On L-close Sperner systems

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Abstract

For a set L of positive integers, a set system $\mathcal{F} \subseteq 2^{[n]}$ is said to be L-close Sperner, if for any pair F, G of distinct sets in $\mathcal{F}$ the skew distance $sd(F,G) = \min\{|F \setminus G|, |G \setminus F|\}$ belongs to L. We reprove an extremal result of Boros, Gurvich, and Milanič on the maximum size of L-close Sperner set systems for $L = \{1\}$ and generalize to $|L| = 1$ and obtain slightly weaker bounds for arbitrary L. We also consider the problem when L might include 0 and reprove a theorem of Frankl, Füredi, and Pach on the size of largest set systems with all skew distances belonging to $L = \{0, 1\}$.

1 Introduction

One of the first results of extremal finite set theory is Sperner's theorem [13] that states that if for any pair F, F' of distinct sets in a set systems $\mathcal{F} \subseteq 2^{[n]}$ we have $\min\{|F \setminus F'|, |F' \setminus F|\} \geq 1$, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ holds. Set systems with this property are called *antichains* or *Sperner systems*. This theorem has lots of generalizations and applications in different areas of mathematics (see the book [7] and Chapter 3 of [11]). Recently, Boros, Gurvich, and Milanič introduced the following notion: given a positive integer k, we say that a set system $\mathcal{F}$ is *k-close Sperner* if every pair $F, G \in \mathcal{F}$ of distinct sets satisfies $1 \leq \min\{|F \setminus G||, |G \setminus F|\} \leq k$. In particular, $\mathcal{F}$ is 1-close Sperner if every pair $F, G \in \mathcal{F}$ of distinct sets satisfies $\min\{|F \setminus G||, |G \setminus F|\} = 1$. (The authors used the unfortunate *k-Sperner* term which, throughout the literature, refers to set systems that are union of k many antichains. That is why we decided to use instead the terminology k-close Sperner systems.) Boros, Gurvich, and Milanič's motivation to study these set systems comes from computer science: they wanted to compare them to other classes of Sperner systems (see also [4] and [6]). They obtained some structural results from which they deduced the following extremal theorem. For a set $F \subseteq [n] = \{1, 2, \ldots, n\}$, its *characteristic vector* v_F is a 0-1 vector of length n with $(v_F)_i = 1$ if and only if $i \in F$.

Theorem 1.1 (Boros, Gurvich, Milanič [5]). *If the set system $\{\emptyset\} \neq \{F_1, F_2 \ldots, F_m\} \subseteq 2^{[n]}$ is 1-close Sperner, then the characteristic vectors $v_{F_1}, v_{F_2}, \ldots, v_{F_m}$ are linearly independent over $\mathbb{R}$. In particular, $m \leq n$.*

In this short note, we reprove the extremal part of Theorem 1.1 via a different linear algebraic approach and generalize the result. For a subset L of $[n]$, we say that a set system $\mathcal{F}$ is *L-close Sperner* if every pair $F, G \in \mathcal{F}$ satisfies $\min\{|F \setminus G|, |G \setminus F|\} \in L$. Our first result is the following.

Theorem 1.2. *If the set system $\{F_1, F_2 \ldots, F_m\} \subseteq 2^{[n]}$ is L-close Sperner for some $L \subseteq [n]$, then we have $m \leq \sum_{h=0}^{|L|} \binom{n}{h}$. Furthermore, if $|L| = 1$, then $m \leq n$ holds.*

Note that if $|L|$ is fixed and n tends to infinity, then the bound is asymptotically sharp as shown by $L = \{1, 2, \ldots, k\}$ (i.e. the k-close Sperner property) and the set system $\binom{[n]}{k} = \{F \subseteq [n] : |F| = k\}$. Observe also that the inequality $m \leq n$ is sharp for $L = \{1\}$ as shown by the family of singletons, but there exist many other 1-close Sperner systems with n sets. Furthermore, if $L = \{q\}$ for some prime power q and $n = q^2 + q + 1$, then the lines of a projective plane of order q form an L-close family of size n, so the bound $m \leq n$ is sharp in this case, too.

Apart from Sperner-type theorems, the other much studied area in extremal finite set theory are intersection properties (see e.g. Chapter 2 of [11]). For a set L of integers, a set system $\mathcal{F}$ is said to be L-intersecting if for any pair F, F' of distinct sets in $\mathcal{F}$ we have $|F \cap F'| \in L$. Frankl and Wilson [10] proved the same upper bound $\sum_{h=0}^{|L|} \binom{n}{h}$ on the size of L-intersecting set systems. Frankl and Wilson used higher incidence matrices to prove their result, but later the polynomial method (see [2] and [1]) turned out to be very effective in obtaining L-intersection theorems. In the proof of the moreover part of Theorem 1.2, an additional idea due to Blokhuis [3] will be used.

We will need the following well-known lemma, we include the proof for sake of completeness. For any field $\mathbb{F}$, we denote by $\mathbb{F}^n[x]$ the vector space over $\mathbb{F}$ of polynomials of n variables with coefficients from $\mathbb{F}$.

Lemma 1.3. *Let $p_1(x), p_2(x), \ldots, p_m(x) \in \mathbb{F}^n[x]$ be polynomials and $v_1, v_2, \ldots, v_m \in \mathbb{F}^n$ be vectors such that $p_i(v_i) \neq 0$ and $p_i(v_j) = 0$ holds for all $1 \leq j < i \leq m$. Then the polynomials are linearly independent.*

Proof. Suppose that $\sum_{i=1}^m c_i p_i(x) = 0$. As $p_i(v_1) = 0$ for all $1 < i$ we obtain $c_1 p_1(v_1) = 0$ and therefore $c_1 = 0$ holds. We proceed by induction on j. If $c_h = 0$ holds for all $h < j$, then using this and $p_i(v_j) = 0$ for all $i > j$, we obtain $c_j p_j(v_j) = 0$ and therefore $c_j = 0$. $\square$

Results on L-intersecting families had some geometric consequences on point sets in $\mathbb{R}^n$ defining only a few distances, in particular on set systems $\mathcal{F}$ with only a few Hamming distance. The *skew distance* $sd(F, G) := \min\{|F \setminus G|, |G \setminus F|\}$ does not define a metric space on $2^{[n]}$ as $sd(F, G) = 0$ holds if and only if $F \subseteq G$ or $G \subseteq F$ and one can easily find triples for which the triangle inequality fails: if A is the set of even integers in $[n]$, C is the set of odd integers in $[n]$, and $B = \{1, 2\}$, then $\lfloor n/2 \rfloor = sd(A, C) \not\leq sd(A, B) + sd(B, C) = 1 + 1$

One can also investigate the case when L includes 0. Then set systems with the required property are not necessarily Sperner, so we will say that $\mathcal{F}$ is $L-$skew distance (or L-sd for short) if $sd(A, B) \in L$ for all pairs of distinct sets $A, B \in \mathcal{F}$. We will write $ex_{sd}(n, L)$ to denote the largest size of an L-skew distance system $\mathcal{F} \subseteq 2^{[n]}$. Observe that $ex_{sd}(n, \{0\})$ asks for the maximum size of a chain in $2^{[n]}$ which is obviously $n + 1$. This shows that the moreover part of Theorem 1.2 does not remain valid in this case. In a different context Frankl, Füredi, and Pach considered the case $L = \{0, 1, \ldots, t\}$. They considered the following construction: let $\emptyset = C_0 \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_n = [n]$ be a maximal chain and let

$$\mathcal{F}_{n,t} = \{F : C_{|F|-t} \subset F\} \cup \{F : |F| \leq t \text{ or } |F| \geq n - t\}.$$

The size of $\mathcal{F}_{n,t}$ is $\binom{n}{t+1} - \binom{2t+1}{t+1} + 2\sum_{i=0}^{t} \binom{n}{i}$ and clearly $\mathcal{F}_{n,t}$ is $\{0, 1, \ldots, t\}$-sd. This gives the lower bounds in the following results.

Theorem 1.4 (Frankl, Füredi, Pach, [9])**.** *If $n \geq 3$, we have $ex_{sd}(n, \{0, 1\}) = \binom{n}{2} + 2n - 1$.*

Theorem 1.5 (Frankl, Füredi, Pach, [9])**.** *For any n, t with $n \geq 2(t + 2)$, we have* $\binom{n}{t+1} - \binom{2t+1}{t+1} + 2\sum_{i=0}^{t} \binom{n}{i} \leq ex_{sd}(n, \{0, 1, \ldots, t\}) < \binom{n}{t+1} + 5(t + 1)^2\binom{n}{t}$.

The authors of [9] conjectured that the lower bound is tight in Theorem 1.5 for large enough n. (There are larger constructions for small n.) We will give a simple, new proof of Theorem 1.4 that proceeds by induction.

2 Proof and remarks

We start by introducing some notation. For two vectors, u, v of length n we denote their scalar product $\sum_{i=1}^{n} u_i v_i$ by $u \cdot v$. We will often use the fact that for any pair F, G of sets we have $v_F \cdot v_G = |F \cap G|$. We will also use that $\min\{|F \setminus G|, |G \setminus F|\} = |F \setminus G|$ if and only if $|F| \leq |G|$ holds.

For two sets $F, L \subseteq [n]$ we define the polynomial $p'_{F,L} \in \mathbb{R}^n[x]$ as

$$p'_{F,L}(x) = \prod_{h \in L} (|F| - v_F \cdot x - h).$$

We obtain $p_{F,L}(x)$ from $p'_{F,L}(x)$ by replacing every x_i^t term by x_i for every $t \geq 2$ and $i = 1, 2, \ldots, n$. As $0 = 0^t$ and $1 = 1^t$ for any $t \geq 2$, we have $p_{F,L}(v_G) = p'_{F,L}(v_G) = \prod_{h \in L}(|F \setminus G| - h)$. Finally, observe that the polynomials $p_{F,L}(x)$ all belong to the subspace $M_{|L|}$ of $\mathbb{R}^n[x]$ spanned by $\{x_{i_1} x_{i_2} \ldots x_{i_l} : 0 \leq l \leq |L|, i_1 < i_2 < \cdots < i_l\}$, where $l = 0$ refers to the constant 1 polynomial $\mathbf{1}$. Note that $\dim(M_{|L|}) = \sum_{i=0}^{|L|} \binom{n}{i}$.

Based on the above, Theorem 1.2 is an immediate consequence of the next result.

Theorem 2.1. *If the set system $\{F_1, F_2 \ldots, F_m\} \subseteq 2^{[n]}$ is L-close Sperner, then the polynomials $p_{F_1,L}(x), p_{F_2,L}(x), \ldots, p_{F_m,L}(x)$ are linearly independent in $\mathbb{R}^n[x]$. In particular, $m \leq \sum_{h=0}^{|L|} \binom{n}{h}$. Moreover, if $|L| = 1$ and $\{F_1, F_2 \ldots, F_m\} \neq \{\emptyset\}$, then the polynomials $p_{F_1,L}(x), p_{F_2,L}(x), \ldots, p_{F_m,L}(x)$ are linearly independent in $\mathbb{R}^n[x]$ even together with $\mathbf{1}$. In particular, $m \leq n$.*

Proof. We claim that if $F_1, F_2, \ldots, F_m$ are listed in a non-increasing order according to the sizes of the sets, then the polynomials $p_{F_1,L}(x), p_{F_2,L}(x), \ldots, p_{F_m,L}(x)$ and the characteristic vectors $v_{F_1}, v_{F_2}, \ldots, v_{F_m}$ satisfy the conditions of Lemma 1.3. Indeed, for any $G \subseteq [n]$ we have $p_{F,k}(G) = \prod_{h \in L}(|F| - |F \cap G| - h) = \prod_{h \in L}(|F \setminus G| - h)$. Therefore $p_{F,L}(v_F) \neq 0$ holds for any $F \subseteq [n]$, while if $|F_j| \leq |F_i|$, then the L-close Sperner property ensures $|F_i \setminus F_j| \in L$ and thus $p_{F_j,L}(v_{F_i}) = 0$.

To prove the moreover part, let $L = \{s\}$, $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ and let us suppose towards a contradiction that $\mathbf{1} = \sum_{i=1}^m c_{F_i} p_{F_i,L}(x)$ holds for some reals c_{F_i}. We claim that if $|F_i| = |F_j|$, then $c_{F_i} = c_{F_j}$ holds and all coefficients are negative. Observe that for any $F \in \mathcal{F}$ using the L-close Sperner property we have

$$1 = c_F p_{F,L}(v_F) + \sum_{\substack{F' \in \mathcal{F} \\ |F'| > |F|}} c_{F'} p_{F',L}(v_F), \tag{1}$$

and $p_{F,L}(v_F) = -s$ for all F. In particular, if F is of maximum size in $\mathcal{F}$, then $c_F = -\frac{1}{s}$ holds. Let m_j denote $|\{F \in \mathcal{F} : |F| = j\}|$ and c_j denote the value of c_F for all $F \in \mathcal{F}$ of size j - once this is proved. By the above, if j^* is the maximum size among sets in $\mathcal{F}$, then c_{j^*} exists. Suppose that for some i we have proved the existence of c_j for all j with $i < j \leq j^*$. If there is no set in $\mathcal{F}$ of size i, there is nothing to prove. If $|F| = i$, then using (1) and the fact $p_{F',L}(v_F) = |F'| - |F| + s - s = |F'| - |F|$ provided $|F'| \geq |F|$, we obtain

$$1 = c_F p_{F,L}(v_F) + \sum_{\substack{F' \in \mathcal{F} \\ |F'| > |F|}} c_{F'} p_{F',L}(v_F) = -sc_F + \sum_{j > i} c_j m_j (j - i). \tag{2}$$

This shows that c_F does not depend on F only on $|F|$ as claimed. Moreover, as s, m_j, $j - i$ are all non-negative and, by induction, all c_j are negative, then in order to satisfy (2), we must have that c_i is negative as well. So we proved that all c_j's are negative. But this contradicts $\mathbf{1} = \sum_{i=1}^m c_{F_i} p_{F_i,L}(x)$, as on the right hand side all coefficients of the variables are positive, so they cannot cancel. (If there are variables. This is where the condition $\{F_1, F_2 \ldots, F_m\} \neq \{\emptyset\}$ is used.) □

Using the original "push-to-the-middle" argument of Sperner, it is not hard to prove that for any k-close Sperner system $\mathcal{F} \subseteq 2^{[n]}$, there exists another one $\mathcal{F}' \subseteq 2^{[n]}$ with $|\mathcal{F}| = |\mathcal{F}'|$ and $\mathcal{F}'$ containing sets of size between k and $n - k$. Is it true that for such set systems we have $\langle p_{F,[k]} : F \in \mathcal{F}' \rangle \cap M_{k-1} = \{\mathbf{0}\}$? This would imply $ex_{sd}(n, [k]) = \binom{n}{k}$.

Let us now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. The lower bound is given by the special case $t = 1$ of the construction given above Theorem 1.4. It remains to prove the upper bound.

We will prove that a $\{0, 1\}$-sd system $\mathcal{F} \subseteq 2^{[n]}$ is of size at most $\binom{n}{2} + 2n - 1$ by induction on n. Since $\binom{3}{2} + 2 \cdot 3 - 1 = 2^3$, the statement is trivially true for $n = 3$. Now assume that $n \geq 4$ and we have already proved the statement for $n - 1$.

Consider the uniform systems $\mathcal{F}_i = \{F \in \mathcal{F} : |F| = i\}$ that are 1-close Sperner. We will define a representative set C_i for all nonempty levels. If $|\mathcal{F}_i| \geq 3$, it is an exercise for the reader (see Lemma 19 in [5]) to see that there exists a set C_i either with $|C_i| = i - 1$ and $C_i \subseteq \cap_{F \in \mathcal{F}_i} F$ or with $|C_i| = i + 1$ and $\cup_{F \in \mathcal{F}_i} F \subseteq C_i$. In the former case we say that $\mathcal{F}_i$ is of type $\vee$, in the latter case we say that $\mathcal{F}_i$ is of type $\wedge$. If $|\mathcal{F}_i| = 2$, then we select one of the two sets to be C_i. If $|\mathcal{F}_i| = 1$, then C_i is the only set in $\mathcal{F}_i$. Finally, if $\mathcal{F}_i = \emptyset$, then C_i is undefined.

Claim 2.2. If $i < j$ and $|\mathcal{F}_i|, |\mathcal{F}_j| > 0$ then $|C_i \backslash C_j| \leq 1$.

Proof. Assume that there are two different elements a, b such that $a, b \in C_i$ but $a, b \notin C_j$. It follows from the definition of the representative sets, that there are sets $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$ such that $a, b \in F_i$ and $a, b \notin F_j$. (This is trivial for levels with one or two sets. If there are 3 or more sets then at most two of them can be wrong.) $\square$

Let $C_{p_1}, C_{p_2}, \ldots C_{p_t}$ $(p_1 < \cdots < p_t)$ denote the representative sets of the nonempty levels among $\mathcal{F}_1, \mathcal{F}_2, \ldots \mathcal{F}_{n-1}$. Since

$$\left| \bigcup_{i=1}^{t-1} C_{p_i} \backslash C_{p_{i+1}} \right| \leq \sum_{i=1}^{t-1} |C_{p_i} \backslash C_{p_{i+1}}| \leq t - 1 \leq n - 2,$$

there will be an element $x \in [n]$ such that $x \notin C_{p_i} \backslash C_{p_{i+1}}$ for any p_i. This implies that there are no nonempty levels $\mathcal{F}_i$ and $\mathcal{F}_j$ such that $i < j$, $x \in C_i$ but $x \notin C_j$. Rearranging the names of the elements, we may assume that $x = n$.

Now we define two families in $2^{[n-1]}$, let

$$\mathcal{G} = \{F \backslash \{n\} \mid F \in \mathcal{F}\}, \quad \mathcal{H} = \{H \in 2^{[n-1]} \mid H, H \cup \{n\} \in \mathcal{F}\}.$$

Note that $|\mathcal{F}| = |\mathcal{G}| + |\mathcal{H}|$. Since $\mathcal{G}$ is a $\{0, 1\}$-sd system in $2^{[n-1]}$, we get an upper bound on its size by induction. We will examine $\mathcal{H}$ to bound its size as well.

Claim 2.3. If $A, B \in \mathcal{H}$ and $|A| < |B|$ then $A \subset B$.

Proof. By the definition of $\mathcal{H}$, we get that $A \cup \{n\} \in \mathcal{F}$ and $n \notin B$. Since $\mathcal{F}$ is a $\{0, 1\}$-sd system, $1 \geq |(A \cup \{n\}) \backslash B| = |A \backslash B| + 1$. Therefore we have $|A \backslash B| = 0$ or equivalently $A \subset B$. $\square$

Claim 2.4. There is at most one level in $\mathcal{H}$ with two or more sets in it.

Proof. Assume that there are two sets of size i and two sets of size j ($i < j$) in $\mathcal{H}$. Then in $\mathcal{F}$ there are two sets of size $i+1$ containing n and two sets of size j that do not contain n. From the definition of the representative sets follows that $n \in C_{i+1}$ but $n \notin C_j$. This is an outright contradiction if $i+1 = j$. If $i+1 < j$, it contradicts the special property of the element n established earlier. $\square$

Claim 2.5. $|\mathcal{H}| \leq n+1$.

Proof. Let $\mathcal{H}_i = \{H \in \mathcal{H} : |H| = i\}$ for all $i = 0, 1, \dots, n-1$. If there is no i such that $|\mathcal{H}_i| > 1$, then $|\mathcal{H}| \leq n$. Assume that $|\mathcal{H}_t| = k > 1$. By Claim 2.4, this is the only level with more than one set. If the level $\mathcal{H}_t$ is of type $\vee$, then the union of its sets is of size $t+k-1$. Claim 2.3 implies that all sets $H \in \mathcal{H}$, $|H| > t$ must contain this union, therefore the levels $\mathcal{H}_{t+1}, \mathcal{H}_{t+2}, \dots, \mathcal{H}_{t+k-2}$ are all empty. If $\mathcal{H}_t$ is of type $\wedge$, then the intersection of its sets is of size $t-k+1$. Claim 2.3 implies that all sets $H \in \mathcal{H}$, $|H| < t$ must be subsets of this intersection, therefore the levels $\mathcal{H}_{t-k+2}, \mathcal{H}_{t-k+3}, \dots, \mathcal{H}_{t-1}$ are all empty. In either case we get that $|\mathcal{H}| \leq k + (k-2)\cdot 0 + (n-k+1)\cdot 1 = n+1$. $\square$

Now we can complete the proof of the theorem:

$$|\mathcal{F}| = |\mathcal{G}| + |\mathcal{H}| \leq \binom{n-1}{2} + 2(n-1) - 1 + n + 1 = \binom{n}{2} + 2n - 1.$$

$\square$

Let us make two final remarks.

- Observe that for the set $L_\ell = \{\ell+1, \ell+2, \dots, n\}$ a system $\mathcal{F} \subseteq 2^{[n]}$ is L_ℓ-close Sperner if and only if for every ℓ-subset Y of $[n]$, the trace $\mathcal{F}_{[n]\setminus Y} = \{F \setminus Y : F \in \mathcal{F}\}$ is Sperner. Set systems with this property are called $(n-\ell)$-trace Sperner and results on the maximum size of such systems can be found in Section 4 of [12].

- A natural generalization arises in $Q^n = \{0, 1, \dots, q-1\}^n$. One can partially order Q^n by $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \dots, n$. We say that $A \subseteq \{0, 1, \dots, q-1\}^n$ is L-close Sperner for some subset $L \subseteq [n]$ if for any distinct $a, b \in A$ we have $sd(a,b) := \min\{|\{i : a_i < b_i\}|, |\{i : a_i > b_i\}|\} \in L$. One can ask for the largest number of points in an L-close Sperner set $A \subseteq Q^n$. Here is a construction for $\{1\}$-close Sperner set: for $2 \leq i \leq n$, $1 \leq h \leq q-1$ let $(v_{i,h})_i = h$, $(v_{i,h})_1 = q-h+1$ and $(v_{i,h})_j = 0$ if $j \neq i$. Then it is easy to verify that $\{v_{i,h} : 2 \leq i \leq n, 1 \leq h \leq q-1\}$ is $\{1\}$-close Sperner of size $(q-1)(n-1)$.

  An easy upper bound on the most number of points in Q^n that form an $\{1\}$-close Sperner system is $O_q(n^{q-1})$. To see this, for any $a = \{a_1, a_2, \dots a_n\} \in Q^n$ let us define the set $F_a \subseteq [(q-1)n]$ as follows.

$$F_a := \bigcup_{i=1}^{n} \bigcup_{j=1}^{a_i} \{(q-1)(i-1)+j\}$$

If $A \subseteq Q^n$ is $\{1\}$-close Sperner, then $A' = \{F_a \mid a \in A\} \subset 2^{[(q-1)n]}$ will be $\{1, 2, \ldots q-1\}$-close Sperner. Theorem 1.2 implies

$$|A| = |A'| \leq \sum_{h=0}^{q-1} \binom{(q-1)n}{h} = O_q(n^{q-1}).$$

We conjecture that for any q there exists a constant C_q such that the maximum number of points in Q^n that form a $\{1\}$-close Sperner system is at most $C_q n$.

Acknowledgement

The research of Nagy was supported by the National Research, Development and Innovation Office - NKFIH under the grants FK 132060 and K 132696.

The research of Patkós was supported partially by the grant of Russian Government N 075-15-2019-1926 and by the National Research, Development and Innovation Office - NKFIH under the grants FK 132060 and SNN 129364.

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