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ROBUST STABILITY OF GLOBAL ATTRACTORS FOR EVOLUTIONARY SYSTEMS WITHOUT UNIQUENESS

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Abstract. We establish the local input-to-state stability of multi-valued evolutionary systems with bounded disturbances with respect to the global attractor of the respective undisturbed system. We apply obtained results to disturbed reaction-diffusion equation.

Key words: global attractor, multi-valued semiprocess, local input-to-state stability, reaction-diffusion equation..

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1. Introduction

Evolutionary systems without uniqueness play an important role in the general infinite-dimensional systems theory because of the large number of applications: 3D Navier-Stokes system and other PDEs, where there is no results about uniqueness of the initial-value problem in the natural phase spase: multidimensional reaction-diffusion systems, nonlinear PDEs with non-smooth nonlinear term, and evolutionary equations with set-valued right-hand part, where it is known that for some initial data more than one solution exist [1–3]. For dissipative infinite-dimensional systems one of the main tools for investigation their qualitative behavior is the global attractors theory [3–5]. The first results on transferring the theory of attractors to evolutionary systems without uniqueness belong to an outstanding Ukrainian mathematician Valery Melnik [6–9]. Later, the theory of global attractors of multi-valued systems was applied to a wide classes of infinite-dimensional problems without uniqueness, including impulsive [10–12], stochastic [13], and general non-autonomous problems [14–16].

One of the important properties of the global attractor is its stability. It is known [4], [9] that under rather general assumption global attractor is stable in

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Lyapunov sense. But from the application point of view it is important to prove robust stability, i.e., stability with respect to disturbances so called Input-to-State Stability (ISS) [17–20]. For single-valued evolutionary systems with non-trivial global attractors ISS theory was developed in [21–23]. In the present paper we generalize these results to general multi-valued case.

2. Setting of the problem

We consider an abstract evolutionary (autonomous) system, which is characterized by a normed phase space $(X, \|\cdot\|)$ and a family of maps (solutions) $K \subset \mathbb{C}([0,+\infty); X)$ such that the following conditions hold:

(K1) $\forall x \in X \ \exists \varphi \in K$ such that $\varphi(0) = x$;

(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in K$, $\forall \tau \geq 0$, $\forall \varphi \in K$.

Than the multi-valued map $G : \mathbb{R}_+ \times X \mapsto 2^X$

$$G(t, x) = \{\varphi(t) |\ \varphi \in K,\ \varphi(0) = x\} \tag{2.1}$$

is called m-semiflow.

Definition 2.1. A compact set $\Theta \subset X$ is called a global attractor of m-semiflow G if

(Θ1) $\Theta \subset G(t, \Theta)$, $\quad \forall t \geq 0$ (semi invariance),

(Θ2) for all bounded $B \subset X$

$$\|G(t, B)\|_\Theta \to 0,\ t \to \infty \text{ (uniform attraction)},$$

where here and after $G(t, B) = \bigcup_{b \in B} G(t, b)$ and for $Y \subset X$

$$\|Y\|_\Theta := \operatorname{dist}(Y, \Theta) = \sup_{y \in Y} \inf_{\theta \in \Theta} \|y - \theta\|.$$

It is known [9] that in the most cases global attractor, if it exists, is stable, i.e.,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \geq 0 \ \ G(t, O_\delta(\Theta)) \subset O_\varepsilon(\Theta), \tag{2.2}$$

where here and after

$$O_\delta(Y) = \{x \in X |\ \operatorname{dist}(x, Y) < \delta\}, \quad \text{for } Y \subset X.$$

In addition to «ε-δ» language, stability property (2.2) can be described in terms of comparison functions [24]. We introduce the following classes:

$\mathcal{K} := \{\gamma : [0,+\infty) \mapsto [0,+\infty) \,|\, \gamma$ is continuous, strictly increasing, $\gamma(0) = 0\}$,

$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \,|\, \gamma$ is unbounded$\}$,

$\mathcal{L} := \{\gamma : [0,+\infty) \mapsto [0,+\infty) \,|\, \gamma$ is continuous, strictly decreasing, $\gamma(t) \to 0,\ t \to \infty\}$,

$\mathcal{KL} := \{\beta : [0,+\infty) \times [0,+\infty) \mapsto [0,+\infty) \,|\, \beta$ is continuous, $\beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(s, \cdot) \in \mathcal{L},\ \forall s > 0\}$.

Under rather general assumptions we will show that for the m-semiflow G with global attractor Θ $\exists\beta \in \mathcal{KL}$ $\forall x \in X,\ \forall t \geq 0$

$$\|G(t, x)\|_\Theta \leq \beta(\|x\|_\Theta, t). \tag{2.3}$$

This property helps us to prove the main result about ISS property of disturbed system. More precisely, we assume that the initial evolutionary system undergoes non-autonomous bounded disturbances $u \in U$, where

(U) $U \subset L^\infty(\mathbb{R}_+),\ 0 \in U$, U is translation-invariant, i.e.,

$$u_h(\cdot) = u(\cdot + h) \in U, \quad \forall h \geq 0,\ \forall u(\cdot) \in U.$$

Denote by $\{S_u : \mathbb{R}^2_\geq \times X \mapsto 2^X\}_{u \in U}$, where $\mathbb{R}^2_\geq = \{(t, s)|\ t \geq s \geq 0\}$, the family of m-semiprocesses (see (3.1) below) generated by solutions of the disturbed evolutionary system (the case $u \equiv 0$ corresponds to the undisturbed system).

Under some additional assumptions we will prove that $\{S_u\}_{u \in U}$ is local ISS w.r.t. the global attractor Θ of the undisturbed system, i.e., $\exists r > 0$, $\exists\beta \in \mathcal{KL}$, $\exists\gamma \in \mathcal{K}$ such that

$$\|x\|_\Theta \leq r,\ \|u\|_\infty \leq r \ \Rightarrow\ \forall t \geq 0\ \|S_u(t, 0, x)\|_\Theta \leq \beta(\|x\|_\Theta, t) + \gamma(\|u\|_\infty), \tag{2.4}$$

where $\|u\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |u(t)|$.

3. M-semiflows and m-semiprocesses

So, let us consider the family of solutions K of undisturbed system under assumptions (K1), (K2). Then the map $G : \mathbb{R}_+ \times X \mapsto 2^X$, defined by (2.1), satisfies semigroup proreties:

(G) $G(0, x) = x,\ G(t + s, x) \subset G(t, G(s, x)),\ \forall x \in X,\ \forall t, s \geq 0$.

Moreover,

$$\varphi(t + s) \in G(t, \varphi(s)), \quad \forall\varphi \in K,\ \forall t, s \geq 0.$$

Additionally, if we assume that

(K3) $\forall\varphi_1, \varphi_2 \in K$ such that $\varphi_2(0) = \varphi_1(s)$ the function

$$\varphi(t) = \begin{cases} \varphi_1(t), & 0 \leq t \leq s, \\ \varphi_2(t - s), & t > s \end{cases}$$

belongs to K, then G is strict, i.e.,

$$G(t + s, x) = G(t, G(s, x)).$$

The last equality allows us to state existence of invariant global attractor.

Lemma 3.1. [25] *Assume that* (K1)–(K3) *hold and*

(G1) *exists bounded* $B_0 \subset X$ *such that for all bounded* $B \subset X$ $\exists T = T(B)$ $\forall t \geq T$ $G(t, B) \subset B_0$ *(dissipativity),*

(G2) $\forall t_n \nearrow \infty$, *for all bounded* $B \subset X$, $\forall \xi_n \in G(t_n, B)$ *the sequence* $\{\xi_n\}$ *is precompact (asymptotic compactness),*

(G3) $\forall t > 0$, $\forall x_n \to x_0$, $\forall \xi_n \in G(t, x_n)$, $\xi_n \to \xi_0$ *we have:* $\xi_0 \in G(t, x_0)$ *(closed graph).*

Then m*-semiflow* G *possesses invariant global attractor* Θ, *i.e.,*

$$\Theta = G(t, \Theta) \ \ \forall t \geq 0.$$

Moreover, if

(G4) $\forall t_n \to t_0 \geq 0$, $\forall x_n \to x_0$, $\forall \xi_n \in G(t_n, x_n)$ *up to sequence* $\xi_n \to \xi_0 \in G(t_0, x_0)$

holds, then Θ *is stable in the sense of* (2.2).

Now assume that our evolutionary system undergoes disturbances $u \in U$, where the set U satisfies (U). Denote by $K_u^\tau \subset \mathbb{C}([\tau, +\infty); X)$ the family of maps satisfying the following properties:

(S1) $\forall x \in X$, $\forall \tau \geq 0$, $\forall u \in U$ $\exists \varphi \in K_u^\tau :$ $\varphi(\tau) = x$,

(S2) $\varphi|_{[s,+\infty)} \in K_u^s$, $\forall \varphi \in K_u^\tau$, $\forall s \geq \tau$,

(S3) $\varphi(\cdot + h) \in K_{u(\cdot + h)}^\tau$, $\forall \varphi \in K_u^{\tau+h}$, $\forall h \geq 0$.

Let us put

$$S_u(t, \tau, x) := \{\varphi(t) | \ \varphi \in K_u^\tau, \ \varphi(\tau) = x\}. \tag{3.1}$$

Then [26] $\{S_u\}_{u \in U}$ generates the family of m-semiprocesses, i.e., $\forall u \in U$, $\forall t \geq s \geq \tau \geq 0$, $\forall x \in X$, $\forall h \geq 0$

$$\begin{aligned} S_u(t, \tau, x) &= x, \\ S_u(t, \tau, x) &\subset S_u(t, s, S_u(s, \tau, x)), \\ S_u(t + h, \tau + h, x) &\subset S_{u(\cdot + h)}(t, \tau, x). \end{aligned}$$

It is easy to verify that $\{S_u\}_{u \in U}$ satisfies cocycle property:

$$S_u(t + h, 0, x) \subset S_u(t + h, h, S_u(h, 0, x)) \subset S_{u(\cdot + h)}(t, 0, S_u(h, 0, x)),$$

and $\forall \varphi \in K_u^\tau$

$$\varphi(t) \in S_u(t, s, \varphi(s)).$$

In particular, $\forall \varphi \in K_u^0$, $\forall t, h \geq 0$

$$\varphi(t + h) \in S_u(t + h, h, \varphi(h)) \subset S_{u(\cdot + h)}(t, 0, \varphi(h)). \tag{3.2}$$

(S4) Moreover, if $\forall s \geq \tau$, $\forall \psi \in K_u^\tau$, $\forall \varphi \in K_u^s$ with $\psi(s) = \varphi(s)$ the function

$$\Theta(p) = \begin{cases} \psi(p), \, p \in [\tau, s], \\ \varphi(p), \, p \geq s \end{cases}$$

belongs to K_u^τ, then inclusion $S_u(t,\tau,x) \subset S_u(t,s,S_u(s,\tau,x))$ takes place.

(S5) If $\forall h \geq 0,\ \forall \varphi \in K^\tau_{u(\cdot+h)}$ we have that $\varphi(\cdot - h) \in K_u^{\tau+h}$, then inclusion $S_u(t+h,\tau+h,x) \subset S_{u(\cdot+h)}(t,\tau,x)$ takes place.

So, under conditions (U), (S1)–(S5) for the semiprocess family $\{S_u\}_{u\in U}$ we have that $\{S_u\}_{u\in U}$ is strict, i.e.,

$$\begin{aligned} S_u(t,\tau,x) &= S_u(t,s,S_u(s,\tau,x)), \\ S_u(t+h,\tau+h,x) &= S_{u(\cdot+h)}(t,\tau,x), \\ S_u(t+h,0,x) &= S_{u(\cdot+h)}(t,0,S_u(h,0,x)). \end{aligned}$$

In particular, in the undisturbed case ($u \equiv 0$)

$$S_0(t+h,0,x) = S_0(t,0,S_0(h,0,x)),$$

so S_0 is a strict m-semiflow.

In the next section we investigate stability property of $\{S_u\}_{u\in U}$ with respect to the global attractor Θ of m-semiflow G of the undisturbed system, i.e.,

$$G(t,x) := S_0(t,0,x).$$

4. Stability of global attractors

Lemma 4.1. *Assume that $G : \mathbb{R}_+ \times X \mapsto 2^X$ is a strict m-semiflow, which has an invariant stable global attractor Θ. Also, assume that*

$$\text{for all bounded } B \subset X \text{ the set } \bigcup_{t\geq 0} G(t,B) \text{ is bounded.} \tag{4.1}$$

Then $\exists \beta \in \mathcal{KL}\ \forall x \in X,\ \forall t \geq 0$

$$\|G(t,x)\|_\Theta \leq \beta(\|x\|_\Theta, t). \tag{4.2}$$

Proof. First let us show that $\exists \alpha \in \mathcal{K}_\infty$ such that

$$\forall x \in X,\ \forall t \geq 0\ \ \|G(t,x)\|_\Theta \leq \alpha(\|x\|_\Theta). \tag{4.3}$$

Using (2.2), let us denote

$$\bar\delta(\varepsilon) := \begin{cases} 0, & \varepsilon = 0, \\ \sup \delta, & (\varepsilon,\delta) \text{ satisfies (2.2).} \end{cases}$$

Then $\bar\delta(\varepsilon) > 0,\ \varepsilon > 0,\ \bar\delta(0) = 0,\ \bar\delta$ is increasing, but not necessary continuous. So, we put for $\kappa \in (0,1)$

$$\xi(\varepsilon) := \begin{cases} \kappa \displaystyle\int_0^\varepsilon \bar\delta(s)\,ds, & \varepsilon \in [0,1], \\ \dfrac{\kappa}{\varepsilon} \displaystyle\int_0^\varepsilon \bar\delta(s)\,ds, & \varepsilon > 1. \end{cases}$$

Then $\xi \in \mathcal{K}$ and $\forall \varepsilon > 0$ $\xi(\varepsilon) \leq \kappa\bar{\delta}(\varepsilon) < \bar{\delta}(\varepsilon)$. Let us prove that $\xi \in \mathcal{K}_\infty$. It is sufficient to show that $\bar{\delta}(\varepsilon) \to \infty$ as $\varepsilon \to \infty$, i.e.,

$$\forall R > 0 \;\; \exists r \;\; \forall \varepsilon > r \;\; \bar{\delta}(\varepsilon) > R.$$

Suppose the contrary:

$$\exists R_0 > 0 \;\; \forall r \;\; \exists \varepsilon > r : \;\; \bar{\delta}(\varepsilon) \leq R_0. \tag{4.4}$$

Due to assumption (4.1) $\exists r_0 \;\; \forall \varepsilon > r_0$

$$\forall t \geq 0 \;\; G(t, O_{R_0+1}(\Theta)) \subset O_\varepsilon(\Theta),$$

so, $\bar{\delta}(\varepsilon) \geq R_0 + 1$, which contradicts (4.4). Now let us put

$$\alpha(r) = \xi^{-1}(r).$$

Then $\forall x \in X$ we put in (2.2) $\varepsilon = \alpha(\|x\|_\Theta)$. Therefore, $\|x\|_\Theta < \bar{\delta}(\varepsilon)$ and $\forall t \geq 0$

$$\|G(t,x)\|_\Theta < \varepsilon = \alpha(\|x\|_\Theta).$$

According to $(\Theta 2)$ $\forall r > 0, \; \forall x \in X : \|x\|_\Theta \leq r$ and $\forall \eta > 0$

$$\exists T = T(\eta, r) > 0 \;\; \forall t \geq T \;\; \|G(t,x)\|_\Theta < \eta. \tag{4.5}$$

We introduce functions

$$\begin{gathered}
\overline{T}(\eta, r) = \inf T(\eta, r), \;\; (\eta, r) \text{ satisfies (4.5)}, \\
W_r(\eta) = \frac{r}{\eta} \int_{\frac{\eta}{r}}^{\eta} \bar{T}(s, r) ds + \frac{r}{\eta}, \\
U_r = W_r^{-1}, \\
\psi(r, s) = \min\{\alpha(r), \inf_{\rho > r} U_\rho(s)\}.
\end{gathered}$$

After that we can repeat without any changes arguments from [24, p. 665] and obtain (4.2) with

$$\beta(r, s) = \int_r^{r+1} \psi(\lambda, s)\, d\lambda + \frac{r}{(r+1)(s+1)}.$$

□

Statement of this lemma allows us to prove the main result of the paper.

Theorem 4.1. *Assume that m-semiflow S_0 is generated by family of maps K satisfying* (K1), (K2), *S_0 is strict, has compact values, and possesses invariant stable global attractor Θ.*

Additionally, exists locally bounded function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\forall r > 0,\ \forall t \geq 0$

$$\|x_1\| \leq r,\ \|x_2\| \leq r \ \Rightarrow\ dist(S_0(t,0,x_1), S_0(t,0,x_2)) \leq e^{c(r)t}\|x_1 - x_2\|. \quad (4.6)$$

Assume that $\{S_u\}_{u\in U}$ is the family of m-semiprocesses satisfying (U), (S1)–(S3)*, where $u \in U$ is disturbances of the initial system S_0.*

Assume that $\exists \sigma \in \mathcal{K}$, exists continuous function $d : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\forall r > 0\ \overline{\lim\limits_{t\to 0+}} \frac{d(r,t)}{t} < \infty$, and $\forall t \geq 0$

$$\|u\|_\infty \leq r,\ \|x\| \leq r \ \Rightarrow\ dist(S_u(t,0,x), S_0(t,0,x)) \leq d(r,t)\sigma(\|u\|_\infty). \quad (4.7)$$

Assume, that

$$\forall r > 0 \text{ the set } \bigcup_{t\geq 0} \bigcup_{\|u\|_\infty \leq r} \bigcup_{\|x\|\leq r} S_u(t,0,x) \text{ is bouded.} \quad (4.8)$$

Then $\{S_u\}_{u\in U}$ is local ISS w.r.t. Θ, i.e., inequality (2.4) *holds.*

Proof. First let us prove that $\forall r > 0\ \exists \underline{\psi}, \overline{\psi}, \alpha \in \mathcal{K}$, exists Lipschitz continuous function V with Lipschitz constatnt equals 1, such that

$$\underline{\psi}(\|x\|_\Theta) \leq V(x) \leq \overline{\psi}(\|x\|_\Theta)\ \ \forall \|x\|_\Theta \leq r, \quad (4.9)$$

$$\dot{V}_0(x) := \overline{\lim_{t\to 0+}} \frac{1}{t} dist(V(S_0(t,0,x)), V(x)) \leq -\alpha(\|x\|_\Theta)\ \forall \|x\|_\Theta \leq r, \quad (4.10)$$

where here and after for $A \subset X,\ V(A) = \bigcup\limits_{a\in A} V(a)$.

For this purpose we choose function β from (4.2), fix $r_0 > 0$ and $\forall \varepsilon > 0$ let $T = T(r_0, \varepsilon)$ be such that

$$\beta(r_0, t) \leq \varepsilon\ \ \forall t \geq T. \quad (4.11)$$

We put

$$V^\varepsilon(x) := e^{-(c_0+c)T} \sup_{t\geq 0}(e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta)), \|x\|_\Theta < r_0,$$

where $c_0 = c(r_0)$ is taken from (4.6), $c > 0$ will be fixed throughout the proof, $\eta_\varepsilon(r) := \max\{0, r - \varepsilon\}$. Due to (4.11)

$$V^\varepsilon(x) = e^{-(c_0+c)T} \sup_{t\in[0,T]} (e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta)).$$

Using elementary properties of η_ε:

$$\eta_\varepsilon(r) \leq r,\ |\eta_\varepsilon(r_1) - \eta_\varepsilon(r_2)| \leq |r_1 - r_2|,$$

we get the following properties of V^ε:

$$V^\varepsilon(x) \leq e^{-c_0 T} \sup_{t\in[0,T]} \eta_\varepsilon(\|S_0(t,0,x)\|_\Theta) \leq \beta(\|x\|_\Theta, 0), \quad \forall \|x\|_\Theta \leq r_0$$

and

$$\begin{aligned}|V^\varepsilon(x)-V^\varepsilon(y)| &\le e^{-(c_0+c)T}\\ &\quad\times \sup_{t\in[0,T]} |e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta)-e^{ct}\eta_\varepsilon(\|S_0(t,0,y)\|_\Theta)|\\ &\le e^{-c_0T}\sup_{t\in[0,T]} |\|S_0(t,0,x)\|_\Theta)-\|S_0(t,0,y)\|_\Theta)|\\ &\le e^{-c_0T}\sup_{t\in[0,T]} \operatorname{dist}(S_0(t,0,x),S_0(t,0,y))\\ &\le e^{-c_0T}e^{c_0T}\|x-y\|\\ &=\|x-y\|,\quad \forall\,\|x\|_\Theta\le r_0,\quad \forall\,\|y\|_\Theta\le r_0.\end{aligned}$$

Here, we utilized the inequality

$$\operatorname{dist}(A,B)\le \operatorname{dist}(A,C)+\operatorname{dist}(C,B)$$

with $A=S_0(t,0,x),\ B=\Theta,\ C=S_0(t,0,y)$.

Due to compactness of Θ we have that $\forall\|x\|_\Theta<r_0$

$$\|x\|_\Theta=\inf_{\xi\in\Theta}\|x-\xi\|=\|x-\xi_0\|,\ \xi_0\in\Theta.$$

Then due to (4.6)

$$\operatorname{dist}(S_0(t,0,x),S_0(t,0,\xi_0))\le e^{c_0t}\|x-\xi_0\|.$$

Invariance of Θ implies the inclusion

$$S_0(t,0,\xi_0)\subset\Theta.$$

Therefore,

$$\operatorname{dist}(S_0(t,0,x),S_0(t,0,\xi_0))\ge\|S_0(t,0,x)\|_\Theta.$$

So, from the sctrict inequality $\|x\|_\Theta<r_0$ we derive that for sufficiently small $\tau>0$

$$\|S_0(\tau,0,x)\|<r_0.$$

Then $\forall\varphi\in K:\ \varphi(0)=x$, we get from the strictness of S_0

$$\begin{aligned}V^\varepsilon(\varphi(\tau)) &= e^{-(c_0+c)T}\sup_{t\ge0}(e^{ct}\eta_\varepsilon(\|S_0(t,0,\varphi(\tau))\|_\Theta))\\ &\le e^{-(c_0+c)T}\sup_{t\ge0}(e^{ct}\eta_\varepsilon(\|S_0(t+\tau,0,x)\|_\Theta))\\ &\le e^{-c\tau}V^\varepsilon(x) \text{ for sufficiently small } \tau>0.\end{aligned}$$

Due to compactness of $S_0(t,0,x)$ we deduce: for every small $\tau>0$ $\exists\varphi\in K,\ \varphi(0)=x$ such that

$$\operatorname{dist}(V^\varepsilon(S_0(\tau,0,x)),V^\varepsilon(x))=V^\varepsilon(\varphi(\tau))-V^\varepsilon(x)\le(e^{-c\tau}-1)V^\varepsilon(x). \tag{4.12}$$

Therefore,

$$\dot{V}_0^{\varepsilon}(x) := \overline{\lim_{t\to 0+}} \frac{1}{t} dist(V^{\varepsilon}(S_0(t,0,x)), V^{\varepsilon}(x)) \le -cV^{\varepsilon}(x), \ \|x\|_{\Theta} < r_0. \tag{4.13}$$

Now, for every $\|x\|_{\Theta} \le r_0$, we put

$$V(x) := \sum_{k=1}^{\infty} 2^{-k} V^{\frac{1}{k}}(x).$$

Then from the previous arguments, we get

$$V(x) \le \beta(\|x\|_{\Theta}, 0), \ \|x\|_{\Theta} \le r_0, \tag{4.14}$$

$$|V(x) - V(y)| \le \|x - y\|, \ \|x\|_{\Theta} \le r_0, \ \|y\|_{\Theta} \le r_0, \tag{4.15}$$

$\forall \varphi \in K, \ \varphi(0) = x$ for sufficiently small $\tau > 0$

$$V(\varphi(\tau)) \le e^{-c\tau} V(x), \text{ and therefore,}$$

$$\text{dist}\,(V(S_0(\tau,0,x)), V(x)) \le (e^{-c\tau} - 1)V(x).$$

So,

$$\dot{V}_0(x) \le -cV(x), \ \|x\|_{\Theta} < r_0. \tag{4.16}$$

Moreover, inequality

$$\sup_{t\ge 0} \left(e^{ct} \eta_{\frac{1}{k}}(\|S_0(t,0,x)\|_{\Theta}) \right) \ge \eta_{\frac{1}{k}}(\|x\|_{\Theta})$$

implies

$$V(x) \ge \sum_{k=1}^{\infty} 2^{-k} e^{-(c_0+c)T(\frac{1}{k})} \eta_{\frac{1}{k}}(\|x\|_{\Theta}), \ \|x\|_{\Theta} \le r_0. \tag{4.17}$$

Finally, denoting

$$\overline{\psi}(r) = \beta(r,0) + r,$$

$$\underline{\psi}(r) = \sum_{k=1}^{\infty} 2^{-k} e^{-(c_0+c)T(\frac{1}{k})} \eta_{\frac{1}{k}}(r),$$

$$\alpha(r) = c\underline{\psi}(r),$$

we obtain (4.9),(4.10).

Then for $\forall \|x\|_{\Theta} < 1, \ \forall u \in U : \ \|u\|_{\infty} \le 1, \ \forall \varphi \in K_u^0 : \varphi(0) = x$, let us consider for $t > 0$ the upper right-hand Dini derivative [27]

$$\overline{D}^{+} V(\varphi(t)) = \overline{\lim_{\tau\to 0+}} \frac{1}{\tau} (V(\varphi(t+\tau)) - V(\varphi(t))).$$

According to property (3.2)

$$\varphi(t+\tau) \in S_u(t+\tau, 0, x) \subset S_{u(\cdot+t)}(\tau, 0, \varphi(t)).$$

From (4.8), for some $r_0 > 0$, $\|\varphi(t)\| < r_0$ $\forall t \geq 0$. We fix such r_0 in all previous arguments. So, in view of (4.7), we can write

$$\begin{aligned} V(\varphi(t+\tau)) - V(\varphi(t)) &\leq \operatorname{dist}\left(V(S_{u(\cdot+t)}(\tau,0,\varphi(t))), V(\varphi(t))\right) \\ &\leq \operatorname{dist}\left(V(S_{u(\cdot+t)}(\tau,0,\varphi(t))), V(S_0(\tau,0,V(S_{u(\cdot+t)}(\tau,0,\varphi(t)))))\right) \\ &\quad + \operatorname{dist}\left(V(S_0(\tau,0,V(S_{u(\cdot+t)}(\tau,0,\varphi(t))))), V(\varphi(t))\right) \\ &\leq d(r_0,\tau)\sigma(\|u\|_\infty) + (e^{-c\tau}-1)V(\varphi(t)). \end{aligned} \tag{4.18}$$

It means that

$$\overline{D}^+ V(\varphi(t)) \leq -cV(\varphi(t)) + \overline{d}\sigma(\|u\|_\infty), \quad \forall t > 0, \tag{4.19}$$

where $\overline{d} = \overline{\lim\limits_{\tau\to 0+}} \frac{d(r_0,\tau)}{\tau}$.

Due to the properties of upper limit, we get from (4.19):

$$\begin{aligned} \overline{D}^+ \left(V(\varphi(t))e^{ct}\right) &\leq -\overline{D}^+ \left(-\frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct}\right), \\ \overline{D}^+ \left(V(\varphi(t))e^{ct} - \frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct}\right) &\leq 0. \end{aligned} \tag{4.20}$$

Then inequality (4.20) implies that (see [27])

$$V(\varphi(t))e^{ct} - \frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct} \leq V(x) - \frac{\overline{d}\sigma(\|u\|_\infty)}{c}, \quad \forall t \geq 0.$$

So,

$$V(\varphi(t)) \leq V(x)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty), \quad \forall t \geq 0. \tag{4.21}$$

Finally,

$$\begin{aligned} \underline{\psi}(\|\varphi(t)\|_\Theta) &\leq \overline{\psi}(\|x\|_\Theta)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty), \\ \|\varphi(t)\|_\Theta &\leq \underline{\psi}^{-1}(\overline{\psi}(\|x\|_\Theta)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty)) \\ &\leq \frac{1}{2}\underline{\psi}^{-1}\left(2\overline{\psi}(\|x\|_\Theta)e^{-ct}\right) + \frac{1}{2}\underline{\psi}^{-1}\left(\frac{2\overline{d}}{c}\sigma(\|u\|_\infty)\right). \end{aligned} \tag{4.22}$$

If we denot

$$\begin{aligned} \beta(r,s) &:= \frac{1}{2}\underline{\psi}^{-1}\left(2\overline{\psi}(\|x\|_\Theta)e^{-cs}\right), \\ \gamma(r) &:= \frac{1}{2}\underline{\psi}^{-1}\left(\frac{2\overline{d}}{c}\sigma(r)\right), \end{aligned}$$

then inequality (4.22) implies the required local ISS property (2.4).

Theorem is proved. □

5. Application to reaction-diffusion equation

We consider the following problem

$$\begin{cases} \frac{\partial y}{\partial t} = \triangle y + g(y) + h(y)u(t), \\ y|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $g \in \mathbb{C}^1(\mathbb{R}), h \in \mathbb{C}(\mathbb{R}),\ \exists \alpha_1, \alpha_2, k, c, \lambda \in (0, +\infty),\ p \geq 2$ such that, $\forall r \in \mathbb{R}$,

$$\begin{aligned} -k - \alpha_1 |r|^p \leq g(r)r \leq k - \alpha_2 |r|^p, \\ g'(r) \leq \lambda, \\ |h(r)| \leq c. \end{aligned} \tag{5.2}$$

Conditions (5.2) allow us to claim that $\forall \tau \geq 0,\ \forall u \in L^\infty(\tau, +\infty),\ \forall y_\tau \in X = L^2(\Omega)$ there exists at least one (but not necessary unique) weak solution $y = y(t, x)$ of (5.1), defined on $(\tau, +\infty)$, such that $y|_{t=\tau} = y_\tau$ [3, 9]. It is known [26] that all weak solutions of (5.1) generate the family of maps $\{K_u^\tau\}$ which satisfies (S1)–(S3), where we choose $U = L^\infty(0, +\infty)$. Moreover, every weak solutions of (5.1) belongs to the class of absolutely continuous functions from $[\tau, T]$ to X for every $T > \tau$, and for a.a. $t > \tau$

$$\frac{d}{dt}\|y(t)\|^2 + \nu \|y(t)\|^2 \leq c_1 + c_2 \|u\|_\infty^2.$$

So,

$$\|y(t)\|^2 \leq \|y(\tau)\|^2 e^{-\nu(t-\tau)} + \frac{1}{\nu}(c_1 + c_2\|u\|_\infty), \quad \forall t \geq \tau. \tag{5.3}$$

In particular, property (4.8) holds.

For $u \equiv 0$ the problem (5.1) is uniquelly resolvable in the phase space X [4], and results from [5] guarantee the existence of invariant stable global attractor Θ of the corresponding semiflow S_0. Moreover, from [21] we deduce that

$$\|S_0(t, 0, y_0^{(1)}) - S_0(t, 0, y_0^{(2)})\| \leq e^{\lambda t} \|y_0^{(1)} - y_0^{(2)}\|,\ \forall y_0^{(1)}, y_0^{(2)} \in X,\ \forall t \geq 0, \tag{5.4}$$

$$y(0) = y_0, \quad \forall y_0 \in X,\ \forall u \in U,\ \forall y \in K_u^0,$$

$$\|y(t) - S_0(t, 0, y_0)\| \leq 2e^{2\lambda} c\mu(\Omega) \|u\|_\infty t, \quad \forall t \geq 0. \tag{5.5}$$

Inequalities (5.3)–(5.5) imply conditions (4.6)–(4.8) of the Theorem 4.1. It means, that the family of m-semiprocess, generated by weak solution of (5.1) is local ISS with respect to the global attractor Θ of the undisturbed system S_0.

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