Emirates Journal for Engineering Research

Volume 27 | Issue 4 Article 4

12-20-2022

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(Received 30th November and Accepted December 20th, 2022)

الحلول العددية للمعادلات التفاضلية العادية غير الخطية الشاذة باستخدام متعددات حدود سعيد– بول

ملخص

في هذا البحث استخدمنا طريقة التجميع بالاعتماد على متعددات حدود سعيد– بول لإيجاد الحل العددي للمعادلات التفاضلية العادية غير الخطية الشاذة. حيث حصلنا على صيغ المصفوفة التشغيلية للحل والمشتقات المختلفة. هذه الطريقة حولت المسألة المعطاة الى نظام من المعادلات الجبرية غير الخطية والتي انهت شذوذ المعادلات التفاضلية العادية. تم حل هذا النظام غير الخطي بطريقة نيوتن للحصول على معاملات حدود متعددات حدود سعيد– بول ومن ثم حصلنا على الحلول التقريبية للمسألة قيد الدراسة. النتائج العددية التي حصلنا عليها أثبتت دقة وفعالية الطريقة المقترحة.

Abstract

The present article uses the collocation method based on Said-Ball polynomials to numerically solve the singular nonlinear ordinary differential equations of various orders. An operational matrix form of these ordinary differential equations is obtained from Said-Ball polynomial with variated solution relations and different derivatives. The presented method reduces the given problem to a nonlinear algebraic equation system, which removes the singularity of ordinary differential equations. The resulting system is solved using Newton's iteration method to get the coefficients of Said-Ball polynomials. We obtained approximate solutions of the problem under study. Numerical results have been obtained and compared with exact and other works. The presented method gives impressive solutions that show the accuracy and reliability of the proposed method.

Keywords: Said-Ball polynomial, collocation method, singular differential equations, operational matrix form.

1. Introduction

We will consider the general form of the singular nonlinear ordinary differential equations of order $n+1$ $(n \geq 1)$:

$$y^{(n+1)} + \frac{\lambda_1}{x} y^{(n)} + \frac{\lambda_2}{x^2} y^{(n-1)} + f(x,y) = 0, \qquad (1)$$

with initial conditions (ICs.)

$$y(0) = \alpha, y(0) = y'(0) = y''(0) \cdots = y^{(n)}(0) = 0, \qquad (2)$$

with λ_1, λ_2 and α are appropriate constants, f is given real values function of two variables x and y.

This type of differential equation with arbitrary values of components of Eq. (1) appears in various fields of science and engineering, for instance, quantum and fluid mechanics, geophysics, chemical reactors, optimal design and so on. See [1, 2, 3, 4, 5, 6] for more details.

Eq. (1) produces different types of famous equations. Some different order types of Emden-Fowler equations can be derived from the following relation:

$$x^{-l} \frac{d}{dx}\left[x^{l-1} \frac{d^k}{dx^k}(x\, y)\right] + f(x,y) = 0 \qquad (3)$$

Where $l, k \geq 1$ and l is called the shape factor.

For $k = 1, 2, 3, \cdots, n$ we obtain the Emden-Fowler equation of the first kind, the second kind, and the third kind up to $(n+1)th$ kind, respectively, as below:

$$y^{(2)} + \frac{l+1}{x} y^{(1)} + \frac{l-1}{x^2} y + f(x,y) = 0 \qquad (4)$$

$$y^{(3)} + \frac{l+2}{x} y^{(2)} + \frac{2(l-1)}{x^2} y^{(1)} + f(x,y) = 0 \qquad (5)$$

$$y^{(4)} + \frac{l+3}{x} y^{(3)} + \frac{3(l-1)}{x^2} y^{(2)} + f(x,y) = 0 \qquad (6)$$

$$\vdots$$

$$y^{(n+1)} + \frac{l+n}{x} y^{(n)} + \frac{n(l-1)}{x^2} y^{(n-1)} + f(x,y) = 0 \qquad (7)$$

Where $\lambda_1 = l + k$ and $\lambda_2 = k(l-1)$, $k = 1, 2, \cdots, n$.

If $n = 1$ and $\lambda_2 = 0$, then Eq. (1) becomes the Lane- Emden equation as the form:

$$y'' + \frac{\lambda_1}{x} y' + f(x,y) = 0 \qquad (8)$$

The standard Lane- Emden equation is produced from Eq. (8) when $f(x,y) = g(y)$. For some appropriate fixed values of λ_1 and $g(y)$, Eq. (8) models many mathematical physics and astrophysics phenomena. Some details you can find in [7-17].

The difficulty in solving these types of equations is in the singularity at $x = 0$.

Various methods are used to solve the Emden - Fowler equations and the Lane- Emden equations numerically and analytically.

Some of these methods, Adomian decomposition method [5, 18, 19, 20, 21],the homotopy analysis method [22, 23, 24],the variational iteration method [25, 26], for more different methods, see [27, 28, 29, 30, 31, 32, 33, 34, 35]. Lastly, Gümgüm [36] used Taylor wavelet method to solve linear and nonlinear Lane-Emden equations and modified Hermite operational matrix method for the nonlinear Lane-Emden problem presented in [37]. In [38], Singh et al. used Haar wavelet quasilinearization method to get the numerical solution of Emden–Fowler-type equations. Khred et al. used Wang-Ball polynomials [39] and DP-Ball polynomials [40] to solve singular ordinary differential equations. [41] Fayek et al. employed Bessel matrix method to solve the linear and nonlinear singular differential equations. Wang et al. [42] solved the nonlinear singular two-point boundary value problems using Chebyshev collocation method. Bhatti and Karim [43] used the least square method based on Wang Ball function to an approximate solution of higher order ODEs by using the control points of Wang Ball curves. Khred et al. [44] solved the linear delay differential equations of the first and second order using Said-Ball Polynomials. In this research, we also used Said-Ball Polynomials, but this time for the purpose of solving singular nonlinear ordinary differential equations of different orders.

This paper is organized as follows: Section 2 presents some concepts of Ball polynomials, Said-Ball polynomials and Said-Ball monomial formulas. Relations of the fundamental matrix are given in section 3. In section 4, numerical examples are presented. The conclusion is presented in section 5.

2. Ball polynomials

A. A. Ball introduced the Ball polynomial in his famous work aircraft design system CONSURF [45], which is defined mathematically as a cubic polynomial from the components of the following polynomials:

$$(1-x)^2, 2x(1-x)^2, 2x^2(1-x), x^2, \quad 0 \le x \le 1 \tag{9}$$

Many studies debated high generalization of Ball polynomial and its properties, for instance, Said-Ball and Wang-Ball, that are known of arbitrary degree [46]. For more Ball polynomial generalization, see the same reference and [47].

2.1 Said-Ball Polynomial and Said-Ball monomial formulas:

Said-Ball Polynomial $S_i^m(x)$ of degree m is defined as [47,48]:

$$S_i^m(x) = \begin{cases} \binom{\left\lfloor \frac{m}{2} \right\rfloor + i}{i} x^i (1-x)^{\left\lfloor \frac{m}{2} \right\rfloor + 1}, & 0 \le i \le \left\lceil \frac{m}{2} \right\rceil - 1, \\ \binom{m}{\frac{m}{2}} x^{\frac{m}{2}} (1-x)^{\frac{m}{2}}, & i = \frac{m}{2}, \\ S_{m-i}^m(1-x), & \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m, \end{cases} \tag{10}$$

Where $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t, and $\lceil t \rceil$ denotes the least integer greater than or equal to t.

Said-Ball curve $S^m(x)$ of degree m with $m+1$ control points, denoted by $\{s_i\}_{i=0}^m$, can be expressed as the following form in power basis:

$$S^m(x) = \sum_{i=0}^{m} \sum_{j=0}^{m} s_{i,j}\, x^j, \quad 0 \le x \le 1, \tag{11}$$

where

$$s_{i,j} = \begin{cases} (-1)^{j-i} \binom{i + \left\lfloor \frac{m}{2} \right\rfloor}{i} \binom{\left\lfloor \frac{m}{2} \right\rfloor + 1}{j-i}, & 0 \le i \le \left\lceil \frac{m}{2} \right\rceil - 1, \\ (-1)^{j-i} \binom{m}{i} \binom{i}{j-i}, & i = \frac{m}{2}, \\ (-1)^{j - \left\lfloor \frac{m}{2} \right\rfloor - 1} \binom{\left\lfloor \frac{m}{2} \right\rfloor + m - i}{m-i} \binom{m-i}{j - \left\lfloor \frac{m}{2} \right\rfloor - 1}, & \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m. \end{cases} \tag{12}$$

The Said-Ball monomial matrix is given by:

$$\mathcal{S}_{(m+1)\times(m+1)} = \begin{bmatrix} s_{0,0} & s_{0,1} & \cdots & \cdots & s_{0,m} \\ s_{1,0} & s_{1,1} & \cdots & \cdots & s_{1,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ s_{m,0} & s_{m,1} & \cdots & \cdots & s_{m,m} \end{bmatrix} \tag{13}$$

where $s_{i,j}$ is defined in Eq. (12).

To obtain approximate solutions of Eq. (1) with some appropriate initial conditions. We will use Said-Ball polynomials in the form:

$$y_m(x) = \sum_{i=0}^{m} a_i\, S_i^m(x) \tag{14}$$

Where $a_i, i = 0,1,\cdots,m$ are unknowns Said-Ball coefficients to be determined, m is any chosen positive integer, and $S_i^m(x)$ are the Said-Ball polynomials.

3. Relations of Fundamental Matrix

Here, we will write an approximate solution (14) as the form:

$$y_m(x) = \boldsymbol{A}^T\, \boldsymbol{S}(x) \tag{15}$$

Where $\boldsymbol{S}(x) = [S_0^m(x)\ \ S_1^m(x)\ \ \cdots\ \ S_m^m(x)]^T$ and $\boldsymbol{A}^T = [a_0\ \ a_1\ \ \cdots\ \ a_m]$.

Eq.(15) can be written as:

$$y_m(x) = \boldsymbol{A}^T\, \mathcal{S}\, \boldsymbol{H_m}(\boldsymbol{x}) = \boldsymbol{A}^T\, \Phi(\boldsymbol{x}) \tag{16}$$

Where $\Phi(\boldsymbol{x}) = \mathcal{S}\, \boldsymbol{H_m}(\boldsymbol{x})$, $\boldsymbol{H_m}(\boldsymbol{x}) = [1\quad x\quad x^2\quad \cdots\quad x^m]^T$ and $\mathcal{S}$ is the monomial matrix, which is given in (13).

3.1. Matrix relation for the first derivative

The derivative of Eq.(15) is given by:

$$\begin{aligned} y_m'(x) &= \boldsymbol{A}^T\, \boldsymbol{S}'(x) \\ &= \boldsymbol{A}^T\, \Phi'(x) \\ &= \boldsymbol{A}^T\, \mathcal{S}\, \boldsymbol{H'_m}(\boldsymbol{x}) \\ &= \boldsymbol{A}^T\, \mathcal{S}\, \frac{d}{dx}(\,\boldsymbol{H_m}(\boldsymbol{x})\,) \\ &= \boldsymbol{A}^T\, \mathcal{S} \begin{bmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ m\, x^{m-1} \end{bmatrix} \\ &= \boldsymbol{A}^T\, \mathcal{S} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^m \end{bmatrix} \\ &= \boldsymbol{A}^T\, \mathcal{S}\, V\, \boldsymbol{H_m}(\boldsymbol{x}) \\ &= \boldsymbol{A}^T\, \mathcal{S}\, V\, \mathcal{S}^{-1}\mathcal{S}\ \boldsymbol{H_m}(\boldsymbol{x}) \\ &= \boldsymbol{A}^T\, D\, \mathcal{S}\ \boldsymbol{H_m}(\boldsymbol{x}) \\ &= \boldsymbol{A}^T\, D\, \Phi(\boldsymbol{x}), \end{aligned}$$

where $D = \mathcal{S}\, V\, \mathcal{S}^{-1}$.

Therefore,

$$y_m'(x) = \boldsymbol{A}^T D^{(1)}\, \Phi(\boldsymbol{x}) \tag{17}$$

where

$$\boldsymbol{V} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & m & 0 \end{bmatrix}_{(m+1)\times(m+1)}$$

Generally, we can deduce that

$$\begin{aligned} y_m^{(\boldsymbol{n})}(x) &= \frac{d^n}{dx^n}\big(y_m(x)\big) = \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}y_m(x)\right) = \boldsymbol{A}^T\, \frac{d^{n-1}}{dx^{n-1}}\Big(D^{(1)}\, \Phi(\boldsymbol{x})\Big) = \boldsymbol{A}^T\ \frac{d^{n-2}}{dx^{n-2}}\Big(D^{(2)}\, \Phi(\boldsymbol{x})\Big) = \cdots \\ &= \boldsymbol{A}^T D^{(n)}\, \Phi(\boldsymbol{x}), \qquad n = 1,2,\cdots \end{aligned} \tag{18}$$

where $D^{(n)}$ is the n^{th} power of D.

Hence, Eq.(1) can be formulated by said-ball polynomials as the following form:

$$\boldsymbol{A}^T D^{(n+1)}\, \Phi(\boldsymbol{x}) + \frac{\lambda_1}{x}\, \boldsymbol{A}^T D^{(n)}\, \Phi(\boldsymbol{x}) + \frac{\lambda_2}{x^2}\, \boldsymbol{A}^T D^{(n-1)}\, \Phi(\boldsymbol{x}) = -F(x, \boldsymbol{A}^T\, \Phi(\boldsymbol{x})) \tag{19}$$

Eq. (19) can be written as the following residual equation:

$$\Re_m(x) = \boldsymbol{A}^T D^{(n+1)}\, \Phi(\boldsymbol{x}) + \frac{\lambda_1}{x}\, \boldsymbol{A}^T D^{(n)}\, \Phi(\boldsymbol{x}) + \frac{\lambda_2}{x^2}\, \boldsymbol{A}^T D^{(n-1)}\, \Phi(\boldsymbol{x}) + F(x, \boldsymbol{A}^T\, \Phi(\boldsymbol{x}))$$

(20)

The nonlinear system (20) components from $(m+1)$ equations that result from $m-(r-1)$ collocation points (r is the number of given initial conditions) with the following appropriate points x_i as:

$$x_i = \frac{1}{2}\left(1-\cos(\frac{(i+1)\pi}{m+1})\right) \quad , i = 1, 2, \cdots, m-r+1, \qquad (21)$$

which can be solved using any method to solve nonlinear systems to obtain unknown coefficients $a_i, i = 0,1,\cdots,m$ of Said-Ball_polynomials. And hence, we finally get the solution of the given problem with some appropriate initial conditions. We applied Newton's iterations method to solve nonlinear systems of unknown coefficients a_i.

4. Residual error estimation and solutions accuracy

In this approach, we used the upper bound of the mean error to test the accuracy of the obtained solutions. Then if we substituted the approximate solution $y_m(x)$ and its derivatives in the residual equation (20), we obtain

$$\mathfrak{R}_m(x_t) \cong 0, \forall\, x_t \in [a,b], t = 0,1, \qquad (22)$$

or

$$\mathfrak{R}_m(x_t) \leq \varepsilon, \qquad (23)$$

where ε is a small positive quantity.

If m is sufficiently large enough, the error decreases, that is, $\mathfrak{R}_m(x_t) \to 0$.

Second hand, we can estimate the accuracy of the solution by using the residual function $\mathfrak{R}_{\mathrm{m}}(x)$ and its mean value of the function $|\mathfrak{R}_{\mathrm{m}}(x)|$ on the interval $[\mathrm{a}, \mathrm{b}]$ as the upper bound of the mean error $\overline{\mathfrak{R}}_{\mathrm{m}}$, which is formed by the following formula [49]:

$$\mathfrak{R}_{\mathrm{m}}(\mathrm{c}) \leq \frac{\int_a^b |\mathfrak{R}_{\mathrm{m}}(x)|\ dx}{b-a} = \overline{\mathfrak{R}}_{\mathrm{m}}, \quad c \in [a,b]. \qquad (24)$$

5. Numerical examples

This section presents five examples and compares our numerical results with the exact solutions and other works [9,18, 34, 37]. See tables 1 and 3. To test the accuracy of the obtained solutions, we presented the upper bound of the mean error $\overline{\mathfrak{R}}_{\mathrm{m}}$on the interval $[0,1]$ for example 2, see table 2.

The absolute error $E(x_r)$ at the point x_r was calculated by using the form

$$E(x_r) = \left|y_{\text{exact solution}}(x_r) - y_{\text{approximate solution}}(x_r)\right|, \qquad x_r \in [0,1]$$

Computations of the examples have been carried out using Maple program.

Example 1:

Consider the following Emden-Fowler equation [18]

$$y'' + \tfrac{5}{x}y' + \tfrac{3}{x^2}y = 15 - x^4 + y^2, \qquad (25)$$

with ICs.

$$y(0) = 0, \qquad y'(0) = 0 \qquad (26)$$

The exact solution of this problem is $y(x) = x^2$.

To apply the method of solution of this problem with $m = 2$, the residual equation of (25) is

$$\left(\boldsymbol{A}^T D^{(2)}\,\Phi(\boldsymbol{x}) + \frac{5}{x}\,\boldsymbol{A}^T D^{(1)}\,\Phi(\boldsymbol{x}) + \frac{3}{x^2}\,\boldsymbol{A}^T\,\Phi(\boldsymbol{x}) - \left(\boldsymbol{A}^T\,\Phi(\boldsymbol{x})\right)^2\right)(1) + x^4 + 15 = 0, \qquad (27)$$

where

$$\mathcal{S} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathcal{S}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

and the operational matrices $\boldsymbol{D}^{(1)}$ and $\boldsymbol{D}^{(2)}$ are given by

$$\boldsymbol{D}^{(1)} = \mathcal{S}\,V\,\mathcal{S}^{-1} = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix},$$

$$\boldsymbol{D}^{(2)} = \begin{bmatrix} 2 & 2 & 2 \\ -4 & -4 & -4 \\ 2 & 2 & 2 \end{bmatrix}.$$

We note that the approximate solution is

$$y(x) \cong y_2(x) = \boldsymbol{A}^T\,\boldsymbol{S}(x) = \boldsymbol{A}^T\,\Phi(\boldsymbol{x}) = \boldsymbol{A}^T\,\mathcal{S}\,\boldsymbol{H_m}(\boldsymbol{x}) \;= [a_0 \;\; a_1 \;\; a_2]\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \qquad (28)$$

The collocation points are $x_0 = \frac{1}{4}, x_1 = \frac{3}{4}, x_2 = 1$. By collocating Eq. (27) with the collocation point $x_0 = \frac{1}{4}$ and using initial conditions in the last two rows in Eq. (27), we get the following system:

$$\begin{cases} -a_0 + 34a_1 + 15a_2 - \left(\dfrac{9}{16}a_0 + \dfrac{3}{8}a_1 + \dfrac{1}{16}a_2\right)^2 - \dfrac{3839}{256} = 0, \\ a_0 = 0, \\ -2a_0 + 2a_1 = 0 \end{cases} \qquad (29)$$

In solving system (29), we get $a_0 = 0$, $a_1 = 0$, and $a_2 = 1$. By substituting these values into (28), we obtain an approximate solution $y_2(x) = x^2$, which is identical to the exact solution, whereas Mutaish and Hasan in [18] have an approximate solution.

Example 2:

Consider the Emden-Fowler equation [34]

$$y'' + \frac{2}{x}y' + y^5 = 0\,, \qquad 0 < x < 1, \qquad (30)$$

with ICs.

$$y(0) = 1, \qquad y'(0) = 0 \qquad (31)$$

The exact solution is $y(x) = \sqrt{\frac{3}{3+x^2}}$.

Tables 1 and 2 indicate the numerical results of the presented method comparing with [34] and the upper bound of the mean error $\overline{\Re}_{\mathrm{m}}$ respectively, for Example 2.

Table 1: Comparison of the approximate solution for Example 2.

x	Exact Sol.	Method in [34] (S2KCWM)	Present method	
		$m = 2$	$m = 2$	$m = 5$
0.0	1.0000	0.9998	1.0000	1.0000
0.2	0.9934	0.9933	0.9937	0.9934
0.4	0.9744	0.9736	0.9746	0.9744
0.6	0.9449	0.9408	0.9429	0.9449
0.8	0.9078	0.8949	0.8985	0.9078
1.0	0.8660	0.8358	0.8414	0.8660

Table 2: The upper bound of the mean error $\overline{\Re}_{\mathrm{m}}$ for Example 2.

m	2	3	4	5	6	7
$\overline{\Re}_{\mathrm{m}}$	1.8690E-01	3.4447E-02	1.4538E-02	3.8690E-04	5.6169E-04	6.5376E-05

It is clear from table 2 that the more increase of m, the fewer errors and the more accuracy of the presented method.

Example 3:
Consider the Lane-Emden equation [37]

$$y'' + \frac{2}{x} y' + \sin y = 0\,, \quad 0 < x < 1, \tag{32}$$

with ICs.

$$y(0) = 1, \quad y'(0) = 0 \tag{33}$$

Table 3 indicates the comparison of the numerical results of the presented method and modified Hermite operational matrix method (MHOMM) [37] for $m = 8$ with Adomian decomposition method (ADM) [9].

Table 3: Comparison of the approximate solution for Example 3.

x	(ADM) [9]	(MHOMM) [37] $m = 8$		Present method $m = 8$	
		Approx. sol.	Absolute error	Approx. sol.	Absolute error
0.0	1.0000000	1.000000	0.00000	1.000000	0.000000
0.1	0.998598	0.998588	0.00001	0.998598	0.000000
0.2	0.994396	0.994395	0.000001	0.994396	0.000000
0.5	0.965178	0.965177	0.000001	0.965178	0.000000
1.0	0.863681	0.863679	0.000002	0.863681	0.000000

Example 4:
Consider the Emden-Fowler equation [26]

$$y''' + \frac{6}{x} y'' + \frac{6}{x^2} y' - 6(10 + 2x^3 + x^6)e^{-3y} = 0, \tag{34}$$

with ICs.

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0 \tag{35}$$

The exact solution is $y(x) = \ln\,(1 + x^3)$.
Figures 1 and 2 illustrate an approximate, exact solution and the absolute error, respectively for this problem.

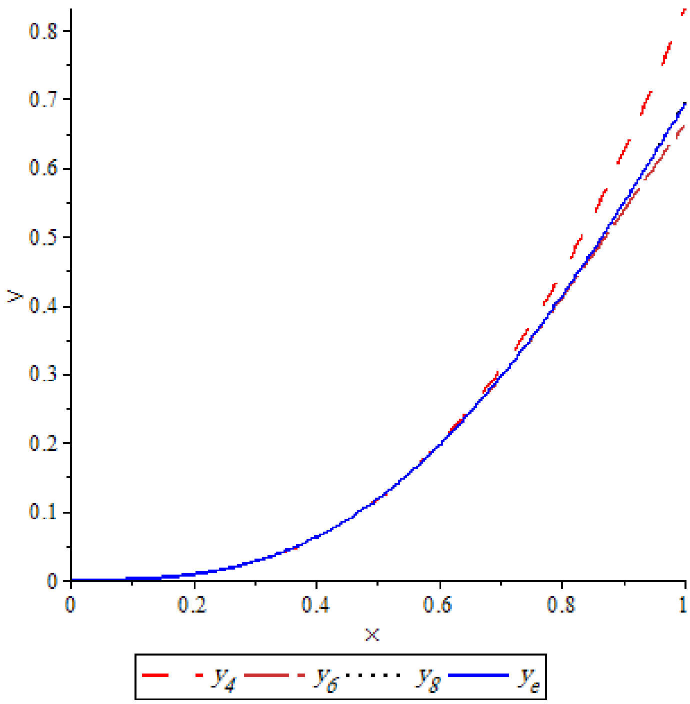

Fig. 1 Comparison of approximate solutions $y_m, m = 4, 6, 8$ with exact solution y_e.

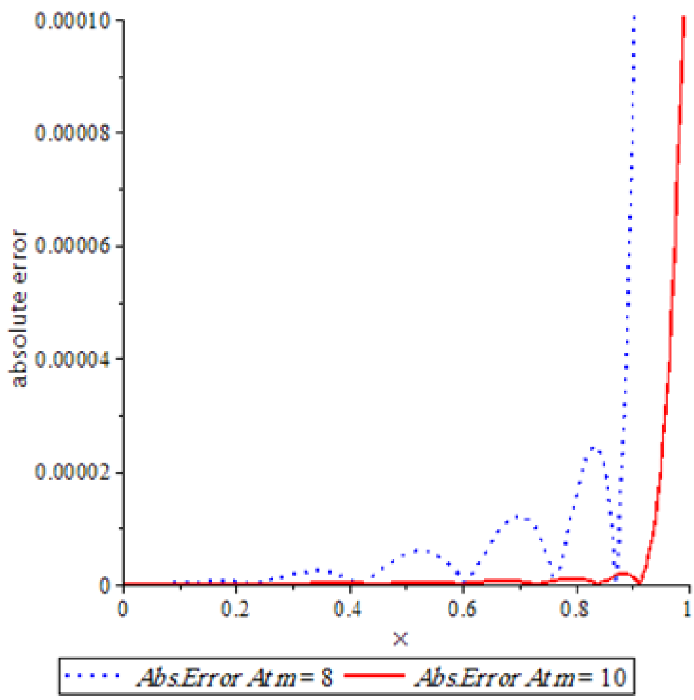

Fig. 2 Plots of the absolute error for example 4 with m =8, 10.

From figures 1 and 2, it is clear that the presented method obtained highly accurate solutions when increasing computational intervals.

Example 5:
Consider the following singular ordinary differential equation [20]

$$y^{(4)} + \frac{2}{x} y^{(3)} - y^2 = 72 - x^8 \,, \tag{36}$$

with ICs.

$$y(0) = 0, \quad y'(0) = 0\,, \qquad y''(0) = 0\,, \qquad y'''(0) = 0 \tag{37}$$

The exact solution of this problem is $y(x) = x^4$.
We applied the method at $m = 4$ to get the approximate solution identical to the exact solution, $y_4(x) = x^4$.

5. Conclusion

Said-Ball Polynomials with collocation method have been employed to solve the singular nonlinear ordinary differential equations of different orders. This method reduces the problem with suitable initial conditions into a system of nonlinear algebraic equations, which we solved by using Newton's method to get the approximate solution. The proposed method gave excellent numerical results compared with the other works and exact solutions, as shown in tables and figures.

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