2022

Student Understanding of the Definite Integral When Solving Calculus Volume Problems

Krista Bresock
West Virginia University

Student Understanding of the Definite Integral When Solving Calculus Volume Problems

Krista Kay Bresock

Dissertation submitted

to the Eberly College of Arts and Sciences
at West Virginia University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in
Mathematics

Vicki Sealey, Ph.D., Chair
Jessica Deshler, Ph.D.
David Miller, Ph.D.
John Goldwasser, Ph.D.
Kasi Jackson, Ph.D.

School of Mathematical and Data Sciences

Morgantown, West Virginia

September 2022

Keywords: calculus, definite integral, volume, student interviews, structuralism

ABSTRACT

Student Understanding of the Definite Integral When Solving Calculus Volume Problems

Krista Kay Bresock

The concept of integration appears in many different scientific fields, and students' understanding of and ability to use the definite integral in applications is important to success in their STEM (science, technology, engineering, and mathematics) classes. One of the first types of application problems that students encounter is finding the volume of a solid using the definite integral. How students approach these problems and how they use the definite integral to find volumes can have an impact on their future use and understanding of the definite integral.

This study involves a deep and thorough investigation of how ten students understand the definite integral when solving two types of volume problems: revolution volume problems and non-revolution volume problems. First, using the Riemann Integral Framework (Sealey, 2014), I analyzed how students understood the underlying structure of the definite integral when solving revolution volume problems. Using Piaget's (1971) learning theory of structuralism, I then examined how students' understanding of the familiar revolution volume problems affected and influenced their solving of novel non-revolution volume problems. The data was collected via one-on-one interviews where students worked through three different volume problems and discussed their thoughts and work.

The findings of this study can be summarized in three parts. First, students can build symbolically correct revolution volume problem integrals without understanding conceptually why their integral is correct. These students relied on memorized formulas without understanding why the formulas worked. Second, students' memorized formulas for revolution volume problems break down when attempting to apply them to non-revolution volume problems. Third, display of or development of conceptual understanding emerged either when being asked deliberate and probing questions about their revolution volume integrals or separately while solving the non-revolution volume problems. The students who were able to discuss their revolution volume problem integrals conceptually accurately had continued success throughout the interview.

Revolution volume problems are a standard application of the definite integral and many textbooks spend a lot of time and pages on them, but as this study has shown, using revolution volume problems alone or without asking conceptual questions is not enough to ensure understanding of how definite integrals work to solve volume problems. Non-revolution volume problems provide an environment that is resistant to students' inclinations to memorize formulas and provides a greater opportunity for students to attend to the underlying structure of the definite integral.

DEDICATION

To Momma Boopsy, Captain, and Hammy.

ACKNOWLEDGEMENTS

First and foremost, I want to thank my advisor and my friend, Dr. Vicki Sealey. Vicki has been with me through more life events, ups and downs, and moderate-to-severe crises than almost anyone I know (except for maybe my mom). She was with me when I was at my lowest, and even though she saw me at my worst, she never gave up on me and always believed in me. There is no other person in the world who would have helped me in the ways she has helped me. Even though we couldn't be more opposite of personality, we somehow just fit together. There was never a single meeting with Vicki where I would leave sad, upset, or discouraged. I always feel better after I talk to her. A few years ago, Vicki gave me a post-it note with "<3 Krista <3" written on it and I still have it to this day. It's a reminder that I always have someone who believes in me, even when I don't believe in myself.

I would like to thank my committee for sticking around through my rollercoaster journey through graduate school. Somehow, I have managed to stack my committee with people who think I'm doing something right. Dr. Jessica Deshler, Dr. David Miller, Dr. John Goldwasser, and Dr. Kasi Jackson have only ever been in my corner and encouraged me through a long and tumultuous journey. I hope my research and the rest of my career make you proud.

I am endlessly obsessed with my two best friends, Keith Gallagher and Natalie Benincosa, who take care of me, make me laugh, listen to me complain, and know me to the core. Keith, you are the Sandi to my Noel, and Natalie, you are the dark to my light. You two have kept me sane, entertained, and supported through this whole process and I am a better person because I know you.

Finally, I want to thank the most important people in my life: my mother, Kay, my sister, Kara, and my husband, Eli. I honestly have no words to describe how important my mother is

and has been on my graduate school journey. She has supported me in every way you can imagine. Everything I do is done with the intention of making her proud. My sister, Kara, is the light of my light. She makes me smile every day, and I wish I could be more like her. I work hard so that someday I will be able to give my sister her best life. And finally, my husband, Eli. My life started the day I met my husband. He helped me begin my sobriety journey and he has been my firm foundation ever since. I have never been more myself since I met him, and this dissertation wouldn't exist if I didn't meet him. It's all for him, and he can't wait to finally be Mr. Dr. Bresock. ☺

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1: INTRODUCTION

Calculus is the foundation upon which the sciences are built, so student understanding of this topic has widespread implications. In their calculus courses, students learn about derivatives, which measure change of quantities, and integrals, which measure accumulation of quantities. They also discover real-world uses of these mathematical concepts, learning applications of calculus to areas such as physics, economics, and biology. Measuring the physical quantity of volume is one of the first applications of integration students encounter when studying integral calculus, which is a main focus of traditional second-semester calculus courses. Calculus can be used to find the volume of any solid, but it is particularly useful for solids that do not have standard geometric shapes. In addition to calculus, volume problems use other mathematical topics and skills such as geometry and visualization. Because of this combination of different mathematical skills, I view the teaching and learning of volume problems as an interesting area to study. This dissertation discusses my investigation into students' approaches to and understandings of calculus volume problems.

Calculus in general, and integration in particular, has been studied extensively in mathematics education research. It has been shown that students can do calculations involving the definite integral but have a hard time conceptualizing the mathematics behind it (Orton, 1983; Thompson, 1994; Sealey, 2008; Jones, 2013; Meredith & Marrongelle, 2008). The major take-away is that students fare better in calculus and calculus-based science courses when they view the integral as the limit of a sum of products of quantities, versus other incomplete conceptions such as area under a curve or antiderivative.

There have also been studies about student understanding of the definite integral in relation to application problems in calculus and physics (Sealey, 2014; Yeatts and Hundhausen, 1992; Cui et al., 2006). These studies examined how students used the information given in the problem to cue their use of a definite integral when problem-solving. Bernard and Jones (2016) studied student set-up of revolution volume integrals by relating their set-up to epistemic games in which one "makes moves" to arrive at a desired form of an answer. Although Bernard and Jones studied students' integral set-up for volume problems, they did not ask students deeper questions about how they understood their integral set-up. Also, Bernard and Jones interviewed students about revolution volume problems only, with no questions in the interview dealing with novel or non-revolution volume problems.

My research adds to the body of mathematics education knowledge by providing a deep dive into students' understanding of integral volume problems and fine-grained analysis of how they use their integral knowledge from revolution volume problems to solve novel non-revolution volume problems.

In this dissertation, I will answer the following research questions:

1. How do students conceptualize revolution volume integrals?
2. How do students use their revolution volume problem conceptions to solve novel volume problems?
3. How can non-revolution volume problems aid in building conceptual understanding of integration?

Chapter 2: LITERATURE REVIEW

Calculus, the study of change, equips students with two invaluable tools for investigating our dynamic world: differentiation for studying rates of change and integration for studying accumulation. Students may be introduced to the definite integral concept via "the area problem" (Stewart, 2007), which involves determining the area of a region contained between a continuous, non-negative curve $y = f(x)$ and the x-axis on a closed interval $[a, b]$. Other physical situations can be used in place of (as in Sealey, 2008) or in addition to the area problem, each of which serves to illustrate different situations in which the definite integral is a useful tool. As researchers and instructors, we want students to recognize a common theme among these examples: the physical quantity that is being determined by the definite integral can be approximated by a sum of products, also known as a Riemann sum, and can be found exactly by taking a limit of this sum.

In the sections that follow, I will discuss relevant literature relating to student understanding in three areas: the general definite integral, integral use in application problems, and geometric aspects related to volumes of solids.

2.1. Definite integrals

In one of the first studies on student understanding of integration, Orton (1983) interviewed 110 students (age 16-22) and asked them to discuss their solutions to problems concerning sequences, limits, convergence, Riemann sums, areas, and volumes of solids. Concerning integration topics, students were skilled at evaluating definite integrals using the Fundamental Theorem of Calculus and antiderivatives. They did, however, encounter difficulties on four questions which required their understanding of integration as a limit of a sum.

> The results suggested that most students had little idea of the procedure of dissecting an area or volume into narrow sections, summing the areas or volumes of the sections, and obtaining an exact answer for the area or volume by narrowing the sections and increasing their number, making use of a limit process. (Orton, 1983, p. 7)

A later study (Pettersson & Scheja, 2008) examined two students' written and verbal explanations of the meaning of the mathematical concepts of limit and integral. In general, the participants tended to describe integration in terms of algorithmic processes and procedures (computing antiderivatives) rather than focusing on underlying conceptual ideas. After a more in-depth analysis of participants' responses, Pettersson and Scheja determined that hints of the necessary conceptual notions were present in students' minds and that the students were aware of their incomplete understanding of these topics. The researchers concluded that this awareness could provide a foundation for the further development of students' conceptual understanding of integration.

As mentioned above, a common way to introduce students to the integral concept is by framing it in terms of the area under a curve. Although this is a valid and convenient way to represent integration, Sealey (2008) warns against focusing students' attention solely on the area conception. Sealey explains that an understanding of the definite integral in terms of area is a powerful tool provided that students have a solid understanding of the underlying limit-of-a-sum-of-products structure. Thompson and Silverman (2008) also warn against over-emphasis on the area conception of integration, stating, "for students to see 'area under a curve' as representing a quantity other than area, it is imperative that they conceive of the quantities being accumulated as being created by accruing incremental bits that are formed multiplicatively" (p. 45).

One area-based integration misconception found in students is that of viewing the definite integral as the *total* area contained between a curve and the *x*-axis (Gonzalez-Martin & Camacho, 2004) rather than viewing it as a net area. This same type of error has been found in several different studies on integration (Kiat, 2005; Rösken & Rolka, 2007; Mahir, 2009; Huang, 2010). Many of these studies (as well as Bezuidenhout & Olivier, 2000) also examined students' understanding of integration as an accumulation function. In an early study on students' images of rate and their understanding of the Fundamental Theorem of Calculus, Thompson (1994) found that students' "images of a Riemann sum [seemed] not to have entailed a sense of motion" which resulted in an insufficient foundation on which to build proper reasoning about a sum's rate of change. Another finding from this study was that students tended to view each component of a Riemann sum (Thompson calls these components "accruals") as solitary objects with no constituent quantities. Holding this type of erroneous view of Riemann sums could lead to major and varied difficulties when attempting to solve volume problems using integration. For example, a student that views an approximating slice of a solid as a solitary object and doesn't pay attention to variations in each slice, may erroneously attribute a constant radius to each slice, disregarding the change in the variable.

In addition to classic research on integration done by Orton (1983) and Thompson (1994), more recent research done by Sealey (2008, 2014) and Jones (2013, 2015a, 2015b) has broadened our understanding of students' conceptions of the definite integral. As mentioned above, Sealey examined many aspects of the definite integral and produced a framework (Sealey, 2014) for characterizing student understanding of Riemann sums and definite integrals. I discuss this framework in detail in Chapter 3. Jones (2013) characterized several different symbolic forms that students have for the definite integral: adding up pieces (later to be known in Jones' (2015b) work as "multiplicatively-based summation" or "MBS"), perimeter and area, function matching (antiderivative), and other, such as "area inbetween." His conclusion was that "student difficulties might not necessarily arise from lack of knowledge, but from the activation of less-productive cognitive resources over others" (p. 138). Jones' next studies (2015a, 2015b) expanded on this idea of symbolic forms for the definite integral and concluded that the area and antiderivative notions are most commonly activated when students are working on definite integral application problems. Even though MBS is the most productive symbolic form for students to have concerning the definite integral, they do not commonly activate it.

Ely (2017) extended the ideas of Jones, theorizing that adding up pieces (AUP) and MBS were actually two conceptions in different integral registers. AUP involves seeing the product $f(x)dx$ as an infinitesimal bit of what is being accumulated; so in the case of volumes, $f(x)dx$ is a little bit of volume: dV. MBS involves the same product, $f(x)dx$, but the function f now represents the rate of change of the quantity being accumulated, which is then multiplied by a little bit of x, represented by dx, resulting in the desired quantity. Ely states that this distinction between different views of the definite integral can help "to distinguish between the acts of modeling and evaluating definite integrals, and to provide tools for students that support these two ways of reasoning" (p. 164).

Jones and Dorko (2015) built on Jones' single-integral work by examining multivariable calculus students' conceptions of multiple integrals. They found that students' understanding of multivariable integrals parallels that of their single-variable integral understanding, with 3-dimensional elements related to function-matching, perimeter and area, and MBS appearing. A take-away finding was that students continue to be linked to integral as area, even in situations involving multiple integrals where quantities other than area are being computed.

2.2. Applications of the definite integral

In addition to studies examining student understanding of the definite integral in general, there are also studies of student understanding of "real-world" integral problems (in particular, those concerned with finding physical quantities other than area) in mathematics education and physics education. In the study by Orton (1983) mentioned previously, students were asked to describe the solid obtained by rotating the 2-dimensional region bounded by the curve $y = x^2$ and the *x*-axis on the interval $1 \leq x \leq 3$ about the *x*-axis (the graph and the shaded region were provided). This problem can be solved using a definite integral, as the situation can be seen as the accumulation of volume pieces. Students performed most poorly on this question, receiving a mean score of less than one on a five-point scale (0–4).

> The vast majority of students could not complete an explanation for this item. A few students managed a partial explanation, the general features of these being that the πy^2 was usually explained, but the reason for integrating was not completely understood. Errors made on Item 19 were structural, and the number of students who really understood integration as the limit of a sum was very small. (Orton, 1983, p. 7)

Gonzalez-Martin and Camacho's (2004) study on student understanding of various aspects of improper integrals included questions concerning volumes of "infinite solids." (An improper integral is an integral in which the bounds are infinite and/or the function has an infinite discontinuity.) Participants were asked to calculate the values of the integrals $\int_2^\infty \frac{1}{x-1}\,dx$ and $\pi \int_2^\infty \frac{1}{(x-1)^2}\,dx$, to interpret their results geometrically, and to discuss any relationship they saw between the two integrals. Although many participants calculated the correct value for both integrals, only 10 (out of 31) expressed that the first integral could represent the area under the function $f(x) = \frac{1}{x-1}$ and the second integral could represent the volume when the function $f(x) = \frac{1}{x-1}$ was rotated about the *x*-axis. These results are in agreement with many other studies that have found that students are adept at evaluating integrals but have very little understanding of the underlying integral concepts (Kiat, 2005; Grundmeier et al., 2006; Mahir, 2009).

Exploring literature outside of mathematics education produced some very interesting and relevant studies on the translation of students' mathematical knowledge to problems in scientific fields such as physics. In a study conducted by Cui et al. (2006), physics students (many of whom were engineering majors) were evaluated on their ability to retain and transfer calculus knowledge when solving introductory physics problems. The researchers also wanted to uncover any specific difficulties students exhibited in the transfer of their calculus knowledge. In Phase 1 of the study, the authors determined that students' difficulties in solving certain physics problems stemmed from their inability to set up the problems and not from deficiencies in the calculus itself. In Phase 2, students were asked to discuss certain variations on the problems from Phase 1, and the researchers used these responses to explore students' understanding of the criteria that determined whether integration was an applicable tool for physics problems. Only three out of the seven students who used integration correctly (there were a total of eight participants) could explain (very roughly) that integration was necessary because of the need to sum up infinitesimally small elements. When examining student difficulties in applying integration to physics problems, Cui et al. (2006) found that students have trouble determining the variable of integration and deciding on the limits of integration.

Another, earlier paper (Yeatts et al., 1992) described an Integrated Calculus and Physics Program (ICP) at a state engineering university and the different types of pedagogical difficulties encountered in attempting to integrate the two subjects. Three general trouble areas were observed and discussed. Within the first area—notation and symbolism—examples were cited ranging from an over-dependence on the symbols x and y as independent and dependent variables (termed "xy-syndrome"), to confusion concerning the variable of integration for integrals in physics contexts (like the integral for the moment of inertia, $\int r^2 dm$). The second area dealt with an effect the authors called "the distraction factor"—making an inadvertent error on one aspect of a problem while focusing on a different aspect. (This can be seen quite frequently in calculus students' algebra and arithmetic mistakes.) Yeatts et al. claimed that this particular difficulty may be "offered as evidence of students' proclivity to grasp at a familiar formula rather than think carefully about the implications of the problem statement and the concepts involved" (p. 719). The third area concerned students' tendency to compartmentalize their knowledge, which can result in students not knowing when or how to apply calculus concepts.

Yeatts et al. (1992) concluded with a small discussion on the topic of mathematical modeling. They claimed that it is necessary for instructors to treat mathematical modeling (developing a mathematical representation of a physical situation) as a separate discipline and to provide students opportunities for honing their modeling skills, apart from the learning of new physics or calculus concepts. With respect to the topic of integration, Yeatts et al. gave the following suggestion.

> …we have found that providing students with a number of exercises in setting up Riemann sums for physical quantities such as mass, center of mass, work, and moment of inertia (complete with sketching and labeling of volume elements, and correctly using the $\sum$-notation) has been quite effective in helping them master both the concepts and symbolism. We would recommend these types of exercises, perhaps coupled with judicious use of technology for evaluating sums and/or integrals, as a very constructive support of the calculus-physics interface. (p.721)

Many studies in physics education research have examined student conceptions of the definite integral in solving physics application problems. Meredith and Marrongelle (2008) investigated how students use their mathematical resources when solving electrostatics problems and what aspects of these problems cued students to use integration. They found that the presence of a "dependence" (e.g., dependence on a changing variable, like x) cued most students to integrate, even though that cue resulted in some misconceptions when building the integral. The "parts-of-a-whole" cue was shown to be the most productive cue, and the authors suggested that instructors encourage students to pay more attention to units, since attention to units can lead students to the parts-of-a-whole line of thinking, as well as provide a self-check opportunity. There are many studies in mathematics and physics education research (e.g., Jones, 2015b; Nguyen & Rebello, 2011a, 2011b; Orton, 1983; Sealey, 2008, 2014; Thompson, 1994; Von Korff & Rebello, 2012; Wagner, 2018) that have shown that students have many difficulties in using the definite integral in practice and that this is a rich area for research.

An area of this "integration in physics" research that has branched off and produced some interested findings is the study of how students and experts view the "*dx*" in integration (Hu & Rebello, 2013; Lucio-Villegas et al., 2015; Sealey & Thompson, 2016; McCarty, 2019). In Hu and Rebello's (2013) study, students were observed having four different resources for differentials: small amount, point, differentiation, and variable of integration, and four different conceptual metaphors: object, location, machine, and motion along a path.

> While activating the small-amount resource and object metaphor, students' solution involved chopping an object into pieces and adding the quantity or effect due to each piece (i.e., chopping-adding pieces approach), which seems to involve more mathematical sense making. Students appeared to be able to translate back and forth between the math and physics concepts. (Hu and Rebello, 2013, p. 12)

These findings in physics education research are in line with math education research findings that students with a deeper understanding of the underlying structure of the definite integral fare better when using the integral to solve application problems.

2.3. Geometric concepts in volume problems

Volume problems require a combination of knowledge from various areas in mathematics: visualization, calculus, and geometry. The obvious geometric concept present in these problems is volume—the amount of 3-dimensional space a solid occupies. Other geometric aspects that can arise, given the type of volume problem encountered, are area, plane sections of a solid, radius of a circle, and relationships between quantities such as angles and side lengths.

In integral volume problems, the two geometric constructs that make up the volume of each successive slice of a solid are (1) the surface area of the cross-sectional slice, and (2) the height of the corresponding cylindrical approximation to the slice (usually denoted as Δx, or dx within the integral). The concept and visualization of cross-sections, or plane sections of a solid, was researched by Davis (1973) in middle school and high school students. The students were asked to select the correct drawing for each cross-section that would have resulted if a knife held in the hand of the experimenter had actually cut through the presented solid in the indicated direction (see Figure 1).

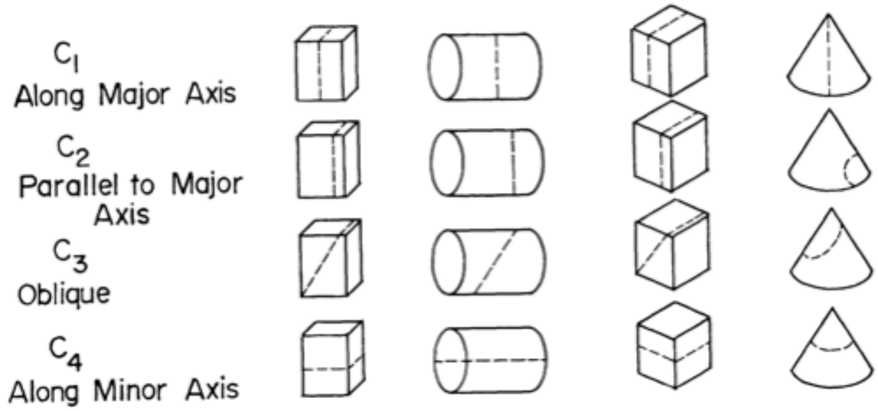


Figure 1. *Davis's (1973) cross-sectional test tasks.*

At the beginning of testing, students were asked to answer a sample question concerning cuts to a foam wedge and the correct answers were illustrated by showing the actual cut performed on the sample solid. Although this study examined performance differences between sex, age, and mathematical ability, a relevant recommendation was made by Davis concerning general student understanding of plane sections with regards to volumes of solids.

> It is recommended that mathematics teachers take the time to provide actual cross-sectioning experiences preceding the study of volume and quadratic functions if they expect the cross sections to add meaning to the learning situations. … This researcher feels that if teachers want to use cross-sectioning experiences to add meaning to the study of a topic such as volume, then students at all grade levels studied do have the ability to function effectively with cuts on the major and minor axes of the rectangular prism, cylinder, cube, and cone. (p. 139)

Another way that students could have issues with geometric aspects of volume problems is if their teacher has difficulties representing or visualizing 2- and 3-dimensional aspects of geometric figures. Moore-Russo and Schroeder (2007) found that many secondary school teachers have difficulty visualizing geometric objects and manipulating 2-dimensional objects in 3-dimensional space, and thus would have problems teaching these geometric skills at the secondary level.

The "height" aspect of a representative volume slice (Δx or dx within the integral) has been discussed in studies on student understanding of integration topics in mathematics education and physics education (Section 2.2). Hobbs and Relf (1998), in their discussion of a fundamental approach to the teaching of the concept of integration, stated that when students are asked to explain integration, they tend to view the dx within the integral as simply a notational indicator of the variable of integration. Although this is true on a superficial level, this view could cause students to rely on finding a formula for cross-sectional area without relating it to the volume of the approximating cylinder, thus causing the student to disregard the idea of summing approximating volumes on small intervals. This lack of understanding of the underlying approximation concepts within integration was also discussed in Orton (1983), in general and in specific relation to volume problems.

The studies mentioned above concerning geometry are relevant to my study for many reasons. In non-revolution volume problems, either the shape of solid is described to the student (Problem 3 of my study) or the cross-sectional and base shapes of the solid are described (Problem 2 of my study) in the statement of the problem. The student must develop the area formula for the cross-section of the solid, which can vary depending on the shape of the cross-section. Since the cross-sectional shape of a geometric solid can take any form, students need to be aware of and comfortable with the formulas for areas of circles, squares, rectangles, triangles, and various other 2-dimensional objects.

In revolution volume problems (Problem 1 in my study), this shape-area aspect is somewhat eliminated, since the revolution of a 2-dimensional region about a line produces cross-sections that are circular in shape–either a solid circle (disk) or a circle with a hole in the middle (washer). Revolution problems lighten the cognitive load on students with respect to finding the area of the cross-sectional surface, but the presence of a non-coordinate-axis line of rotation (e.g., $x = 1$ or $y = -2$) can add an additional level of difficulty that geometric volume problems generally lack. Rotation of a 2-dimensional region (situated on the Cartesian plane) about a coordinate axis results in a solid with cross-sectional disks or washers whose radii are determined by the region's bounding functions. Rotating the region about a non-coordinate-axis line also forms a solid whose cross-sections are disks or washers, but their radii are no longer solely determined by the region's bounding functions—the formula for the radii must account for the shift away from the axis. It is easy to see how students that have difficulties visualizing and representing graphical transformations of functions could have difficulties with this aspect of revolution problems. In relation to this concept, Lean and Clements (1981) found that students with low spatial abilities might also have problems with geometrical transformations such as translations, reflections, rotations, dilations, and expansions.

A final geometric consideration that students must attend to in solving volume problems is that of geometric relationships between quantities. This comes up specifically in geometric volume problems where the bounding functions of the solid are not stated explicitly in the problem (Problem 3 of my study). In order to build a valid volume integral for the solid in these types of problems, students must develop a bounding function on their own, which would require them to observe and describe relationships between the physical quantities of the solid (e.g., length, height, etc). These same types of relationships are used in relating changing quantities in related rates problems. In a study on identifying students' conceptual barriers when solving related rates problems, Engelke (2004) observed the following result.

> Students had particular problems recognizing when to use the similar triangle relationship; they did not understand the power of substitution and function composition; and they were not effective in determining what algebraic procedures to implement to arrive at the most appropriate defining relationship. Computational errors led to incorrect solutions; geometric misconceptions led to incorrect models. (p. 3)

In another study on related rates, Engelke (2007) examined mathematicians' solution processes for related rates problems and developed a framework to help assess student understanding of these types of problems. She found that the most important aspects of mathematicians' knowledge that allowed them to solve related rates problems successfully were their richly connected understanding of the concepts of geometry, variable, function, and derivative, and their abundant content knowledge.

Chapter 3: THEORETICAL PERSPECTIVE

This chapter will cover the two parts that comprise my theoretical perspective: the learning theory structuralism and the more calculus-specific Riemann Integral Framework.

3.1. Learning theory: Structuralism

The lens through which I analyze interview data comes from Jean Piaget's (1971) philosophy of structuralism. According to Piaget, "The notion of structure is comprised of three key ideas: the idea of wholeness, the idea of transformation, and the idea of self-regulation" (1971, p. 5). Wholeness refers to students' knowledge as a network of interconnected, communicating pieces, as opposed to an aggregate of elements that are independent of the system into which they enter. These interconnected pieces are constantly being adjusted and transformed through actions the student performs on mathematical objects. Mental and physical actions involved in working through mathematics problems and discussing mathematical ideas allow students to undergo *reflective abstraction*: "a mode of thought that does not derive properties from the *things* but from our ways of *acting on things*, the operations we perform on them" (Piaget, 1971; p. 19). This concept of reflective abstraction lends much credence to the ideas of group work and other types of active learning.

Schema (also called cognitive structures) are collections of mental and physical actions that help us organize our understandings of and reactions to the world. The system of self-regulation and adaptation of schema has equilibrium as its goal and consists of two processes that are running continuously throughout the lives of all living organisms: assimilation and accommodation (Piaget, 1977).

According to Piaget, *assimilation* is defined as "the incorporation of an external element, for example, an object or an event, into a sensorimotor or conceptual scheme of the subject" (p.

5). Piaget also went on to note that "every assimilatory scheme tends to incorporate external elements that are compatible with it" (p. 6). Thus, assimilation does not cause a change in the scheme, but adds elements to it. Unfortunately, Piaget did not explicitly define *accommodation*, but he stated that accommodation occurs when assimilation "must take account of the particularities of the elements being assimilated" (Piaget, 1977, p. 6). He went on to say that "because it is assimilatory schemes that are accommodated, accommodation is always secondary to assimilation" and that it is required for there to be "an equilibrium between assimilation and accommodation" (p. 6). Piaget's definitions and descriptions tend to be a bit dense and abstract, so I will add descriptions of these concepts given by Wadsworth (1979).

> *Assimilation* is the cognitive process by which the person integrates new perceptual matter or stimulus into existing schemata or patterns of behavior. … This process of assimilation allows for growth of schemata. … *Accommodation* is the creation of new schemata or the modification of old schemata. … Accommodation accounts for development (a qualitative change), and assimilation accounts for growth (a quantitative change); together they account for intellectual adaptation and the development of intellectual structures. (p. 14-16)

When students encounter volume problems in second-semester calculus, they have already started to build their schema for integration and have seen some basic application problems dealing with physical concepts like area and distance. When the concept of the definite integral as a measurement of volume arises, students can process this new information in many different ways. If the student has built a conceptually accurate understanding of integration, it is possible for them to assimilate the new information into their accurate integration schema with little need for accommodation. If their understanding of integration is inaccurate or weak (for example, only knowing that the integral measures area) this new information about volumes can be brought into their schema by accommodation and there can be an adjustment of the old, incorrect knowledge.

Unfortunately, students can also assimilate new data into a given scheme even if they have an inaccurate or weak understanding of integration. For example, if a student sees integral applications as exercises in memorizing formulas, they could assimilate the volume integral problems as further instances of integral formulas they need to memorize. It is our goal as teachers to provide students with educational situations that foster rich reflective abstraction so that they do not have the opportunity to mindlessly assimilate new information.

3.2. Calculus: Riemann Integral Framework

Additionally, I use Sealey's (2014) Riemann Integral Framework (RIF) to describe and characterize students' levels of understanding of the underlying structure of the definite integral. The RIF was developed with the goal of understanding and evaluating students' cognitive progress as they build the structure of the Riemann integral concept. This framework was based on Zandieh's (2000) derivative framework, which described student understanding of the limit definition of the derivative of a function, $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$. Zandieh viewed student understanding of the derivative as existing in layers corresponding to the mathematical layers present in the structure of the derivative: Difference, Ratio, Limit, and Function. Similarly, Sealey described student understanding of the Riemann integral in terms of the corresponding mathematical layers: Product, Summation, Limit, and Function. Sealey's RIF also includes a preliminary layer called "Orienting" during which students make sense of the wording and topics encountered within the statement of the problem.

Pre-layer: Orienting

As mentioned above, the Orienting phase consists of instances in which students engage in making sense of the wording and topics encountered within a problem. This is also the step in which graphs or diagrams can be constructed or adjusted. Since basic geometric figures and rotations have rich visual representations and these visualizations directly guide the integral set-up, the Orienting layer is very important in the volume problem-solving schema. Once an acceptable representation of a solid is sketched or visualized, the student may then identify given values and define unknowns and variables.

Sealey stated that, symbolically, this layer is represented by two pieces within the Product – one containing a function $f(x_i)$ and one containing the small increment Δx. This means that students spend time familiarizing themselves with the physical quantities represented by these pieces separately. In volume problems, the portion that contains the function $f(x_i)$ is the area of the base of an approximating cylinder (or the surface area of a cylindrical shell). Once students recognize the base's geometric shape, they then determine the formula for the area of this cross-section in general terms (e.g., area of a circle = πr^2, area of triangle = $\frac{1}{2}bh$, etc.). The quantity Δx represents the height of each approximating cylinder (or thickness of each approximating shell). From my experience teaching calculus students, I have observed that students tend to neglect thinking about the meaning of this quantity, and instead view it as a necessary component of any integral or as merely an indicator of the variable of integration. I believe that the failure to assign meaning to the symbol Δx (or *dx* within the integral) in this step could be a source of students' overall misconceptions about integration.

The symbolic representation of the volume of an approximating cylinder as a simple product $f(x_i)\Delta x$ is somewhat deceiving, considering that the underlying area formulas for $f(x_i)$

can get quite complex, depending on the shape of the cross-sections and the axis of rotation. In non-revolution volume problems—where the 3-dimensional shape of the solid, or the shapes of the cross-sections and the base of the solid, are given—$f(x_i)$ can represent the area of any 2-dimensional geometric shape. In revolution volume problems—where the bounding functions of a 2-dimensional region on the coordinate plane are given—the shapes of the cross-sections are circular (i.e., either disks or washers). In revolution problems requiring concentric slicing of the solid, the "cross-sections" are shells with rectangular surface areas. When trying to determine the formula for $f(x_i)$ in this step, students can encounter various obstacles depending on the type of volume problem they are solving.

The fact that the Orienting layer is called a "preliminary" layer does not imply that it only occurs at the beginning of the problem-solving process. Sealey stated that it was often "necessary for students to reassess their understanding of the meaning of the terms in the problem as well as the goal of the activity" (2008, pp. 165-166). The Orienting pre-layer is necessary for successful problem-solving of any nature, so obviously this is an applicable and essential step for students in their understanding and solving of volume problems.

Layer 1: Product

Symbolically, this layer involves the product $f(x_i)\Delta x$ found within the definition of the Riemann integral, $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. When using the Riemann integral to find the volume of a solid that is situated on the interval $[a, b]$, the product $f(x_i)\Delta x$ represents the volume of a (general) cylinder with surface area $f(x_i)$ and height Δx. The volume of this cylinder is approximately equal to the volume of a Δx-width sliver of the solid at the x-value x_i. As was observed in Sealey's study, this is the layer with which students have the most trouble when reasoning through a definite integral problem because it requires students to form a new quantity from two other quantities.

Layer 2: Summation

Symbolically, this layer involves the Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$, which, in the context of volume problems, represents an approximation of the exact volume of the given solid. In this layer, students should be attending to the solid as a whole and to the idea of the sum of the approximating volumes for each subinterval as an approximation of the volume of the solid. In Sealey's (2014) study, students tended to enter the framework through the Summation layer. The students were presented with a situation where the relevant physical quantity could only be approximated accurately by measuring it on small intervals and adding up the approximations. This is the foundation of the concept underlying the definite integral and as students begin learning topics like techniques of integration, this idea could get diminished or forgotten. The participants in my study had already seen integrals in many different contexts, so Summation layer thinking may not be present, or it may require some prompting from the interviewer.

Layer 3: Limit

Symbolically, this layer involves the limit of a sum, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. Students may show evidence of a basic understanding of this layer as early as the first Orienting phase, where they may sketch an approximating slice or cylinder within (or near) the sketch of the solid. In doing so, they are exhibiting an understanding of the need for approximations in the development of a Riemann integral. By the time calculus students encounter volume problems, they should be fairly familiar with the basic concepts of integration. Thus, associating integration with approximations on small subintervals and in turn entering the integral framework through the Limit layer is completely plausible.

Layer 4: Function

Symbolically, this layer involves the definite integral as a function where the input is the upper limit (i.e. right endpoint) of the interval on which the solid is situated, and the output is the value of the definite integral: $g(x) = \int_a^x f(t)dt$. As in Sealey's study, the integrals involved in solving volume problems are associated with specified intervals (in particular, the intervals on which the solids are situated), but viewing the integral as a dynamic accumulation of approximating volumes may be advantageous for further application of integration in finding volumes of "infinite solids" (as in Gonzalez-Martin & Camacho, 2004). Given the nature of volume problems, specifically that volume is always positive and the solids are situated on a specific interval, I do not expect much discussion from students involving the Function layer. I include this information here to fully describe Sealey's framework.

3.3. How I use these frameworks in my study

I chose to use two different frameworks for my study because as I analyzed data, there were two different levels of integral understanding that were interesting and that I wanted to focus on: a more detailed level involving how students understood the definite integral while solving volume problems and a more general level involving how they approached novel volume problems. I used the RIF to discuss and categorize how students understood the definite integral as it applied to calculus volume problems (Problems 1-3). What I did not take into account when planning my study, though, was that most of the students (nine of ten) had not encountered volume problems like Problems 2 and 3 (all of the participants were familiar with Problem 1). It is because of this that I decided to add a second framework so I could also analyze how they use their prior integration understanding (from Problem 1) for the novel integration volume problems (Problems 2 and 3).

Chapter 4: PILOT STUDIES

In the year leading up to my dissertation prospectus, I conducted two different pilot studies (Bresock & Sealey, 2018a, Bresock & Sealey, 2018b) with the aim of solidifying my research questions, interview protocol, and data analysis techniques.

4.1. Pilot Study 1

In this study, my main goal was to decide on the problems I wanted to ask during the interviews. I was examining if the problems produced robust data and if they covered the calculus concepts I was interested in studying. I was also working out how to analyze the data in a way that allowed me to answer my research questions.

The participants in this study were seven calculus students attending West Virginia University. Four of the participants were recruited from a Calculus 2 class and three were recruited from an Elementary Differential Equations class taught by the researcher. For all seven participants, one-on-one, video-recorded interviews were conducted outside of the classroom setting. The participants were asked to solve three different solid of revolution volume problems and to think out loud and discuss their problem-solving strategies. The problems are listed below.

1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ about the line $x = -1$.
2. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2 + 1$, $x = 2$, $x = 3$, and $y = 0$ about the y-axis.
3. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 4x$, $y = \frac{1}{4}x$, and $y = \frac{1}{x}$, $(x > 0)$, about the y-axis.

The initial analysis of the Differential Equations students' data showed that students used double and triple integrals in their discussions and written work. Since the purpose of this study was to understand how students conceptualize single-integral volume problems, I chose to exclude these students from the remainder of the data analysis. Furthermore, this influenced my decision to recruit my dissertation study participants solely from Calculus 2 students that have already learned about volume problems in their Calculus 2 class and Calculus 3 students that have not encountered multiple integrals yet.

Data Analysis

Two aspects of students' work and discussions were analyzed: their ability to correctly set up the volume integrals and their explanation of their volume integral. In the Integral Setup columns of Table 1, a designation of "correct" (green check mark) was made if the student set up the volume integral completely correctly. A designation of "almost-correct" (blue squiggle) was made if the student set up the integral correctly except for one small mistake, (e.g., bounds in wrong variable or shell height as bottom minus top instead of top minus bottom) or if the student corrected their incorrect integral with prompting from the interviewer. A designation of "incorrect" (red X) was made if the student made more than one mistake or made some sort of major error like incorrect variable of integration or incorrect formula. The entries in the Problem 1 Integral Setup column have two parts because, after their first integral setup, students were then asked to set up the volume integral for the same solid using a different method. Student T has a blank for the second entry of Problem 1 because I forgot to ask T to set up a second integral.

	Problem 1		Problem 2	
	Integral Setup	Explanation	Integral Setup	Explanation
R	✓/✓	✓	✓	✓
V	≈/≈	✗	✓	✗
T	✗/-	✗	✓	✗
S	≈/✗	✗	✗	✗

Table 1. *Pilot study 1 correct and incorrect responses chart*

In Table 1, the Explanation column information came from the students discussing why their integral set up gave a volume after being asked by the interviewer. A designation of "correct" (green check mark) in this column was made if the student could accurately describe their integral as a sum of small volume approximations with given measurements and corresponding formulas. To receive a green check mark here, students would have to discuss their volume integral in the context of the layers of Sealey's (2014) Riemann Integral Framework. In particular, they needed to mention at least one concept situated in the product layer and one concept situated in the summation layer. A designation of "incorrect" (red X) was made if a student's discussion did not include concepts related to products and summations.

Conclusions

In this study, I had a general research protocol, and I was able to extract some significant data from students. Unfortunately, I did not do the best job sticking to the interview protocol, so there were many instances of missed opportunities where I could have probed students deeper for their understanding of the concepts. For example, I neglected to ask Student T to set up an additional integral in Problem 1 using a different method, so I did not get information about their understanding of the differences in and similarities between the two methods when used in solving the same problem. Due to these instances of neglect of the protocol, I decided to make the protocol for my dissertation much more methodical and in the form of a checklist. This shortened and more technical format allowed me to not have to read as much mid-interview and to visualize what questions were done and what questions still needed to be attended to.

Also, in addition to asking about the volume integral formulas, I learned that I would have liked to ask about integrals in general, in order to extract further evidence of student understanding of the underlying limit of a sum of products structure of the definite integral. Finally, I decided that I would like to have more variety in the questions asked to students during the interview. Problem 3 was not analyzed because it became clear that three revolution problems was too many and it was not the right choice of problems for what I was trying to study. This was when I decided to include non-revolution volume problems so as to have the opportunity to ask more general questions about the connection between volume and the definite integral.

4.2. Pilot Study 2

The aim of the second pilot study was to refine the interview problems, interview protocol, and data analysis used in my dissertation study. In Pilot Study 1, students were only asked about revolution volume problems of varying difficulties. The participants tended to have very formulaic approaches to solving these types of problems, so in response to this, I added a geometric volume problem – one in which the geometric figure is given as a 3-dimensional solid and not formed by a revolution of a bounded 2-dimensional region in the *xy*-plane.

Another adjustment that was made was the method of recording students' written responses. In Pilot Study 1, it was difficult to see when the participant was transitioning their attention between their symbolic work and their drawings. In response to this, I tested out two different methods in this study: 1) symbolic and integral work on a sheet of paper and drawings on a separate sheet of paper, and 2) symbolic and integral work on a sheet of paper and drawings on a white board.

The participants in this study were two second-semester calculus students attending West Virginia University (pseudonyms: Carrie and Kevin). Volunteers were recruited from two different Calculus 2 classes, and one-on-one, video-recorded interviews were conducted outside of the classroom setting. The participants were asked to solve three different integral volume problems – two traditional revolution problems and one geometric volume problem – and to think out loud and discuss their problem-solving strategies. The problems are listed below.

1. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2 + 1$, $x = 2$, $x = 3$, and $y = 0$, about the y-axis.
2. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ about the line $y = -1$.
3. Find the volume of a pyramid whose base is a square with side length L and whose height is h.

Each of the two participants was asked to do their symbolic mathematical work on a sheet of paper and their drawings (graphs, pictures, etc.) on a separate writing surface. The first student (Carrie) drew on a separate piece of paper and the second student (Kevin) drew on a white board.

Data Analysis

In the first pass at analyzing Pilot Study 2 data, I looked at the participants' accuracy in producing the correct volume integral and their subsequent explanations of those volume integrals. Their performance on these measures is summarized in Table 2.

	Problem 1		Problem 2		Problem 3	
	Integral Setup	Explanation	Integral Setup	Explanation	Integral Setup	Explanation
C	✓/✓	✗	✓/✓	✗	✗	✓
K	✗/✗	✗	✓/-	✗	✗	✓

Table 2. *Pilot study 2 correct and incorrect responses chart*

The distinction between correct and incorrect in the Explanation column was more nuanced and was judged differently for the revolution problems versus the geometric solid problems. A red X ("incorrect") in the Explanation column was given as a result of the student not being able to describe or inaccurately describing the volume integral as a summation of small volume pieces with certain measurements. A green checkmark ("correct") in the Explanation column only appeared in Problem 3. To get a green checkmark for the revolution volume problem Explanation column, the student must correctly identify all pieces of the three dimensions that comprised the volume measurement, along with the correct formulas for each. For the geometric solid problems, the participants never accurately developed formulas to work with, so they could discuss the measurements in more general terms (without formulas), and still achieve a designation of “correct”.

Conclusions

As a result of Pilot Study 2, I realized several things that led to my decision to change my interview problems for my dissertation. First, I was able to get plenty of information about student understanding of revolution problems from one revolution problem; I did not need two. I decided to keep Problem 2 and discard Problem 1. Second, the pyramid problem seemed very challenging compared to the revolution volume problem so I added another non-revolution volume problem that included a given bounding function. My intention here was to provide a non-revolution volume problem, but with more information given than what is given explicitly in the pyramid problem.

Chapter 5: METHODS

To obtain a sufficiently deep and thorough investigation of students' understanding of definite integrals in the context of volume problems, semi-structured, one-on-one interviews were selected to be the mode of data collection for this qualitative study. I will first discuss the participants and how they were recruited. Then, I will describe the interview questions and protocol, along with the interview environment. Two different instances and layers of analysis were done, so I will also describe the process of data analysis and provide examples.

Participants

Participants were recruited from summer Calculus 2 and Calculus 3 (Multivariable Calculus) classes that were held at a large, public university in the Northeast. The Calculus 2 students were recruited after learning about volume applications of integration, and the Calculus 3 students were recruited before learning about multiple integrals. The goal in recruiting at these specific benchmarks was to have participants who had experience solving single-integral volume problems, but not multiple-integral volume problems. In an earlier pilot study (see Chapter 4), differential equations students solved the problems contained in this study, and their prior experience with finding volumes via multiple integrals served as a confounding factor that produced data outside the scope of the research questions. For the current study, ten students were recruited – five from Calculus 2 (Ali, Blair, Casey, Dana, and Erron) and five from Calculus 3 (Francis, Glenn, Hao, Iris, and Jay). Three of the Calculus 3 students had previously taken Calculus 2 with the researcher, but none of the participants were in a class with the researcher at the time of the interviews.

Interviews

Student interviews were conducted over the course of four weeks during the summer 2018 semester. These interviews were one-on-one with the interviewer, and the sessions were video recorded. The participants were not compensated monetarily or academically for their time, but they were provided with light refreshments during the interview.

During each interview, the participants worked through the three single-variable calculus volume problems listed below. The pictures in Figure 2 are provided for the reader's convenience in visualizing the regions and solids but were not provided to the students during the interviews.

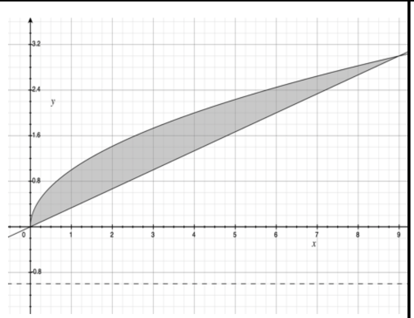
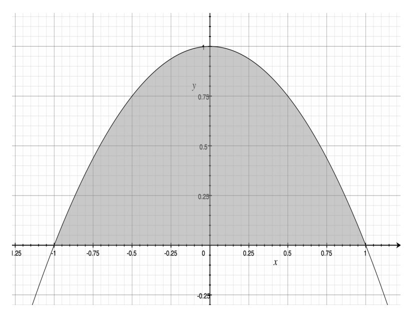
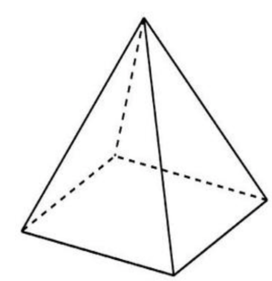
		
Problem 1	Problem 2	Problem 3

Figure 2. *Regions associated with the volume problems.*

<u>Problem 1</u>: Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ about the line $y = -1$.

<u>Problem 2</u>: Find the volume of the solid *S* whose base is the region enclosed by the parabola $y = 1 - x^2$ and the *x*-axis.
(2a) Cross-sections parallel to the *y*-axis are squares.
(2b) Cross-sections perpendicular to the *y*-axis are squares.

<u>Problem 3</u>: Find the volume of a pyramid whose base is a square with side length *L* and whose height is *h*.

Problem 1 was chosen because it was predicted to be familiar and relatively low stress to students as a problem to start the interview. It also acted as a tool to extract students' baseline understanding of the relationship between the definite integral and volume. Problems 2 and 3 were chosen as problems that required a more detailed focus on the underlying structure of the definite integral as a sum of small volume approximations. Problem 2 was chosen specifically because it was a non-revolution volume problem that gave an explicit function that bounded the base of the solid. Problem 3 was chosen because it was a familiar shape, but an explicit bounding function was not provided.

As students worked through the problems, they were asked to explain their work and reasoning aloud as they wrote and thought about the problems. In addition to what students naturally spoke about, there was an interview protocol that was followed by the interviewer and consisted of a checklist of questions that were to be asked by the interviewer in order to collect consistent data across interviews. The checklist varied somewhat between problems, but in general it served as a guide for the interviewer to ask all participants the same type of questions. Below are some examples of questions from the checklist (see Appendix A for full protcol).

- Why does an integral give a volume?
- Why does that integral give the volume of that solid?
- What does the dx or dy in the integral mean?
- What is the shape of one slice?
- Why did your previous (revolution) volume integral contain π and this one does not?

To capture students' written work, drawings, and graphs, students were asked to do their symbolic work on paper and any drawings or graphs on a separate large, portable white board. The written surfaces (paper and white board) were separate so that I could more effectively note when students were attending to their symbolic work and when they were attending to their sketches.

Data analysis

The data was analyzed in two phases, all according to the thematic analysis (Braun and Clarke, 2006) method. This is a qualitative analysis method in which themes are identified from the data either from an inductive or deductive (theoretical) approach. This is different from a grounded theory (Strauss and Corbin, 1998) approach in that I did not code the data with the intention of building a new theory. Instead, my intent was to extract and examine various themes from student interviews and discuss them in relation to the definite integral.

After interviews were transcribed, the first phase of data analysis involved coding for students' correct and incorrect volume integrals. Because there was a specific element of the data that I was attending to, this was considered to be theoretical (deductive) thematic analysis. As coding began, I discovered that there were different levels of correctness that I wanted to attend to, so I named these levels "symbolic structure," "symbolic details," and "conceptual understanding." *Symbolic structure* pertains to the placement of elements within and the configuration of students' integral representations. This category is equivalent to Sherin's (2001) "symbolic template," a schema in which a student knows that symbolic expressions go in certain areas or boxes. For example, a student would get a designation of "symbolic structure correct" using the washer method for Problem 1 if their written response had the form $\int_{\blacksquare}^{\blacksquare} \pi(\blacksquare^2 - \blacksquare^2)\, d\blacksquare$, regardless of the accuracy of the mathematical expressions that appeared in the boxes. *Symbolic details* are the actual symbolic expressions that reside in the boxes of the symbolic structure. Thus, it was impossible for a student to get a designation of "symbolic structure incorrect" and "symbolic details correct" for any problem.

Conceptual understanding was determined from students' verbal explanations about their understanding of the volume integral in general. Because this level of understanding is not necessarily linked to the symbols in the integral directly, it was possible for a student to receive a designation of "conceptual understanding correct," even if they got neither the symbolic structure nor symbolic details correct. For example, if a student indicated that they knew a pyramid's volume could be approximated by the sum of the volumes of very thin stacked boxes, but they could not build a corresponding integral, this student would have correct conceptual understanding but incorrect symbolic structure and details. It was necessary, but not sufficient, for a student to discuss their understanding of the definite integral using at least one of Sealey's (2014) Riemann Integral Framework (RIF) layers of Product, Summation, and/or Limit. See below for a more detailed discussion of the RIF Framework layers.

The second phase of data analysis involved coding interview transcriptions using Sealey's RIF. Again, as I went into the data analysis looking for specific types of student understanding, this phase was also considered thematic (deductive) analysis. In phase two, I looked for any phrases that would indicate the student was thinking of the problem from within a certain Riemann integral layer. When in the Orienting pre-layer, students attended to the integrand and/or the dx separately and not as elements of a product. The Product layer required students to see the integrand and dx interacting to form a product. A designation of Summation layer occurred when a student discussed the integral as signifying a summation of pieces or quantities. Lastly, students were within the Limit layer when they explicitly mentioned limits or infinity in the context of their integral. Also considered Limit layer thinking was if a student mentioned tiny pieces (size of x approaching zero, or $\Delta x \rightarrow 0$) or many pieces (number of pieces growing without bound, or $n \rightarrow \infty$). No students in this study reflected on their volume integral as representing a function, so the Function layer will not be discussed in this paper. Some example coded transcript excerpts follow.

<u>*Orienting layer*</u> (Ali, Problem 3): Because I know that I want to solve for, I want to solve for the areas of the squares (cross-sections of the pyramid). And I know that whatever I get is squared because, obviously it's a square. And I know that I'm going to be doing it in *dy*.

The above excerpt was coded as Orienting because Ali was attending only to the cross-sectional squares' areas in their discussion, which is the volume integral's integrand. Although they are talking about the pieces of a product, they do not discuss the pieces as a product, but as separate entities. They also mention that the differential *dy* acts as a signal for the variable of integration rather than an element with a physical component.

<u>*Product layer*</u> (Francis, Problem 1): Well, volume is just length times width times height. [*pause*] *dx*? That (*dx*) is going to be the width. This ($\sqrt{x} - \frac{1}{3}x$) is like the height. [*In V=lwh, labels h as* $\sqrt{x} - \frac{1}{3}x$ *and w as dx.*]

The excerpt above, which is mathematically incorrect, was coded as Product due to Francis discussing volume as being composed of products and then labeling the pieces of the product with their integrand and *dx*. Francis's written work at this point in their problem solving did not actually involve an integral sign yet, but they were in the process of building it from the inside out.

<u>*Summation layer*</u> (Francis, Problem 3): Because you use integrals to sum information together. That's the simplest way to put it. No matter what you're solving, the integral will give you the sum, because it comes from the Riemann sum.

The Summation layer discussions were relatively easy to identify because of students' use of words like "sum" and "add." Participants in this study approached their discussion of the underlying sum from two views: *sum* as a static noun and *sum* as a dynamic verb. In the quote above, Francis used both of these views in their discussion.

> Limit layer (Dana, Problem 1): OK, I had to think through it, but it turns into the volume of a rectangular prism, where you have thickness *dy* which would just become infinitely thin.

Here, Dana was working through Problem 1 using the shell method and discussing how it is really just finding the volume of a rectangular prism that is rolled around to form a shell. Dana also mentioned that *dy* has a physical component (thickness) that gets infinitely thin, which indicated that they were thinking within the Limit layer. When discussing the integral from within the Limit layer, there were two views: infinitely many and infinitely small. This makes sense given that those two ideas work in tandem. These two views also show up in different definitions of the definite integral – one in which the number of approximating intervals approaches infinity ($n \to \infty$) and one in which the size of the intervals approaches zero ($\Delta x \to 0$).

After phase one and phase two of data coding were complete, the data was organized into several different charts. The symbolic structure, symbolic detail, and conceptual understanding codes (phase one of coding) were collected in a visual chart according to student and problem number (see Tables 5, 6, and 7 in Chapter 6). In this chart, I looked for patterns across students and across problems. The RIF codes (phase two of coding) were also collected in a visual chart, organized according to student and problem number. Finally, these two charts were combined with the goal of finding connections and patterns between students' performance on symbolic and conceptual aspects compared to their abilities to discuss their work from within the layers of the RIF.

Chapter 6: DATA AND RESULTS

This research study was conducted with the intent of examining student understanding of the definite integral when solving calculus volume problems. Problem 1 provided data on how students understand and use the definite integral in volume problems that are familiar to them (Section 5.1), while Problems 2 and 3 provided data on how students use (or do not use) the definite integral for volume problems they are not familiar with (Sections 5.2 and 5.3). I was also able to examine how their definite integral conceptions from Problem 1 carried over into Problems 2 and 3, by using Piaget's (1977) assimilation and accommodation to describe these occurrences. All participants will be referred to using the singular "they," as preferred pronoun usage was not asked during the interview and is not a factor in this study.

6.1. Revolution volume problem (Problem 1)

As discussed in Chapter 5, there were three designated levels of correctness that were analyzed for each problem. *Symbolic structure* was the form of the volume integral (for example, a disk method revolution volume integral would have symbolic structure $\int_{\blacksquare}^{\blacksquare} \pi(\blacksquare)^2 d\blacksquare$). *Symbolic details* involved the numerical bounds, the function(s) in the integrand, and the variable of integration. Finally, *conceptual understanding* was knowledge and use of the underlying structure of the volume integral.

In the sections that follow, I discuss results related to two combinations of symbolic and conceptual understanding. In Section 6.1.2.1, I present data for students who exhibited correct symbolic structure and detail knowledge along with having accurate conceptual understanding. This combination is of interest for two reasons: it allows us to see how students verbalize their correct conceptions, and it allows us to have a baseline to compare their performance on and approach to the unfamiliar non-revolution volume problems (Sections 6.2 and 6.3). In Section

6.1.2.2, I present data for students who had correct symbolic structure and detail knowledge, but had inaccurate conceptual understanding. This combination is what we hope to avoid when it comes to student learning. The data in 6.1.2.2 serves as evidence that students can produce symbolically correct revolution volume integrals, but have inaccurate or incomplete understanding of why their answer is correct. This data can also allow us to see what types of incorrect conceptions students hold, so that we can work to interrupt or challenge those misconceptions before they become part of students' definite integral schemes.

Problem 1: Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ about the line $y = -1$.

6.1.1. Problem 1 – Symbolic structure and symbolic details

The revolution problem (Problem 1) was recognized by all participants as a type of problem they had seen in a current or past calculus class. All students had either used or heard of the two methods (washers and shells) for solving revolution volume problems. Eight of ten students interviewed were able to get at least some symbolic part (either structure or details) of Problem 1 correct. Six of ten students were able to set up a completely correct volume integral for Problem 1 using the washer method, while only three of ten were able to set up a correct shell method volume integral. Students' detailed volume integral written responses are given in Table 3, and a summary including symbolic structure and symbolic details performance is given in Table 4.

	Washer Method	*Cylindrical Shell Method*
Correct→	$\int_0^9 \pi\left[\left(\sqrt{x}+1\right)^2-\left(\frac{1}{3}\,x+1\right)^2\right]dx$ *inner radius* *outer radius*	$\int_0^3 2\pi(y+1)(3y-y^2)dy$ *radius* *height*
Ali	correct	correct
Blair	correct	$\int_0^3 2\pi(1+3y)(y^2)dy$ incorrect shell height & radius
Casey	correct	$2\pi\int_0^3 (y+1)(y^2-3y)dy$ incorrect shell height
Dana	correct	correct
Erron	correct	correct
Francis	$\int_0^9 \pi r^2\left(\sqrt{x}-\frac{1}{3}x\right)dx$ Was not sure of method, incorrect for both	
Glenn	$\int_0^9 \left(\sqrt{x}-1\right)^2-\left(\frac{1}{3}\,x-1\right)^2\;dx$ incorrect washer radii, no π	"can't be done with shells"
Hao	correct	$2\pi\int_0^3 (3y+1)(3y-y^2)dy$ incorrect shell radius
Iris	$\int_0^9 \left(\left(\sqrt{x}\right)^2-\left(\frac{1}{3}x\right)^2\right)(y+1)(2\pi)dy$ incorrect washer radii, extra term, 2π instead of π, incorrect variable	"I don't believe it can be done with shells"
Jay	$\pi\int_{-2}^0 \left[x\left(\frac{1}{3}x-1\right)+1\right]^2-[x+1]^2$ incorrect bounds, incorrect washer radii, no *dx*	$2\pi\int_0^2 y(y^2-3y)dy$ incorrect bounds, incorrect shell radius, incorrect shell height

Table 3. *Detailed student symbolic structure and symbolic detail work on Problem 1 (revolution problem)*

As a reminder, Ali, Blair, Casey, Dana, and Erron were Calculus 2 students, and Francis, Glenn, Hao, Iris, and Jay were Calculus 3 students at the time of the interviews. The interviews were conducted right after the Calculus 2 students covered volumes and right before the Calculus 3 students covered multiple integrals. It can be seen in Tables 3 and 4 that the Calculus 2 students performed significantly better than the Calculus 3 students on Problem 1.

	Students:	Ali	Blair	Casey	Dana	Erron	Francis	Glenn	Hao	Iris	Jay
Structure	Problem 1 (washers)										
Details											
Structure	Problem 1 (shells)						unsure method	not valid		not valid	
Details											
Correct											
Incorrect											

Table 4. *Summarized student symbolic structure and symbolic detail work on Problem 1 (revolution problem).*

As can be seen in Table 4, Ali, Dana, and Erron got both washer and shell method volume integral setups completely symbolically correct. Blair, Casey, and Hao got their washer method volume integral symbolically correct, but their shell method integral incorrect. Blair's shell-method integral had two mistakes: the shell radius and shell height. Blair had a radius of $1 + 3y$ instead of the correct $1 + y$, and a height of y^2 instead of the correct $3y - y^2$. Blair seemed to have neglected the fact that the height of the shell was bounded by two functions, rather than just one. Casey had a small error in their shell height, putting $y^2 - 3y$ instead of the correct $3y - y^2$. This flipping of the shell height was common, also occurring with Hao and Jay.

Only the Calculus 3 students made errors of the symbolic structure type. For example, Francis produced the integral $\int_0^9 \pi r^2 \left(\sqrt{x} - \frac{1}{3}x\right) dx$ which does not fit into one specific correct structure but instead contains elements reminiscent of both the washer method (πr^2 as disk area) and shell method ($\sqrt{x} - \frac{1}{3}x$ as a shell height). Similarly, Iris's washer method integral $\int_0^9 \left(\left(\sqrt{x}\right)^2 - \left(\frac{1}{3}x\right)^2\right)(y+1)(2\pi)dy$ also has elements that resemble both washer and shell method.

6.1.2. Problem 1 – Conceptual understanding

When setting up their revolution integrals, all students were highly formula-focused and did not initially go into detail concerning their deeper understanding of the underlying structure of their volume integral. Thus, after they set up their integrals, the interviewer then questioned them on their understanding of the concepts (some of these questions are mentioned in Chapter 4 Methods).

For Problem 1, a student obtained a designation of "correct conceptual understanding" if they were able to describe how their integral (or an integral in general) can represent a volume. A necessary (but not sufficient) condition for the "correct conceptual" designation was that the student had to discuss their integral from within at least one layer of the Riemann Integral Framework (RIF). Below, I will present two instances of interest that arose during the interviews: (1) students having correct symbolic structure/details with accurate conceptual understanding, and (2) students having correct symbolic structure/details with inaccurate conceptual understanding. See Table 5 for summarized information on symbolic structure, symbolic details, and conceptual understanding results for Problem 1 and Table 6 for students' discussions of their understanding of Problem 1 from within the different layers of the RIF.

	Students:	Ali	Blair	Casey	Dana	Erron	Francis	Glenn	Hao	Iris	Jay
Structure	Revolution Problem (washers)	■	■	■	■	■		■	■		
Details		■	■	■	■	■			■		
Concept				■							
Structure	Revolution Problem (shells)	■	■	■	■	■	unsure of method	"shells not valid"	■	"shells not valid"	■
Details		■			■	■					
Concept				■	■						
Correct	■										
Incorrect											

Table 5. *Data on student performance on symbolic structure, symbolic details, and conceptual aspects of Problem 1 (revolution problem)*

Student	Ali	Blair	Casey	Dana	Erron	Francis	Glenn	Hao	Iris	Jay
RIF Layers (Prob 1)	Orienting	Orienting	Orienting Product	Orienting Product Sum Limit	n/a	Product Sum Limit	n/a	Orienting	Orienting	Orienting Sum

Table 6. *Student discussions of Problem 1 from within the different Riemann Integral Framework Layers.*

6.1.2.1. Problem 1 – Correct symbolic structure/details and accurate conceptual understanding (Casey and Dana)

Two students (Casey and Dana) were able to successfully build correct revolution volume integrals while also giving accurate responses to questions involving the underlying concepts of the definite integral. Casey was able to do so while talking through the washer method, but it took Dana until their shell method setup to make progress on explaining the conceptual side of their integral.

Casey started Problem 1 using the washer method and discussed their understanding by focusing on a general relationship between integrals, derivatives, and antiderivatives. Casey was unsure while engaging in this discussion, so the interviewer steered the conversation toward the meanings of the specific integrand pieces. As Casey talked through the meaning of the pieces, they began to get more confident in their answers and began to see the connection between the area formulas and the volume.

Interviewer: You took this [*points to outer radius function*] and put it there and squared it, and then you took this [*points to inner radius function*] and put it there and squared it. So, what are those two pieces representative of?

Casey: Those two pieces are representative of a distance on the graph. So, the first one, the big R, is the outer circumference of your overall volume sphere bit. Um, and you're subtracting the smaller piece, the inner radius, because you don't need that, you just need that outside piece.

Interviewer: OK, how come they're squared?

Casey: That one I don't know. Um, (pause) If you were to prove this, or (pause) I feel like it has something to do with the area of a circle on the coordinate plane, when you're building that equation.

Interviewer: What is the area of a circle?

Casey: $2\pi r$. Or r (pause) πr^2 is the area, $2\pi r$ is the circumference.

Interviewer: OK, so, one more question, we kind of talked about all the things, what is the *dx*?

Casey: The *dx* is the width of the, your disk. Which is more easily represented on the graph as the change in *x*.

Interviewer: OK, so how when you put that stuff all together, how does that give you a volume? Say, for example, the volume of one disk.

Casey: Um, so, I actually understand this much more now that you're making me answer these questions. Um, so, it's a small cylinder is what we're representing with the graph. So, the area of the circle is the outer face, and the *dx* is the width, your height of the cylinder. So, you're multiplying that one face of the circle all the way through the entire cylinder, that's a prism.

Casey started this conversation focusing on the specific parts that comprise the integral because the interviewer led them in that direction. The interviewer also led them to consider the *dx* as a separate piece. These two parts of the discussion would fall within the Orienting pre-layer, which is where a student considers the pieces of the product as separate entities. The interviewer then asked Casey to consider what happens "when you put all that stuff together," meaning considering the area formula and the *dx* together. Considering the area formula and the *dx* together as a product that produces a volume lies within the Product layer of the RIF. So technically, Casey did not consider these things on their own. This being said, although Casey did not make this journey alone, they did so comfortably and mathematically soundly, by translating the interviewer's prompts into correct observations connecting the integral and volume. Moreover, after attending to the Product layer, the student expressed, "I actually understand this much more now that you're making me answer these questions." This is evidence that having students think about and discuss their integral set-ups can result in a deeper understanding of the underlying structure of the definite integral.

During this discussion, Casey described the *dx* as having a physical property ("width of the disk") and as a change in the *x*-value. Earlier in the interview, Casey mentioned the *dx* as signaling the variable of integration, but they were able to move between different conceptions of the *dx* depending on which aspect of the problem they were discussing. Although they did not mention anything about the size of *dx* (which could possibly signal Limit layer thinking), they were able to think of multiplicative interactions between the integrand and the *dx*, signaling Product layer thinking (Table 6).

Another aspect related to *dx* that came up in both Casey's washer and shell method problem-solving was how they decided which method to use. In particular, Casey's choice of washer method or shell method stemmed from the orientation of the rectangle to the line of rotation (Figure 3).

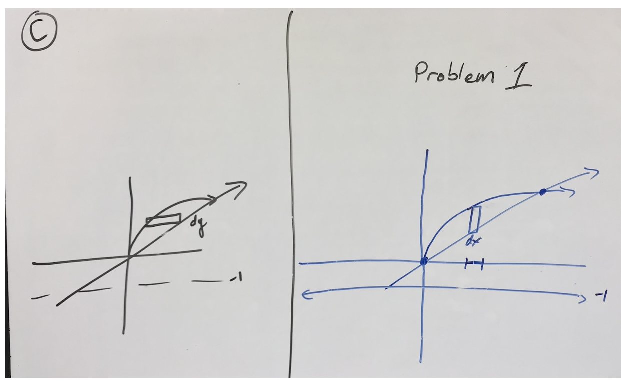


Figure 3. *Casey's sketches for Problem 1.*

Interviewer: OK, why did you choose this (washer) method?

Casey: Uh, because the shape (the rectangle) is perpendicular to the axis of rotation. Um, even though disk/washer and the other one, they're the same thing, it's just depending on which way your shape (rectangle) is faced.

Interviewer: So when you say your shape, you mean that little guy right there? [*points to vertical rectangle*]

Casey: Yeah, yeah, the rectangle. It depends on which way he's oriented on the graph.

Interviewer: But with respect to the axis of rotation?

Casey: Yes. Yeah, so if he's parallel, you'd do the other one I didn't do.

Interviewer: Does the other one have a name?

Casey: Yeah, it does. Disk/washer, and um, cone.

Interviewer: Cylindrical shells.

Casey: Yeah, shells. Shells, that's it.

For the shell method of Problem 1, Casey built an almost-correct integral, with only the height component being incorrect, having $y^2 - 3y$ instead of the correct $3y - y^2$. Casey discussed the pieces of the shell method integral in detail ($2\pi r$ as a circumference, h as a height, dy as a thickness), but they did not go as far as to put the pieces together to discuss a volume from the view of the Product layer. The interviewer did not probe as deeply for this part as they did for the washer method.

In summary, Casey was designated as "conceptually correct" for Problem 1 because they exhibited Product layer thinking of the integrand and the *dx*; they were able to consider *dx* as a flexible notation, standing for the variable of integration and the width of a slice, depending on what context it was being discussed; and because they attended to the volume of a representative slice of the solid. As mentioned above, Casey started out in Problem 1 with a function-matching conception of volume, but it evolved to a conceptually robust conception during the process of the interview, which is evidence of Casey accommodating his previous understanding with newer, more accurate knowledge.

Like Casey, Dana was also successful in their discussion of Problem 1, but their success was not as quick and continuous as Casey's. In fact, Dana never completely described the washer method integral in a way that would be considered mathematically accurate. It took them until their following shell method attempt to put together a coherent, accurate description of the underlying concepts.

Dana began Problem 1 using the washer method and set the integral up symbolically correctly in their first attempt. As Dana was discussing their thought process, they were connected to a conception that the definition of the integral was specifically connected to area.

Dana said, “because, thinking of the definition of the integral, to find the area under the graph, you split it into rectangles, infinitely smaller rectangles. So you could do that with this region.” This conception lingered through the duration of their washer method explanation. As seen in the following excerpt, Dana struggled to assimilate the idea of “integrals give volumes” with their prior knowledge that “integrals give areas.” They eventually began to accommodate this new information by bringing in a third variable in the form of *dz* to account for the third dimension that their 2D area integral was missing.

Interviewer: So, you jumped to that general formula pretty quick, how do you know that an integral actually gives a volume?

Dana: The way I think about it is, I have this region, which I can use an integral to find the area of. And then, if I’m revolving that region around this point, that means I’m just finding the area of this region, over and over, infinitely many times, for infinitely smaller changes in d-something, maybe *dz*, because it goes to the third dimension around this.

Dana’s discussion frequently referenced the Limit layer. Below is their response to the interviewer’s question, “why does an integral give a volume?,” first asked during their working of the washer method for Problem 1.

Dana: And the area of the circle I know is πr^2, so then that’s the, Area 1 would be the area of this big circle. Circle 1. And then Area 2 is also πr^2. And I can change this *r* in area 1 to big *R*. So, to find the area of this cross-section of the washer, take that big area A1 minus A2.

Interviewer: OK, so those are areas, how come your integral gives a volume then?

Dana: Because like I said, this is just one sort of small, infinitely small piece of the whole washer, and then, like in my mind I want to think about like, you need to use another integral to find like. So, if this is the washer from the side, you almost need to use another integral to find all of these up through here. But I don’t, and I really don’t understand that part.

Next, the interviewer asked Dana about *dx*, to which Dana had connected two common conceptions: variable of integration and orientation of the rectangle to the axis of rotation. It wasn't until later in the interview that Dana discussed *dx* as representing a physical quantity. Dana then worked on Problem 1 using cylindrical shells method and continued to have a hard time juggling the fact that an integral, which represents an area, can also represent a volume.

As before, Dana built a correct integral from a memorized formula and the interviewer asked them about their conceptual understanding of the formula. The quotes that follow are sequential, but parts of the transcript that did not illustrate Dana's forward progress were omitted (see Figure 4 for Dana's sketch with the measurement *r* inserted for clarity).

Interviewer:	So, why does all that stuff give you a volume? How do you know that integral gives you a volume? What are the pieces that would contribute?
Dana:	… the shell method has its roots from the volume of a cylinder. (pause) Here is my little region. *dy* and *h*. And I'll revolve around this line. It kind of makes a cylinder.
Dana:	… I'm pretty sure the way that this one works, is that it takes this cylinder and sort of unwraps it. So that instead of a cylinder you have like this rectangle, and you find the area of that, which is the *r* times *h*, and then times, you also have to multiply by the area of the circle on either side. So that's, again, that's πr^2. I might be wrong about this actually.
Dana:	No, it has something to do with circumference, not area, of the circles. This is as far as I know for sure, you take this cylinder, or you think about it as like a cylinder when it's done, sort of unwrap it, and you um. So, I guess this rectangle would have a thickness of *dy*.
Dana:	You can find that circle's circumference with $2\pi r$, so I know that's where this 2π in the formula comes from.
Dana:	OK, I had to think through it, but it turns into the volume of a rectangular prism, where you have thickness *dy* which would just become infinitely thin. And then *r* is sort of your height, I guess if you look at it like this, it's your height, but it's this distance [motions from rectangle to line of rotation] times your height, and that's the volume of the rectangle prism which when you wrap it around this line forms a shell.
Interviewer:	So did you know that before going into that, or did you just work that out right here?
Dana:	I knew beforehand sort of thinking about a cylinder and unwrapping it to find the area or the volume of that. … I took myself down the wrong path thinking about a rectangle instead of a rectangular prism, so that much, I had to work that out.

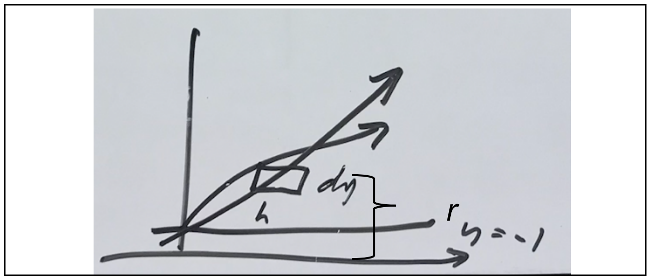


Figure 4. *Dana's sketch for Problem 1.*

Dana's evolution of understanding of the cylindrical shell conception took place over the span of about five minutes, during which they talked from within many different layers of the RIF. In particular, the Limit layer appeared once again, illustrating how strong that idea was for Dana and their understanding of the definite integral. Another thing to note in this excerpt is Dana recognized that they had to adjust their understanding of "integrals give areas" using rectangles to accommodate the new knowledge that "integrals give volumes" using rectangular prisms.

Dana's understanding for the shells method of Problem 1 was labeled as "conceptually correct" because they exhibited Product and Limit layer thinking (Table 6), and they considered the *dy* as the thickness of a representative shell. Dana already had an understanding of the underlying structure of the volume integral for shells, but in the process of talking through their understanding, they solidified and expanded their understanding, showing evidence of accommodation of current schema with newer knowledge.

6.1.2.2. Problem 1 – Correct symbolic structure/details and inaccurate conceptual understanding

The most common combination of results for Problem 1 was from students who demonstrated an incorrect conceptual understanding, even though they were able to produce a symbolically correct integral. The students I will focus on in this section are those who got at least one instance of correct symbolic structure *and* symbolic details, along with incorrect conceptual understanding: Ali, Blair, Erron, and Hao. Francis and Iris were unable to get any part of the symbolic integral setup correct, while Glenn and Jay were only able to get the symbolic structure correct.

Ali was able to produce correct symbolic answers for both the washer method and cylindrical shell method for Problem 1. When asked why the integral gives a volume, Ali focused solely on the area functions in the integrand, which was evidence of only Orienting pre-layer thinking (Table 6).

Interviewer: So, what do you know about the relationship between the integral and volume? How do you know that an integral gives you a volume?
Ali: Um, whenever I'm looking for the volume, I tend to just remember the fact that the area of a circle is πr^2. And that the area of, like the surface area of the cylinder is um oh $2\pi r$, 2π height times the radius. And then I just kind of figure out which one (washer or shell method) I think is going to be easier to solve.

Like all participants in this study, Ali started out their discussion focusing on the details of the area functions under the integral rather than the conceptual idea behind the volume integrals. Ali was correct with their statements about the area functions having the structures of πr^2 for washer method and $2\pi rh$ for shells, but more information was needed to discover if Ali was considering a third dimension. The interviewer then asked Ali about the meaning of the *dx* associated with the integral.

Interviewer: OK, so what part does the *dx* play?
Ali: The *dx* is, it's in every integral. It's just the derivative of *x* and when you take it out, it's, obviously its integral is *x*.
Interviewer: OK. So you're saying that that *dx* is an indicator of your variable.
Ali: Yeah, yeah.

Ali's conception of the volume integral consisted of the area function integrand and the differential *dx* as separate pieces that did not interact with each other aside from *dx* being "in every integral." Ali's inattention to the Product layer of the integral does not affect their ability to build the correct volume integral for Problem 1. This inaccurate conception carried through Ali's work, but it did not impede their ability to get correct symbolic answers for most of the problems. As we will see in Sections 6.2 and 6.3, Ali has an "adding areas" conception of the volume integral that works well with setting up a symbolically-correct integral, but hides the idea behind why a volume integral gives a volume.

Blair was successful in setting up the correct volume integral using the washer method but was not able to set up a correct integral using cylindrical shells method (incorrect shell radius and height). Blair discussed their conception of the integral early in the interview.

Interviewer:	So it seems like you had a formula memorized (for washer method), your πr^2.
Blair:	Yes, the volume is basically the integral of the area.
Interviewer:	OK, so um so why, why an integral? Why would we use integral to calculate volume?
Blair:	…Since we have circles here, we're doing it as, as the integrals of areas.
Interviewer:	OK, so how come when you integrate areas you get volume? What's that jump?
Blair:	Um, since the derivative of the volume is the area, so, I know it that way.

Blair knew of a derivative-antiderivative connection between area and volume but was unable to give any insight into why those connections occurred. This view is consistent with Jones' (2013) "function-matching" symbolic form of the definite integral in which students connected certain functions via a derivative-antiderivative relationship. For this study, I will state this in a way that is more specific to volume problems by calling it "integrating areas". This was a conception that was less refined than Ali's "adding areas" because, as we will see in Problems

2 and 3, Blair viewed the integral as somewhat of a machine that turned area into volume, regardless of which area function served as the integrand. Blair's understanding was considered to be conceptually inaccurate because their thinking resided solely in the Orienting pre-layer; they considered the definite integral only as a producer of function-matching between area and volume; and because they viewed *dx* only as a notational indicator of variable and having to do with the orientation of a representative rectangle (versus as being a component of a representative rectangle).

Erron got both the washers and shells versions of Problem 1 structurally and symbolically correct due to being very skilled with manipulating the memorized disk and washer volume formulas. When asked why an integral gives a volume, Erron's first response was, "um… because the formula does it?" When asked to go into more detail, they stated that if each part of the integrand was measured in miles (labeled as "*m*" above each part of the integral in Figure 5), then when you multiply the pieces of the integral together it "would be a unit cubed which is a volume" (labeled as "m^3" in Figure 5). This units-based discussion continued throughout Erron's Problem 1 discussion, hinging on "squared things are areas" and "cubed things are volumes." Erron contributed this focus on units to being a physics major.

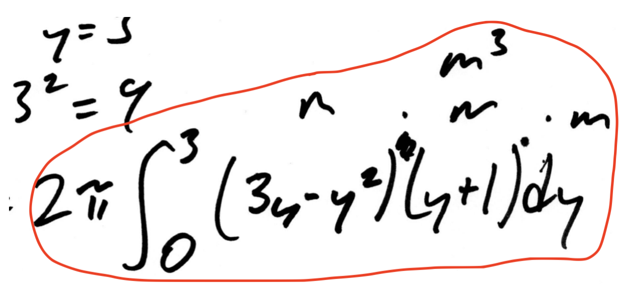


Figure 5. *Erron's shells integral for Problem 1 with "miles" marked above each part of the integral.*

Erron only referred to *dx* as "the derivative of *x*" or an indicator of the variable of integration throughout Problem 1 even though they assigned a unit measure to it in their units discussion. They also considered the *dx* (or *dy*) to be part of the product of terms under the integral, which is indicative of Product layer thinking, but not necessarily what the product represents.

Erron also had a visual understanding of the washers and shells, but those shapes never played a part in their discussion of why an integral gives a volume.

> Erron: You just kind of like have to make a mental image in your mind and rotate it (the rectangle) around the point and then say "oh yeah, that would look more like a shell" or "that would look like a little disk."

When asked to give more information about the revolution volume integral concept beyond units, Erron did not go deeper than over-explaining the relationship between their memorized formulas and distances (heights, radii, etc.) in their sketches. If not exposed to any other type of integral volume problem besides revolution volume problems, depending on memorized formulas and being able to manipulate them in different mathematical situations would work very well, as it has for Erron.

At noted in Table 6, Francis knew of general ideas of products, summations, and limits relating to the integral, but they were never able to fully decipher how those general ideas related to the specific solid in Problem 1. In Francis's volume integral of $\int_0^9 \pi r^2 \left(\sqrt{x} - \frac{1}{3}x\right) dx$, they were cognizant of their integrand and *dx* being physical pieces of a washer that were multiplied together (Product layer), but they could not accurately describe how they formed a volume. Francis also used phrases like "infinitely small piece of the total" and "the sum of spaces", which were tagged as Limit layer and Summation layer thinking, respectively, but they were not robust and accurate enough to warrant Francis a distinction of "conceptually accurate" because Francis could not pin these ideas down to the specific washers and solid in Problem 1. Francis serves as an example of how working within a certain layer of the RIF does not necessarily mean that the layer is fully understood in relation to the definite integral.

Hao set up the washer method volume integral correctly on their first try. When asked why they started with the disk method, Hao stated that it had to do with the orientation of the rectangle with respect to the line of rotation (in this case, perpendicular). The following is Hao's response to, "why does an integral give a volume?".

Hao: I am actually not really sure why. In this class was the first time I was introduced to an integral to find volume. Every other way, I've just known that it was length times width times height. And it's always a "cubed" as my answer. So besides that, the integral is, this is the first time I've seen it. I guess you could say the integral is used to find the volume of more complicated geometric things more or less.

Hao then went on to over-focus on the washer formula structure but gave no meaningful information to answer the interviewer's question. When asked about the *dx*, Hao stated that it was to indicate the variable of integration, and the *dx* above the rectangle in their picture served as a "visual aid" to help them know which method to use.

For the shell method, Hao got everything in the integral symbolically correct except for the shell radius, where Hao put $1 + 3y$ instead of the correct radius of $1 + y$. (Recall that one of the given functions was $y = \frac{1}{3}x$, so $x = 3y$.) This mistake is not uncommon for students, as the radii in the disk/washer method depend on the function, but for the shell method, the radius is related to the independent variable. Once probed more about the *dx* and *dy*, Hao stated that the rectangles themselves were the *dx* and the *dy* (as opposed to the rectangles having width *dx* and *dy*). Hao was never able to discuss their volume integral beyond the Orienting pre-layer. Their views on integration were very rigid and superficial. Once again, not being exposed to any other types of volume problems would lead to this kind of dependence solely on memorized formulas without the need to go deeper.

6.2. Non-revolution volume problem with given bounding function (Problem 2)

Problem 2: Find the volume of the solid S whose base is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. (2a) Cross-sections **parallel** to the y-axis are squares. (2b) Cross-sections **perpendicular** to the y-axis are squares.

As discussed in Chapter 4, Problem 2 was chosen with the intent of being a "gateway" problem to the pyramid problem (Problem 3). The solid in Problem 2 was free-standing – not the result of a rotation – and the statement of the problem contained explicit information about the functions bounding the 2-dimensional base of the solid. Table 7 gives detailed symbolic work for students on Problem 2, and Table 8 gives the symbolic structure, symbolic details, and conceptual performance summaries of the students for Problem 2. There are dashed lines for Dana under Problem 2b because they were not asked about the second half of Problem 2, so there is no data.

In contrast to Problem 1, Problems 2 and 3 were novel to all participants except Ali and Francis (this information was self-reported). The researcher of this study had two of the other Calculus 3 students in her Calculus 2 classes in a previous semester, and problems like Problem 2 and Problem 3 were covered. This discrepancy was not discussed during the course of the interviews.

Due to most participants viewing Problems 2 and 3 as novel, they had the opportunity to produce volume integrals from reasoning with their understanding of volumes and integrals rather than depending on memorized formulas. Six of ten participants arrived at a symbolically correct answer for Problem 2a, but none were successful on Problem 2b. Casey and Dana were once again successful in their conceptual understanding (for 2a), and as we will see in the next section, this success can be seen as building on their understanding of and success with Problem 1. The most common issue with Problem 2 was students' inability to understand and/or visualize the solid. I will first discuss Casey and Dana, then I will briefly discuss the four students who obtained symbolically correct integrals but who demonstrated inaccurate conceptual understanding.

	Problem 2a	*Problem 2b*
Correct→	$\int_{-1}^{1}(1-x^2)^2dx$	$\int_{0}^{1}\left(2\sqrt{1-y}\right)^2dy$
Ali	correct	$\int_{0}^{1} 2\left(\sqrt{1-y}\right)^2dy$
Blair	$\int_{-1}^{1}\left(x-\frac{x^3}{3}\right)dx$	No change from 2a
Casey	$\int_{-1}^{1} z(1-x^2)dx$ (correct)	No changes from 2a
Dana	correct	--------------
Erron	$V(z,r) = hzr$	$V(x) = 3\pi\int_{z}^{y}()()()dy$
Francis	correct	$\int_{0}^{1}\left(\sqrt{1-y}\right)^2dy$
Glenn	correct	No change from 2a
Hao	$\int_{-1}^{1}(1-x^2)dx$	$\int_{0}^{1}\sqrt{-y+1}\ dy$
Iris	$\int_{-1}^{1}(1-x^2)^2$	No change from 2a
Jay	$\int_{0}^{1}(1-x^2)-x\ dx$	$\int_{0}^{1}\sqrt{1-y}-y\ dy$

Table 7. *Detailed student performance on symbolic structure and symbolic details of Problem 2 (non-revolution problem with given bounding functions).*

	Students	Ali	Blair	Casey	Dana	Erron	Francis	Glenn	Hao	Iris	Jay
Structure	Problem 2a	■		■	■	■	■	■		■	
Details		■		■	■		■	■		■	
Concept				■	■						
Structure	Problem 2b	■		■	--------		■				
Details					--------						
Concept					--------						
Correct	■										
Incorrect											
Not applicable	------------										

Table 8. *Student performance on symbolic structure and symbolic details of Problem 2 (non-revolution problem with given bounding functions).*

Due to most participants viewing Problems 2 and 3 as novel, they had the opportunity to produce volume integrals from reasoning with their understanding of volumes and integrals rather than depending on memorized formulas. Six of ten participants arrived at a symbolically correct answer for Problem 2a, but none were successful on Problem 2b. Casey and Dana were once again successful in their conceptual understanding (for 2a), and as we will see in the next section, this success can be seen as building on their understanding of and success with Problem 1. The most common issue with Problem 2 was students' inability to understand and/or visualize the solid. I will first discuss Casey and Dana, then I will briefly discuss the four students who obtained symbolically correct integrals but who demonstrated inaccurate conceptual understanding.

6.2.1. Problem 2 – Accurate conceptual understanding (Casey and Dana)

Casey began Problem 2 not quite understanding the statement of the problem, so there were several minutes of discussion between Casey and the interviewer concerning the solid and what the problem was asking. This happened with most students in this study. The interviewer and Casey used a whiteboard eraser to illustrate the orientation of a slice of the solid, to which Casey responded, "an infinite amount of vertical cuts … I was assuming it wanted an integral, because it's calculus." To clarify, Casey was stating that to them, there was a connection between “an infinite amount of cuts” and “integral”. In Figure 6, the salience of the “infinite amount of cuts” are evident in Casey’s sketches. This shows that Casey puts a strong emphasis on the connection between the Limit layer and integration. From this point forward, Casey began building an integral from the pieces they extracted from the statement of the problem.

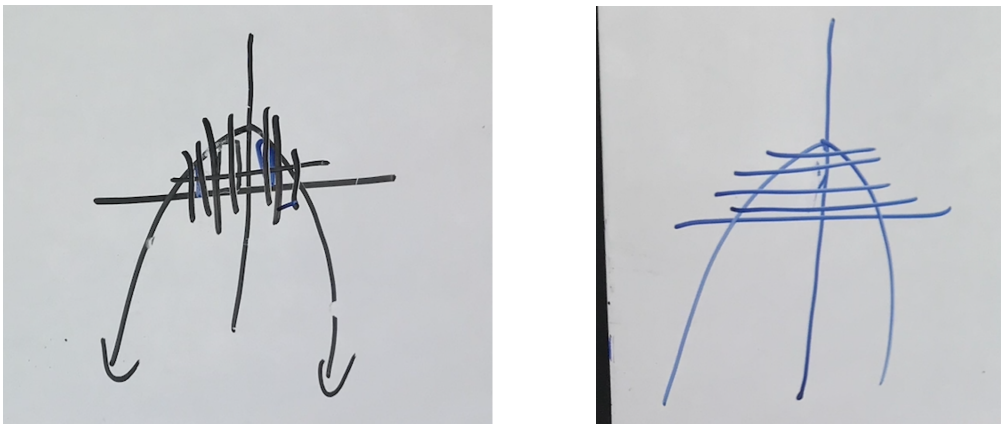


Figure 6. *Casey’s sketches for Problems 2a (left) and 2b (right).*

At first, Casey went into building the integrand as if they were using the washer method, but they corrected themselves without assistance from the interviewer.

Casey: It (the rectangle) is parallel (pause) or it’s perpendicular, so I’m going to be using the big *R* and small *r*. … No, I’m not going to use that equation. Because this isn’t ro— this, the problem doesn’t ask me about a rotating shape. It’s asking about a stationary shape. So I have *dx*. The height of the shape is the parabola, the equation of the parabola. One minus *x*-squared. And we have the bounds. The last thing we don’t have is the actual, the *z* dimension [*makes motion upward from the board*]. And I guess I’ll just label that *z*.

$$V=\int_{-1}^{1} z(1-x^2)\,dx$$

Figure 7. *Casey’s volume integral for Problem 2.*

The initial reaction of Casey starting the problem using washer method is indication of Casey attempting to assimilate this problem into their previous experience with revolution volume problems. As Casey talked through the problem and realized that there was no rotation, they had to accommodate by seeing Problem 2 as a different type of volume problem that needed to be solved by attending to each slice, rather than relying on a memorized integration formula. This could be seen as evidence of a carryover between Casey's discussions with the interviewer on Problem 1 concerning focusing on one washer, the specific pieces, and how they work together to form the volume of a slice. Casey then focused on finding three dimensions that could comprise the bounds of the slice: dx for width, the equation of the parabola for the height, and z for the 3rd dimension. Casey was paying attention to separate slices here, versus the whole solid, as indicated by them saying that the width of "the shape" is dx, as opposed to the width being the distance between the left and right bounding x-values (resulting in a width of 2 – the width of the entire solid).

For Problem 2b (slicing the solid perpendicular to the y-axis with cross-sectional shape of squares), Casey erroneously assumed that the solid had the same volume as the solid in Problem 2a so they did not build a new integral for 2b. Casey was also hesitant to consider horizontal, dy-thickness slices for Problem 2b because "you're representing a difference between the same equation" when considering the horizontal distance between the edges of the parabola.

Like many other participants in this study, Dana had significant trouble understanding the statement of Problem 2 – specifically, the meaning of "cross-sections (of the solid) are squares." As Dana talked through this issue of understanding the problem, they made the following statement, which could be viewed as the foundation of an understanding of multiple integrals.

Dana: I'll just take this, this whole area (region in the *xy*-plane under the parabola) and once I have an integral to find that area, then I would think, obviously this solid would have some height. And I guess I could think of that as, like another integral?

Dana obtained assistance from the interviewer about the visual details of the solid and cross-sections (the discussion consisted of comparing the solid to a loaf of bread and the slices to slices of the bread loaf), and the interviewer prompted Dana to consider one single slice of the solid. After that, Dana started to make more progress.

Dana: So that would be a square [*draws 2D square on board*]. And you said you slice it, so obviously it has some thickness [*gives square thickness to form a rectangular prism* (Figure 8)], so I guess it wouldn't really be a cube. But that thickness would be, the way I think about thickness is that it would be some difference in *x* [*writes dx along thickness of rectangular prism*], if you think of it on the coordinate plane. So just that, *dx* times the area of the face of the square.

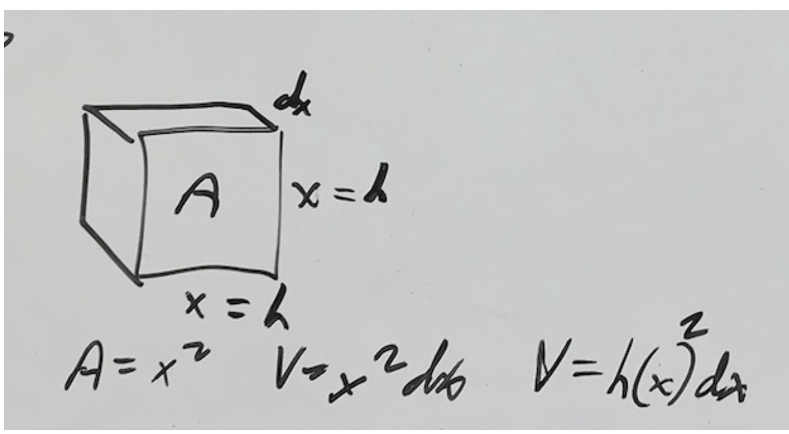


Figure 8. *Dana's drawing of an approximating slice of the solid in Problem 2.*

The idea of the slice relating to the volume showed up in Dana's last sentence, as they discussed the pieces via the Product layer ("*dx* times the area of the face of the square"). Although Dana had the correct idea generally and visually, they continued to have trouble relating the measurements of the slice to the function given in the statement of the problem. As seen in Figure 8, Dana started out by calling the side lengths of the square face x, then adjusted them to be $h(x)$. After some discussion with the interviewer, Dana ended up concluding that $h(x) = 1 - x^2$, allowing them to finish the problem by building the correct volume integral.

In both Problem 1 and Problem 2, the interviewer had to jump in and focus Dana on one slice of the solid, and they were able to continue correctly from there. The clearer evidence that Dana used knowledge from a previous problem comes in their work with Problem 3. A note: the interviewer decided to skip Problem 2b with Dana because the time was running long (49 minutes at this point), and it was clear Dana was starting to get fatigued.

6.2.2. Problem 2 – Inaccurate conceptual understanding

Four students in the study were able to obtain symbolically correct integrals for Problem 2a but without accurate conceptual understanding: Ali, Francis, Glenn, and Iris.

As with Problem 1, Ali was comfortable with building the area function integrand from the information given in the problem. But after further questioning from the interviewer, Ali's inaccurate view on the concept of the volume integral began to emerge.

> Interviewer: OK, so you're saying a bunch of stuff about the areas of the squares, but what I'm asking you to find is the volume. So how does that integral give you a volume? Because you're telling me about area.
>
> Ali: Because it's um, when you add all of the areas of each individual square up, you get the volume of the end shape.

This concept of the integral is in line with Jones and Dorko's (2015) "adding up slices without thickness" as well as Czarnocha's (2001) "indivisibles". Ali's view of the integral as "adding areas" ran throughout their interview, also showing up in their sketches. In Ali's sketch for Problem 2a (Figure 9), a representative 2D square (drawn by Ali in black, highlighted here in red) is drawn toward the middle of the region with no indication of thickness. The red square is meant to look as if it is extending out and upward from the 2D whiteboard.

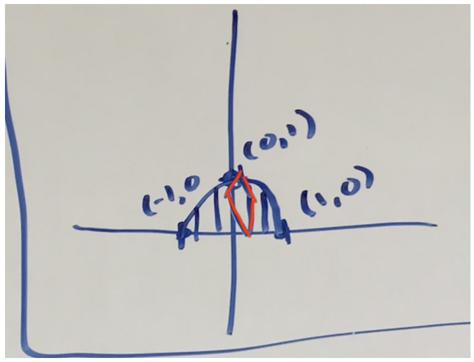


Figure 9. *Ali's sketch for Problem 2a.*

Ali's strong "adding areas" conception worked very well for slicing volume problems, and there is no evidence of their need to assimilate/accommodate due to Ali seeing these problems before in Calculus 2.

Blair's "integrating areas" conception did not fare as well for Problem 2 as it did (symbolically) for Problem 1. Blair immediately had trouble because they took "square cross-sections" to mean that the 2D base region was being "diced" into squares instead of sliced into rectangles (Figure 10). Blair even said, "Cross-sections are squares. Squares? That will take forever? Rectangles are easier." It was common in these interviews for students to misunderstand what "cross-sections of a solid" meant.

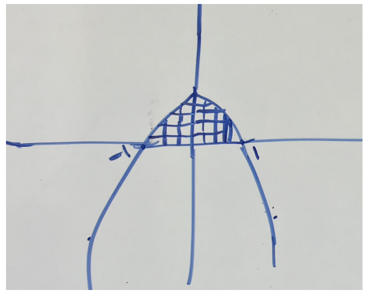


Figure 10. *Blair's sketch for Problem 2a.*

After the interviewer explained the solid and the slicing in more detail, Blair leaned back on their "integrating area" conception, doing so primarily symbolically. Blair started by recalling from Calculus 1 that to find the area of a 2D region, one integrates "the function of the top minus the function of the bottom," obtaining an area function: $A = \int_{-}^{-}(1 - x^2)dx = x - \frac{x^3}{3}$. From here, Blair employed their "integrating area" conception of volume to get $V = \int_{-1}^{1}\left(x - \frac{x^3}{3}\right)dx$ (Figure 11).

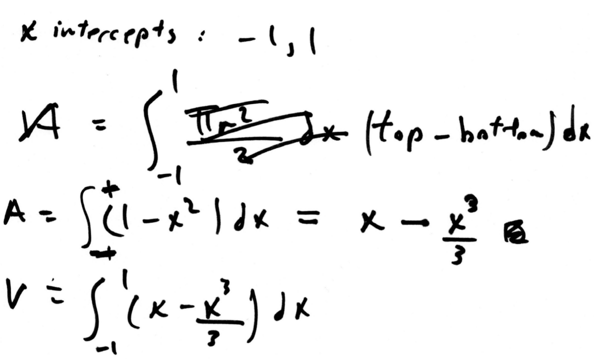


Figure 11. *Blair's volume integral for Problem 2a.*

Blair's "integrating area" conception was so general, that they chose any area at all to plug into the integral in order to produce the requested volume. They saw this as a problem that could be solved using an integral, but their assimilation of this problem into a scheme containing a shallow understanding of how the integral works resulted in an incorrect solution. Blair assimilated any volume problem into their integrating areas conception without second thought.

Glenn tried to assimilate Problem 2 into their washer method schema, talking about "bigger *R* and smaller *r*" (radii of washers) even though these concepts were irrelevant to the square cross-section slices of the solid. Strangely enough, Glenn made a mistake that ended up producing a symbolically correct volume integral as their final answer. In the top right corner of Figure 12, Glenn wrote that the outer radius R was $1 - x^2$ and the inner radius *r* was 0. Just as in Problem 1, Glenn forgot to put the factor of π, making the integral correct, even though it was arrived at by their schema for disk method for a volume by revolution.

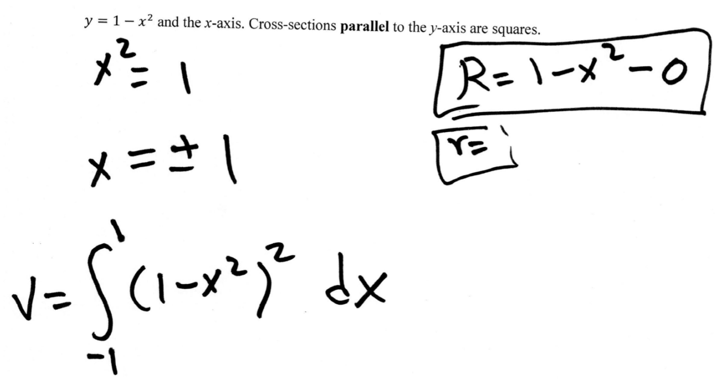


Figure 12. *Glenn's volume integral for Problem 2a.*

For Problem 2b, Glenn said that the integral would not change because "it will have the same area."

Iris's statements during Problem 2 hinted at a basis of a conceptual understanding but they were never really able to state it accurately. Iris stated, "You can approximate the volume by taking … however many *n* slices of cross sections, and you know the areas of those, so string a bunch of those together you can approximate the volume, which is the integral, I think. The integral part of it is however many slices." Here, Iris talked about stringing areas together to approximate the volume. This is reminiscent of Ali's adding areas without thickness conception.

Iris ended up with the correct volume integral for Problem 2a (but omitted the dx), and they said the following as an explanation.

> Iris: I know that to find volume, you need to have a length, width, and height that's going to exist in 3 dimensions [inaudible] x, y, and z's. So this part [*points to integrand*] gives me my, this is my length and width. And I'm not sure what my height is.

Iris has a $V = LWH$ conception that works for solids with square cross-sections, although they did not attend to the dx as part of the three-part product.

6.3. Non-revolution volume problem (Problem 3, pyramid problem)

Problem 3: Find the volume of a pyramid whose base is a square with side length L and whose height is h.

The pyramid problem posed two common problems for students in this study. First, the familiar nature of the shape of a pyramid caused many students to start by attempting to come up with a memorized formula from geometry. Although a few students did mention the correct formula, they were not confident in their answer, so they did not settle on the formula as their final answer. Some students also attempted a "cube volume minus non-pyramid volume" method, but none were comfortable or confident with that either. For this, they tried to develop the formula for the volume of a pyramid by subtracting out the volume of the "non-pyramid solid" from the volume of the cube with side length L. The second and more troublesome issue that arose with Problem 3 was the absence of an explicit bounding function for the edges of the pyramid. Because students were not able to come up with the correct function, none were able to build a symbolically correct volume integral for the pyramid.

		Ali	Blair	Casey	Dana	Erron	Francis	Glenn	Hao	Iris	Jay
Structure	Problem 3 (pyramid problem)	■		■	■	■					
Details											
Concept				■	■	■					
Correct	■										
Incorrect											

Table 9. *Student performance on symbolic structure, symbolic details, and conceptual aspects of Problem 3 (pyramid problem).*

In Table 9, you can see that although no participants were able to arrive at an integral with correct symbolic details, many were able to discuss their solution in a conceptually accurate way – even Erron, who had not been able to do so in Problems 1 and 2. Below, I will discuss the conceptions of Casey and Dana first, as their conceptions have continued and refined throughout the span of the interview. Next will be Erron, as their accurate conception arose just in Problem 3 and nowhere else. Last, I will discuss Ali and Blair's persistent incorrect conceptions.

6.3.1. Problem 3 – Casey and Dana's accurate conceptions

Casey's accurate integral conception took a while to develop due to their trying to assimilate this problem first using geometric methods (trying to recall memorized formulas from geometry) and then using their function-matching schema first mentioned in Problem 1.

> Casey: I have position, velocity, and acceleration down pat, I haven't figured out how to apply it to other related stuff. Obviously area relates to, or area or surface area, the area of a 2-dimensional thing can relate to the area of a 3-dimensional thing because they're close enough, I imagine. And taking a derivative or an antiderivative will give you one or the other. Probably the antiderivative since that gives you more x's, or more variables to work with.

Casey had an understanding of the derivative/antiderivative relationship between the physical quantities of position, velocity, and acceleration, so they were attempting to extend that to area/volume by recognizing that taking the antiderivative produces "more *x*'s" (therefore adding a dimension). Although there is an antiderivative/derivative relationship between area and volume, Casey's next steps indicated that their understanding was incomplete: they used the surface area of the entire pyramid as the area function. Casey continued down this route incorrectly until the interviewer stopped and suggested they consider the pyramid situated on the *xy*-plane.

After this suggestion, it took Casey 6 minutes to visualize the pyramid and slicing the pyramid in a way that they could move forward with solving the problem. In particular, they were focusing on their drawn triangle (Figure 13) in two dimensions (so horizontal slices produced trapezoids) rather than being a representation of the 3D pyramid (where horizontal slices formed square cross-sections).

Figure 13. *Casey's sketch for Problem 3.*

Casey: So I'm picturing an infinite amount of now squares going up into the peak of the pyramid. So you've got the integral and you're going to have your, you're going to have an *x*-squared, since you have the face of the pyramid. The area of that is *x*-squared. Not the face of the pyramid, the face of the square. And that's going to be multiplied by the thickness of the square, which is labeled as *dx*.

Casey then corrected the *dx* to *dy* (after prompting from the interviewer) but could not make much more meaningful progress beyond this. Casey's accommodation of the pyramid problem as a “slicing” problem took much more time and prompting from the interviewer than in Problem 2. A hypothesis I have concerning this is that this effortful accommodation was due to the lack of an explicit function given in Problem 3. This is how it went with the other participants with correct conceptual understanding – they could discuss the idea of small volume boxes combined to approximate the volume of the pyramid, but they were unable to mathematize it into an integral.

As with other students in this study, Dana first noticed the familiar geometric shape of the pyramid and considered calculating the volume by finding the volume of the cube and subtracting “all of the volume of the cube that is not the pyramid.” Dana instantly saw that this proved difficult. After prompting from the interviewer to consider past problems and integration, Dana began by considering the pyramid as being placed with its base on the *xy*-plane, like the solid in Problem 2 (Figure 14 – top left is the base of the pyramid on the *xy*-plane, bottom right is the base of the solid in Problem 2 on the *xy*-plane).

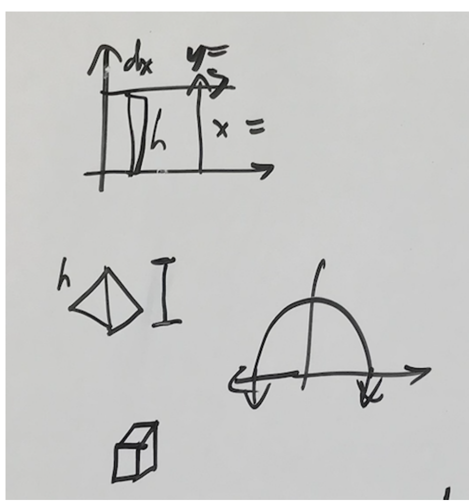


Figure 14. *Dana’s sketches at the beginning of Problem 3.*

Dana assimilated the pyramid problem into their schema from Problem 2 but had some confusion because "I know this (the size of the square cross-section) is going to change as I go up the pyramid." Dana had a choice at this point to follow the square cross-section route (slices parallel to the *xy*-plane) or to slice perpendicular to the *x*-axis, like in Problem 2; they chose the latter.

Dana was very good at drawing the 3D aspects of the solids but in this situation, they missed the fact that the faces of the vertical slices would be trapezoids rather than squares. Dana did understand, though, that slicing perpendicular to the *x*-axis would form slices that were not perfect rectangular prisms (Figure 15).

> Dana: I just drew a rectangular prism. But I know it's going to have, like the top is going to be slanted. So I guess one part would, if I wanted to find the total volume of that, you add up the volumes of the rectangular prism on the bottom, and then the triangular prism on the top.

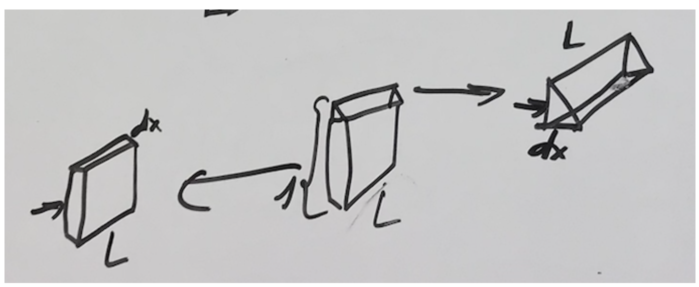


Figure 15. *Dana's drawing of a slice of the pyramid.*

Dana was even possibly unconsciously aware that they were incorrect in their slice shape because they stated, "this would be a lot easier for me to do if we had a little shapes" – meaning manipulatives to assist in visualizing.

Another aspect of Figure 15 to note is that Dana did understand that the height of each slice would be changing, so they labeled the height of the slice with an *f* (middle slice of Figure 15). Dana was not able to make much more progress from here, but they were certainly aware at this point how focusing on the slices of the solid can aid in building a volume integral.

6.3.2. Problem 3 – Erron's accurate conception

This section will include more detailed descriptions due to the students in this section arriving at their accurate conceptions in Problem 3 and at no point before.

Erron began by drawing a very clear 3D pyramid and focusing on areas of general triangles and squares (Figure 16). They then considered multiplying these two formulas together, but "multiplying these things together isn't going to get you a volume. Because a unit squared times a unit squared is going to get you two… no it's not, it's going to get you that unit to the fourth, because it's two plus two." Even though Erron's method was incorrect for other reasons, the unit mismatch was what caused them to try a different method.

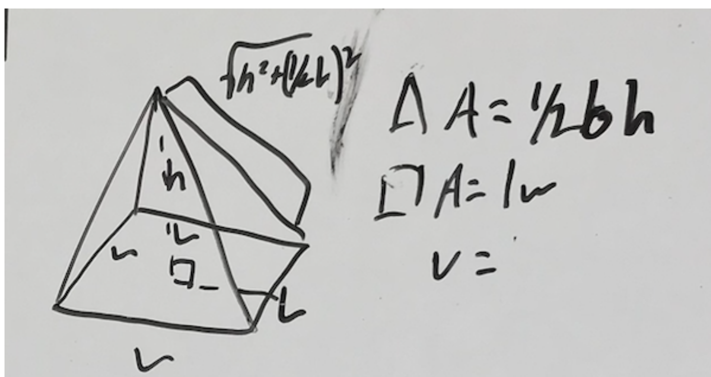


Figure 16. *Erron's 3D drawing of the pyramid for Problem 3.*

After prompting from the interviewer to consider doing this problem using integrals, Erron drew the *xy*-plane with an isosceles triangle sitting on the *x*-axis in the first quadrant (Figure 17). Erron ruled out a revolution helping with this problem, stating, "if I would just take this and swirl it, it wouldn't get me what I want. It wouldn't get me that [pyramid] shape." From here, Erron noticed that they could find the length of the edge of the triangle using the Pythagorean theorem (Figure 17).

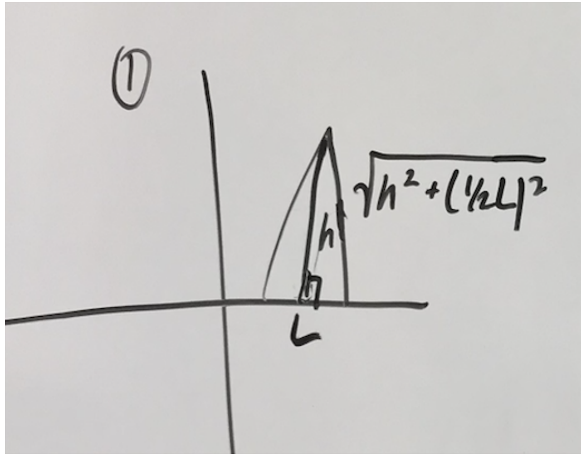


Figure 17. *Erron's first xy-plane drawing for Problem 3.*

There is no real method to Erron's work here, they were talking aloud and seeing what they could come up with mathematically for which they could recognize formulas. Figure 18 is a more detailed picture that Erron drew as they were following the Pythagorean theorem line of reasoning (labeled (2) because Erron was very helpful by numbering their work chronologically).

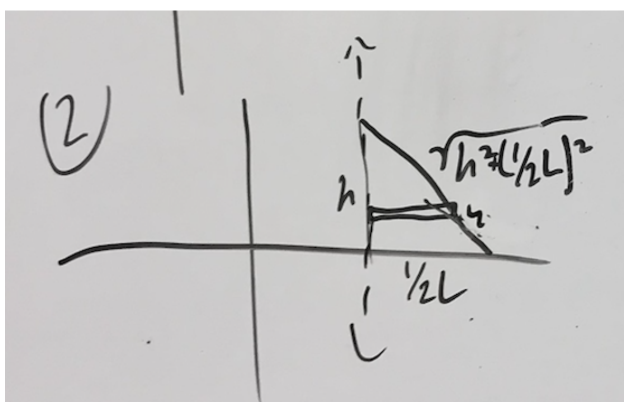


Figure 18. *Erron's second xy-plane drawing for Problem 3.*

Erron then drew a horizontal rectangle in the 2D triangular region, and had the following line of reasoning, showing their attempt to assimilate the pyramid problem into their revolution problem schema.

Erron: So that would be with respect to *y*, *dy* because it's like that [*gestures horizontally*]. So then if I would want to find the area of this entire thing, this would actually just be *y*. And then, this right here (length of hypotenuse) doesn't matter at all. So like, I would erase it, but I want you all to continue having that. So this right here, this little square (horizontal rectangle), whenever I take it, and I rotate it like this (around leg of right triangle to form a cone) it's going to get me the volume. So then if I do that from the bottom (of the pyramid) all the way to the top, it's going to get me the volume (pause) that way. Yeah. Except it's going to get me the volume of a circle. So it would get me the volume of a pyramid that's like a circle on the bottom. That doesn't work.

Although this line of reasoning doesn't give the correct volume (and Erron knew that), it ended up giving Erron an idea and sending them down a better conceptual path. Erron hinged the remainder of their discussion about Problem 3 on stacking "volume squares" to form the pyramid (Figure 19). (Erron called rectangles squares several times throughout the interview.)

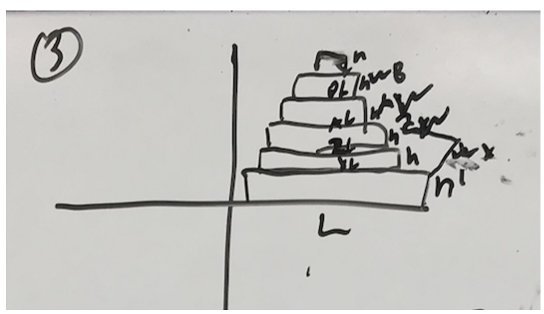


Figure 19. *Erron's third drawing for Problem 3.*

At this point, Erron got very animated, talked a lot, and moved to writing on paper in order to try to mathematize what they were visualizing concerning the volume squares (Figure 20).

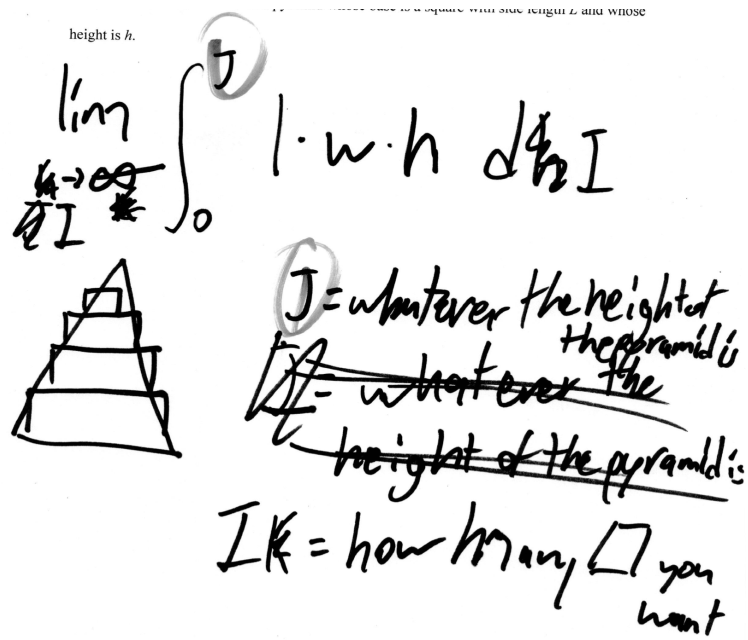


Figure 20. *Erron's mathematization of the "volume squares" conception for Problem 3.*

There was a lot of scribbling in Erron's final work for Problem 3 because they were having a hard time juggling the variables. What can be extracted, though, is that Erron understood that many "volume squares" were needed ($I \to \infty$, $I =$ how many squares you want), and that they were comfortable with the Limit layer of this integral. Also they knew that the upper bound of the integral (*J*) needed to be the height of the pyramid. You can also see that their "multiply three things to get a volume" concept arose again, as the integrand contains $l \cdot w \cdot h$. The differential is reduced to indicator of variable of integration rather than playing a part in the product, as Erron stated, "and then, I don't know, I'll just put it as *dh* because *h* is on the outside, and you have to have a *dh* when you're taking the limit of an integral. You have to have it with respect to something else." It is unclear if this *h* was the *h* from the statement of the problem. A little later, Erron changed their mind and decided it should be *dI* instead of *dh*.

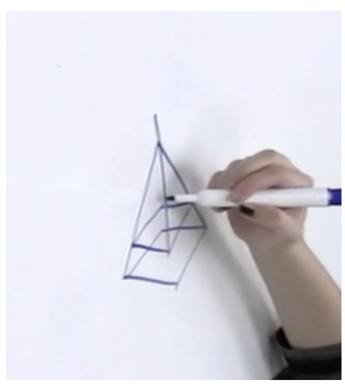
Erron had the right idea for how to visualize the pyramid as being approximated by square cross-sectional slices and they knew this meant they could build an integral to represent the volume of the pyramid, but they fell short on the symbolic details.

6.3.3. Problem 3 – Ali and Blair's inaccurate conceptions

Ali's "adding areas" and Blair's "integrating areas" conceptions carried through into Problem 3. Once again, Ali's ability to visualize the correct cross-sectional slices worked in their favor and allowed them to get close to a symbolically correct final answer (although ultimately incorrect).

Ali started out by drawing a free-standing 3D pyramid, including the square slice with no thickness, before even verbally deciding to slice it in the direction that produced square cross-sections.

Ali:	(24:28) [*Draws 3D pyramid exterior lines with vertical line down the middle*]
Ali:	(24:45) [*Draws interior 2D representative slice with no thickness*]



Ali:	(24:48) With this one I'm not really sure whether it would be easier for me to solve in terms of like the triangles… well, I guess not. Um, it seems like it would be best to solve with the squares.

Ali then went on to label one of the base sides of the pyramid as *L*, the height as *H*, and jumped directly to setting up an integral.

> Ali: Let's see, each of these is *L*. And this is *H*. So, this is definitely going to be [*writes integral symbol on paper*] from 0 to *H*. And then, L^2? Um, no. … I know that is definitely 0 to *H*. Because I know that I want to solve for, I want to solve for the areas of the squares. And I know that whatever I get is squared because, obviously it's a square. And I know that I'm going to be doing it in *dy*.

Once again, Ali was attending to the area of the square and "doing it in *dy*" as pieces of the integral that do not interact with each other. Ali's strong and effective conception of adding areas without thickness in combination with having previous encounters with free-standing solid problems led Ali to assimilate the pyramid problem into their schemes of integration ("adding") and representative slice ("area without thickness"). Ali was unable to finish the problem completely because they used an incorrect linear function for the side lengths of the square cross-sections.

Ali's conception of adding areas without thickness is deceptive because although it allowed Ali to perform well in with the volume problems contained in this study, it masks the concept of the volume integral adding up small pieces of volume, not small pieces of area. In fact, other than when reading the problems, Ali did not say the word "volume" at all. I hypothesize that Ali would have trouble transferring this integration understanding to other types of integral application problems. For example, hydrostatic force problems require students to build an integral from a product of a pressure and an area. Here, the *dx* is part of the area portion, and the integrand itself (without the *dx*) does not have a physical meaning. Thus, to build an accurate integral, you must attend to all parts of the Product layer, not just the part represented by the integrand, as Ali does with their volume integrals.

Blair began Problem 3 by saying "I'm not going to actually integrate it" and tried to rely on geometric properties and memorized geometric formulas. During this process, Blair was imagining that the pyramid was formed from slicing the sides off a cube, so they asserted that the height of the pyramid was L. Even though this is not necessarily true, this assertion stuck throughout Blair's working of Problem 3.

Once Blair realized that they were not making progress using geometric formulas, they assimilated the pyramid problem into their "integrating areas" integral conception. Blair began by drawing a 2D triangle to represent the pyramid, and this became Blair's area in their "integrating area" work (Figure 21). Blair found the area of a triangle with base L and height h ($\frac{1}{2}Lh$), but they then changed the h to an L ($\frac{1}{2}L^2$) due to their previous assertion that the pyramid came from a cube with all side lengths L. Exactly which triangle area Blair was referencing became clear after they did their next calculation (line 4 in Figure 21), writing "area of all 4 $\triangle$: $4 \cdot \frac{1}{2}L^2 = 2L^2$", meaning that Blair was attending to the four faces of the pyramid as the requisite "area" that they needed to integrate.

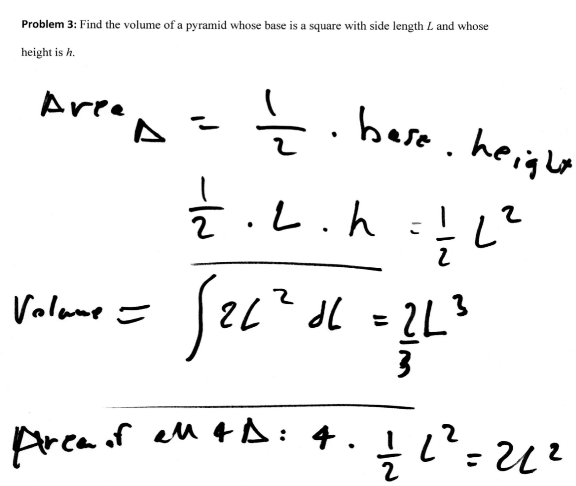


Figure 21. *Blair's volume integral for Problem 3.*

Chapter 7: DISCUSSION

The data presented in the previous chapter are based on student responses to calculus volume problems of three types:

- Problem 1: a revolution volume problem with explicit bounding functions given
- Problem 2: a non-revolution volume problem with explicit bounding functions given
- Problem 3: a non-revolution volume problem with explicit bounding functions not given

Problem 1 was familiar to all participants, while Problems 2 and 3 were novel to all participants except one. This difference in exposure allowed me to not only analyze their problem-solving strategies and conceptions for each problem, but also how they approached new types of volume problems in relation to their conceptions of revolution volume problems. Once again, these were the research questions guiding this study.

1. How do students conceptualize revolution volume integrals?
2. How do students use their revolution volume problem conceptions to solve novel volume problems?
3. How can non-revolution volume problems aid in building conceptual understanding of integration?

7.1. Revolution volume problem (Problem 1)

7.1.1. Students' conceptions of revolution volume problems

The revolution volume problem was familiar to all ten students in this study, so initially there was no disequilibration until they were asked to explain their integral setup. Since all students used memorized formulas to come up with an answer, the disequilibrium came about as a result of having to tie the memorized formula to a concept. Students in this study explained their revolution volume integrals in a way similar to participants in Jones and Dorko's (2015) study of student understanding of multiple integrals. Ali's 2-dimensional slices are consistent with students in Jones and Dorko's study who considered a multiple integral as calculating volume by "adding up slices without thickness". This conception can also be related to Czarnocha's (2001) "indivisibles," as well as Oehrtman's (2009) "collapsing dimension." Ali never considered their slices as having a third dimension, which indicated that they were not considering their integral as having a Product layer, although Summation and Limit layer ideas were present in Ali's discussion. This misconception could possibly be remedied with a focus on the Δx and the dx, their relationship to each other (Riemann sum versus integral), and what part they play in the physical quantity that the integral is measuring.

Casey began their discussion of Problem 1 with an antiderivative-related conception of integration, which coincides with "function-matching" (Jones and Dorko, 2015). Blair had a persistent function-matching conception ("integrating area gives volume") that worked well in Problem 1 but failed in Problems 2 and 3 because they were using *any* area they could pin down with an explicit formula. This is consistent with previous studies on integration where students solely related integrating to the action of taking the antiderivative (Orton, 1983; Pettersson & Scheja, 2008). Blair was unable to consider any of the integrals from the Product, Summation, or Limit layers due to this very rigid, entrenched, and limited view of integration. It should also be noted that at the beginning of the interview, Blair stated, "I didn't prepare well for that [applications of integration] exam, but I still got an 88%." Thus, the effects of superficial knowledge, understanding, and assessing can be seen.

Several students in this study exhibited "perimeter and area" conceptions when they discussed the volume of the pyramid as being composed of the sum of surface areas. This specific view of volume (volume as sum of surface areas) was also discussed in Dorko's (2013) study of secondary school students' understanding of general volume (not in a calculus context). In this study, the volume as sum of surface areas conception only arose as students were working on the pyramid problem. The combination of a widely recognizable solid and the absence of bounding functions had students relying on geometric arguments to find the volume and not considering integration as an option (without interviewer intervention).

Lastly, Casey and Dana exhibited the multiplicatively-based summation (MBS) conception (Jones, 2015b) after questioning and guidance by the interviewer during Problem 1. The ability to see that the volume is approximated by a sum of products (meaning, being able to conceive of the integrand from the Product Layer and understand what physical quantities the pieces of the product are representative of) is evidence of a thorough understanding of the concept of the definite integral. Casey and Dana carried this conception throughout, and it helped assist in their understanding of and progress with the subsequent novel volume problems.

These previously discussed constructs related to general integration continuing to show up in applications of integration means that instructors can bring attention to them and discuss why they are inaccurate conceptions. For the incorrect conceptions to show up in multiple integral-related mathematical situations indicates that they are strong and have worked for students in the past. Volume problems are usually one of the first types of integral application problems students are exposed to, so discussions on inaccurate views of integration at this point could be very powerful in reducing the perpetuation of integral misconceptions.

7.1.2. Revolution volume problem general findings

The overarching findings concerning the revolution volume problem, Problem 1, are twofold. First, it is possible for students to get traditional revolution problems symbolically correct but not understand why their answer is correct. Table 5 in Chapter 6 shows that five (Ali, Blair, Dana, Erron, and Hao) of the ten participants were able to arrive at fully correct integral setups for the revolution volume problem without being able to give a conceptually sound explanation of their volume integral. This is concerning for two reasons. First, most traditional and widely used calculus textbooks (Stewart, 2020, for example, which is in its 9th edition) spend a large amount of space and exercises on these types of problems. If students are working through these types of problems with inaccurate understanding, getting them correct due to memorized formulas, and not being challenged about their understanding, it will reinforce that using memorized formulas is good and is enough. A recommendation is that instructors be cognizant of this and make sure to give equal time and attention to various types of integral volume problems. Another recommendation is for instructors to include and develop revolution volume problems that require more deep thought than just setting up an integral. The intent here is to bring about disequilibrium in their thinking in order to either confirm (by assimilating) or refine (by accommodating) their understanding of integration as it relates to volume problems.

Now the good news: students can arrive at accurate conceptions of integration when they are pushed to explain and think about their revolution volume integral set-up. This happened for Casey and Dana during Problem 1. Once Casey was questioned about their first revolution volume integral and they began putting a correct understanding together by verbalizing it, they said, “I actually understand this much more now that you’re making me answer these questions.” Dana took a little more time than Casey, but Dana was greatly helped by their sketches and was able to piece together their understanding that way. A recommendation here would be to include more diverse aspects to the classroom learning environment, such as incorporating more visual aids, active learning, discussion, and open-ended questions that provide students with the opportunity to connect the symbolic components within the integral with the quantities they represent in three dimensions.

7.2. Non-revolution volume problem with given bounding function (Problem 2)

Students had more trouble conceptually with Problem 2 than with Problem 3, which was unexpected. Problem 2 was included in this study specifically because it contained an explicit bounding function of the form $y = f(x)$, which ended up being somewhat of an issue for students. Specifically, the presence of the bounding function influenced some students to rely on their memorized revolution volume formulas, which were not applicable to Problem 2.

The biggest hurdles to student understanding and completion of Problem 2 involved the problem in general: what the cross-sections were (or even what the term "cross-section of a solid" meant), what the solid looked like, and how the cross-sections combined to make the solid. These difficulties with visualizing cross-sections and solids are consistent with past work on student understanding of 3-dimensional solids and cross-sections (Davis, 1973; Moore-Russo & Schroeder, 2007). The interviewer had to intervene more frequently and more intensely to help students understand Problem 2 because it was not a solid that was familiar to them (like the pyramid in Problem 3). Many students also assumed that the solid with vertically-cut square cross-sections (Problem 2a) and the solid with horizontally-cut square cross-sections (Problem 2b) would have the same shape and volume.

Casey and Dana, the most successful students in this study, even struggled with this problem at first and required extended intervention by the interviewer. Dana was very connected to their pictures and drawings, so the inability to understand the solid was a major issue. At one point in the interview, Dana even said, "This would be a lot easier for me to do if we had little shapes," meaning physical manipulatives to decrease the mental load of visualizing the solid.

Once the interviewer suggested that they consider one small slice of the solid, Casey and Dana were able to make progress. This suggestion by the interviewer was very powerful for Casey and Dana specifically because they were able to discuss the volume of representative pieces of their solids of revolution. This suggestion was less effective with the other participants because they did not have that foundation of understanding from the previous problem, plus they had a difficult time visualizing and understanding the solid.

The takeaway from student performance on Problem 2 concerns the weakness of some students' visualizing skills and the strength of the "representative slice" conception. Visualization is historically a hard task for students of all ages (Lean & Clements, 1981; Battista, 1990; Boothe & Thomas, 1999; Stylianou and Silver, 2004), and with most students in this study, the inability to visualize and understand the solid resulted in a hurdle that they were unable to overcome. They got stuck in their disequilibrium and in order to re-equilibrate (which may just mean 'finish the problem'), some students leaned back on assimilation of this problem into their revolution volume problem schema and relied on their memorized formulas from the revolution problems. This resulted in incorrect answers because revolution volume formulas were not relevant to Problem 2 – the solid in Problem 2 was not the result of a rotation and did not have circular cross-sections. The key to getting some students past this solid-visualizing issue was to focus on one representative slice of the solid.

7.3. Non-revolution volume problem (Problem 3, pyramid problem)

The pyramid problem resulted in zero correct final integrals, but it is not a story of failure, it is a story of promise. Given that a pyramid was recognizable to all participants in this study, the initial hurdle of "understanding the solid" was not present like it was in Problem 2. This gave students time and mental space to consider different ways to approach the problem. In fact, two students (Erron and Francis) who had conceptual issues throughout the interview had seemingly "aha!" moments that did not appear until the pyramid problem. It can be said that another name for an "aha! moment" is a "re-equilibration event".

Several students were able to arrive at an accurate 'approximate the pyramid by stacked boxes' idea, but none were able to convert that idea into a mathematically accurate volume integral. The issue for Problem 3, then, was less of a calculus problem and more of a translation problem. Using terms developed by Duval (1999), the 'approximate the pyramid by stacked boxes' idea could be described as residing in the graphical/geometric register (a visual/graphical representation of the mathematical situation), and the symbolic volume integral – in particular, the bounding function – would be in the symbolic register (a notational representation of the mathematical situation). Thus, students tended to have issues converting between the graphical and the symbolic registers for Problem 3. Even though the fewest students produced an accurate integral for Problem 3, the greatest number of students could understand how an integral would work to solve this problem. The calculus "big idea" is there, but the steps in getting to a symbolically accurate final answer are missing.

One way to reduce the cognitive load of translation from graphical to symbolic register (finding the bounding function required for the integral) is to build a general volume integral instead of a completely finalized volume integral. For example, if a solid has half-circle-shaped cross-sections (so cross-section areas would be $\frac{1}{2}\pi r^2$) but a student was unable to come up with the function that represented the circles' radii, they could build the general volume integral $\int_a^b \frac{1}{2}\pi(f(x))^2 dx$ without needing to explicitly find the function formula. Another recommendation would be to do prep work based on finding symbolic representations of geometric situations, such as developing linear functions given information about points and slopes. This would help students with volume problems that require them to build functions in order to solve the problem.

7.4. Teaching implications summary

The teaching implications discussed above can be summarized in general as: less traditional exercises and more active, deep, discussion-based activities will be beneficial to students in their learning and understanding of volumes. Traditional exercises, although easy to do and to grade, can mask and reinforce pseudo-conceptual understanding (Vinner, 1997) of the definite integral. In this study, this was evident in Problem 1 with the students who were able to arrive at a symbolically correct volume integral but were not able to explain the underlying concepts. Non-traditional activities, like group work, discussions, oral interviews, and presentations, allow students to continually have disequilibrium events and the opportunity to achieve re-equilibrium in ways that support true understanding.

Given that volume is a very visual topic, another way to enhance student understanding would be to incorporate physical models and manipulatives. As Dana mentioned during the pyramid problem, having a model to inspect may help ease the cognitive load of trying to visualize a solid, understand the slices, and attend to the new concept of integrals measuring volume. Physical models can help students see the solid as whole and can also aid in students visualizing shapes of cross-sections of the solid. These physical models can be as sophisticated as 3D printed models of slices that go together to form the solid, or as basic as solids formed from foam or clay.

The interview protocol was used to assist in my organization and comprehensiveness when interviewing students, so it could also be used to develop step-by-step activities that could help students organize and check their work. Having a scaffolded activity based off the interview protocol could provide students with forced stopping points in the problem-solving process that are intended to provide embedded self-check moments. The protocol-based activity could also provide students with instances of discussion with classmates, by asking questions like, “Why *dx* and not *dy*?”, “What part of your picture provides you with information about radius/height?”, and “Why does your integral give the volume of this particular solid?” Questions like this will lead students to think deeply about notation, their visualization, and their understanding of the definite integral while verbalizing, discussing, and externalizing it. As Casey said during Problem 1, “I actually understand this much more now that you’re making me answer these questions.”

The results of this study could also be used to assist instructors in common misconceptions that students have when solving volume problems. When starting out, new calculus instructors may not have a wealth of knowledge about student misconceptions, and perhaps they only have their own experience in calculus from which to build. This information could be particularly useful for graduate student teaching assistants.

Finally, I would like to discuss how these studies (the pilot studies and my dissertation) have influenced, informed, and changed the way I teach this topic. When I began teaching calculus, I started the volume section by covering volumes of revolution first (because I saw that as the “easier” material), then I moved onto the “harder” volumes of non-revolution solids. My belief was that starting students out on easier (less cognitively demanding) revolution problems would allow them to step up to the more cognitively demanding non-revolution problems. I have changed my ways and I now start the entire volume unit with the pyramid problem. I do this because it is a recognizable solid (I can even draw it on the board) and the cross-sections are basic shapes, but they still need to think about the best way to slice the pyramid. I then present all remaining volume problems as having the same underlying idea of slicing into cross-sections. This way, it lessens the tendency of seeing revolution volume problems as the easy ones (memorizable formulas!) and non-revolution problems as the hard ones.

Another way I have changed my instruction is I focus much more on the representative slice and how the word “representative” means something very specific: its measurements must represent any slice I would take throughout the entirety of the solid. This focus aims to enhance students’ focus on both the Summation and Limit layers of the volume integral.

Considering student learning as being formed from assimilation and accommodation of new knowledge into schemas has influenced me to continually discuss the “big ideas” of integration. It is easy to get buried in the details and minutiae in a calculus class as a student, so in each section, I always try to bring it back to the overarching concepts. In particular, I have the following fill-in-the-blank short activity.

Derivatives measure __________________. Some examples of this are ________________.

Integrals measure ___________________. Some examples of this are _________________.

The big idea of derivatives measuring rates of change and integrals measuring accumulation are the schemas, while the examples of those (slope, velocity, optimization, etc., for derivative; area, volume, arc length, etc., for integral) serve to illustrate that derivative is not just velocity and integral is not just area.

7.5. Limitations

This study was conducted with students who were registered for summer calculus courses at one single university, so it is possible that the results discussed in this study are not necessarily representative of all possible students that take calculus. Another limiting factor concerning the participants is that they were volunteers and not chosen at random, so this study does not necessarily represent an even distribution of calculus students.

The interview environment was another aspect that could bring confounding factors into the research process. Elements like a video camera, being questioned by a perceived math expert, and not understanding the statement of the problem could make participants feel uncomfortable (frustrated, embarrassed, etc.), which could influence their mathematical performance. Thompson, Carlson, Byerley, and Hatfield (2014) coined the terms “in-the-moment understanding” and “stable understanding” pertaining to different levels of understanding portrayed by students when they are discussing mathematical concepts. In situations where students are uncomfortable, they may make more mental actions in the moment in order to produce a response quickly, rather than access a stable conception. This distinction was not analyzed in this study, and if it was, it may have brought about different conclusions.

A final limitation that I had not considered before my dissertation prospectus was the impact that I as the interviewer have on the interview process. A personality trait of mine that I believe had the strongest impact on the interview situation is my discomfort with seeing people agitated, frustrated, or annoyed. It is well known that students can feel strong negative emotions associated with mathematics, and when those arise, it is in my nature to try to assuage those negative feelings. In order to keep that to a minimum, I tried my very best to stick to the protocol (which was definitely helpful to have!), but in certain situations, my desire to make people in distress feel better shone through. This happened in general when the interview was getting long and I could tell students were getting tired. I wrapped up problems when I may have been able to go longer and get more data. A specific instance of this was with Dana during Problem 2. Dana had been working on Problem 2a for a long time and I could tell that they were getting fatigued with the problem. Because of this, I did not ask Dana about Problem 2b. This introduced a limitation in that it removed an opportunity in which I could collect data to compare with other participants in the study.

7.6. Future research

One frustration that continually arose in the process of analyzing data was that I never had the participants re-visit their revolution volume integrals after working through the non-revolution problems. A future study I would love to conduct as an addendum to this one involves students' updated conceptions after a successful re-equilibration event and how they use that to adjust and/or refine previous inaccurate conceptions. For example, I would love to ask Erron (who had the aha! moment at the very end of the pyramid problem) to revisit their revolution volume integral and talk about how the pyramid problem related to Problem 1.

Volume problems for most students involve pictures, graphs, and visualizations. Although this study touched on those aspects of the interviews briefly, there is much more to look at in this area. Related to visualization is the concept of gesture and embodied cognition. Given that volume problems have such a dynamic nature (rotations, slicing, limits, etc.), investigating student use of gestures while solving volume problems is another interesting area for future research.

Many of the teaching implications mentioned above involve using non-traditional types of problems to enhance student understanding and learning. A natural next step for this research would be to develop these types of problems and perform teaching experiments on their efficacy. Since volume problems are an entry to other, more complex integral application problems (many that do not have the visual quality that volume problems do), creating bridge activities could be productive in establishing strong connections between different types of integral application problems.

BIBLIOGRAPHY

Battista, M. (1990). Spatial visualization and gender differences in high school geometry. *Journal for Research in Mathematics Education, 21*(1), 47-60.

Bernard, A, & Jones, S. (2016). Student problem solving in the context of volumes of revolution. Proceedings of the *19th special interest group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: MAA.

Bezuidenhout, J., & Olivier, A. (2000). Students' conceptions of the integral. In *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 73-80). Athens, GA: PME.

Booth, R., & Thomas, M. (1999). Visualization in mathematics learning: Arithmetic problem solving and student difficulties. *Journal of Mathematical Behavior, 18*(*2*), 169–190.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77–101.

Bresock, K. & Sealey, V. (2018a). Integrals, volumes, and visualizations. In *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education. Conference on Research in Undergraduate Mathematics Education*, San Diego, CA.

Bresock, K. & Sealey, V. (2018). Calculus students' use of visualizations when solving volume problems. In *Proceedings of the 42nd Annual Meeting of the International Group for the Psychology of Mathematics Education. Conference for the International Group for the Psychology of Mathematics Education*, Umeå, Sweden.

Cui. L., Rebello, N., Fletcher, P., & Bennett, A. (2006). Transfer of learning from college calculus to physics courses. In *Proceedings of the National Association for Research in Science Teaching 2006 Annual Meeting* (pp. 1-7). San Francisco, CA: unpublished.

Czarnocha, B., Dubinsky, E., Loch, S., Prabhu, V., & Vidakovic, D. (2001) Conceptions of area: In students and in history. *The College Mathematics Journal, 32(2)*, 99-109, DOI: 10.1080/07468342.2001.11921861

Davis, E. (1973). A study of the ability of selected school pupils to perceive the plane sections of selected solid figures. *Journal for Research in Mathematics Education, 4*(*3*), 132–140.

Dorko, A. (2013). Calculus students' understanding of volume. *Investigations in Mathematics Learning, 6, 48.* 10.1080/24727466.2013.11790332.

Duval, R. (1999). Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 3-26). Cuernavaca, Morelos, Mexico: PME.

Ely, R. (2017). Definite integral registers using infinitesimals. *Journal of Mathematical Behavior, 48,* 152–167.

Engelke, N. (2004). *Related rates problems: Identifying conceptual barriers*. Paper presented at the 26th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Engelke, N. (2007). A framework to describe the solution process for related rates problems in calculus. In Lamberg, T., & Wiest, L. R. (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Lake Tahoe, NV, Oct. 25–28, 2007), 185–191.

Gonzalez-Martin, A, & Camacho, M. (2004). What is first-year mathematics students' actual knowledge about improper integrals? *International Journal of Mathematical Education in Science and Technology*, 35(1), 73-89.

Grundmeier, T., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral calculus. *PRIMUS*, 16(2), 178-191.

Hobbs, D., & Relf, S. (1998). The concept of integration: A fundamental approach. *The Mathematical Gazette, 82*(*494*), 167–182.

Hu, D. & Rebello, S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. *Physics Review Special Topics – Physics Education Research,* 9, 020118.

Huang, C. (2010). Conceptual and procedural abilities of engineering students in integration. In *Proceedings of the Conference of the Joint International IGIP-SEFI Annual Conference.* Trnava, Slovakia.

Jones, S. (2013). Understanding the integral: Students' symbolic forms. *Journal of Mathematical Behavior*, *32*(2), 122–141.

Jones, S. (2015a). Areas, antiderivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behavior, 38,* 9-28.

Jones, S. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals. *International Journal of Mathematical Education in Science and Technology*, *46*(5), 721–736.

Jones, S. & Dorko, A. (2015). Students' understandings of multivariate integrals and how they may be generalized from single integral conceptions. *Journal of Mathematical Behavior, 40*, 154–170.

Kiat, S. (2005). Analysis of students' difficulties in solving integration problems. *The Mathematics Educator*, 9(1), 39-59.

Lean, G. & Clements, M.A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics, 12*(*3*), 267–299.

Lucio-Villegas, R., Sáez, J., & Martinez Torregrosa, J. (2015). Obstacles to Mathematization in Physics: The Case of the Differential. Science & Education. 10.1007/s11191-015-9757-7.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, *40*(*2*), 201–211.

McCarty, T. S., "Analyzing Mathematicians' Concept Images of Differentials" (2019). Graduate Theses, Dissertations, and Problem Reports. 4062.

Meredith, D., & Marrongelle, K. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics*, *76*(6), 570–578.

Moore-Russo, D., & Schroeder, T. (2007). Preservice and inservice secondary mathematics teachers' visualization of 3-dimensional objects and their relationships. In Lamberg, T., & Wiest, L. R. (Eds.), *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Lake Tahoe, NV, Oct. 25–28, 2007), 133–136.

Nguyen, D. & Rebello, S. (2011a). Students' difficulties with integration in electricity. *Physics Review Special Topics – Physics Education Research,* 7(1), 010113.

Nguyen, D. & Rebello, S. (2011b). Students' understanding and application of the area under the curve concept in physics problems. *Physics Review Special Topics – Physics Education Research,* 7(1), 010112.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*. 10.5951/jresematheduc.40.4.0396.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics, 14*(1), 1-18.

Pettersson, K. & Scheja, M. (2008). Algorithmic contexts and learning potentiality: A case study of students. *International Journal of Mathematical Education in Science and Technology,* (39)6, 767-784.

Piaget, J. (1971). *Structuralism*. New York: Basic Books, Inc.

Piaget, J. (1977). The development of thought: Equilibration of cognitive structures. (Trans A. Rosin). Viking.

Rösken, B. and Rolka, K. (2007). Integrating intuition: The role of concept image and concept definition for students' learning of integral calculus. *The Montana Mathematics Enthusiast*, Monograph, 3, 181–204.

Sealey, V. (2008). *Calculus students' assimilation of the Riemann integral into a previously established limit structure*. (Unpublished doctoral dissertation). Arizona State University, Phoenix, AZ.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *Journal of Mathematical Behavior, 33,* 230-245.

Sealey, V., & Thompson, J. (2016). Students' interpretation and justification of "backward" definite integrals. In *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics* (pp. 1275-1281). Pittsburgh, PA: MAA.

Sherin, B. (2001). How Students Understand Physics Equations. *Cognition and Instruction, 19(4)*, 479-541.

Stewart, J. (2020). *Essential Calculus: Early Transcendentals, 9th Edition*. Pacific Grove, CA: Brooks-Cole Publishing.

Strauss, A., & Corbin, J. (1998). Basics of qualitative research techniques.

Stylianou, D., & Silver, E. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning, 6*(*4*), 353–387.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, *26*(*2*), 229–274.

Thompson, P, Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming. http:/bit.ly/1aNquwz.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America.

Von Korff, J., & Rebello, N. (2012). Teaching integration with layers and representations: A case study. *Physics Review Special Topics – Physics Education Research, 8*(1), 010125.

Wagner, J.F. (2018). Students' obstacles to using riemann sum interpretations of the definite integral. *International Journal of Research in Undergraduate Mathematics Education*, 4, 327–356.

Wadsworth, B. (1979). *Piaget's theory of cognitive development*. New York: David McKay Co., Inc.

Yeatts, F., & Hundhausen, J. (1992). Calculus and physics: Challenges at the interface. *American Journal of Physics*, *60*(*8*), 716-721.

Zandieh, M. (2000). A theoretical framework for analysing student understanding of the concept of derivative. *Research in Collegiate Mathematics Education IV,* 8, 103-126.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytical strategies: A study of students' understanding of the group D_4. *Journal for Research in Mathematics Education, 27*(*4*), 435–457.

APPENDIX A: Interview Protocol Checklist

Problem 1: Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ about the line $y = -1$.

	Curves	Sketched correctly	2D region correct	
	Method	Why that method/formula?		
	Integral gives volume	Why integral?	Why that integral?	
	Parts of integral	What do the integral pieces stand for?	What does the *dx* or *dy* mean?	How did you choose *dx* or *dy*?
	Parts of picture	What part of your picture gives radius/height/etc?	What part gives dx or *dy*?	
	Different method	Set up integral using different/other method?		

Notes:

Problem 2a: Find the volume of the solid S whose base is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections **parallel** to the y-axis are squares.

	Curves/3D	Sketched correctly	2D region correct	Representation of solid or piece of solid
	dx* or *dy	Why *dx* or *dy*?		
	One slice of solid	What is the *shape* of one slice?	What is the *volume* of one slice?	
	π in integral	Why is there a π in your integral?	Why is there NOT a π in your integral?	
	Parts of picture	What part of your picture gives radius/height/etc?	What part gives dx or *dy*?	

Notes:

Problem 2b: Find the volume of the solid S whose base is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. Cross-sections **perpendicular** to the y-axis are squares.

	Curves/3D	Sketched correctly	2D region correct	Representation of solid or piece of solid
	dx* or *dy	Why *dx* or *dy*?		
	One slice of solid	What is the *shape* of one slice?	What is the *volume* of one slice?	
	π in integral	Why is there a π in your integral?	Why is there NOT a π in your integral?	
	Parts of picture	What part of your picture gives radius/height/etc.?	What part gives *dx* or *dy*?	
	Cuts change problem	How does this change in the problem change the integral?		

Notes:

Problem 3: Find the volume of a pyramid whose base is a square with side length L and height h.

	Solid	Sketched correctly	Solid as 2D	Placed on *xy*-plane (give options for orientation if nec)
	Slicing solid	Which way to slice the solid?	Shapes of cross-sections for each direction?	
	Bounding functions	How to come up with bounding function?		
	Volume of slice	Volume of one slice		
	Parts of picture	What part of your picture gives radius/height/etc?	What part gives *dx* or *dy*?	
	Bounds	Bounds of integral		

Notes:

Calculus history: **156** **251** **First time** **Repeating**

Why taking calculus over the summer? **Get ahead** **Catch up** **Other**

APPENDIX B: Consent Form

WestVirginiaUniversity
OFFICE OF RESEARCH INTEGRITY & COMPLIANCE

Definite Integral Study

Principal Investigator	Dr. Vicki Sealey
Co-Investigators	Dr. John Thompson, Krista Bresock, Cody Hood, Tim McCarty
Department	Mathematics
Protocol Number	1511923724
Study Title	Definite Integrals in STEM
Sponsor (if any)	N/A

Contact Persons

If you have any questions, concerns, or complaints about this research, you should contact Dr. Vicki Sealey at (304) 293-5329. For information regarding your rights as a research subject, to discuss problems, concerns, or suggestions related to the research, to obtain information or offer input about the research, contact the Office of Research Integrity & Compliance at (304) 293-7073.

Introduction

You, ______________________, have been asked to participate in this research study, which has been explained to you by ________________________. This study is being conducted by Dr. Vicki Sealey in the Department of Mathematics at West Virginia University.

Purpose(s) of the Study

Various uses of integrals in science, engineering, and mathematics and the ways in which students and experts reason about these concepts are being analyzed in this study.

Description of Procedures

This study involves videotaping of interviews with participants, where we will discuss integration. Interviews are expected to last approximately 45 minutes. Researchers will ask you a series of questions about integrals. You will be asked to think out loud and share the ways in which you think about the topics. You are welcome to use pen and paper, if you wish. Excerpts of written work may be published, and it is possible that your handwriting could be recognized.

Alternatives

You do not have to participate in this study.

Benefits

You may not receive any direct benefit from this study. The knowledge gained from this study may eventually benefit others in the teaching and learning of concepts involving integrals.

Discomforts

There are no known or expected risks from participating in this study.

Financial Considerations

There are no fees or payment for participating in this study.

Confidentiality

Any information about you that is obtained as a result of your participation in this research will be kept as confidential as legally possible. Videotapes will be kept locked up and/or stored on a password protected, secure service. Data will be kept for a minimum of 3 years and will be destroyed as soon as the research is finished. In any publications that result from this research, your name will not be published, but it is possible that your handwriting could be recognized.

Voluntary Participation

Participation in this study is voluntary. You are free to withdraw your consent to participate in this study at any time. Refusal to participate or withdrawal will involve no penalty to you. If you are a student, participation in this study will not affect your class standing or course grade in any way. In the event new information becomes available that may affect your willingness to participate in this study, this information will be given to you so that you can make an informed decision about whether or not to continue your participation.

You have been given the opportunity to ask questions about the research, and you have received answers concerning areas you did not understand. By signing below, you acknowledge that you willingly consent to participate in this research.

Signatures

Signature of Subject

____

Printed Name Date Time

____

The participant has had the opportunity to have questions addressed. The participant willingly agrees to be in the study.

Signature of Investigator or Co-Investigator

____

Printed Name Date Time

____

Phone: 304-293-7073
Fax: 304-293-3098
http://oric.research.wvu.edu

Chestnut Ridge Research Building
886 Chestnut Ridge Road
PO Box 6845
Morgantown, WV 26506-6845

Subject's Initials________________
Date________________