^2\right)^{c-1} exp\left[-\left(\frac{\lambda x+\frac{\theta}{2}x^2}{\gamma}\right)^{c}\right] \\
\times \sum_{j=0}^{n-r}\sum_{k=0}^{r+j-1}\sum_{l=0}^{\infty}\binom{n-r}{j}\left(-1\right)^{j+k}\frac{\left[k\left(\frac{\lambda x+\frac{\theta}{2}x^2}{\gamma}\right)^{c}\right]^{l}}{l!}.
\end{aligned}
\tag{38}
$$

5 Parameters Estimation

In this section, we discuss the estimation of the model parameters θ, γ, c and λ of WLED by using the method of maximum likelihood in the presence of two different types of data. The first type is a complete sample and the second type is the progressive Type-II censored sample.

5.1 The complete sample

Let $X_1, X_2, ..., X_n$ be a simple random sample from WLED with unknown parameter vector $\Omega = (\theta, \gamma, c, \lambda)$. The log-likelihood function $L(x;\Omega)$ for Ω is

$$
\begin{aligned}
l(x;\lambda,\theta,\gamma,c) = \left(n\log c\right) - \left(nc\log\gamma\right) + \left(\sum_{i=1}^{n}\log(\lambda+\theta x)\right) \\
+\left((c-1)\sum_{i=1}^{n}\log\left(\lambda x+(\frac{\theta}{2})x^2\right)\right) - \left(\sum_{i=1}^{n}(\frac{\lambda x+\frac{\theta}{2}x^2}{\gamma})^{c}\right).
\end{aligned}
\tag{39}
$$

The components of the score vector $U(\Omega) = \left(U_\theta, U_\gamma, U_c, U_\lambda\right)^T$ are

$$
\begin{aligned}
U_\theta &= (\sum_{i=1}^{n}\log(\frac{x}{(\lambda+\theta x)})) \\
&+(c-1)\sum_{i=1}^{n}\log(\frac{x^2}{2(\lambda x+(\frac{\theta}{2})x^2)}) \\
&+\sum_{i=1}^{n}(\frac{c}{\gamma^c})(\lambda x+x^2)(\lambda x+(\frac{\theta}{2})x^2)^{c-1} = 0,
\end{aligned}
\tag{40}
$$

$$
U_\gamma = (\frac{-nc}{\gamma}) + n\gamma^{-c(n-1)}\sum_{i=1}^{n}(\lambda x+\frac{\theta}{2}x^2) = 0, \tag{41}
$$

$$
U_c = \frac{n}{c} - n\log\gamma + \sum_{i=0}^{n}\log\left((\lambda x+\frac{\theta}{2}x^2)\right) - c^n\sum_{i=0}^{n}(\frac{(x+\frac{\theta}{2}x^2)}{\gamma})^{c-1} = 0, \tag{42}
$$

$$
\begin{aligned}
U_\lambda &= \sum_{i=1}^{n}\log\left(\frac{\theta x}{\lambda+\theta x}\right) \\
&+(c-1)\sum_{i=0}^{n}\log\left(\frac{(x+\frac{\theta}{2}x^2)}{(\lambda x+\frac{\theta}{2}x^2)}\right) \\
&-\sum_{i=0}^{n}\frac{c}{\gamma^c}(x+\frac{\theta}{2}x^2)(x+\frac{\theta}{2}x^2)^{c-1} = 0.
\end{aligned}
\tag{43}
$$

Setting these expressions to zero, $U(\Omega)=0$, and solving them simultaneously gives the maximum likelihood estimate (MLE) $\hat{\Omega}=\left(\hat{\theta},\hat{\gamma},\hat{c},\hat{\lambda}\right)^T$ of the four parameters. The system of these four nonlinear equations do not have explicit solutions and they have to be obtained numerically. However, mathematical or statistical software should apply to get a numerical solution via iterative techniques such as the Newton–Raphson method. For more details about the iteration algorithm see [44].

For asymptotic interval estimation of the four parameters θ, γ, c and λ, we obtain the observed Fisher information matrix. The observed Fisher information matrix has second partial derivatives of log-likelihood function as the entries, which easily can be obtained. Hence, the observed information matrix is given by

$$I\left(\hat{\Omega}\right)=\begin{pmatrix}-U_{\theta\theta} & -U_{\theta\gamma} & -U_{\theta c} & -U_{\theta\lambda}\\ -U_{\gamma\theta} & -U_{\gamma\gamma} & -U_{\gamma c} & -U_{\gamma\lambda}\\ -U_{c\theta} & -U_{c\gamma} & -U_{cc} & -U_{c\lambda}\\ -U_{\lambda\theta} & -U_{\lambda\gamma} & -U_{\lambda c} & -U_{\lambda\lambda}\end{pmatrix}_{\downarrow\Omega=\hat{\Omega}}. \tag{44}$$

The elements of the main daigonal in observed information matrix are given as follows

$$\begin{aligned}U_{\theta\theta}=&(\sum_{i=1}^{n}\log(\frac{x}{(\lambda+\theta x)}))\\ &+(c-1)\sum_{i=1}^{n}\log(\frac{x^2}{2(\lambda x+(\frac{\theta}{2})x^2)})\\ &+\sum_{i=1}^{n}(\frac{c}{\gamma^c})(\lambda x+x^2)(\lambda x+(\frac{\theta}{2})x^2)^{c-1}=0,\end{aligned} \tag{45}$$

$$U_{\gamma\gamma}=(\frac{-nc}{\gamma})+n\gamma^{-c(n-1)}\sum_{i=1}^{n}(\lambda x+\frac{\theta}{2}x^2)=0, \tag{46}$$

$$\begin{aligned}U_{\lambda\lambda}=&\sum_{i=1}^{n}\log\left(\frac{\theta x}{\lambda+\theta x}\right)\\ &+(c-1)\sum_{i=0}^{n}\log\left(\frac{(x+\frac{\theta}{2}x^2)}{(\lambda x+\frac{\theta}{2}x^2)}\right)\\ &-\sum_{i=0}^{n}\frac{c}{\gamma^c}(x+\frac{\theta}{2}x^2)(x+\frac{\theta}{2}x^2)^{c-1}=0,\end{aligned} \tag{47}$$

$$U_{cc}=\frac{n}{c}-n\log\gamma+\sum_{i=0}^{n}\log\left((\lambda x+\frac{\theta}{2}x^2)\right)-c^n\sum_{i=0}^{n}\left(\frac{(x+\frac{\theta}{2}x^2)}{\gamma}\right)^{c-1}=0. \tag{48}$$

The asymptotic variance-covariance matrix $I^{-1}\left(\hat{\Omega}\right)$ for the MLEs is obtained by inverting the observed information matrix $I\left(\hat{\Omega}\right)$ or equivalent

$$I^{-1}\left(\hat{\Omega}\right)=\begin{pmatrix}var(\hat{\theta}) & Cov(\hat{\theta}\hat{\gamma}) & Cov(\hat{\theta}\hat{c}) & Cov(\hat{\theta}\hat{\lambda})\\ Cov(\hat{\gamma}\hat{\theta}) & var(\hat{\gamma}) & Cov(\hat{\gamma}\hat{c}) & Cov(\hat{\gamma}\hat{\lambda})\\ Cov(\hat{c}\hat{\theta}) & Cov(\hat{c}\hat{\gamma}) & var(\hat{c}) & Cov(\hat{c}\hat{\lambda})\\ Cov(\hat{\lambda}\hat{\theta}) & Cov(\hat{\lambda}\hat{\gamma}) & Cov(\hat{\lambda}\hat{c}) & var(\hat{\lambda})\end{pmatrix}. \tag{49}$$

It is well known that under some regularity conditions, see [45], $\hat{\Omega}$ is approximately distributed as multivariate normal with mean Ω and covariance matrix $I^{-1}\left(\hat{\Omega}\right)$. Thus, the asymptotic $100(1-\delta)\%$ confidence interval (ACI) of Ω can be given by

$$(\hat{\Omega}_L,\hat{\Omega}_U)=\hat{\Omega}\pm z_{\frac{\delta}{2}}\sqrt{var(\hat{\Omega})},\ \hat{\Omega}=\left(\hat{\theta},\hat{\gamma},\hat{c},\hat{\lambda}\right), \tag{50}$$

where $z_{\delta/2}$ is the percentile of the standard normal distribution with right-tail probability $\delta/2$. Furthermore, to construct the asymptotic confidence interval of the reliability function $S(t)$ and hazard function $h(t)$, we need to find the variances of them. In order to calculate the approximate estimates of the variance of $S(t)$ and $h(t)$, we use the delta method discussed by [46]. Hence, the asymptotic $100(1-\delta)\%$ confidence intervals (ACIs) of $S(t)$ and $h(t)$ are

$$\hat{S}(t) \pm z_{\frac{\delta}{2}} \sqrt{var(\hat{S}(t))}, \ \hat{h}(t) \pm z_{\frac{\delta}{2}} \sqrt{var(\hat{h}(t))}. \tag{51}$$

5.2 The progressive Type-II censoring sample

In industrial life testing and medical survival analysis, very often the object of interest is lost or withdrawn before failure, or the object lifetime is only known within an interval. Hence, the obtained sample is called a censored sample or an incomplete sample. Some of the major reasons for the removal of the experimental units are saving the working experimental units for future use, reducing the total time on the test, and lowering the cost associated with these. For this reason, a progressive Type II censoring is a useful scheme in which a specific fraction of individuals at risk may be removed from the experiment at each of several ordered failure times. A thorough overview of the subject of progressive censoring and the excellent review article is given in [47].

A progressively Type II censored sample can be described as follows, Suppose that n independent items are put on a life test with continuous identically distributed failure times $X_1, X_2, ..., X_n$. Suppose further that a censoring scheme $(R_1, R_2, ..., R_m)$ is previously fixed such that immediately following the first failure X_1, R_1 surviving items are removed from the experiment at random, and immediately following the second failure X_2, R_2 surviving items are removed from the experiment at random. This process continues until, at the time of the $m\ th$ observed failure X_m, the remaining R_m surviving items are removed from the test. The m ordered observed failure times denoted by $X_{1:m:n}$, $X_{2:m:n}, ..., X_{m:m:n}$ are called progressively Type II censored of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, ..., R_m)$, see Figure 2. It is clear that $n = m + \sum_{i=1}^{m} R_i$.

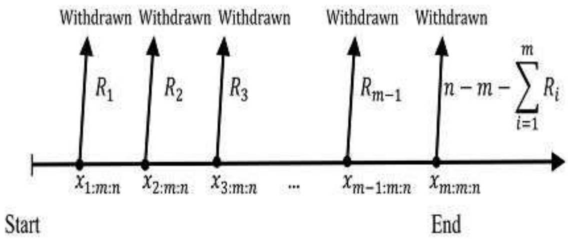


Fig. 2: Diagram of experiment under progressively Type II censored.

Now, let $X_{1:m:n}$, $X_{2:m:n}, ..., X_{m:m:n}$ be a progressively Type II censored sample when the lifetime of the tested unites follow the WLED with unknown parameter vector $\Omega = (\theta, \gamma, c, \lambda)$ and censoring scheme $(R_1, R_2, ..., R_m)$. The log-likelihood function $L(x; \Omega)$ for Ω without normalized constant is

$$f_{x_1, x_2, ..., x_m}(X1, X2, ..., Xm) = A \prod_{i=0}^{m} f(x_i; \gamma; \lambda; \theta; c)[1 - F(x_i; \gamma; \lambda; \theta; c)]^{R_i} \tag{52}$$

Calculating the first partial derivatives of $L(x; \Omega)$ with respect to Ω and equating each to zero, we get the likelihood equations. By solving them simultaneously gives the MLE $\hat{\Omega} = \left(\hat{\theta}, \hat{\gamma}, \hat{c}, \hat{\lambda}\right)^T$ of the four parameters.

Similarly, as in Subsection 5.1 the system of these four nonlinear equations do not have closed form solutions, the Newton-Raphson iteration method is used to obtain the estimates. Also, we can use the Fisher information matrix and delta method to calculate the approximate confidence intervals of Ω, $S(t)$ and $h(t)$.

6 Application to Real-Life Data

In this section, we provide data analysis to demonstrate the performance of the WLED in practice. Consider the real data set in [48] which was analyzed also in [49] and [50]. These data are the duration of remission of 20 leukemia patients who are treated with one drug. Table 1 gives the ordered duration of remission (in years).

Table 1. Lifetimes of 20 patients suffering from Leukemia. .

1.013	1.034	1.109	1.169	1.226	1.509	1.533	1.563	1.716	1.929
1.965	2.061	2.344	2.546	2.626	2.778	2.951	3.413	4.118	5.136

Table 2. MLE, 95% ACI and the interval length using the complete sample.

Parameter	Mean	Interval	Length
γ	1.8352	$[1.2385, 2.4319]$	1.1933
c	1.2593	$[0.9892, 1.5294]$	0.5402
λ	0.1389	$[-0.1252, 0.4031]$	0.5283
θ	0.4657	$[0.2666, 0.6647]$	0.3981
$\boldsymbol{S(0.44)}$	0.9831	$[0.9397, 1.0266]$	0.0869
$\boldsymbol{h(0.44)}$	0.0872	$[-0.0821, 0.2565]$	0.3386

Before progressing further, the 20 values were used to verify that the data set follow WLED. Based on Kolmogorov-Smirnov (K-S) test, we checked the data distribution fit as WLED or not. The calculated value of the K-S test is 0.1313 for the WLED and this value is smaller than their corresponding values expected at 5% significance level, which is 0.2850 at $n = 20$ and P-value equal 0.8808 is quite high. So, we cannot reject the null hypothesis that the data set is coming from WLED. Figure 3 shows the empirical and fitted survival function of the WLED. It is evident that the WLED can be a good model fitting these data.

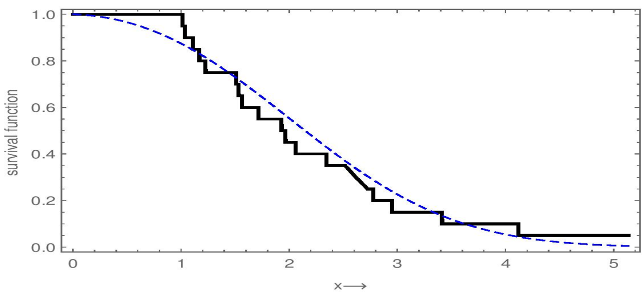


Fig. 3: Empirical and fitted survival function of the WLED

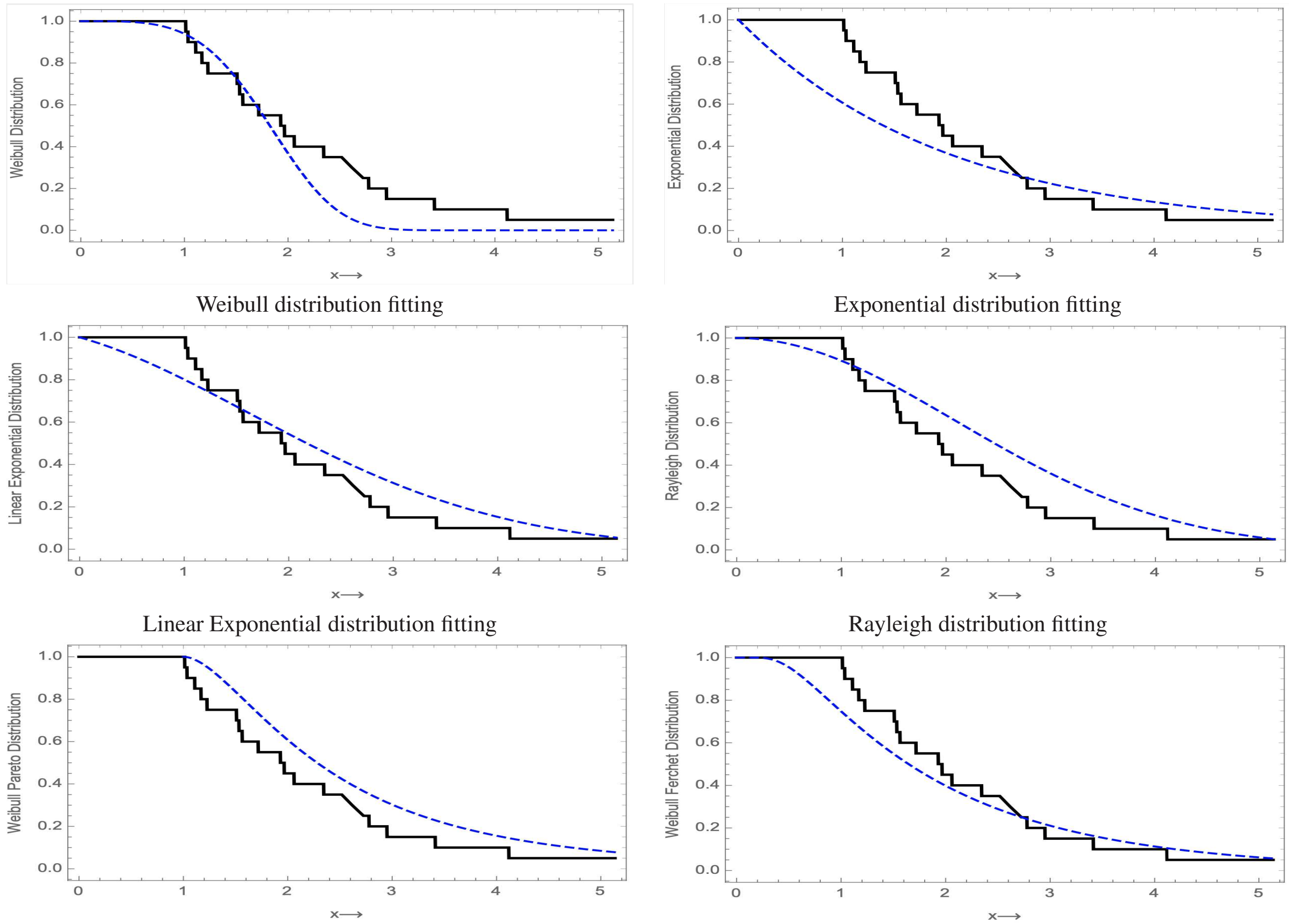
Based on these data the MLE, 95% ACI and the interval length for θ, γ, c, λ, $S(0.44)$ and $h(0.44)$ are computed and presented in Table 2. In our example, let us consider the following progressively Type II censored sample of size $m = 9$ generated randomly from the $n = 20$ observations, using the algorithm described in [51]. The observations and the tree-stage removal pattern applied are reported in Table 3. Using these sample (progressive Type II censored) the MLE, 95% ACI and the interval length for θ, γ, c, λ, $S(0.44)$ and $h(0.44)$ are calculated and presented in Table 4.

Table 3. Progressively censored sample generated from Lifetimes of 20 patients.

i	1	2	3	4	5	6	7	8	9
$x_{i,9,20}$	1.013	1.034	1.109	1.169	1.226	1.509	2.546	2.778	3.413
R_i	3	6	1	1	0	0	0	0	0

Table 4. MLE, 95% ACI and the interval length using progressively censored sample.

Parameter	Mean	Interval	Length
γ	4.5039	$[4.4163, 4.5916]$	0.1753
c	1.0136	$[0.3841, 1.643]$	1.2589
λ	0.2413	$[-0.6268, 1.1094]$	1.7362
θ	0.4552	$[-0.036, 0.9464]$	0.9823
$\boldsymbol{S(0.44)}$	0.9682	$[0.9024, 1.034]$	0.1316
$\boldsymbol{h(0.44)}$	0.0953	$[0.0761, 0.1146]$	0.0385



Weibull distribution fitting   Exponential distribution fitting

Linear Exponential distribution fitting   Rayleigh distribution fitting

Weibull Pareto distribution fitting   Weibull Ferchet distribution fitting

Figure 4. Plots of fitting using real data with many distributions such as, Weibull, Exponential, Linear Exponential, Rayleigh, Weibull Pareto and Weibull Ferchet distributions respectively.

7 Comparison

In this section p-value and k-test are used to provide a comparison between WLED and other distributions. Some of the given distribution depends on one parameter such as Exponential distribution and Rayleigh distribution. On the other hand, there are distributions that have two parameters such as Linear Exponential Distribution. Furthermore, the Weibull Pareto distribution has three parameters. In addition, Weibull Ferchet distribution and Weibull distribution both depend on four parameters.

The K-S and p-value results show that WLED has the lowest k-test value and the greatest p-value, which means that WLED is the best fitting for this data but not at all, as it is shown in Table 5 and Figure 4.

Table 5. The K-S values and p-values for patients data.

Models	p-Value	K-S
WLED	0.8808	0.1313
Weibull	0.0744126	0.277641
Linear Exponential	0.3875	0.2021
Exponential	0.1048	0.2625
Rayleigh	0.2371	0.2225
Weibull Pareto	0.3664	0.2056
Weibull Ferchet	0.1392	0.2581

8 Conclusion

A new four-parameter model called the Weibull Linear Exponential distribution (WLED) is proposed. We provide some of its structural and statistical properties including quartiles, moments, mean waiting time, mean residual lifetime, Renyi entropy, mode, and order statistics. Plots for the PDF, CDF, reliability function, and failure rate function with its discussion are presented. The maximum likelihood method is used for estimating the model parameters as well as some of the lifetime parameters (reliability and failure rate functions). We calculate the observed information matrix in addition to the delta method. Finally, the goodness of fit test using Kolmogorov-Smirnov (K-S) and p-value Statistics for a real-life data set provides better fits than some other distributions. The new distribution can be considered an alternative model to other lifetime distributions which can be fit for modeling positive real data in many fields. We trust our new distribution has the ability to attract large sets of applications in reliability analysis and lifetime data

Abbreviations
WLED: Weibull-Linear Exponential Distribution; K-S: Kolmogorov-Smirnov; PDF: Probability density function; CDF: Cumulative distribution function; MLE: Maximum Likelihood Estimate; ACI: Adaptive Confidence Interval; LED: Linear Exponential Distribution; FIFO: First-In, First-Out; qf: Quantile Function; h(x): Hazard Rate Function; S(x): Survival Function;
Availability of data and materials
The datasets are available in Wiley Online Library at http://doi.org/10.1002/9781118033005 ,[48]

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