Remarks on Graphons

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Abstract

L. Lovász and B. Szegedy proved in 2006 that the limits of convergent graph sequences can be described by measurable symmetric functions $W : [0,1] \times [0,1] \to [0,1]$ called graphons. In our present paper we investigate the structure of the set of all graphons within the semigroup $(\mathfrak{F}([0,1]^2); \circ)$ of all fuzzy subsets of the unit square $[0,1]^2 = [0,1] \times [0,1]$, where the operation $\circ$ is defined by: for every $f, g \in \mathfrak{F}([0,1]^2)$ and every $s \in [0,1]^2$, $(f \circ g)(s) = \vee_{x \in [0,1]^2}(f(x) \wedge g(s))$.

Mathematics Subject Classification: 20M10; 08A72; 05C99.

Keywords: fuzzy subset, graphon, semigroup.

1 Introduction and motivation

Let G_n be a sequence of finite simple graphs whose number of nodes tends to infinity. For every fixed finite simple graph F, let $hom(F, G_n)$ denote the number of all homomorphisms from F into G_n, that is, the edge-preserving functions from $V(F)$ into $V(G_n)$. Put

$$t(F, G_n) = \frac{hom(F, G_n)}{|V(G_n)|^{|V(F)|}}.$$

Clearly, $t(F, G_n)$ is the probability that a random mapping from $V(F)$ into $V(G_n)$ should be a homomorphism. The sequence G_n is called convergent if $lim_{n\to\infty} t(F, G_n)$ exists for every finite simple graph F. Let

$$t(F) = lim_{n\to\infty} t(F, G_n).$$

[1]This work was supported by the National Research, Development and Innovation Office – NKFIH, 115288.

Then t is a graph parameter, that is, a function on simple graphs that is invariant under isomorphism. In [4], the authors given characterizations of graph parameters that arise in this manner; that is, the authors characterize the set $\mathfrak{T}$ of graph parameters t for which there is a convergent sequence of simple graphs G_n such that $t(F) = lim_{n\to\infty} t(F, G_n)$ for every simple graph F. In the characterization of $\mathfrak{T}$, the symmetric and measurable functions $W : [0,1]^2 = [0,1] \times [0,1] \mapsto [0,1]$ called graphons play an important role. Recall that a function $W : [0,1]^2 \mapsto [0,1]$ is said to be symmetric if $W(x,y) = W(y,x)$ is satisfied for all $x, y \in [0,1]$. A graph is said to be *k-labelled* (k is a positive integer) if the graph has k nodes labelled by $1, 2, \ldots, k$. For a k-labelled simple graph F and a graphon W, the integral

$$t(F,W) = \int_{[0,1]^k} \prod_{ij\in E(F)} W(x_i, x_j) dx_1 dx_2 \cdots dx_k$$

is called the *density of the graph F in the graphon W* ([5]), where $E(F)$ denotes the set of all edges of F. In [4, Theorem 2.2] it was shown that a graph parameter t belongs to $\mathfrak{T}$ if and only if there is a graphon W such that $t(F) = t(F,W)$ for all simple graphs F.

A function of a non-empty set S into the real unit interval $[0,1]$ is called a *fuzzy subset* of S (see [11]). By [3] and [7], if $*$ is an associative operation on a non-empty set S, then the set $\mathfrak{F}(S)$ of all fuzzy subsets of S form a semigroup under the operation $\circ$ defined by the following way: for arbitrary $f, g \in \mathfrak{F}(S)$ and $s \in S$,

$$(f \circ g)(s) = \begin{cases} \vee_{s=x*y}(f(x) \wedge g(y)), & \text{if } s \in S^2 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

As every graphon is a fuzzy subset of the unit square $[0,1]^2$, the following problem seems interesting from a semigroup theory perspective.

Problem: *If an associative operation $*$ is given on the unit square $[0,1]^2$, what can we say about the structure of the set $\mathcal{W}_0$ of all graphons in the semigroup $(\mathfrak{F}([0,1]^2); \circ)$? Is it true that $\mathcal{W}_0$ forms a substructure of $(\mathfrak{F}([0,1]^2); \circ)$? If so, what kind of substructure is it?*

In this paper we deal with this problem in a special case: the given associative operation $*$ on $[0,1]^2$ satisfies the identity $(x,y) * (u,v) = (u,v)$. A semigroup $(S; *)$ is called a *right zero semigroup* if it satisfies the identity $a * b = b$. With this terminology, the above problem is examined in that case when $[0,1]^2$ is a right zero semigroup.

We note that if S is a non-empty set (and so it is a right zero semigroup), then the operation $\circ$ defined in (1) has the following form:

$$(f \circ g)(s) = \vee_{x\in S}(f(x) \wedge g(s)). \tag{2}$$

Throughout the paper, for a non-empty set S, $(\mathfrak{F}(S); \circ)$ will denote the semigroup in which the operation $\circ$ is defined by (2). Thus the purpose of this paper is to examine the structure of the set $\mathcal{W}_0$ of all graphons in the semigroup $(\mathfrak{F}([0,1]^2); \circ)$. Our studies consist of two parts. In Section 2 we describe the structure of the semigroup $(\mathfrak{F}(S); \circ)$ for an arbitrary non-empty set S, in Section 3 we focus on the semigroup $(\mathfrak{F}([0,1]^2); \circ)$ and its subset $\mathcal{W}_0$. A semigroup S is called a *band* if every element e of S is an idempotent element, that is, $e^2 = e$. A band satisfying the identity $axa = xa$ is called a *right regular band* ([9]). In Section 2 we prove that if S is an arbitrary non-empty set, then the semigroup $(\mathfrak{F}(S); \circ)$ is a right regular band (Theorem 2.6). In Section 3, applying the above result for the right regular band $(\mathfrak{F}([0,1]^2); \circ)$, we show that the set $\mathcal{W}_0$ of all graphons is a left ideal of $(\mathfrak{F}([0,1]^2); \circ)$. By this result, if W is a graphon and f is a fuzzy subset of $[0,1]^2$, then $f \circ W$ is a graphon. Thus, for arbitrary simple graphs F, we can consider the densities $t(F; W)$ and $t(F; f \circ W)$ of F in W and in $f \circ W$, respectively. In Section 3 we give an upper bound to $|t(F; W) - t(F; f \circ W)|$. In Theorem 3.6 we show that $|t(F; W) - t(F; f \circ W)| \leq |E(F)|(\sup(W) - \sup(f))\Delta(\{W > \sup(f)\})$, where $\Delta(\{W > \sup(f)\})$ denotes the area of the set $\{W > \sup(f)\} = \{(x, y) \in [0,1]^2 : W(x, y) > \sup(f)\}$.

For notations and notions not defined here, we refer to the paper [4] and the books [1], [6], [8], and [9].

2 On the semigroup $(\mathfrak{F}(S); \circ)$, where S is an arbitrary non-empty set

For a fuzzy subset f and a subset X of a non-empty set S, let $\sup_X(f) = \vee_{x \in X} f(x)$. Especially, let $\sup(f) = \sup_S(f)$. If f and g are arbitrary fuzzy subsets of S, then let g_f and g_f^* denote the following fuzzy subsets of S: for an arbitrary $s \in S$, let

$$g_f(s) = \begin{cases} \sup(f), & \text{if } g(s) > \sup(f) \\ g(s), & \text{otherwise} \end{cases}$$

and

$$g_f^*(s) = \begin{cases} g(s) - \sup(f), & \text{if } g(s) > \sup(f) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.1 By the above definitions, $g_f + g_f^* = g$ for every fuzzy subsets f and g of a non-empty set S.

Remark 2.2 Let f and g be arbitrary fuzzy subsets of a non-empty set S. It is clear that $\sup(g) \leq \sup(f)$ implies $g(s) \leq \sup(f)$ for every $s \in S$ and so $g_f = g$. In case $\sup(g) > \sup(f)$, there is an element $s \in S$ such that $g(s) > \sup(f)$ and so $g_f(s) = \sup(f) < g(s)$. Hence $g_f \neq g$. Thus, for every fuzzy subsets f and g of S, the equation $g_f = g$ holds if and only if $\sup(g) \leq \sup(f)$.

By Remark 2.2, the following lemma holds.

Lemma 2.3 *For arbitrary fuzzy subsets f and g of a non-empty set S, the equations $g_f = g$ and $f_g = f$ together hold if and only if* $\sup(g) = \sup(f)$.

The next lemma will be used in Lemma 3.1.

Lemma 2.4 *If f and g are fuzzy subsets of a non-empty set S such that* $\sup(f) \leq \sup(g)$ *then* $\sup(g_f) = \sup(f)$ *and* $\sup(g_f^*) = \sup(g) - \sup(f)$.

Proof. By the definition of g_f and g_f^*, it is obvious. □

Theorem 2.5 *Let S be a non-empty set. For every fuzzy subsets f and g of S, we have $f \circ g = g_f$.*

Proof. Let f and g be arbitrary fuzzy subsets of a non-empty set S. By the above, $(\mathfrak{F}(S); \circ)$ is a semigroup. Let s be an arbitrary element of S. If $g(s) > \sup(f)$, then $f(x) \wedge g(s) = f(x)$ for every $x \in S$, and so $(f \circ g)(s) = \vee_{x \in S} f(x) = \sup(f)$. If $g(s) \leq \sup(f)$, then we have two subcases.
Case 1: If $g(s) = \sup(f)$, then $f(x) \wedge g(s) = f(x)$ for all $x \in S$, and so $(f \circ g)(s) = \vee_{x \in S} f(x) = \sup(f) = g(s)$.
Case 2: If $g(s) < \sup(f)$, then there is an $x_0 \in S$ such that $f(x_0) > g(s)$ and so $f(x_0) \wedge g(s) = g(s)$. Moreover, for arbitrary $x \in S \setminus \{x_0\}$, we have

$$f(x) \wedge g(s) = \begin{cases} g(s), & \text{if } g(s) < f(x) \\ f(x), & \text{if } f(x) \leq g(s), \end{cases}$$

and so $(f \circ g)(s) = (f(x_0) \wedge g(s)) \vee (\vee_{x \in S \setminus \{x_0\}} (f(x) \wedge g(s)) = g(s)$. Summarizing our results, we get

$$(f \circ g)(s) = \begin{cases} \sup(f), & \text{if } g(s) > \sup(f) \\ g(s), & \text{otherwise,} \end{cases}$$

that is, $(f \circ g)(s) = g_f(s)$, which proves our assertion. □

A commutative band is called a *semilattice*. A congruence α on a semigroup A is said to be a *semilattice congruence* if the factor semigroup A/α is a semilattice. A semigroup A is said to be *semilattice indecomposable* if the universal relation is the only semilattice congruence on A. It is known ([10]) that every semigroup has a least semilattice congruence η; the classes of η are semilattice indecomposable. By [9, II.3.12. Proposition], a band is a right regular band if and only if its η-classes are right zero semigroups.

Theorem 2.6 *For an arbitrary non-empty set S, the semigroup $(\mathfrak{F}(S); \circ)$ is a right regular band. The η-classes of $\mathfrak{F}(S)$ are right zero semigroups. Two fuzzy subsets f and g of S are in the same η-class if and only if* $\sup(f) = \sup(g)$.

Proof. Let S be an arbitrary non-empty set. Then S is a right zero semigroup, and so $(\mathfrak{F}(S); \circ)$ is a semigroup under the operation $\circ$ defined in (2), that is, $(f \circ g)(s) = \vee_{x \in S}(f(x) \wedge g(s))$ for every fuzzy subsets f and g of S and every element $s \in S$. By Theorem 2.5, it is clear that $f \circ f = f$ for every $f \in \mathfrak{F}(S)$, and so $(\mathfrak{F}(S); \circ)$ is a band. Using also Theorem 2.5, we have $g \circ f \circ g = g \circ g_f$. As $\sup(g) \geq \sup(g_f)$, we have $g \circ g_f = g_f$. Thus $g \circ f \circ g = g_f = f \circ g$. Hence $(\mathfrak{F}(S); \circ)$ is a right regular band. Let η denote the least semilattice congruence on $(\mathfrak{F}(S); \circ)$. The η-classes of $(\mathfrak{F}(S); \circ)$ are right zero semigroups by [9, II.3.12. Proposition]. Let f and g be arbitrary fuzzy subsets of S. By [9, II.1.1. Proposition], $(f, g) \in \eta$ if and only if $f \circ g \circ f = f$ and $g \circ f \circ g = g$. As $(\mathfrak{F}(S); \circ)$ is a right regular band, we have $f \circ g \circ f = g \circ f$ and $g \circ f \circ g = f \circ g$. Thus $(f, g) \in \eta$ if and only if $g \circ f = f$ and $f \circ g = g$. Using Theorem 2.5, $(f, g) \in \eta$ if and only if $f_g = f$ and $g_f = g$. By Lemma 2.3, we get $(f, g) \in \eta$ if and only if $\sup(f) = \sup(g)$. □

3 On the structure of the set of all graphons in the semigroup $(\mathfrak{F}([0, 1]^2); \circ)$

Let $(S, \mathcal{A}, \mu)$ be a measurable space ([2]). For a fuzzy subset h of S and a real number A, let $\{h > A\} = \{s \in S : \ h(s) > A\}$. A fuzzy subset h of S is said to be *measurable* if, for every real number A, the subset $\{h > A\}$ of S is measurable (that is, $\{h > A\} \in \mathcal{A}$).

Lemma 3.1 *Let $(S, \mathcal{A}, \mu)$ be a measurable space. Then, for an arbitrary fuzzy subset f and an arbitrary measurable fuzzy subset g of S, the fuzzy subsets g_f and g_f^* are measurable.*

Proof. Let f and g be arbitrary fuzzy subsets of S such that g is measurable. If $\sup(f) \geq \sup(g)$, then $g_f = f \circ g = g$ and $g_f^* = 0$. In this case the fuzzy subsets g_f and g_f^* are measurable. Consider the case when $\sup(f) < \sup(g)$.

Then $\sup(g_f) = \sup(f)$ and $\sup(g_f^*) = \sup(g) - \sup(f)$ by Lemma 2.4. Let A be an arbitrary real number. It is easy to see that

$$\{g_f > A\} = \begin{cases} \emptyset, & \text{if } A \geq \sup(f) \\ \{g > A\}, & \text{otherwise} \end{cases}$$

and

$$\{g_f^* > A\} = \begin{cases} \emptyset, & \text{if } A \geq \sup(g) - \sup(f) \\ \{g > A + \sup(f)\}, & \text{if } 0 \leq A < \sup(g) - \sup(f) \\ S, & \text{if } A < 0 \end{cases}$$

from which it follows that g_f and g_f^* are measurable fuzzy subsets of S. □

A fuzzy subset f of $[0,1]^2$ is said to be *symmetric* if $f(x,y) = f(y,x)$ is satisfied for all $x, y \in [0,1]$.

Lemma 3.2 *If f is an arbitrary fuzzy subset and g is a symmetric fuzzy subset of $[0,1]^2$, then g_f and g_f^* are symmetric fuzzy subsets of $[0,1]^2$.*

Proof. It is obvious by the definition of g_f and g_f^*. □

Lemma 3.3 *If W is a graphon and f is a fuzzy subset of $[0,1]^2$, then W_f and W_f^* are graphons.*

Proof. By Lemma 3.1 and Lemma 3.2, it is obvious. □

The following theorem provides an answer to the question raised in Problem in the case, where the given operation $\cdot$ on $[0,1]^2$ satisfies the identity $a \cdot b = b$.

Theorem 3.4 *The set $\mathcal{W}_0$ of all graphons is a left ideal of the right regular band $(\mathfrak{F}([0,1]^2); \circ)$ of all fuzzy subsets of $[0,1]^2$. Thus the semigroup $(\mathcal{W}_0; \circ)$ of all graphons is a right regular band, and so it is a semilattice I of right zero subsemigroups S_i $(i \in I)$. Two graphons W_1 and W_2 are in the same S_i if and only if* $\sup(W_1) = \sup(W_2)$.

Proof. Let W be a graphon and f be a fuzzy subset of $[0,1]^2$. By Theorem 2.5, $f \circ W = W_f$. Then $f \circ W$ is a graphon by Lemma 3.3. Thus the set $\mathcal{W}_0$ of all graphons is a left ideal of the semigroup $(\mathfrak{F}([0,1]^2); \circ)$ of all fuzzy subsets of $[0,1]^2$. By Theorem 2.6, the semigroup $(\mathfrak{F}([0,1]^2); \circ)$ and so its subsemigroup $(\mathcal{W}_0; \circ)$ is a right regular band. Moreover, the η-classes of $\mathcal{W}_0$ are right zero semigroups; two graphons W_1 and W_2 are in the same η-class if and only if $\sup(W_1) = \sup(W_2)$. □

Let σ denote the equivalence relation on the set $\mathcal{W}_0$ of all graphons defined by $(W_1, W_2) \in \sigma$ if and only if $W_1 = W_2$ almost everywhere in $[0,1]^2$.

Proposition 3.5 *The equivalence relation $\sigma \cap \eta$ is a congruence on the right regular band $(\mathcal{W}_0; \circ)$ of all graphons, where η is the least semilattice congruence on $(\mathcal{W}_0; \circ)$.*

Proof. Let W_1 and W_2 be two graphons with $(W_1, W_2) \in \sigma \cap \eta$. Then, using Theorem 3.4, we have $\sup(W_1) = \sup(W_2)$ and $W_1 = W_2$ almost everywhere in $[0,1]^2$. Let W be an arbitrary graphon. As $\sup(W_1) = \sup(W_2)$, we have $W_1 \circ W = W_2 \circ W$. Thus $(W_1 \circ W, W_2 \circ W) \in \sigma \cap \eta$. Hence $\sigma \cap \eta$ is a right congruence on $(\mathcal{W}_0; \circ)$. Let $T = \{(x,y) \in [0,1]^2 | \ W_1(x,y) \neq W_2(x,y)\}$. As $(W_1, W_2) \in \sigma$, the area of T is 0. It is clear that $\{(x,y) \in [0,1]^2 : (W \circ W_1)(x,y) \neq (W \circ W_2)(x,y)\} \subseteq T$ and so $(W \circ W_1, W \circ W_2) \in \sigma$. As $(W_1, W_2) \in \eta$ and η is a congruence on $(\mathcal{W}_0; \circ)$, we have $(W \circ W_1, W \circ W_2) \in \eta$. Thus $(W \circ W_1, W \circ W_2) \in \sigma \cap \eta$ and so $\sigma \cap \eta$ is a left congruence on $(\mathcal{W}_0; \circ)$. Thus $\sigma \cap \eta$ is a congruence on $(\mathcal{W}_0; \circ)$. □

Let W be a graphon and f a fuzzy subset of $[0,1]^2$. By Theorem 3.4, $f \circ W$ is a graphon. Thus, for arbitrary simple graphs F, we can consider the densities $t(F; W)$ and $t(F; f \circ W)$ of F in W and $f \circ W$, respectively. The next theorem gives an upper bound to $|t(F; W) - t(F; f \circ W)|$.

Theorem 3.6 *Let W be an arbitrary graphon. Then, for an arbitrary fuzzy subset f of $[0,1]^2$ and an arbitrary finite simple graph F,*

$$|t(F; W) - t(F; f \circ W)| \leq |E(F)|(\sup(W) - \sup(f))\Delta(\{W > \sup(f)\}),$$

where $E(F)$ denotes the set of all edges of F and $\Delta(\{W > \sup(f)\})$ denotes the area of the set $\{W > \sup(f)\} = \{(x,y) \in [0,1]^2 : \ W(x,y) > \sup(f)\}$.

Proof. Let W be an arbitrary graphon and f an arbitrary fuzzy subset of $[0,1]^2$. By Theorem 3.4, $f \circ W$ is a graphon. If $\sup(W) \leq \sup(f)$, then $W = f \circ W$ and $\{W > \sup(f)\} = \emptyset$. Thus $|t(F; W) - t(F; f \circ W)| = 0 = |E(F)|(\sup(W) - \sup(f))\Delta(\{W > \sup(f)\})$. Consider the case when $\sup(W) > \sup(f)$. By Remark 2.1, $W - (f \circ W) = W_f^*$. As W is a graphon, $W_f = f \circ W$ and W_f^* are graphons by Lemma 3.3. Thus W, $f \circ W$ and W_f^* are integrable functions on $[0,1]^2$. Using [4, Lemma 4.1], $|t(F; W) - t(F; f \circ W)| \leq |E(F)| \cdot ||W_f^*||_\square$, where $||W_f^*||_\square = \sup_{\substack{A \subseteq [0,1] \\ B \subseteq [0,1]}} \left| \int_A \int_B W_f^*(x,y) dx dy \right|$. As W_f^* is a non-negative function, $||W_f^*||_\square = ||W_f^*||_1$, where $||W_f^*||_1 = \int_0^1 \int_0^1 |W_f^*(x,y)| dx dy$. Thus $|t(F; W) - t(F; f \circ W)| \leq |E(F)| \cdot ||W_f^*||_1$. As $W_f^*(x,y) = 0$ for all $(x,y) \in [0,1]^2 \setminus \{W > \sup(f)\}$, we have $||W_f^*||_1 = \int_0^1 \int_0^1 W_f^*(x,y) dx dy \leq (\sup(W) - \sup(f))\Delta(\{W > \sup(f)\}$, because $\sup(W_f^*) = \sup(W) - \sup(f)$ by Lemma 2.4. Consequently $|t(F; W) - t(F; f \circ W)| \leq |E(F)|(\sup(W) - \sup(f))\Delta(\{W > \sup(f)\})$. □

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