Belief Revision in Expressive Knowledge Representation Formalisms

Dissertation

zur Erlangung des akademischen Grades
Doctor of Philosophy (Ph.D)

vorgelegt an der
Technischen Universität Dresden
Fakultät Informatik

eingereicht von
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eingereicht am 26. September 2022
verteidigt am 13. Dezember 2022

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Dresden, im Dezember 2022

Abstract

We live in an era of data and information, where an immeasurable amount of discoveries, findings, events, news, and transactions are generated every second. Governments, companies, or individuals have to employ and process all that data for knowledge-based decision-making (i.e. a decision-making process that uses predetermined criteria to measure and ensure the optimal outcome for a specific topic), which then prompt them to view the knowledge as valuable resource. In this knowledge-based view, the capability to create and utilize knowledge is the key source of an organization or individual's competitive advantage. This dynamic nature of knowledge leads us to the study of belief revision (or belief change), an area which emerged from work in philosophy and then impacted further developments in computer science and artificial intelligence.

In belief revision area, the AGM postulates by Alchourrón, Gärdenfors, and Makinson continue to represent a cornerstone in research related to belief change. Katsuno and Mendelzon (K&M) adopted the AGM postulates for changing belief bases and characterized AGM belief base revision in propositional logic over finite signatures. In this thesis, two research directions are considered. In the first, by considering the semantic point of view, we generalize K&M's approach to the setting of (multiple) base revision in arbitrary Tarskian logics, covering all logics with a classical model-theoretic semantics and hence a wide variety of logics used in knowledge representation and beyond. Our generic formulation applies to various notions of "base", such as belief sets, arbitrary or finite sets of sentences, or single sentences. The core result is a representation theorem showing a two-way correspondence between AGM base revision operators and certain "assignments": functions mapping belief bases to total — yet not transitive — "preference" relations between interpretations. Alongside, we present a companion result for the case when the AGM postulate of syntax-independence is abandoned. We also provide a characterization of all logics for which our result can be strengthened to assignments producing transitive preference relations (as in K&M's original work), giving rise to two more representation theorems for such logics, according to syntax dependence vs. independence. The second research direction in this thesis explores two approaches for revising description logic knowledge bases under fixed-domain semantics, namely model-based approach and individual-based approach. In this logical setting, models of the knowledge bases can be enumerated and can be computed to produce the revision result, semantically. We show a characterization of the AGM revision operator for this logic and present a concrete model-based revision approach via distance between interpretations. In addition, by weakening the KB based on certain domain elements, a novel individual-based revision operator is provided as an alternative approach.

Acknowledgements

ALHAMDULILLAH, all praises and thanks to Allah.

I thank LPDP (INDONESIA ENDOWMENT FUND FOR EDUCATION) for providing such a great opportunity with the four years scholarship funding to pursue a PhD in Germany.

I would like to express my gratitude to my supervisor SEBASTIAN RUDOLPH for his support, insightful discussions and comments, patience, wisdom, as well as extra financial support via university funding and SCADS.AI project after the scholarship run out. I will never regret my first meeting with him in 2016 Computational Logic Summer School in Thailand. I would say that he has all best qualities as a supervisor. He has my long life respect and admiration.

I am especially indebted to KAI SAUERWALD for his great help, sound advice, supportive collaboration, and interesting discussions on belief revision.

I am fortunate and honored to work at the Computational Logic group. Thank you to all members of the group for providing kind environment, especially to HANNES STRASS, JONAS KARGE, TIM LYON for proofreading this dissertation, and LUKAS SCHWEIZER for helpful discussions on fixed-domain semantics DL.

I would also like to thank GABRIELE KERN-ISBERNER for taking time to review this dissertation and to come to Dresden from Dortmund for my defense. I thank all doctoral committee members.

Moreover, I also thank my Indonesian friends in Germany for the joy and for helping me in difficult times, especially ADRIAN NURADIANSYAH and SATYADHARMA TIRTARASA for proofreading this dissertation and helping me answered logic-related questions.

And last but never least, I thank my family for endless support, pray, and patience.

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Chapter 1

Introduction

We live in an era of data and information, where an immeasurable amount of discoveries, findings, events, news, and transactions are generated every second. Governments, companies, or individuals have to employ and process all that data for knowledge-based decision-making (i.e. a decision-making process that uses predetermined criteria to measure and ensure the optimal outcome for a specific topic), which then prompt them to view the knowledge as valuable resource. In this knowledge-based view, the capability to create and utilize knowledge is the key source of an organization or individual's competitive advantage [TPW03]. This dynamic nature of knowledge leads us to the study of belief revision (or belief change, cf. Section 1.1), an area which emerged from work in philosophy and then impacted further developments in computer science and artificial intelligence.

In belief revision area, knowledge are represented in many forms ranging from databases [FUV83; Bor85; Wil97; GPW03], classical propositional logic [AGM85; Han99; KM91; Dal88a], to the more expressive data representation such as ontologies [QLB06b; HK06a; DDL17; RWF+13; RW14a; Flo06]. The change process is defined as a function or an operator, where the inputs are the knowledge that is specified in one the forms mentioned earlier and the output is a new knowledge in the same form of the inputs. The forms are usually called the knowledge bases or knowledge sets. To be a rational revision operator, the change function should satisfy a set of constraints called postulates. Beyond the constraints, there is no hint on how to construct a specific revision operator. Overall, the methodology in studying belief revision is to approach a change operator from two directions: (1) a set of *postulates* to define the qualities of a change operator that should be satisfied, and (2) a formal *construction* to characterize the collection of instances of the change operator. Then, a *representation theorem* is provided to show that these two approaches coincide, i.e. both approaches capture the same class of operators.

When we focus on the construction side, we will find many equivalent ways in the literature to characterize the revision operators, which generally can be categorized into two types: (1) syntax-based approaches (or formula-based approaches, e.g. partial meet [AGM85; RW14b], kernel [Han94; RW14b; QHH+08], and epistemic entrentchment [GM88; Fer00; Rot03]) and (2) semantics-based approaches (or model-based approaches, e.g. spheres system [Gro88] and faithful assignment [KM91]). A syntax-based approach includes modification on sentences or formula of the knowledge formalism, whereas a semantics-based approach performs calculation on the interpretation or possible worlds level. However, when working with ontologies, a syntax-based approach is lack of suitable semantic justification in general [WWT15] and for two different ontologies $\mathcal{O}$ and $\mathcal{O}'$ that have same meaning, revision may lead to different results [DDL17]. On the opposite side, in the semantics-based approach, the result of the revision is independent of the initial input knowledge, which is known as Dalal's Principle of Irrelevance of Snytax [Dal88b]. This principle follows the argument by Levesque [Lev84] that one should define the operations on knowledge bases at the knowledge (semantics) level. This mainly motivates us to approach belief revision problem more from the semantics point of view. We will investigate the semantics approaches for logics which cover a wide range of modern (and expressive) knowledge representation formalisms, namely Tarskian logics.

In the later part of this thesis, we consider Description Logics (DLs) to be investigated as they provide logical formalisms for the ontologies and the Semantic Web that have been standardized since 2004. As well as for Tarskian logics, we study the semantics-based revision approaches for DLs. We found that a number of works have been carried out to find specific semantic revision approaches for particular DL families [WWT10; CSG14; WWQ+14; WWT15; ZWW+14; DDL17]. However, as they consider standard semantics for DLs, they are all struggling with infinitely many models and inexpressibility problems. This mainly motivates us to explore particular DLs with non-standard semantics, called fixed-domain semantics [GRS16] (cf. Section 1.2), where the domain is fixed and finite, providing a feasible computation over models.

1.1 Belief Revision

Belief revision can be described as the problem of how a rational agent should change her beliefs in the light of new information. An agent can be a human being, a computer machine, or any kind of system which is able to store beliefs and perform reasoning over beliefs. The set of information or knowledge that is believed by an agent is represented as a *belief state*. This problem is crucial in several areas, including AI systems.

We provide an example where an agent's belief state contains the following pieces of information, where the knowledge are represented in classical propositional logic: "Alice is a university student" (p), "Only few people booked the appointment for vaccination from the university" (q), "If Alice is a university student and there are only few people who booked the appointment for vaccination from the university, then Alice is able to book the vaccination appointment" ($p \wedge q \rightarrow r$), and "The appointment booking system is developed by the university IT department" (s).

From the belief state, one can infer the following consequence: "Alice is able to book the appointment for vaccination" (r). However, suppose it turns out that *in fact* Alice could not book an appointment slot ($\neg r$) and we want to add the incoming fact to the belief state. Unfortunately, the belief state would be inconsistent, i.e. there occurs a contradiction among sentences. Since the inconsistent belief state would be "useless" (one can infer any sentence from it), we want to maintain its consistency – we need to *revise* it. This is exactly our main focus on belief revision. In consequence, some sentence(s) should be removed from the belief state. When removing sentences, we want to keep *the minimality principle*, i.e. we do not want to retract too much information from the belief state, or in other words, we want to do the changes as minimal as possible. If a sentence at the end should be dropped, then the sentence must be really involved in the inconsistency. If a previously believed sentence does not contradict the new information, then there is no reason to give it up.

The many possibilities of choosing which sentence(s) to give up (or to retain) manifest the non-triviality of the belief revision problem. For example, to revise our belief state, we can choose which sentence(s) to retract: p, q, or $p \wedge q \rightarrow r$. Of course, one can decide to drop all the three sentences, but she will lose all of the information and thus violate the principle of minimal change. Choosing one of the three sentences is sufficient to maintain the consistency of the belief state. Assuming unavailable information as false (closed world assumption), if we trust the conditional statement ($p \wedge q \rightarrow r$), we can still believe that either Alice is not a university student or there are too many people that book the appointment. On the other hand, if we decide that the rule $p \wedge q \rightarrow r$ can not be trusted any longer, we can keep both facts that Alice is a university student and there are only few people who book the slot. Note also that the sentence s is not relevant to the inconsistency. Then, by the minimality principle, one should not touch it for modification of the initial belief state.

This area of research has been massively influenced by the AGM paradigm of Alchourrón, Gärdenfors, and Makinson [AGM85]. The AGM theory assumes that an agent's beliefs are represented by a deductively closed set of sentences (commonly referred to as a *belief set*). A revision operator for belief sets is required to satisfy appropriate postulates in order to qualify as a rational revision operator. While the contribution of AGM is widely accepted as solid and inspiring foundation, it lacks support for certain relevant aspects: it provides no immediate solution on how to deal with multiple inputs (i.e., several sentences instead of just one), with *bases* (i.e., arbitrary collections of sentences, not necessarily deductively closed), or with the problem of iterated belief revision.

Katsuno and Mendelzon [KM91] – henceforth abbreviated *K&M* – deal with the issues of belief bases and multiple inputs in an elegant way: as in propositional logic, every set of sentences (including an infinite one) is equivalent to one single sentence, belief states and multiple inputs are considered as such single sentences. In this setting, K&M provided a set of postulates, derived from (and equivalent to) the AGM revision postulates, capturing the principles mentioned earlier in the example. They nicely characterized the AGM revision operator via mappings of every propositional logic knowledge base to a specific relation over interpretations.

While the AGM paradigm is axiomatic, much of its success originated from operationalisations via representation theorems. Yet, most existing characterizations of AGM revision require the underlying logic to fulfil additional assumptions such as compactness, closure under standard connectives, deduction, or supra-classicality [RWF+13]. Leaving the safe grounds of these assumptions complicates matters and gives rise to the main challenge: representation theorems do not easily generalize to arbitrary logics. This has sparked investigations into tailored characterizations of AGM belief change for non-classical logics, such as the work by Ribeiro, Wassermann, and colleagues [RWF+13; Rib13; RW14a], Delgrande, Peppas, and Woltran [DPW18], Pardo, Dellunde, and Godo [PDG09], or Aiguier, Atif, Bloch, and Hudelot [AAB+18]. Approaches to specific logics were also proposed, such as Horn logic [DP15], temporal logics [Bon07], action logics [SPL+11], first-order logic [ZWW+19], and description logics (DLs) [QLB06b; HK06a; DDL17].

1.2 Description Logics under Fixed-Domain Semantics

Description Logics (DLs) are a family of knowledge representation formalisms that can be used to represent knowledge of an application domain in a structured and well-understood way [BHL+17]. DLs are more expressive than classical propositional logic, but are still decidable, as opposed to first order logic. With these important properties, they serve as foundation for the popular Web Ontology Language (OWL) [Gro12] for exchanging knowledge to support the Semantic Web Technologies. Many DL-based ontologies are being extensively developed and used in the biology and medical areas, such as NCI Thesaurus[1], the Foundational Model of Anatomy (FMA)[2], SNOMED CT[3], GALEN[4], and GENE ONTOLOGY[5].

Furthermore, *fixed-domain semantics* for description logics have been introduced to accommodate the scenario where the knowledge bases represent constraint-type or configuration problems [GRS16; RS17]. In this setting, the domain is explicitly given and thus is finite and fixed *a priori*. Consequently, this particular DL accomodates the use of knowledge bases adapting a closed-world assumption, i.e. if an information is not in the knowledge

[1]https://ncithesaurus.nci.nih.gov/ncitbrowser/
[2]https://bioportal.bioontology.org/ontologies/FMA
[3]https://www.snomed.org/
[4]https://bioportal.bioontology.org/ontologies/GALEN
[5]http://geneontology.org/docs/download-ontology/

base, it should be concluded that it is false. A reasoner called `Wolpertinger`[6] has been developed to support typical reasoning tasks over knowledge bases under the fixed-domain semantics, which includes satisfiability checking and model enumeration. With respect to the revision problem, these features would enable us to directly deal with the models of the knowledge bases to obtain the revision outcome, as opposed to description logics with standard semantics. However, to the best of our knowledge, the study of belief revision in description logics under this setting has not been explored so far.

1.3 Research Directions

In view of the above challenges in the area of belief revision (cf. Section 1.1 and Section 1.2), the main goal of the present work is to find:

1. Appropriate characterizations of AGM revision operators in general logics satisfying monotonicity. In this regard, we consider *Tarskian logics*, a class of logics capturing many well-known logical formalisms, such as classical propositional logic, first-order and second-order predicate logic, modal logics, and description logics. Our considerations do, however, **not** apply to non-monotonic formalisms, such as default logic, circumscription, or logic programming frameworks using negation as failure.
2. Approaches and concrete revision operations for description logics under fixed-domain semantics. As DLs are also Tarskian logics, results from the first research direction can be applied to investigate the characterization of revision operators in this logical setting and then be used to develop their instances. The finiteness of the domain is also an interesting property to be explored towards alternative revision approaches.

1.4 Contributions and Outline of the Thesis

Chapter 1 motivates this work, discusses its background, and outlines its objectives. Chapter 2 introduces the basic logical setting for AGM revision and notions used in this thesis, including: description logics, fixed-domain semantics, and Answer Set Programming (ASP). Chapter 3 is dedicated to exploring the first research direction. It contains the following contributions:

- We introduce the notion of *base logics* to uniformly capture all popular ways of defining belief states by certain sets of sentences over Tarskian logics. Among others, this includes the cases where belief states are *arbitrary sets of sentences* and where belief states are *belief sets*.

[6] https://github.com/wolpertinger-reasoner

- We extend K&M's semantic approach from the setting of singular base revision in propositional logic to multiple base revision in arbitrary base logics.
- For this setting, we provide a representation theorem characterizing AGM belief change operators via appropriate assignments.
- We provide a variant of the characterization dealing with the case where the postulate of syntax-independence is not imposed.
- We characterize all those logics for which every AGM operator can even be captured by preorder assignments (i.e., in the classical K&M way). In particular, this condition applies to all logics supporting disjunction and hence all classical logics. For those logics, we provide one representation theorem for the syntax-independent and one for the syntax-dependent setting.

The content of Chapter 3, some parts of Chapter 1 and some parts of Chapter 2 were developed in joint work with Kai Sauerwald and Sebastian Rudolph which has been peer-reviewed at RuleML+RR conference:

- Faiq Miftakhul Falakh, Sebastian Rudolph, Kai Sauerwald: 'Semantic Characterizations of AGM Revision for Arbitrary Tarskian Logics'. In *Proceeding of the 6th International Joint Conference on Rules and Reasoning*, Berlin (Virtual), Germany, 2022. (Best Student Paper Awards)

A preliminary report of this joint work has been peer-reviewed and published at the FCR Workshop 2021 [FRS21]:

- Faiq Miftakhul Falakh, Sebastian Rudolph, Kai Sauerwald: 'A KatsunoMendelzon-Style Characterization of AGM Belief Base Revision for Arbitrary Monotonic Logics (Preliminary Report)'. In *Proceedings of the 7th Workshop on Formal and Cognitive Reasoning co-located with the 44th German Conference on Artificial Intelligence (KI 2021), September 28, 2021*.

My specific contribution was to extend the idea, develop examples, and conduct most proofs of the main representation theorem in this chapter (Section 3.1–Section 3.5). For the remaining sections of this chapter (Section 3.6–Section 3.10), while Kai Sauerwald developed the notions introduced here (e.g. critical loop), conceptualized and conducted the proofs of the propositions, I provided support in proofreading and making proposals for the presentation of the proofs. I also initiated the discussion of related work and developed the comparison table in Section 3.9.

Chapter 4 discusses the second research direction. In this chapter, we show the semantic representation theorem for knowledge base revision in DL under the fixed-domain semantics. Alongside, we present two concrete approaches for revising the knowledge bases. The first approach is a semantics-based revision approach, which is inspired from the approach by Katsuno and Mendelzon [KM91] for revising KBs in finite-signature propositional logic. We provide a representation theorem characterizing AGM revision operators via appropriate assignments. We also provide a concrete revision operator using the notion of distance between interpretations and show that the proposed operator satisfies all standard AGM postulates for DL. The models as the outcome of this operation are expressed in a knowledge base using our axiom constructor. The second approach is a novel revision operator based on the notion of exceptional individual set. This individual set serves as a basis to weaken the prior KB whenever inconsistency occurs. The revision result of this approach is a union of the weakened prior KB with the new incoming KB. The content of this chapter has been peer-reviewed at DL Workshop 2022:

- Faiq Miftakhul Falakh, Sebastian Rudolph: 'AGM Revision in Description Logics under Fixed-Domain Semantics'. In *Proceeding of the 35th International Workshop on Description Logics*, Haifa, Israel, 2022.

Finally, Chapter 5 summarizes the contributions of this work and discusses some ideas for future work.

Chapter 2

Preliminaries

We introduce the logical and algebraic notions used in the thesis.

2.1 Logics with Classical Model-Theoretic Semantics

We consider logics endowed with a classical model-theoretic semantics. The syntax of such a logic $\mathbb{L}$ is given syntactically by a (possibly infinite) set $\mathcal{L}$ of *sentences*, while its model theory is provided by specifying a (potentially infinite) class Ω of *interpretations* (also called *worlds*) and a binary relation $\models$ between Ω and $\mathcal{L}$ where $\mathcal{I} \models \varphi$ indicates that $\mathcal{I}$ is a model of φ. Hence, a logic $\mathbb{L}$ is identified by the triple $(\mathcal{L}, \Omega, \models)$. We let $[\![\varphi]\!] = \{\mathcal{I} \in \Omega \mid \mathcal{I} \models \varphi\}$ denote the set of all models of $\varphi \in \mathcal{L}$. Logical entailment is defined as usual (overloading "$\models$") via models: for two sentences φ and ψ we say φ *entails* ψ (written $\varphi \models \psi$) if $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

Notions of modelhood and entailment can be easily lifted from single sentences to sets. We obtain the models of a set $\mathcal{K} \subseteq \mathcal{L}$ of sentences via $[\![\mathcal{K}]\!] = \bigcap_{\varphi \in \mathcal{K}} [\![\varphi]\!]$. For $\mathcal{K} \subseteq \mathcal{L}$ and $\mathcal{K}' \subseteq \mathcal{L}$ we say $\mathcal{K}$ *entails* $\mathcal{K}'$ (written $\mathcal{K} \models \mathcal{K}'$) if $[\![\mathcal{K}]\!] \subseteq [\![\mathcal{K}']\!]$. We write $\mathcal{K} \equiv \mathcal{K}'$ to express $[\![\mathcal{K}]\!] = [\![\mathcal{K}']\!]$. A (set of) sentence(s) is called *consistent with* another (set of) sentence(s) if the two have models in common. Unlike many other belief revision frameworks, we impose no further requirements on $\mathcal{L}$ (like closure under certain operators or compactness).

The existence of a classical model-theoretic semantics as above is equivalent to the logic being *Tarskian* [Tar56; SSC97]. We start by providing the definition of Tarskian logics.

Definition 2.1. *Let $\mathcal{L}$ be a set. A function $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is called a* Tarskian *consequence operator (on $\mathcal{L}$) if it is a closure operator, i.e., it satisfies the following properties for all subsets $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{L}$:*

$$\mathcal{K} \subseteq Cn(\mathcal{K}) \qquad \text{(extensive)}$$

$$\textit{if } \mathcal{K}_1 \subseteq \mathcal{K}_2, \textit{ then } Cn(\mathcal{K}_1) \subseteq Cn(\mathcal{K}_2) \qquad \text{(monotone)}$$

$$Cn(\mathcal{K}) = Cn(Cn(\mathcal{K})) \qquad \text{(idempotent)}$$

Any Tarskian consequence operator $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ gives rise to a Tarskian consequence relation $\Vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ *defined by $\mathcal{K} \Vdash \varphi$ if $\varphi \in Cn(\mathcal{K})$. Each $(\mathcal{L}, \Vdash)$ obtained from a Tarskian consequence operator $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ will be called a* Tarskian logic *here.*

We proceed to show that the existence of a model-theoretically defined semantics is sufficient and necessary for a logic being Tarskian.

Proposition 2.2. *For every model theory $(\mathcal{L}, \Omega, \models)$ there exists a Tarskian logic $(\mathcal{L}, \Vdash)$ with $\mathcal{K} \Vdash \varphi$ if and only if $\mathcal{K} \models \varphi$ for all $\varphi \in \mathcal{L}$ and $\mathcal{K} \in \mathcal{P}(\mathcal{L})$.*

Proof. Given $(\mathcal{L}, \Omega, \models)$, let $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ be defined by $\mathcal{K} \mapsto \{\varphi \in \mathcal{L} \mid [\![\mathcal{K}]\!] \subseteq [\![\varphi]\!]\}$. We will show that Cn is a Tarskian consequence operator.

For extensivity, consider some arbitrary $\psi \in \mathcal{K}$. Then we obtain $[\![\mathcal{K}]\!] = \bigcap_{\varphi \in \mathcal{K}} [\![\varphi]\!] \subseteq [\![\psi]\!]$ and hence $\psi \in Cn(\mathcal{K})$. Hence, since ψ was chosen arbitrarily, we obtain $\mathcal{K} \subseteq Cn(\mathcal{K})$.

For monotonicity, suppose $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then $[\![K_2]\!] = \bigcap_{\varphi \in \mathcal{K}_2} [\![\varphi]\!] \subseteq \bigcap_{\varphi \in \mathcal{K}_1} [\![\varphi]\!] = [\![K_1]\!]$. Therefore, we obtain $Cn(\mathcal{K}_1) = \{\varphi \in \mathcal{L} \mid [\![\mathcal{K}_1]\!] \subseteq [\![\varphi]\!]\} \subseteq \{\varphi \in \mathcal{L} \mid [\![\mathcal{K}_2]\!] \subseteq [\![\varphi]\!]\} = Cn(\mathcal{K}_2)$.

For idempotency, we show bidirectional inclusion. $Cn(\mathcal{K}) \subseteq Cn(Cn(\mathcal{K}))$ is an immediate consequence of extensivity already shown. For the other direction, consider an arbitrary $\psi \in Cn(Cn(\mathcal{K}))$. Then, we obtain $[\![Cn(\mathcal{K})]\!] \subseteq [\![\psi]\!]$. On the other hand, we have

$$[\![Cn(\mathcal{K})]\!] = \bigcap_{\substack{\varphi \in \mathcal{L} \\ [\![\mathcal{K}]\!] \subseteq [\![\varphi]\!]}} [\![\varphi]\!] = \bigcap_{\varphi \in \mathcal{K}} [\![\varphi]\!] = [\![\mathcal{K}]\!],$$

and therefore, we obtain $[\![\mathcal{K}]\!] \subseteq [\![\psi]\!]$ and finally $\psi \in Cn(\mathcal{K})$. Hence, since ψ was chosen arbitrarily, we obtain $Cn(Cn(\mathcal{K})) \subseteq Cn(\mathcal{K})$.

Let now $\Vdash$ denote the Tarskian consequence relation induced by Cn. Then we obtain for all $\mathcal{K} \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ the following:

$$\mathcal{K} \Vdash \varphi \iff \varphi \in Cn(\mathcal{K}) \iff [\![\mathcal{K}]\!] \subseteq [\![\varphi]\!] \iff \mathcal{K} \models \varphi. \qquad \square$$

As last step, we show that for each Tarskian logic there is a canonical model-theoretic semantics for this Tarskian logic.

Proposition 2.3. *For every Tarskian logic $(\mathcal{L}, \Vdash)$ there exists a model theory $(\mathcal{L}, \Omega, \models)$ such that $\mathcal{K} \Vdash \varphi$ if and only if $\mathcal{K} \models \varphi$ holds for all for all $\varphi \in \mathcal{L}$ and $\mathcal{K} \in \mathcal{P}(\mathcal{L})$.*

Proof. Let $(\mathcal{L}, \Vdash)$ be a Tarskian logic and let $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ be the corresponding Tarskian consequence operator. We now define an appropriate $(\mathcal{L}, \Omega, \models)$ as follows: Let $\Omega = \{Cn(T) \mid T \subseteq \mathcal{L}\}$. Define the model relation $\models \subseteq \Omega \times \mathcal{L}$ such that some $Cn(T) \in \Omega$ is a model of some $\varphi \in \mathcal{L}$ whenever $\varphi \in Cn(T)$.

Then we obtain for all $\mathcal{K} \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ the following:

$$\begin{aligned}
\mathcal{K} \models \varphi \quad &\Longleftrightarrow \quad [\![\mathcal{K}]\!] \subseteq [\![\varphi]\!] \\
&\Longleftrightarrow \quad \textstyle\bigcap_{\kappa \in \mathcal{K}} [\![\kappa]\!] \subseteq [\![\varphi]\!] \\
&\Longleftrightarrow \quad \{Cn(T) \mid T \subseteq \mathcal{L},\ \mathcal{K} \subseteq Cn(T)\} \subseteq \{Cn(T) \mid T \subseteq \mathcal{L},\ \varphi \in Cn(T)\} \\
&\Longleftrightarrow \quad \forall T \subseteq \mathcal{L} : \mathcal{K} \subseteq Cn(T) \Rightarrow \varphi \in Cn(T) \qquad (*)
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
(*) \quad &\Longrightarrow \quad \mathcal{K} \subseteq Cn(\mathcal{K}) \Rightarrow \varphi \in Cn(\mathcal{K}) && \text{instantiate } T = \mathcal{K} \\
&\Longrightarrow \quad \varphi \in Cn(\mathcal{K}) && \text{extensivity of } Cn \\
&\Longrightarrow \quad \mathcal{K} \Vdash\!\models \varphi,
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\mathcal{K} \Vdash\!\models \varphi \quad &\Longrightarrow \quad \varphi \in Cn(\mathcal{K}) \\
&\Longrightarrow \quad \forall S \subseteq \mathcal{L} : Cn(\mathcal{K}) \subseteq S \Rightarrow \varphi \in S \\
&\Longrightarrow \quad \forall T \subseteq \mathcal{L} : Cn(\mathcal{K}) \subseteq Cn(T) \Rightarrow \varphi \in Cn(T) && \text{restriction to closed sets} \\
&\Longrightarrow \quad \forall T \subseteq \mathcal{L} : Cn(\mathcal{K}) \subseteq Cn(Cn(T)) \Rightarrow \varphi \in Cn(T) && \text{idempotency of } Cn \\
&\Longrightarrow \quad (*) && \text{monotonicity of } Cn
\end{aligned}$$

Concluding, we have established that for all $\mathcal{K} \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$ the following holds:

$$\mathcal{K} \models \varphi \Longleftrightarrow (*) \Longleftrightarrow \mathcal{K} \Vdash\!\models \varphi. \qquad \square$$

Among others, this means that all logics considered here are monotonic, i.e., they satisfy the following condition:

$$\text{If } \mathcal{K}_1 \models \varphi \text{ and } \mathcal{K}_1 \subseteq \mathcal{K}_2, \text{ then } \mathcal{K}_2 \models \varphi. \qquad \text{(monotonicity)}$$

The notion of *Tarskian logic* captures many well-known classical logical formalisms and in the following we will give some examples.

We start by providing an example, where sentences and interpretations are finite sets, which allows us to specify them (as well as the models relation) explicitly. We note that this is an extension of an example given by Delgrande et al. [DPW18], which will serve as a running example throughout this thesis.

Example 2.4 (based on [DPW18]). *Let* $\mathbb{L}_{\mathrm{Ex}} = (\mathcal{L}_{\mathrm{Ex}}, \Omega_{\mathrm{Ex}}, \models_{\mathrm{Ex}})$ *be the logic defined by* $\mathcal{L}_{\mathrm{Ex}} = \{\psi_0, \ldots, \psi_5, \varphi_0, \varphi_1, \varphi_2, \chi, \chi'\}$ *and* $\Omega_{\mathrm{Ex}} = \{\omega_0, \ldots, \omega_5\}$*, with the models relation* $\models_{\mathrm{Ex}}$ *implicitly given by:*

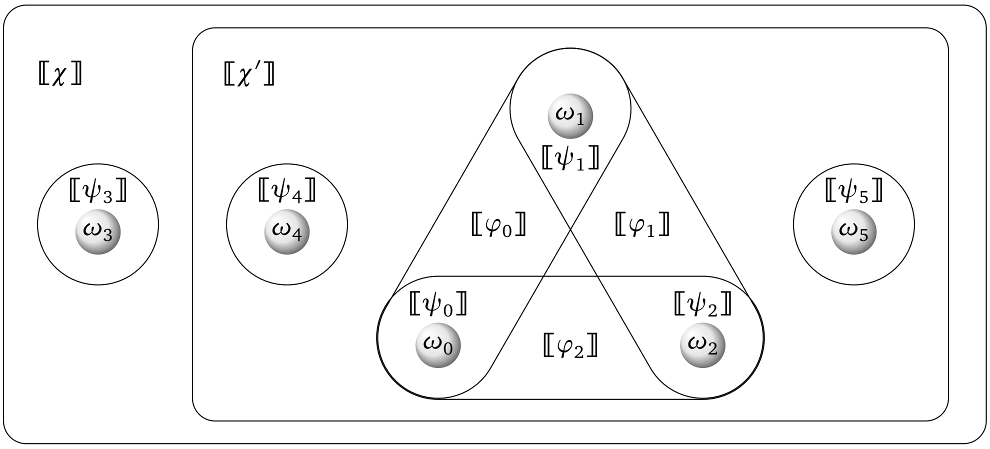


Figure 2.5: Illustration of the logic $\mathbb{L}_{\mathrm{Ex}}$, including the modelhood relations where the solid borders represent the set of models.

$$
\begin{aligned}
[\![\psi_i]\!] &= \{\omega_i\} & [\![\varphi_0]\!] &= \{\omega_0, \omega_1\} \\
[\![\chi]\!] &= \{\omega_0, \ldots, \omega_5\} & [\![\varphi_1]\!] &= \{\omega_1, \omega_2\} \\
[\![\chi']\!] &= \{\omega_0, \omega_1, \omega_2, \omega_4, \omega_5\} & [\![\varphi_2]\!] &= \{\omega_2, \omega_0\}
\end{aligned}
$$

Since $\mathbb{L}_{\mathrm{Ex}}$ is defined in the classical model-theoretic way, $\mathbb{L}_{\mathrm{Ex}}$ is a Tarskian logic. Note that logic $\mathbb{L}_{\mathrm{Ex}}$ has no connectives. Figure 2.5 illustrates the logic setting $\mathbb{L}_{\mathrm{Ex}}$.

Next we turn to propositional logic, observing that the distinction between a finite or infinite set of propositional symbols leads to differences that we will revisit later on.

Example 2.6 ($\mathbb{PL}_n$, propositional logic over n propositional atoms). *The logic is defined by $\mathbb{PL}_n = (\mathcal{L}_{\mathbb{PL}_n}, \Omega_{\mathbb{PL}_n}, \models_{\mathbb{PL}_n})$ in the usual way: Given a finite set $\Sigma_{\mathrm{p}} = \{p_1, \ldots, p_n\}$ of atomic propositions, we let $\mathcal{L}_{\mathbb{PL}_n}$ be the set of Boolean expressions built from $\Sigma_{\mathrm{p}} \cup \{\top, \bot\}$ using the usual set of propositional connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$). We then let the set $\Omega_{\mathbb{PL}_n}$ of interpretations contain all functions from Σ_{p} to $\{$**true**,**false**$\}$. The relation $\models_{\mathbb{PL}_n}$ is then defined inductively over the structure of sentences in the usual way.*

Notably, finiteness of Σ implies finiteness of $\Omega_{\mathbb{PL}_n}$ (more specifically, $|\Omega_{\mathbb{PL}_n}| = 2^n$). This in turn ensures that, despite $\mathcal{L}_{\mathbb{PL}_n}$ being infinite, there are only finitely many (namely 2^n) sentences which are pairwise semantically distinct. Even more so: for every (finite or infinite) set $\mathcal{K}$ of $\mathbb{PL}_n$ sentences, there exists some sentence $\varphi \in \mathcal{L}_{\mathbb{PL}_n}$ with $\varphi \equiv \mathcal{K}$.

Example 2.7 ($\mathbb{PL}_\infty$, propositional logic over infinite signature). *The basic definitions for $\mathbb{PL}_\infty = (\mathcal{L}_{\mathbb{PL}_\infty}, \Omega_{\mathbb{PL}_\infty}, \models_{\mathbb{PL}_\infty})$ are just like in the previous example, with the notable difference of $\Sigma_{\mathrm{p}} = \{p_1, p_2, \ldots\}$ being countably infinite. This implies immediately that $\Omega_{\mathbb{PL}_\infty}$ is infinite (in fact, even uncountable), implying that there are infinitely many sentences that are pairwise* *non-equivalent (e.g., all the atomic ones). Also, there exist infinite sets of sentences for which no single equivalent sentence from* $\mathcal{L}_{\mathbb{PL}_\infty}$ *exists, e.g.,* $\{p_2, p_4, p_6, \ldots\}$.

Many more (and more expressive) logics are captured by the model-theoretic framework assumed by us, e.g. first-order and second-order predicate logic, modal logics, and description logics.

2.2 Relation over Interpretations

For describing belief revision on the semantic level, it is expedient to endow the interpretation space Ω with some structure. In particular, we will employ binary relations $\preceq$ over Ω (formally: $\preceq \subseteq \Omega \times \Omega$), where the intuitive meaning of $\mathcal{I}_1 \preceq \mathcal{I}_2$ is that $\mathcal{I}_1$ is "equally good or better" than $\mathcal{I}_2$ when it comes to serving as a model. We call $\preceq$ *total* if $\mathcal{I}_1 \preceq \mathcal{I}_2$ or $\mathcal{I}_2 \preceq \mathcal{I}_1$ for any $\mathcal{I}_1, \mathcal{I}_2 \in \Omega$ holds. We write $\mathcal{I}_1 \prec \mathcal{I}_2$ as a shorthand, whenever $\mathcal{I}_1 \preceq \mathcal{I}_2$ and $\mathcal{I}_2 \not\preceq \mathcal{I}_1$ (the intuition being that $\mathcal{I}_1$ is "strictly better" than $\mathcal{I}_2$). For a selection $\Omega' \subseteq \Omega$ of interpretations, an $\mathcal{I} \in \Omega'$ is called $\preceq$*-minimal in* Ω' if there is no $\mathcal{I}'' \in \Omega'$ with $\mathcal{I}'' \prec \mathcal{I}$.[1] We let $\min(\Omega', \preceq)$ denote the set of $\preceq$-minimal interpretations in Ω'. We call $\preceq$ a *preorder* if it is transitive and reflexive. For a relation $R \subseteq \Omega \times \Omega$, the transitive closure of R is the relation $TC(R) = \bigcup_{i=0}^{\infty} R^i$, where $R^0 = R$ and $R^{i+1} = R^i \cup \{(\mathcal{I}_1, \mathcal{I}_3) \mid \exists \mathcal{I}_2.(\mathcal{I}_1, \mathcal{I}_2) \in R^i \text{ and } (\mathcal{I}_2, \mathcal{I}_3) \in R^i\}$.

2.3 Bases

This article addresses the revision of and by *bases*. In the belief revision community, the term of base commonly denotes an arbitrary (possibly infinite) set of sentences. However, in certain scenarios, other assumptions might be more appropriate. Hence, for the sake of generality, we define the notion of *base logic*, which enable us to employ bases on an abstract level with minimal requirements (just as we introduced our notion of *logic*), allowing for its instantiation in many ways.

Definition 2.8. *A* base logic *is a quintuple* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$, *where*

- $(\mathcal{L}, \Omega, \models)$ *is a logic,*
- $\mathfrak{B} \subseteq \mathcal{P}(\mathcal{L})$ *is a family of sets of sentences, called* bases*, and*
- $\Cup : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ *is a binary operator over bases, called the* abstract union*, satisfying* $[\![\mathcal{B}_1 \Cup \mathcal{B}_2]\!] = [\![\mathcal{B}_1]\!] \cap [\![\mathcal{B}_2]\!]$.

Next, we will demonstrate how, for some logic $\mathbb{L} = (\mathcal{L}, \Omega, \models)$, a corresponding base logic can be chosen depending on one's preferences regarding what bases should be.

[1] If $\preceq$ is total, this definition is equivalent to the $\mathcal{I} \preceq \mathcal{I}'$ for all $\mathcal{I}' \in \Omega'$.

Arbitrary Sets. If all (finite and infinite) sets of sentences should qualify as bases, one can simply set $\mathfrak{B} = \mathcal{P}(\mathcal{L})$. In that case, $\Cup$ can be instantiated by set union $\cup$, then the claimed behavior follows by definition.

Finite Sets. In some settings, it is more convenient to assume bases to be finite (e.g. when computational properties or implementations are to be investigated). In such cases, one can set $\mathfrak{B} = \mathcal{P}_{\text{fin}}(\mathcal{L})$, i.e., all (and only) the finite sets of sentences are bases. Again, $\Cup$ can be instantiated by set union $\cup$ (as a union of two finite sets will still be finite).

Belief Sets. This setting is closer to the original framework, where the "knowledge states" to be modified were assumed to be deductively closed sets of sentences. We can capture such situations by accordingly letting $\mathfrak{B} = \{\mathcal{B} \subseteq \mathcal{L} \mid \forall \varphi \in \mathcal{L} : \mathcal{B} \models \varphi \Rightarrow \varphi \in \mathcal{B}\}$. In this case, the abstract union operator needs to be defined via $\mathcal{B}_1 \Cup \mathcal{B}_2 = \{\varphi \in \mathcal{L} \mid \mathcal{B}_1 \cup \mathcal{B}_2 \models \varphi\}$.

Single Sentences. In this popular setting, one prefers to operate on single sentences only (rather than on proper collections of those). For this to work properly, an additional assumption needs to be made about the underlying logic $\mathbb{L} = (\mathcal{L}, \Omega, \models)$: it must be possible to express conjunction on a sentence level, either through the explicit presence of the Boolean operator $\wedge$ or by some other means. Formally, we say that $\mathbb{L} = (\mathcal{L}, \Omega, \models)$ *supports conjunction*, if for any two sentences $\varphi, \psi \in \mathcal{L}$ there exists some sentence $\varphi \owedge \psi \in \mathcal{L}$ satisfying $[\![\varphi \owedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$ (if $\wedge$ is available within the logic, we would simply have $\varphi \owedge \psi = \varphi \wedge \psi$). For such a logic, we can "implement" the single-sentence setting by letting $\mathfrak{B} = \{\{\varphi\} \mid \varphi \in \mathcal{L}\}$ and defining $\{\varphi\} \Cup \{\psi\} = \{\varphi \owedge \psi\}$.

For any of the four different notions of bases, one can additionally choose to disallow or allow the empty set as a base, while maintaining the required closure under abstract union.

In the following, we will always operate on the abstract level of "base logics"; our notions, results and proofs will only make use of the few general properties specified for these. This guarantees that our results are generically applicable to any of the four described (and any other) instantiations, and hence, are independent of the question what the right notion of bases ought to be. The cognitive overload caused by this abstraction should be minimal; e.g., readers only interested in the case of arbitrary sets can safely assume $\mathfrak{B} = \mathcal{P}(\mathcal{L})$ and mentally replace any $\Cup$ by $\cup$.

2.4 Base Change Operators

In this thesis we use base change operators to model multiple revision, which is the process of incorporating multiple new beliefs into the present beliefs held by an agent in a consistent way (whenever that is possible). We define change operators over a base logic as follows.

Definition 2.9. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic. A function $\circ : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ is called a* multiple base change operator over $\mathbb{B}$.

We will use multiple base change operators in the "standard" way of the belief change community: the first parameter represents the actual beliefs of an agent, the second parameter contains the new beliefs. The operator then yields the agent's revised beliefs. The term "multiple" refers the fact that the second input to $\circ$ is not just a single sentence, but a belief base that may consist of several sentences [FH18]. For convenience, we will henceforth drop the term "multiple" and simply speak of base change operators instead.

So far, the pure notion of base change operator is unconstrained and can be instantiated by an arbitrary binary function over bases. Obviously, this does not reflect the requirements or expectations one might have when speaking of a revision operator. Hence, in line with the traditional approach, we will consider additional constraints (so-called "postulates") for base change operators, in order to capture the gist of revisions.

2.5 Postulates for Revision

We consider multiple revision focusing on *package revision*, which is that *all* given sentences have to be incorporated, i.e. given a base $\mathcal{K}$ and new information Γ (also a base here), we demand success of revision ($\mathcal{K} \circ \Gamma \models \Gamma$)[2]. Besides the success condition, the belief change community has brought up and discussed several further requirements for belief change operators to make them *rational*, for summaries see, e.g., [Han99; FH18]. This has led to the now famous AGM approach of revision [AGM85], originally proposed through a set of rationality postulates, which correspond to the postulates (KM1)–(KM6) by K&M as follows, where $\varphi, \varphi_1, \varphi_2, \alpha$, and β are propositional sentences, and $\circ$ is a base revision operator:

(KM1) $\varphi \circ \alpha \models \alpha$.
(KM2) If $\varphi \wedge \alpha$ is consistent, then $\varphi \circ \alpha \equiv \varphi \wedge \alpha$.
(KM3) If α is consistent, then $\varphi \circ \alpha$ is consistent.
(KM4) If $\varphi_1 \equiv \varphi_2$ and $\alpha \equiv \beta$, then $\varphi_1 \circ \alpha \equiv \varphi_2 \circ \beta$.
(KM5) $(\varphi \circ \alpha) \wedge \beta \models \varphi \circ (\alpha \wedge \beta)$.
(KM6) If $(\varphi \circ \alpha) \wedge \beta$ is consistent, then $\varphi \circ (\alpha \wedge \beta) \models (\varphi \circ \alpha) \wedge \beta$.

In our work, we will make use of the K&M version of the AGM postulates adjusted to our generic notion of a base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$:

(G1) $\mathcal{K} \circ \Gamma \models \Gamma$.

[2] Another type of multiple revision is *choice revision* [Zha19], in which the agent could accept some sentences of Γ, and could reject some others.

(G2) If $[\![\mathcal{K} \Cup \Gamma]\!] \neq \emptyset$ then $\mathcal{K} \circ \Gamma \equiv \mathcal{K} \Cup \Gamma$.

(G3) If $[\![\Gamma]\!] \neq \emptyset$ then $[\![\mathcal{K} \circ \Gamma]\!] \neq \emptyset$.

(G4) If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\Gamma_1 \equiv \Gamma_2$ then $\mathcal{K}_1 \circ \Gamma_1 \equiv \mathcal{K}_2 \circ \Gamma_2$.

(G5) $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \models \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)$.

(G6) If $[\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!] \neq \emptyset$ then $\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \models (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2$.

Together, the postulates implement the paradigm of minimal change, stating that a rational agent should change her beliefs as little as possible in the process of belief revision. We consider the postulates in more detail: (G1) guarantees that the newly added belief must be a logical consequence of the result of the revision. (G2) says that if the expansion of φ by α is consistent, then the result of the revision is equivalent to the expansion of φ by α. (G3) guarantees the consistency of the revision result if the newly added belief is consistent. (G4) is the principle of the irrelevance of the syntax, stating that the revision operation is independent of the syntactic form of the bases. (G5) and (G6) ensure more careful handling of (abstract) unions of belief bases. In particular, together, they enforce that $\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \equiv (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2$, unless Γ_2 contradicts $\mathcal{K} \circ \Gamma_1$.

We can see that, item by item, (G1)–(G6) tightly correspond to (KM1)–(KM6) presented in the introduction. Note also that further formulations similar to (G1)–(G6) are given in multiple particular contexts, e.g. in the context of belief base revision specifically for Description Logics [QLB06b], for parallel revision [DJ12] and investigations on multiple revision [Zha96; Pep04; KH17]. An advantage of the specific form of the postulates (G1)–(G6) chosen for our presentation is that it does not require $\mathcal{L}$ to support conjunction (while, of course, conjunction on the sentence level is still implicitly supported via (abstract) union of bases).

2.6 Description Logics

We very briefly review the description logic $\mathcal{SROIQ}$ (which is the logical counterpart of the standard Web Ontology Language) with its standard syntax and semantics [HKS06; Rud11]. Let N_I, N_C, and N_R be finite and pairwise disjoint sets of individual names, concept names, and role names, respectively. Using these entities, *concept expressions* and *axioms* are built according to the standard $\mathcal{SROIQ}$ constructors. A $\mathcal{SROIQ}$ axiom is either a general concept inclusion (GCI), a concept assertion, a role assertion, an individual (in)equality assertion, a role inclusion, a role composition, a role disjointness, a role transitivity, a role (a)symmetry, or a role (ir)reflexivity. Their forms are given in Table 2.11. A $\mathcal{SROIQ}$ *knowledge base* is a (finite) set of $\mathcal{SROIQ}$ axioms, which are in the form of ABox, TBox, or RBox axioms.

Given a $\mathcal{SROIQ}$ knowledge base $\mathcal{K}$, we essentially determine the size of $\mathcal{K}$ by counting the number of symbols it takes to write the knowledge base. We start by inductively defining the size of $\mathcal{SROIQ}$ concepts[3] and axioms as shown in Table 2.12 and Table 2.13. Then, the size of $\mathcal{K}$ is the sum of the size of all axioms in $\mathcal{K}$, i.e. $\mathsf{size}(\mathcal{K}) = \sum_{\alpha \in \mathcal{K}} \mathsf{size}(\alpha)$.

Name	Syntax	Semantics
individual name	a	$a^{\mathcal{I}}$
atomic role	r	$r^{\mathcal{I}}$
inverse role	r^-	$\{(x,y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (y,x) \in r^{\mathcal{I}}\}$
universal role	u	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
atomic concept	A	$A^{\mathcal{I}}$
top concept	$\top$	$\Delta^{\mathcal{I}}$
bottom concept	$\bot$	$\emptyset$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{x \mid \exists y.(x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
universal restriction	$\forall r.C$	$\{x \mid \forall y.(x,y) \in r^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\}$
at-least restriction	$\geq n\ R.C$	$\{x \mid \#\{y \in C^{\mathcal{I}} \mid (x,y) \in r^{\mathcal{I}}\} \geq n\}$
at-most restricion	$\leq n\ R.C$	$\{x \mid \#\{y \in C^{\mathcal{I}} \mid (x,y) \in r^{\mathcal{I}}\} \leq n\}$
local reflexivity	$\exists r.Self$	$\{x \mid (x,x) \in r^{\mathcal{I}}\}$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$

Table 2.10: Syntax and semantics of concept and role constructors in $\mathcal{SROIQ}$, where $a \in N_I$, $r \in N_R$, and $C, D \in N_C$.

Name	Axiom α	$\mathcal{I} \models \alpha$ if
general concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$r(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$
role inclusion	$r \sqsubseteq s$	$r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$
role composition	$r_1 \circ \ldots \circ r_n \sqsubseteq r$	$r_1^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \sqsubseteq r^{\mathcal{I}}$
role disjointness	$Dis(s,r)$	$s^{\mathcal{I}} \cap r^{\mathcal{I}} = \emptyset$
individual equality assertion	$a \doteq b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
individual inequality assertion	$a \not\doteq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$

Table 2.11: Syntax and semantics of $\mathcal{SROIQ}$ axioms.

The semantic of $\mathcal{SROIQ}$ is defined through an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, which consists of a non-empty set $\Delta^{\mathcal{I}}$ called domain of $\mathcal{I}$ and a function $\cdot^{\mathcal{I}}$ that maps each individual $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, each concept $C \in N_C$ to a subset of $\Delta^{\mathcal{I}}$, and each role name $r \in N_R$ to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The function $\cdot^{\mathcal{I}}$ is extended to arbitrary $\mathcal{SROIQ}$ concept and role

[3]We assume that the number n in the qualified number restriction concept is written in binary encoding.

Table 2.12: Size of concepts in a knowledge base

$\mathsf{size}(A)$	$= 1$ for any $A \in N_C$(including $\top, \bot$, and *Self*)
$\mathsf{size}(\{a_1, \ldots, a_n\})$	$= n$ for any nominal concept, where $a_1, \ldots, a_n \in N_I$
$\mathsf{size}(C \sqcap D) = \mathsf{size}(C \sqcup D)$	$= 1 + \mathsf{size}(C) + \mathsf{size}(D)$ for any $C, D \in N_C$
$\mathsf{size}(\neg C)$	$= 1 + \mathsf{size}(C)$ for any $C \in N_C$
$\mathsf{size}(\exists r.C) = \mathsf{size}(\forall r.C)$	$= 2 + \mathsf{size}(C)$ for any $C \in N_C$ and any $r \in N_R$
$\mathsf{size}(\leq n\, r.C)$	$= \mathsf{size}(\geq n\, r.C) = 2 + log(n) + \mathsf{size}(C)$ for any $C \in N_C$ and any $r \in N_R$
$\mathsf{size}(r) = \mathsf{size}(r^-)$	$= 1$ for any $r \in N_R$

Table 2.13: Size of axioms in a knowledge base

$\mathsf{size}(C \sqsubseteq D)$	$= 1 + \mathsf{size}(C) + \mathsf{size}(D)$ for any axiom $C \sqsubseteq D$ in TBox
$\mathsf{size}(r \sqsubseteq s)$	$= 3$ for any $r \sqsubseteq s$ in RBox
$\mathsf{size}(C(a))$	$= 1 + \mathsf{size}(C)$ for any concept assertion in ABox
$\mathsf{size}(r(a, b))$	$= 3$ for any role assertion in ABox
$\mathsf{size}(r_1 \circ \ldots \circ r_n \sqsubseteq r_{n+1})$	$= n + 2$ for any role chain axiom in RBox
$\mathsf{size}(Dis(r, s))$	$= 3$ for any role disjointness axiom in RBox

expressions as defined in Table 2.10 and used to define satisfaction of axioms as shown in Table 2.11. We say that $\mathcal{I}$ *satisfies* a knowledge base $\mathcal{K}$ (or $\mathcal{I}$ is a model of $\mathcal{K}$) if it satisfies all axioms of $\mathcal{K}$, denoted as $\mathcal{I} \models \mathcal{K}$. A knowledge base $\mathcal{K}$ *entails* an axiom α if all models of $\mathcal{K}$ are models of α. We use $\mathcal{L}$ to denote the DL language, i.e. the set of all possible DL axioms and Ω to denote the set of all interpretations.

2.7 Fixed-Domain Semantics

Let $\Delta \subseteq N_I$ be a non empty finite set called the fixed domain. An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is said to be Δ-fixed, if

- $\Delta^{\mathcal{I}} = \Delta$, and
- $a^{\mathcal{I}} = a$ for all $a \in \Delta$.

For a DL knowledge base $\mathcal{K}$, an interpretation $\mathcal{I}$ is a Δ-model of $\mathcal{K}$ ($\mathcal{I} \models_\Delta \mathcal{K}$), if $\mathcal{I}$ is a Δ-fixed interpretation and $\mathcal{I} \models \mathcal{K}$. A knowledge base $\mathcal{K}$ is called Δ-consistent (or Δ-satisfiable) if it has at least one Δ-model. A knowledge base $\mathcal{K}$ Δ-entails an axiom α ($\mathcal{K} \models_\Delta \alpha$) if $\mathcal{I} \models \alpha$ for every $\mathcal{I} \models_\Delta \mathcal{K}$. Two KBs $\mathcal{K}$ and $\mathcal{K}'$ are Δ-semantically equivalent (written as $\mathcal{K} \equiv_\Delta \mathcal{K}'$) iff $\mathcal{K} \models_\Delta \mathcal{K}'$ and $\mathcal{K}' \models_\Delta \mathcal{K}$. Instead of Δ-consistent, Δ-entail, or Δ-semantically equivalent, we will just say *consistent*, *entail*, or *equivalent* respectively, if Δ is clear from the context. The set of all Δ-models of $\mathcal{K}$ is denoted by $[\![\mathcal{K}]\!]_\Delta$. We denote $\mathcal{L}_\Delta$ as a set of all possible axioms in DL under fixed-domain semantics. Note that we only consider finite knowledge bases in the process of revision.

Fixed domain semantics can be seen as a further restriction of finite model reasoning [GRS16]. This approach restricts reasoning to a domain known *a priori*. The restriction gives us not only a computational complexity advantage, but arguably more intuitive models of a knowledge base in some cases (for more about the reasoning complexity, see [RST17; RS17]). Previous studies have provided a practical reasoner [RST17], SPARQL querying [RSY19], and justification framework [RST18] under this approach. The models that are generated from a knowledge base under fixed-domain semantics, however, can be exponentially many. In particular, given a KB $\mathcal{K}$ under fixed domain Δ, one can possibly have a number of $2^{|\Delta|\times|N_C(\mathcal{K})|} \times 2^{|\Delta|^2\times|N_R(\mathcal{K})|}$ models, where $N_C(\mathcal{K})$ and $N_R(\mathcal{K})$ are the set of concept names and role names occurring in $\mathcal{K}$, respectively.

We are working with $\mathcal{SROIQ}$ knowledge bases under some assumptions on the axiom side. The original definition of $\mathcal{SROIQ}$ RBox contains axioms expressing role hierarchy ($r \sqsubseteq s$), role chains ($r_1 \circ \ldots \circ r_n \sqsubseteq r$), role disjointness ($Dis(r,s)$), transitivity ($Tra(r)$), symmetry ($Sym(r)$), asymmetry ($Asy(r)$), reflexivity ($Ref(r)$), and irreflexivity ($Irr(r)$). In this article, we will only consider the first three axiom expressions since the remaining forms can be syntactically rewritten into other known axioms: $Sym(r)$ can be translated as $r^- \sqsubseteq r$, $Asy(r)$ can be expressed as $Dis(r, r^-)$, and $Tra(r)$ can be rewritten into role chain $r \circ r \sqsubseteq r$. For (ir)reflexivity axioms, $Ref(r)$ and $Irr(r)$ can be translated as $\top \sqsubseteq \exists r.Self$ and $\top \sqsubseteq \neg\exists r.Self$, respectively. Moreover, as opposed to the standard $\mathcal{SROIQ}$ definition, we do not impose the global restriction called *regularity* since any KB with unrestricted role hierarchies under the fixed-domain semantics is always guaranteed to be decidable [GRS16].

Considering practical application for revision in Description Logics under fixed-domain semantics, we are going to encode the knowledge bases into ASP programs (cf. Section 4.2 and Section 4.4). To this end, we assume that the knowledge bases are in simplified forms. In general, the simplification is based on previous work by Gaggl and colleagues [GRS16]. For a fixed domain Δ and a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$, the simplification of $\mathcal{K}$ comprises several preprocessing steps: (1) ABox simplification, (2) nominal concepts replacement, and (3) KB normalization. First, we review the notion of a simplified ABox $\mathcal{A}$.

Definition 2.14 (Simplified ABox). *Let $\mathcal{A}$ be an ABox in $\mathcal{K}$. $\mathcal{A}$ is* simplified *if:*

- *it only refers to domain individuals from Δ,*
- *it does not mention elements from $N_I(\mathcal{K}) \setminus \Delta$ (except in nominal concepts), and*
- *it is free of (in)equality statements.*

To obtain the above requirements, one can apply the following model-preserving transformations. For each $\alpha \in \mathcal{A}$, we consider several possible forms of α, where $a, b \in N_I(\mathcal{K}) \setminus \Delta$ and $c, d \in \Delta$:

$$\begin{array}{ll} C(a) \rightsquigarrow \{a\} \sqsubseteq C & a \doteq b \rightsquigarrow \{a\} \sqsubseteq \{b\} \\ r(a,b) \rightsquigarrow \{a\} \sqsubseteq \exists r.\{b\} & a \not\doteq b \rightsquigarrow \{a\} \sqsubseteq \neg\{b\} \\ r(c,b) \rightsquigarrow \exists r.\{b\}(c) & c \doteq b \rightsquigarrow \{b\}(c) \\ r(a,d) \rightsquigarrow \exists r^-.\{a\}(d) & c \not\doteq b \rightsquigarrow \neg\{b\}(c) \end{array}$$

Second, we assume that $\mathcal{K}$ does not contain any nominal concepts. One can apply the following replacement to obtain the desired requirement:

- Replace every occcurence of nominal concept $\{a\}$ in $\mathcal{K}$ with a fresh auxiliary concept $O_{\{a\}}$.
- Add two axioms $\top \sqsubseteq \leq 1u.O_{\{a\}}$ and $\top \sqsubseteq \geq 1u.O_{\{a\}}$ to the TBox. These axioms enforce the concept $O_{\{a\}}$ to be interpreted as a singleton set.
- If $a \in \Delta$, add $O_{\{a\}}(a)$ to the ABox.

Finally, the third preprocessing step is KB normalization.

Definition 2.15 (Normalized Knowledge Base). *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$ be a $\mathcal{SROIQ}$ knowledge base under fixed-domain semantics.*

- *A GCI σ is normalized if it is of the form $\top \sqsubseteq \bigsqcup_{i=1}^{n} C_i$, where C_i is of the form B, $\{a\}$, $\forall r.B$, $\exists r.Self$, $\neg\exists r.Self$, $\geq nr.B$, or $\leq nr.B$, for B a literal concept, r a role, and n a positive integer.*
- *A TBox $\mathcal{T}$ is normalized if each GCI $\sigma \in \mathcal{T}$ is normalized.*
- *An ABox $\mathcal{A}$ is normalized if:*
  - *each concept assertion in $\mathcal{A}$ contains only a literal concept,*
  - *each role assertion in $\mathcal{A}$ contains only an atomic role, and*
  - *$\mathcal{A}$ contains at least one assertion.*
- *An RBox $\mathcal{R}$ is normalized if each role inclusion axiom is of the form $r \sqsubseteq s$ or $r_1 \circ r_2 \sqsubseteq r_3$.*

We say $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$ is normalized if $\mathcal{T}$, $\mathcal{A}$, and $\mathcal{R}$ are normalized. Given an input $\mathcal{K}$, we can obtain the normalization of $\mathcal{K}$ by using the transformation $\Omega(\mathcal{K})$ as described in Table 2.16.

2.8 Answer Set Programming

Answer Set Programming (ASP) is a language that can be used for knowledge representation and reasoning based on the *answer set* or *stable model semantics* for logic programs [Gel08]. It is a form of declarative programming oriented towards difficult, primarily NP-hard, search problems [Lif08; Lif19]. ASP is mainly used to solve combinatoric problems by modelling

$\Omega(\mathcal{K})$	$=$	$\bigcup_{\alpha \in \mathcal{R} \cup \mathcal{A}} \Omega(\alpha) \cup \bigcup_{C_1 \sqsubseteq C_2 \in \mathcal{T}} \Omega(\top \sqsubseteq \mathsf{nnf}(\neg C_1 \sqcup C_2))$
$\Omega(\top \sqsubseteq \mathbf{C} \sqcup C')$	$=$	$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \alpha_{C'}) \cup \bigcup_{1 \le i \le n} \Omega(\top \sqsubseteq \dot{\neg}\alpha_{C'} \sqcup C_i)$ for C' of the form $C' = C_1 \sqcap \ldots \sqcap C_n$ and $n \ge 2$
$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \exists r.D)$	$=$	$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \ge 1 r.D)$
$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \forall r.D)$	$=$	$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \forall r.\alpha_D) \cup \Omega(\top \sqsubseteq \dot{\neg}\alpha_D \sqcup D)$
$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \ge n\, r.D)$	$=$	$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \ge n\, r.\alpha_D) \cup \Omega(\top \sqsubseteq \dot{\neg}\alpha_D \sqcup D)$
$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \le n\, r.D)$	$=$	$\Omega(\top \sqsubseteq \mathbf{C} \sqcup \le n\, r.\dot{\neg}\alpha_{\dot{\neg}D}) \cup \Omega(\top \sqsubseteq \dot{\neg}\alpha_{\dot{\neg}D} \sqcup \dot{\neg}D)$
$\Omega(D(s))$	$=$	$\{\alpha_D(s)\} \cup \Omega(\top \sqsubseteq \dot{\neg}\alpha_D \sqcup \mathsf{nnf}(D))$
$\Omega(r^-(s,t))$	$=$	$r(t,s)$
$\Omega(r_1 \circ \ldots \circ r_n \sqsubseteq r)$	$=$	$\{r_1 \circ r_2 \sqsubseteq r_{(r_1 \circ r_2)}\} \cup \Omega(r_{(r_1 \circ r_2)} \circ r_3 \circ \ldots \circ r_n \sqsubseteq r)$ for any RIA with $n > 2$
$\Omega(\beta)$	$=$	$\{\beta\}$ for any other axiom β

$\alpha_C = \begin{cases} Q_C, & if\, \mathsf{pos}(C) = \mathsf{true} \\ \neg Q_C, & if\, \mathsf{pos}(C) = \mathsf{false} \end{cases}$,where Q_C is a fresh concept name unique for C.

$\mathsf{pos}(\top)$	$=$	false	$\mathsf{pos}(\bot)$	$=$	false
$\mathsf{pos}(A)$	$=$	true	$\mathsf{pos}(\neg A)$	$=$	false
$\mathsf{pos}(\exists r.Self)$	$=$	true	$\mathsf{pos}(\neg \exists r.Self)$	$=$	false
$\mathsf{pos}(C_1 \sqcap C_2)$	$=$	$\mathsf{pos}(C_1) \vee \mathsf{pos}(C_2)$	$\mathsf{pos}(C_1 \sqcup C_2)$	$=$	$\mathsf{pos}(C_1) \vee \mathsf{pos}(C_2)$
$\mathsf{pos}(\forall r.C_1)$	$=$	$\mathsf{pos}(C_1)$	$\mathsf{pos}(\le n\, r.C_1)$	$=$	$\begin{cases} \mathsf{pos}(\dot{\neg}(C_1)) & if\, n = 0 \\ \mathsf{true} & \text{otherwise} \end{cases}$
$\mathsf{pos}(\ge n\, r.C_1)$	$=$	true			

Note: A is an atomic concept, $C_{(i)}$ are arbitrary concept expressions, $\mathbf{C}$ is a possibly empty disjunction of concept expressions, D is **not** a literal concept. The function $\dot{\neg}$ is defined as $\dot{\neg}(\neg A) = A$ and $\dot{\neg}(A) = \neg A$ for some atomic concept A.

Table 2.16: Ω-normalization of knowledge base axioms.

them in logic programs and then by using an engine called *answer set solver* to compute the stable models representing the solutions for the problems. The answer set solver can load the program and return the answer (the "stable model") consisting of all facts that can be derived using the rules of the program.

ASP is purely declarative and the program always terminates, where the order of the program rules does not matter. Although ASP is quite close to SAT where the answer sets are particular classical models of the program, ASP provides more expressive and high-level features such as transitive closures, negation as failure, and variables. In the following, we present some basic notions in ASP based on previous work [EIK09; GKK+12].

Syntax Let $\mathcal{U}$ be a fixed countable set of (*domain*) elements or *constants* and $<$ be a total order over the domain elements. A *(predicate) atom* is an expression $p(t_1, \ldots, t_n)$, where p is a *predicate* of arity $n \ge 0$ and each t_i is either a variable or an element from $\mathcal{U}$. An atom is *ground* if it is free of variables. We denote $B_{\mathcal{U}}$ as the set of all ground atoms over $\mathcal{U}$. A (*normal*) *rule* ρ is of the form

$$a \leftarrow b_1, \ldots, b_k, not\ b_{k+1}, \ldots, not\ b_m.$$

with $m \geq k \geq 0$, where $a, b_1, \ldots, b_m$ are atoms and "not" denotes *default negation*. The *head* of ρ is the singleton set $H(\rho) = \{a\}$ and the *body* of ρ is $B(\rho) = \{b_1, \ldots, b_k, not\ b_{k+1}, \ldots, not\ b_m\}$. Furthermore, we denote $B^+(\rho) = \{b_1, \ldots, b_k\}$ and $B^-(\rho) = \{not\ b_{k+1}, \ldots, not\ b_m\}$. A rule ρ is *safe* if each variable in ρ occurs in $B^+(\rho)$. A rule ρ is *ground* if no variable occurs in ρ. A *fact* is a ground rule with empty body. An (*input*) *database* is a set of facts. A (normal) *program* is a finite set of normal rules. For a program Π and an input database D, we often write $\Pi(D)$ instead of $\Pi \cup D$. For any program Π, let U_Π be the set of all constants appearing in Π. $Gr(\Pi)$ is the set of all rules $\rho\sigma$ obtained by applying, to each rule $\rho \in \Pi$, all possible substitutions σ from the variables in ρ to elements of U_Π.

Semantics An *interpretation* $I \subseteq B_{\mathcal{U}}$ *satisfies* a ground rule ρ iff $H(\rho) \cap I \neq \emptyset$ whenever $B^+(\rho) \subseteq I$, $B^-(\rho) \cap I = \emptyset$. I satisfies a ground program Π, if each $\rho \in \Pi$ is satisfied by I. A non-ground rule ρ (respectively, a program Π) is satisfied by an interpretation I iff I satisfies all groundings of ρ (respectively, $Gr(\Pi)$). $I \subseteq B_{\mathcal{U}}$ is an *answer set* (also called *stable model*) of Π iff it is a subset-minimal set satisfying the *Gelfond-Lifschitz reduct* $\Pi^I = \{H(\rho) \leftarrow B^+(\rho) \mid I \cap B^-(\rho) = \emptyset, \rho \in Gr(\Pi)\}$. For a program Π, we denote the set of its answer sets by $\mathcal{AS}(\Pi)$.

We make use of further syntactic extensions, namely integrity constraints, count expressions, and sum expressions, which all of them can be recast to ordinary normal rules as described in [GKK+12]. An integrity constraint is a rule ρ where $H(\rho) = 0$, which intuitively represents an undesirable situation, i.e. to avoid $B(\rho)$ evaluates positively. Count expressions are of the form $\#\mathsf{count}\{l : l_1, \ldots, l_n\} \bowtie u$, where l is an atom and $l_j = p_j$ or $l_j = not\ p_j$, for p_j an atom, $1 \leq j \leq i$, u a non-negative integer, and $\bowtie \in \{\leq, <, =, >, \geq\}$. The expression $\{l : l_1, \ldots, l_n\}$ denotes the set of all ground instantiations of l, governed through $\{l_1, \ldots, l_n\}$. The sum expression are of the form $\#\mathsf{sum}\{w_1 : l_1, \ldots, w_n : l_n\} \bowtie u$, where $\{w_1 : l_1, \ldots, w_n : l_n\}$ denotes the set of all ground instantiations with their weights (or cost) and $w_1, \ldots, w_n$ are positive integers. We restrict the occurrence of count expressions and sum expressions in a rule ρ to $B^+(\rho)$ only. Intuitively, an interpretation satisfies count expression $\#\mathsf{count}\{l : l_1, \ldots, l_n\} \bowtie u$ if $N \bowtie u$ holds, where N is the cardinality of the set of ground instantiations of l, i.e. $N = |\{l \mid l_1, \ldots, l_n\}|$, for $\bowtie \in \{\leq, <, =, >, \geq\}$ and u a non-negative integer. Accordingly, an interpretation intuitively satisfies the sum expression $\#\mathsf{sum}\{w_1 : l_1, \ldots, w_n : l_n\} \bowtie u$ if $M \bowtie u$ holds, where $M = \sum_{1 \leq w_i \leq n} w_i$.

ASP Solver `Clingo` In this work, we use an ASP solver `Clingo` from the Potassco[4] suite to generate the answer sets for ASP programs. The description of the tool and its features are based on Potassco Guide version 2.2.0[5]. Given a program in a text file "file.pl", the solver

[4]Potsdam Answer Set Solving Collection (https://potassco.org/)

[5]https://github.com/potassco/guide/releases/download/v2.2.0/guide.pdf

can be easily run by calling the program "`clingo file.pl`" from the command line. We will exploit certain `Clingo` features to solve revision problems considered in this thesis, in particular:

- *Minimize statement*. The minimize statement is a way to express an optimization problem. A minimize statement of the form

  $$\#minimize\{w_1 : l_1, \ldots, w_n : l_n\}.$$

  represents the following n special integrity constraints called *weak constraints*:

  $$\leftarrow l_1.[w_1]. \quad \ldots \quad \leftarrow l_n.[w_n].$$

  The semantics of a program with weak constraints is intuitive: an answer set is minimal if the obtained weight w (or cost) is minimal among all answer sets of the given program.

- *Interval*. `Clingo` supports integer interval $i..j$, which represents each integer k such that $i \leq k \leq j$ is expanded during grounding. In the head of a rule, an interval is expanded conjunctively, while in the body of a rule, it is expanded disjunctively. For example, we could simply write "`num(1..5).`" to represent five facts.

- *Heuristic*. `Clingo` provides means for incorporating domain-specific heuristics into ASP solving. This allows for modifying the heuristic of the solver from within a logic program or from the command line. We will use one heuristic specifically to compute subset minimal answer sets. To achieve this task, we append the program with heuristic directive

  $$\#heuristic \quad a(X).[1, false],$$

  which guarantees that the first answer set produced is subset minimal with respect to the atoms of predicate $a/1$ (irrespective of the value of variable X). The heuristic can be activated using option `--heuristic=Domain` from the command line when running the `Clingo` application.

Chapter 3

Representation Theorem in Tarskian Logics

In this chapter, we consider (multiple) base revision in arbitrary Tarskian logics, i.e., logics exhibiting a classically defined model theory. We thereby refine and generalise the popular approach by Katsuno and Mendelzon [KM91] which was tailored to belief base revision in propositional logic with a finite signature. K&M start out from belief bases, assigning to each a total preorder on the interpretations, which expresses – intuitively speaking – a degree of "modelishness". The models of the result of any AGM revision then coincide with the preferred (i.e., preorder-minimal) models of the injected information.

Our approach generalises this idea of preferences over interpretations to the general setting, which necessitates adjusting the nature of the "modelishness-indicating" assignments: We have to explicitly require that minimal models always exist (*min-completeness*) and that they can be described in the logic (*min-expressibility*). Moreover, we show that demanding preference relations to be preorders is infeasible in our setting; we have to waive transitivity and retain only a weaker property (*min-retractivity*).

3.1 Base Revision in Propositional Logic

A well-known and by now popular characterization of base revision has been described by Katsuno and Mendelzon [KM91] for the special case of propositional logic. To be more specific and apply our terminology, K&M's approach applies to the base logic

$$\mathbb{PL}_n = (\mathcal{L}_{\mathbb{PL}_n}, \Omega_{\mathbb{PL}_n}, \models_{\mathbb{PL}_n}, \mathcal{P}_{\text{fin}}(\mathcal{L}_{\mathbb{PL}_n}), \cup)$$

for arbitrary, but fixed n (cf. Example 2.6). The assumption of the finiteness on the underlying signature of atomic propositions is not overtly explicit in K&M's paper, but it becomes apparent upon investigating their arguments and proofs – we will see shortly that their characterization fails as soon as this assumption is dropped. K&M's approach also hinges on other particularities of this setting: As discussed earlier, any propositional belief base $\mathcal{K}$ can be equivalently written as a single propositional sentence. Consequently, in their approach, belief bases are actually represented by single sentences, without loss of expressivity.

One key contribution of K&M is to provide an alternative characterization of the propositional base revision operators satisfying (KM1)–(KM6) by model-theoretic means, i.e. through comparisons between propositional interpretations. In the following, we present their results in a formulation that facilitates later generalization. One central notion for the characterization is the notion of *faithful assignment*.

Definition 3.1 (assignment, faithful). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic. An* assignment *for $\mathbb{B}$ is a function $\preceq_{(.)} \colon \mathfrak{B} \to \mathcal{P}(\Omega \times \Omega)$ that assigns to each belief base $\mathcal{K} \in \mathfrak{B}$ a total binary relation $\preceq_{\mathcal{K}}$ over Ω. An assignment $\preceq_{(.)}$ for $\mathbb{B}$ is called* faithful *if it satisfies the following conditions for all $\mathcal{I}, \mathcal{I}' \in \Omega$ and all $\mathcal{K}, \mathcal{K}' \in \mathfrak{B}$:*

(F1) If $\mathcal{I}, \mathcal{I}' \models \mathcal{K}$, then $\mathcal{I} \prec_{\mathcal{K}} \mathcal{I}'$ does not hold.
(F2) If $\mathcal{I} \models \mathcal{K}$ and $\mathcal{I}' \not\models \mathcal{K}$, then $\mathcal{I} \prec_{\mathcal{K}} \mathcal{I}'$.
(F3) If $\mathcal{K} \equiv \mathcal{K}'$, then $\preceq_{\mathcal{K}} = \preceq_{\mathcal{K}'}$.

An assignment $\preceq_{(.)}$ is called a preorder assignment *if $\preceq_{\mathcal{K}}$ is a preorder for every $\mathcal{K} \in \mathfrak{B}$.*

Intuitively, faithful assignments provide information about which of the two interpretations is "closer to $\mathcal{K}$-modelhood". Consequently, the actual $\mathcal{K}$-models are $\preceq_{\mathcal{K}}$-minimal. The next definition captures the idea of an assignment adequately representing the behaviour of a revision operator.

Definition 3.2 (compatible). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ a base logic. A base change operator $\circ$ for $\mathbb{B}$ is called* compatible *with some assignment $\preceq_{(.)}$ for $\mathbb{B}$ if it satisfies*

$$[\![\mathcal{K} \circ \Gamma]\!] = \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$$

for all bases $\mathcal{K}$ and Γ from $\mathfrak{B}$.

With these notions in place, K&M's representation result can be smoothly expressed as follows:

Theorem 3.3 (Katsuno and Mendelzon 1991). *A base change operator $\circ$ for $\mathbb{PL}_n$ satisfies (G1)–(G6) if and only if it is compatible with some faithful preorder assignment for $\mathbb{PL}_n$.*

In the next section, we discuss and provide a generalization of the overall approach to the setting of arbitrary base logics.

3.2 Approach for Arbitrary Base Logics

In this section, we prepare our main result by revisiting K&M's concepts for propositional logic and investigating their suitability for our general setting of base logics. The result by Katsuno and Mendelzon established an elegant combination of the notions of preorder assignments, faithfulness, and compatibility in order to semantically characterize AGM base change operators. However, as we mentioned before and will make more precise in the following, K&M's characterization hinges on features of signature-finite propositional logic that do not generally hold for Tarskian logics. So far, attempts to find similar formulations for less restrictive logics have made good progress for understanding the nature of AGM revision (cf. Section 3.9). Here we go further, by extending the K&M approach by novel notions to the very general setting of base logics.

3.2.1 First Problem: Non-Existence of Minima

The first issue with K&M's original characterization when generalizing to arbitrary base logics is the possible absence of $\preceq_{\mathcal{K}}$-minimal elements in $[\![\Gamma]\!]$.

Observation 3.4. *For arbitrary base logics, the minimum from Definition 3.2, required in Theorem 3.3, might be empty.*

One way this might happen is due to infinite descending $\preceq_{\mathcal{K}}$-chains of interpretations. To illustrate this problem (and to show that it arises even for propositional logic, if the signature is infinite but bases are finite), consider the base logic

$$\mathbb{PL}_\infty = (\mathcal{L}_{\mathbb{PL}_\infty}, \Omega_{\mathbb{PL}_\infty}, \models_{\mathbb{PL}_\infty}, \mathcal{P}_{\text{fin}}(\mathcal{L}_{\mathbb{PL}_\infty}), \cup),$$

i.e., propositional logic with finite bases, but countably infinitely many distinct atomic propositions $\Sigma = \{p_1, p_2, \ldots\}$ (cf. Example 2.7). We will exhibit a base change operator that is compatible with a faithful preorder assignment, yet does violate one of the postulates, due to the problem mentioned above.

Example 3.5. *We define $\circ^{\cup}$ by simply letting $\mathcal{K} \circ^{\cup} \Gamma = \mathcal{K} \cup \Gamma$. Obviously $\circ^{\cup}$ violates (G3) as one can see by picking, say $\mathcal{K} = \{p_1\}$ and $\Gamma = \{\neg p_1\}$. Nevertheless, for this operator, a compatible assignment exists, as we will show next. Assume a base $\mathcal{K}$ and two propositional interpretations $\mathcal{I}_1, \mathcal{I}_2 : \Sigma \to \{\mathbf{true}, \mathbf{false}\}$. Let $\mathcal{I}_k^{\mathbf{true}}$ denote $\{p_i \in \Sigma \mid \mathcal{I}_k(p_i) = \mathbf{true}\}$ for $k \in \{1, 2\}$, i.e., the set of atomic symbols that $\mathcal{I}_k$ maps to* **true**. *Then we let $\mathcal{I}_1 \preceq_{\mathcal{K}}^{\cup} \mathcal{I}_2$ if at least one of the following is the case:*

(1) $\mathcal{I}_1 \models \mathcal{K}$
(2) $\mathcal{I}_2 \not\models \mathcal{K}$ *and* $\mathcal{I}_2^{\mathbf{true}}$ *is infinite*
(3) $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$, *both* $\mathcal{I}_1^{\mathbf{true}}$ *and* $\mathcal{I}_2^{\mathbf{true}}$ *are finite, and* $|\mathcal{I}_1^{\mathbf{true}}| \geq |\mathcal{I}_2^{\mathbf{true}}|$

The following proposition shows that $\preceq_{\mathcal{K}}^{\cup}$ is a faithful preorder assignment that is compatible with $\circ^{\cup}$. This shows that Theorem 3.3 by K&M does not straightforwardly generalize to $\mathbb{PL}_\infty$.

Proposition 3.6. *The relation $\preceq^{\cup}_{\mathcal{K}}$ is a faithful preorder assignment and is compatible with the base change operator $\circ^{\cup}$ for $\mathbb{PL}_{\infty}$, yet $\circ^{\cup}$ does not satisfy (G3).*

Proof. We show that $\preceq^{\cup}_{\mathcal{K}}$ is a preorder assignment.

(Totality) For totality, assume the contrary, i.e. there are two interpretations $\mathcal{I}_1, \mathcal{I}_2$ with $\mathcal{I}_1 \not\preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$ and $\mathcal{I}_2 \not\preceq^{\cup}_{\mathcal{K}} \mathcal{I}_1$. From the definition of $\preceq^{\cup}_{\mathcal{K}}$, we have $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$ where both $\mathcal{I}_1$ and $\mathcal{I}_2$ are finite with $|\mathcal{I}_1^{\mathbf{true}}| \not\geq |\mathcal{I}_2^{\mathbf{true}}|$ and $|\mathcal{I}_2^{\mathbf{true}}| \not\geq |\mathcal{I}_1^{\mathbf{true}}|$. Since $\geq$ is total over integers, this is a contradiction. Reflexivity follows from totality.

(Transitivity) For transitivity, suppose $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$ and $\mathcal{I}_2 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$. We make a case distinction by $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$ and the definition of $\preceq^{\cup}_{\mathcal{K}}$:

(1) The case of $\mathcal{I}_1 \models \mathcal{K}$. Then $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$ follows immediately.

(2) The case of $\mathcal{I}_2 \not\models \mathcal{K}$ and $\mathcal{I}_2^{\mathbf{true}}$ is infinite. As $\mathcal{I}_2 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$, we consider three subcases:

(2.1) $\mathcal{I}_2 \models \mathcal{K}$. This contradicts the prior assumption, and hence this case is not possible.

(2.2) $\mathcal{I}_3 \not\models \mathcal{K}$ with infinite $\mathcal{I}_3^{\mathbf{true}}$. Then $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$ follows.

(2.3) $\mathcal{I}_2^{\mathbf{true}}$ and $\mathcal{I}_3^{\mathbf{true}}$ are finite. This is also impossible due to immediate contradiction.

(3) The case of $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$, both $\mathcal{I}_1^{\mathbf{true}}$ and $\mathcal{I}_2^{\mathbf{true}}$ are finite and $|\mathcal{I}_1^{\mathbf{true}}| \geq |\mathcal{I}_2^{\mathbf{true}}|$. From $\mathcal{I}_2 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$ we consider three subcases:

(3.1) $\mathcal{I}_2 \models \mathcal{K}$. This is not possible, immediate contradiction.

(3.2) $\mathcal{I}_3 \not\models \mathcal{K}$ with infinite $\mathcal{I}_3^{\mathbf{true}}$. This implies $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$.

(3.3) $\mathcal{I}_2, \mathcal{I}_3 \not\models \mathcal{K}$, both $\mathcal{I}_2^{\mathbf{true}}$ and $\mathcal{I}_3^{\mathbf{true}}$ are finite with $|\mathcal{I}_2^{\mathbf{true}}| \geq |\mathcal{I}_3^{\mathbf{true}}|$. Since $|\mathcal{I}_1^{\mathbf{true}}| \geq |\mathcal{I}_2^{\mathbf{true}}|$ and $|\mathcal{I}_2^{\mathbf{true}}| \geq |\mathcal{I}_3^{\mathbf{true}}|$, from transitivity of $\geq$ over integers, we have $|\mathcal{I}_1^{\mathbf{true}}| \geq |\mathcal{I}_3^{\mathbf{true}}|$ and finally $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_3$.

We show that $\preceq^{\cup}_{\mathcal{K}}$ is faithful and that $\preceq^{\cup}_{(.)}$ is compatible with $\circ^{\cup}$.

(Faithfulness) The first condition of faithfulness, the Condition (F1), follows from the assumption $\mathcal{I}_1, \mathcal{I}_2 \models \mathcal{K}$ and case (1) of the definition of $\preceq^{\cup}_{\mathcal{K}}$, given in Example 3.5.

For (F2), let $\mathcal{I}_1 \models \mathcal{K}$ and $\mathcal{I}_2 \not\models \mathcal{K}$. From the case (1) of the definition, $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$ holds. Now assume, by way of contradiction, that $\mathcal{I}_2 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_1$. Following the definition of $\preceq^{\cup}_{\mathcal{K}}$, we consider three cases. (Case 1) $\mathcal{I}_2 \models \mathcal{K}$ contradicts our assumption. (Case 2) and (Case 3) are not applicable because they require $\mathcal{I}_1 \not\models \mathcal{K}$. Hence, $\mathcal{I}_2 \not\preceq^{\cup}_{\mathcal{K}} \mathcal{I}_1$ and therefore $\mathcal{I}_1 \prec^{\cup}_{\mathcal{K}} \mathcal{I}_2$ holds.

For (F3), assume $\mathcal{K} \equiv \mathcal{K}'$ (i.e. $[\![\mathcal{K}]\!] = [\![\mathcal{K}']\!]$) and let $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$. We consider three cases. (Case 1) $\mathcal{I}_1 \models \mathcal{K}$. Then it also holds $\mathcal{I}_1 \models \mathcal{K}'$, and hence $\mathcal{I}_1 \preceq_{\mathcal{K}'} \mathcal{I}_2$. (Case 2) $\mathcal{I}_2 \not\models \mathcal{K}$ and $\mathcal{I}_2^{\mathbf{true}}$ is infinite. Then $\mathcal{I}_2 \not\models \mathcal{K}'$ and hence $\mathcal{I}_1 \preceq_{\mathcal{K}'} \mathcal{I}_2$. (Case 3) where $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$ we also have $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}'$ and consequently $\mathcal{I}_1 \preceq_{\mathcal{K}'} \mathcal{I}_2$. Therefore, we have $\preceq^{\cup}_{\mathcal{K}} = \preceq_{\mathcal{K}'}$ (i.e. $\mathcal{I}_1 \preceq^{\cup}_{\mathcal{K}} \mathcal{I}_2$ if and only if $\mathcal{I}_1 \preceq_{\mathcal{K}'} \mathcal{I}_2$).

(Compatibility with $\circ$*)* For the compatibility with $\circ^{\cup}$, we show that $[\![\mathcal{K} \circ^{\cup} \Gamma]\!] = \min([\![\Gamma]\!], \preceq^{\cup}_{\mathcal{K}})$. For any inconsistent Γ, we have $[\![\mathcal{K} \circ^{\cup} \Gamma]\!] = \emptyset = \min([\![\Gamma]\!], \preceq^{\cup}_{\mathcal{K}})$. If $\mathcal{K} \cup \Gamma$ is consistent, then we have $[\![\mathcal{K} \circ \Gamma]\!] = [\![\mathcal{K} \cup \Gamma]\!]$. Because $\preceq^{\cup}_{\mathcal{K}}$ is faithful, we directly obtain $[\![\mathcal{K} \circ \Gamma]\!] = \min([\![\Gamma]\!], \preceq^{\cup}_{\mathcal{K}})$. Thus, for the remaining steps of the proof, we assume that $\mathcal{K} \cup \Gamma$ is inconsistent and Γ is consistent.

We show in the following that $\min([\![\Gamma]\!], \preceq^{\cup}_{\mathcal{K}}) = \emptyset$ holds by contradiction, i.e., there exists some $\mathcal{I}_1 \in \min([\![\Gamma]\!], \preceq^{\cup}_{\mathcal{K}})$. This means, that $\mathcal{I}_1 \in [\![\Gamma]\!]$ and there is no other $\mathcal{I}_2 \in [\![\Gamma]\!]$ such that $\mathcal{I}_2 \prec^{\cup}_{\mathcal{K}} \mathcal{I}_1$. Note that from the definition of $\circ^{\cup}$ and our case assumption, we have $[\![\mathcal{K} \circ^{\cup} \Gamma]\!] = [\![\mathcal{K} \cup \Gamma]\!] = \emptyset$, and hence $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$. Let $\Sigma_\Gamma \subseteq \Sigma$ be a set of atomic symbols occurring in Γ. Clearly, because Γ is finite, we have that Σ_Γ contains finitely many atoms. We have two cases: $\mathcal{I}_1^{\mathbf{true}}$ can be finite or infinite.

($\mathcal{I}_1^{\mathbf{true}}$ is finite) Then, there exists an atomic symbol q such that $q \in \Sigma \setminus (\mathcal{I}_1^{\mathbf{true}} \cup \Sigma_\Gamma)$ (as both $\mathcal{I}_1^{\mathbf{true}}$ and Σ_Γ are finite and Σ is infinite). Then we could define another interpretation $\mathcal{I}_2$ such that $\mathcal{I}_2(q) = \mathbf{true}$ and $\mathcal{I}_2(p_i) = \mathcal{I}_1(p_i)$ for all $p_i \in \Sigma \setminus \{q\}$. Since q does not occur in Γ, we have $\mathcal{I}_2 \in [\![\Gamma]\!]$ and $|\mathcal{I}_2^{\mathbf{true}}| = |\mathcal{I}_1^{\mathbf{true}}| + 1$. Hence, $\mathcal{I}_2 \prec^{\cup}_{\mathcal{K}} \mathcal{I}_1$, a contradiction to the minimality of $\mathcal{I}_1$.

($\mathcal{I}_1^{\mathbf{true}}$ is infinite) We define another interpretation $\mathcal{I}_2$ such that for all $p_i \in \Sigma$ we set $\mathcal{I}_2(p_i) = \mathbf{true}$ if $p_i \in (\Sigma_\Gamma \cap \mathcal{I}_1^{\mathbf{true}})$ and $\mathcal{I}_2(p_i) = \mathbf{false}$ otherwise. As $\mathcal{I}_1$ and $\mathcal{I}_2$ coincide on Σ_Γ, we obtain $\mathcal{I}_2 \in [\![\Gamma]\!]$. Since $\mathcal{I}_2^{\mathbf{true}}$ is finite while $\mathcal{I}_1^{\mathbf{true}}$ is infinite, we have $\mathcal{I}_2 \prec^{\cup}_{\mathcal{K}} \mathcal{I}_1$, which again is a contradiction to the minimality of $\mathcal{I}_1$. □

Fact 3.7. *The base change operator* $\circ^{\cup}$ *for* $\mathbb{PL}_\infty$ *violates (G3) despite being compatible with the faithful preorder assignment* $\preceq^{\cup}_{(.)}$.

To remedy the problem exposed above, one needs to impose the requirement that minima exist whenever needed, as specified in the notion of *min-completeness*, defined next.

Definition 3.8 (min-complete). *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *be a base logic. A binary relation* $\preceq$ *over* Ω *is called* min-complete *(for* $\mathbb{B}$*) if* $\min([\![\Gamma]\!], \preceq) \neq \emptyset$ *holds for every* $\Gamma \in \mathfrak{B}$ *with* $[\![\Gamma]\!] \neq \emptyset$.

The following example demonstrates that for a binary relation it depends on the base logic whether the relation is min-complete or not.

Example 3.9. *Consider two base logics* $\mathbb{B}_{\mathbb{Z}\leq}$ *and* $\mathbb{B}_{\mathbb{Z}\geq}$ *with*

$$\mathbb{B}_{\mathbb{Z}\leq} = (\mathcal{L}_1, \mathbb{Z}, \models, \mathcal{P}_{\text{fin}}(\mathcal{L}_1) \setminus \{\emptyset\}, \cup), \text{ and}$$
$$\mathbb{B}_{\mathbb{Z}\geq} = (\mathcal{L}_2, \mathbb{Z}, \models, \mathcal{P}_{\text{fin}}(\mathcal{L}_2) \setminus \{\emptyset\}, \cup),$$

where $\mathcal{L}_1 = \{[\leq n] \mid n \in \mathbb{Z}\}$ *and* $\mathcal{L}_2 = \{[\geq n] \mid n \in \mathbb{Z}\}$. *Furthermore let* $m \models [\leq n]$ *if* $m \leq n$ *and* $m \models [\geq n]$ *if* $n \leq m$, *assuming the usual meaning of* $\leq$ *for integers. In words, these logics talk about the domain of integers by means of comparisons with a fixed integer. We now define the relation* $\preceq$ *over* Ω *by letting* $m_1 \preceq m_2$ *if and only if* $m_1 \leq m_2$. *It can be verified that the relation is transitive and for any consistent base* $\Gamma \in \mathcal{P}_{\text{fin}}(\mathcal{L}_1)$, *respectively for* $\Gamma \in \mathcal{P}_{\text{fin}}(\mathcal{L}_2)$, *we have infinitely many models* $[\![\Gamma]\!]$.

Note that for each set of sentences of the form $[\leq n] \in \mathcal{L}_1$, *there are no minimal models* $\min([\![\Gamma]\!], \preceq)$, *and thus,* $\preceq$ *is* not *min-complete for* $\mathbb{B}_{\mathbb{Z}\leq}$. *However, for* $\mathbb{B}_{\mathbb{Z}\geq}$, *the relation* $\preceq$ *is min-complete.*

In the special case of $\preceq$ being transitive and total, min-completeness trivially holds whenever Ω is finite (as, e.g., in the case of propositional logic over n propositional atoms; cf. Example 2.6). In the infinite case, however, it might need to be explicitly imposed, as already noted in earlier works [DPW18] (cf. also the notion of *limit assumption* by Lewis [Lew73]). Note that min-completeness does not entirely disallow infinite descending chains (as well-foundedness would), it only ensures that minima exist inside all model sets of consistent belief bases.

3.2.2 Second Problem: Transitivity of Preorder

When generalizing from the setting of propositional to arbitrary base logics, the requirement that assignments must produce preorders (and hence transitive relations) turns out to be too restrictive.

Observation 3.10. *Transitivity of the relation produced by the assignment, as required in Theorem 3.3, is too strict of a property for characterizing arbitrary Tarskian logics.*

In fact, it has been observed before that the incompatibility between transitivity and K&M's approach already arises for propositional Horn logic [DP15]. The following example builds on Example 2.4 and provides an operator and a belief base for which no compatible transitive assignment exists.

Example 3.11 (continuation of Example 2.4). *Consider the base logic $\mathbb{B}_{\mathrm{Ex}} = (\mathcal{L}_{\mathrm{Ex}}, \Omega_{\mathrm{Ex}}, \models_{\mathrm{Ex}}, \mathcal{P}(\mathcal{L}_{\mathrm{Ex}}), \cup)$. Let $\mathcal{K}_{\mathrm{Ex}} = \{\psi_3\}$ and let $\circ_{\mathrm{Ex}}$ be the base change operator defined as follows:*

$$\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma = \begin{cases} \mathcal{K}_{\mathrm{Ex}} \cup \Gamma & \text{if } [\![\mathcal{K}_{\mathrm{Ex}} \cup \Gamma]\!] \neq \emptyset, \\ \Gamma \cup \{\psi_4\} & \text{if } [\![\mathcal{K}_{\mathrm{Ex}} \cup \Gamma]\!] = \emptyset \text{ and } [\![\{\psi_4\} \cup \Gamma]\!] \neq \emptyset, \\ \Gamma \cup \{\psi_0\} & \text{if } [\![\mathcal{K}_{\mathrm{Ex}} \cup \Gamma]\!] = \emptyset \text{ and } [\![\{\psi_0\} \cup \Gamma]\!] \neq \emptyset \text{ and } [\![\{\psi_2\} \cup \Gamma]\!] = \emptyset, \\ \Gamma \cup \{\psi_1\} & \text{if } [\![\mathcal{K}_{\mathrm{Ex}} \cup \Gamma]\!] = \emptyset \text{ and } [\![\{\psi_1\} \cup \Gamma]\!] \neq \emptyset \text{ and } [\![\{\psi_0\} \cup \Gamma]\!] = \emptyset, \\ \Gamma \cup \{\psi_2\} & \text{if } [\![\mathcal{K}_{\mathrm{Ex}} \cup \Gamma]\!] = \emptyset \text{ and } [\![\{\psi_2\} \cup \Gamma]\!] \neq \emptyset \text{ and } [\![\{\psi_1\} \cup \Gamma]\!] = \emptyset, \\ \Gamma & \text{if none of the above applies,} \end{cases}$$

Moreover, for all $\mathcal{K}'$ with $\mathcal{K}' \equiv \mathcal{K}_{\mathrm{Ex}}$ we define $\mathcal{K}' \circ_{\mathrm{Ex}} \Gamma = \mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma$ and for all $\mathcal{K}'$ with $\mathcal{K}' \not\equiv \mathcal{K}_{\mathrm{Ex}}$ we define

$$\mathcal{K}' \circ_{\mathrm{Ex}} \Gamma = \begin{cases} \mathcal{K}' \cup \Gamma & \text{if } \mathcal{K}' \cup \Gamma \text{ consistent} \\ \Gamma & \text{otherwise.} \end{cases}$$

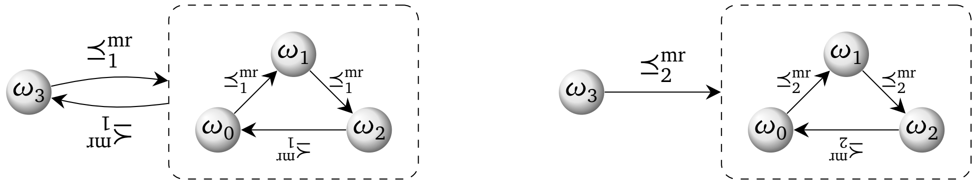
For all $\mathcal{K}'$ with $\mathcal{K}' \not\equiv \mathcal{K}_{\mathrm{Ex}}$, there is no violation of the postulates (G1)–(G6) since we obtain a trivial revision, which satisfies (G1)–(G6) (cf. Example 3.33). For the case of $\mathcal{K}' \equiv \mathcal{K}_{\mathrm{Ex}}$, the satisfaction of (G1)–(G6) can be shown case by case or using Theorem 3.31 in Section 3.4.

Now assume there were a preorder assignment $\preceq_{(.)}$ compatible with $\circ_{\mathrm{Ex}}$. This means that for all bases $\mathcal{K}$ and Γ from $\mathcal{P}(\mathcal{L}_{\mathrm{Ex}})$, the relation $\preceq_{\mathcal{K}}$ is a preorder and $[\![\mathcal{K} \circ_{\mathrm{Ex}} \Gamma]\!] = \min([\![\Gamma]\!], \preceq_{\mathcal{K}_{\mathrm{Ex}}})$. Now consider $\Gamma_0 = \{\varphi_0\}$, $\Gamma_1 = \{\varphi_1\}$, and $\Gamma_2 = \{\varphi_2\}$. From the definition of $\circ_{\mathrm{Ex}}$ and compatibility, we obtain $[\![\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_0]\!] = \{\mathcal{I}_0\} = \min([\![\Gamma_0]\!], \preceq_{\mathcal{K}_{\mathrm{Ex}}})$, $[\![\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_1]\!] = \{\mathcal{I}_1\} = \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}_{\mathrm{Ex}}})$, and $[\![\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_2]\!] = \{\mathcal{I}_2\} = \min([\![\Gamma_2]\!], \preceq_{\mathcal{K}_{\mathrm{Ex}}})$. Recall that $[\![\Gamma_0]\!] = \{\mathcal{I}_0, \mathcal{I}_1\}$, $[\![\Gamma_1]\!] = \{\mathcal{I}_1, \mathcal{I}_2\}$, and $[\![\Gamma_2]\!] = \{\mathcal{I}_2, \mathcal{I}_0\}$. Yet, this implies $\mathcal{I}_0 \prec_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_1$, $\mathcal{I}_1 \prec_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_2$, and $\mathcal{I}_2 \prec_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_0$, contradicting the assumption that $\preceq_{\mathcal{K}_{\mathrm{Ex}}}$ is transitive. Hence it cannot be a preorder.

As a consequence, we cannot help but waive transitivity (and hence the property of the assignment providing a preorder) if we want our characterization result to hold for all Tarskian logics. However, for our result, we need to retain a new, weaker property (which is implied by transitivity) defined next.

Definition 3.12 (min-retractive). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic. A binary relation $\preceq$ over Ω is called* min-retractive *(for $\mathbb{B}$) if, for every $\Gamma \in \mathfrak{B}$ and $\mathcal{I}', \mathcal{I} \in [\![\Gamma]\!]$ with $\mathcal{I}' \preceq \mathcal{I}$, $\mathcal{I} \in \min([\![\Gamma]\!], \preceq)$ implies $\mathcal{I}' \in \min([\![\Gamma]\!], \preceq)$.*

Note that min-retractivity prevents minimal elements from being $\preceq$-equivalent to elements with $\prec$-lower neighbours, for instance elements lying on a "$\prec$-cycle" or elements being part of an infinite descending chain. Consider the following illustrative example.



(a) Not min-retractive relation $\preceq^{\mathrm{mr}}_1$ for $\mathbb{B}_{\mathrm{mr}}$. (b) Min-retractive relation $\preceq^{\mathrm{mr}}_2$ for $\mathbb{B}_{\mathrm{mr}}$.

Figure 3.13: Illustration of the two relations $\preceq^{\mathrm{mr}}_1$ and $\preceq^{\mathrm{mr}}_2$ from Example 3.14.

Example 3.14. *Let $\mathbb{B}_{\mathrm{mr}} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic with just one base $\mathfrak{B} = \{\Gamma_{\mathrm{mr}}\}$ and four interpretations $\Omega = \{\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ such that $[\![\Gamma_{\mathrm{mr}}]\!] = \Omega$. Now consider the following total relation $\preceq^{\mathrm{mr}}_1$ on Ω illustrated in Figure 3.13a and given by*

$$\begin{aligned}
&\omega_i \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_i, \quad 0 \leq i \leq 3, && \omega_0 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_1,\\
&\omega_3 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_i, \quad 0 \leq i \leq 2, && \omega_1 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_2,\\
&\omega_i \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_3, \quad 0 \leq i \leq 2, && \omega_2 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \omega_0.
\end{aligned}$$

We show that $\preceq^{\mathrm{mr}}_1$ is not min-retractive for $\mathbb{B}_{\mathrm{mr}}$. The $\preceq^{\mathrm{mr}}_1$-minimal models of Γ_{mr} are given by $\min([\![\Gamma_{\mathrm{mr}}]\!], \preceq^{\mathrm{mr}}_1) = \{\omega_3\}$. *Observe that ω_0 is a non-minimal model of Γ_{mr} while being $\preceq^{\mathrm{mr}}_1$-equivalent to ω_3, and in particular $\omega_0 \preceq^{\mathrm{mr}}_1 \omega_3$. This is a violation of min-retractivity.*

Let $\preceq^{\mathrm{mr}}_2$ be the same relation as $\preceq^{\mathrm{mr}}_1$, except that $\preceq^{\mathrm{mr}}_2$ strictly prefers ω_3 over all over interpretations, i.e., $\preceq^{\mathrm{mr}}_2 = \preceq^{\mathrm{mr}}_1 \setminus \{(\omega, \omega_3) \mid \omega \neq \omega_3\}$. An illustration of $\preceq^{\mathrm{mr}}_2$ is given in Figure 3.13b. Indeed, we have that $\preceq^{\mathrm{mr}}_2$ is min-retractive for $\mathbb{B}_{\mathrm{mr}}$. In particular, observe that the prior counterexample for $\preceq^{\mathrm{mr}}_1$ does not apply to $\preceq^{\mathrm{mr}}_2$, as we have $\omega_0 \not\preceq^{\mathrm{mr}}_2 \omega_3$.

As an aside, let us note that, if $\preceq$ is total but not transitive, min-completeness can be violated even in the setting where Ω is finite, by means of strict cyclic relationships.

Example 3.15. *Let $\mathbb{B}_{\mathrm{rps}} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \cup)$ be the base logic defined by $\mathcal{L} = \{$*ALL-THREE*$\}$ and $\Omega = \{$✊, ✋, ✌$\}$, with the models relation $\models$ given by $[\![$*ALL-THREE*$]\!] = \Omega$. We now define the relation $\preceq^{\mathrm{rps}}$ as the common game "rock-paper-scissors": paper beats rock (✋ $\prec^{\mathrm{rps}}$ ✊), scissors beat paper (✌ $\prec^{\mathrm{rps}}$ ✋), and rock beats scissors (✊ $\prec^{\mathrm{rps}}$ ✌). Clearly, the set of interpretations Ω is finite and the relation $\preceq^{\mathrm{rps}}$ is total, but not transitive. It is, however vacuously min-retractive. By considering a consistent base Γ containing the only sentence* ALL-THREE, *we find that* $\min([\![\Gamma]\!], \preceq^{\mathrm{rps}}) = \emptyset$, *and hence a violation of min-completeness.*

As a last act in this section, we conveniently unite the two identified properties into one notion.

Definition 3.16 (min-friendly). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic. A binary relation $\preceq$ over Ω is called* min-friendly *(for $\mathbb{B}$) if it is both min-retractive and min-complete. An assignment $\preceq_{(.)} \colon \mathfrak{B} \to \mathcal{P}(\Omega \times \Omega)$ is called min-friendly if $\preceq_{\mathcal{K}}$ is min-friendly for all $\mathcal{K} \in \mathfrak{B}$.*

3.3 One-Way Representation Theorem

We are now ready to generalize K&M's representation theorem from propositional to arbitrary Tarskian logics, by employing the notion of compatible min-friendly faithful assignments.

Theorem 3.17. *Let $\circ$ be a base change operator for some base logic $\mathbb{B}$. Then, $\circ$ satisfies (G1)–(G6) if and only if it is compatible with some min-friendly faithful assignment for $\mathbb{B}$.*

We show Theorem 3.17 in three steps. First, we provide a canonical way of obtaining an assignment for a given revision operator. Next, we show that our construction indeed yields a min-friendly faithful assignment that is compatible with the revision operator. Finally, we show that the notion of min-friendly compatible assignment is adequate to capture the class of base revision operators satisfying (G1)–(G6).

3.3.1 From Postulates to Assignments

Very central for the original result by Katsuno and Mendelzon 1991 is a constructive way to obtain the assignment from a revision operator. In their proof for Theorem 3.3, they provided the following way of extracting the preference relations from the revision operator:

$$\mathcal{I}_1 \leq_{\mathcal{K}} \mathcal{I}_2 \text{ if } \mathcal{I}_1 \models \mathcal{K} \text{ or } \mathcal{I}_1 \models \mathcal{K} \circ \mathit{form}(\mathcal{I}_1, \mathcal{I}_2) \tag{3.1}$$

where $\mathit{form}(\mathcal{I}_1, \mathcal{I}_2) \in \mathcal{L}$ denotes a sentence with $[\![\mathit{form}(\mathcal{I}_1, \mathcal{I}_2)]\!] = \{\mathcal{I}_1, \mathcal{I}_2\}$. Unfortunately, this method for obtaining a canonical encoding of the revision strategy of $\circ$ does not generalize to the general setting here. This is because a belief base Γ satisfying $[\![\Gamma]\!] = \{\mathcal{I}_1, \mathcal{I}_2\}$ may not exist.

As a recourse, we suggest the following construction, which we consider one of this thesis's core contributions. It realizes the idea that one should (strictly) prefer ω_1 over ω_2 only if there is a witness belief base Γ that certifies that $\circ$ prefers ω_1 over ω_2. Should no such witness exist, ω_1 and ω_2 will be deemed equally preferable.

Definition 3.18. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic, let $\circ$ be a base change operator for $\mathbb{B}$ and let $\mathcal{K} \in \mathfrak{B}$ be a belief base. The relation $\sqsubseteq_{\mathcal{K}}^{\circ}$ over Ω is defined by*

$$\mathcal{I}_1 \sqsubseteq_{\mathcal{K}}^{\circ} \mathcal{I}_2 \;\; \textit{if} \;\; \mathcal{I}_2 \models \mathcal{K} \circ \Gamma \textit{ implies } \mathcal{I}_1 \models \mathcal{K} \circ \Gamma \textit{ for all } \Gamma \in \mathfrak{B} \textit{ with } \mathcal{I}_1, \mathcal{I}_2 \in [\![\Gamma]\!].$$

Definition 3.18 already yields an adequate encoding strategy for many base logics. However, to also properly cope with certain "degenerate" base logics, we have to hard-code that the prior beliefs of an agent are prioritized in all cases, that is, only models of the prior beliefs are minimal. In Section 3.8.2 we will analyze this in more detail. The following relation builds upon the relation $\sqsubseteq_{\mathcal{K}}^{\circ}$ and takes explicit care of handling prior beliefs.

Definition 3.19. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic, let $\circ$ be a base change operator for $\mathbb{B}$ and let $\mathcal{K} \in \mathfrak{B}$ be a belief base. The relation $\preceq^{\circ}_{\mathcal{K}}$ over Ω is then defined by*

$$\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2 \text{ if } \mathcal{I}_1 \models \mathcal{K} \text{ or } \big(\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K} \text{ and } \mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2\big).$$

Let $\preceq^{\circ}_{(.)} \colon \mathfrak{B} \to \mathcal{P}(\Omega \times \Omega)$ denote the mapping $\mathcal{K} \mapsto \preceq^{\circ}_{\mathcal{K}}$.

In the following, we apply the relation encoding given in Definition 3.19 to our running example and show that the relation is not transitive, yet min-friendly.

Example 3.20 (continuation of Example 3.11). *Applying Definition 3.19 to $\mathcal{K}_{\text{Ex}}$ and $\circ_{\text{Ex}}$ yields the following relation $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ on Ω_{Ex} (where $\omega \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega'$ denotes $\omega \preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega'$ and $\omega' \not\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega$):*

$$\begin{array}{lll}
\omega_i \preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_i, \quad 0 \leq i \leq 5 & \omega_0 \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_1 & \omega_4 \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_i, \quad i \in \{0,1,2,5\} \\
\omega_3 \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_i, \quad i \in \{0,1,2,4,5\} & \omega_1 \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_2 & \omega_i \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_5, \quad 0 \leq i < 4 \\
 & \omega_2 \prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega_0 &
\end{array}$$

Observe that $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ is not transitive, since $\omega_0, \omega_1, \omega_2$ form a $\prec^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$-circle (see Figure 3.21). Yet, one can easily verify that $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ is a total and min-friendly relation. In particular, as Ω_{Ex} is finite, min-completeness can be checked by examining minimal model sets of all consistent bases in $\mathbb{L}_{\text{Ex}}$. Moreover, there is no belief base $\Gamma \in \mathcal{P}(\mathcal{L}_{\text{Ex}})$ such that there is some $\omega \notin \min(\llbracket \Gamma \rrbracket, \preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}})$ and $\omega' \in \min(\llbracket \Gamma \rrbracket, \preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}})$ with $\omega \preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}} \omega'$. Note that such a situation could appear in $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ if an interpretation ω would be $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$-equivalent to ω_0, ω_1 and ω_2 and there would be a belief base Γ satisfied in all these interpretations, e.g., if $\omega = \omega_5$ would be equal to ω_0, ω_1 and ω_2, and $\llbracket \Gamma \rrbracket = \{\omega_0, \omega_1, \omega_2, \omega_5\}$. However, this is not the case in $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ and such a belief base Γ does not exist in $\mathbb{B}_{\text{Ex}}$. Therefore, the relation $\preceq^{\circ_{\text{Ex}}}_{\mathcal{K}_{\text{Ex}}}$ is min-retractive.

As a first insight, we obtain that the construction in Definition 3.19 is strong enough for always obtaining a relation that is total and reflexive.

Lemma 3.22 (totality). *If $\circ$ satisfies (G5) and (G6), the relations $\preceq^{\circ}_{\mathcal{K}}$ and $\sqsubseteq^{\circ}_{\mathcal{K}}$ are total (and hence reflexive) for every $\mathcal{K} \in \mathfrak{B}$.*

Proof. Note that by construction, totality of $\preceq^{\circ}_{\mathcal{K}}$ is an immediate consequence of totality of $\sqsubseteq^{\circ}_{\mathcal{K}}$. We show the latter by contradiction: Assume the contrary, i.e. there are $\sqsubseteq^{\circ}_{\mathcal{K}}$-incomparable $\mathcal{I}_1$ and $\mathcal{I}_2$. Due to Definition 3.18, there must exist $\Gamma_1, \Gamma_2 \in \mathfrak{B}$ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma_1$ and $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma_2$, such that $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma_1$ and $\mathcal{I}_2 \not\models \mathcal{K} \circ \Gamma_1$ whereas $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma_2$ and $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma_2$. Since $\mathcal{I}_1 \in \llbracket \mathcal{K} \circ \Gamma_1 \rrbracket \cap \llbracket \Gamma_2 \rrbracket = \llbracket (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \rrbracket$ and thus $\llbracket (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \rrbracket \neq \emptyset$, (G5) and (G6) jointly entail $\llbracket (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \rrbracket = \llbracket \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \rrbracket$. From commutativity of $\Cup$, $\llbracket \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \rrbracket = \llbracket \mathcal{K} \circ (\Gamma_2 \Cup \Gamma_1) \rrbracket$ follows. Now again, since $\mathcal{I}_2 \in \llbracket \mathcal{K} \circ \Gamma_2 \rrbracket \cap \llbracket \Gamma_1 \rrbracket = \llbracket (\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1 \rrbracket$ and hence $\llbracket (\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1 \rrbracket \neq \emptyset$,

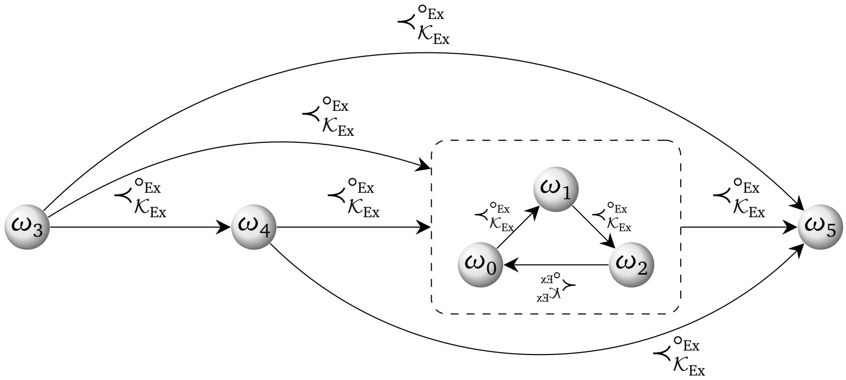


Figure 3.21: The structure of relation $\preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}}$ on Ω_{Ex}, where a solid arrow represents $\omega \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}} \omega'$ for any $\omega, \omega' \in \Omega_{\mathrm{Ex}}$.

(G5) and (G6) together entail $[\![\mathcal{K} \circ (\Gamma_2 \Cup \Gamma_1)]\!] = [\![(\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1]\!]$. So, together, we obtain $\mathcal{I}_1 \in [\![(\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1]\!] = [\![\mathcal{K} \circ \Gamma_2]\!] \cap [\![\Gamma_1]\!]$ which directly contradicts our assumption $\mathcal{I}_1 \notin [\![\mathcal{K} \circ \Gamma_2]\!]$.

Reflexivity follows immediately from totality. $\square$

We proceed with an auxiliary lemma about belief bases and $\preceq^{\circ}_{\mathcal{K}}$.

Lemma 3.23. *Let $\circ$ satisfy (G2), (G5) and (G6) and let $\mathcal{K} \in \mathfrak{B}$. Then the following hold:*

(a) If $\mathcal{I}_1 \npreceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ and $\mathcal{I}_2 \not\models \mathcal{K}$, then there exists some Γ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma$ as well as $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma$ and $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma$.

(b) If there is a Γ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma$ such that $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma$, then $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$.

(c) If there is a Γ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma$ such that $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma$ and $\mathcal{I}_2 \not\models \mathcal{K} \circ \Gamma$, then $\mathcal{I}_1 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_2$.

Proof. For the proofs of all statements, recall that by Lemma 3.22, the relation $\preceq^{\circ}_{\mathcal{K}}$ is total.

(a) By totality of $\preceq^{\circ}_{\mathcal{K}}$, guarateed by Lemma 3.22 , we obtain $\mathcal{I}_2 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$. By definition of $\preceq^{\circ}_{\mathcal{K}}$, this together with $\mathcal{I}_2 \not\models \mathcal{K}$ entails $\mathcal{I}_1 \not\models \mathcal{K}$. Therefore, again by definition, we obtain $\mathcal{I}_1 \not\sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. Consequently, in view of Definition 3.18, there must exist some $\Gamma \in \mathfrak{B}$ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma$ such that $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma$ does not imply $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma$. Yet this can only be the case if $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma$ and $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma$, as claimed.

(b) Let Γ and $\mathcal{I}_1, \mathcal{I}_2$ be as assumed. We proceed by case distinction:

$\mathcal{I}_2 \models \mathcal{K}$. Then $\mathcal{I}_2 \in [\![\mathcal{K}]\!] \cap [\![\Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$ and thus $[\![\mathcal{K} \Cup \Gamma]\!] \neq \emptyset$. Therefore, by (G2), we obtain $[\![\mathcal{K} \circ \Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!] = [\![\mathcal{K}]\!] \cap [\![\Gamma]\!]$ and consequently $\mathcal{I}_1 \models \mathcal{K}$. By Definition 3.19, we conclude $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$.

$\mathcal{I}_2 \not\models \mathcal{K}$. Toward a contradiction, suppose $\mathcal{I}_1 \npreceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. Then, by part (a) above, there is a Γ' with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma'$, $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma'$ and $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma'$. Thus $\mathcal{I}_1$ and $\mathcal{I}_2$ witness non-emptiness of $[\![(\mathcal{K} \circ \Gamma) \Cup \Gamma']\!]$ and $[\![(\mathcal{K} \circ \Gamma') \Cup \Gamma]\!]$, respectively. Then, using (G5) and

(G6) twice, we obtain $(\mathcal{K}\circ\Gamma')\Cup\Gamma \equiv \mathcal{K}\circ(\Gamma\Cup\Gamma') \equiv (\mathcal{K}\circ\Gamma)\Cup\Gamma'$. But this allows to conclude $\mathcal{I}_1 \in [\![\mathcal{K}\circ\Gamma]\!]\cap[\![\Gamma']\!] = [\![(\mathcal{K}\circ\Gamma)\Cup\Gamma']\!] = [\![(\mathcal{K}\circ\Gamma')\Cup\Gamma]\!] = [\![\mathcal{K}\circ\Gamma']\!]\cap[\![\Gamma]\!] \subseteq [\![\mathcal{K}\circ\Gamma']\!]$, and thus $\mathcal{I}_1 \models \mathcal{K}\circ\Gamma'$, which contradicts $\mathcal{I}_1 \not\models \mathcal{K}\circ\Gamma'$ above. $\square$

(c) Let Γ and $\mathcal{I}_1,\mathcal{I}_2$ be as assumed. We already know $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ due to part (b). It remains to show $\mathcal{I}_2 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$. We proceed by case distinction:

$\mathcal{I}_1 \models \mathcal{K}$. Then $\mathcal{I}_1 \in [\![\mathcal{K}]\!]\cap[\![\Gamma]\!] = [\![\mathcal{K}\Cup\Gamma]\!]$ and thus $[\![\mathcal{K}\Cup\Gamma]\!] \neq \emptyset$. Therefore, by (G2), we obtain $[\![\mathcal{K}\circ\Gamma]\!] = [\![\mathcal{K}\Cup\Gamma]\!] = [\![\mathcal{K}]\!]\cap[\![\Gamma]\!]$. Since $\mathcal{I}_2 \not\models \mathcal{K}\circ\Gamma$ but $\mathcal{I}_2 \models \Gamma$ we can infer $\mathcal{I}_2 \not\models \mathcal{K}$. Consequently, by Definition 3.19, we obtain $\mathcal{I}_2 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$.

$\mathcal{I}_1 \not\models \mathcal{K}$. Since we already established $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$, Definition 3.19 ensures $\mathcal{I}_2 \not\models \mathcal{K}$. Yet, by Definition 3.18, the existence of Γ implies $\mathcal{I}_2 \not\sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_1$, and thus Definition 3.19 yields $\mathcal{I}_2 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$.

We show that our construction indeed yields a compatible assignment.

Lemma 3.24 (compatibility). *If $\circ$ satisfies (G1)–(G3), (G5), and (G6), then it is compatible with $\preceq^{\circ}_{(.)}$.*

Proof. We have to show that $[\![\mathcal{K}\circ\Gamma]\!] = \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$. In the following, we show inclusion in both directions.

($\subseteq$) Let $\mathcal{I} \in [\![\mathcal{K}\circ\Gamma]\!]$. By (G1), we obtain $\mathcal{I} \in [\![\Gamma]\!]$. But then, using Lemma 3.23(b), we can conclude $\mathcal{I} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}'$ for any $\mathcal{I}' \in [\![\Gamma]\!]$, hence $\mathcal{I} \in \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$.

($\supseteq$) Let $\mathcal{I} \in \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$. Due to $[\![\Gamma]\!] \neq \emptyset$ and (G3), there exists an $\mathcal{I}' \in [\![\mathcal{K}\circ\Gamma]\!]$. From the ($\subseteq$)-proof follows $\mathcal{I}' \in \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$. Then, by (G1) and Lemma 3.23(b), we obtain $\mathcal{I}' \preceq^{\circ}_{\mathcal{K}} \mathcal{I}$ from $\mathcal{I} \in [\![\Gamma]\!]$ and $\mathcal{I}' \in [\![\Gamma]\!]$ and $\mathcal{I}' \in [\![\mathcal{K}\circ\Gamma]\!]$. From $\mathcal{I} \in \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$ and $\mathcal{I}' \in [\![\Gamma]\!]$ follows $\mathcal{I} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}'$. We proceed by case distinction:

$\mathcal{I} \models \mathcal{K}$. Then $\mathcal{I} \in [\![\mathcal{K}]\!]\cap[\![\Gamma]\!] = [\![\mathcal{K}\Cup\Gamma]\!]$ and thus $[\![\mathcal{K}\Cup\Gamma]\!] \neq \emptyset$. Therefore, by (G2), we obtain $[\![\mathcal{K}\circ\Gamma]\!] = [\![\mathcal{K}\Cup\Gamma]\!] = [\![\mathcal{K}]\!]\cap[\![\Gamma]\!]$ and hence $\mathcal{I} \in [\![\mathcal{K}\circ\Gamma]\!]$.

$\mathcal{I} \not\models \mathcal{K}$. Then by Definition 3.19, $\mathcal{I} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}'$ requires $\mathcal{I}' \not\models \mathcal{K}$ and therefore $\mathcal{I} \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}'$ must hold. Consequently, by Definition 3.18, $\mathcal{I},\mathcal{I}' \in [\![\Gamma]\!]$ and $\mathcal{I}' \in [\![\mathcal{K}\circ\Gamma]\!]$ imply $\mathcal{I} \in [\![\mathcal{K}\circ\Gamma]\!]$. $\square$

For min-friendliness, we have to show that each $\preceq^{\circ}_{\mathcal{K}}$ is min-complete and min-retractive.

Lemma 3.25 (min-friendliness). *If $\circ$ satisfies (G1)–(G3), (G5), and (G6), then $\preceq^{\circ}_{\mathcal{K}}$ is min-friendly for every $\mathcal{K} \in \mathfrak{B}$.*

Proof. Observe that min-completeness is a consequence of (G3) and the compatibility of $\preceq^{\circ}_{(.)}$ with $\circ$ from Lemma 3.24.

For min-retractivity, suppose towards a contradiction that it does not hold. That means there is a belief base Γ and interpretations $\mathcal{I}',\mathcal{I} \models \Gamma$ with $\mathcal{I}' \preceq^{\circ}_{\mathcal{K}} \mathcal{I}$ and $\mathcal{I} \in \min([\![\Gamma]\!],\preceq^{\circ}_{\mathcal{K}})$ but

$\mathcal{I}' \notin \min([\![\Gamma]\!], \preceq^{\circ}_{\mathcal{K}})$. From Lemma 3.24 we obtain $\mathcal{I} \models \mathcal{K} \circ \Gamma$ and $\mathcal{I}' \not\models \mathcal{K} \circ \Gamma$. Now, applying Lemma 3.23(c) yields $\mathcal{I} \prec^{\circ}_{\mathcal{K}} \mathcal{I}'$, contradicting $\mathcal{I}' \preceq^{\circ}_{\mathcal{K}} \mathcal{I}$. □

We show that $\preceq^{\circ}_{(.)}$ yields faithful relations for every belief base.

Lemma 3.26 (faithfulness). *If $\circ$ satisfies (G2), (G4), (G5), and (G6), the assignment $\preceq^{\circ}_{(.)}$ is faithful.*

Proof. We show satisfaction of the three conditions of faithfulness, (F1)–(F3).

(F1) Let $\mathcal{I}, \mathcal{I}' \in [\![\mathcal{K}]\!]$. Then $\mathcal{I}' \preceq^{\circ}_{\mathcal{K}} \mathcal{I}$ is an immediate consequence of Definition 3.19. This implies $\mathcal{I} \not\prec^{\circ}_{\mathcal{K}} \mathcal{I}'$.

(F2) Let $\mathcal{I} \in [\![\mathcal{K}]\!]$ and $\mathcal{I}' \notin [\![\mathcal{K}]\!]$. By Definition 3.19 we obtain $\mathcal{I} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}'$ and $\mathcal{I}' \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}$.

(F3) Let $\mathcal{K} \equiv \mathcal{K}'$ (i.e. $[\![\mathcal{K}]\!] = [\![\mathcal{K}']\!]$). From Definition 3.19 and (G4) follows $\preceq^{\circ}_{\mathcal{K}} = \preceq^{\circ}_{\mathcal{K}'}$, i.e., $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ if and only if $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}'} \mathcal{I}_2$.

The previous lemmas can finally be used to show that the construction of $\preceq^{\circ}_{(.)}$ according to Definition 3.19 yields an assignment with the desired properties.

Proposition 3.27. *If $\circ$ satisfies (G1)–(G6), then $\preceq^{\circ}_{(.)}$ is a min-friendly faithful assignment compatible with $\circ$.*

Proof. Assume (G1)–(G6) are satisfied by $\circ$. Then $\preceq^{\circ}_{(.)}$ is an assignment since every $\preceq_{\mathcal{K}}$ is total by Lemma 3.22; it is min-friendly by Lemma 3.25; it is faithful by Lemma 3.26; and it is compatible with $\circ$ by Lemma 3.24. □

3.3.2 From Assignments to Postulates

Now, it remains to show the "if" direction of Theorem 3.17.

Proposition 3.28. *If there exists a min-friendly faithful assignment $\preceq_{(.)}$ compatible with $\circ$, then $\circ$ satisfies (G1)–(G6).*

Proof. Let $\preceq_{(.)}: \mathcal{K} \mapsto \preceq_{\mathcal{K}}$ be as described. We now show that $\circ$ satisfies all of (G1)–(G6).

(G1) Let $\mathcal{I} \in [\![\mathcal{K} \circ \Gamma]\!]$. Since $[\![\mathcal{K} \circ \Gamma]\!] = \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$, we have that $\mathcal{I} \in \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$. Then, we also have that $\mathcal{I} \in [\![\Gamma]\!]$. Thus, we have that $[\![\mathcal{K} \circ \Gamma]\!] \subseteq [\![\Gamma]\!]$ as desired.

(G2) Assume $[\![\mathcal{K} \Cup \Gamma]\!] \neq \emptyset$. We prove $[\![\mathcal{K} \circ \Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$ by showing inclusion in both directions.

  (⊆) Let $\mathcal{I} \in [\![\mathcal{K} \circ \Gamma]\!]$. By compatibility, we obtain $\mathcal{I} \in \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$ and thus trivially also $\mathcal{I} \in [\![\Gamma]\!]$. Since $[\![\mathcal{K} \Cup \Gamma]\!] \neq \emptyset$, there exists some other $\mathcal{I}' \in [\![\mathcal{K} \Cup \Gamma]\!] = [\![\mathcal{K}]\!] \cap [\![\Gamma]\!]$, which implies $\mathcal{I}' \in [\![\mathcal{K}]\!]$ and $\mathcal{I}' \in [\![\Gamma]\!]$. Therefore, $\mathcal{I} \in \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$ implies $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$, which means that $\mathcal{I}' \prec_{\mathcal{K}} \mathcal{I}$ cannot hold and therefore, by contraposition, (F2) ensures $\mathcal{I} \in [\![\mathcal{K}]\!]$. Yet then $\mathcal{I} \in [\![\mathcal{K}]\!] \cap [\![\Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$ as desired.

($\supseteq$) Let $\mathcal{I} \in [\![\mathcal{K} \Cup \Gamma]\!] = [\![\mathcal{K}]\!] \cap [\![\Gamma]\!]$, i.e. $\mathcal{I} \in [\![\mathcal{K}]\!]$ and $\mathcal{I} \in [\![\Gamma]\!]$. Since $\mathcal{I} \in [\![\mathcal{K}]\!]$, we obtain from (F1) and (F2) that $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$ must hold for all $\mathcal{I}' \in [\![\Gamma]\!]$. Hence, $\mathcal{I} \in \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$, and by compatibility $\mathcal{I} \in [\![\mathcal{K} \circ \Gamma]\!]$.

(G3) Assume $[\![\Gamma]\!] \neq \emptyset$. By min-completeness, we have $\min([\![\Gamma]\!], \preceq_{\mathcal{K}}) \neq \emptyset$. Since $[\![\mathcal{K} \circ \Gamma]\!] = \min([\![\Gamma]\!], \preceq_{\mathcal{K}})$ by compatibility, we obtain $[\![\mathcal{K} \circ \Gamma]\!] \neq \emptyset$.

(G4) Suppose there exist $\mathcal{K}_1, \mathcal{K}_2, \Gamma_1, \Gamma_2 \in \mathfrak{B}$ with $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\Gamma_1 \equiv \Gamma_2$. Then, $[\![\mathcal{K}_1]\!] = [\![\mathcal{K}_2]\!]$ and $[\![\Gamma_1]\!] = [\![\Gamma_2]\!]$. From (F3), we conclude $\preceq_{\mathcal{K}_1} = \preceq_{\mathcal{K}_2}$. Now assume some $\mathcal{I} \in [\![\mathcal{K}_1 \circ \Gamma_1]\!]$, then by compatibility $\mathcal{I} \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}_1}) = \min([\![\Gamma_2]\!], \preceq_{\mathcal{K}_2})$. Therefore, again by compatibility, $\mathcal{I} \in [\![\mathcal{K}_2 \circ \Gamma_2]\!])$. Thus, $[\![\mathcal{K}_1 \circ \Gamma_1]\!] \subseteq [\![\mathcal{K}_2 \circ \Gamma_2]\!]$ holds. Inclusion in the other direction follows by symmetry. Therefore, we have $\mathcal{K}_1 \circ \Gamma_1 \equiv \mathcal{K}_2 \circ \Gamma_2$.

(G5) Let $\mathcal{I} \in [\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!] = [\![\mathcal{K} \circ \Gamma_1]\!] \cap [\![\Gamma_2]\!]$. This means that $\mathcal{I} \in [\![\Gamma_2]\!]$ but – since $[\![\mathcal{K} \circ \Gamma_1]\!] = \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}})$ by compatibility – we also obtain $\mathcal{I} \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}})$, meaning that $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$ holds for all $\mathcal{I}' \in [\![\Gamma_1]\!]$. Yet then $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$ holds particularly for all $\mathcal{I}' \in [\![\Gamma_1]\!] \cap [\![\Gamma_2]\!]$ and hence $\mathcal{I} \in \min([\![\Gamma_1]\!] \cap [\![\Gamma_2]\!], \preceq_{\mathcal{K}}) = \min([\![\Gamma_1 \Cup \Gamma_2]\!], \preceq_{\mathcal{K}})$. By compatibility follows $\mathcal{I} \in [\![\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)]\!]$. Thus $[\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!] \subseteq [\![\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)]\!]$ as desired.

(G6) Let $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \neq \emptyset$, thus $\mathcal{I}' \in [\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!] = [\![\mathcal{K} \circ \Gamma_1]\!] \cap [\![\Gamma_2]\!]$ for some $\mathcal{I}'$. By compatibility, we then obtain $\mathcal{I}' \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}})$. Now consider an arbitrary $\mathcal{I}$ with $\mathcal{I} \in [\![\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)]\!]$. By compatibility we obtain $\mathcal{I} \in \min([\![\Gamma_1 \Cup \Gamma_2]\!], \preceq_{\mathcal{K}})$ and therefore, since $\mathcal{I}' \in [\![\Gamma_1]\!] \cap [\![\Gamma_2]\!] = [\![\Gamma_1 \Cup \Gamma_2]\!]$, we can conclude $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$. This and $\mathcal{I}' \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}})$ imply $\mathcal{I} \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}})$ by min-retractivity. Hence every $\mathcal{I} \in [\![\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)]\!]$ satisfies $\mathcal{I} \in \min([\![\Gamma_1]\!], \preceq_{\mathcal{K}}) = [\![\mathcal{K} \circ \Gamma_1]\!]$ but also $\mathcal{I} \in [\![\Gamma_2]\!]$, whence $[\![\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)]\!] \subseteq [\![\mathcal{K} \circ \Gamma_1]\!] \cap [\![\Gamma_2]\!] = [\![(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2]\!]$ as desired.

The proof of Theorem 3.17 follows from Proposition 3.27 and 3.28.

3.4 Two-Way Representation Theorem

Theorem 3.17 establishes the correspondence between operators and assignments under the assumption that $\circ$ is given and therefore known to exist. What remains unsettled is the question if generally **every** min-friendly faithful assignment is compatible with some base change operator that satisfies (G1)–(G6). It is not hard to see that this is not the case.

Example 3.29. *Consider the base logic $\mathbb{B}_{\text{nb}} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \cup)$ with $\mathcal{L} = \{none, both\}$ and $\Omega = \{\mathcal{I}_1, \mathcal{I}_2\}$ satisfying $[\![none]\!] = \emptyset$ and $[\![both]\!] = \{\mathcal{I}_1, \mathcal{I}_2\} = \Omega$. There are four bases in this logic, satisfying $\{none\} \equiv \{none, both\}$ and $\emptyset \equiv \{both\}$. Let the assignment $\preceq^{\text{nb}}_{(.)}$ be such that $\preceq^{\text{nb}}_{\{\}} = \preceq^{\text{nb}}_{\{both\}} = \Omega \times \Omega$ and $\preceq^{\text{nb}}_{\{none\}} = \preceq^{\text{nb}}_{\{none,both\}} = \{(\mathcal{I}_1, \mathcal{I}_1), (\mathcal{I}_1, \mathcal{I}_2), (\mathcal{I}_2, \mathcal{I}_2)\}$. It is straightforward to check that $\preceq^{\text{nb}}_{(.)}$ is a min-friendly faithful assignment. Note that any $\circ$*

compatible with $\preceq^{\mathrm{nb}}_{(.)}$ *would have to satisfy* $[\![\{none\}\circ\{both\}]\!] = \min([\![\{both\}]\!], \preceq^{\mathrm{nb}}_{\{none\}}) = \{\mathcal{I}_1\}$, *yet, as we have seen, no base with this model set exists, therefore such a* $\circ$ *is impossible.*

Therefore, toward a full, two-way correspondence, we have to provide an additional condition on assignments, capturing the operator existence.

As indicated by the example, for the existence of an operator, it will turn out to be essential that any minimal model set of a belief base obtained from an assignment corresponds to some belief base, a property which is formalized by the following notion.

Definition 3.30 (min-expressible). *Let* $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ *be a base logic. A binary relation* $\preceq$ *over* Ω *is called* min-expressible *if for each* $\Gamma \in \mathfrak{B}$ *there exists a belief base* $\mathcal{B}_{\Gamma,\preceq} \in \mathfrak{B}$ *such that* $[\![\mathcal{B}_{\Gamma,\preceq}]\!] = \min([\![\Gamma]\!], \preceq)$. *An assignment* $\preceq_{(.)}$ *will be called min-expressible, if for each* $\mathcal{K} \in \mathfrak{B}$, *the relation* $\preceq_{\mathcal{K}}$ *is min-expressible. Given a min-expressible assignment* $\preceq_{(.)}$, *let* $\circ_{\preceq_{(.)}}$ *denote the base change operator defined by* $\mathcal{K} \circ_{\preceq_{(.)}} \Gamma = \mathcal{B}_{\Gamma,\preceq_{\mathcal{K}}}$.

It should be noted that min-expressibility is a straightforward generalization of the notion of *regularity* by Delgrande, Peppas, and Woltran 2018 to base logics. By virtue of this extra notion, we now find the following bidirectional relationship between assignments and operators, amounting to a full characterization.

Theorem 3.31. *Let* $\mathbb{B}$ *be a base logic. Then the following hold:*

- *Every base change operator for* $\mathbb{B}$ *satisfying (G1)–(G6) is compatible with some min-expressible min-friendly faithful assignment.*
- *Every min-expressible min-friendly faithful assignment for* $\mathbb{B}$ *is compatible with some base change operator satisfying (G1)–(G6).*

Proof. For the first item, let $\circ$ be the corresponding base change operator. Then, by Proposition 3.27, the assignment $\preceq^{\circ}_{(.)}$ as given in Definition 3.19 is min-friendly, faithful, and compatible with $\circ$. As for min-expressibility, recall that, by compatibility, $[\![\mathcal{K} \circ \Gamma]\!] = \min([\![\Gamma]\!], \preceq^{\circ}_{\mathcal{K}})$ for every Γ. As $\mathcal{K} \circ \Gamma$ is a belief base, min-expressibility follows immediately.

For the second item, let $\preceq_{(.)}$ be the corresponding min-expressible assignment and $\circ_{\preceq_{(.)}}$ as provided in Definition 3.30. By construction, $\circ_{\preceq_{(.)}}$ is compatible with $\preceq_{(.)}$. Proposition 3.28 implies that $\circ_{\preceq_{(.)}}$ satisfies (G1)–(G6). $\square$

As an aside, note that the above theorem also implies that every min-expressible min-friendly faithful assignment is compatible **only** with AGM base change operators. This is due to the fact that, one the one hand, any such assignment fully determines the corresponding compatible base change operator model-theoretically and, on the other hand, (G1)–(G6) are purely model-theoretic conditions.

Continuing our running example, we observe that $\preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}}$ is also a min-expressible relation.

Example 3.32 (continuation of Example 3.20). *Consider again $\preceq_{(.)}^{\circ_{\mathrm{Ex}}}$, and observe that $\preceq_{(.)}^{\circ_{\mathrm{Ex}}}$ is compatible with $\circ_{\mathrm{Ex}}$, e.g. $[\![\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma]\!] = \min([\![\Gamma]\!], \preceq_{\mathcal{K}_{\mathrm{Ex}}}^{\circ_{\mathrm{Ex}}})$. Thus, for every belief base $\Gamma \in \mathcal{P}(\mathcal{L}_{\mathrm{Ex}})$, the minimum $\min(\Gamma, \preceq_{\mathcal{K}_{\mathrm{Ex}}}^{\circ_{\mathrm{Ex}}})$ yields a set expressible by a belief base. Theorem 3.31 guarantees us that $\circ_{\mathrm{Ex}}$ satisfies (G1)–(G6), as $\preceq_{(.)}^{\circ_{\mathrm{Ex}}}$ is a faithful min-expressible and min-friendly assignment.*

As a last step of this section, we will apply the theory developed here to demonstrate that the standard operator of trivial revision[1] [Han99; FH18] indeed satisfies (G1)–(G6) in the general setting of base logics.

Example 3.33. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathcal{P}(\mathcal{L}), \Cup)$ be an arbitrary base logic. We define the* trivial revision operator *$\circ^{\mathrm{fm}}$ for $\mathbb{B}$ by*

$$\mathcal{K} \circ^{\mathrm{fm}} \Gamma = \begin{cases} \mathcal{K} \Cup \Gamma & \text{if } [\![\mathcal{K} \Cup \Gamma]\!] \text{ is consistent} \\ \Gamma & \text{otherwise} \end{cases}$$

To show satisfaction of (G1)–(G6) we construct a min-expressible min-friendly faithful assignment $\preceq_{(.)}^{\mathrm{fm}}$ compatible with $\circ^{\mathrm{fm}}$. For each $\mathcal{K} \in \mathfrak{B}$ let $\mathcal{I}_1 \preceq_{\mathcal{K}}^{\mathrm{fm}} \mathcal{I}_2$ if $\mathcal{I}_1 \models \mathcal{K}$ or $\mathcal{I}_2 \not\models \mathcal{K}$. Obviously, the relation $\preceq_{\mathcal{K}}^{\mathrm{fm}}$ is a total preorder where $\mathcal{I}_1, \mathcal{I}_2$ are $\preceq_{\mathcal{K}}^{\mathrm{fm}}$-equivalent, if either $\mathcal{I}_1, \mathcal{I}_2 \models \mathcal{K}$ or $\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$ holds. Moreover, it is not hard to see that the relation $\preceq_{\mathcal{K}}^{\mathrm{fm}}$ is min-complete and min-retractive. By construction of $\preceq_{(.)}^{\mathrm{fm}}$ we obtain that $\min([\![\Gamma]\!], \preceq_{\mathcal{K}}^{\mathrm{fm}}) = [\![\Gamma]\!]$ if $\mathcal{K} \Cup \Gamma$ is inconsistent. If $\mathcal{K} \Cup \Gamma$ is consistent, we obtain $\min([\![\Gamma]\!], \preceq_{\mathcal{K}}^{\mathrm{fm}}) = [\![\mathcal{K}]\!] \cap [\![\Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$. In summary, the assignment $\preceq_{(.)}^{\mathrm{fm}}$ is min-expressible and min-friendly, and the base change operator $\circ^{\mathrm{fm}}$ is compatible with it.

In Section 3.2 to Section 3.4, we discussed how K&M's result about semantically characterizing AGM belief revision in finite-signature propositional logic can be generalized to arbitrary base logics. Thereby, we cover all Tarskian logics and support any notion of bases that are closed under "abstract union". We demonstrated certain central aspects by our running example (see Example 2.4, Example 3.11, Example 3.20, Example 3.32), which can be summarized as follows.

Fact 3.34. *The operator $\circ_{\mathrm{Ex}}$ for the base logic $\mathbb{B}_{\mathrm{Ex}}$ satisfies (G1)–(G6) and is compatible with the faithful min-friendly and min-expressible assignment $\preceq_{(.)}^{\circ_{\mathrm{Ex}}}$. That is, for any base $\mathcal{K}$ of $\mathbb{B}_{\mathrm{Ex}}$, the relation $\preceq_{\mathcal{K}}^{\circ_{\mathrm{Ex}}}$ is min-friendly and min-expressible. However there is a base $\mathcal{K}_{\mathrm{Ex}}$, such that $\preceq_{\mathcal{K}_{\mathrm{Ex}}}^{\circ}$ is not transitive. In fact, no transitive faithful min-friendly and min-expressible assignment compatible with $\circ_{\mathrm{Ex}}$ exists, whatsoever.*

By now, our rationale has been to cover the most general setting of base logics possible, while sticking to the complete set of the AGM postulates and without adding further conditions.

[1] Note that trivial revision is known to coincide with full meet revision in many logical settings.

However, one might remark that the AGM postulates were specifically designed for describing the change of belief sets, i.e., deductively closed theories, which naturally include all syntactic variants. As opposed to this, approaches to describing the change of (not necessarily deductively closed) bases might take the syntax into account [Han99]. Under such circumstances, the syntax-independence expressed by (G4) might be called into question.

Another aspect is that, for the sake of generality, we had to replace the stronger requirement of transitivity by the weaker notion of min-retractivity inside the assignments. Waiving transitivity (and hence preorders) might be considered unconventional, as a transitive preference relation is often deemed to be the actual motivation behind the postulates (G1) and (G6). This raises the question for which Tarskian logics the existence of a compatible **preorder** assignment for any AGM revision operator can be guaranteed.

In the following sections we will discuss these aspects as variations of the approach we presented in the preceding sections, showing that exact characterizations exist for these cases as well. Moreover, we will discuss some aspects of the notion of base logic, and the role of disjunctions in decomposability.

3.5 Base Changes and Syntax-Independence

Up to this point, we have been considering base change operators fulfilling the full set of postulates (G1)–(G6). The research on base changes deals with syntax-dependent changes, and in our approach the postulate (G4) implies that a base change operator yields semantically the same result on all semantically equivalent bases. As consequence, one might conclude that the base change operators considered here have only limited freedom when it comes to taking the syntactic structure into account when changing.

However, note that neither the postulates (G1)–(G6) nor our representation results make assumptions about the specific syntactic structure of a base obtained by a base change operator. Thus, for syntactically different bases Γ_1 and Γ_2 that are semantically equivalent, we might obtain syntactically different results after revision, which are semantically equivalent.

Example 3.35. *Consider the logic $\mathbb{PL}_2$ (cf. Example 2.6), e.g. propositional logic over the signature $\{p,q\}$ as follows. Given $\mathcal{K}_1 = \{p,q\}, \mathcal{K}_2 = \{p \wedge q\}, \Gamma_1 = \{p, p \rightarrow \neg q\}$, and $\Gamma_2 = \{p \wedge \neg q\}$. We have $\mathcal{K}_1$ and $\mathcal{K}_2$, as well as Γ_1 and Γ_2, which are two semantically equivalent bases with different syntax. By applying the trivial revision operation $\circ^{\mathrm{fm}}$ (cf. Example 3.33) to $\mathcal{K}_1$ by Γ_1 and to $\mathcal{K}_2$ by Γ_2, we obtain $\mathcal{K}_1 \circ \Gamma_1 = \{p, p \rightarrow \neg q\}$ and $\mathcal{K}_2 \circ \Gamma_2 = \{p \wedge \neg q\}$. The two revision results are different syntactically, yet semantically equivalent (i.e. $[\![\mathcal{K}_1 \circ \Gamma_1]\!] = [\![\mathcal{K}_2 \circ \Gamma_2]\!] = \{\mathcal{I} : p \mapsto \textbf{true}, q \mapsto \textbf{false}\}$).*

Moreover, the semantic viewpoint developed here in this article is flexible and is eligible for further liberation regarding syntax-dependence of a base change operator. In particular, our approach allows us to drop (G4). As an alternative to (G4), consider the following weaker version [Han99]:

(G4w) If $\Gamma_1 \equiv \Gamma_2$, then $\mathcal{K} \circ \Gamma_1 \equiv \mathcal{K} \circ \Gamma_2$.

The main difference between (G4w) and (G4) is that by (G4w) a base change operator is not restricted to treat semantically equivalent prior belief bases equivalently. When considering the extended AGM postulates (G5) and (G6) it turns out that postulate (G4w) is a baseline of syntax-independence, as (G1), (G5) and (G6) together already imply (G4w), which is a generalization of a result by Aiguier et al. 2018, Prop. 3.

Proposition 3.36. *Let $\circ$ be a base change operator for a base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$. If $\circ$ satisfies (G1), (G5) and (G6), then $\circ$ satisfies (G4w).*

Proof. Let $\mathcal{K}, \Gamma_1, \Gamma_2 \in \mathfrak{B}$ be belief bases such that $\Gamma_1 \equiv \Gamma_2$. By (G1), the postulate (G4w) holds if Γ_1 is inconsistent. For the remaining parts of the proof, we assume consistency of Γ_1. First observe that $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \equiv \mathcal{K} \circ \Gamma_1$ by (G1) and analogously $(\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1 \equiv \mathcal{K} \circ \Gamma_2$. By (G5) we obtain $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \models \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)$. Moreover, because $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2$ is consistent, we obtain $\mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \models (\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2$ by (G6). In summary we obtain $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \equiv \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2)$. By an analogous line of arguments we obtain $(\mathcal{K} \circ \Gamma_1) \Cup \Gamma_2 \equiv \mathcal{K} \circ (\Gamma_1 \Cup \Gamma_2) \equiv (\mathcal{K} \circ \Gamma_2) \Cup \Gamma_1$. Using our prior observations this expands to $\mathcal{K} \circ \Gamma_1 \equiv \mathcal{K} \circ \Gamma_2$. □

To obtain a representation theorem for base change operators without (G4), relaxing the constraint on the syntactic side requires the relation of the conditions on the semantic side. For dropping (G4), we weaken the notion of faithfulness to the notion of quasi-faithfulness.

Definition 3.37 (quasi-faithful). *An assignment $\preceq_{(.)}$ is called* quasi-faithful *if it satisfies the following conditions:*

(F1) If $\mathcal{I}, \mathcal{I}' \models \mathcal{K}$, then $\mathcal{I} \prec_{\mathcal{K}} \mathcal{I}'$ does not hold.

(F2) If $\mathcal{I} \models \mathcal{K}$ and $\mathcal{I}' \not\models \mathcal{K}$, then $\mathcal{I} \prec_{\mathcal{K}} \mathcal{I}'$.

Note that quasi-faithful assignments might assign to every belief base a different order, independent from whether they are semantically equivalent or not. Thus, this enables a base change operator to treat base differently depending on their syntactic structure.

Luckily, our canonical assignment $\preceq^{\circ}_{(.)}$ (cf. Definition 3.19) carries over to the setting where (G4) is not satisfied. The following lemma attests that $\preceq^{\circ}_{(.)}$ yields a quasi-faithful assignment for this case.

Lemma 3.38. *If $\circ$ satisfies (G2), (G5), and (G6), then the assignment $\preceq^{\circ}_{(.)}$ is quasi-faithful.*

Proof. The proof of the two conditions of quasi-faithfulness, (F1) and (F2), is identical to the proof of (F1) and (F2) in Lemma 3.26. □

Using the notion of quasi-faithfulness and $\preceq^{\circ}_{(.)}$ (cf. Definition 3.19) we obtain the following characterization result, which is similar to a result already provided by Aiguier et al. 2018, Thm. 2.

Proposition 3.39. *Let $\circ$ be a base change operator. The operator $\circ$ satisfies (G1)–(G3), (G5), and (G6) if and only if it is compatible with some min-friendly quasi-faithful assignment.*

Proof (Sketch). The proof is the nearly the same as for Theorem 3.17. Note that the proof of Theorem 3.17, which shows correspondence between (G1)–(G6) and compatible min-friendly faithful assignments uses (G4) and (F3) only in special situations. In particular, observe that condition (F3) is only used to show satisfaction of (G4) in the proof of Proposition 3.28. Moreover, note that $\preceq^{\circ}_{\mathcal{K}}$ from Definition 3.19 is a total min-friendly relation due to Lemma 3.22 and Lemma 3.25 for each $\mathcal{K} \in \mathfrak{B}$; compatibility of $\preceq^{\circ}_{(.)}$ with $\circ$ is ensured by Lemma 3.24 while satisfaction of quasi-faithfulness is ensured by Lemma 3.38. □

In view of this, we can now present the syntax-dependent version of our two-way representation theorem.

Theorem 3.40. *Let $\mathbb{B}$ be a base logic. Then the following hold:*

- *Every base change operator for $\mathbb{B}$ satisfying (G1)–(G3), (G5), and (G6) is compatible with some min-expressible min-friendly quasi-faithful assignment.*
- *Every min-expressible min-friendly quasi-faithful assignment for $\mathbb{B}$ is compatible with some base change operator satisfying (G1)–(G3), (G5), and (G6).*

In research on base revision, various special postulates for the changing of bases have been considered, e.g. in the seminal research on belief revision by Hansson, special postulates for base changes are proposed, e.g., see [Han99]. Of course, an interesting and open question is, which of them could be characterized or reconstructed by the approach of this thesis.

3.6 Total Preorder-Representability

As we have shown, regrettably, not every AGM belief revision operator in every Tarskian logic can be described by a total preorder assignment. Yet, we also saw that, for some logics (like $\mathbb{PL}_n$), this correspondence does indeed hold. Consequently, this section is dedicated to find a characterization of precisely those logics wherein every AGM base change operator is representable by a compatible min-complete faithful preorder assignment. The following definition captures the notion of operators that are well-behaved in that sense.

Definition 3.41 (total-preorder-representable). *A base change operator $\circ$ for some base logic is called* total-preorder-representable *if there is a min-complete quasi-faithful preorder assignment compatible with $\circ$.*

Recall that transitivity implies min-retractivity, and thus, every min-complete preorder is automatically min-friendly. Moreover, in view of Section 3.5, our definition uses the more lenient notion of quasi-faithfulness to accommodate the syntax-dependent setting. However, as the following lemma shows, the same definition of total-preorder-representability is adequate in the syntax-independent setting.

Lemma 3.42. *For any base change operator $\circ$ that satisfies (G4), total-preorder-representability coincides with the existence of a min-complete* faithful *preorder assignment compatible with $\circ$.*

Proof. Any compatible min-complete *faithful* preorder assignment is also *quasi-faithful* and hence the existence of such an assignment implies total-preorder-representability. For the other direction, let $\preceq_{(.)}$ be a min-complete quasi-faithful preorder assignment compatible with $\circ$. We then define $\preceq_{(.)}^{\mathrm{ff}}$ as $\mathcal{K} \mapsto \preceq_{\sigma([\mathcal{K}]_{\equiv})}$ where σ is a selection function mapping every $\equiv$-equivalence class of $\mathfrak{B}$ to one of its elements (i.e., $\sigma([\mathcal{K}]_{\equiv}) \in [\mathcal{K}]_{\equiv}$). Then, the property of being a min-complete quasi-faithful preorder assignment compatible with $\circ$ carries over pointwise from $\preceq_{(.)}$ to $\preceq_{(.)}^{\mathrm{ff}}$, while the construction ensures that $\preceq_{(.)}^{\mathrm{ff}}$ also satisfies (F3) from Definition 3.1 and hence is faithful. $\square$

In the next section, we will provide a necessary and sufficient criterion for a logic such that universal total-preorder-representablity is guaranteed.

The following definition describes the occurrence of a certain relationship between several bases. Such an occurrence will turn out to be the one and only reason to prevent total-preorder-representability.

Definition 3.43 (critical loop). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic. Three or more bases $\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0} \in \mathfrak{B}$ are said to form a* critical loop of length $(n+1)$ for $\mathbb{B}$ *if there exists a base $\mathcal{K} \in \mathfrak{B}$ and consistent bases $\Gamma_0, \ldots, \Gamma_n \in \mathfrak{B}$ such that*

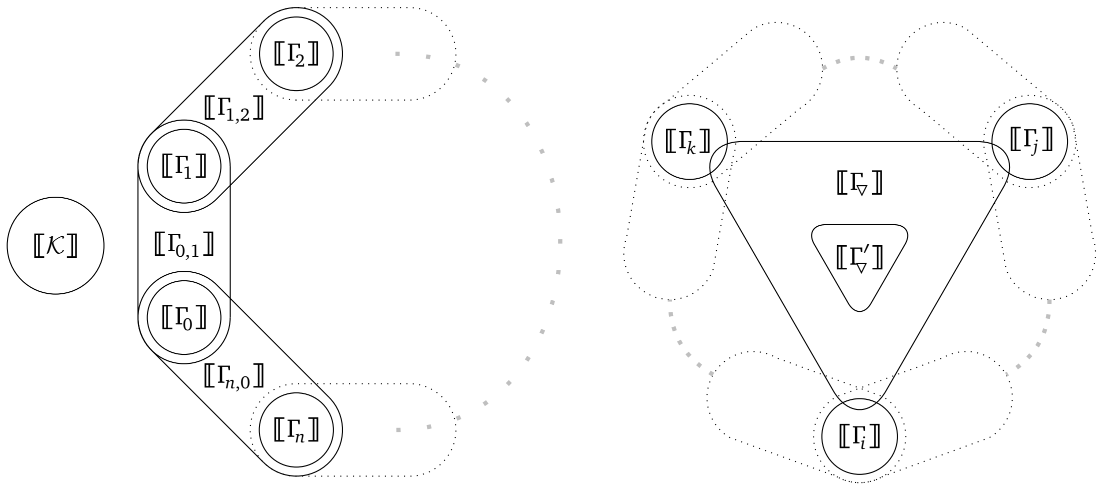
(1) $[\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!] = \emptyset$ for every $i \in \{0, \ldots, n\}$, where $\oplus$ is addition $\bmod (n+1)$,

(2) $[\![\Gamma_i]\!] \cup [\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ and $[\![\Gamma_j \Cup \Gamma_i]\!] = \emptyset$ for each $i, j \in \{0, \ldots, n\}$ with $i \neq j$, and

(3) for each $\Gamma_{\triangledown} \in \mathfrak{B}$ that is consistent with at least three bases from $\Gamma_0, \ldots, \Gamma_n$, there exists some $\Gamma_{\triangledown}' \in \mathfrak{B}$ such that $[\![\Gamma_{\triangledown}']\!] \neq \emptyset$ and $[\![\Gamma_{\triangledown}']\!] \subseteq [\![\Gamma_{\triangledown}]\!] \setminus \big([\![\Gamma_{0,1}]\!] \cup \ldots \cup [\![\Gamma_{n,0}]\!]\big)$.

The three conditions in Definition 3.43 describe the canonic situation brought about by some bases $\Gamma_{0,1}, \ldots, \Gamma_{n,0}$ allowing for the construction of a revision operator that unavoidably gives rise to a circular compatible relation. Note that due to Condition (3), every three of $\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0}$ together are inconsistent, but each two of them which have an index in common are consistent, i.e. $\Gamma_{i,i\oplus 1} \Cup \Gamma_{i\oplus 1,i\oplus 2}$ is consistent for each $i \in \{0, \ldots, n\}$.

In the following, we provide some intuition for the notion of critical loop. The bases $\Gamma_0, \ldots, \Gamma_n$ provide model sets that are pairwise disjoint (cf. the second part of Condition (2))



(a) By Condition (2), the models of each base $\Gamma_{i,i\oplus 1}$ encompass the models of Γ_i and of $\Gamma_{i\oplus 1}$, while by Condition (1), all these model sets are disjoint from the models of $\mathcal{K}$.

(b) By Condition (3), for each $\Gamma_\triangledown$ that is consistent with at least three distinct elements of the circle (e.g. $\Gamma_i, \Gamma_j, \Gamma_k \in \{\Gamma_0, \ldots, \Gamma_n\}$), there exists a base $\Gamma'_\triangledown$ that is subsumed by $\Gamma_\triangledown$ but inconsistent with all $\Gamma_{0,1}, \ldots, \Gamma_{n-1,n}, \Gamma_{n,0}$.

Figure 3.44: Illustrations of the Conditions (1)–(3) of a critical loop given in Definition 3.43.

and can be thought of as arranged in a circle, while the bases $\Gamma_{0,1}, \ldots, \Gamma_{n,0}$ overlap any two adjacent model sets as indicated by their indices (cf. the first part of Condition (2)). Exploiting this situation, we now want to define the result of revising $\mathcal{K}$ such that the circular arrangement governs the choice of the "$\mathcal{K}$-preferred" models as follows: the models of $\mathcal{K} \circ \Gamma_{i,i\oplus 1}$, obtained by revising $\mathcal{K}$ with $\Gamma_{i,i\oplus 1}$, encompass all models of Γ_i, but no model of $\Gamma_{i\oplus 1}$. Consequently, for any i, the revision $\mathcal{K} \circ \Gamma_{i,i\oplus 1}$ provides a preference of Γ_i over $\Gamma_{i\oplus 1}$. Thus, a relation compatible to $\circ$ has to contain a "preference-loop" of interpretations. In order to guarantee that this arrangement technique is applicable, Condition (1) and Condition (3) from Definition 3.43 are ruling out all cases, where other bases of $\mathbb{B}$ together with (G1)–(G6) prevent our intended construction from working:

Condition (1) ensures that none of the bases $\Gamma_{0,1}, \ldots, \Gamma_{n,0}$ has models in common with the current belief base $\mathcal{K}$ (c.f. Figure 3.44a). If one base $\Gamma_{i,i\oplus 1}$ would have a model in common with $\mathcal{K}$, then the postulate (G2) would prevent a circular situation. Thus, this condition is necessary for admitting circular situations.

Condition (3) comes into play if a belief base $\Gamma_\triangledown$ "covers" three or more elements of the circle, meaning that three or more interpretations of a circle are models of this base $\Gamma_\triangledown$. For any such $\Gamma_\triangledown$, there is a consistent belief base $\Gamma'_\triangledown$ which shares all of its models with $\Gamma_\triangledown$ but no model with any of the $\Gamma_{i,i\oplus 1}$ (c.f. Figure 3.44b). This is crucial for the presence of circles: if no such $\Gamma'_{\triangledown}$ would exist, the operator would (by min-completeness and min-expressibility) choose models of the circle, e.g., the bases $\Gamma_i \Cup \Gamma_{\triangledown}$, as the result of the revision by $\Gamma_{\triangledown}$. In the end, this would give one base Γ_i preference over $\Gamma_{i\oplus 1}, \ldots, \Gamma_{i\oplus n}$ and thus, would prevent creation of a circle. Therefore, Condition (3) rules out the cases where min-completeness and min-expressibility and non-existence of such a $\Gamma'_{\triangledown}$ together would prevent formation of a circle.

Definition 3.43 is inspired by our running example. Before explicating this link, we continue with the presentation of the general results.

The next theorem is the central result of this section, stating that the notion of critical loop captures exactly those base logics for which some operator exists that is not total-preorder-representable. By contraposition, this just means that for all base logics $\mathbb{B}$, the absence of critical loops from $\mathbb{B}$ is a necessary and sufficient criterion for universal total-preorder-representability and hence for the existence of a characterization result for $\mathbb{B}$ that is based on total preorders. This characterization result will not only hold for base change operators that satisfy (G1)–(G6), but also for operators that do not satisfy (G4), but the remaining postulates (G1)–(G3), (G5), and (G6). To provide a result applicable to both groups of postulates, we will show for the necessary and sufficient direction the respectively stronger result, i.e., if our base logic exhibits a critical loop we provide a construction for a non-total-preorder-representable base change operator that satisfies (G1)–(G6), and for the other direction, we show that in the absence of critical loops every operator that satisfies (G1)–(G3), (G5), and (G6) is total-preorder-representable.

Theorem 3.45. *For all base logics $\mathbb{B}$, the following statements hold:*

(I) If $\mathbb{B}$ exhibits a critical loop, then there exists a base change operator for $\mathbb{B}$ that satisfies (G1)–(G6) and is not total-preorder-representable.

(II) If $\mathbb{B}$ does not admit a critical loop, then every base change operator for $\mathbb{B}$ that satisfies (G1)–(G3), (G5), and (G6) is total-preorder-representable.

We dedicate Section 3.6.1 to the first statement of Theorem 3.45 while the second statement is shown in Section 3.6.2.

3.6.1 Total Preorder-Representability Implies Absence of Critical Loops

We proceed to show (by contraposition) that the absence of critical loops is necessary for total-preorder-representability of all AGM change operators. To this end, we will provide a construction which, given a critical loop $\mathfrak{C}$ in some base logic $\mathbb{B}$, yields an AGM change operator $\circ_{\mathfrak{C}}$ for $\mathbb{B}$ that is demonstrably not total-preorder-representable.

Definition 3.46. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic with a critical loop $\mathfrak{C} = (\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0})$ and let $\Gamma_0, \ldots, \Gamma_n \in \mathfrak{B}$ and $\mathcal{K}$ as in Definition 3.43.*

Let $\mathcal{C}$ denote the set of all $\Gamma'_\triangledown$ guaranteed by Condition (3) from Definition 3.43, i.e. $\Gamma'_\triangledown \in \mathcal{C}$ if there is some $\Gamma_\triangledown$ with $\emptyset \neq [\![\Gamma'_\triangledown]\!] \subseteq [\![\Gamma_\triangledown]\!] \setminus \big([\![\Gamma_{0,1}]\!] \cup \ldots \cup [\![\Gamma_{n,0}]\!]\big)$ and $\Gamma_\triangledown$ is consistent with three (or more) bases from $\{\Gamma_0, \ldots, \Gamma_n\}$. Now let $\mathcal{C}' = \{\Gamma'_\triangledown \in \mathcal{C} \mid [\![\Gamma'_\triangledown \Cup \mathcal{K}]\!] = \emptyset\}$, i.e., all belief bases from $\mathcal{C}$ that are inconsistent with $\mathcal{K}$. Let $\leqslant_{\mathcal{C}'}$ be an arbitrary linear order on $\mathcal{C}'$ with respect to which every non-empty subset of $\mathcal{C}'$ has a minimum.[2]

We now define $\circ_{\mathfrak{C}}$ as follows: for every $\mathcal{K}' \not\equiv \mathcal{K}$ and any Γ, let $\mathcal{K}' \circ_{\mathfrak{C}} \Gamma = \mathcal{K}' \Cup \Gamma$ if $\mathcal{K}' \Cup \Gamma$ is consistent, otherwise $\mathcal{K}' \circ_{\mathfrak{C}} \Gamma = \Gamma$. For $\mathcal{K}' \equiv \mathcal{K}$, we define:

$$\mathcal{K}' \circ_{\mathfrak{C}} \Gamma = \begin{cases} \Gamma \Cup \mathcal{K}' & \text{if } [\![\mathcal{K}' \Cup \Gamma]\!] \neq \emptyset, \\ \Gamma \Cup \Gamma_{\min}^{\mathcal{C}'} & \text{if } [\![\mathcal{K}' \Cup \Gamma]\!] = \emptyset, \text{ and } [\![\Gamma \Cup \Gamma'_\triangledown]\!] \neq \emptyset \text{ for some } \Gamma'_\triangledown \in \mathcal{C}', \\ \Gamma \Cup \Gamma_i & \text{if none of the above applies, } [\![\Gamma_i \Cup \Gamma]\!] \neq \emptyset, \text{ and } \bigcup_{j \in \{0,\ldots,n\} \setminus \{i, i \oplus 1\}} [\![\Gamma_j \Cup \Gamma]\!] = \emptyset, \\ \Gamma & \text{if none of the cases above apply,} \end{cases}$$

where $\Gamma_{\min}^{\mathcal{C}'} = \min(\{\Gamma'_\triangledown \in \mathcal{C}' \mid [\![\Gamma'_\triangledown \Cup \Gamma]\!] \neq \emptyset\}, \leqslant_{\mathcal{C}'})$.

In the following, we show that $\circ_{\mathfrak{C}}$ from Definition 3.46 is indeed an AGM revision, but not total-preorder-representable.

Proposition 3.47. *For a base logic $\mathbb{B}$ with a critical loop $\mathfrak{C}$, the operator $\circ_{\mathfrak{C}}$ for $\mathbb{B}$ satisfies (G1)–(G6) and is not total-preorder-representable.*

Proof. We will first show that $\circ_{\mathfrak{C}}$ satisfies (G1)–(G6). For $\mathcal{K}' \not\equiv \mathcal{K}$ we obtain a trivial revision which satisfies (G1)–(G6) (cf. Example 3.33). Consider the remaining case of $\mathcal{K}$ (and any equivalent base):

Postulates (G1)–(G4). The satisfaction of (G1)–(G3) follows directly from the construction of $\circ_{\mathfrak{C}}$. For (G4) observe that, when computing $\mathcal{K} \circ_{\mathfrak{C}} \Gamma$, the case distinction above only considers the model sets of the participating bases rather than their syntax. Thus, for $\mathcal{K} \equiv \mathcal{K}'$ and $\Gamma_1^* \equiv \Gamma_2^*$ we always obtain $\mathcal{K} \circ_{\mathfrak{C}} \Gamma_1^* \equiv \mathcal{K}' \circ_{\mathfrak{C}} \Gamma_2^*$.

Postulate (G5) and (G6). Consider two belief bases Γ_1^* and Γ_2^*. If Γ_2^* is inconsistent with $\mathcal{K} \circ_{\mathfrak{C}} \Gamma_1^*$, then we obtain satisfaction of (G5) immediately. For the remaining case of (G5) and for (G6) we assume $\mathcal{K} \circ_{\mathfrak{C}} \Gamma_1^*$ to be consistent with Γ_2^*, i.e., $[\![(\mathcal{K} \circ_{\mathfrak{C}} \Gamma_1^*) \Cup \Gamma_2^*]\!] \neq \emptyset$. Consequently, there exists some interpretation $\mathcal{I}$ such that $\mathcal{I} \in [\![\mathcal{K} \circ_{\mathfrak{C}} \Gamma_1^*]\!]$ and $\mathcal{I} \in [\![\Gamma_2^*]\!]$. The postulate (G1) implies that $\mathcal{I} \in [\![\Gamma_1^*]\!]$ and hence $\Gamma_1^* \Cup \Gamma_2^*$ is consistent. We now inspect all different cases from the definition of $\circ_{\mathfrak{C}}$ above that may apply when revising $\mathcal{K}$ by Γ_1^*:

[2] Such a $\leqslant_{\mathcal{C}'}$ exists due to the well-ordering theorem, by courtesy of the *axiom of choice* [Zer04].

If Γ_1^* is consistent with $\mathcal{K}$, then we obtain from $[\![(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^*]\!] \neq \emptyset$ and (G2) that $\mathcal{K}$ is consistent with $\Gamma_1^*\Cup\Gamma_2^*$. This implies $(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^* \equiv (\mathcal{K}\Cup\Gamma_1^*)\Cup\Gamma_2^* \equiv \mathcal{K}\Cup(\Gamma_1^*\Cup\Gamma_2^*) \equiv \mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*)$; yielding satisfaction of (G5) and (G6).

Next, consider the second case of the definition, where Γ_1^* is inconsistent with $\mathcal{K}$, but consistent with some $\Gamma'_\triangledown \in \mathcal{C}'$ and assume $\Gamma'_\triangledown$ is the $\leqslant_{\mathcal{C}'}$-minimal such base, i.e., $\Gamma'_\triangledown = (\Gamma_1^*)^{\mathcal{C}'}_{\min}$. Then, from the construction of $\circ_{\mathfrak{C}}$ and the consistency of $(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^*$ we obtain $[\![(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^*]\!] = [\![\Gamma_1^*\Cup\Gamma'_\triangledown\Cup\Gamma_2^*]\!] \neq \emptyset$. Consequently, the set $\Gamma_1^*\Cup\Gamma_2^*$ is also consistent with $\Gamma'_\triangledown$, which, together with $\Gamma'_\triangledown = (\Gamma_1^*)^{\mathcal{C}'}_{\min}$, implies $\Gamma'_\triangledown = (\Gamma_1^*\Cup\Gamma_2^*)^{\mathcal{C}'}_{\min}$. For determining $\mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*)$, note that from $\mathcal{K}$ being inconsistent with Γ_1^*, it follows that $\mathcal{K}$ must also be inconsistent with $\Gamma_1^*\Cup\Gamma_2^*$, therefore, due to the existence of $\Gamma'_\triangledown$, the second line of the definition of $\circ_{\mathfrak{C}}$ must apply. We obtain $(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^* \equiv ((\Gamma_1^*)^{\mathcal{C}'}_{\min}\Cup\Gamma_1^*)\Cup\Gamma_2^*$ $\equiv \Gamma'_\triangledown\Cup\Gamma_1^*\Cup\Gamma_2^* \equiv (\Gamma_1^*\Cup\Gamma_2^*)^{\mathcal{C}'}_{\min}\Cup(\Gamma_1^*\Cup\Gamma_2^*) \equiv \mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*)$; establishing (G5) and (G6) for this case.

We now inspect the third case from the definition, i.e., we consider some Γ_1^* that is inconsistent with $\mathcal{K}$ and with all elements from $\mathcal{C}'$. If Γ_1^* is consistent with Γ_i and inconsistent with all Γ_j, where $j \in \{0,\ldots,n\}\setminus\{i,i\oplus 1\}$, then by the construction of $\circ_{\mathfrak{C}}$ and the consistency of $(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^*$ we have $[\![(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^*]\!] = [\![\Gamma_1^*\Cup\Gamma_i\Cup\Gamma_2^*]\!] \neq \emptyset$. Then, likewise $\Gamma_1^*\Cup\Gamma_2^*$ is consistent with Γ_i and inconsistent with all Γ_j with $j \in \{0,\ldots,n\}\setminus\{i,i\oplus 1\}$. Moreover, if Γ_1^* is inconsistent with $\mathcal{K}$ and with all elements from $\mathcal{C}'$, then so is $\Gamma_1^*\Cup\Gamma_2^*$, i.e., when determining $\mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*)$, the third case of the definition applies. Hence, by the definition of $\circ_{\mathfrak{C}}$ we obtain $(\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^* \equiv \Gamma_1^*\Cup\Gamma_2^*\Cup\Gamma_i \equiv \mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*)$.

If none of the conditions above applies to Γ_1^*, then they also do not apply to $\Gamma_1^*\Cup\Gamma_2^*$. From the construction of $\circ_{\mathfrak{C}}$ we obtain $\mathcal{K}\circ_{\mathfrak{C}}(\Gamma_1^*\Cup\Gamma_2^*) \equiv (\mathcal{K}\circ_{\mathfrak{C}}\Gamma_1^*)\Cup\Gamma_2^* \equiv \Gamma_1^*\Cup\Gamma_2^*$.

In summary, we obtain that $\circ_{\mathfrak{C}}$ satisfies (G5) and (G6) in all cases. It remains to show that $\circ_{\mathfrak{C}}$ is not total-preorder-representable. Towards a contradiction suppose the contrary, i.e., there is a min-complete faithful preorder assignment $\preceq_{(.)}$, such that $\circ_{\mathfrak{C}}$ is compatible with $\preceq_{(.)}$. Transitivity and min-completeness imply that $\preceq_{(.)}$ is min-friendly. As all $\Gamma_0,\ldots,\Gamma_n$ are consistent, there are $\mathcal{I}_i \in [\![\Gamma_i]\!]$ for all $i \in \{0,\ldots,n\}$. By construction of $\circ_{\mathfrak{C}}$ and Condition (2) of Definition 3.43, we have $\mathcal{K}\circ_{\mathfrak{C}}\Gamma_{i,i\oplus 1} = \Gamma_{i,i\oplus 1}\Cup\Gamma_i \equiv \Gamma_i$, and consequently $\mathcal{I}_i \models \mathcal{K}\circ_{\mathfrak{C}}\Gamma_{i,i\oplus 1}$ and $\mathcal{I}_{i\oplus 1} \not\models \mathcal{K}\circ_{\mathfrak{C}}\Gamma_{i,i\oplus 1}$ for each $i \in \{0,\ldots,n\}$. As $\circ_{\mathfrak{C}}$ is compatible with $\preceq_{(.)}$, we obtain $[\![\mathcal{K}\circ_{\mathfrak{C}}\Gamma_{i,i\oplus 1}]\!] = \min([\![\Gamma_{i,i\oplus 1}]\!],\preceq_{\mathcal{K}})$. In particular, the definition of $\circ_{\mathfrak{C}}$ yields $\mathcal{I}_i \in \min([\![\Gamma_{i,i\oplus 1}]\!],\preceq_{\mathcal{K}})$ and $\mathcal{I}_i,\mathcal{I}_{i\oplus 1} \models \Gamma_{i,i\oplus 1}$ and $\mathcal{I}_{i\oplus 1} \notin \min([\![\Gamma_{i,i\oplus 1}]\!],\preceq_{\mathcal{K}})$. We obtain thereof the strict relationship $\mathcal{I}_i \prec_{\mathcal{K}} \mathcal{I}_{i\oplus 1}$. In summary, we get $\mathcal{I}_0 \prec_{\mathcal{K}} \mathcal{I}_1 \prec_{\mathcal{K}} \ldots \prec_{\mathcal{K}} \mathcal{I}_n \prec_{\mathcal{K}} \mathcal{I}_0$, which contradicts the presumed transitivity of $\preceq_{\mathcal{K}}$. $\square$

This establishes that the absence of critical loops is a necessary condition for universal total-preorder-representability in any Tarskian logic, because Theorem 3.45 (I) is an immediate consequence of Proposition 3.47.

3.6.2 Absence of Critical Loops Implies Total Preorder-Representability

We will now show that the identified criterion of critical loop (Definition 3.43) is also sufficient, even in the more general, syntax-dependent setting. That is, we will demonstrate in the following that Theorem 3.45 (II) holds. To this end, we need to argue that any base change operator $\circ$ that satisfies (G1)–(G3), (G5), and (G6) for any critical-loop-free $\mathbb{B}$ gives rise to a compatible min-complete quasi-faithful preorder assignment $\preccurlyeq^{\circ}_{(.)}$. We will show how to obtain $\preccurlyeq^{\circ}_{(.)}$ via a step-wise transformation of the assignment $\preceq^{\circ}_{(.)}$ from Definition 3.19.

The transformation from $\preceq^{\circ}_{(.)}$ to $\preccurlyeq^{\circ}_{(.)}$ consists of three steps. To begin with, recall that $\preceq^{\circ}_{(.)}$ is a min-complete quasi-faithful assignment compatible with $\circ$ by Proposition 3.39. This means that $\preceq^{\circ}_{\mathcal{K}}$ is a total relation for each $\mathcal{K}$, whence transitivity is the only condition that $\preceq^{\circ}_{\mathcal{K}}$ fails to meet to qualify as a total preorder.

For the first step, we will identify a group of interpretation pairs $\mathfrak{D}^{\circ}_{\mathcal{K}} \subseteq \preceq^{\circ}_{\mathcal{K}}$ such that at least one pair from $\mathfrak{D}^{\circ}_{\mathcal{K}}$ is involved whenever $\preceq^{\circ}_{\mathcal{K}}$ violates transitivity. The first step then consists in removing all $\mathfrak{D}^{\circ}_{\mathcal{K}}$ from $\preceq^{\circ}_{\mathcal{K}}$, resulting in $\preceq^{\circ}_{\mathcal{K}}{}'$. The relation $\preceq^{\circ}_{\mathcal{K}}{}'$ will be a non-transitive and non-total relation, but minima of models of bases will be preserved. We will then extend $\preceq^{\circ}_{\mathcal{K}}{}'$ to a transitive relation $\preceq^{\circ}_{\mathcal{K}}{}''$ in the second step, by taking the transitive closure. We will show that only elements from $\mathfrak{D}^{\circ}_{\mathcal{K}}$ can be added back by the transitive closure, which guarantees that, again, minima of models of bases are preserved. In a last step, we obtain the final result $\preccurlyeq^{\circ}_{(.)}$ by "linearizing" $\preceq^{\circ}_{\mathcal{K}}{}'$ to a total preorder in a way that minima of models of bases are again preserved.

Step I: Removing detached pairs. Let $\circ$ be a base change operator that satisfies (G1)–(G3), (G5), and (G6). Then, for any two bases $\mathcal{K}, \Gamma \in \mathfrak{B}$, all quasi-faithful assignments $\preceq_{(.)}$ compatible with $\circ$ yield the same set of minimal interpretations of $[\![\Gamma]\!]$ with respect to $\preceq_{\mathcal{K}}$. This property already stipulates much of $\preceq_{\mathcal{K}}$ for each $\mathcal{K}$ (for some base logics $\preceq_{\mathcal{K}}$ is even completely determined by that property). Still, in the general case, when forming a compatible assignment, there is certain freedom on relating those interpretations for which the given base change operator gives no hint about how to order them. The following notion formally defines such pairs of interpretations.

Definition 3.48. *Let $\circ$ be a base change operator for $\mathbb{B}$ and $\mathcal{K}$ a base of $\mathbb{B}$. A pair $(\mathcal{I}, \mathcal{I}') \in \Omega \times \Omega$ is called* detached from $\circ$ in $\mathcal{K}$, *if $\mathcal{I}, \mathcal{I}' \not\models \mathcal{K} \circ \Gamma$ for all $\Gamma \in \mathfrak{B}$ with $\mathcal{I}, \mathcal{I}' \models \Gamma$. With $\mathfrak{D}^{\circ}_{\mathcal{K}}$ we denote the set of all pairs $(\mathcal{I}, \mathcal{I}')$ which are detached from $\circ$ in $\mathcal{K}$ and satisfy $\mathcal{I} \neq \mathcal{I}'$.*

Note that detachment is a symmetric property, i.e., $(\mathcal{I}, \mathcal{I}')$ is detached if and only if $(\mathcal{I}', \mathcal{I})$ is. It so happens that $\preceq^{\circ}_{\mathcal{K}}$ may contain too many of such detached pairs, i.e., in some cases, $\preceq^{\circ}_{\mathcal{K}}$ is not a total preorder even if the base change operator $\circ$ is total preorder-representable (see also Section 3.8.2). In the following, we show that every violation of transitivity in $\preceq^{\circ}_{\mathcal{K}}$ involves a detached pair (as illustrated in Figure 3.49).

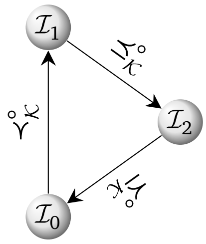


Figure 3.49: Illustration of a critical loop-situation of length 3 on the semantic side. If $\mathbb{B}$ does not exhibit a critical loop, then this situation is, due to Lemma 3.50, only possible when ($\mathcal{I}_1$,$\mathcal{I}_2$) or ($\mathcal{I}_2$,$\mathcal{I}_0$) is a detached pair.

Lemma 3.50. *Assume $\mathbb{B}$ be a base logic which does not admit a critical loop and $\circ$ a base change operator for $\mathbb{B}$ which satisfies (G1)–(G3), (G5), and (G6). If $\mathcal{I}_0 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$ and $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ with $\mathcal{I}_0 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$, then $(\mathcal{I}_0, \mathcal{I}_1)$ or $(\mathcal{I}_1, \mathcal{I}_2)$ is detached from $\circ$ in $\mathcal{K}$.*

Proof. Let $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ such that a violation of transitivity is obtained as given above, i.e. $\mathcal{I}_0 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$ and $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ with $\mathcal{I}_0 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. By Definition 3.19, we have that $\mathcal{I}_0 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ is only possible if $\mathcal{I}_0 \not\models \mathcal{K}$. From Definition 3.19 and $\mathcal{I}_0 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_1$, we obtain $\mathcal{I}_1 \not\models \mathcal{K}$. By an analogue argument we obtain $\mathcal{I}_2 \not\models \mathcal{K}$. Thus, for the rest of the proof we have $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$.

Towards a contradiction, assume that $(\mathcal{I}_0, \mathcal{I}_1)$ and $(\mathcal{I}_1, \mathcal{I}_2)$ are both not detached from $\circ$ in $\mathcal{K}$. By Lemma 3.22 the relation $\preceq^{\circ}_{\mathcal{K}}$ is total, and thus we have that $\mathcal{I}_2 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_0$. As $\mathcal{I}_2 \not\models \mathcal{K}$ and $\mathcal{I}_0 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$, due to Lemma 3.23(a), there is a base $\Gamma_{2,0} \in \mathfrak{B}$ with $\mathcal{I}_0, \mathcal{I}_2 \models \Gamma_{2,0}$ such that $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma_{2,0}$ and $\mathcal{I}_0 \not\models \mathcal{K} \circ \Gamma_{2,0}$. By $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$ and Definition 3.19 we obtain $\mathcal{I}_0 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_1$ and $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ (cf. Definition 3.18). Because $(\mathcal{I}_0, \mathcal{I}_1)$ is not detached, there is some $\Gamma_{0,1} \in \mathfrak{B}$ with $\mathcal{I}_0, \mathcal{I}_1 \models \Gamma_{0,1}$ such that $\mathcal{I}_0 \models \mathcal{K} \circ \Gamma_{0,1}$ or $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma_{0,1}$. By Definition 3.18 and $\mathcal{I}_0 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_1$ we obtain that $\mathcal{I}_0 \models \mathcal{K} \circ \Gamma_{0,1}$. Using an analogue argumentation, there exist $\Gamma_{1,2} \in \mathfrak{B}$ satisfying $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma_{1,2}$ and $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma_{1,2}$.

Recall that $\preceq^{\circ}_{(.)}$ is compatible, min-retractive and quasi-faithful by Lemma 3.24 and by the proof of Lemma 3.38. Let $\Gamma_i = (\mathcal{K} \circ \Gamma_{i,i\oplus 1}) \Cup \Gamma_{i\oplus 2,i}$ for each $i \in \{0,1,2\}$. Note that each Γ_i is a consistent base, since we have $\mathcal{I}_i \in [\![\Gamma_i]\!]$. We now show that Conditions (1) and Condition (2) from Definition 3.43 are satisfied:

(1) Towards a contradiction, assume that $\mathcal{K}$ is consistent with some $\Gamma_{i,i\oplus 1}$. From (G2) we obtain $[\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] = [\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!]$ for some $i \in \{0,1,2\}$. Since $\mathcal{I}_i \in [\![\Gamma_i]\!]$, by the definition of Γ_i we have $\mathcal{I}_i \in [\![(\mathcal{K} \circ \Gamma_{i,i\oplus 1}) \Cup \Gamma_{i\oplus 2,i}]\!] = [\![(\mathcal{K} \Cup \Gamma_{i,i\oplus 1}) \Cup \Gamma_{i\oplus 2,i}]\!]$ and obtain $\mathcal{I}_i \in [\![\mathcal{K}]\!]$ for some $i \in \{0,1,2\}$, which contradicts $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$.

(2) By the postulate (G1) we have $[\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ for each $i \in \{0,1,2\}$. The definition of Γ_i yields $[\![\Gamma_i]\!] \subseteq [\![\Gamma_{i,i\oplus 1} \Cup \Gamma_{i\oplus 2,i}]\!]$ for each $i \in \{0,1,2\}$. Substituting i by $i \oplus 1$ yields $[\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i\oplus 1,i\oplus 2} \Cup \Gamma_{i,i\oplus 1}]\!]$; showing that $[\![\Gamma_i]\!] \cup [\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ holds for each $i \in \{0,1,2\}$.

We show that each $\Gamma_i \Cup \Gamma_j$ is inconsistent, by assuming the contrary, i.e., there are some $i, j \in \{0,1,2\}$ such that $i \neq j$ and $\Gamma_i \Cup \Gamma_j$ is consistent, i.e. there exists some $\mathcal{I}^* \in [\![\Gamma_i]\!] \cap [\![\Gamma_j]\!]$. From the definition of Γ_i and the definition of Γ_j, we obtain $\mathcal{I}^* \in [\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] \cap [\![\Gamma_{i\oplus 2,i}]\!] \cap [\![\mathcal{K} \circ \Gamma_{j,j\oplus 1}]\!] \cap [\![\Gamma_{j\oplus 2,j}]\!]$. Hence, we obtain $\mathcal{I}^* \in [\![\Gamma_{i\oplus 2,i} \Cup \Gamma_{j\oplus 2,j}]\!]$ and from compatibility of $\preceq^{\circ}_{(.)}$ with $\circ$, we obtain $\mathcal{I}^* \in \min([\![\Gamma_{i,i\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$ and $\mathcal{I}^* \in \min([\![\Gamma_{j,j\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$. Now observe that $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2 \in [\![\Gamma_{i,i\oplus 1}]\!] \cup [\![\Gamma_{j,j\oplus 1}]\!]$ holds; this is because we have $[\![\Gamma_k]\!] \subseteq [\![\Gamma_{k,k\oplus 1}]\!] \cup [\![\Gamma_{k\oplus n,k}]\!]$ for each $k \in \{0,1,2\}$. Hence, independent of the specific i and j, we obtain $\mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_k$ from $\mathcal{I}^* \in [\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!]$ and Lemma 3.23(b) for each $k \in \{0,1,2\}$. Together, $\mathcal{I}_i \in [\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!]$, $\mathcal{I}_j \in [\![\mathcal{K} \circ \Gamma_{j,j\oplus 1}]\!]$, and compatibility, imply $\mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$. Because of $[\![\Gamma_i]\!] \cup [\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$, we have that $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}^* \in [\![\Gamma_{i,i\oplus 1}]\!]$ or $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}^* \in [\![\Gamma_{j,j\oplus 1}]\!]$ holds. For the case $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}^* \in [\![\Gamma_{i,i\oplus 1}]\!]$, since $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}^* \in \min([\![\Gamma_{i,i\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$, from min-retractivity we obtain $\mathcal{I}_j \in \min([\![\Gamma_{i,i\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$. As $\mathcal{I}_i \in \min([\![\Gamma_{i,i\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$, we obtain $\mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_j$ and $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_i$. By an analogue argumentation, we obtain for the case of $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}^* \in [\![\Gamma_{j,j\oplus 1}]\!]$ the same conclusion, i.e., $\mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_j$ and $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_i$. This shows that $\mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_j$ and $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_i$ must hold in general.

We consider in the following all possible choices for i and j. For the case of $i = 0$ and $j = 2$, we obtain a contradiction to $\mathcal{I}_2 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_0$. We next consider the case of $i = 1$ and $j = 2$. Because of $[\![\Gamma_0]\!] = [\![\mathcal{K} \circ \Gamma_{0,1}]\!] \cap [\![\Gamma_{2,0}]\!] = \min([\![\Gamma_{0,1}]\!], \preceq^{\circ}_{\mathcal{K}}) \cap [\![\Gamma_{2,0}]\!]$, we have that $\mathcal{I}_0, \mathcal{I}_2, \mathcal{I}^* \in [\![\Gamma_{2,0}]\!]$ holds. As $\mathcal{I}_0 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}^* \in \min([\![\Gamma_{2,0}]\!], \preceq^{\circ}_{\mathcal{K}})$ holds, min-retractivity of $\preceq^{\circ}_{\mathcal{K}}$ yields $\mathcal{I}_0 \in \min([\![\Gamma_{2,0}]\!], \preceq^{\circ}_{\mathcal{K}})$. Consequently, we obtain that $\mathcal{I}_0 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ holds, which is a contradiction to $\mathcal{I}_2 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_0$. The proof for the case of $i = 2$ and $j = 1$ is analogous to the case of $i = 1$ and $j = 2$. We obtain that Condition (2) from Definition 3.43 is satisfied.

Recall that by assumption, the base logic $\mathbb{B}$ does not exhibit a critical loop. Yet $\Gamma_{0,1}$, $\Gamma_{1,2}, \Gamma_{2,0}$ satisfy Conditions (1) and Condition (2) of a a critical loop, hence Condition (3) of Definition 3.43 must be violated. This means that there exists some $\Gamma_{\triangledown} \in \mathfrak{B}$ such that $[\![\Gamma_i \Cup \Gamma_{\triangledown}]\!] \neq \emptyset$ for every $i \in \{0,1,2\}$, but no required base $\Gamma'_{\triangledown} \in \mathfrak{B}$ such that Condition (3) is satisfied. Consequently, for all $\Gamma \in \mathfrak{B}$ holds

$$[\![\Gamma]\!] \neq \emptyset \text{ implies } [\![\Gamma]\!] \not\subseteq [\![\Gamma_{\triangledown}]\!] \setminus ([\![\Gamma_{0,1}]\!] \cup [\![\Gamma_{1,2}]\!] \cup [\![\Gamma_{2,0}]\!]). \qquad (\star 1)$$

For the remaining parts of the proof, let $\mathcal{I}_i^{\triangledown} \in \Omega$ be an interpretation with $\mathcal{I}_i^{\triangledown} \in [\![\Gamma_i]\!] \cap [\![\Gamma_{\triangledown}]\!]$ for each $i \in \{0,1,2\}$. Because $\circ$ satisfies (G1) and (G3), we obtain $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \subseteq [\![\Gamma_{\triangledown}]\!]$ and consistency of $\mathcal{K} \circ \Gamma_{\triangledown}$. Together with ($\star$1) we obtain that there exists $k \in \{0,1,2\}$ with $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!] \neq \emptyset$. We consider each of the two cases $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] \neq \emptyset$ and $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] = \emptyset$ independently.

The case of $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] \neq \emptyset$. As first step, we show that

$$\mathcal{I}_0^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_2^{\triangledown} \text{ and } \mathcal{I}_2^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_1^{\triangledown} \text{ and } \mathcal{I}_1^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_0^{\triangledown} \tag{$\star$2}$$

holds for this case. Clearly, $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] \neq \emptyset$ implies that there exists some $\mathcal{I}_k^{\star} \in \Omega$ such that $\mathcal{I}_k^{\star} \in [\![\mathcal{K} \circ \Gamma_{\triangledown}]\!]$ and $\mathcal{I}_k^{\star} \in [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!]$. From the compatibility of $\circ$ with $\preceq_{(.)}^{\circ}$, we obtain $\mathcal{I}_k^{\triangledown} \in \min([\![\Gamma_{k,k\oplus 1}]\!], \preceq_{\mathcal{K}}^{\circ})$, implying that $\mathcal{I}_k^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\star}$ holds. Remember that $\mathcal{I}_k^{\triangledown}, \mathcal{I}_k^{\star} \in [\![\Gamma_{\triangledown}]\!]$ and $\mathcal{I}_k^{\star} \in \min([\![\Gamma_{\triangledown}]\!], \preceq_{\mathcal{K}}^{\circ})$, by min-retractivity we obtain $\mathcal{I}_k^{\triangledown} \in \min([\![\Gamma_{\triangledown}]\!], \preceq_{\mathcal{K}}^{\circ})$. From this last observation and from $\mathcal{I}_{k\oplus 1}^{\triangledown}, \mathcal{I}_{k\oplus 2}^{\triangledown} \in [\![\Gamma_{\triangledown}]\!]$ we obtain that $\mathcal{I}_k^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{k\oplus 1}^{\triangledown}$ and $\mathcal{I}_k^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{k\oplus 2}^{\triangledown}$ holds. Remember that by Condition (2) we have $\mathcal{I}_k^{\triangledown}, \mathcal{I}_{k\oplus 2}^{\triangledown} \in [\![\Gamma_{k\oplus 2,k}]\!]$ and by compatibility we obtain $\mathcal{I}_{k\oplus 2}^{\triangledown} \in \min([\![\Gamma_{k\oplus 2,k}]\!], \preceq_{\mathcal{K}}^{\circ})$. This last observation, together with $\mathcal{I}_k^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{k\oplus 2}^{\triangledown}$, $\mathcal{I}_k \models \mathcal{K} \circ \Gamma$ and min-retractivity, yields $\mathcal{I}_k^{\triangledown} \in \min([\![\Gamma_{k\oplus 2,k}]\!], \preceq_{\mathcal{K}}^{\circ})$. Thus, we have $\mathcal{I}_{k\oplus 2}^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\triangledown}$. By a symmetric argument, we have $\mathcal{I}_{k\oplus 1}^{\triangledown}, \mathcal{I}_{k\oplus 2}^{\triangledown} \in [\![\Gamma_{k\oplus 1,k\oplus 2}]\!]$ and $\mathcal{I}_{k\oplus 1}^{\triangledown} \in [\![\mathcal{K} \circ \Gamma_{k\oplus 1,k\oplus 2}]\!]$. Thus, we obtain $\mathcal{I}_{k\oplus 1}^{\triangledown} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{k\oplus 2}^{\triangledown}$ from Lemma 3.23(b). By combination of these observations with $\mathcal{I}_{k\oplus 1}^{\triangledown}, \mathcal{I}_{k\oplus 2}^{\triangledown} \in [\![\Gamma_{\triangledown}]\!]$ and $\mathcal{I}_k^{\triangledown} \in \min([\![\Gamma_{\triangledown}]\!], \preceq_{\mathcal{K}}^{\circ})$, we obtain $\mathcal{I}_k^{\triangledown}, \mathcal{I}_{k\oplus 1}^{\triangledown}, \mathcal{I}_{k\oplus 2}^{\triangledown} \in \min([\![\Gamma_{\triangledown}]\!], \preceq_{\mathcal{K}}^{\circ})$ from min-retractivity. As direct consequence, we obtain that ($\star$2) holds.

We will now show that a contradiction with $\mathcal{I}_2 \prec_{\mathcal{K}}^{\circ} \mathcal{I}_0$ is unavoidable. Recall that $\mathcal{I}_0, \mathcal{I}_2 \in [\![\Gamma_{2,0}]\!]$ and $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma_{2,0}$, but $\mathcal{I}_0 \not\models \mathcal{K} \circ \Gamma_{2,0}$. The last observation together with the compatibility of $\preceq_{(.)}^{\circ}$ with $\circ$ implies that $\mathcal{I}_2^{\triangledown} \in \min([\![\Gamma_{2,0}]\!], \preceq_{\mathcal{K}}^{\circ})$ holds. Because ($\star$2) holds, we obtain $\mathcal{I}_0^{\triangledown} \in \min([\![\Gamma_{2,0}]\!], \preceq_{\mathcal{K}}^{\circ})$ from min-retractivity of $\preceq_{(.)}^{\circ}$. Similarly, we obtain $\mathcal{I}_0, \mathcal{I}_0^{\triangledown} \in \min([\![\Gamma_{0,1}]\!], \preceq_{\mathcal{K}}^{\circ})$ from compatibility and $\mathcal{I}_0, \mathcal{I}_0^{\triangledown} \in [\![\mathcal{K} \circ \Gamma_{0,1}]\!]$; showing that $\mathcal{I}_0 \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_0^{\triangledown}$ holds. Because of $\mathcal{I}_0^{\triangledown}, \mathcal{I}_0, \mathcal{I}_2 \in [\![\Gamma_{2,0}]\!]$, we obtain $\mathcal{I}_0 \in \min([\![\Gamma_{2,0}]\!], \preceq_{\mathcal{K}}^{\circ})$ from $\mathcal{I}_0^{\triangledown} \in \min([\![\Gamma_{2,0}]\!], \preceq_{\mathcal{K}}^{\circ})$ and min-retractivity, and consequently, we obtain the contradiction $\mathcal{I}_0 \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_2$.

The case of $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] = \emptyset$. Using $[\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!] \neq \emptyset$ yields that there exist some $\mathcal{I}^* \in [\![\mathcal{K} \circ \Gamma_{\triangledown}]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!]$. From Lemma 3.23(c) and $\mathcal{I}^*, \mathcal{I}_k^{\triangledown} \in [\![\Gamma_{k,k\oplus 1}]\!]$ and $\mathcal{I}_k^{\triangledown} \in [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!]$ and $\mathcal{I}^* \notin [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!]$ we obtain $\mathcal{I}_k^{\triangledown} \prec_{\mathcal{K}}^{\circ} \mathcal{I}^*$. Because (G1) is satisfied by $\circ$, we have that $\mathcal{I}^* \in [\![\mathcal{K} \circ \Gamma_{\triangledown}]\!]$ implies $\mathcal{I}^* \in [\![\Gamma_{\triangledown}]\!]$. We obtain the contradiction $\mathcal{I}^* \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\triangledown}$ from $\mathcal{I}^*, \mathcal{I}_k^{\triangledown} \in [\![\Gamma_{\triangledown}]\!]$ and $\mathcal{I}^* \in [\![\mathcal{K} \circ \Gamma_{\triangledown}]\!]$ by using Lemma 3.23(b).

In summary, this shows that Conditions (1)–(3) from Definition 3.43 are satisfied, i.e., $\Gamma_{0,1}, \Gamma_{1,2}, \Gamma_{2,0}$ form a critical loop. This contradicts the assumption that $\mathbb{B}$ does not exhibit a critical loop and consequently, $(\mathcal{I}_0, \mathcal{I}_1)$ or $(\mathcal{I}_1, \mathcal{I}_2)$ is detached from $\circ$ in $\mathcal{K}$. □

Lemma 3.50 provides the rationale for the first transformation step: For every $\mathcal{K} \in \mathfrak{B}$, we obtain $\preceq_{\mathcal{K}}^{\circ\prime}$ by removing all non-reflexive detached pairs from $\preceq_{\mathcal{K}}^{\circ}$, that is, $\preceq_{\mathcal{K}}^{\circ\prime} = \preceq_{\mathcal{K}}^{\circ} \setminus \mathfrak{D}_{\mathcal{K}}^{\circ}$. The resulting $\preceq_{\mathcal{K}}^{\circ\prime}$ is not guaranteed to be total anymore, and it is not necessarily transitive. But we will show that $\preceq_{\mathcal{K}}^{\circ\prime}$ inherits other important properties from $\preceq_{\mathcal{K}}^{\circ}$.

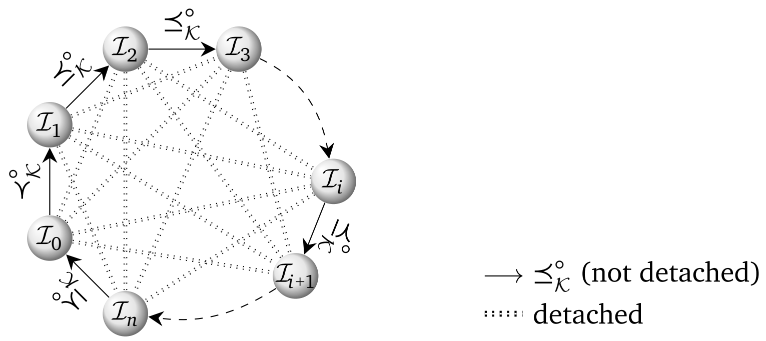


Figure 3.52: Illustration of a critical loop-situation of length n on the semantic side. This situation is due to Lemma 3.53 impossible for $\preceq^{\circ}_{\mathcal{K}}$ if $\mathbb{B}$ does not exhibit a critical loop. If $\mathbb{B}$ does not exhibit a critical loop, then this situation is due to Lemma 3.53 only possible when there is some $i \in \{1, \ldots, n\}$ such that $(\mathcal{I}_i, \mathcal{I}_{i \oplus 1})$ is a detached pair.

Lemma 3.51. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic which does not admit a critical loop, let $\circ$ be a base change operator satisfying (G1)–(G3), (G5), and (G6) and let $\preceq^{\circ}_{\mathcal{K}}$ be a quasi-faithful min-friendly assignment compatible with $\circ$. For each $\mathcal{K}, \Gamma \in \mathfrak{B}$, $\min(\llbracket \Gamma \rrbracket, {\preceq^{\circ}_{\mathcal{K}}}') = \min(\llbracket \Gamma \rrbracket, \preceq^{\circ}_{\mathcal{K}})$ holds and ${\preceq^{\circ}_{\mathcal{K}}}'$ is min-complete and reflexive.*

Proof. By definition of ${\preceq^{\circ}_{\mathcal{K}}}'$ we have $\mathcal{I} \,{\preceq^{\circ}_{\mathcal{K}}}'\, \mathcal{I}'$ if and only if $\mathcal{I} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}'$ for all $(\mathcal{I}, \mathcal{I}') \in \Omega \times \Omega$ which are not detached pairs. Because for every $\mathcal{I}, \mathcal{I}' \in \llbracket \Gamma \rrbracket$ with $\mathcal{I} \in \min(\llbracket \Gamma \rrbracket, \preceq^{\circ}_{\mathcal{K}})$ we have $\mathcal{I} \models \mathcal{K} \circ \Gamma$ by compatibility of $\preceq^{\circ}_{(.)}$ with $\circ$. Consequently, the pair $(\mathcal{I}, \mathcal{I}')$ is not detached and thus $\min(\llbracket \Gamma \rrbracket, {\preceq^{\circ}_{\mathcal{K}}}') = \min(\llbracket \Gamma \rrbracket, \preceq^{\circ}_{\mathcal{K}})$. The latter implies that min-completeness of $\preceq^{\circ}_{\mathcal{K}}$ carries over to ${\preceq^{\circ}_{\mathcal{K}}}'$. Reflexivity of ${\preceq^{\circ}_{\mathcal{K}}}'$ is obtained by construction, the reflexivity of $\preceq^{\circ}_{\mathcal{K}}$ and by the definition of $\mathfrak{D}^{\circ}_{\mathcal{K}}$. □

Step II: Taking the transitive closure. In this step, for every $\mathcal{K} \in \mathfrak{B}$, we obtain ${\preceq^{\circ}_{\mathcal{K}}}''$ by taking the transitive closure of ${\preceq^{\circ}_{\mathcal{K}}}'$, i.e., we have ${\preceq^{\circ}_{\mathcal{K}}}'' = TC({\preceq^{\circ}_{\mathcal{K}}}') = TC(\preceq^{\circ}_{\mathcal{K}} \setminus \mathfrak{D}^{\circ}_{\mathcal{K}})$. The resulting ${\preceq^{\circ}_{\mathcal{K}}}''$ is still not guaranteed to be total, but it is reflexive and transitive by construction, and it inherits further important properties from ${\preceq^{\circ}_{\mathcal{K}}}'$. It will turn out that the transitive closure will only add pairs to ${\preceq^{\circ}_{\mathcal{K}}}'$ that are detached pairs. This means that ${\preceq^{\circ}_{\mathcal{K}}}''$ contains only elements from ${\preceq^{\circ}_{\mathcal{K}}}'$ and from $\mathfrak{D}^{\circ}_{\mathcal{K}}$. Because adding detached pairs does not influence minimal sets of models of a base Γ with respect to $\preceq_{\mathcal{K}}$, we will obtain that these sets are preserved when taking the transitive closure.

If the transitive closure would (hypothetically) add non-detached pairs to ${\preceq^{\circ}_{\mathcal{K}}}'$, then the relation $\preceq^{\circ}_{\mathcal{K}}$ would contain a circle of interpretations consisting only of non-detached pairs (such as the circle illustrated in Figure 3.52). The following lemma shows that for base logics without critical loops such circles do not exist in $\preceq^{\circ}_{\mathcal{K}}$.

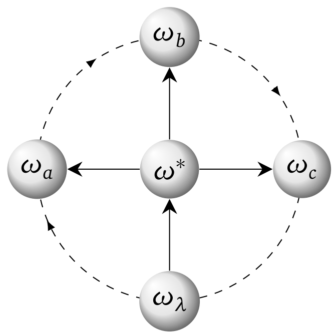


Figure 3.55: Exemplary situation of Lemma 3.56. Four interpretations lying on a strict circle, connected by another interpretation $\mathcal{I}^*$.

Lemma 3.53. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic which does not admit a critical loop, let $\mathcal{K} \in \mathfrak{B}$ be a base, and let $\circ$ be a base change operator for $\mathbb{B}$ that satisfies (G1)–(G3), (G5), and (G6). If there are three or more interpretations $\mathcal{I}_0, \ldots, \mathcal{I}_n \in \Omega$, i.e. $n \geq 2$, such that*

(a) $\mathcal{I}_0 \prec^\circ_{\mathcal{K}} \mathcal{I}_1$,

(b) $\mathcal{I}_i \preceq^\circ_{\mathcal{K}} \mathcal{I}_{i\oplus 1}$ *for all $i \in \{1, \ldots, n\}$, where $\oplus$ is addition* $\mathrm{mod}(n+1)$,

then there is some $i \in \{1, \ldots, n\}$ such that $(\mathcal{I}_i, \mathcal{I}_{i\oplus 1})$ is a detached pair.

The proof will make use of circles of interpretations which are violating the situation given in Lemma 3.53. To make such situations easier to handle, we introduce the following notion which makes implicit use of $\preceq^\circ_{(.)}$, defined in Definition 3.19.

Definition 3.54. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic, let $\mathcal{K} \in \mathfrak{B}$ be a base, and let $\circ$ be a base change operator for $\mathbb{B}$ that satisfies (G1)–(G3), (G5), and (G6). A sequence of interpretations $\circlearrowleft = \mathcal{I}_0, \ldots, \mathcal{I}_n, \mathcal{I}_0$ from Ω is said to form a* strict circle of length $n+1$ (with respect to $\circ$ and $\mathcal{K}$) *if*

- *$\mathcal{I}_0, \ldots, \mathcal{I}_n$ are satisfying Condition (a) and Condition (b) from Lemma 3.53, and*
- *$(\mathcal{I}_i, \mathcal{I}_{i+1})$ is not a detached pair for each $i \in \{0, \ldots, n\}$, where $\oplus$ is addition* $\mathrm{mod}(n+1)$.

We will also substitute elements in a strict circle $\circlearrowleft$ and use therefore the following notion. For a substitution $\sigma = \{\mathcal{I}_{i_1} \mapsto x_1,\ \mathcal{I}_{i_2} \mapsto x_2,\ \ldots\}$, we denote by $\circlearrowleft[\sigma]$ the simultaneous replacement of $\mathcal{I}_{i_j}$ by x_j in $\circlearrowleft$ for all $\mathcal{I}_{i_j} \mapsto x_j \in \sigma$.

The following lemma will be useful, and describes situations like in Figure 3.55.

Lemma 3.56 (cross lemma). *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic with no critical loop, let $\mathcal{K} \in \mathfrak{B}$ be a base, and let $\circ$ be a base change operator for $\mathbb{B}$ that satisfies (G1)–(G3), (G5), and (G6). If there are $\mathcal{I}_0, \ldots, \mathcal{I}_n \in \Omega$, with $n > 3$, and pairwise distinct $\lambda, a, b, c \in \{0, \ldots, n\}$, such that*

(a) $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n, \mathcal{I}_0$ *is a strict circle of length* $n+1$,

(b) *there exists an interpretation* $\mathcal{I}^*$ *such that*

$$\mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_a \qquad \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_b \qquad \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_c \qquad \mathcal{I}_\lambda \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* \text{ , and}$$

(c) *every pair of* $\preceq^{\circ}_{\mathcal{K}}$ *considered in (b) is not detached from* $\circ$ *in* $\mathcal{K}$,

then there is a strict circle of length m *with* $3 \leq m \leq n$.

Proof. We assume $a < b < c$, and we assume that the path $\mathcal{I}_c, \ldots, \mathcal{I}_\lambda$ does not contain $\mathcal{I}_a$ and $\mathcal{I}_b$ (when seeing $\preceq^{\circ}_{\mathcal{K}}$ as a graph). All other cases will follow by symmetry. We continue by considering several cases:

The case of $\mathcal{I}_\lambda \prec^{\circ}_{\mathcal{K}} \mathcal{I}^*$. We obtain $\mathcal{I}_\lambda \prec^{\circ}_{\mathcal{K}} \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_\lambda$, which yields that $\circlearrowleft_{\lambda c} = \mathcal{I}_\lambda, \mathcal{I}^*, \mathcal{I}_c, \ldots, \mathcal{I}_\lambda$ is a strict circle. Note that because $\circlearrowleft_{\lambda c}$ contains $\mathcal{I}^*$ and in addition only elements of $\{\mathcal{I}_0, \ldots, \mathcal{I}_n\} \setminus \{\mathcal{I}_a, \mathcal{I}_b\}$, we have that $\circlearrowleft_{\lambda c}$ has a length of at most n.

The case of $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_c$ *and no prior case applies.* If $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_c$, then we obtain $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_\lambda \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$, yielding that $\circlearrowleft_{c\lambda} = \mathcal{I}^*, \mathcal{I}_c, \ldots, \mathcal{I}_\lambda, \mathcal{I}^*$ is a strict circle. Note that because $\circlearrowleft_{c\lambda}$ contains $\mathcal{I}^*$ and in addition only elements of $\{\mathcal{I}_0, \ldots, \mathcal{I}_n\} \setminus \{\mathcal{I}_a, \mathcal{I}_b\}$, we have that $\circlearrowleft_{c\lambda}$ has a length of at most n.

The case of $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_b$ *and no prior case applies.* In this case we have $\mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$. We obtain $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_b \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$, which yields that $\circlearrowleft_{bc} = \mathcal{I}^*, \mathcal{I}_b, \ldots, \mathcal{I}_c, \mathcal{I}^*$ is a strict circle. Note that because $\circlearrowleft_{bc}$ contains, beside of $\mathcal{I}^*$, only elements of $\{\mathcal{I}_0, \ldots, \mathcal{I}_n\} \setminus \{\mathcal{I}_a, \mathcal{I}_\lambda\}$, we have that $\circlearrowleft_{bc}$ has a length of at most n.

The case of $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_a$ *and no prior case applies.* In this case we have $\mathcal{I}_b \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$. We obtain $\mathcal{I}^* \prec^{\circ}_{\mathcal{K}} \mathcal{I}_a \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_b \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$, which yields that $\circlearrowleft_{ab} = \mathcal{I}^*, \mathcal{I}_a, \ldots, \mathcal{I}_b, \mathcal{I}^*$ is a strict circle. Note that because $\circlearrowleft_{ab}$ contains, beside of $\mathcal{I}^*$, only elements of $\{\mathcal{I}_0, \ldots, \mathcal{I}_n\} \setminus \{\mathcal{I}_c, \mathcal{I}_\lambda\}$, we have that $\circlearrowleft_{ab}$ has a length of at most n.

If none of the cases above applies, then we have that $\mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_\lambda$ and $\mathcal{I}_a \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}_b \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ holds. For the following line of arguments, recall that $a < b < c$ holds. We consider the case of $0 < \lambda < a$; for all other cases (where $0 < \lambda < a$ does not hold), the line of arguments is symmetric to the proof we present here in the following for the case of $0 < \lambda < a$. Because $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n, \mathcal{I}_0$ is a strict circle of length $n+1$, we obtain that $\mathcal{I}_0 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_\lambda \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_c \preceq^{\circ}_{\mathcal{K}} \ldots \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_0$. This show that $\circlearrowleft_{0\lambda c} = \mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_\lambda, \mathcal{I}^*, \mathcal{I}_c, \ldots, \mathcal{I}_0$ is a strict circle. Because $\circlearrowleft_{0\lambda c}$ contains $\mathcal{I}^*$ and additionally only elements from $\mathcal{I}_0, \ldots, \mathcal{I}_n$, but not $\mathcal{I}_a$ and $\mathcal{I}_b$, we obtain that $\circlearrowleft_{0\lambda c}$ has a length of at most n.

In summary, we obtain a strict circle of length m with $3 \leq m \leq n$ for each case. $\square$

Note that it is not necessary to assume that $\mathcal{I}^*$ is distinct from $\mathcal{I}_0, \ldots, \mathcal{I}_n$ in Lemma 3.56. We now give a full proof of Lemma 3.53.

Proof of Lemma 3.53 . Let $\mathcal{I}_0, \ldots, \mathcal{I}_n \in \Omega$ such that Condition (a) and Condition (b) of Lemma 3.53 are satisfied. With $\oplus$ we denote addition $\mathrm{mod}(n+1)$. The proof will be by induction. Note that for $n = 2$ we obtain the result by Lemma 3.50. We proceed with the proof for the case of $n > 2$ and assume that Lemma 3.53 already holds for all m with $2 \leq m < n$. A consequence of the induction hypothesis is that there is no strict circle of length c for $3 \leq c \leq n$.

We are striving for a contradiction. Therefore, we assume $\circlearrowleft_{0n} = \mathcal{I}_0, \ldots, \mathcal{I}_n, \mathcal{I}_0$ is a strict circle of length $n+1$, which is, due to Condition (a) and Condition (b) from Lemma 3.53, equivalent to assuming that $(\mathcal{I}_i, \mathcal{I}_{i\oplus 1})$ is not a detached pair for each $i \in \{1, \ldots, n\}$. The remaining parts of the proof show that the existence of the strict circle $\circlearrowleft_{0n}$ implies existence of a critical loop.

As first step, we show that $\mathcal{I}_0, \ldots, \mathcal{I}_n \notin [\![\mathcal{K}]\!]$ holds. If $\mathcal{I}_1 \in [\![\mathcal{K}]\!]$, then due Definition 3.19, we obtain $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_0$, which contradicts Condition (a). If $\mathcal{I}_i \in [\![\mathcal{K}]\!]$ for some $i \in \{0, 2, 3, \ldots, n\}$, then, because of Condition (b), there is some j with $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_{j\oplus 1}$ and $\mathcal{I}_j \notin [\![\mathcal{K}]\!]$ and $\mathcal{I}_{j\oplus 1} \in [\![\mathcal{K}]\!]$; which is again impossible due to Definition 3.19. Thus, we have $\mathcal{I}_0, \ldots, \mathcal{I}_n \notin [\![\mathcal{K}]\!]$ for the remaining parts of the proof.

We continue by showing the existence of several bases, which will form a critical loop. Definition 3.19 and Definition 3.48 together implies that for each $i \in \{1, \ldots, n\}$ exists a base $\Gamma_{i,i\oplus 1} \in \mathfrak{B}$ such that

$$\mathcal{I}_i, \mathcal{I}_{i\oplus 1} \models \Gamma_{i,i\oplus 1} \text{ and } \mathcal{I}_i \models \mathcal{K} \circ \Gamma_{i,i\oplus 1} \tag{\#1}$$

holds. Moreover, by $\mathcal{I}_0 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_1$ from Condition (1) and $\mathcal{I}_1 \not\models \mathcal{K}$ and Lemma 3.23(a), there exists a base $\Gamma_{0,1} \in \mathfrak{B}$ such that the following holds:

$$\mathcal{I}_0, \mathcal{I}_1 \models \Gamma_{0,1} \text{ and } \mathcal{I}_0 \models \mathcal{K} \circ \Gamma_{0,1} \text{ and } \mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma_{0,1}. \tag{\#2}$$

We show that $\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0}$ is forming a critical loop. To this end we are setting $\Gamma_i = (\mathcal{K} \circ \Gamma_{i,i\oplus 1}) \Cup \Gamma_{i\oplus n,i}$ for each $i \in \{0, \ldots, n\}$. By (#1) and (#2) each Γ_i is a consistent base with $\mathcal{I}_i \in [\![\Gamma_i]\!]$. We continue by verifying that Conditions (1)–(3) from Definition 3.43 are satisfied.

(1) If $\mathcal{K}$ is inconsistent, then Condition (1) is immediately satisfied. We consider the case where $\mathcal{K}$ is consistent and $[\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!] \neq \emptyset$ for some $i \in \{0, \ldots, n\}$. From (G2) we obtain $[\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] = [\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!]$. From $\mathcal{I}_i \in [\![\Gamma_i]\!]$ and the definition of Γ_i, we obtain $\mathcal{I}_i \in [\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] \cap [\![\Gamma_{i\oplus n,i}]\!]$. As $[\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] = [\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!]$, we obtain $\mathcal{I}_i \in [\![\mathcal{K} \Cup \Gamma_{i,i\oplus 1}]\!] \cap [\![\Gamma_{i\oplus n,i}]\!]$. Consequently, there exists some $i \in \{0, \ldots, n\}$ such that $\mathcal{I}_i \in [\![\mathcal{K}]\!]$, yielding a contradiction to $\mathcal{I}_0, \ldots, \mathcal{I}_n \notin [\![\mathcal{K}]\!]$.

(2) By the postulate (G1) we have $[\![\mathcal{K} \circ \Gamma_{i,i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ for each $i \in \{0,\ldots,n\}$. The definition of Γ_i yields $[\![\Gamma_i]\!] \subseteq [\![\Gamma_{i,i\oplus 1} \Cup \Gamma_{i\oplus n,i}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ for each $i \in \{0,1,2\}$. Substitution of i by $i \oplus 1$ yields $[\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i\oplus 1,i\oplus 2} \Cup \Gamma_{i,i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$; showing that $[\![\Gamma_i]\!] \cup [\![\Gamma_{i\oplus 1}]\!] \subseteq [\![\Gamma_{i,i\oplus 1}]\!]$ holds for each $i \in \{0,\ldots,n\}$.

We show that each $\Gamma_i \Cup \Gamma_j$ is inconsistent, by assuming the contrary, i.e. there are some $i,j \in \{0,\ldots,n\}$ such that $i \neq j$ and $\Gamma_i \Cup \Gamma_j$ is consistent. Because of the commutativity of $\Cup$, we assume $i < j$ without loss of generality. By compatibility and definition of Γ_i and by definition of Γ_j, there exists some $\mathcal{I}^* \in [\![\Gamma_{i\oplus n,i} \Cup \Gamma_{j\oplus n,j}]\!]$ with $\mathcal{I}^* \in \min([\![\Gamma_{i,i\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$ and $\mathcal{I}^* \in \min([\![\Gamma_{j,j\oplus 1}]\!], \preceq^{\circ}_{\mathcal{K}})$. Recall that $\mathcal{I}_i, \mathcal{I}_{i\oplus 1} \in [\![\Gamma_{i,i\oplus 1}]\!]$ and $\mathcal{I}_j, \mathcal{I}_{j\oplus 1} \in [\![\Gamma_{j,j\oplus 1}]\!]$. Consequently, for all $k \in \{i, i\oplus 1, j, j\oplus 1\}$ holds $\mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_k$. Moreover, because of (#1) and (#2) we obtain $\mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ from Lemma 3.23(b). From $\mathcal{I}^* \in [\![\Gamma_{i\oplus n,i} \Cup \Gamma_{j\oplus n,j}]\!]$ we obtain, by an analogous argument, that $\mathcal{I}_{i\oplus n} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ and $\mathcal{I}_{j\oplus n} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^*$ holds. In summary, we have:

$$\begin{array}{cccc} \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_i & \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_j & \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_{i\oplus 1} & \mathcal{I}_{i\oplus n} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* \\ \mathcal{I}_i \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* & \mathcal{I}_j \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* & \mathcal{I}^* \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_{j\oplus 1} & \mathcal{I}_{j\oplus n} \preceq^{\circ}_{\mathcal{K}} \mathcal{I}^* \end{array} \tag{⊛1}$$

Note that all pairs $(\mathcal{I}^*, \mathcal{I}_\xi)$, $(\mathcal{I}^*, \mathcal{I}_{\xi\oplus 1})$ and $(\mathcal{I}_{\xi\oplus n}, \mathcal{I}^*)$ with $\xi \in \{i,j\}$ are not detached.

We are now striving for a contradiction by showing the existence of a strict circle with a length of at most n. Recall that $\circlearrowleft_{0n} = \mathcal{I}_0, \ldots, \mathcal{I}_n, \mathcal{I}_0$ is a strict circle of length $n+1$. At first, we consider two particular cases:

($\mathcal{I}_i = \mathcal{I}_{j\oplus 1}$) We obtain a strict circle of length of at most n from Lemma 3.56 by using $\circlearrowleft_{0n}$ and setting $\lambda = i$, $a = j$, $b = i \oplus 1$, and $c = j \oplus n$. Note that λ, a, b, c are pairwise distinct indices.

($\mathcal{I}_j = \mathcal{I}_{i\oplus 1}$) We obtain a strict circle of length of at most n from Lemma 3.56 by using $\circlearrowleft_{0n}$ and setting $\lambda = i$, $a = j$, $b = j \oplus 1$, and $c = i \oplus n$. Note that λ, a, b, c are pairwise distinct indices.

For all situations not covered by the cases above, we obtain that $i, i\oplus 1, j, j\oplus 1$ are pairwise distinct. Because of (⊛1), we can apply Lemma 3.56 by using $\circlearrowleft_{0n}$ and setting $\lambda = i$, $a = i \oplus 1$, $b = j$, and $c = j \oplus n$. This yields a strict circle with a length of at most n.

In summary, for every possible case we obtain a contradiction, which shows that Condition (2) of critical loops (cf. Definition 3.43) is satisfied.

(3) We show Condition (3) from Definition 3.43 by contradiction. Therefore, assume there is a base $\Gamma_\triangledown \in \mathfrak{B}$ such that for

$$B = \{\Gamma_i \mid [\![\Gamma_\triangledown \Cup \Gamma]\!] \neq \emptyset\} \subseteq \{\Gamma_0, \ldots, \Gamma_n\} \tag{⋆1}$$

holds $|B| \geq 3$ and there exists no base $\Gamma'_\triangledown \in \mathfrak{B}$ as required by Condition (3). Consequently, for each base $\Gamma \in \mathfrak{B}$ we have

$$[\![\Gamma]\!] \neq \emptyset \text{ implies } [\![\Gamma]\!] \not\subseteq [\![\Gamma_\triangledown]\!] \setminus \big([\![\Gamma_{0,1}]\!] \cup \ldots \cup [\![\Gamma_{n,0}]\!]\big). \qquad (\star 2)$$

From ($\star$1) we obtain that $\Gamma_\triangledown$ is consistent, and thus, by (G3), that $\mathcal{K} \circ \Gamma_\triangledown$ is consistent and from satisfaction of (G1) we obtain $[\![\mathcal{K} \circ \Gamma_\triangledown]\!] \subseteq [\![\Gamma_\triangledown]\!]$. Consequently, because of ($\star$2), we have $[\![\mathcal{K} \circ \Gamma_\triangledown]\!] \not\subseteq [\![\Gamma_\triangledown]\!] \setminus \big([\![\Gamma_{0,1}]\!] \cup \ldots \cup [\![\Gamma_{n,0}]\!]\big)$. This implies that $[\![\mathcal{K} \circ \Gamma_\triangledown]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!] \neq \emptyset$ for some $k \in \{0, \ldots, n\}$. Let $\mathcal{I}_k^\star$ be an interpretation with $\mathcal{I}_k^\star \in [\![\mathcal{K} \circ \Gamma_\triangledown]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!]$. From (G5) and (G6), $\mathcal{I}_k^\star \in [\![\mathcal{K} \circ \Gamma_\triangledown]\!] \cap [\![\Gamma_{k,k\oplus 1}]\!]$ and commutativity of $\Cup$ we obtain

$$\mathcal{I}_k^\star \in \min([\![\Gamma_\triangledown]\!], \preceq^\circ_{\mathcal{K}}) \cap [\![\Gamma_{k,k\oplus 1}]\!] = \min([\![\Gamma_{k,k\oplus 1} \Cup \Gamma_\triangledown]\!], \preceq^\circ_{\mathcal{K}}) \text{ and } \mathcal{I}_k \preceq^\circ_{\mathcal{K}} \mathcal{I}_k^\star, \qquad (\star 3)$$

whereby the latter is a direct consequence of $\mathcal{I}_k^\star \in [\![\Gamma_{k,k\oplus 1}]\!]$ and $\mathcal{I}_k \in \min([\![\Gamma_{k,k\oplus 1}]\!], \preceq^\circ_{\mathcal{K}})$. Furthermore, let $\mathcal{I}_i^\triangledown$ be some interpretation with $\mathcal{I}_i^\triangledown \in [\![\Gamma_\triangledown \Cup \Gamma_i]\!]$ for each $\Gamma_i \in B$. We show as next for each $\Gamma_\xi \in B$ and for each $\mathcal{I}^*_{\xi\oplus n} \in [\![\Gamma_{\xi\oplus n}]\!]$ and for each $\mathcal{I}^*_{\xi\oplus 1} \in [\![\Gamma_{\xi\oplus 1}]\!]$ that we have

$$\begin{array}{ccc} \mathcal{I}_k^\star \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown & \mathcal{I}^*_{\xi\oplus n} \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown & \mathcal{I}_\xi \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown \\ & \mathcal{I}_\xi^\triangledown \preceq^\circ_{\mathcal{K}} \mathcal{I}^*_{\xi\oplus 1} & \mathcal{I}_\xi^\triangledown \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi\ . \end{array} \qquad (\star 4)$$

We obtain $\mathcal{I}_k^\star \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown$ from Lemma 3.23(b), because $\mathcal{I}_k^\star, \mathcal{I}_\xi^\triangledown \in [\![\Gamma_\triangledown]\!]$ and $\mathcal{I}_k^\star \in [\![\mathcal{K} \circ \Gamma_\triangledown]\!]$ holds. From $\mathcal{I}_\xi, \mathcal{I}_\xi^\triangledown \in \min([\![\Gamma_{\xi,\xi\oplus 1}]\!], \preceq^\circ_{\mathcal{K}})$ we directly obtain $\mathcal{I}_\xi \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown$ and $\mathcal{I}_\xi^\triangledown \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi$. Compatibility of $\preceq^\circ_{(.)}$ with $\circ$, together with the definitions of Γ_ξ and Condition (2), yields the remaining statements of ($\star$4).

Moreover, as next step, we show for each $\Gamma_\xi \in B$ and for each $\mathcal{I}^*_{\xi\oplus n} \in [\![\Gamma_{\xi\oplus n}]\!]$ and for each $\mathcal{I}^*_{\xi\oplus 1} \in [\![\Gamma_{\xi\oplus 1}]\!]$ the following holds:

$$\begin{array}{rcl} \mathcal{I}^*_{\xi\oplus n} \prec^\circ_{\mathcal{K}} \mathcal{I}_\xi & \text{if and only if} & \mathcal{I}^*_{\xi\oplus n} \prec^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown \\ \mathcal{I}_\xi \prec^\circ_{\mathcal{K}} \mathcal{I}^*_{\xi\oplus 1} & \text{if and only if} & \mathcal{I}_\xi^\triangledown \prec^\circ_{\mathcal{K}} \mathcal{I}^*_{\xi\oplus 1} \end{array} \qquad (\star 5)$$

Observe that ($\star$5) holds, otherwise, we would obtain a strict circle of length 3. These strict circles are directly obtainable from ($\star$4): if $\mathcal{I}^*_{\xi\oplus n} \prec^\circ_{\mathcal{K}} \mathcal{I}_\xi$ and $\mathcal{I}_\xi^\triangledown \preceq^\circ_{\mathcal{K}} \mathcal{I}^*_{\xi\oplus n}$, obtain the strict circle $\mathcal{I}^*_{\xi\oplus n} \prec^\circ_{\mathcal{K}} \mathcal{I}_\xi \preceq^\circ_{\mathcal{K}} \mathcal{I}_\xi^\triangledown \preceq^\circ_{\mathcal{K}} \mathcal{I}^*_{\xi\oplus n}$ with a length of 3. For all other cases, we obtain analogously a strict circle of length 3.

Now let $\ell_{\min}, \ell_{\mathrm{med}}, \ell_{\max}$ be integers with $0 \leq \ell_{\min} < \ell_{\mathrm{med}} < \ell_{\max} \leq n$ such that $\Gamma_{k\oplus\ell_{\min}}$, $\Gamma_{k\oplus\ell_{\mathrm{med}}}, \Gamma_{k\oplus\max} \in B$ and $\ell_{\min}$ is the smallest number from $\{0, \ldots, n\}$ with $\Gamma_{k\oplus\ell_{\min}} \in B$, and $\ell_{\max}$ is the greatest number from $\{0, \ldots, n\}$ with $\Gamma_{k\oplus\max} \in B$. For convenience, we will sometimes write ℓ_x and $\mathcal{I}_x$, instead of $\ell_{k\oplus\ell_x}$ and $\mathcal{I}_{k\oplus\ell_x}$, respectively, for any $x \in \{\min, \mathrm{med}, \max\}$.

We now establish that replacing ω_i in $\circlearrowright_{0n}$ by $\mathcal{I}_i^{\triangledown}$ for some $\Gamma_i \in B$ yields again a strict circle. Remember that each pair given in ($\star$4) and ($\star$5) is a non-detached pair. Because of this and because $\circlearrowright_{0n}$ is a strict circle of length $n+1$, we obtain from ($\star$4) and ($\star$5) that $\circlearrowright_{0n}[\sigma]$ is also a strict circle of length $n+1$ for each substitution σ with

$$\sigma \subseteq \{\mathcal{I}_{\text{min}} \mapsto \mathcal{I}_{\text{min}}^{\triangledown},\ \mathcal{I}_{\text{med}} \mapsto \mathcal{I}_{\text{med}}^{\triangledown},\ \mathcal{I}_{\text{max}} \mapsto \mathcal{I}_{\text{max}}^{\triangledown}\}\ ,$$

i.e., substituting each $\mathcal{I}_x$ by $\mathcal{I}_x^{\triangledown}$ in $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_n$, for some of $x \in \{\text{min}, \text{med}, \text{max}\}$, yields a strict circle of length $n+1$.

We consider two cases, the case where $\Gamma_k \uplus \Gamma_{\triangledown}$ is inconsistent and the case where $\Gamma_k \uplus \Gamma_{\triangledown}$ is consistent.

The case of $\Gamma_k \uplus \Gamma_{\triangledown}$ *is inconsistent.* For this case we have that $\Gamma_k \notin B$ holds. Remember that by ($\star$3) and ($\star$4) the following holds:

$$\begin{array}{ll} \mathcal{I}_k \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\star} & \qquad \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{\text{med}}^{\triangledown} \\ \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{\text{min}}^{\triangledown} & \qquad \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{\text{max}}^{\triangledown} \end{array}$$

We obtain that there exists a strict circle with a length of at most n by using Lemma 3.56 when setting $\lambda = k$, $a = k \oplus \ell_{\text{min}}$, $b = k \oplus \ell_{\text{med}}$ and $c = k \oplus \ell_{\text{max}}$, using the strict circle $\circlearrowright_{0n}[\mathcal{I}_a \mapsto \mathcal{I}_p^{\triangledown}, \mathcal{I}_b \mapsto \mathcal{I}_\tau^{\triangledown}, \mathcal{I}_c \mapsto \mathcal{I}_q^{\triangledown}]$.

The case of $\Gamma_k \uplus \Gamma_{\triangledown}$ *is consistent.* This case is equivalent to having $\ell_{\text{min}} = 0$, i.e., $\Gamma_{\text{min}} = \Gamma_k \in B$. Consequently, we have that $\mathcal{I}_k^{\triangledown} \in [\![\Gamma_k \uplus \Gamma_{\triangledown}]\!]$.

From the definition of Γ_k, and from $\mathcal{I}_k^{\triangledown}, \mathcal{I}_k^{\star} \in [\![\Gamma_{k,k\oplus 1}]\!]$ with $\mathcal{I}_k^{\triangledown} \in [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!]$, and from compatibility and min-retractivity we also obtain $\mathcal{I}_k^{\star}, \mathcal{I}_k^{\triangledown} \in [\![\mathcal{K} \circ \Gamma_{k,k\oplus 1}]\!] \cap [\![\Gamma_{\triangledown}]\!] = \min([\![\Gamma_{k,k\oplus 1}]\!], \preceq_{\mathcal{K}}^{\circ}) \cap [\![\Gamma_{\triangledown}]\!]$. Consequently, all observations for $\mathcal{I}_k^{\star}$ do also hold for $\mathcal{I}_k^{\triangledown}$; in particular, this applies to ($\star$3)–($\star$5). Thus, we assume $\mathcal{I}_k^{\star} = \mathcal{I}_k^{\triangledown}$ in the following.

Together with ($\star$4) and ($\star$5) we can summarize as follows:

$$\begin{array}{lll} \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{\text{max}}^{\triangledown} & \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{k\oplus 1} & \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k \\ \mathcal{I}_k^{\star} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_{\text{med}}^{\triangledown} & \mathcal{I}_{k\oplus n} \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\star} & \mathcal{I}_k \preceq_{\mathcal{K}}^{\circ} \mathcal{I}_k^{\star} \end{array} \tag{$\star$6}$$

We are striving for a contradiction by showing the existence of a strict circle of length that is strictly smaller than $n+1$. Therefore, we will make use of Lemma 3.56, whenever that is possible. We consider three cases in the following, depending on the values of ℓ_{med} and ℓ_{max}. Recall that $1 \leq \ell_{\text{med}} < \ell_{\text{max}} \leq n$ holds.

($\ell_{\mathrm{med}} \neq 1$) Because of ($\star$6), we can directly apply Lemma 3.56 by setting $\lambda = k$, $a = k \oplus 1$, $b = k \oplus \ell_{\mathrm{med}}$ and $c = k \oplus \ell_{\max}$, and by using the strict circle $\circlearrowright_{0n}[\mathcal{I}_\tau \mapsto \mathcal{I}_\tau^\triangledown, \mathcal{I}_q \mapsto \mathcal{I}_q^\triangledown]$ for Lemma 3.56, which yields a strict circle with a length of at most n.

($\ell_{\max} \neq n$) We apply Lemma 3.56 by setting $\lambda = k$, $a = k \oplus \ell_{\mathrm{med}}$, $b = k \oplus \ell_{\max}$ and $c = k \oplus n$, and by using the strict circle $\circlearrowright_{0n}[\mathcal{I}_p \mapsto \mathcal{I}_p^\triangledown, \mathcal{I}_\tau \mapsto \mathcal{I}_\tau^\triangledown]$, which yields again a strict circle with a length of at most n.

($\ell_{\mathrm{med}} = 1$ and $\ell_{\max} = n$) Because of $\ell_{\max} = n$, we have that $\mathcal{I}_{\max}^\triangledown \in [\![\Gamma_{k\oplus n}]\!]$. Together with $\mathcal{I}_k^\star \in [\![\Gamma_k]\!]$, we obtain from ($\star$4) that $\mathcal{I}_{\max}^\triangledown \preceq_{\mathcal{K}}^\circ \mathcal{I}_k^\star$. From the min-retractivity of $\preceq_{\mathcal{K}}^\circ$ and $\Gamma_{k\oplus n} \in B$, we obtain $\mathcal{I}_{\max}^\triangledown \in \min([\![\Gamma_\triangledown]\!], \preceq_{\mathcal{K}}^\circ)$, which implies $\mathcal{I}_{\max}^\triangledown \preceq_{\mathcal{K}}^\circ \mathcal{I}_{\mathrm{med}}^\triangledown$. If $\mathcal{I}_{\max}^\triangledown \prec_{\mathcal{K}}^\circ \mathcal{I}_{\mathrm{med}}^\triangledown$, then we obtain the strict circle $\mathcal{I}_{\max}^\triangledown, \mathcal{I}_{\mathrm{med}}^\triangledown, \mathcal{I}_{k\oplus(\ell_{\mathrm{med}}+1)}, \ldots, \mathcal{I}_{k\oplus(\ell_{\max}+n)}, \mathcal{I}_{\max}^\triangledown$ which has a length of at most n. Due to the induction hypothesis there is no such strict circle, and thus, $\mathcal{I}_{\max}^\triangledown \prec_{\mathcal{K}}^\circ \mathcal{I}_{\mathrm{med}}^\triangledown$ is impossible. From totality of $\preceq_{\mathcal{K}}^\circ$ we obtain that $\mathcal{I}_{\mathrm{med}}^\triangledown \preceq_{\mathcal{K}}^\circ \mathcal{I}_{\max}^\triangledown$ holds. If $k \oplus \ell_{\max} = 0$, then we obtain the strict circle $\mathcal{I}_{\max}^\triangledown, \mathcal{I}_k, \mathcal{I}_{\mathrm{med}}^\triangledown, \mathcal{I}_{\max}^\triangledown$ of length 3. If $k = 0$, then we obtain the strict circle $\mathcal{I}_k, \mathcal{I}_{\mathrm{med}}^\triangledown, \mathcal{I}_{\max}^\triangledown, \mathcal{I}_k$ of length 3. If none of the prior cases applies, then $\circlearrowright = \mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_{\max}^\triangledown, \mathcal{I}_{\mathrm{med}}^\triangledown, \ldots, \mathcal{I}_0$ is a strict circle. Note that $\mathcal{I}_k$ is not part of the strict circle $\circlearrowright$, and consequently the length of $\circlearrowright$ is bounded by n.

We obtain a contradiction in every case, which shows that Condition (3) of Definition 3.43 is satisfied.

In summary, assuming that each $(\mathcal{I}_i, \mathcal{I}_{i\oplus 1})$ is not a detached pair leads to formation of a critical loop by $\Gamma_{0,1}, \Gamma_{1,2}, \ldots, \Gamma_{n,0}$; contradicting the critical loop-freeness of $\mathbb{B}$. Consequently, at least one $(\mathcal{I}_i, \mathcal{I}_{i\oplus 1})$ has to be a detached pair. □

By employing Lemma 3.53 we show now that transformation of ${\preceq_{\mathcal{K}}^\circ}'$ to ${\preceq_{\mathcal{K}}^\circ}''$ by taking the transitive closure only adds detached pairs.

Lemma 3.57. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic which does not admit a critical loop, let $\mathcal{K} \in \mathfrak{B}$ be a base, and let $\circ$ be a base change operator for $\mathbb{B}$ that satisfies (G1)–(G3), (G5), and (G6). The following holds:*

$$ {\preceq_{\mathcal{K}}^\circ}' \subseteq {\preceq_{\mathcal{K}}^\circ}'' \subseteq \preceq_{\mathcal{K}}^\circ $$

Proof. By construction of ${\preceq_{\mathcal{K}}^\circ}''$, we have ${\preceq_{\mathcal{K}}^\circ}' \subseteq {\preceq_{\mathcal{K}}^\circ}''$, and by construction of ${\preceq_{\mathcal{K}}^\circ}'$ we have ${\preceq_{\mathcal{K}}^\circ}' \subseteq \preceq_{\mathcal{K}}^\circ$. To show that ${\preceq_{\mathcal{K}}^\circ}'' \subseteq \preceq_{\mathcal{K}}^\circ$ holds, we assume the contrary, i.e., there exists a pair $(\mathcal{I}_1, \mathcal{I}_0) \in {\preceq_{\mathcal{K}}^\circ}''$ such that $\mathcal{I}_1 \not\preceq_{\mathcal{K}}^\circ \mathcal{I}_0$. From ${\preceq_{\mathcal{K}}^\circ}' \subseteq \preceq_{\mathcal{K}}^\circ$ we obtain $\mathcal{I}_1 \,{\not\preceq_{\mathcal{K}}^\circ}'\, \mathcal{I}_0$ and because $\preceq_{\mathcal{K}}^\circ$ is a total relation, we have that $\mathcal{I}_0 \prec_{\mathcal{K}}^\circ \mathcal{I}_1$ holds. By the definition of transitive closure (cf. Section 2.2), there exist $\mathcal{I}_2, \ldots, \mathcal{I}_n \in \Omega$, for some $n \in \mathbb{N}$, such that $\mathcal{I}_1 \,{\preceq_{\mathcal{K}}^\circ}'\, \mathcal{I}_2$ and $\mathcal{I}_n \,{\preceq_{\mathcal{K}}^\circ}'\, \mathcal{I}_0$ and $\mathcal{I}_i \,{\preceq_{\mathcal{K}}^\circ}'\, \mathcal{I}_{i+1}$ for each $i \in \{2, \ldots, n-1\}$. From ${\preceq_{\mathcal{K}}^\circ}' \subseteq \preceq_{\mathcal{K}}^\circ$, we obtain $\mathcal{I}_0 \prec_{\mathcal{K}}^\circ \mathcal{I}_1 \preceq_{\mathcal{K}}^\circ \mathcal{I}_2 \ldots \preceq_{\mathcal{K}}^\circ \mathcal{I}_n \preceq_{\mathcal{K}}^\circ \mathcal{I}_0$. We obtain a contradiction, because ${\preceq_{\mathcal{K}}^\circ}'$ does not contain any detached pairs, but due to Lemma 3.53 there is some $i \in \{2, \ldots, n-1\}$ such that $(\mathcal{I}_i, \mathcal{I}_{i+1})$ is a detached pair. Consequently, we obtain $\preceq_{\mathcal{K}}^{\circ}{}'' \subseteq \preceq_{\mathcal{K}}^{\circ}$. □

Combining Lemma 3.51 and Lemma 3.57 yields that $\preceq_{\mathcal{K}}^{\circ}{}''$ is a (possibly non-total) preorder with useful properties. In particular, the sets of minimal models for every base $\Gamma \in \mathfrak{B}$ coincide for $\preceq_{\mathcal{K}}^{\circ}{}'$ and $\preceq_{\mathcal{K}}^{\circ}{}''$.

Lemma 3.58. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic which does not admit a critical loop, let $\mathcal{K} \in \mathfrak{B}$ and let $\circ$ be a base change operator for $\mathbb{B}$ which satisfies (G1)–(G3), (G5), and (G6). Then $\preceq_{\mathcal{K}}^{\circ}{}''$ is a min-complete preorder and for any $\Gamma \in \mathfrak{B}$ holds $\min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}'') = \min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}')$.*

Proof. Because of Lemma 3.57 we have $\min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}'') = \min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}')$ for any $\Gamma \in \mathfrak{B}$, since $\preceq_{\mathcal{K}}^{\circ}{}'' \setminus \mathfrak{D}_{\mathcal{K}}^{\circ} = \preceq_{\mathcal{K}}^{\circ} \setminus \mathfrak{D}_{\mathcal{K}}^{\circ}$. Recall that by Lemma 3.51 we have that $\preceq_{\mathcal{K}}^{\circ}{}'$ is min-complete and reflexive. Consequently, the transitive closure $\preceq_{\mathcal{K}}^{\circ}{}''$ of $\preceq_{\mathcal{K}}^{\circ}{}'$ is a preorder. Moreover, as in the proof of Lemma 3.51, from $\min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}'') = \min(\llbracket\Gamma\rrbracket, \preceq_{\mathcal{K}}^{\circ}{}')$ we obtain that min-completeness carries over from $\preceq_{\mathcal{K}}^{\circ}{}'$ to $\preceq_{\mathcal{K}}^{\circ}{}''$. □

Step III: Linearizing. As last step, we extend $\preceq_{\mathcal{K}}^{\circ}{}''$ to a total relation without losing transitivity. In order to obtain totality, we make use of the following result. Note that this theorem requires the *axiom of choice*.[3]

Theorem 3.59 (Hansson 1968, Lemma 3). *For every preorder $\leq$ on a set X there exists a total preorder $\leq^{\mathrm{lin}}$ on X such that*

- *if $x \leq y$, then $x \leq^{\mathrm{lin}} y$, and*
- *if $x \leq y$ and $y \not\leq x$, then $x \leq^{\mathrm{lin}} y$ and $y \not\leq^{\mathrm{lin}} x$.*

As stated in Lemma 3.58, the relation $\preceq_{\mathcal{K}}^{\circ}{}''$ is a preoder. Thus, we can safely apply Theorem 3.59 to obtain $\preccurlyeq_{\mathcal{K}}^{\circ}$ from $\preceq_{\mathcal{K}}^{\circ}{}''$ through extension, i.e., $\preccurlyeq_{\mathcal{K}}^{\circ} = (TC(\preceq_{\mathcal{K}}^{\circ} \setminus \mathfrak{D}_{\mathcal{K}}^{\circ}))^{\mathrm{lin}}$. The resulting relation $\preccurlyeq_{\mathcal{K}}^{\circ}$ is then a total preorder, while it still coincides with $\preceq_{\mathcal{K}}^{\circ}{}''$ regarding the relevant properties. Combining Theorem 3.59, Lemma 3.51 and Lemma 3.58 we obtain the desired result.

Proposition 3.60. *If $\mathbb{B}$ does not admit a critical loop, then, for any given base change operator $\circ$ for $\mathbb{B}$ satisfying (G1)–(G3), (G5), and (G6), the mapping $\preccurlyeq_{(.)}^{\circ} : \mathcal{K} \mapsto \preccurlyeq_{\mathcal{K}}^{\circ}$ is a min-friendly quasi-faithful preorder assignment compatible with $\circ$.*

Proof. From Lemma 3.58 we obtain that $\preceq_{(.)}^{\circ}{}''$ is a min-complete preorder assigment. Application of Theorem 3.59 yields a total preorder $\preccurlyeq_{\mathcal{K}}^{\circ}$. Observe that linearization by Theorem 3.59 retains strict relations, i.e., if $\mathcal{I}_1 \preceq_{\mathcal{K}}^{\circ}{}'' \mathcal{I}_2$ and $\mathcal{I}_2 \not\preceq_{\mathcal{K}}^{\circ}{}'' \mathcal{I}_1$, then $\mathcal{I}_1 \preccurlyeq_{\mathcal{K}}^{\circ} \mathcal{I}_2$ and

[3] In set theory, *axiom of choice* states that for any collection of (non-empty) sets, one can construct a new set containing an element from each set in the original collection.

$\mathcal{I}_2 \not\preccurlyeq^{\circ}_{\mathcal{K}} \mathcal{I}_1$. Thus, we have $\mathcal{I}_1 \in \min(\llbracket\Gamma\rrbracket, \preceq^{\circ}_{\mathcal{K}}{}'')$ and only if $\mathcal{I}_1 \in \min(\llbracket\Gamma\rrbracket, \preccurlyeq^{\circ}_{\mathcal{K}})$, which yields $\min(\llbracket\Gamma\rrbracket, \preceq^{\circ}_{\mathcal{K}}{}'') = \min(\llbracket\Gamma\rrbracket, \preccurlyeq^{\circ}_{\mathcal{K}})$ for each base $\Gamma \in \mathbb{B}$. Consequently, min-completeness carries over from $\preceq^{\circ}_{\mathcal{K}}{}''$ to $\preccurlyeq^{\circ}_{\mathcal{K}}$. Moreover, by Lemma 3.51 and Lemma 3.58 we obtain $\min(\llbracket\Gamma\rrbracket, \preccurlyeq^{\circ}_{\mathcal{K}}) = \min(\llbracket\Gamma\rrbracket, \preceq^{\circ}_{\mathcal{K}})$ for each base Γ of $\mathbb{B}$. As every $\preccurlyeq^{\circ}_{\mathcal{K}}$ is transitive and total, we obtain that $\preccurlyeq^{\circ}_{\mathcal{K}}$ is min-retractive and thus, $\preccurlyeq^{\circ}_{(.)}$ is a min-friendly assignment. Because $\preceq^{\circ}_{(.)}$ is a quasi-faithful assignment which is compatible with $\circ$ and we have $\min(\llbracket\Gamma\rrbracket, \preccurlyeq^{\circ}_{\mathcal{K}}) = \min(\llbracket\Gamma\rrbracket, \preceq^{\circ}_{\mathcal{K}})$ for each $\mathcal{K} \in \mathfrak{B}$, we also obtain that $\preccurlyeq^{\circ}_{(.)}$ is a quasi-faithful assignment which is compatible with $\circ$. $\square$

This completes the argument regarding the correspondence between the absence of critical loops and total-preorder-representability, by establishing that the former is also sufficient for the latter. Obviously, Theorem 3.45 (II) is a direct consequence of Proposition 3.60.

3.7 Characterization Theorems and Example

Combining the two arguments presented in Section 3.6.1 and Section 3.6.2, we establish that the absence of critical loop coincides with universal total-preorder-representability, i.e., Theorem 3.45 holds.

We will now employ the novel notion of critical loop (cf. Definition 3.43) and our representation theorem for total-preorder-representability (Theorem 3.45) to show that there is no (total) preorder assignment for the operator $\circ_{\mathrm{Ex}}$ from our running example.

Example 3.61 (continuation of Example 3.20). *We will now see that the base logic $\mathbb{B}_{\mathrm{Ex}} = (\mathcal{L}_{\mathrm{Ex}}, \Omega_{\mathrm{Ex}}, \models_{\mathrm{Ex}}, \mathcal{P}(\mathcal{L}_{\mathrm{Ex}}), \cup)$ from Example 3.11 constructed from $\mathbb{L}_{\mathrm{Ex}}$ exhibits a critical loop.*

For this, choose $\Gamma_{i,i\oplus 1} = \{\varphi_i\}$, and $\mathcal{K} = \mathcal{K}_{\mathrm{Ex}} = \{\psi_3\}$ (as in Example 3.11) and $\Gamma_i = \{\psi_i\}$ for $i \in \{0,1,2\}$, where $\oplus$ denotes addition mod 3. *We consider each of the three conditions of Definition 3.43 as a separate case:*

Condition (1). *Observe that $\mathcal{K}_{\mathrm{Ex}}$ is inconsistent with $\Gamma_{0,1}$, $\Gamma_{1,2}$ and $\Gamma_{2,0}$. Thus, Condition (1) is satisfied.*

Condition (2). *For each $i \in \{0,1,2\}$, the models of bases Γ_i and $\Gamma_{i\oplus 1}$ are contained in $\llbracket\Gamma_{i,i\oplus 1}\rrbracket$, but Γ_i is inconsistent with Γ_j with $i \neq j$, e.g. $\llbracket\{\psi_0\}\rrbracket \cup \llbracket\{\psi_1\}\rrbracket \subseteq \llbracket\{\varphi_0\}\rrbracket$ and $\{\psi_0\}$ is not consistent with neither $\{\psi_1\}$ nor $\{\psi_2\}$.*

Condition (3). *The belief base $\Gamma_{\triangledown} = \{\chi'\}$ is the only belief base consistent with Γ_0, Γ_1, and Γ_2. For the satisfaction of Condition (3) observe that $\Gamma'_{\triangledown} = \{\psi_4\}$ fulfils the required condition $\emptyset \neq \llbracket\Gamma'_{\triangledown}\rrbracket \subseteq \llbracket\Gamma_{\triangledown}\rrbracket \setminus (\llbracket\Gamma_{0,1}\rrbracket \cup \llbracket\Gamma_{1,2}\rrbracket \cup \llbracket\Gamma_{2,0}\rrbracket)$.*

In summary $\Gamma_{0,1}$, $\Gamma_{1,2}$, and $\Gamma_{2,0}$ form a critical loop for $\mathbb{B}_{\mathrm{Ex}}$ (see Figure 3.62). As given by Theorem 3.45 (I), every min-complete faithful preorder assignment compatible with $\circ_{\mathrm{Ex}}$ is not

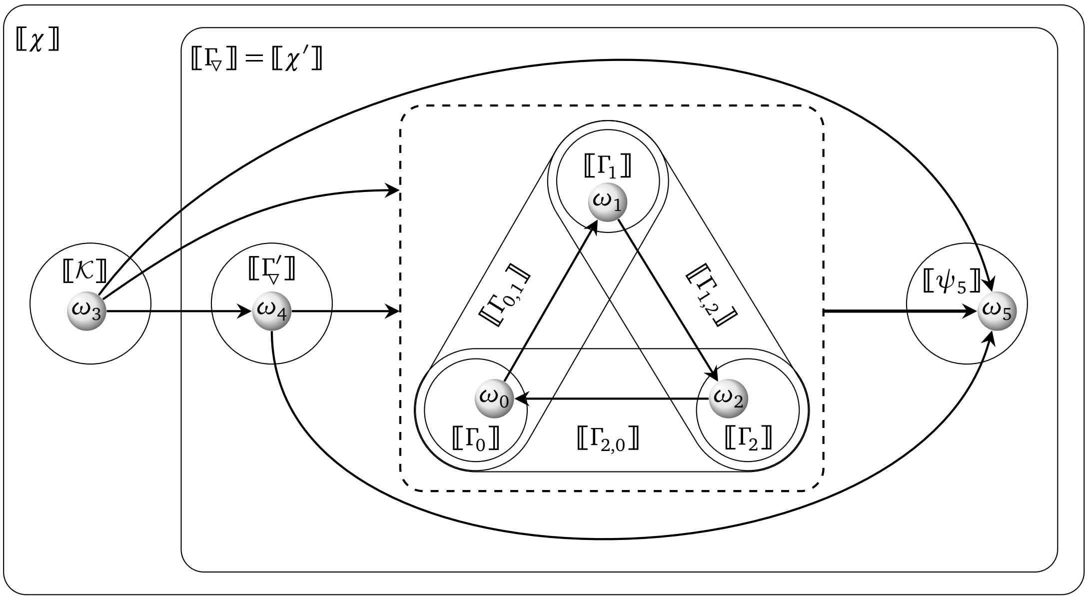


Figure 3.62: Critical loop situation in $\mathbb{B}_{\mathrm{Ex}}$ presented in Example 3.61. The solid borders represent the sets of models and each arrow depicts the relation $\prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}}$ between models.

transitive. To illustrate this, we use here the assignment $\preceq^{\circ_{\mathrm{Ex}}}_{(.)}$ defined in Example 3.20 and sketched in Figure 3.21 for $\mathcal{K}_{\mathrm{Ex}}$ (see also Figure 3.62). Consider the revisions $\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_{0,1}$, $\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_{1,2}$, and $\mathcal{K}_{\mathrm{Ex}} \circ_{\mathrm{Ex}} \Gamma_{2,0}$. From the construction of $\circ_{\mathrm{Ex}}$ given in Definition 3.46 and compatibility of $\preceq^{\circ_{\mathrm{Ex}}}_{(.)}$ with $\circ_{\mathrm{Ex}}$, we have

$$\mathcal{I}_0 \in \min(\llbracket \Gamma_{0,1} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}}),\ \textit{but}\ \mathcal{I}_1 \notin \min(\llbracket \Gamma_{0,1} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}})\ ,$$
$$\mathcal{I}_1 \in \min(\llbracket \Gamma_{1,2} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}}),\ \textit{but}\ \mathcal{I}_2 \notin \min(\llbracket \Gamma_{1,2} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}})\ ,\ \textit{and}$$
$$\mathcal{I}_2 \in \min(\llbracket \Gamma_{2,0} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}}),\ \textit{but}\ \mathcal{I}_0 \notin \min(\llbracket \Gamma_{2,0} \rrbracket, \preceq^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}})\ ,$$

showing $\mathcal{I}_0 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_1 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_2 \prec^{\circ_{\mathrm{Ex}}}_{\mathcal{K}_{\mathrm{Ex}}} \mathcal{I}_0$, which is impossible for a transitive relation.

Moreover, observe that the construction of $\circ_{\mathrm{Ex}}$ presented in Example 3.11 illustrates the construction given by Definition 3.46 and used in the proof of Proposition 3.47. In particular, for the example presented here one would obtain $\mathcal{C}' = \{\Gamma'_{\triangledown}\} = \{\{\psi_4\}\}$ when following the outline of the construction.

Having established the necessary and sufficient criterion for total-preorder-representability, we can now provide two more versions of the two-way representation theorem. The first representation theorem is one where the base change operator satisfies (G4), thus abstracting from the syntactic form of the belief bases. Note that transitivity implies min-retractivity, and thus a transitive min-complete relation is automatically min-friendly.

Theorem 3.63. *Let $\mathbb{B}$ be a base logic which does not admit a critical loop. Then the following hold:*

- *Every base change operator for $\mathbb{B}$ satisfying (G1)–(G6) is compatible with some min-expressible min-complete faithful preorder assignment.*
- *Every min-expressible min-complete faithful preorder assignment for $\mathbb{B}$ is compatible with some base change operator satisfying (G1)–(G6).*

Proof. The first statement is a consequence of statement (II) of Theorem 3.45 together with Lemma 3.42. The second statement is an immediate consequence of the second statement of Theorem 3.31. □

The second representation theorem is for base change operators which do not necessarily satisfy (G4), and thus might be sensitive to the syntax of the prior belief base.

Theorem 3.64. *Let $\mathbb{B}$ be a base logic which does not admit a critical loop. Then the following hold:*

- *Every base change operator for $\mathbb{B}$ satisfying (G1)–(G3), (G5), and (G6) is compatible with some min-expressible min-complete quasi-faithful preorder assignment.*
- *Every min-expressible min-complete quasi-faithful preorder assignment for $\mathbb{B}$ is compatible with some base change operator satisfying (G1)–(G3), (G5), and (G6).*

Proof. The first statement is a consequence of statement (II) of Theorem 3.45. The second statement is an immediate consequence of the second statement of Theorem 3.40. □

We close this section with an implication of Theorem 3.45. A base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ is called *disjunctive*, if for every two bases $\Gamma_1, \Gamma_2 \in \mathfrak{B}$ there is a base $\Gamma_1 \varovee \Gamma_2 \in \mathfrak{B}$ such that $[\![\Gamma_1 \varovee \Gamma_2]\!] = [\![\Gamma_1]\!] \cup [\![\Gamma_2]\!]$. This includes the case of any (base) logic allowing disjunction to be expressed on the sentence level, i.e., when for every $\gamma, \delta \in \mathcal{L}$ there exists some $\gamma \varovee \delta \in \mathcal{L}$ with $[\![\gamma \varovee \delta]\!] = [\![\gamma]\!] \cup [\![\delta]\!]$, such that $\Gamma_1 \varovee \Gamma_2$ can be obtained as $\{\gamma \varovee \delta \mid \gamma \in \Gamma_1, \delta \in \Gamma_2\}$.

Corollary 3.65. *In a disjunctive base logic, every belief change operator satisfying (G1)–(G6) is total-preorder-representable.*

Proof. A disjunctive base logic never exhibits a critical loop; Condition (3) would be violated by picking $\Gamma = ((\Gamma_0 \varovee \Gamma_1) \ldots) \varovee \Gamma_n$. □

As a consequence, for a vast amount of well-known logics, including all classical logics such as first-order and second order predicate logic, one directly obtains total-preorder-representablility of every AGM base change operator by Corollary 3.65.

3.8 Discussion on The General Approach

In this chapter, we discuss some more specific aspects and noteworthy implications of our approach. First, we will discuss the notion of base logics and demonstrate that the decision how to define bases affects the applicability of certain notions. Next, we explore the novel notion of min-retractivity (in comparison to transitivity) and discuss its relationship to decomposability of disjunctions. We give additional insights into our way of encoding operators as assignments in the next section. In the last section, we compare the existing related works with our approach.

3.8.1 On the Notion of Min-Retractivity

As a primary ingredient to our results, the novel notion of min-retractivity has been introduced and motivated in Section 3.2.2 and proven to serve its purpose later on. As noted earlier, it is immediate that any preorder over Ω is min-retractive, irrespective of the choice of the other components of the underlying base logic. On the other hand, we have exposed examples of min-retractive relations that are not transitive for certain base logics. This raises the question if there are conditions, under which the two notions *do* coincide, at least when presuming min-completeness (a condition already known and generally accepted). We start by formally defining this notion of coincidence.

Definition 3.66. *A base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ is called* preorder-enforcing*, if every binary relation over Ω that is total and min-friendly for $\mathbb{B}$ is also transitive (and hence a total preorder).*

As an aside, we note that being preorder-enforcing implies the absence of critical loops.

Proposition 3.67. *Every preorder-enforcing base logic does not have critical loops.*

Proof. Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a preorder-enforcing base logic. By Theorem 3.45, absence of critical loops would follow from the fact that every base change operator for $\mathbb{B}$ satisfying (G1)–(G6) is total-preorder-representable. To show the latter, consider an arbitrary base change operator $\circ$ of that kind. By Proposition 3.27, $\preceq^{\circ}_{(.)}$ is a min-friendly faithful assignment compatible with $\circ$. In particular, for every $\mathcal{K} \in \mathfrak{B}$, the corresponding $\preceq^{\circ}_{\mathcal{K}}$ is total and min-friendly for $\mathbb{B}$. Yet then, by assumption, any such $\preceq^{\circ}_{\mathcal{K}}$ is also a total preorder, and therefore $\preceq^{\circ}_{(.)}$ is even a preorder assignment. Hence, $\circ$ is total-preorder-representable. □

The question remains, which base logics are actually preorder-enforcing. We will next present a simple criterion and then show that it is indeed necessary and sufficient for being preorder-enforcing.

Definition 3.68. *A base logic $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ is called* trio-expressible*, if for any three interpretations $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3 \in \Omega$ there is a base $\Gamma_{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}$ satisfying $[\![\Gamma_{\mathcal{I}_1\mathcal{I}_2\mathcal{I}_3}]\!] = \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$.*

Theorem 3.69. *A base logic is preorder-enforcing if and only if it is trio-expressible.*

Proof. We show the "if" direction followed by the "only if" one.

"$\Leftarrow$" Let $\preceq$ be a min-friendly total relation over Ω. Toward a contradiction, assume $\preceq$ is not transitive, i.e., there exist interpretations $\mathcal{I},\mathcal{I}',\mathcal{I}'' \in \Omega$ such that $\mathcal{I} \preceq \mathcal{I}'$ and $\mathcal{I}' \preceq \mathcal{I}''$ but $\mathcal{I} \not\preceq \mathcal{I}''$. By totality, the latter implies $\mathcal{I}'' \prec \mathcal{I}$. Now consider $\Gamma_{\mathcal{I}\mathcal{I}'\mathcal{I}''}$ and note that $\min([\![\Gamma_{\mathcal{I}\mathcal{I}'\mathcal{I}''}]\!],\preceq) \neq \emptyset$ thanks to min-completeness, but then, by min-retractivity, $\min([\![\Gamma_{\mathcal{I}\mathcal{I}'\mathcal{I}''}]\!],\preceq) = \{\mathcal{I},\mathcal{I}',\mathcal{I}''\}$ follows. However, $\mathcal{I} \in \min([\![\Gamma_{\mathcal{I}\mathcal{I}'\mathcal{I}''}]\!],\preceq)$ contradicts $\mathcal{I} \not\preceq \mathcal{I}''$.

"$\Rightarrow$" We actually show the contraposition: starting from a base logic $\mathbb{B} = (\mathcal{L},\Omega,\models,\mathfrak{B},\Cup)$ that is not trio-expressible, we show violation of being preorder-enforcing by exhibiting a total, min-friendly relation over Ω that is not transitive. From non-trio-expressibility, the existence of $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3 \in \Omega$ with $[\![\Gamma]\!] \neq \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ for every $\Gamma \in \mathfrak{B}$ follows. Let now $\preceq^-$ be an arbitrary well-order[4] over $\Omega^- = \Omega \setminus \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$, i.e., it is total, transitive (hence min-retractive) and min-complete, therefore also min-friendly. We now define

$$\preceq \;=\; \preceq^- \cup (\Omega^- \times \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}) \cup \{(\mathcal{I}_1,\mathcal{I}_2),(\mathcal{I}_1,\mathcal{I}_3),(\mathcal{I}_3,\mathcal{I}_1),(\mathcal{I}_2,\mathcal{I}_3)\}.$$

It is easy to see that $\preceq$ is still a total relation. It is min-complete (for $\mathbb{B}$) by case distinction: on one hand, if $[\![\Gamma]\!] \not\subseteq \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$, then $\min([\![\Gamma]\!],\preceq) \neq \emptyset$ follows from min-completeness of $\preceq^-$, on the other hand, for any two- or one-element subset of $\{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ also a minimum clearly exists (note that by assumption $[\![\Gamma]\!] = \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ cannot occur). We proceed to show min-retractivity of $\preceq$, again by case-distinction: If $[\![\Gamma]\!] \not\subseteq \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ then, due to min-completeness and antisymmetry of $\preceq^-$, the set $\min([\![\Gamma]\!],\preceq)$ contains exactly one element and is strictly smaller than any other element from $[\![\Gamma]\!]$, thus min-retractivity is vacuously satisfied. If $[\![\Gamma]\!] \subset \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3\}$ min-retractivity is easily verified case by case. We finish our argument by showing that $\preceq$ is not transitive: we have $\mathcal{I}_2 \preceq \mathcal{I}_3$ as well as $\mathcal{I}_3 \preceq \mathcal{I}_1$, but $\mathcal{I}_2 \preceq \mathcal{I}_1$ fails to hold. □

The preceding theorem provides yet another argument why preorders can be used as preference relations for finite-signature propositional logic (as in fact, they are the **only** preference relations arising in that setting). However, note that the result also applies to more complex logics such as first-order logic under the finite model semantics.[5]

[4]As discussed earlier, existence of such a $\preceq^-$ is assured by the well-ordering theorem, depending on the axiom of choice.

[5]Strictly speaking, this requires a slightly non-standard (but semantically equivalent) model theory which abstracts from the domains used and considers isomorphic models as equal.

3.8.2 On the Encoding of Operators

We will now discuss how revision operators are encoded into a preference relation. Recall that for K&M's encoding, presented in Equation (3.1), the existence of a sentence $\mathit{form}(\mathcal{I}_1,\mathcal{I}_2)$ satisfying $[\![\mathit{form}(\mathcal{I}_1,\mathcal{I}_2)]\!] = \{\mathcal{I}_1,\mathcal{I}_2\}$ is required for any interpretations $\mathcal{I}_1,\mathcal{I}_2$ in the considered logic. The problem in a general Tarskian logical setting is that there might not be such a sentence or base.

Therefore, generalizing this idea to our case (using just the bases that do exist) bears the danger that the relation between certain pairs of elements is left undetermined: depending on the shape of the logic (and its model theory) as well as the operator, there might be no preference between certain elements (because there is no revision which provides information on the preference). We called these pairs of interpretations *detached* (cf. Definition 3.48). In particular, when one wants to obtain a total relation, these elements have to be ordered in a certain way, and the appropriate selection of a *"preference"* between these two interpretations is a *"non-local"* choice (as it may have ramifications for other "ordering choices"). As a solution, we came up with Definition 3.19, providing an encoding $\preceq^{\circ}_{(.)}$ different from the approach by Katsuno and Mendelzon. This definition solves the problem with the detached pairs of interpretations by treating them as equally preferable.

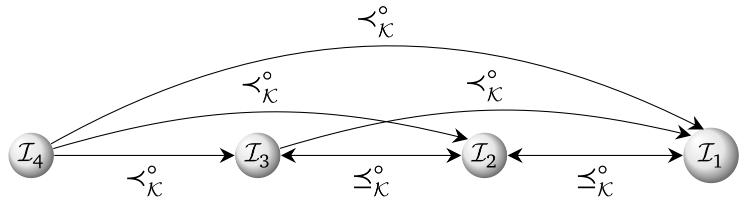
This uniform treatment of all detached pairs may produce a non-preorder assignment even in cases where an encoding by means of a preorder assignment were actually possible as demonstrated next.

Example 3.70. *Let* $\mathbb{B} = (\mathcal{L},\Omega,\models,\mathfrak{B},\Cup)$ *with* $\mathcal{L} = \{\bot,\varphi,\psi,\gamma_1,\ldots,\gamma_4\}$ *and* $\Omega = \{\mathcal{I}_1,\ldots,\mathcal{I}_4\}$*, such that:*

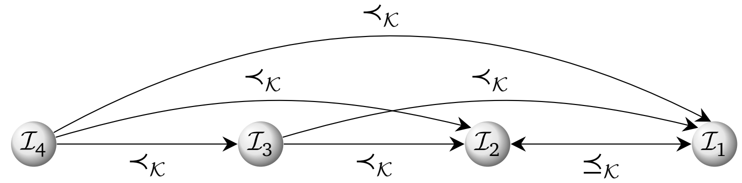
$$[\![\bot]\!] = \{\} \qquad [\![\varphi]\!] = \{\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_4\} \qquad [\![\psi]\!] = \{\mathcal{I}_1,\mathcal{I}_3\} \qquad [\![\gamma_i]\!] = \{\mathcal{I}_i\}$$

Moreover let $\mathfrak{B} = \{\{\chi\} \mid \chi \in \mathcal{L}\}$ *and let* $\Cup$ *be the idempotent, commutative binary function over* $\mathfrak{B}$ *satisfying* $\{\varphi\}\Cup\{\psi\} = \{\varphi\}\Cup\{\gamma_1\} = \{\psi\}\Cup\{\gamma_1\} = \{\gamma_1\}$ *and producing* $\{\bot\}$ *in all other cases. Let* $\circ$ *be as defined in the following operator table.*

$\circ$	$\{\bot\}$	$\{\varphi\}$	$\{\psi\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\bot\}$	$\{\bot\}$	$\{\varphi\}$	$\{\psi\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\varphi\}$	$\{\bot\}$	$\{\varphi\}$	$\{\gamma_1\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\psi\}$	$\{\bot\}$	$\{\gamma_1\}$	$\{\psi\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\gamma_1\}$	$\{\bot\}$	$\{\gamma_1\}$	$\{\gamma_1\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\gamma_2\}$	$\{\bot\}$	$\{\gamma_2\}$	$\{\psi\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\gamma_3\}$	$\{\bot\}$	$\{\varphi\}$	$\{\gamma_3\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$
$\{\gamma_4\}$	$\{\bot\}$	$\{\gamma_4\}$	$\{\gamma_3\}$	$\{\gamma_1\}$	$\{\gamma_2\}$	$\{\gamma_3\}$	$\{\gamma_4\}$



(a) Relation $\preceq^{\circ}_{\mathcal{K}}$ for $\mathcal{K} = \{\gamma_4\}$. No preorder as $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ and $\mathcal{I}_2 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_3$, yet $\mathcal{I}_1 \npreceq^{\circ}_{\mathcal{K}} \mathcal{I}_3$.



(b) Appropriate preorder encoding $\preceq_{\mathcal{K}}$ of preference relation with respect to $\mathcal{K} = \{\gamma_4\}$ for $\circ$.

Figure 3.71: Illustrations of the relations used in Example 3.70.

In particular, for $\mathcal{K} = \{\gamma_4\}$, we thus obtain

$$\begin{aligned} \mathcal{K} \circ \{\varphi\} &= \{\gamma_4\} & \qquad [\![\mathcal{K} \circ \{\varphi\}]\!] &= \{\mathcal{I}_4\} \\ \mathcal{K} \circ \{\psi\} &= \{\gamma_3\} & \qquad [\![\mathcal{K} \circ \{\psi\}]\!] &= \{\mathcal{I}_3\} \\ \mathcal{K} \circ \{\gamma_i\} &= \{\gamma_i\} & \qquad [\![\mathcal{K} \circ \{\gamma_i\}]\!] &= \{\mathcal{I}_i\} \end{aligned}$$

Figure 3.71a shows that the assignment $\preceq^{\circ}_{(.)}$ derived from $\circ$ is not a preorder assignment, while Figure 3.71b demonstrates that such an assignment for $\circ$ indeed exists.

Still, while failing to yield preorder assignments whenever possible, Definition 3.19 is unique in another respect: $\preceq^{\circ}_{(.)}$ turns out to be the (set-inclusion-)maximal canonical representation for the preferences of an operator – a property the encoding approaches given by Equation (3.1) do not have.

Proposition 3.72. *Let $\circ$ be a base change operator satisfying (G1)-(G6). If $\preceq_{(.)}$ is a min-friendly faithful assignment compatible with $\circ$, then $\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2$ implies $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ for every $\mathcal{I}_1, \mathcal{I}_2 \in \Omega$ and every belief base $\mathcal{K} \in \mathfrak{B}$.*

Proof. Toward a contradiction, assume there were $\mathcal{I}_1, \mathcal{I}_2 \in \Omega$ such that $\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2$ but $\mathcal{I}_1 \npreceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ (hence, by totality $\mathcal{I}_2 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_1$).

Let us first consider the case $\mathcal{I}_2 \models \mathcal{K}$. Then $\mathcal{I}_2 \prec^{\circ}_{\mathcal{K}} \mathcal{I}_1$ and faithfulness of $\prec^{\circ}_{(.)}$ imply $\mathcal{I}_1 \not\models \mathcal{K}$. But this contradicts $\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2$, as $\preceq_{(.)}$ is also faithful by assumption.

It remains to consider the case $\mathcal{I}_2 \not\models \mathcal{K}$. Then, by Lemma 3.23(a), there is a belief base Γ with $\mathcal{I}_1, \mathcal{I}_2 \models \Gamma$ such that $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma$ and $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma$. Therefore, by compatibility, $\mathcal{I}_2 \in \min([\![\Gamma]\!], \preceq_{\mathcal{K}}) = [\![\mathcal{K} \circ \Gamma]\!]$ and $\mathcal{I}_1 \notin \min([\![\Gamma]\!], \preceq_{\mathcal{K}}) = [\![\mathcal{K} \circ \Gamma]\!]$, a contradiction to $\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2$ due to min-retractivity. □

As a last discussion item in this section, we would like to point out that the more smoothly and economically defined relation $\sqsubseteq^{\circ}_{(.)}$ (Definition 3.18) is very close to already serving the purpose of the somewhat more "tinkered" $\preceq^{\circ}_{(.)}$ (Definition 3.19). In fact, the very natural assumption of the existence of an "non-constraining" base that covers all interpretations makes the two relation encodings coincide. In most logics, such a base is trivially available (for instance, the empty base).

Proposition 3.73. *Let $\mathbb{B} = (\mathcal{L}, \Omega, \models, \mathfrak{B}, \Cup)$ be a base logic and $\circ$ be a base change operator for $\mathbb{B}$ satisfying (G1)–(G6). If there exists a base $\Gamma_\Omega \in \mathfrak{B}$ such that $[\![\Gamma_\Omega]\!] = \Omega$, then $\sqsubseteq^{\circ}_{\mathcal{K}} = \preceq^{\circ}_{\mathcal{K}}$ for any $\mathcal{K} \in \mathfrak{B}$, i.e. $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ if and only if $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ for any $\mathcal{I}_1, \mathcal{I}_2 \in \Omega$.*

Proof. Let $\mathcal{I}_1, \mathcal{I}_2$ be two interpretations and assume there exists a base $\Gamma_\Omega \in \mathfrak{B}$ such that $[\![\Gamma_\Omega]\!] = \Omega$. Then, for any $\mathcal{K} \in \mathfrak{B}$, we have $[\![\mathcal{K} \Cup \Gamma_\Omega]\!] = [\![\mathcal{K}]\!] \cap \Omega = [\![\mathcal{K}]\!]$. We show the equivalence of $\sqsubseteq^{\circ}_{\mathcal{K}}$ and $\preceq^{\circ}_{\mathcal{K}}$ in two directions:

"$\Rightarrow$" Let $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. Assume for a contradiction that $\mathcal{I}_1 \not\preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. From Definition 3.19, we have $\mathcal{I}_1 \not\models \mathcal{K}$ and three cases: $\mathcal{I}_1 \models \mathcal{K}$, $\mathcal{I}_2 \models \mathcal{K}$ or $\mathcal{I}_1 \not\sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. The case $\mathcal{I}_1 \models \mathcal{K}$ immediately contradicts $\mathcal{I}_1 \not\models \mathcal{K}$ and the third case $\mathcal{I}_1 \not\sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$ contradicts our assumption $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. For the remaining case $\mathcal{I}_2 \models \mathcal{K}$, since $\mathcal{I}_2 \in [\![\mathcal{K}]\!] = [\![\mathcal{K} \Cup \Gamma_\Omega]\!]$, from postulate (G2) we obtain $[\![\mathcal{K} \circ \Gamma_\Omega]\!] = [\![\mathcal{K} \Cup \Gamma_\Omega]\!]$. From Definition 3.18, we have two subcases:

- $\mathcal{I}_1 \models \mathcal{K} \circ \Gamma_\Omega$. Since $[\![\mathcal{K} \circ \Gamma_\Omega]\!] = [\![\mathcal{K} \Cup \Gamma_\Omega]\!]$, we have $\mathcal{I}_1 \in [\![\mathcal{K} \Cup \Gamma_\Omega]\!] = [\![\mathcal{K}]\!]$, which contradicts $\mathcal{I}_1 \not\models \mathcal{K}$.
- $\mathcal{I}_2 \not\models \mathcal{K} \circ \Gamma_\Omega$. Since $[\![\mathcal{K} \circ \Gamma_\Omega]\!] = [\![\mathcal{K} \Cup \Gamma_\Omega]\!]$, we have $\mathcal{I}_2 \notin [\![\mathcal{K} \Cup \Gamma_\Omega]\!]$, and hence $\mathcal{I}_2 \notin [\![\mathcal{K}]\!]$, which contradicts our case assumption $\mathcal{I}_2 \models \mathcal{K}$. □

"$\Leftarrow$" Let $\mathcal{I}_1 \preceq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. In view of Definition 3.19, we consider two cases: $\mathcal{I}_1 \models \mathcal{K}$ or ($\mathcal{I}_1, \mathcal{I}_2 \not\models \mathcal{K}$ and $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$). The second case immediately yields the desired $\mathcal{I}_1 \sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. For the former case, $\mathcal{I}_1 \models \mathcal{K}$, assume for a contradiction $\mathcal{I}_1 \not\sqsubseteq^{\circ}_{\mathcal{K}} \mathcal{I}_2$. Then, there exists $\Gamma \in \mathfrak{B}$ with $\mathcal{I}_1, \mathcal{I}_2 \in [\![\Gamma]\!]$ such that $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma$ and $\mathcal{I}_2 \models \mathcal{K} \circ \Gamma$. Since $\mathcal{I}_1 \in [\![\mathcal{K}]\!] \cap [\![\Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$, from postulate (G2) we have $[\![\mathcal{K} \circ \Gamma]\!] = [\![\mathcal{K} \Cup \Gamma]\!]$. This implies $\mathcal{I}_1 \in [\![\mathcal{K} \circ \Gamma]\!]$, which contradicts $\mathcal{I}_1 \not\models \mathcal{K} \circ \Gamma$.

3.9 Related Work

In settings beyond propositional logic, we are aware of three closely related approaches that propose model-based frameworks for revision of belief bases (or sets) without fixing a particular logic or the internal structure of interpretations, and characterize revision operators via minimal models à la K&M with some additional assumptions. In the following, we discuss these results and their relationship to our approach. Table 3.74 summarizes the three approaches and compares them with K&M's result and our approach, which comes in four variants.

Grove [Gro88]. One semantics-based approach related to the one of K&M was proposed by Grove 1988 in the setting of Boolean-closed logics. Instead of a preorder relation $\preceq_{\mathcal{K}}$, he originally characterized AGM revision operators via *systems of spheres*, collections $\mathbf{S}$ of sets of interpretations satisfying certain conditions. The notion of a system of spheres is closely related to that of a faithful preorder relation $\preceq_{\mathcal{K}}$ as the latter can be generated from the former [GR95]. Given a faithful preorder $\preceq_{\mathcal{K}}$, for each $\mathcal{I} \in \Omega$, one can define $S_{\mathcal{I}}$ as a set of interpretations $\mathcal{I}'$ such that $\mathcal{I}' \preceq_{\mathcal{K}} \mathcal{I}$. However, the set $\mathbf{S}$ of all such sets might not satisfy min-completeness in general (Grove [Gro88] denotes min-completeness with *(S4)* – one of the four conditions that must be satisfied for a sphere system). Delgrande and colleagues [DPW18] then reformulated Grove's representation theorem stating that (expressed in our terminology) any AGM revision operator can be obtained from a compatible min-complete faithful preorder assignment, provided the set of interpretations is Ω-expressible, i.e. for any subset $\Omega' \subseteq \Omega$ there exists a base Γ such that $[\![\Gamma]\!] = \Omega'$. In this formulation, Groves result also holds for logics with infinite Ω.

Grove's result constitutes a special case of our representation theorem: from the assumption of Boolean-closedness, it follows that the considered logics are disjunctive and therefore free of critical loops (cf. Theorem 3.45 and Corollary 3.65). The assumption of Ω-expressibility immediately implies min-expressibility for all relations. In the light of these observations, Grove's result turns out to be a special case of the third variant of our result in Table 3.74.

Delgrande, Peppas, and Woltran [DPW18]. The representation result of Delgrande et al. [DPW18] confines the considered logics to those where the set Ω of interpretations (or possible worlds) is finite[6] and where any two different interpretations $\omega, \omega' \in \Omega$ can be distinguished by some sentence $\varphi \in \mathcal{L}$, i.e., $\omega \in [\![\varphi]\!]$ and $\omega' \notin [\![\varphi]\!]$. Moreover, they extend the AGM postulates by the following extra one, denoted (Acyc):

> For any base $\mathcal{K}$ and all $\Gamma_1, \ldots, \Gamma_n \in \mathcal{P}(\mathcal{L})$ with $[\![\Gamma_i \cup \mathcal{K} \circ \Gamma_{i+1}]\!] \neq \emptyset$ for each $1 \leq i < n$ as well as $[\![\Gamma_n \cup \mathcal{K} \circ \Gamma_1]\!] \neq \emptyset$ holds $[\![\Gamma_1 \cup \mathcal{K} \circ \Gamma_n]\!] \neq \emptyset$.

With these ingredients in place, Delgrande and colleagues [DPW18] establish that, for the logics they consider, there is a two-way correspondence between those AGM revision operators satisfying (Acyc) and min-expressible faithful preorder assignments. Instead of the term "min-expressible", they use the term *regular*.

[6]Note that this precondition excludes more complex logics such as first-order or modal logics and most of their fragments, but also propositional logic with infinite signature. On the positive side, this choice guarantees min-completeness of any preorder.

logic setting	belief bases	assignment	postulates	relation encoding	notes
Katsuno and Mendelzon [KM91]					
propositional logic over finite signature	$\mathcal{P}(\mathcal{L})$, $\mathcal{P}_{\text{fin}}(\mathcal{L})$, $\mathcal{L}$	preorder, faithful	`(G1)-(G6)`	Eq. (3.1)	logic natively free of critical loops; natively min-friendly; two-way representation theorem
Grove [Gro88], reformulated by Delgrande, Peppas, and Woltran [DPW18]					
Boolean-closed logics that are Ω-expressible	$\mathcal{P}(\mathcal{L})$	preorder, faithful, min-complete	`(G1)-(G6)`	Eq. (3.1)	all such logics natively free of critical loops; any assignment min-expressible; two-way representation theorem
Delgrande, Peppas, and Woltran [DPW18]					
Tarskian logics with finite Ω, any ω, ω' distinguishable by some sentence	$\mathcal{P}(\mathcal{L})$	preorder, faithful, min-expressible	`(G1)-(G6)`, `(Acyc)`	Eq. (3.2)	extra postulate (Acyc) rules out "non-preorder operators"; two-way representation theorem
Aiguier, Atif, Bloch, Hudelot [AAB+18]					
Tarskian logics	$\mathcal{P}(\mathcal{L})$	quasi-faithful, min-complete	`(G1)-(G3)`, `(G5)`,`(G6)`	Eq. (3.3)	non-standard notion of inconsistency; additional adhoc constraint on compatibility; one-way representation theorem
our approach					
Tarskian logics	arbitrary, closed under abstract union	faithful, min-complete, min-retractive, min-expressible	`(G1)-(G6)`	Def. 3.19	most general (syntax-independent version); two-way representation theorem (Theorem 3.31)
Tarskian logics		quasi-faithful, min-complete, min-retractive, min-expressible	`(G1)-(G3)`, `(G5)`,`(G6)`	Def. 3.19	most general (syntax-dependent version); two-way representation theorem (Theorem 3.40)
Tarskian logics without critical loop (e.g., with disjunction)		preorder, faithful, min-complete, min-expressible	`(G1)-(G6)`	Def. 3.19	preorder preference (syntax-independent version); two-way representation theorem (Theorem 3.63)
Tarskian logics without critical loop (e.g., with disjunction)		preorder, quasi-faithful, min-complete, min-expressible	`(G1)-(G3)`, `(G5)`,`(G6)`	Def. 3.19	preorder preference (syntax-dependent version); two-way representation theorem (Theorem 3.64)

Table 3.74: Overview of our characterization results and comparison with related work.

The approach of [DPW18] can be seen as complementary to ours. As we saw before, in logics exhibiting critical loops, one cannot hope for a characterization of all AGM revision operators by means of assignments producing preorders. Our proposal is to relinquish the requirement of using preorders, giving up transitivity and merely retaining min-retractivity. As an alternative to this approach, one might argue that those AGM revision operators **not** corresponding to some preorder assignment are somewhat "unnatural" and should be ruled out from the consideration. The additional postulate (Acyc) serves precisely this purpose: it allows for a preorder characterization even in logics with critical loops, by disallowing some "unnatural" AGM revision operators.

Aiguier, Atif, Bloch, and Hudelot [AAB+18]. The approach of Aiguier, Atif, Bloch, and Hudelot [AAB+18] considers AGM belief base revision in logics with a possibly infinite set Ω of interpretations. Notably, they propose to consider certain bases that actually *do* have models as inconsistent (and thus in need of revision). While, in our view, this is at odds with the foundational assumptions of belief revision (revision should be union/conjunction of the inputs unless facing unsatisfiability), this appears to be a design choice immaterial to the established results. To avoid confusion, we will ignore it in our presentation. As far as the postulates are concerned, Aiguier et. al. [AAB+18] decide to rule out (KM4)/(G4), arguing in favor of syntax-dependence. Consequently, they re-define the notion of faithfulness of assignments, eliminating (F3), and arriving at the notion that we call quasi-faithfulness. Like us, Aiguier and colleagues [AAB+18] propose to drop the requirement that assignments have to yield preorders. Also, their representation result (Theorem 1) features a condition that corresponds to min-completeness (second bullet point). In addition to the standard notion of compatibiliy, their result hinges on the following additional correspondence between the assignment and the preorder (third bullet point), for every $\mathcal{K}, \Gamma_1, \Gamma_2 \in \mathcal{P}(\mathcal{L})$:

$$\text{If } (\mathcal{K} \circ \Gamma_1) \cup \Gamma_2 \text{ is consistent, then } \min(\llbracket \Gamma_1 \rrbracket, \preceq_{\mathcal{K}}) \cap \llbracket \Gamma_2 \rrbracket = \min(\llbracket \Gamma_1 \cup \Gamma_2 \rrbracket, \preceq_{\mathcal{K}}).$$

A closer inspection of this extra condition shows that it is essentially a translation of a combination of the postulates (G5) and (G6) into the language of assignments and minima. It remains somewhat unclear to us what the intuitive justification of this (arguably rather technical and unwieldy) extra condition should be, beyond providing the missing ingredient to make the result work. Possibly, this is the reason why the presented result is just one-way: it does not provide a characterization of exactly those assignments for which a compatible AGM revision operator exists. Rather it pre-assumes existence of a revision operator under consideration.

We think that our approach provides improvements regarding ways to construct an appropriate assignment from a given belief revision operator. For comparison, Delgrande et. al.

[DPW18] solve this problem by simultaneously revising with all sentences satisfying $\mathcal{I}_1$ and $\mathcal{I}_2$, in order to "simulate" the revision by the desired formula *form*$(\mathcal{I}_1, \mathcal{I}_2)$: for every base $\mathcal{K}$,

$$\preceq_{\mathcal{K}} \text{ is the transitive closure of } \{(\mathcal{I}_1, \mathcal{I}_2) \mid \mathcal{I}_1 \models \mathcal{K} \circ (t(\{\mathcal{I}_1\}) \cap t(\{\mathcal{I}_2\}))\} \tag{3.2}$$

where $t(\{\omega\})$ is the set of all sentences satisfied by ω. Aiguier, Atif, Bloch, and Hudelot 2018 use a similar approach by revising with all sentences at once: they let, for every base $\mathcal{K}$,

$$\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2 \text{ if } \mathcal{I}_1 \models \mathcal{K} \text{ or } \mathcal{I}_1 \models \mathcal{K} \circ \{\mathcal{I}_1, \mathcal{I}_2\}^* \tag{3.3}$$

where $\{\omega_1, \omega_2\}^*$ is the set of all sentences satisfied by both ω_1 and ω_2. In summary, neither Aiguier et al. nor Delgrande et al. use an encoding approach in the spirit of Katsuno and Mendelzon, attempting to establish a relation between two interpretations, whenever the revision operator provides evidence. As discussed in Section 3.8.2, we take a somewhat dual approach: we establish a relation between any two worlds, unless the considered revision operator provides evidence to the contrary.

3.10 Summary

The central objective of our treatise was to provide an exact model-theoretic characterization of AGM belief revision in the most general reasonable sense, i.e., one that uniformly applies

- to every logic with a classical model theory (i.e., every Tarskian logic),
- to any notion of bases that allows for taking some kind of "unions" (including the cases of belief sets, sets of sentences, finite sets of sentences, and single sentences), and
- to all base change operators adhering to the unaltered AGM postulates (without imposing further restrictions through additional postulates).

To this end, we followed the well-established approach of using *assignments*: functions that map every base $\mathcal{K}$ to a binary relation $\preceq_{\mathcal{K}}$ over the interpretations Ω of the considered logic, where $\mathcal{I}_1 \preceq_{\mathcal{K}} \mathcal{I}_2$ intuitively expresses a preference, i.e., that $\mathcal{I}_1$ is "closer" than (or at least as close as) $\mathcal{I}_2$ to being a model of $\mathcal{K}$ (even if it is not a proper model). With this notion in place, the *compatibility* between a revision operator and an assignment then requires that the result of revising a base $\mathcal{K}$ by some base Γ yields a base whose models are the $\preceq_{\mathcal{K}}$-minimal interpretations among the models of Γ.

The original result of K&M for signature-finite propositional logic established a two-way-correspondence between AGM revision operators and faithful assignments that yield total preorders. We showed that in the general case considered by us, this original result fails in many ways and needs substantial adaptations. In particular, aside from delivering total relations and being faithful, the assignment now needs to satisfy

- *min-expressibility*, guaranteeing existence of a describing base for any model set obtained by taking minimal interpretations among some base's models,
- *min-completeness*, ensuring that minimal interpretations exist in every base's model set, and
- *min-retractivity* instead of transitivity, making sure that minimality is inherited to more preferable elements.

While the first two adjustments have been recognized and described in prior work, the notion of min-retractivity (and the decision to replace transitivity by this weaker notion and thus give up on the requirement that preferences be preorders) is novel to the best of our knowledge. Yet, it turns out to be the missing piece for establishing the desired two-way compatibility-correspondency between AGM revision operators and preference assignments of the described kind (cf. Theorem 3.31).

In view of the fact that the requirement of syntax-independence – as expressed in Postulate (G4) – may be (and has been) put into question, we also established a syntax-dependent version of our characterization (cf. Theorem 3.40). Crucial to this result is the observation that (G4) is exactly mirrored by the third faithfulness condition on the semantic side; thus removing it (going from faithfulness to *quasi-faithfulness*) yields the class of assignments compatibility-corresponding to revision operators satisfying the postulates (G1)–(G3), (G5), and (G6).

Conceding that transitivity is a rather natural choice for preferences and preorder assignments might be held dear by members of the belief revision community, we went on to investigate for which logics our general result holds even if assignments are required to yield preorders. We managed to pinpoint a specific logical phenomenon (called *critical loop*), the absence of which in a logic is necessary and sufficient for *total-preorder-representability*. While the criterion by itself maybe somewhat technical and unwieldy, it can be shown to subsume all logics featuring disjunction and therefore all classical logics. This justifies to formulate these findings in two theorems: a syntax independent version (Theorem 3.63) and a syntax-dependent one (Theorem 3.64).

Chapter 4

Revision in Description Logics under Fixed-Domain Semantics

In general, approaches for revising DL knowledge bases are classified into *syntax-based* and *semantics-based* approaches. In syntax-based approaches, the operators directly modifiy the axioms in the knowledge bases. Existing work on syntactic approaches could not satisfy all AGM postulates [QLB06a; AAB+18], considered only semi-revision [HK06b; RW09a], or proposed additional postulates (different from the AGM's) for capturing the minimality principle [RW09b; RW14b].

In contrast, semantics-based revision approaches investigate the models of KBs, search for the most plausible set of models to become the revision result, and generate a KB which corresponds to the produced model set. However, it has been shown that in DL with standard semantics, there are two main issues: (1) the models of the knowledge bases can be infinitely many and (2) even if we can somehow "compute" the model-based revision, the set of models as the result of the revision might not be able to be expressed in a knowledge base (this is known as the inexpressibility problem [LLM+06]). Investigations were carried out to find alternative semantic characterizations [WWT15; ZWW+14; DDL17] or to consider a hybrid approach for lightweight DL families [ZKN+19a]. However, these approaches required a new set of completely translated postulates to be satisfied, rather than the standard postulates for DL knowledge bases.

This chapter discusses the revision problem in description logics under fixed-domain semantics, where we follow the standard postulates for revision. Besides semantic characterization, we present two concrete approaches to revise knowledge bases expressed in these logics. The first approach is inspired from distance-based revision over models, while the second one is a syntax-based approach which exploits the individual elements. We start by instantiating the base logic for DL $\mathcal{SROIQ}$ under fixed-domain semantics as follows:

$$\mathbb{DL}_{\Delta} = (\mathcal{L}_{\mathbb{DL}_{\Delta}}, \Omega_{\mathbb{DL}_{\Delta}}, \models_{\Delta}, \mathcal{P}_{\text{fin}}(\mathcal{L}_{\mathbb{DL}_{\Delta}}), \cup)$$

As the model-based revision operator would be defined semantically, we will require some way to obtain a knowledge base from the set of 'preferred' models. For this objective, we introduce a method to construct a $\mathcal{SROIQ}$ axiom under fixed-domain semantics from a given set of interpretations such that the models of the axiom are exactly the given interpretations. This construction is useful to express the result of our model-based revision approach (see Section 4.1) into a DL knowledge base and to show that our semantic characterization is indeed compatible with the revision operator (see Definition 4.8). Let $\{\mathcal{I}_1, \ldots, \mathcal{I}_n\} \subseteq \Omega$ be a set of Δ-interpretations and $\mathcal{I}_i \in \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ be one of the interpretations, we define

$$\begin{aligned}\tau(\mathcal{I}_i) = & \Big(\bigsqcap_{C \in N_C} \bigsqcap_{d \in \Delta \text{ and } d \in C^{\mathcal{I}_i}} \exists u.(\{d\} \sqcap C) \Big) \sqcap \Big(\bigsqcap_{r \in N_R} \bigsqcap_{d,e \in \Delta \text{ and } (d,e) \in R^{\mathcal{I}_i}} \exists u.(\{d\} \sqcap \exists r.\{e\}) \Big) \sqcap \\ & \Big(\bigsqcap_{C \in N_C} \bigsqcap_{d \in \Delta \text{ and } d \notin C^{\mathcal{I}_i}} \exists u.(\{d\} \sqcap \neg C) \Big) \sqcap \Big(\bigsqcap_{r \in N_R} \bigsqcap_{d,e \in \Delta \text{ and } (d,e) \notin r^{\mathcal{I}_i}} \exists u.(\{d\} \sqcap \neg \exists r.\{e\}) \Big) \sqcap \\ & \Big(\bigsqcap_{a \in N_I(\mathcal{K}) \setminus \Delta, d \in \Delta \text{ and } a^{\mathcal{I}_i} = d} \exists u.(\{a\} \sqcap \{d\}) \Big),\end{aligned}$$

where u is a universal role. Then, we construct a $\mathcal{SROIQ}$ axiom as follows:

$$form(\{\mathcal{I}_1, \ldots, \mathcal{I}_n\}) = \top \sqsubseteq \bigsqcup_{1 \leq i \leq n} (\tau(\mathcal{I}_i)) \tag{4.1}$$

Lemma 4.1. *For any Δ-interpretation $\mathcal{I}$ and concept expression C, it holds that $(\exists u.C)^{\mathcal{I}} = \emptyset$ if $C^{\mathcal{I}} = \emptyset$ and $(\exists u.C)^{\mathcal{I}} = \Delta^{\mathcal{I}}$ if $C^{\mathcal{I}} \neq \emptyset$.*

Proof. For the first part, assume $C^{\mathcal{I}} = \emptyset$, i.e., there is no $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$. By definition $(\exists u.C)^{\mathcal{I}} = \{e \mid \exists d.(e,d) \in (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \wedge d \in C^{\mathcal{I}}\}$ and therefore $(\exists u.C)^{\mathcal{I}} = \emptyset$. For the second part, assume $C^{\mathcal{I}} \neq \emptyset$, i.e., there is some $d \in \Delta^{\mathcal{I}}$ with $d \in C^{\mathcal{I}}$. By definition $(\exists u.C)^{\mathcal{I}} = \{e \mid \exists d.(e,d) \in (\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}) \wedge d \in C^{\mathcal{I}}\}$ and therefore $(\exists u.C)^{\mathcal{I}} = \Delta^{\mathcal{I}}$. □

Lemma 4.2. *$\mathcal{I} \models form(\mathcal{I})$ for any Δ-interpretation $\mathcal{I}$.*

Proof. We show $\mathcal{I} \models form(\mathcal{I})$, i.e. $\Delta^{\mathcal{I}} \subseteq \tau(\mathcal{I})^{\mathcal{I}}$. Let $c \in \Delta^{\mathcal{I}}$. From the definition of $\tau(\mathcal{I})$, we have $\tau(\mathcal{I})^{\mathcal{I}} = \tau_C(\mathcal{I})^{\mathcal{I}} \cap \tau_{\neg C}(\mathcal{I})^{\mathcal{I}} \cap \tau_r(\mathcal{I})^{\mathcal{I}} \cap \tau_{\neg r}(\mathcal{I})^{\mathcal{I}} \cap \tau_a(\mathcal{I})^{\mathcal{I}}$, where

$$\tau_C(\mathcal{I})^{\mathcal{I}} = (\bigsqcap_{C\in N_C} \bigsqcap_{d\in\Delta \text{ and } d\in C^{\mathcal{I}}} \exists u.(\{d\}\sqcap C))^{\mathcal{I}},$$

$$\tau_{\neg C}(\mathcal{I})^{\mathcal{I}} = (\bigsqcap_{C\in N_C} \bigsqcap_{d\in\Delta \text{ and } d\notin C^{\mathcal{I}}} \exists u.(\{d\}\sqcap \neg C))^{\mathcal{I}},$$

$$\tau_r(\mathcal{I})^{\mathcal{I}} = (\bigsqcap_{r\in N_R} \bigsqcap_{d,e\in\Delta \text{ and } (d,e)\in r^{\mathcal{I}}} \exists u.(\{d\}\sqcap \exists r.\{e\}))^{\mathcal{I}},$$

$$\tau_{\neg r}(\mathcal{I})^{\mathcal{I}} = (\bigsqcap_{r\in N_R} \bigsqcap_{d,e\in\Delta \text{ and } (d,e)\notin r^{\mathcal{I}}} \exists u.(\{d\}\sqcap \neg\exists r.\{e\}))^{\mathcal{I}}, and$$

$$\tau_a(\mathcal{I})^{\mathcal{I}} = (\bigsqcap_{a\in N_I(\mathcal{K})\setminus\Delta, d\in\Delta \text{ and } a^{\mathcal{I}}=d} \exists u.(\{a\}\sqcap \{d\}))^{\mathcal{I}}.$$

We split the proof into several parts w.r.t concept interpretation sets involved in the intersection:

Part 1: $\tau_C(\mathcal{I})^{\mathcal{I}}$. We have $\tau_C(\mathcal{I})^{\mathcal{I}} = \bigcap_{C\in N_C}\bigcap_{d\in\Delta \text{ and } d\in C^{\mathcal{I}}}(\exists u.(\{d\}\sqcap C))^{\mathcal{I}}$. Now consider $(\{d\}\sqcap C)^{\mathcal{I}} = \{d\}^{\mathcal{I}} \cap C^{\mathcal{I}}$. From the definition of $\tau_C(\mathcal{I})^{\mathcal{I}}$, for every $C\in N_C$, for every $d\in\Delta$ and $d\in C^{\mathcal{I}}$, we have $\{d\}^{\mathcal{I}}\cap C^{\mathcal{I}} \neq \emptyset$, and hence $(\{d\}\sqcap C)^{\mathcal{I}} \neq \emptyset$. Since $c\in\Delta^{\mathcal{I}}$ and $(\{d\}\sqcap C)^{\mathcal{I}} \neq \emptyset$, from Lemma 4.1 we have $c\in(\exists u.(\{d\}\sqcap C))^{\mathcal{I}}$ for every $C\in N_C$ and for every $d\in\Delta$ and $d\in C^{\mathcal{I}}$. Hence, $c\in\tau_C(\mathcal{I})^{\mathcal{I}}$.

Part 2: $\tau_{\neg C}(\mathcal{I})^{\mathcal{I}}$. We have $\tau_{\neg C}(\mathcal{I})^{\mathcal{I}} = \bigcap_{C\in N_C}\bigcap_{d\in\Delta \text{ and } d\notin C^{\mathcal{I}}}(\exists u.(\{d\}\sqcap\neg C))^{\mathcal{I}}$. Now consider $(\{d\}\sqcap\neg C)^{\mathcal{I}} = \{d\}^{\mathcal{I}}\cap\Delta^{\mathcal{I}}\setminus C^{\mathcal{I}}$. From the definition of $\tau_{\neg C}(\mathcal{I})^{\mathcal{I}}$, for every $C\in N_C$, for every $d\in\Delta$ and $d\notin C^{\mathcal{I}}$, we have $\{d\}^{\mathcal{I}}\cap\Delta^{\mathcal{I}}\setminus C^{\mathcal{I}}\neq\emptyset$, and hence $(\{d\}\sqcap\neg C)^{\mathcal{I}}\neq\emptyset$. Since $c\in\Delta^{\mathcal{I}}$ and $(\{d\}\sqcap\neg C)^{\mathcal{I}}\neq\emptyset$, from Lemma 4.1 we have $c\in(\exists u.(\{d\}\sqcap\neg C))^{\mathcal{I}}$ for every $C\in N_C$ and for every $d\in\Delta$ and $d\notin C^{\mathcal{I}}$. Hence, $c\in\tau_{\neg C}(\mathcal{I})^{\mathcal{I}}$.

Part 3: $\tau_r(\mathcal{I})^{\mathcal{I}}$. We have $\tau_r(\mathcal{I})^{\mathcal{I}} = \bigcap_{r\in N_R}\bigcap_{d,e\in\Delta \text{ and } (d,e)\in r^{\mathcal{I}}}(\exists u.(\{d\}\sqcap\exists r.\{e\}))^{\mathcal{I}}$. Now consider $(\{d\}\sqcap\exists r.\{e\})^{\mathcal{I}} = \{d\}^{\mathcal{I}}\cap(\exists r.\{e\})^{\mathcal{I}}$. From the definition of $\tau_r(\mathcal{I})^{\mathcal{I}}$, for every $r\in N_R$, for every $d,e\in\Delta$ and $(d,e)\in r^{\mathcal{I}}$, we have $\{d\}^{\mathcal{I}}\cap(\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$, and hence $(\{d\}\sqcap\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$. Since $c\in\Delta^{\mathcal{I}}$ and $(\{d\}\sqcap\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$, from Lemma 4.1 we have $c\in(\exists u.(\{d\}\sqcap\exists r.\{e\}))^{\mathcal{I}}$ for every $r\in N_R$ and for every $d,e\in\Delta$ and $(d,e)\in r^{\mathcal{I}}$. Hence, $c\in\tau_r(\mathcal{I})^{\mathcal{I}}$.

Part 4: $\tau_{\neg r}(\mathcal{I})^{\mathcal{I}}$. We have $\tau_r(\mathcal{I})^{\mathcal{I}} = \bigcap_{r\in N_R}\bigcap_{d,e\in\Delta \text{ and } (d,e)\notin r^{\mathcal{I}}}(\exists u.(\{d\}\sqcap\neg\exists r.\{e\}))^{\mathcal{I}}$. Now consider $(\{d\}\sqcap\neg\exists r.\{e\})^{\mathcal{I}} = \{d\}^{\mathcal{I}}\cap\Delta^{\mathcal{I}}\setminus(\exists r.\{e\})^{\mathcal{I}}$. From the definition of $\tau_{\neg r}(\mathcal{I})^{\mathcal{I}}$, for every $r\in N_R$, for every $d,e\in\Delta$ and $(d,e)\notin r^{\mathcal{I}}$, we have $\{d\}^{\mathcal{I}}\cap\Delta^{\mathcal{I}}\setminus(\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$, and hence $(\{d\}\sqcap\neg\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$. Since $c\in\Delta^{\mathcal{I}}$ and $(\{d\}\sqcap\neg\exists r.\{e\})^{\mathcal{I}}\neq\emptyset$, from Lemma 4.1 we have $c\in(\exists u.(\{d\}\sqcap\neg\exists r.\{e\}))^{\mathcal{I}}$ for every $r\in N_R$ and for every $d,e\in\Delta$ and $(d,e)\in r^{\mathcal{I}}$. Hence, $c\in\tau_{\neg r}(\mathcal{I})^{\mathcal{I}}$.

Part 5: $\tau_a(\mathcal{I})^{\mathcal{I}}$. We have $\tau_a(\mathcal{I})^{\mathcal{I}} = \bigcap_{a\in N_I(\mathcal{K})\setminus\Delta, d\in\Delta \text{ and } a^{\mathcal{I}}=d}(\exists u.(\{d\}\sqcap\{a\}))^{\mathcal{I}}$. Now consider $(\{d\}\sqcap\{a\})^{\mathcal{I}} = \{d\}^{\mathcal{I}}\cap\{a\}^{\mathcal{I}}$. From the definition of $\tau_a(\mathcal{I})^{\mathcal{I}}$, for every $a\in N_I(\mathcal{K})\setminus\Delta$, for every

$d \in \Delta$ and $a^{\mathcal{I}} = d$, we have $\{d\}^{\mathcal{I}} \cap \{a\}^{\mathcal{I}} \neq \emptyset$, and hence $(\{d\} \sqcap \{a\})^{\mathcal{I}} \neq \emptyset$. Since $c \in \Delta^{\mathcal{I}}$ and $(\{d\} \sqcap \{a\})^{\mathcal{I}} \neq \emptyset$, from Lemma 4.1 we have $c \in (\exists u.(\{d\} \sqcap \{a\}))^{\mathcal{I}}$ for every $a \in N_I(\mathcal{K}) \setminus \Delta$, such that $d \in \Delta$ and $a^{\mathcal{I}} = d$. Hence, $c \in \tau_a(\mathcal{I})^{\mathcal{I}}$.

From all parts above, we have $c \in \tau_C(\mathcal{I})^{\mathcal{I}} \wedge c \in \tau_{\neg C}(\mathcal{I})^{\mathcal{I}} \wedge c \in \tau_r(\mathcal{I})^{\mathcal{I}} \wedge c \in \tau_{\neg r}(\mathcal{I})^{\mathcal{I}} \wedge c \in \tau_a(\mathcal{I})^{\mathcal{I}}$. This implies $c \in (\tau_C(\mathcal{I})^{\mathcal{I}} \cap \tau_{\neg C}(\mathcal{I})^{\mathcal{I}} \cap \tau_r(\mathcal{I})^{\mathcal{I}} \cap \tau_{\neg r}(\mathcal{I})^{\mathcal{I}} \cap \tau_a(\mathcal{I})^{\mathcal{I}})$. Hence, we have $c \in \tau(\mathcal{I})^{\mathcal{I}}$. □

Lemma 4.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be two Δ-interpretations. For any $d \in \Delta^{\mathcal{I}}$, $d \in \tau(\mathcal{J})^{\mathcal{I}}$ if and only if $\mathcal{I} = \mathcal{J}$.*

Proof. (Only if). Let $d \in \tau(\mathcal{J})^{\mathcal{I}}$. Then $d \in (\tau_C(\mathcal{J})^{\mathcal{I}} \cap \tau_{\neg C}(\mathcal{J})^{\mathcal{I}} \cap \tau_r(\mathcal{J})^{\mathcal{I}} \cap \tau_{\neg r}(\mathcal{J})^{\mathcal{I}} \cap \tau_a(\mathcal{J})^{\mathcal{I}})$, where

$$
\begin{aligned}
\tau_C(\mathcal{J})^{\mathcal{I}} &= (\textstyle\bigsqcap_{C \in N_C} \bigsqcap_{d_j \in \Delta \text{ and } d_j \in C^{\mathcal{J}}} \exists u.(\{d_j\} \sqcap C))^{\mathcal{I}},\\
\tau_{\neg C}(\mathcal{J})^{\mathcal{I}} &= (\textstyle\bigsqcap_{C \in N_C} \bigsqcap_{d_j \in \Delta \text{ and } d_j \notin C^{\mathcal{J}}} \exists u.(\{d_j\} \sqcap \neg C))^{\mathcal{I}},\\
\tau_r(\mathcal{J})^{\mathcal{I}} &= (\textstyle\bigsqcap_{r \in N_R} \bigsqcap_{d_j, e_j \in \Delta \text{ and } (d_j, e_j) \in R^{\mathcal{J}}} \exists u.(\{d_j\} \sqcap \exists r.\{e_j\}))^{\mathcal{I}},\\
\tau_{\neg r}(\mathcal{J})^{\mathcal{I}} &= (\textstyle\bigsqcap_{r \in N_R} \bigsqcap_{d_j, e_j \in \Delta \text{ and } (d_j, e_j) \notin R^{\mathcal{J}}} \exists u.(\{d_j\} \sqcap \neg \exists r.\{e_j\}))^{\mathcal{I}}, \text{ and}\\
\tau_a(\mathcal{J})^{\mathcal{I}} &= (\textstyle\bigsqcap_{a \in N_I(\mathcal{K}) \setminus \Delta, d_j \in \Delta \text{ and } a^{\mathcal{J}} = d_j} \exists u.(\{a\} \sqcap \{d_j\}))^{\mathcal{I}}.
\end{aligned}
$$

This implies $d \in \tau_C(\mathcal{J})^{\mathcal{I}} \wedge d \in \tau_{\neg C}(\mathcal{J})^{\mathcal{I}} \wedge d \in \tau_r(\mathcal{J})^{\mathcal{I}} \wedge d \in \tau_{\neg r}(\mathcal{J})^{\mathcal{I}} \wedge d \in \tau_a(\mathcal{J})^{\mathcal{I}}$. We divide the proof into several parts, whose each showing arguments based on every conjunct:

Part 1: $d \in \tau_C(\mathcal{J})^{\mathcal{I}}$. Then for all $C \in N_C$, for all $d_j \in C^{\mathcal{J}}$, we have $d \in (\exists u.(\{d_j\} \sqcap C))^{\mathcal{I}}$. Since $(\exists u.(\{d_j\} \sqcap C))^{\mathcal{I}} \neq \emptyset$, then from the semantical definition of a concept we have $(\{d_j\} \sqcap C)^{\mathcal{I}} \neq \emptyset$. Consequently, there exists $y \in (\{d_j\} \sqcap C)^{\mathcal{I}}$, and thus $y \in \{d_j\}^{\mathcal{I}} \cap C^{\mathcal{I}}$ with $y = d_j$ and $d_j \in C^{\mathcal{I}}$, yield $C^{\mathcal{J}} \subseteq C^{\mathcal{I}}$.

Now we show the other inclusion. Since $d \in \tau_{\neg C}(\mathcal{J})^{\mathcal{I}}$, then for all $C \in N_C$, for all $d_j \notin C^{\mathcal{J}}$, we have $d \in (\exists u.(\{d_j\} \sqcap \neg C))^{\mathcal{I}}$. Since $(\exists u.(\{d_j\} \sqcap \neg C))^{\mathcal{I}} \neq \emptyset$, then from the concept semantic definition we have $(\{d_j\} \sqcap \neg C)^{\mathcal{I}} \neq \emptyset$. There exists $y \in (\{d_j\} \sqcap \neg C)^{\mathcal{I}}$. We have $y \in \{d_j\}^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}}$ with $y = d_j$ and $d_j \in (\neg C)^{\mathcal{I}}$, yield $(\neg C)^{\mathcal{J}} \subseteq (\neg C)^{\mathcal{I}}$. Now let $d_i \in C^{\mathcal{I}}$, then $d_i \notin (\neg C)^{\mathcal{I}}$. Since $(\neg C)^{\mathcal{J}} \subseteq (\neg C)^{\mathcal{I}}$, it holds $d_i \notin (\neg C)^{\mathcal{J}}$. Thus $d_i \in C^{\mathcal{J}}$ holds and hence $C^{\mathcal{I}} \subseteq C^{\mathcal{J}}$. We have $C^{\mathcal{J}} = C^{\mathcal{I}}$ as desired.

Part 2: $d \in \tau_r(\mathcal{J})^{\mathcal{I}}$. Then for all $r \in N_R$, for all $(d_j, e_j) \in r^{\mathcal{J}}$, we have $d \in (\exists u.(\{d_j\} \sqcap \exists r.\{e_j\}))^{\mathcal{I}}$. Since $(\exists u.(\{d_j\} \sqcap \exists r.\{e_j\}))^{\mathcal{I}} \neq \emptyset$, then from the concept semantic definition we have $(\{d_j\} \sqcap \exists r.\{e_j\})^{\mathcal{I}} \neq \emptyset$. There exists $y \in (\{d_j\} \sqcap \exists r.\{e_j\})^{\mathcal{I}}$. We have $y \in \{d_j\}^{\mathcal{I}} \cap (\exists r.\{e_j\})^{\mathcal{I}}$ with $y = d_j$ and $y \in \{r \mid \exists s.(r, s) \in r^{\mathcal{I}} \wedge s \in \{e_j\}^{\mathcal{I}}\}$. Hence, $\exists s.(y, s) \in r^{\mathcal{I}} \wedge s \in \{e_j\}^{\mathcal{I}}$. Consequently, $s = e_j$ and $(d_j, e_j) \in r^{\mathcal{I}}$, yield $r^{\mathcal{J}} \subseteq r^{\mathcal{I}}$.

Now we show the other inclusion. Since $d \in \tau_{\neg r}(\mathcal{J})^{\mathcal{I}}$, then for all $r \in N_R$, for all $(d_j, e_j) \notin r^{\mathcal{J}}$, we have $d \in (\exists u.(\{d_j\} \sqcap \neg \exists r.\{e_j\}))^{\mathcal{I}}$. Since $(\exists u.(\{d_j\} \sqcap \neg \exists r.\{e_j\}))^{\mathcal{I}} \neq \emptyset$, then from the concept semantic definition, it holds $(\{d_j\} \sqcap \neg \exists r.\{e_j\})^{\mathcal{I}} \neq \emptyset$. There exists $y \in (\{d_j\} \sqcap \neg \exists r.\{e_j\})^{\mathcal{I}}$. Thus, $y \in \{d_j\}^{\mathcal{I}} \cap (\neg \exists r.\{e_j\})^{\mathcal{I}}$ with $y = d_j$ and $y \notin \{r \mid \exists s.(r, s) \in r^{\mathcal{I}} \wedge s \in \{e_j\}^{\mathcal{I}}\}$. Then, $s = e_j$ and $(d_j, e_j) \notin r^{\mathcal{I}}$, yield $u^{\mathcal{J}} \backslash r^{\mathcal{J}} \subseteq u^{\mathcal{I}} \backslash r^{\mathcal{I}}$.

Now let $(d_i, e_i) \in r^{\mathcal{I}}$, then $(d_i, e_i) \notin u^{\mathcal{I}} \backslash r^{\mathcal{I}}$. Since $u^{\mathcal{J}} \backslash r^{\mathcal{J}} \subseteq u^{\mathcal{I}} \backslash r^{\mathcal{I}}$, we have $(d_i, e_i) \notin u^{\mathcal{J}} \backslash r^{\mathcal{J}}$. Thus $(d_i, e_i) \in r^{\mathcal{J}}$ holds, and hence $r^{\mathcal{I}} \subseteq r^{\mathcal{J}}$. Finally we have $r^{\mathcal{J}} = r^{\mathcal{I}}$ as desired.

Part 3: $d \in \tau_a(\mathcal{J})^{\mathcal{I}}$. Then for all $a \in N_I(\mathcal{K}) \backslash \Delta$, $d_j \in \Delta$ and $a^{\mathcal{J}} = d_j$, we have $d \in (\exists u.(\{a\} \sqcap \{d_j\}))^{\mathcal{I}}$. Since $(\exists u.(\{a\} \sqcap \{d_j\}))^{\mathcal{I}} \neq \emptyset$, then from the concept semantic definition we have $(\{a\} \sqcap \{d_j\})^{\mathcal{I}} \neq \emptyset$. There exists $y \in (\{a\} \sqcap \{d_j\})^{\mathcal{I}}$. Thus, $y \in \{a\}^{\mathcal{I}} \cap \{d_j\}^{\mathcal{I}}$, implies $y = a^{\mathcal{J}} = d_j = a^{\mathcal{I}}$.

From the three parts above, we have $C^{\mathcal{J}} = C^{\mathcal{I}}$, $r^{\mathcal{J}} = r^{\mathcal{I}}$, and $a^{\mathcal{J}} = a^{\mathcal{I}}$ for all $C \in N_C$, $r \in N_R$, and $a \in N_I(\mathcal{K}) \backslash \Delta$. This implies $\mathcal{I} = \mathcal{J}$.

(If). Let $\mathcal{I} = \mathcal{J}$. Then $\mathit{form}(\mathcal{I}) = \mathit{form}(\mathcal{J})$, consequently $(\top \sqsubseteq \tau(\mathcal{I})) \equiv (\top \sqsubseteq \tau(\mathcal{J})$. From Lemma 4.2, we have $\mathcal{I} \models (\top \sqsubseteq \tau(\mathcal{I}))$. Then, $\Delta^{\mathcal{I}} \subseteq \tau(\mathcal{I})^{\mathcal{I}}$ is true. This implies for any $d \in \Delta^{\mathcal{I}}$ we have $d \in \tau(\mathcal{I})^{\mathcal{I}}$. Since $\mathcal{I} = \mathcal{J}$, we have $d \in \tau(\mathcal{J})^{\mathcal{I}}$. □

Proposition 4.4. *Let $\{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ be a set of Δ-interpretations. $Mod_\Delta(form(\mathcal{I}_1, \ldots, \mathcal{I}_n)) = \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ holds.*

Proof. ($\subseteq$) We show $Mod_\Delta(form(\mathcal{I}_1, \ldots, \mathcal{I}_n)) \subseteq \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$. Let $\mathcal{I} \in Mod_\Delta$ $(form(\mathcal{I}_1, \ldots, \mathcal{I}_n))$. Then, $\mathcal{I} \models_\Delta \top \sqsubseteq \bigsqcup_{1 \leq i \leq n}(\tau(\mathcal{I}_i))$. This implies $\Delta^{\mathcal{I}} \subseteq (\tau(\mathcal{I}_1))^{\mathcal{I}} \cup \ldots \cup (\tau(\mathcal{I}_n))^{\mathcal{I}}$. Then, For any $d \in \Delta^{\mathcal{I}}$, we have $d \in \tau(\mathcal{I}_1)^{\mathcal{I}} \vee \ldots \vee d \in \tau(\mathcal{I}_n)^{\mathcal{I}}$. From Lemma 4.3, we have $(\mathcal{I}_1 = \mathcal{I}) \vee \ldots \vee (\mathcal{I}_n = \mathcal{I})$. Then, there exist $\mathcal{I}_i \in \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ with $1 \leq i \leq n$ such that $\mathcal{I}_i = \mathcal{I}$. Hence, $\mathcal{I} \in \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$.

($\supseteq$) We show $Mod_\Delta(form(\mathcal{I}_1, \ldots, \mathcal{I}_n)) \supseteq \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$. Let $\mathcal{I} \in \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$. Then, there exist $\mathcal{I}_i \in \{\mathcal{I}_1, \ldots, \mathcal{I}_n\}$ with $1 \leq i \leq n$ such that $\mathcal{I}_i = \mathcal{I}$. We have $(\mathcal{I}_1 = \mathcal{I}) \vee \ldots \vee (\mathcal{I}_n = \mathcal{I})$. From Lemma 4.3, we have for any $d \in \Delta^{\mathcal{I}}$, $d \in \tau(\mathcal{I}_1)^{\mathcal{I}} \vee \ldots \vee d \in \tau(\mathcal{I}_n)^{\mathcal{I}}$. This implies $\Delta^{\mathcal{I}} \subseteq (\tau(\mathcal{I}_1))^{\mathcal{I}} \cup \ldots \cup (\tau(\mathcal{I}_n))^{\mathcal{I}}$ and consequently $\mathcal{I} \models \top \sqsubseteq \tau(\mathcal{I}_1) \sqcup \ldots \sqcup \tau(\mathcal{I}_n)$. Hence, $\mathcal{I} \in Mod_\Delta(form(\mathcal{I}_1, \ldots, \mathcal{I}_n))$. □

We provide a semantic characterization of AGM revision operator in $\mathbb{DL}_\Delta$ as follows.

Theorem 4.5 (Adaptation of Theorem 3.3. in [KM91]). *Let $\circ_\Delta$ be a revision operator for $\mathbb{DL}_\Delta$ (i.e. DL under fixed-domain semantics). Then, $\circ_\Delta$ satisfies (G1)–(G6) if and only if it is compatible with some preorder faithful assignment.*

Proof. The proof is similar to the one of the Representation Theorem by Katsuno and Mendelzon [KM91, Theorem 3.3.]. For the "if" direction, the arguments are similar and straightforward. For the "only if" direction, we assume the existence of a revision operator $\circ_\Delta$ which satisfies postulates (G1)–(G6). Then, for any knowledge base $\mathcal{K}$, one can obtain a faithful preorder assignment compatible with $\circ_\Delta$ by employing relation encoding $\preceq_{\mathcal{K}}$ as: $\mathcal{I} \preceq_{\mathcal{K}} \mathcal{I}'$ if and only if either $\mathcal{I} \in [\![\mathcal{K}]\!]_\Delta$ or $\mathcal{I} \in [\![\mathcal{K} \circ_\Delta \mathit{form}_\Delta(\mathcal{I}, \mathcal{I}')]\!]_\Delta$ for any interpretations $\mathcal{I}$ and $\mathcal{I}'$. □

We also remark that $\mathbb{DL}_\Delta$ is a disjunctive (base) logic, such that for every bases $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{L}_{\mathbb{DL}_\Delta}$, a base $\mathcal{K} \varovee \mathcal{K}'$ can be obtained using the axiom constructor as $\{\mathit{form}([\![\mathcal{K}]\!]_\Delta \cup [\![\mathcal{K}']\!]_\Delta)\}$.

4.1 Model-based Approach

In this section, we present our first approach to perform model-based revision in the fixed-domain semantics setting. Our concrete revision operator is adapted from Dalal's operator [Dal88b]. The original Dalal operator works for two propositional formulas ψ and μ. The *difference set* between their models consists of propositional variables that are interpreted differently by them (e.g. when comparing two models, each from the models of ψ and μ, a variable p is interpreted as true in the model of ψ, but it is interpreted as false the model of μ). Then, the *distance* between them is defined as the minimal cardinality of the *difference sets* between models of ψ and μ. The set of models of revising ψ by μ consists of models of μ such that there exists a model of ψ such that the cardinality of the difference set between the two models is the same as the distance between ψ and μ. In [KM91], it has been shown that Dalal's revision operator can be defined as the set of minimal models of μ w.r.t a faithful preorder relation $\preceq_\psi$.

To adapt Dalal's revision operator to DL under fixed-domain semantics, we need to define the "difference set" between two models. Thanks to the finitely many known elements in the domain, we can characterize the Δ-models of the knowledge bases and then we can define the difference between two Δ-models in a similar way as the difference set between two models in propositional logic based on the grounded form of the interpretations.

Definition 4.6 (Grounded interpretation). *Let $\mathcal{K}$ be a KB and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a Δ-fixed interpretation. The* ground representation *of $\mathcal{I}$ is the following: $Gr(\mathcal{I}) = \{C(d) \mid d \in \Delta$ and $d \in C^{\mathcal{I}}\} \cup \{r(d,e) \mid d,e \in \Delta$ and $(d,e) \in r^{\mathcal{I}}\} \cup \{a = d \mid a \in N_{\mathcal{I}}(\mathcal{K}), d \in \Delta$ and $a^{\mathcal{I}} = d\}$.*

In the following, we introduce a distance between two models based on the operator of symmetric difference, denoted with $\oplus$, which is defined as $S \oplus S' = (S \cup S') \setminus (S \cap S')$ for any set S and S'.

Definition 4.7 (Distance). *Let $[\![\mathcal{K}]\!]_\Delta$ be a set of all Δ-models of knowledge base $\mathcal{K}$ and $\mathcal{I}'$ be a Δ-interpretation. The* distance *between $[\![\mathcal{K}]\!]_\Delta$ and $\mathcal{I}'$ is defined as:*

$$dist([\![\mathcal{K}]\!]_\Delta, \mathcal{I}') = \min_{\mathcal{I} \in [\![\mathcal{K}]\!]_\Delta} dist(\mathcal{I}, \mathcal{I}'),$$

where $dist(\mathcal{I}, \mathcal{I}') = |diff(\mathcal{I}, \mathcal{I}')|$ and $diff(\mathcal{I}, \mathcal{I}') = Gr(\mathcal{I}) \oplus Gr(\mathcal{I}')$.

Now we are ready to introduce a *model-based revision operator* for Description Logic under fixed-domain semantics.

Definition 4.8. *Let $\mathcal{K}$ and $\mathcal{K}'$ be any two knowledge bases. Let $\preceq^\Delta_{(.)}: \mathcal{K} \mapsto \preceq^\Delta_{\mathcal{K}}$ be an assignment, where a binary relation $\preceq^\Delta_{\mathcal{K}}$ is defined as*

$$\mathcal{I}_1 \preceq^\Delta_{\mathcal{K}} \mathcal{I}_2 \text{ if and only if } dist([\![\mathcal{K}]\!]_\Delta, \mathcal{I}_1) \leq dist([\![\mathcal{K}]\!]_\Delta, \mathcal{I}_2) \text{ for all interpretations } \mathcal{I}_1 \text{ and } \mathcal{I}_2.$$

We define the model-based revision operator $\circ^d_\Delta$ *as follows:*

$$[\![\mathcal{K} \circ^d_\Delta \mathcal{K}']\!] = \min([\![\mathcal{K}']\!]_\Delta, \preceq^\Delta_\mathcal{K}).$$

Proposition 4.9. *The model-based change operator* $\circ^d_\Delta$ *satisfies postulates (G1)–(G6).*

Proof. Similar to the assignment presented in [KM91, Section 4.1.], we have that the assignment $\preceq^\Delta_{(.)}$ is a preorder faithful assignment. From Proposition 4.4, we have $[\![\mathcal{K} \circ^d_\Delta \mathcal{K}']\!]_\Delta = \min([\![\mathcal{K}']\!]_\Delta, \preceq^\Delta_\mathcal{K})$, which shows compatibilty. Finally from Theorem 4.5, we have $\circ^d_\Delta$ satisfies postulates (G1)-(G6). □

Note that up to this point, we just defined the revision operator semantically. As the output of a revision operator should be a knowledge base, one can simply use the axiom construction to obtain the desired revision outcome as $\mathcal{K} \circ_\Delta \mathcal{K}' = \{form(min([\![\mathcal{K}']\!]_\Delta, \preceq^\Delta_\mathcal{K}))\}$.

4.2 Model-Based Approach via ASP Encoding

We now introduce an encoding of the model-based revision approach via answer-set programming (ASP). The KB-to-program encoding is based on the naïve encoding [GRS16], which has already been implemented in the `Wolpertinger` reasoner. The main idea is to generate the models of the two knowledge bases involved in the revision using the naïve encoding, compute the distance between models (which are now in the form of answer sets), then find the model set of the new information $\mathcal{K}'$ with minimal distance to the model set of the prior KB $\mathcal{K}$. To this end, we assume that both knowledge bases $\mathcal{K}$ and $\mathcal{K}'$ are in normalized forms [GRS16].

First, we generate all possible interpretations and then check for each interpretation whether it violates any axiom in both knowledge bases or not. Note that we are dealing with two knowledge bases, we translate the axioms from both $\mathcal{K}$ and $\mathcal{K}'$. Since the domain is fixed, the number of interpretations generated from this step are bounded by finitely many elements. Let Δ be a fixed domain, $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ and $\mathcal{K}' = (\mathcal{A}', \mathcal{T}', \mathcal{R}')$ be two knowledge bases, we have

$$\Pi_{gen}(\mathcal{K} \cup \mathcal{K}', \Delta) = \{A\_1(X) \leftarrow not\ \neg A\_1(X), thing(X) \mid A \in N_C(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.2}$$

$$\{\neg A\_1(X) \leftarrow not\ A\_1(X), thing(X) \mid A \in N_C(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.3}$$

$$\{r\_1(X,Y) \leftarrow not\ \neg r\_1(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.4}$$

$$\{\neg r\_1(X,Y) \leftarrow not\ r\_1(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.5}$$

$$\{A\_2(X) \leftarrow not\ \neg A\_2(X), thing(X) \mid A \in N_C(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.6}$$

$$\{\neg A\_2(X) \leftarrow not\ A\_2(X), thing(X) \mid A \in N_C(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.7}$$

$$\{r\_2(X,Y) \leftarrow not\ \neg r\_2(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.8}$$

$$\{\neg r\_2(X,Y) \leftarrow not\ r\_2(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K} \cup \mathcal{K}')\} \cup \tag{4.9}$$

$$\{thing(\delta) \mid \delta \in \Delta\}. \tag{4.10}$$

We append indices "1" and "2" to the predicate name which represents the concept and role name to distinguish the concept and role name from the prior KB (e.g. A_1(X) or r_1(X,Y)) or from the new information (e.g. A_2(X) or r_2(X,Y)). Intuitively, the ASP program in Equation (4.2)–Equation (4.10) spans all candidate interpretations based on the knowledge bases signature. The next step is checking if there exists some axiom violation. To find the violation, again for both knowledge bases, we encode for each ABox, TBox, and RBox. The following Equation (4.11)–Equation (4.14) are the encodings for the ABox from $\mathcal{K}$:

$$\Pi_{chk}(\mathcal{A}) = \quad \{A\_1(a) \mid A(a) \in \mathcal{A}\} \cup \tag{4.11}$$

$$\{\neg A\_1(a) \mid \neg A(a) \in \mathcal{A}\} \cup \tag{4.12}$$

$$\{r\_1(a,b) \mid r(a,b) \in \mathcal{A}\} \cup \tag{4.13}$$

$$\{\neg r\_1(a,b) \mid \neg r(a,b) \in \mathcal{A}\}. \tag{4.14}$$

As we can see, the encoding of ABox is straightforward. Every individual assertion $C(a)$ and every role assertion $r(a,b)$ in the ABox is encoded in the program as a fact. Any contradiction ($A\_i(X)$ and $\neg A\_i(X)$) is handled by semantics of strong negations in ASP. For the ABox from KB $\mathcal{K}'$, the encoding are the same, except that the appended index after each concept/role name is changed to 2 and each predicate represent concept/role assertion in $\mathcal{A}'$.

Now for the TBox, remember that the TBox is in normalized form, i.e. each GCI in the TBox is of the form $\top \sqsubseteq \bigsqcup_{1 \le i \le n} C_i$. From the DL semantics, the axiom means that each individual should be a member of a concept C_i for some $1 \le i \le n$. The axiom is violated if there exists an individual which is not a member of any C_i. This axiom is directly translated into a constraint in ASP in the Equation (4.15) below. As for TBox $\mathcal{K}'$, we simply modify the index and employ $\mathcal{T}'$ for the rule.

$$\Pi_{chk}(\mathcal{T}) = \quad \{\leftarrow trans(C_1), \ldots, trans(C_n) \mid \top \sqsubseteq \bigsqcup_{1 \le i \le n} C_i \in \mathcal{T}\}. \tag{4.15}$$

The RBox might consist of role axioms of the form $r \sqsubseteq s$, $r_1 \circ r_2 \sqsubseteq r_3$, or $Dis(r,s)$. Similar to the TBox encoding, we search for any violation of RBox axioms as ASP constraints in the Equation (4.16)–Equation (4.18) below. Again, RBox from $\mathcal{K}'$ is encoded similarly with simple replacement of index 1 by 2 and $\mathcal{R}$ by $\mathcal{R}'$.

$$\Pi_{chk}(\mathcal{R}) = \{\leftarrow r\_1(X,Y), not\ s\_1(X,Y) \mid r \sqsubseteq s \in \mathcal{R}\} \cup \tag{4.16}$$

$$\{\leftarrow r\_1(X,Y), s\_1(X,Y) \mid Dis(r,s) \in \mathcal{R}\} \cup \tag{4.17}$$

$$\{\leftarrow s_1\_1(X,Y), s_2\_1(Y,Z), not\ r\_1(X,Z) \mid s_1 \circ s_2 \sqsubseteq r \in \mathcal{R}\}. \tag{4.18}$$

Now, we have complete translations of axioms in $\mathcal{K}$ and $\mathcal{K}'$ as the following equations:

$$\Pi_{chk}(\mathcal{K}, \Delta) = \Pi_{chk}(\mathcal{A}) \cup \Pi_{chk}(\mathcal{T}) \cup \Pi_{chk}(\mathcal{R}). \tag{4.19}$$

$$\Pi_{chk}(\mathcal{K}', \Delta) = \Pi_{chk}(\mathcal{A}') \cup \Pi_{chk}(\mathcal{T}') \cup \Pi_{chk}(\mathcal{R}'). \tag{4.20}$$

Gaggl and colleagues [GRS16, Theorem 10] have shown that the candidate generation (Equation (4.2)–Equation (4.10)) and the axiom encoding (Equation (4.11)–Equation (4.18)) correctly produce a tight one-to-one correspondence between Δ-models of the knowledge bases and answer sets of its ASP translation. Therefore, up to this stage, we have the collection of both models from the KB $\mathcal{K}$ and $\mathcal{K}'$.

Second, for every answer set (which is a union of two single models from the two KBs), we count the number of differences between grounded instances for each concept name A and each role name r. The difference increases whenever $A(X)$ occurs in the model of $\mathcal{K}$ but $\neg A(X)$ occurs in the model of $\mathcal{K}'$, or we have $\neg A(X)$ in the model of $\mathcal{K}$ but $A(X)$ is in the model of $\mathcal{K}'$. In particular, we collect the number of the concepts interpreted differently by the two models in the atom $count(A, M)$. Since we are only interested in overall differences between two interpretations, we sum all concept differences in the atom $total(Z)$. We use the aggregate construct $\#count$ and $\#sum$ from `Clingo` to perform the counting and the summation.

$$\begin{aligned}\Pi_{diff}(\mathcal{K} \cup \mathcal{K}', \Delta) = \{count(A, M) \leftarrow K = \#count\{X : A\_1(X), not\ A\_2(X)\},\\ L = \#count\{X : not\ A\_1(X), A\_2(X)\},\\ M = K + L. \mid A \in N_C(\mathcal{K} \cup \mathcal{K}'))\} \cup\end{aligned} \tag{4.21}$$

$$\begin{aligned}\{count(r, M) \leftarrow K = \#count\{X, Y : r\_1(X,Y), not\ r\_2(X,Y)\},\\ L = \#count\{X, Y : not\ r\_1(X,Y), r\_2(X,Y)\},\\ M = K + L. \mid r \in N_R(\mathcal{K} \cup \mathcal{K}'))\}.\end{aligned} \tag{4.22}$$

$$total(Z) \leftarrow Z = \#sum\{Y, (X,Y) : count(X,Y)\}. \tag{4.23}$$

Now, given two KBs $\mathcal{K}$ and $\mathcal{K}'$, we can encode the model-based revision approach as the following program:

$$\Pi(\mathcal{K} \circ^d_\Delta \mathcal{K}', \Delta) = \Pi_{gen}(\mathcal{K} \cup \mathcal{K}', \Delta) \cup \Pi_{chk}(\mathcal{K}, \Delta) \cup \Pi_{chk}(\mathcal{K}', \Delta) \\ \cup \Pi_{diff}(\mathcal{K} \cup \mathcal{K}', \Delta). \tag{4.24}$$

Note that the answer sets of the program $\Pi(\mathcal{K} \circ^d_\Delta \mathcal{K}', \Delta)$ (i.e. Equation (4.24)) contain all possible models of both knowledge bases together with the total number of difference in each answer set. As we are only interested in the models $\mathcal{K}'$ with the smallest "distance" to the models of $\mathcal{K}$, we have to pick the minimal models of $\mathcal{K}'$ based on the smallest number Z captured in the atom $total(Z)$ in each answer set. We utilize the optimization statement $\#minimize$ from `Clingo` to get the desired results by appending "$\#minimize\{C : total(C)\}$." in the program file.

We also remark that each generated answer set contains models from both knowledge bases $\mathcal{K}$ and $\mathcal{K}'$. To come up with the final result of the revision, we only require the models of $\mathcal{K}'$ which have minimal differences to the model of $\mathcal{K}$. Since we already have index "2" in the predicates of the answer set, we can only consider the predicates which represent the models of the new information $\mathcal{K}'$. Thus, an answer set $\mathbf{A}$ of $\Pi_{gen}(\mathcal{K} \cup \mathcal{K}', \Delta)$ corresponds to an interpretation $\mathcal{I}'_{\mathbf{A}} = (\Delta, \cdot^{\mathcal{I}'_{\mathbf{A}}})$ of knowledge base $\mathcal{K}'$ over the fixed-domain Δ as follows:

$$A^{\mathcal{I}'_{\mathbf{A}}} = \{\delta \mid A\_2(\delta) \in \mathbf{A}\}, \text{for all } A \in N_C(\mathcal{K} \cup \mathcal{K}'),$$
$$r^{\mathcal{I}'_{\mathbf{A}}} = \{(\delta, \delta') \mid r\_2(\delta, \delta') \in \mathbf{A}\}, \text{for all } r \in N_R(\mathcal{K} \cup \mathcal{K}'),$$
$$a^{\mathcal{I}'_{\mathbf{A}}} = a.$$

Given $\{\mathcal{I}'_{\mathbf{A}_1}, \dots, \mathcal{I}'_{\mathbf{A}_n}\} \subseteq \mathcal{AS}(\Pi(\mathcal{K} \circ^d_\Delta \mathcal{K}', \Delta))$ as the minimal models of $\mathcal{K}'$, one can construct a knowledge base as a revision result using the axiom constructor in Equation (4.1), i.e. $form(\{\mathcal{I}'_{\mathbf{A}_1}, \dots, \mathcal{I}'_{\mathbf{A}_n}\})$.

4.3 Individual-based Approach

In this section, we present the second approach to revise our DL knowledge bases. The main idea is that instead of removing the whole axiom(s) whenever inconsistency occurs, the axioms are modified by adding some exceptions. We call these modified axioms *weakened axioms*. Different from the previous approach (cf. Section 4.1) which computes the interpretations, this approach exploits the prior given domain elements. In particular, we will focus on the set of exceptional individuals which serves as a basis to weaken the knowledge base. For the weakening process, we impose the assumption that the knowledge base $\mathcal{K}$ is free of RBox axioms. This assumption will enable simpler weakening steps as we only consider TBox and ABox axioms. To this end, we introduce an equivalent transformation for an arbitrary knowledge base into a KB without RBox axioms. This transformation is possible as we are working with fixed-domain semantics. The idea is to keep the TBox and ABox axioms unchanged and to "partially ground" any RBox axiom into a set of GCIs involving existential restriction with nominal concepts.

Definition 4.10 (KB transformation). *Let Δ be a fixed domain and $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be a KB under the fixed-domain semantics, where $\mathcal{A}$ is an ABox, $\mathcal{T}$ is a TBox, and $\mathcal{R}$ is an RBox. The* KB transformation *is* $\mathsf{trans}_\Delta(\mathcal{K}) = \bigcup_{\alpha \in \mathcal{K}} \mathsf{trans}_\Delta(\alpha)$*, where:*

- $\mathsf{trans}_\Delta(\alpha) = \{\alpha\}$ *for any* $\alpha \in \mathcal{T} \cup \mathcal{A}$.
- $\mathsf{trans}_\Delta(\alpha) = \bigcup_{d \in \Delta}\{\exists r.\{d\} \sqsubseteq \exists s.\{d\}\}$ *for any* $\alpha = r \sqsubseteq s \in \mathcal{R}$.
- $\mathsf{trans}_\Delta(\alpha) = \bigcup_{d \in \Delta}\{\exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{(n+1)}.\{d\}\}$ *for any* $\alpha = r_1 \circ \ldots \circ r_n \sqsubseteq r_{(n+1)} \in \mathcal{R}$.
- $\mathsf{trans}_\Delta(\alpha) = \bigcup_{d \in \Delta}\{(\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot\}$ *for any* $\alpha = Dis(r,s) \in \mathcal{R}$.

We observe that the new RBox-free KB is semantically equivalent to the original one.

Lemma 4.11. *Let Δ be a fixed domain and $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be a KB under the fixed-domain semantics and $\mathsf{trans}(\mathcal{K})$ be the transformation of $\mathcal{K}$ (cf. Definition 4.10). $\mathsf{trans}(\mathcal{K}) \equiv_\Delta \mathcal{K}$ holds.*

Proof. We show $\mathsf{trans}_\Delta(\mathcal{K}) \models_\Delta \mathcal{K}$ and $\mathcal{K} \models_\Delta \mathsf{trans}_\Delta(\mathcal{K})$.

($\mathsf{trans}_\Delta(\mathcal{K}) \models_\Delta \mathcal{K}$) Let $\mathcal{I}_{tr} \models_\Delta \mathsf{trans}_\Delta(\mathcal{K})$, i.e. $\mathcal{I}_{tr} \models_\Delta \alpha_{tr}$ for any $\alpha_{tr} \in \mathsf{trans}_\Delta(\mathcal{K})$. We show $\mathcal{I}_{tr} \models_\Delta \alpha$ for any $\alpha \in \mathcal{K}$. We consider cases based on the form of axiom $\alpha \in \mathcal{K}$:

(1) $\alpha \in \mathcal{A}$ or $\alpha \in \mathcal{T}$. Then, $\mathsf{trans}_\Delta(\alpha) = \{\alpha\}$ and hence $\mathcal{I}_{tr} \models_\Delta \alpha$.

(2) $\alpha \in \mathcal{R}$. Note that $\mathsf{trans}_\Delta(\alpha) \subseteq \mathsf{trans}_\Delta(\mathcal{K})$ and $\mathcal{I}_{tr} \models_\Delta \mathsf{trans}_\Delta(\mathcal{K})$, then $\mathcal{I}_{tr} \models_\Delta \sigma$ for any $\sigma \in \mathsf{trans}_\Delta(\alpha)$.

We consider subcases based on the type of the RBox axiom:

(2.1) $\alpha = r \sqsubseteq s$. Then, $\mathsf{trans}_\Delta(\alpha) = \{\exists r.\{d\} \sqsubseteq \exists s.\{d\} \mid d \in \Delta\}$. Let $(x,d) \in r^{\mathcal{I}_{tr}}$. Then, $x \in (\exists r.\{d\})^{\mathcal{I}_{tr}}$. Since $\mathcal{I}_{tr} \models_\Delta \sigma$ for any $\sigma \in \mathsf{trans}_\Delta(\alpha)$, we have $\mathcal{I}_{tr} \models_\Delta \exists r.\{d\} \sqsubseteq \exists s.\{d\}$ for any $d \in \Delta$. It holds that $(\exists r.\{d\})^{\mathcal{I}_{tr}} \subseteq (\exists s.\{d\})^{\mathcal{I}_{tr}}$. Then, we have $x \in (\exists s.\{d\})^{\mathcal{I}_{tr}}$. From the definition of the semantics, there exists some $y \in \Delta$ such that $(x,y) \in s^{\mathcal{I}_{tr}}$ with $y = d$. Then, $(x,d) \in s^{\mathcal{I}_{tr}}$. Hence, $r^{\mathcal{I}_{tr}} \subseteq s^{\mathcal{I}_{tr}}$ and $\mathcal{I}_{tr} \models_\Delta r \sqsubseteq s$ as desired.

(2.2) $\alpha = r_1 \circ \ldots \circ r_n \sqsubseteq r_{(n+1)}$. Then, $\mathsf{trans}_\Delta(\alpha) = \{\exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{(n+1)}.\{d\} \mid d \in \Delta\}$. Let $(x, x_i) \in r_1^{\mathcal{I}_{tr}}$, $(x_i, x_{(i+1)}) \in r_i^{\mathcal{I}_{tr}}$, and $(x_{(n-1)}, d) \in r_n^{\mathcal{I}_{tr}}$ for $1 \leq i \leq n$. Then, $x \in (\exists r_1 \ldots \exists r_n.\{d\})^{\mathcal{I}_{tr}}$. Since $\mathcal{I}_{tr} \models_\Delta \sigma$ for any $\sigma \in \mathsf{trans}_\Delta(\alpha)$, we have $\mathcal{I}_{tr} \models_\Delta \exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{(n+1)}.\{d\}$ for any $d \in \Delta$. It holds that $(\exists r_1 \ldots \exists r_n.\{d\})^{\mathcal{I}_{tr}}$

$\subseteq (\exists r_{(n+1)}.\{d\})^{\mathcal{I}_{tr}}$. Then, we have $x \in (\exists r_{(n+1)}.\{d\})^{\mathcal{I}_{tr}}$. From the definition of the semantics, there exists some $y \in \Delta$ such that $(x,y) \in r_{(n+1)}^{\mathcal{I}_{tr}}$ with $y = d$. Then, $(x,d) \in r_{(n+1)}^{\mathcal{I}_{tr}}$. Hence, $r_1^{\mathcal{I}_{tr}} \circ \ldots \circ r_n^{\mathcal{I}_{tr}} \subseteq r_{(n+1)}^{\mathcal{I}_{tr}}$ and $\mathcal{I}_{tr} \models_\Delta r_1 \circ \ldots \circ r_n \sqsubseteq r_{(n+1)}$ as desired.

(2.3) $\alpha = Dis(r,s)$. Then, $\mathsf{trans}_\Delta(\alpha) = \{(\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot \mid d \in \Delta\}$. Since $\mathcal{I}_{tr} \models_\Delta \sigma$ for any $\sigma \in \mathsf{trans}_\Delta(\alpha)$, we have $\mathcal{I}_{tr} \models_\Delta (\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot$ for any $d \in \Delta$. It holds that $(\exists r.\{d\})^{\mathcal{I}_{tr}} \cap (\exists s.\{d\})^{\mathcal{I}_{tr}} \subseteq \emptyset$. Then, for any $x \in \Delta$, it holds $x \notin (\exists r.\{d\})^{\mathcal{I}_{tr}}$ or $x \notin (\exists s.\{d\})^{\mathcal{I}_{tr}}$. From the semantic definition, $x \notin (\exists r.\{d\})^{\mathcal{I}_{tr}}$ means there is no $y \in \Delta$ such that $(x,y) \in r^{\mathcal{I}_{tr}}$ with $y = d$ and $x \notin (\exists s.\{d\})^{\mathcal{I}_{tr}}$ means there is no $y \in \Delta$ such that $(x,y) \in s^{\mathcal{I}_{tr}}$ with $y = d$. Then, we have $(x,d) \notin r^{\mathcal{I}_{tr}}$ or $(x,d) \notin s^{\mathcal{I}_{tr}}$. Hence, $r^{\mathcal{I}_{tr}} \cap s^{\mathcal{I}_{tr}} = \emptyset$ and $\mathcal{I}_{tr} \models_\Delta Dis(r,s)$ as desired.

($\mathcal{K} \models_\Delta \mathsf{trans}_\Delta(\mathcal{K})$) Let $\mathcal{I} \models_\Delta \mathcal{K}$, i.e. $\mathcal{I} \models_\Delta \alpha$ for any $\alpha \in \mathcal{K}$. We show $\mathcal{I} \models_\Delta \alpha_{tr}$ for any axiom $\alpha_{tr} \in \mathsf{trans}_\Delta(\mathcal{K})$. We consider several cases of the axiom set $\mathsf{trans}_\Delta(\mathcal{K})' \subseteq \mathsf{trans}_\Delta(\mathcal{K})$:

(1) $\mathsf{trans}_\Delta(\mathcal{K})' = \{\alpha\}$, where $\alpha \in \mathcal{T}$ or $\alpha \in \mathcal{A}$. Then, $\mathcal{I} \models_\Delta \alpha_{tr}$ for any $\alpha_{tr} \in \mathsf{trans}_\Delta(\mathcal{K})'$.

(2) $\mathsf{trans}_\Delta(\mathcal{K})' = \{\exists r.\{d\} \sqsubseteq \exists s.\{d\} \mid d \in \Delta\}$ and $\mathsf{trans}_\Delta(\mathcal{K})' \nsubseteq \mathcal{T}$. Then, $r \sqsubseteq s \in \mathcal{K}$. Since $\mathcal{I} \models_\Delta \mathcal{K}$, then $\mathcal{I} \models_\Delta r \sqsubseteq s$. It holds that $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. Let $x, d \in \Delta$ with $x \in (\exists r.\{d\})^{\mathcal{I}}$. Then, there exists $y \in \Delta$ such that $(x,y) \in r^{\mathcal{I}}$ and $y = d$. We have $(x,d) \in r^{\mathcal{I}}$. Since $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, we also have $(x,d) \in s^{\mathcal{I}}$. From the definition of the semantics, we have $x \in (\exists s.\{d\})^{\mathcal{I}}$. We have $(\exists r.\{d\})^{\mathcal{I}} \subseteq (\exists s.\{d\})^{\mathcal{I}}$ and $\mathcal{I} \models_\Delta \exists r.\{d\} \sqsubseteq \exists s.\{d\}$ for any $d \in \Delta$. Hence, $\mathcal{I} \models_\Delta \alpha_{tr}$ for any $\alpha_{tr} \in \{\exists r.\{d\} \sqsubseteq \exists s.\{d\} \mid d \in \Delta\}$ as desired.

(3) $\mathsf{trans}_\Delta(\mathcal{K})' = \{\exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{(n+1)}.\{d\} \mid d \in \Delta\}$ and $\mathsf{trans}_\Delta(\mathcal{K})' \nsubseteq \mathcal{T}$. Then, $r_1 \circ \ldots \circ r_n \sqsubseteq r_{(n+1)} \in \mathcal{K}$. Since $\mathcal{I} \models_\Delta \mathcal{K}$, then $\mathcal{I} \models_\Delta r_1 \circ \ldots \circ r_n \sqsubseteq r_{(n+1)}$. It holds that $r_1^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \subseteq r_{n+1}^{\mathcal{I}}$. Let $x, d \in \Delta$ with $x \in (\exists r_1 \ldots \exists r_n.\{d\})^{\mathcal{I}}$. Then, there exist $x_1, \ldots, x_n \in \Delta$ such that $(x, x_1) \in r_1^{\mathcal{I}}, (x_i, x_{(i+1)}) \in r_i^{\mathcal{I}}, (x_{(n-2)}, x_{(n-1)}) \in r_{(n-1)}^{\mathcal{I}}$ and $x_{(n-1)} \in (\exists r_n.\{d\})^{\mathcal{I}}$. Then, there exists $y \in \Delta$ such that $(x_{(n-1)}, y) \in r_n^{\mathcal{I}}$ with $y = d$. Since $r_1^{\mathcal{I}} \circ \ldots \circ r_n^{\mathcal{I}} \subseteq r_{(n+1)}^{\mathcal{I}}$, we also have $(x,d) \in r_{(n+1)}^{\mathcal{I}}$. From the definition of the semantics, we have $x \in (\exists r_{(n+1)}.\{d\})^{\mathcal{I}}$. We have $(\exists r_1 \ldots \exists r_n.\{d\})^{\mathcal{I}} \subseteq (\exists r_{(n+1)}.\{d\})^{\mathcal{I}}$ and $\mathcal{I} \models_\Delta \exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{n+1}.\{d\}$ for any $d \in \Delta$. Hence, $\mathcal{I} \models_\Delta \alpha_{tr}$ for any $\alpha_{tr} \in \{\exists r_1 \ldots \exists r_n.\{d\} \sqsubseteq \exists r_{(n+1)}.\{d\} \mid d \in \Delta\}$ as desired.

(4) $\mathsf{trans}_\Delta(\mathcal{K})' = \{(\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot \mid d \in \Delta\}$ and $\mathsf{trans}_\Delta(\mathcal{K})' \nsubseteq \mathcal{T}$. Then, $Dis(r,s) \in \mathcal{K}$. Since $\mathcal{I} \models_\Delta \mathcal{K}$, then $\mathcal{I} \models_\Delta Dis(r,s)$. It holds that $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$. Then, there is no $(x,y) \in \Delta \times \Delta$, such that $(x,y) \in r^{\mathcal{I}}$ and $(x,y) \in s^{\mathcal{I}}$. Now assume for a contradiction that $(\exists r.\{d\})^{\mathcal{I}} \cap (\exists s.\{d\})^{\mathcal{I}} \neq \emptyset$. Then, there exists $x \in \Delta$ such that $x \in (\exists r.\{d\})^{\mathcal{I}}$ and $x \in (\exists s.\{d\})^{\mathcal{I}}$. From semantic definition, there exists a $y \in \Delta$ such that $(x,y) \in r^{\mathcal{I}}$ with $y = d$ and there exists $z \in \Delta$ such that $(x,z) \in s^{\mathcal{I}}$ with $z = d$. We have $(x,d) \in r^{\mathcal{I}}$

and $(x,d) \in s^{\mathcal{I}}$, which is a contradiction. Then, we have $(\exists r.\{d\})^{\mathcal{I}} \cap (\exists s.\{d\})^{\mathcal{I}} = \emptyset$. Therefore, $\mathcal{I} \models_{\Delta} (\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot$ for any $d \in \Delta$ and hence $\mathcal{I} \models_{\Delta} \alpha_{tr}$ for any $\alpha_{tr} \in \{(\exists r.\{d\}) \sqcap (\exists s.\{d\}) \sqsubseteq \bot \mid d \in \Delta\}$.

From the proof of the both directions above, we finally have $\mathsf{trans}(\mathcal{K}) \equiv_{\Delta} \mathcal{K}$. □

While preserving the semantics of the original KB $\mathcal{K}$, one might notice that the new KB $\mathsf{trans}_{\Delta}(\mathcal{K})$ is "bigger" than $\mathcal{K}$. Let $n_{\mathcal{K}}$ be the size of some KB $\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$. Since $\mathsf{trans}_{\Delta}(\alpha)$ produces the same axiom for each axiom $\alpha \in \mathcal{A}$ or $\alpha \in \mathcal{T}$, the size of the transformed ABox and TBox are equal to the size of the original ABox and TBox in $\mathcal{K}$. For an RBox axiom $\alpha \in \mathcal{R}$, $\mathsf{trans}_{\Delta}(\alpha)$ generates $|\Delta|$ number of transformed axioms. Then, the size of $\mathsf{trans}_{\Delta}(\mathcal{K})$ is linearly bounded by $n_{\mathcal{K}} \times |\Delta|$.

Given the knowledge base is in the transformed form, now we are ready to weaken the axioms in the knowledge base.

Definition 4.12 (Weakened knowledge base). *Let Δ be a fixed domain and $\mathcal{L}_{\Delta}$ be a set of possible axioms in DL under fixed-domain semantics, $\mathcal{K}$ be a transformed knowledge base, C, D be any two concept names, r be a role name, and $\Delta' = \{a_1, \ldots, a_n\}$ be a set of individual elements, i.e. $\Delta' \subseteq \Delta$. Consider an axiom $\sigma \in \mathcal{K}$:*

(1) If σ is a general concept inclusion $C \sqsubseteq D$, then the weakened GCI $\sigma^{-\Delta'}$ w.r.t Δ' is $C \sqcap \neg\{a_1\} \sqcap \ldots \sqcap \neg\{a_n\} \sqsubseteq D$.

(2) If σ is a concept assertion $C(a_i)$, then the weakened concept assertion $\sigma^{-\Delta'}$ w.r.t Δ' is $\top(a_i)$ if $a_i \in \Delta'$ and $C(a_i)$ otherwise.

(3) If σ is a role assertion $r(a,b)$, then the weakened role assertion $\sigma^{-\Delta'}$ w.r.t Δ' is $u(a,b)$ if $a \in \Delta'$, and $r(a,b)$ otherwise. The same rule also applies for any inverse role assertion $r^{-}(a,b)$.

The weakened knowledge base $\mathcal{K}^{-\Delta'}$ *of $\mathcal{K}$ w.r.t Δ' is $\mathcal{K}^{-\Delta'} = \{\sigma^{-\Delta'} \mid \sigma \in \mathcal{K}\}$, i.e. the set of all weakened axioms of $\mathcal{K}$.*

Definition 4.12 describes the way to weaken any axiom in a KB $\mathcal{K}$ given the set $\Delta' \subseteq \Delta$ of individuals. We note that our definition of weakening is syntax-dependent. For two semantically equivalent knowledge bases, the weakening process might produce two non-equivalent results, even if we weaken both knowledge bases based on the exact same individuals. For instance, let $\Delta = \{c, d\}, \mathcal{K}_1 = \{A \sqsubseteq \forall r.B, A(c), r(c,d)\}$ and $\mathcal{K}_2 = \{\exists r^{-}.A \sqsubseteq B, A(c), r^{-}(d,c)\}$. It can be checked that $\mathcal{K}_1 \equiv_{\Delta} \mathcal{K}_2$. Suppose we weaken the two KBs w.r.t. $\Delta' = \{c\}$, then the results are $\mathcal{K}_1^{-\Delta'} = \{A \sqcap \neg\{c\} \sqsubseteq \forall r.B, \top(c), u(c,d)\}$ and $\mathcal{K}_2^{-\Delta'} = \{\exists r^{-}.A \sqcap \neg\{c\} \sqsubseteq B, \top(c), r^{-}(d,c)\}$. Consider a Δ-interpretation $\mathcal{I}$ such that $A^{\mathcal{I}} = \{c, d\}, B^{\mathcal{I}} = \{c\}$, and $r^{\mathcal{I}} = \{(c,d)\}$. We observe that $\mathcal{I}$ is a model of $\mathcal{K}_1^{-\Delta'}$, but it is not a model of $\mathcal{K}_2^{-\Delta'}$. This shows that $\mathcal{K}_1^{-\Delta'}$ and $\mathcal{K}_2^{-\Delta'}$ are not semantically equivalent. Next, we proceed by defining the exceptional individual set as follows.

Definition 4.13 (Exceptional individual set). *Let $\mathcal{K}$ and $\mathcal{K}'$ be two knowledge bases.* A set of exceptional individuals w.r.t $\mathcal{K}$ and $\mathcal{K}'$ *is a set $Exc \subseteq \Delta$ such that $\mathcal{K}^{-Exc} \cup \mathcal{K}'$ is consistent. We use $\mathcal{E}(\mathcal{K},\mathcal{K}')$ to denote the set of all sets of exceptional individuals w.r.t $\mathcal{K}$ and $\mathcal{K}'$.*

The following lemma shows that an exceptional individual set always exists w.r.t any two consistent knowledge bases.

Lemma 4.14. *Let $\Delta = \{a_1,\ldots,a_n\}$ be a set of fixed-domain elements. For any two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$ which are consistent and in the transformed forms (w.l.o.g), we have $\mathcal{E}(\mathcal{K},\mathcal{K}') \neq \emptyset$.*

Proof. We can always pick $Exc = \Delta$ (i.e. all domain elements to be exceptional individuals) to satisfy the consistency requirement. Then, each axiom $\sigma^{-\Delta} \in \mathcal{K}^{-\Delta}$ is in the form of one of the following:

(1) $\top(a)$ for any $C(a) \in \mathcal{K}$.

(2) $u(a_i, a_j)$ for any $r(a_i, a_j) \in \mathcal{K}$.

(3) $C \sqcap \neg\{a_1\} \sqcap \ldots \sqcap \neg\{a_n\} \sqsubseteq D$ for any $C \sqsubseteq D \in \mathcal{K}$.

Since $\mathcal{K}'$ is consistent, there exists some $\mathcal{I}' \models_\Delta \mathcal{K}'$. We show that $\mathcal{K}^{-\Delta} \cup \mathcal{K}'$ is consistent by showing that $\mathcal{I}'$ also satisfies each axiom $\sigma^{-\Delta} \in \mathcal{K}^{-\Delta}$. We consider several cases based on the form of $\sigma^{-\Delta}$:

(1) $\sigma^{-\Delta} = \top(a)$. Since any $\mathcal{I}$ satisfies $\top(a)$, it follows that $\mathcal{I}' \models_\Delta \top(a)$.

(2) $\sigma^{-\Delta} = u(a_i, a_j) \in \mathcal{K}^{-\Delta}$. Since any $\mathcal{I}$ satisfies $u(a_i, a_j)$, it follows that $\mathcal{I}' \models_\Delta u(a_i, a_j)$.

(3) $\sigma^{-\Delta} = C \sqcap \neg\{a_1\} \sqcap \ldots \sqcap \neg\{a_n\} \sqsubseteq D$. From the definition of the semantics, $(C \sqcap \neg\{a_1\} \sqcap \ldots \sqcap \neg\{a_n\})^{\mathcal{I}}$ $= C^{\mathcal{I}} \cap \Delta \setminus \{a_1\} \cap \ldots \cap \Delta \setminus \{a_n\} = C^{\mathcal{I}} \cap \emptyset = \emptyset$ for any $\mathcal{I}$. As $\emptyset \subseteq D^{\mathcal{I}}$ for any concept $D \in N_C$ and any $\mathcal{I}$, we obtain $\mathcal{I}' \models_\Delta C \sqcap \neg\{a_1\} \sqcap \ldots \sqcap \neg\{a_n\} \sqsubseteq D$.

Since $\mathcal{I}' \models_\Delta \mathcal{K}'$ and $\mathcal{I}' \models_\Delta \sigma^{-\Delta}$ for each axiom $\sigma^{-\Delta} \in \mathcal{K}^{-\Delta}$, we have $\mathcal{I}' \models_\Delta \mathcal{K}^{-\Delta}$ and hence $\mathcal{K}^{-\Delta} \cup \mathcal{K}'$ is consistent. □

We show that our exceptional-individual-based weakening is monotonic in terms of Δ-entailment between two weakened knowledge bases.

Lemma 4.15. *Let $\mathcal{K}$ be a knowledge base that is consistent and w.l.o.g in a transformed form. Let $\Delta_1, \Delta_2 \subseteq \Delta$ be two sets of individuals. If $\Delta_1 \subseteq \Delta_2$, then $\mathcal{K}^{-\Delta_1} \models_\Delta \mathcal{K}^{-\Delta_2}$.*

Proof. We show that for each $\mathcal{J} \models_\Delta \mathcal{K}^{-\Delta_1}$, it also holds $\mathcal{J} \models_\Delta \beta$ for each axiom $\beta \in \mathcal{K}^{-\Delta_2}$. Let $\Delta_1 \subseteq \Delta_2$ and $\mathcal{J} \models_\Delta \mathcal{K}^{-\Delta_1}$. Then, $\mathcal{J} \models_\Delta \alpha$ for every axiom $\alpha \in \mathcal{K}^{-\Delta_1}$. We consider several cases of β:

(1) $\beta = C \sqcap \neg\{b_1\} \sqcap \ldots \sqcap \neg\{b_n\} \sqsubseteq D$, where $\{b_1, \ldots, b_n\} = \Delta_2$. Then, we have axiom $C \sqsubseteq D \in \mathcal{K}$. Since $\Delta_1 \subseteq \Delta_2$, we have $\alpha = C \sqcap \neg\{b_i\} \sqcap \ldots \sqcap \neg\{b_j\} \sqsubseteq D \in \mathcal{K}^{-\Delta_1}$, for some $b_i, \ldots, b_j \in \Delta_2$ with $1 \leq i < j \leq n$. Since $\mathcal{J} \models_\Delta \alpha$, it holds that $C^{\mathcal{J}} \cap \Delta \setminus \{b_i\} \cap \ldots \cap \Delta \setminus \{b_j\} \subseteq D^{\mathcal{J}}$. Let $x \in \Delta$ such that $x \in (C^{\mathcal{J}} \cap \Delta \setminus \{b_1\} \cap \ldots \cap \Delta \setminus \{b_n\})$. Then, $x \in C^{\mathcal{J}}$ and $x \in \Delta \setminus \{b_k\}$ for all $k \in \{1, \ldots, n\}$. In particular, we have $x \in (C^{\mathcal{J}} \cap \Delta \setminus \{b_i\} \cap \ldots \cap \Delta \setminus \{b_j\})$ with $1 \leq i < j \leq n$. Since $(C^{\mathcal{J}} \cap \Delta \setminus \{b_i\} \cap \ldots \cap \Delta \setminus \{b_j\}) \subseteq D^{\mathcal{J}}$, we also have $x \in D^{\mathcal{J}}$. Hence, $(C^{\mathcal{J}} \cap \Delta \setminus \{b_1\} \cap \ldots \cap \Delta \setminus \{b_n\}) \subseteq D^{\mathcal{J}}$ and $\mathcal{J} \models_\Delta \beta$.

(2) β is a concept assertion. We consider subcases $\beta = C(a)$ or $\beta = \top(a)$:

  (2.1) $\beta = C(a)$. Then, $a \notin \Delta_2$. We have $C(a) \in \mathcal{K}$. Since $\Delta_1 \subseteq \Delta_2$ and $a \notin \Delta_2$, we have $a \notin \Delta_1$. Then, $\alpha = C(a) \in \mathcal{K}^{-\Delta_1}$. Since $\mathcal{J} \models_\Delta \alpha$ and $\beta = \alpha$, we have $\mathcal{J} \models_\Delta \beta$.

  (2.2) $\beta = \top(a)$. Then, $\mathcal{J} \models_\Delta \beta$ holds since any $\mathcal{I}$ satisfies $\top(a)$.

(3) β is a role assertion. We consider subcases $\beta = r(a, b)$ or $\beta = u(a, b)$:

  (3.1) $\beta = r(a, b)$. Then, both $a, b \notin \Delta_2$. We have $r(a, b) \in \mathcal{K}$. Since $\Delta_1 \subseteq \Delta_2$ and $a, b \notin \Delta_2$, we have $a, b \notin \Delta_1$. Then, $\alpha = r(a, b) \in \mathcal{K}^{-\Delta_1}$. Since $\mathcal{J} \models_\Delta \alpha$ and $\beta = \alpha$, we have $\mathcal{J} \models_\Delta \beta$.

  (3.2) $\beta = u(a, b)$. Then, $\mathcal{J} \models_\Delta \beta$ holds since any $\mathcal{I}$ satisfies $u(a, b)$.

Since for any $\mathcal{J} \models_\Delta \mathcal{K}^{-\Delta_1}$ we have that $\mathcal{J} \models_\Delta \beta$ for any $\beta \in \mathcal{K}^{-\Delta_2}$, it follows that $\mathcal{J} \models_\Delta \mathcal{K}^{-\Delta_2}$. Hence, $\mathcal{K}^{-\Delta_1} \models_\Delta \mathcal{K}^{-\Delta_2}$ as desired. □

Using the notion of the exceptional individual set, we present the individual-based revision operator for any two knowledge bases under the fixed-domain semantics. Whenever the incoming KB is inconsistent with the prior KB, the operator chooses one of the minimal exceptional individual sets so that the weakened prior KB is consistent with the incoming one.

Definition 4.16 (Individual-based Revision). *Let $\mathcal{K}$ and $\mathcal{K}'$ be two knowledge bases and $\mathcal{E}(\mathcal{K}, \mathcal{K}')$ be the set of exceptional individual sets.* An individual-based revision operator *is a revision operator $\circ^\pi_\Delta$ such that for any knowledge base $\mathcal{K}$ and $\mathcal{K}'$:*

$$\mathcal{K} \circ^\pi_\Delta \mathcal{K}' = \begin{cases} \mathsf{trans}_\Delta(\mathcal{K})^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}' & \text{if } \mathcal{K}' \text{ is consistent,} \\ \mathcal{K}' & \text{otherwise,} \end{cases}$$

where $\pi : \mathcal{P}(\mathcal{P}(\Delta)) \to \mathcal{P}(\Delta)$ is a selection function such that: (1) π retrieves subset-minimal elements, i.e. $\pi(\mathcal{X}) \in \mathcal{X}$ and there is no $Y \in \mathcal{X}$ such that $Y \subset \pi(\mathcal{X})$, and (2) for every $\mathcal{K}''$ with $\mathcal{K}'' \models_\Delta \mathcal{K}'$, $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}'')) \subseteq \pi(\mathcal{E}(\mathcal{K}, \mathcal{K}'))$.

The result of the revision $\mathcal{K} \circ^\pi_\Delta \mathcal{K}'$ is linearly bigger than the inputs $\mathcal{K}$ and $\mathcal{K}'$. The only size change is for the prior KB $\mathcal{K}$ (i.e. to be transformed and weakened), while $\mathcal{K}'$ remains unchanged. In the weakening process, the size of the axioms changes only whenever the GCIs are weakened. As $\pi(\mathcal{E}(\mathcal{K},\mathcal{K}')) \subseteq \Delta$, every GCI weakening adds at most n_Δ negated nominal concepts which represent exceptional individuals, where $n_\Delta = |\Delta|$. Hence, the size growth from $\mathsf{trans}_\Delta(\mathcal{K})$ to $\mathsf{trans}_\Delta(\mathcal{K})^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))}$ is only linearly bounded by n_Δ. Overall, in the worst case scenario, when we revise an arbitrary knowledge base $\mathcal{K}$ (with the size of $n_\mathcal{K}$) by some KB $\mathcal{K}'$ (with the size of $n_{\mathcal{K}'}$), the result of the revision $\mathcal{K} \circ^\pi_\Delta \mathcal{K}'$ has the size of $(n_\mathcal{K} \times n^2_\Delta) + n_{\mathcal{K}'}$. Note that n^2_Δ comes from transformation and weakening procedures.

This individual-based revision operator works on the syntactic level by weakening the axioms of the original knowledge base. Recall that the weakening process is syntax dependent, this revision operation also depends on the syntax of the knowledge base. For two knowledge bases which are semantically equivalent but syntactically different, there is no guarantee that the revision would result in two equivalent weakened knowledge bases. For instance, assume we have $\Delta = \{c,d\}$ and two equivalent knowledge bases as previously defined $\mathcal{K}_1 = \{A \sqsubseteq \forall r.B, A(c), r(c,d)\}$ and $\mathcal{K}_2 = \{\exists r^-.A \sqsubseteq B, A(c), r^-(d,c)\}$. Suppose we want to revise each $\mathcal{K}_1$ and $\mathcal{K}_2$ by an incoming KB $\mathcal{K}_3 = \{\neg B(d)\}$. Since both unions $\mathcal{K}_1 \cup \mathcal{K}_3$ and $\mathcal{K}_2 \cup \mathcal{K}_3$ are inconsistent, we search for the minimal set of exceptional individuals that would make the weakened version of the two prior KBs consistent with $\mathcal{K}_3$. Then, we find $\pi(\mathcal{E}(\mathcal{K}_1,\mathcal{K}_3)) = \{c\}$ and $\pi(\mathcal{E}(\mathcal{K}_2,\mathcal{K}_3)) = \{d\}$. The result of the revision $\mathcal{K}_1 \circ^\pi_\Delta \mathcal{K}_3 = \{A \sqcap \neg\{c\} \sqsubseteq \forall r.B, \neg B(d)\}$, while for the other one $\mathcal{K}_2 \circ^\pi_\Delta \mathcal{K}_3 = \{\exists r^-.A \sqcap \neg\{d\} \sqsubseteq B, A(c), \neg B(d)\}$. Hence, we observe that $\mathcal{K}_1 \circ^\pi_\Delta \mathcal{K}_3 \not\equiv_\Delta \mathcal{K}_2 \circ^\pi_\Delta \mathcal{K}_3$. These observation can be considered a counter example to show that the revision operator $\circ^\pi_\Delta$ fails to satisfy postulate (G4) which guarantees the irrelevance of syntax principle. For the satisfaction of the five remaining postulates, the following proposition shows positive results.

Proposition 4.17. *The individual-based change operator $\circ^\pi_\Delta$ satisfies postulates (G1)–(G3), (G5), and (G6).*

Proof. For inconsistent $\mathcal{K}'$, (G1-G3) are immediately satisfied since $[\![\mathcal{K} \circ^\pi_\Delta \mathcal{K}']\!]_\Delta = [\![\mathcal{K}']\!]_\Delta = [\![\mathcal{K} \cup \mathcal{K}']\!]_\Delta = \emptyset$. For (G5-G6), since $\mathcal{K}'$ is inconsistent, we have $[\![\mathcal{K} \circ^\pi_\Delta \mathcal{K}' \cup \mathcal{K}'']\!]_\Delta = \emptyset = [\![\mathcal{K} \circ^\pi_\Delta (\mathcal{K}' \cup \mathcal{K}'')]\!]_\Delta$. Now we assume $\mathcal{K}'$ is consistent.

(G1). Let $\mathcal{I} \in [\![\mathcal{K} \circ^\pi_\Delta \mathcal{K}']\!]_\Delta$. By Definition 4.16, we have $\mathcal{I} \in [\![\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}']\!]_\Delta$. This means $\mathcal{I} \in [\![\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))}]\!]_\Delta$ and $\mathcal{I} \in [\![\mathcal{K}']\!]_\Delta$. Therefore, $\mathcal{K} \circ^\pi_\Delta \mathcal{K}' \models_\Delta \mathcal{K}'$.

(G2). Let $\mathcal{K} \cup \mathcal{K}'$ be consistent. Then, $\emptyset \in \mathcal{E}(\mathcal{K}, \mathcal{K}')$. Consequently, $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}')) = \emptyset$ and hence $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \equiv \mathcal{K}$. Therefore, $[\![\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}']\!]_\Delta = [\![\mathcal{K} \cup \mathcal{K}']\!]_\Delta$ and we have $\mathcal{K} \circ_\Delta \mathcal{K}' \equiv \mathcal{K} \cup \mathcal{K}'$.

(G3). Since $\mathcal{K}'$ is consistent, from Definition 4.16, we have that $\mathcal{K} \circ^\pi_\Delta \mathcal{K}' = \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}'$. From the definition of of exceptional individual set (chosen by π), $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}'$ is consistent, i.e. $[\![\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}']\!]_\Delta \neq \emptyset$.

(G5–G6). For (G5) and (G6), assume $(\mathcal{K} \circ^\pi_\Delta \mathcal{K}') \cup \mathcal{K}''$ is consistent. By the definition 4.16 and the associativity of the $\cup$ operator, we have that $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}' \cup \mathcal{K}''$ is also consistent. By the associativity of the $\cup$ operator, we can see the knowledge base as $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup (\mathcal{K}' \cup \mathcal{K}'')$. Since the knowledge base is consistent, we have $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}')) \in \mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')$. Now consider $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')) \in \mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')$. From Condition (2) of π, we have $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')) \subseteq \pi(\mathcal{E}(\mathcal{K}, \mathcal{K}'))$. From Lemma 4.15, it holds $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}' \cup \mathcal{K}''))} \models_\Delta \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))}$. From Defini-tion 4.13, we also have $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}' \cup \mathcal{K}''))} \cup \mathcal{K}' \cup \mathcal{K}''$ is consistent. Then, $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')) \in \mathcal{E}(\mathcal{K}, \mathcal{K}')$. As $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}'))$ is a subset-minimal set and $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}'')) \subseteq \pi(\mathcal{E}(\mathcal{K}, \mathcal{K}'))$, we also have $\pi(\mathcal{E}(\mathcal{K}, \mathcal{K}')) \subseteq \pi(\mathcal{E}(\mathcal{K}, \mathcal{K}' \cup \mathcal{K}''))$. Again by Lemma 4.15, we have $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \models_\Delta \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}' \cup \mathcal{K}''))}$. Hence, $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \equiv_\Delta \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}' \cup \mathcal{K}''))}$. We have $\mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}' \cup \mathcal{K}'' \equiv \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}' \cup \mathcal{K}''))} \cup \mathcal{K}' \cup \mathcal{K}''$. Therefore, by Definition 4.16, we have $(\mathcal{K} \circ^\pi_\Delta \mathcal{K}') \cup \mathcal{K}'' \equiv \mathcal{K} \circ^\pi_\Delta (\mathcal{K}' \cup \mathcal{K}'')$. $\square$

4.4 Individual-Based Approach via ASP Encoding

In the following, we introduce an ASP encoding for the individual-based revision approach described in the previous section. Given a fixed domain Δ and two knowledge bases $\mathcal{K} = (\mathcal{T}, \mathcal{A}, \mathcal{R})$ and $\mathcal{K}' = (\mathcal{T}', \mathcal{A}', \mathcal{R}')$, the goal of this encoding is to find the minimal set of exceptional individuals $\Delta' \subseteq \Delta$ such that if we weaken the knowledge base $\mathcal{K}$ based on those set, i.e. $\mathcal{K}^{-\Delta'}$, we have that $\mathcal{K}^{-\Delta'} \cup \mathcal{K}'$ is consistent. We begin by generating the set of all possible sets of exceptional individuals ($\mathcal{E}(\mathcal{K}, \mathcal{K}')$) using the interval function from `Clingo` of the form $1..|\Delta|$. As the interval function occurs in the body, it is expanded disjunctively in the process of grounding, providing all possible sets of the individuals. We capture all generated elements as exceptional individuals $exc(X)$.

$$\Pi_{exc}(\mathcal{K} \cup \mathcal{K}', \Delta) = \{exc(X) \leftarrow X = 1..|\Delta|\}. \tag{4.25}$$

As in the model-based revision encoding, we generate all possible interpretations based on the signature of the union of both knowledge bases.

$$
\begin{aligned}
\Pi_{gen}(\mathcal{K}\cup\mathcal{K}',\Delta) = \quad & \{A(X) \leftarrow \mathit{not}\ \neg A(X), thing(X) \mid A \in N_C(\mathcal{K}\cup\mathcal{K}')\}\cup \\
& \{\neg A(X) \leftarrow \mathit{not}\ A(X), thing(X) \mid A \in N_C(\mathcal{K}\cup\mathcal{K}')\}\cup \\
& \{r(X,Y) \leftarrow \mathit{not}\ \neg r(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K}\cup\mathcal{K}')\}\cup \\
& \{\neg r(X,Y) \leftarrow \mathit{not}\ r(X,Y), thing(X), thing(Y) \mid r \in N_R(\mathcal{K}\cup\mathcal{K}')\}\cup \\
& \{thing(i) \mid \delta_i \in \Delta\}.
\end{aligned}
\tag{4.26}
$$

Next, for each answer set representing an interpretation, we search for a set of exceptional individuals within the answer set. The rule for the TBox is similar to the constraint we used for a violation checking in the model-based revision encoding, except that we add an atom $exc(X)$ in the body. Intuitively, if an individual is involved in a violation of some axiom, it is marked as an exceptional individual. Note that we do this treatment only for the TBox from the prior knowledge base $\mathcal{K}$ as our aim is to weaken $\mathcal{K}$ only.

$$
\begin{aligned}
\Pi_{chk}(\mathcal{T},\mathcal{T}',\Delta) = \quad & \{\leftarrow \text{not}\ exc(X), trans(C_1), ..., trans(C_n) \mid \top \sqsubseteq \bigsqcup_{i=1}^{n} C_i \in \mathcal{T}\}\cup \\
& \{\leftarrow trans(C_1), ..., trans(C_n) \mid \top \sqsubseteq \bigsqcup_{i=1}^{n} C_i \in \mathcal{T}'\}.
\end{aligned}
\tag{4.27}
$$

We encode the checking rule for the ABox as the same as in model-based revision encoding. Every individual assertion $C(a)$ and every role assertion $r(a,b)$ in the ABox is encoded in the program as a fact.

$$
\begin{aligned}
\Pi_{chk}(\mathcal{A}\cup\mathcal{A}',\Delta) = \quad & \{A(a_i) \mid A(a_i) \in \mathcal{A}\cup\mathcal{A}'\}\cup \\
& \{\neg A(a_i) \mid \neg A(a_i) \in \mathcal{A}\cup\mathcal{A}'\}\cup \\
& \{r(a_i,b_j) \mid r(a_i,b_j) \in \mathcal{A}\cup\mathcal{A}'\}\cup \\
& \{\neg r(a_i,b_j) \mid \neg r(a_i,b_j) \in \mathcal{A}\cup\mathcal{A}'\}.
\end{aligned}
\tag{4.28}
$$

For the RBox, we also add atom $exc(X)$ to capture the exceptional individuals violating the RBox axioms of the knowledge base $\mathcal{K}$. Again, we consider the RBox axioms in the form of $r \sqsubseteq s$, $r_1 \circ r_2 \sqsubseteq r_3$, or $Dis(r,s)$ for both $\mathcal{K}$ and $\mathcal{K}'$.

$$
\begin{aligned}
\Pi_{chk}(\mathcal{R},\mathcal{R}',\Delta) = \quad & \{\leftarrow \text{not } exc(X), \text{not } exc(Y), r(X,Y), s(X,Y) \mid \text{Dis}(r,s) \in \mathcal{R}\} \cup \\
& \{\leftarrow \text{not } exc(X), \text{not } exc(Y), r(X,Y), not\_s(X,Y) \mid r \sqsubseteq s \in \mathcal{R}\} \cup \\
& \{\leftarrow \text{not } exc(X), \text{not } exc(Y), s_1(X,Y), s_2(Y,Z), not\_r(X,Z) \mid \\
& s_1 \circ s_2 \sqsubseteq r \in \mathcal{R}\} \cup \\
& \{\leftarrow r(X,Y), s(X,Y) \mid \text{Dis}(r,s) \in \mathcal{R}'\} \cup \\
& \{\leftarrow r(X,Y), not\_s(X,Y) \mid r \sqsubseteq s \in \mathcal{R}'\} \cup \\
& \{\leftarrow s_1(X,Y), s_2(Y,Z), not\_r(X,Z) \mid s_1 \circ s_2 \sqsubseteq r \in \mathcal{R}'\}.
\end{aligned}
\tag{4.29}
$$

We summarize the axioms encoding from the knowledge base $\mathcal{K} \cup \mathcal{K}'$ as follows:

$$
\Pi_{chk}(\mathcal{K},\mathcal{K}',\Delta) = \Pi_{chk}(\mathcal{T},\mathcal{T}',\Delta) \cup \Pi_{chk}(\mathcal{A} \cup \mathcal{A}',\Delta) \cup \Pi_{chk}(\mathcal{R},\mathcal{R}',\Delta) \tag{4.30}
$$

Given two knowledge bases $\mathcal{K}$ and $\mathcal{K}'$, the complete encoding of the individual-based revision approach is the following program:

$$
\Pi(\mathcal{K} \circ^{\pi}_{\Delta} \mathcal{K}',\Delta) = \Pi_{exc}(\mathcal{K} \cup \mathcal{K}',\Delta) \cup \Pi_{gen}(\mathcal{K} \cup \mathcal{K}',\Delta) \cup \Pi_{chk}(\mathcal{K},\mathcal{K}',\Delta). \tag{4.31}
$$

Given the set $\mathcal{AS}(\Pi(\mathcal{K} \circ^{\pi}_{\Delta} \mathcal{K}',\Delta))$, we obtain the answer sets containing many exceptional individual set possibilities to weaken the KB $\mathcal{K}$ (i.e. the set $\mathcal{E}(\mathcal{K},\mathcal{K}')$). Remember that we want to only retrieve the answer sets where the exceptional individual sets are (subset) minimal. We use the *heuristic* feature from `Clingo` to obtain the subset minimal answer sets. The application of this feature requires the additional statement below to be appended in the logic program. To activate the heuristic, we call a particular command option when running the `Clingo` program on the command line.

$$
\{\#\textit{heuristic } exc(X).\ [1, \textit{false}]\ \}
$$

The directive `#heuristic` represents that the statement is a heuristic program to be activated when the solver is running. The statement "`exc(X). [1,false]`" assigns modifier `false` with value 1 to the atom $exc(X)$ to minimize. We employ the line option "`--heuristic=Domain`" when executing `Clingo` to run the whole program (i.e. "`clingo --heuristic=Domain file.pl`"). Note that since it is possible to have many subset-minimal answer sets, the final choice for KB weakening (i.e. the set $\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))$) is left to user preference. Finally, the individual-based revision result can be obtained by weakening $\mathcal{K}$ w.r.t Δ' as $\mathcal{K} \circ^{\pi}_{\Delta} \mathcal{K}' = \mathcal{K}^{-\pi(\mathcal{E}(\mathcal{K},\mathcal{K}'))} \cup \mathcal{K}'$.

We observe that the computation of individual-based approach via the ASP encoding is exponential in the size of individual elements in the domain and thus causes non-scalability in practical applications, i.e. the solution space is incomprehensible. In particular, as we could see in the ASP encoding earlier, the program requires to generate *all* possible exceptional individual sets before finding the appropriate one to be chosen as the underlying set for weakening process. Given n number of individuals, the program generates 2^n possible sets. To reduce the exponential magnitude, one can limit the number of individual elements for the generation step. Some individuals can be prioritized not to be exceptional. This could be achieved by setting a ranking or preference over the individuals in the domain by users. For example, in the ABox, suppose we want to maintain some concept or role assertions not to be weakened (or removed) from the knowledge base. Based on the individual ranking, those individuals occurred in the assertions would not be involved when generating exceptional individual sets. In other words, we only generate the solution space based on the other non-prioritized individuals.

Another way to help with the incomprehensibility issue is by applying a navigation framework for the answer sets, which was introduced by Fichte, Gaggl, and Rusovac [FGR22]. Using their approach, users are able to explore the solution space (e.g. all possible sets of exceptional individuals) by consciously zooming in or out of sub-solutions at a certain configurable pace. As we are only interested in a subset-minimal solutions, we can zoom in on particular exceptional individuals that still produce proper revision results, which leads to smaller sub-space of solutions. On the implementation side, this navigation framework employs an additional layer on top of `Clingo` solver, which supports the usability of our encoding-based revision practice.

4.5 Related Work

Syntax-based approaches for revision in DLs directly modify the axioms occuring in the knowledge base. The modification includes dropping a set of axioms [HK06b; RW09a; RW08; ZKN+19b] or by weakening axioms [QLB06a; AAB+18]. However, applying the original AGM postulates [AGM85] to a syntax-based approach for revision in DL is found to have a main issue: while AGM used axiom negation for their syntax-based revision construction, DL axioms are not closed under negation. Earlier approaches [HK06b; RW09a] implemented *semi-revision* in the DL family $\mathcal{SHOIN}$, where the *consistency* postulate (corresponding to (G3)) and the *success* postulate (corresponding to (G1)) can not be guaranteed simultaneously. Later, Ribeiro and Wasserman [RW09b; RW14b] introduced alternative constructions for revision in general negation-free logics. However, they did not consider postulates (G5) and (G6) in their representation theorem. Instead, they proposed some special postulates for base change inspired by Hansson [Han99], namely *core-retainment* and *relevance* to capture the minimal change principles. Our individual-based approach can be regarded as a syntax-based approach since the outcome of the revision is generated by axioms weakening.

Table 4.18: Overview of our approach and comparison with related work.

Approach	Class	DL setting	Postulates	Method for generating result
Qi et al. [QLB06a]	syntax-based	$\mathcal{ALC}$	(G1)-(G3), (G5), (G6)	axioms weakening
Aiguier et al. [AAB+18]		$\mathcal{ALC}$	(G1)-(G3), (G5), (G6)	axioms relaxation
Halaschek-Wiener et al. [HK06b]		$\mathcal{SHOIN}$	semi-revision postulates	axioms removal
Ribeiro and Wassermann [RW09a; RW08]		$\mathcal{SHOIN}(\mathcal{D})$; $\mathcal{SROIQ}$	semi-revision postulates	axioms removal
Zheleznyakov et al. [ZKN+19b]		DL-Lite	customized postulates	axioms removal
Our individual-based approach		$\mathcal{SROIQ}$; fixed-domain	(G1)-(G3), (G5), (G6)	axioms weakening
Wang et al. [WWT10; WWT15]	semantic-based	DL-Lite$^{N}_{bool}$	(G1)-(G5)	distance between features; approximation
Chang et al. [CSG14]		$\mathcal{EL}_{\perp}$	(G1)-(G5)	graph-based justification; axioms removal
Zhuang et al. [WWQ+14; ZWW+14]		DL-Lite$_{core}$	customized postulates	type-based axioms removal
Dong et al. [DDL17]		$\mathcal{SHIQ}$	customized postulates	distance between completion graphs; approximation
Our model-based approach		$\mathcal{SROIQ}$; fixed-domain	(G1)-(G6)	distance between models; direct axiom construction from models (c.f. Equation (4.1))

To deal with the possibility of infinitely many models in DL knowledge bases under standard semantics, many studies in semantic-based approaches [WWT10; CSG14; WWQ+14; WWT15; ZWW+14; DDL17] investigate alternative semantic characterizations for specific DL families. As a consequence, their model-based revision operators work with finitely many "characterized" interpretations. To address the inexpressibility problem, the notion of *a maximal approximation* was introduced to capture the revision result into a knowledge base [GLP+06; WWT15; DDL17; ZKN+19a]. A maximal approximation of a result of revision $\mathcal{K} \circ \mathcal{K}'$ is a new knowledge base $\mathcal{K}''$ such that $Mod(\mathcal{K} \circ \mathcal{K}') \subseteq Mod(\mathcal{K}'')$ and there is no other $\mathcal{K}^*$ with $Mod(\mathcal{K} \circ \mathcal{K}') \subseteq Mod(\mathcal{K}^*)$ and $Mod(\mathcal{K}') \subset Mod(\mathcal{K}^*)$. In our fixed-domain semantics setting, both above issues can be resolved naturally. The most plausible (the minimal) models can be computed as the interpretations are finite and the revised knowledge base can be obtained as these models can be expressed into axioms. Table 4.18 summarizes the related approaches and compares them with our model-based and individual-based approach.

4.6 Summary

We have presented two approaches for revising knowledge bases in Description Logics under the fixed-domain semantics, where the models of the knowledge bases are guaranteed to be finite. The approaches are: (1) a model-based approach and (2) an individual-based approach. For both revision approaches, we proposed an encoding into an ASP program to exhibit the possibility for practical implementations.

For our model-based approach, we provided an axiom construction from a given set of interpretations where the axiom's models are exactly the given interpretation set. We adapted the K&M's semantic approach and provided a representation theorem for the AGM revision operator in DL under fixed-domain semantics, as well as a concrete model-based revision operation using the notion of distance. To find the the revision result, we first compute the distances between models of the new information and the models of the prior knowledge base. The models of the new information which have least distance to the models of the initial knowledge are becoming the models of the revision outcome. Then, an axiom constructor is used to represent the minimal models into a knowledge base.

The second approach is a novel revision technique for this particular DL by axioms weakening based on exceptional individual sets. For the general concept inclusion axioms, weakenings are performed by expanding the left hand side concept expression of the subsumption with conjunctions of nominal concepts representing exceptional individuals. For the concept or role assertions, we replace the concept or role name with the top concept, or in other words, we drop the assertions from the knowledge base. The revision outcome is achieved by the union of the incoming knowledge base and the weakened prior knowledge base with regard to certain subset-minimal exceptional individual set. We showed that this revision approach does not satisfy one particular postulate (G4) and hence syntax-independent results from applying this method can not be guaranteed.

In the preliminaries, we proposed a DL axiom constructor from a given set of interpretations. Despite the constructor at the end serves our intention to express the revision models back into a knowledge base, the structure is arguably rather technical and unwieldy: it contains only one axiom with possibly "very big" concept expression on the right hand side of the subsumption. Finding a new axiom constructor from a given set of Δ-interpretations may be a viable path for the future.

Chapter 5

Conclusions and Outlook

5.1 Conclusions

In this thesis, we investigated AGM belief revision in general Tarskian logics from the semantic point of view. Tarskian logics are logics satisfying extensivity, idempotence, and monotonicity. This class of logics captures many well-known knowledge representation formalisms such as classical propositional logic, first-order logic, and description logics. In the preliminaries, we provided the proofs showing that these logics are equivalent to the logics endowed with a classical model-theoretical semantics. We aim to find appropriate semantic characterization for AGM revision operators in these logics. To achieve our main goal, we introduced a generic notion of base logic which is uniformly applicable to every Tarskian logic allowing union over bases without additional restriction on postulates side.

While the semantic characterization of AGM revision operators in finite-signature propositional logic has been well studied by Katsuno and Mendelson [KM91] via the notion of assignment, there remains several challenges to find an appropriate semantic representation theorem for arbitrary Tarskian logics. The identified problems are: (1) the possibility of knowledge bases to have infinitely many models, (2) the possibility of AGM revision operators coincide with non-transitive (hence not a preorder) assignment, and (3) the possibility of the minimal models as the revision outcome cannot be expressed into any base in the considered logic. To address those challenges, we provided a two-ways representation theorem which shows that in general Tarskian logics case, in addition to totality and faithfulness, now an assignment should also satisfy min-completeness, min-retractivity, and min-expressibility. We also established a syntax-dependent version of the theorem, where we showed that revision operators satisfying the AGM postulates except (G4) coincide with an assignment satisfying all above general conditions except the third condition of faithfulness.

We also investigated further to answer the question of what properties of logics where every AGM revision operator is compatible with some faithful assignment that only yields total preorders. We managed to identify a situation involving several bases in the logic that is called a critical loop. The absence of critical loop indicates that any AGM revision operator in the considered logic is guaranteed to be compatible with some preorder assignments.

In addition to the results for general logics, we also studied the AGM revision in a particular logic family that is called description logics under fixed-domain semantics. This logical setting enables us to have a direct computations over models (e.g. counting distance between models) and always provides us with some knowledge base from any given set of models. Although the logics are enjoying those nice properties, study on belief revision in this logical setting has not been carried out yet. We aimed to find a semantic revision approach and other potential methods to revise knowledge bases in these logics. Continuing from the general results, we started by presenting a semantic characterization of AGM revision operators in these logics, which showed that the representation theorem is similar to the one of Katsuno and Mendelzon via total preorder assignment. Using the notion of distance between interpretations, a concrete model-based revision operator was proposed and showed to satisfy all standard AGM postulates. In this model-based approach, we search the models of the incoming knowledge which have minimal distance to the models of the initial knowledge. Thanks to the finiteness of the domain elements, using axiom constructor introduced in the preliminaries, we could obtain a knowledge base as a revision outcome from the minimal models of the new information. An encoding of this approach in ASP was presented for a practical showcase.

As the axiom constructor is rather complex and might not be intuitive for human observers, we proposed an alternative approach to revise knowledge bases using the axioms weakening method so that the result can be relatively "more readable". We called this approach individual-based revision. The revision outcome is the union of the new information with weakened prior knowledge base. The weakening process is based on the exceptional individuals set that is subset-minimal over all set possibilities. As the revision process manipulates the axioms, this individual-based approach is categorized as a syntax-based approach. We provided a case where two semantically equivalent knowledge bases might produce non-equivalent results even though we revise both initial KBs by the exact same piece of information. To confirm this observation, we showed that the revision operators applying this approach satisfy all AGM postulates except (G4). We also presented an ASP encoding of this individual-based approach to support practical applications.

5.2 Outlook

Historically, the AGM theory departed from a belief change problem that is called belief *contraction*. In contracting a sentence α from the knowledge base $\mathcal{K}$, denoted by $\mathcal{K} - \alpha$, the agent no longer believes α (while not necessarily believing $\neg\alpha$). Formally, if α is not a tautology, then one requires that $\alpha \notin \mathcal{K} - \alpha$. Interestingly, in the classical AGM approach, the revision and contraction operation are interdefinable. Given a contraction operator $-$, a revision operator can be defined via Levi identity: $\mathcal{K} \circ \alpha = (\mathcal{K} - \neg\alpha) + \alpha$, where $+$ is called *expansion* operator (e.g. simply conjunction in propositional logic). Analogously, given a revision operator $\circ$, via Harper identity, a contraction operator can be defined as: $\mathcal{K} - \alpha = \mathcal{K} \cap (\mathcal{K} \circ \neg\alpha)$. Unfortunately, when we extend to general logics, both operators are independent of each others, e.g. description logics sentences (or axioms) are not closed under negation. As the whole thesis focuses only on revision, we think that working on finding a semantic characterization for contraction is an interesing direction to pursue.

The traditional AGM revision framework that is considered throughout this work focuses only on one-step transition and does not provide any further strategy for iterated revision. As one has observed in our semantic approach, we can obtain the corresponding assignment that is compatible with the AGM revision operator. For instance, we want to revise $\mathcal{K}$ by some base Γ and expect the revision outcome $\mathcal{K} \circ \Gamma$. Then, we can obtain the corresponding relation $\preceq_{\mathcal{K}}$ compatible with $\circ$. Suppose now we have another new information Γ' to be added into the first revision result $\mathcal{K} \circ \Gamma$. We expect that we can produce $\mathcal{K} \circ \Gamma \circ \Gamma'$ from what we already know ($\mathcal{K}$, Γ, and $\preceq_{\mathcal{K}}$). However, the previously obtained $\preceq_{\mathcal{K}}$ cannot be used again to have the desired result of the second revision. We need a new appropriate relation $\preceq_{\mathcal{K} \circ \Gamma}$ that is in some sense rationally related to the previous one. Unfortunately, the AGM postulates give us no clue about how to produce $\preceq_{\mathcal{K} \circ \Gamma}$. To address this issue, Darwiche and Pearl [DP97] introduced an additional set of postulates, called (DP1)–(DP4), to govern the iterated revision. They used a different structure from AGM framework: revision operators apply to *epistemic states* rather than to a belief base (or a set). Further investigation is needed for examining the general setting of base logics presented here in the line of the iterated base revision. Apart from that, in research on base revision, various special postulates for changing the bases have been considered, e.g. in the seminal research on belief base revision by Hansson [Han99]. For example, to capture the principle of minimal change, he proposed two additional postulates, namely *relevance* and *core-retainment*. It would be interesting to see how semantic approach in this work could characterize the special postulates of base revision.

In a wider perspective, we think that the semantic framework for revising beliefs can be instrumental in understanding phenomena such as different attitudes between entities in the society towards knowledge incorporation. Suppose we have two different entities which possess the same initial knowledge. Moreover, we assume that the two entities receive the same external input. It is not always the case that the two entities adjust their knowledge in the same way. This can happen because they both have a different 'change mechanism' – their revision operators are not necessarily the same. From the semantic point of view, for instance, the same incoming input possibly has different meaning or importance for the two parties. Consequently, both entities would have distinct ranking (or preference) towards the interpretations, which leads them to display different revision policies.

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