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SOLVABILITY ISSUES FOR SOME NONCOERCIVE AND NONMONOTONE PARABOLIC EQUATIONS ARISING IN THE IMAGE DENOISING PROBLEMS

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Abstract. This paper is devoted to the solvability of an initial-boundary value problem for second-order parabolic equations in divergence form with variable order of nonlinearity. The characteristic feature of the considered class of Cauchy-Neumann parabolic problem is the fact that the variable exponent $p(t,x)$ and the anisotropic diffusion tensor $D(t,x)$ are not well predefined a priori, but instead these characteristic depend on a solution of this problem, i.e., $p = p(t,x,u)$ and $D = D(t,x,u)$. Recently, it has been shown that the similar models appear in a natural way as the optimality conditions for some variational problems related to the image restoration technique. However, from practical point of view, in this case some principle difficulties can appear because of the absence of the corresponding rigorous mathematically substantiation. Thus, in this paper, we study the solvability issues of the Cauchy-Neumann parabolic boundary value problem for which the corresponding principle operator is strongly non-linear, non-monotone, has a variable order of nonlinearity, and satisfies a nonstandard coercivity and boundedness conditions that do not fall within the scope of the classical method of monotone operators. To construct a weak solution, we apply the technique of passing to the limit in a special approximation scheme and the Schauder fixed-point theorem.

Key words: Weak solution, parabolic equation, variable order of nonlinearity, noncoercive problem, compensated compactness technique..

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1. Introduction

This paper is devoted to the proof of existence of weak solutions $u(t,x)$ to the following Neumann-Cauchy initial-boundary value problem (IBVP) for anisotropic parabolic equation of $p(t,x,u)$-Laplacian type with variable exponent of nonlinearity

$$\frac{\partial u}{\partial t} - div\left(|D(t,x,u)\nabla u|^{p_u(t,x)-2} D(t,x,u)\nabla u\right) + u = f \quad \text{in } \ Q_T := (0,T)\times\Omega, \tag{1.1}$$

$$\partial_\nu u = 0 \quad \text{on } \ (0,T)\times\partial\Omega, \tag{1.2}$$

$$u(0,\cdot) = f_0(\cdot) \quad \text{in } \ \Omega. \tag{1.3}$$

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Here, $\Omega \subset \mathbb{R}^N$ is a bounded simple-connected open set with a sufficiently smooth boundary $\partial\Omega$, $T > 0$ is a positive value, $f \in L^2(0,T;L^2(\Omega))$ and $f_0 \in L^2(\Omega)$ are given distributions, $D = D(t,x,u)$ is the anisotropic diffusion tensor, and the exponent $p_u : Q_T \to \mathbb{R}$ is defined by the rule

$$p_u(t,x) := 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(x)\right|^2 d\tau\right), \quad \forall\,(t,x) \in Q_T, \tag{1.4}$$

where

$$g(s) = \frac{a}{a+s}, \tag{1.5}$$

$$G_\sigma(x) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^N} \exp\left(-\frac{|x|^2}{2\sigma^2}\right), \quad \sigma > 0, \tag{1.6}$$

$$\left(G_\sigma * \widetilde{u}(t,\cdot)\right)(x) = \int_{\mathbb{R}^N} G_\sigma(x-y)\widetilde{u}(t,y)\,dy, \tag{1.7}$$

$\widetilde{u}$ denotes zero extension of u from Q_T to $\mathbb{R} \times \mathbb{R}^N$, and $h > 0$ and $a > 0$ are given small positive values.

The main characteristic feature of the proposed IBVP is the fact that the matrix of anisotropy D and the exponent p depend not only on (t,x) but also on $u(t,x)$. It is well-known that the variable character of exponent p causes a gap between the monotonicity and coercivity conditions. Because of this gap, equations of the type (1.1) can be termed equations with nonstandard growth conditions, and it can be viewed as a generalization of the evolutional version of $p(t,x)$-Laplacian equation

$$\frac{\partial u}{\partial t} = div\left(|\nabla u|^{p(t,x)-2}\nabla u\right) \tag{1.8}$$

with an exponent that depends only on t and x. During the last decades equation (1.8) was intensively studied by many authors. There is extensive literature devoted to equation (1.8). We limit ourselves by referring here to the following ones [6, 7, 13, 37, 39, 43] which provide an excellent insight to the theory of evolutional $p(t,x)$-Laplacian equations.

PDEs with variable nonlinearity are rather interesting from the purely mathematical point of view. On the other hand, their study is motivated by various applications where the problem (1.1)–(1.3), or some special cases of it, appear in the most natural way [1,2,12,15]. It was recently shown that the model (1.1)–(1.3) naturally appears as the Euler-Lagrange equation in the problem of restoration of cloud contaminated satellite optical images [18, 26]. Moreover, the above mentioned problem can be considered as a model for the deblurring and denoising of multi-spectral images. In particular, this model has been proposed in [19, 30] in order to avoid the blurring of edges and other localization problems presented by linear diffusion models in images processing. We also refer to [28], where the authors study some optimal control problems associated with a special case of the model (1.1)–(1.3) and show the given class of optimal control problems is well posed.

It is also worth to notice that the model (1.1)–(1.3) can be considered as a natural generalization of the well-know Perona-Malik model [38]. In spite of the fact that Perona-Malik model reduces the diffusivity of color in places having higher likelihood of being edges, its major defect is that this model is ill-posed and there are no results of existence and its consistency (see [28]). To overcome this problem it has been proposed to modify this model by applying a Gaussian filter on the gradient (we can refer to the pioneering works [5, 14]).

However, to the best of our knowledge, the above reported results on the solvability of parabolic equations of the type (1.1) mainly concern the equations with variable exponent and matrix of anisotropy depending on (t, x) only, whereas less attention has been paid to IBVP of the form (1.1) with $D = D(t, x, u)$ and the exponent $p_u(t, x)$ given by the rule (1.4). Besides, in contrast to the majority of the existing results, we do not predefine the variable exponent p a priori, but instead we associate this characteristic with a particular solution of IBVP (1.1)–(1.3). Thus, the rate of nonlinearity p and the tensor D can be affected by the values of the unknown solution u. It is worth to emphasize that we do not assume here that the dependency of p and D on u is local whereas it is the crucial assumption in the most of existing publications (see for instance [6,8]). In fact, we show that each weak solution to this problem lives in the corresponding 'personal' functional space, and, in view of our assumptions on the structure of $D(t, x, u)$ and $p_u(t, x)$, the problem (1.1)–(1.3) can admit the weak solutions that may not possess the usual properties of solutions to parabolic equations. In particular, it would be rather questionable assertion that a weak solution to the above is unique and it satisfies the standard energy equality.

The remainder of the paper is organized as follows: In Section 2 we give some preliminaries and introduce the main assumptions on the structure of anisotropic diffusion tensor $D(t, x, u)$ and variable exponent $p_u(t, x)$. We also give here the main auxiliary results concerning the Orlicz spaces and Sobolev-Orlicz spaces with variable exponent. In Section 3 we focus on the solvability issues for IBVP (1.1)–(1.3). With that in mind we follows the indirect approach using the technique of passing to the limit in some special approximation scheme.

2. Main Assumptions and Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set with a sufficiently smooth boundary $\partial\Omega$, and let $T > 0$ be a given value. We suppose that the unit outward normal $\nu = \nu(x)$ is well-defined for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$, where a.e. means here with respect to the $N-1$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. We set $Q_T = (0, T) \times \Omega$. For any measurable subset $D \subset \Omega$ we denote by $|D|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(D)$. We denote its closure by $\overline{D}$ and its boundary by ∂D.

Let χ_D be the characteristic function of D, i.e.,

$$\chi_D(x) := \begin{cases} 1, & \text{for } \ x \in D, \\ 0, & \text{otherwise.} \end{cases}$$

For vectors $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$, $(\xi, \eta) = \xi^t \eta$ denotes the standard vector inner product in $\mathbb{R}^N$, where ^{t} stands for the transpose operator. The norm $|\xi|$ is the Euclidean norm given by $|\xi| = \sqrt{(\xi, \xi)}$. We also make use of the following notation $\operatorname{diam} \Omega = \sup_{x,y \in \Omega} |x - y|$.

2.1. Functional Spaces

Let X denote a real Banach space with norm $\|\cdot\|_X$, and let X' be its dual. Let $\langle \cdot, \cdot \rangle_{X';X}$ be the duality form on $X' \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and weak* convergence in normed spaces X and X', respectively.

For given $1 \leqslant p \leqslant +\infty$, the space $L^p(\Omega; \mathbb{R}^N)$ is defined by

$$L^p(\Omega; \mathbb{R}^N) = \left\{ f : \Omega \to \mathbb{R}^N \ : \ \|f\|_{L^p(\Omega;\mathbb{R}^N)} < +\infty \right\},$$

where $\|f\|_{L^p(\Omega;\mathbb{R}^N)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$ for $1 \leqslant p < +\infty$. The inner product of two functions f and g in $L^p(\Omega; \mathbb{R}^N)$ with $p \in [1, \infty)$ is given by

$$(f, g)_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega (f(x), g(x)) \, dx = \int_\Omega \sum_{k=1}^N f_k(x) g_k(x) \, dx.$$

We denote by $C_c^\infty(\mathbb{R}^N)$ the locally convex space of all infinitely differentiable functions with compact support in $\mathbb{R}^N$. We recall here some functional spaces that will be used throughout this paper. We define the Banach space $H^1(\Omega)$ as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|y\|_{H^1(\Omega)} = \left(\int_\Omega \left(y^2 + |\nabla y|^2 \right) \, dx \right)^{1/2}.$$

We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$.

Given a real Banach space X, we will denote by $C([0,T]; X)$ the space of all continuous functions from $[0,T]$ into X. We recall that a function $u : [0,T] \to X$ is said to be Lebesgue measurable if there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ of step functions (i.e., $u_k = \sum_{j=1}^{n_k} a_j^k \chi_{A_j^k}$ for a finite number n_k of Borel subsets $A_j^k \subset [0,T]$ and with $a_j^k \in X$) converging to u almost everywhere with respect to the Lebesgue measure in $[0,T]$.

Then for $1 \leqslant p < \infty$, $L^p(0,T;X)$ is the space of all measurable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(0,T;X)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in[0,T]} \|u(t)\|_X < \infty.$$

The presentation of this topic can be found in [21].

Let us recall that, for $1 \leqslant p \leqslant \infty$, $L^p(0,T;X)$ is a Banach space. Moreover, if X is separable and $1 \leqslant p < \infty$, then the dual space of $L^p(0,T;X)$ can be identified with $L^{p'}(0,T;X')$.

For our purpose X will mainly be either the Lebesgue space $L^p(\Omega)$ or $L^p(\Omega;\mathbb{R}^N)$ or the Sobolev space $W^{1,p}(\Omega)$ with $1 \leqslant p < \infty$. Since, in this case, X is separable, we have that $L^p(0,T;L^p(\Omega)) = L^p(Q_T)$ is the ordinary Lebesgue space defined in $Q_T = (0,T)\times\Omega$. As for the space $L^p(0,T;W^{1,\alpha}(\Omega))$ with $1 \leqslant \alpha, p < +\infty$, it consists of all functions $u : [0,T]\times\Omega \to \mathbb{R}$ such that u and $|\nabla u|$ belongs to $L^p(0,T;L^\alpha(\Omega))$. Moreover,

$$\left(\int_0^T \left[\left(\int_\Omega |u|^\alpha\,dx\right)^{\frac{p}{\alpha}} + \left(\int_\Omega |\nabla u|^\alpha\,dx\right)^{\frac{p}{\alpha}}\right] dt\right)^{\frac{1}{p}}$$

defines the norm in $L^p(0,T;W^{1,\alpha}(\Omega))$.

Now we state the well-known Gagliardo-Nirenberg embedding theorem (see [36, Lecture II]) that will plays an important role in our further analysis (for more details and other types of embedding, we refer to [9–11]).

Theorem 2.1 (Gagliardo-Nirenberg). *Let v be a function in $W^{1,q}(\Omega)\cap L^\rho(\Omega)$ with $q \geqslant 1$, $\rho \geqslant 1$. Then there exists a positive constant C, depending on N, q, and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)} \leqslant C\|\nabla v\|^\theta_{L^q(\Omega;\mathbb{R}^N)}\|v\|^{1-\theta}_{L^\rho(\Omega)},$$

for every θ and γ satisfying

$$0 \leqslant \theta \leqslant 1, \quad 1 \leqslant \gamma \leqslant +\infty, \quad \frac{1}{\gamma} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{\rho}. \tag{2.1}$$

As a consequence of this assertion, we have the following embedding result.

Corollary 2.1. *Let $v \in L^\alpha(0,T;W^{1,\alpha}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $\alpha = 1+\delta$ where $0<\delta<<1$. Then $v \in L^\gamma(Q_T)$ with $\gamma = \alpha\frac{N+2}{N}$ and*

$$\|v\|_{L^\gamma(Q_T)} \leqslant C\|v\|^{\frac{2}{(N+2)}}_{L^\infty(0,T;L^2(\Omega))}\|\nabla v\|^{\frac{N}{N+2}}_{L^\alpha(0,T;L^\alpha(\Omega))}. \tag{2.2}$$

Proof. Setting $q=\alpha$, $p=2$, and $\theta = \frac{N}{N+2}$, we deduce from (2.14) that

$$\gamma = \frac{(N+2)\alpha}{N}, \quad \gamma\theta = \alpha, \quad \gamma(1-\theta) = \frac{2\alpha}{N}.$$

Hence, utilizing the Gagliardo-Nirenberg inequality, we can write

$$\|v(t)\|^{\gamma}_{L^{\gamma}(\Omega)} = \int_{\Omega} |v(t)|^{\gamma}\, dx \leqslant C \|\nabla v(t)\|^{\theta\gamma}_{L^{\alpha}(\Omega;\mathbb{R}^N)} \|v(t)\|^{(1-\theta)\gamma}_{L^2(\Omega)},$$

and, therefore,

$$\begin{aligned}\int_0^T \int_{\Omega} |v(t)|^{\gamma}\, dxdt &\leqslant C \int_0^T \|\nabla v(t)\|^{\alpha}_{L^{\alpha}(\Omega;\mathbb{R}^N)} \|v(t)\|^{\frac{2\alpha}{N}}_{L^2(\Omega)}\, dt \\ &\leqslant C \|v\|^{\frac{2\alpha}{N}}_{L^{\infty}(0,T;L^2(\Omega))} \int_0^T \|\nabla v(t)\|^{\alpha}_{L^{\alpha}(\Omega;\mathbb{R}^N)}\, dt. \end{aligned} \tag{2.3}$$

As a result, the announced inequality (2.2) easily follows from (2.3). □

2.2. Variable Exponent

Let $u \in C([0,T]; L^2(\Omega))$ be a given function. Let $u_{\sigma}(t,x)$ be its convolution with N-dimensional Gaussian of width (standard deviation) $\sigma > 0$ (see (1.7)). We associate with a function $u : Q_T \mapsto \mathbb{R}$ the exponent $p_u : Q_T \to \mathbb{R}$ defined by the rule (1.4).

Since $G_{\sigma} \in C^{\infty}(\mathbb{R}^N)$, it follows from (1.4) and absolute continuity of the Lebesgue integral that $1 < p_u(t,x) \leqslant 2$ in Q_T and $p_u \in C^1([0,T]; C^{\infty}(\mathbb{R}^N))$ even if u is just an absolutely integrable function in Q_T. Moreover, for each $t \in [0,T]$, $p_u(t,x) \approx 1$ in those places of Ω where some discontinuities are present in $u(t,\cdot)$, and $p_u(t,x) \approx 2$ in places where $u(t,x)$ is smooth or contains homogeneous features. In view of this, $p_u(t,x)$ can be interpreted as a characteristic of the sparse texture of the function u.

The following result plays a crucial role in the sequel (for comparison, we refer to [17, Lemma 2.1], [18]).

Lemma 2.1. *Let $\{u_k\}_{k\in\mathbb{N}} \subset C([0,T]; L^2(\Omega))$ be a sequence of measurable functions such that each element of this sequence is extended by zero outside of Q_T and*

$$\begin{gathered}\sup_{k\in\mathbb{N}} \|u_k\|_{C([0,T];L^2(\Omega))} < +\infty, \\ u_k \to u \ \ \textit{weakly in } L^2(0,T;L^2(\Omega)) \textit{ for some } \ u \in C([0,T]; L^2(\Omega)).\end{gathered} \tag{2.4}$$

Let

$$\left\{ p_{u_k} = 1 + g\left(\frac{1}{h} \int_{t-h}^{t} |(\nabla G_{\sigma} * \widetilde{u}_k(\tau,\cdot))|^2\, d\tau \right) \right\}_{k\in\mathbb{N}}$$

be the corresponding sequence of variable exponents. Then there exists a constant

$C > 0$ depending on Ω, G, and $\sup_{k\in\mathbb{N}} \|u_k\|_{C([0,T];L^2(\Omega))}$ such that

$$\alpha := 1 + \delta \leqslant p_{u_k}(t,x) \leqslant \beta := 2, \quad \forall\,(t,x) \in Q_T,\ \forall\, k \in \mathbb{N}, \tag{2.5}$$

$$\{p_{u_k}(\cdot)\} \subset \mathfrak{S} = \left\{ q \in C^{0,1}(Q_T) \;\middle|\; \begin{array}{c} |q(t,x) - q(s,y)| \leqslant C\left(|x-y| + |t-s|\right), \\ \forall\,(t,x),(s,y) \in \overline{Q_T}, \\ 1 < \alpha \leqslant q(\cdot,\cdot) \leqslant \beta \ \text{in } \overline{Q_T}. \end{array} \right\} \tag{2.6}$$

$$p_{u_k} \to p_u = 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(\cdot)\right|^2 \, d\tau\right) \quad \text{uniformly in } \overline{Q_T} \text{ as } k \to \infty, \tag{2.7}$$

where

$$\delta = ah\left[ah + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} |\Omega| \sup_{k\in\mathbb{N}} \|u_k\|^2_{L^2(0,T;L^2(\Omega))}\right]^{-1}, \tag{2.8}$$

$$\begin{aligned} \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} &= \max_{\substack{z=x-y \\ x\in\overline{\Omega}, y\in\overline{\Omega}}} \Big[|G_\sigma(z)| + |\nabla G_\sigma(z)|\Big] \\ &= \frac{e^{-1}}{\left(\sqrt{2\pi}\sigma\right)^N}\left[1 + \frac{1}{\sigma^2}\operatorname{diam}\Omega\right]. \end{aligned} \tag{2.9}$$

Proof. In view of the initial assumptions, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$. Hence, by smoothness of the Gaussian filter kernel G_σ, it follows that

$$\begin{aligned} \frac{1}{h}\int_{t-h}^{t} \Big|(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot))(x)\Big|^2 d\tau &\leqslant \frac{1}{h}\int_{t-h}^{t}\left(\int_\Omega |\nabla G_\sigma(x-y)|\,|\widetilde{u}_k(\tau,y)|\,dy\right)^2 d\tau \\ &\leqslant \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} \frac{|\Omega|}{h} \|u_k\|^2_{L^2(0,T;L^2(\Omega))}, \\ 2 \geqslant p_{u_k}(x) = 1 + &\frac{ah}{ah + \displaystyle\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\right|^2 \, d\tau} \\ &\geqslant 1 + \frac{ah}{ah + |\Omega| \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} \|u_k\|^2_{L^2(0,T;L^2(\Omega))}}, \\ &\qquad \forall\,(t,x) \in Q_T, \end{aligned}$$

where $\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}$ is defined in (2.9). Then L^2-boundedness of $\{u_k\}_{k\in\mathbb{N}}$ guarantees the existence of a positive value $\delta \in (0,1)$ (see (2.8)) such that estimate (2.5) holds true for all $k \in \mathbb{N}$.

Moreover, as follows from the relations

$$
\begin{aligned}
&\left|p_{u_k}(t,x)-p_{u_k}(t,y)\right|\\
&\leqslant ah\left|\frac{\displaystyle\int_{t-h}^{t}\left(\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(x)\right|^2-\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2\right)d\tau}{\left(ah+\displaystyle\int_{t-h}^{t}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(x)\right|^2d\tau\right)\left(ah+\displaystyle\int_{t-h}^{t}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau\right)}\right|\\
&\leqslant\frac{1}{ah}\left(\int_0^T\left(\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(x)\right|+\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|\right)^2d\tau\right)^{1/2}\\
&\quad\times\left(\int_0^T\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(x)-\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau\right)^{1/2}\\
&\leqslant\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\|u_k\|_{L^2(0,T;L^2(\Omega))}}{ah}\\
&\quad\times\left(\int_\Omega\left|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)\right|^2dz\int_0^T\int_\Omega u^2(\tau,z)\,dz\,d\tau\right)^{1/2}\\
&=\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\sqrt{|\Omega|}\gamma_1}{ah}\left(\int_\Omega\left|\nabla G_\sigma(x-z)-\nabla G_\sigma(y-z)\right|^2dz\right)^{1/2},\\
&\qquad\qquad\forall\,x,y\in\overline{\Omega}
\end{aligned}
\tag{2.10}
$$

with $\gamma_1=\sup_{k\in\mathbb{N}}\|u_k\|^2_{L^2(0,T;L^2(\Omega))}$, and smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a positive constant $C_G>0$ independent of k such that, for each $t\in[0,T]$, we have the following estimate

$$
|p_{u_k}(t,x)-p_{u_k}(t,y)|\leqslant\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1\sqrt{|\Omega|}C_G}{ah}|x-y|,\ \forall\,x,y\in\overline{\Omega}.
$$

Arguing in a similar manner, we see that

$$
\begin{aligned}
&\left|p_{u_k}(t,y)-p_{u_k}(s,y)\right|\\
&\leqslant\frac{1}{ah}\left|\int_{t-h}^{t}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau-\int_{s-h}^{s}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau\right|\\
&=\frac{1}{ah}\left|\int_{s}^{t}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau-\int_{s-h}^{t-h}\left|\left(\nabla G_\sigma*\widetilde{u}_k(\tau,\cdot)\right)(y)\right|^2d\tau\right|\\
&\leqslant\frac{\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}}{ah}\left(\left|\int_s^t\left(\int_\Omega u_k(\tau,z)\,dz\right)^2d\tau\right|+\left|\int_{s-h}^{t-h}\left(\int_\Omega\widetilde{u}_k(\tau,z)\,dz\right)^2d\tau\right|\right)\\
&\leqslant\frac{2\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega|\gamma_2}{ah}|t-s|,\quad\forall\,t,s\in[0,T],
\end{aligned}
\tag{2.11}
$$

where $\gamma_2=\sup_{k\in\mathbb{N}}\|u_k\|^2_{C([0,T];L^2(\Omega))}$.

As a result, utilizing the estimates (2.10)–(2.11), and setting

$$
C:=\frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1\sqrt{|\Omega|}C_G}{ah}+\frac{2\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega|\gamma_2}{ah},
\tag{2.12}
$$

we see that

$$
\begin{aligned}
|p_{u_k}(t,x) - p_{u_k}(s,y)| &\leqslant |p_{u_k}(t,x) - p_{u_k}(t,y)| + |p_{u_k}(t,y) - p_{u_k}(s,y)| \\
&\leqslant C\left[|x-y| + |t-s|\right], \\
&\forall\,(t,x),(s,y) \in \overline{Q_T} := [0,T]\times\overline{\Omega}.
\end{aligned}
\tag{2.13}
$$

Thus, $\{p_{u_k}\} \subset \mathfrak{S}$. Since $\max_{(t,x)\in\overline{Q_T}} |p_{u_k}(t,x)| \leqslant \beta$ and each element of the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ has the same modulus of continuity, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the sequence $\{p_{u_k}\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\overline{Q_T})$. Taking into account the estimate (2.13) and the fact that the set $\mathfrak{S}$ is closed with respect to the uniform convergence and

$$
\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}_k(\tau,\cdot)\right)(x)\right|^2\, d\tau \to \frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(x)\right|^2\, d\tau
$$
$$
\text{as } k\to\infty,\ \forall\,(t,x)\in Q_T
$$

by definition of the weak convergence in $L^2(0,T;L^2(\Omega))$, we deduce: $p_{u_k} \to p_u$ uniformly in $\overline{Q_T}$ as $k\to\infty$, where

$$
p_u(t,x) = 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|\left(\nabla G_\sigma * \widetilde{u}(\tau,\cdot)\right)(\cdot)\right|^2\, d\tau\right)
$$

in Q_T. The proof is complete. □

2.3. Anisotropic Diffusion Tensor

Let $\mathbb{S}^N$ be the set of all real-valued symmetric matrices $A = [a_{ij}]_{i,j=1}^N$, ($a_{ij} = a_{ji} \in \mathbb{R}$). We suppose that $\mathbb{S}^N$ is endowed with the Euclidian scalar product $A\cdot B = \mathrm{tr}(A\,B) := \sum_{i,j=1}^N a_{ij}b_{ij}$ and with the corresponding Euclidian norm $\|A\|_{\mathbb{S}^N} = (A\cdot A)^{1/2} = \sqrt{\mathrm{tr}(A^2)}$. We also make use of the spectral norm $\|A\|_2 := \sup\left\{|A\xi|\ :\ \xi\in\mathbb{R}^N\ \text{ with }\ |\xi| = 1\right\}$ of matrices $A\in\mathbb{S}^N$, which is different from the Euclidean norm $\|A\|_{\mathbb{S}^N}$. However, the relation $\|A\|_2 \leqslant \|A\|_{\mathbb{S}^N} \leqslant \sqrt{N}\|A\|_2$ holds true for all $A\in\mathbb{S}^N$.

Let $u\in C([0,T];L^1(\Omega))$ be a given function and we assume that this function is zero-extended outside of Q_T. Being mainly motivated by practical implementations in image processing [24, 33], we define the so-called structure tensor $J_\rho(u_\sigma)$ associated with the function $u : Q_T \mapsto \mathbb{R}$ as follows

$$
J_\rho(u_\sigma) := G_\rho * \left(\nabla u_\sigma \otimes \nabla u_\sigma\right) = G_\rho * \left(\nabla u_\sigma\,(\nabla u_\sigma)^t\right), \tag{2.14}
$$

where G_ρ is the Gaussian convolution kernel, and

$$
\nabla u_\sigma = \nabla u_\sigma(t,x) := \left(\nabla G_\sigma * \widetilde{u}(t,\cdot)\right)(x).
$$

It is not very hard to verify that the symmetric matrix $J_\rho(u_\sigma) = \begin{bmatrix} j_{11} & \cdots & j_{1N} \\ \cdots & \cdots & \cdots \\ j_{1N} & \cdots & j_{NN} \end{bmatrix}$ is positively semi-definite and uniformly bounded in Ω. Indeed, for any $\xi \in \mathbb{R}^N$, we have

$$\begin{aligned} \xi^t J_\rho(u_\sigma)\xi &\leqslant N \int_\Omega G_\rho(x-y)|\nabla u_\sigma(t,\cdot)|^2|\xi|^2\,dx \\ &\leqslant \frac{2e^{-1}}{\left(\sqrt{2\pi\rho}\right)^N}|\Omega|\|u(t,\cdot)\|^2_{L^1(\Omega)}\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\xi|^2, \\ &\leqslant \frac{2e^{-1}}{\left(\sqrt{2\pi\rho}\right)^N}|\Omega|\|u\|^2_{C([0,T];L^1(\Omega))}\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\xi|^2, \quad \forall\,(t,x)\in Q_T, \end{aligned} \tag{2.15}$$

$$\xi^t J_\rho(u_\sigma)\xi = \int_\Omega G_\rho(x-y)\left(\nabla u_\sigma(t,y),\xi\right)^2_{\mathbb{R}^N}\,dy \geqslant 0, \quad \forall\,(t,x)\in Q_T. \tag{2.16}$$

Taking this fact into account, we introduce the so-called relaxed version of the anisotropic diffusion tensor $D(t,x,u)$ and define it as follows (for comparison we refer to [1,2])

$$D(t,x,u) := \varepsilon I + J_\rho(u_\sigma), \tag{2.17}$$

where $\varepsilon > 0$ is a small parameter, and $I \in (\mathbb{R}^N,\mathbb{R}^N)$ is the unit matrix.

It is easy to infer from (2.15)–(2.16) and (2.17) that, for any $u \in C([0,T];L^1(\Omega))$, the following two-side estimate

$$d_1^2|\xi|^2 \leqslant \xi^t\left[D(t,x,u)\right]^2\xi \leqslant d_2^2|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^N, \quad \forall\,(t,x)\in Q_T. \tag{2.18}$$

holds true with

$$\begin{aligned} d_1 &= \varepsilon, \\ d_2 &= \varepsilon + \frac{2e^{-1}}{\left(\sqrt{2\pi\rho}\right)^N}|\Omega|\|u\|^2_{C([0,T];L^1(\Omega))}\|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}. \end{aligned}$$

For the further convenience, we suppose that $d_2 \geqslant 1$.

Then arguing as in the proof of Lemma 2.1, we can establish the following result.

Lemma 2.2. *Let $\{u_k\}_{k\in\mathbb{N}} \subset C([0,T];L^2(\Omega))$ and $u \in C([0,T];L^2(\Omega))$ be measurable and extended by zero outside of Q_T functions with properties (2.4). Let $\{D(t,x,u_k)\}_{k\in\mathbb{N}}$ be the corresponding sequence of anisotropic diffusion tensors. Then*

$$d_1^2|\xi|^2 \leqslant \xi^t\left[D(t,x,u_k)\right]^2\xi \leqslant d_2^2|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^N, \quad \forall\,(t,x)\in Q_T,\ \forall\,k\in\mathbb{N}, \tag{2.19}$$

$$D(t,x,u_k) \to D(t,x,u) \text{ uniformly in } \overline{Q_T} \text{ as } k\to\infty, \tag{2.20}$$

$$\{D(\cdot,\cdot,u_k)\} \subset \mathfrak{D}, \tag{2.21}$$

where

$$d_1 = \varepsilon, \quad , d_2 = \varepsilon + \frac{2e^{-1}}{\left(\sqrt{2\pi}\rho\right)^N} |\Omega| \sup_{k\in\mathbb{N}} \|u_k\|^2_{C([0,T];L^1(\Omega))} \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})},$$

$$\mathfrak{D} = \left\{ \Lambda \in C^{0,1}(Q_T;\mathbb{R}^{N\times N}) \;\middle|\; \begin{array}{c} \|\Lambda(t,x) - \Lambda(s,y)\|_2 \leqslant C\left(|x-y| + |t-s|\right), \\ \forall\,(t,x),(s,y) \in \overline{Q_T}. \end{array} \right\}$$

2.4. On Orlicz Spaces

Let $w \in C([0,T];L^2(\Omega))$ be a given function. Let $p_w : Q_T \to \mathbb{R}$ be the corresponding variable exponent which is defined by the rule (1.4). Then

$$1 < \alpha \leqslant p_w(t,x) \leqslant \beta < \infty \ \text{ a.e. in } \ Q_T$$

(see Lemma 2.1), where the constants α and β are given by (2.5). Let $p'_w(t,x) = \frac{p_w(t,x)}{p_w(t,x)-1}$ be the corresponding conjugate exponent. It is clear that

$$2 = \underbrace{\frac{\beta}{\beta-1}}_{\beta'} \leqslant p'_w(t,x) \leqslant \underbrace{\frac{\alpha}{\alpha-1}}_{\alpha'} = \frac{\alpha}{\delta} \quad \textit{a.e. in } Q_T, \tag{2.22}$$

where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p_w(\cdot)}(Q_T)$ the set of all measurable functions $f : Q_T \to \mathbb{R}$ such that

$$\int_{Q_T} |f(t,x)|^{p_w(t,x)}\,dxdt < \infty.$$

Then $L^{p_w(\cdot)}(Q_T)$ is a reflexive separable Banach space with respect to the Luxemburg norm (see [20, 22] for the details)

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} = \inf\left\{ \lambda > 0 \ : \ \int_{Q_T} |\lambda^{-1} f(t,x)|^{p_w(t,x)}\,dxdt \leqslant 1 \right\}. \tag{2.23}$$

Its dual is $L^{p'_w(\cdot)}(Q_T)$, that is, any continuous functional $F = F(f)$ on $L^{p_w(\cdot)}(Q_T)$ has the form (see [43, Lemma 13.2])

$$F(f) = \int_{Q_T} fg\,dxdt, \quad \text{with } \ g \in L^{p'_w(\cdot)}(Q_T).$$

Notice that

$$\begin{gathered} \|f\|^{\alpha}_{L^{p_w(\cdot)}(Q_T)} - 1 \leqslant \int_{Q_T} |f(t,x)|^{p_w(t,x)}\,dxdt \leqslant \|f\|^{\beta}_{L^{p_w(\cdot)}(Q_T)} + 1, \\ \forall\, f \in L^{p_w(\cdot)}(Q_T), \end{gathered} \tag{2.24}$$

$$\begin{gathered} \|f_k - f\|_{L^{p_w(\cdot)}(Q_T)} \to 0 \quad \Longleftrightarrow \quad \int_{Q_T} |f_k(t,x) - f(t,x)|^{p_w(t,x)}\,dxdt \to 0 \\ \text{as } k \to \infty, \end{gathered} \tag{2.25}$$

and the estimates: if $f \in L^{p_w(\cdot)}(Q_T)$ then

$$\|f\|_{L^\alpha(Q_T)} \leqslant (1+T|\Omega|)^{1/\alpha}\, \|f\|_{L^{p_w(\cdot)}(Q_T)}, \tag{2.26}$$

$$\|f\|_{L^{p_w(\cdot)}(Q_T)} \leqslant (1+T|\Omega|)^{1/\beta'}\, \|f\|_{L^\beta(Q_T)}, \quad \beta' = \frac{\beta}{\beta-1},\ \forall\, f \in L^\beta(Q_T), \tag{2.27}$$

are well-known (see, for instance, [20, 22, 42]).

The following result can be viewed as an analogous of the Hölder inequality in Lebesgue spaces with variable exponents (for the details we refer to [20, 22]).

Proposition 2.1. If $f \in L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)$ and $g \in L^{p'_w(\cdot)}(Q_T;\mathbb{R}^N)$, then $(f,g) \in L^1(Q_T)$ and

$$\int_{Q_T} (f,g)\; dxdt \leqslant 2\|f\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)}\|g\|_{L^{p'_w(\cdot)}(Q_T;\mathbb{R}^N)}. \tag{2.28}$$

Let $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$, with some $\widehat{\delta} \in (0,1]$, be a given sequence of exponents. Hereinafter in this subsection we assume that

$$\begin{aligned} &p, p_k \in C^{0,\widehat{\delta}}(\overline{Q_T}) \quad \text{for } k = 1,2,\dots, \text{ and}\\ &p_k(\cdot) \to p(\cdot) \quad \text{uniformly in } \overline{Q_T} \text{ as } k \to \infty. \end{aligned} \tag{2.29}$$

We associate with this sequence the another one $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. The characteristic feature of this set of functions is that each element f_k lives in the corresponding Orlicz space $L^{p_k(\cdot)}(Q_T)$. So, we have a sequence in the scale of spaces $\left\{L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$. We say that the sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is bounded if

$$\limsup_{k\to\infty} \int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\, dxdt < +\infty. \tag{2.30}$$

Definition 2.1. A bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ is weakly convergent in the variable Orlicz space $L^{p_k(\cdot)}(Q_T)$ to a function $f \in L^{p(\cdot)}(Q_T)$, where $p \in C^{0,\delta}(\overline{Q_T})$ is the limit of $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\widehat{\delta}}(\overline{Q_T})$ in the uniform topology of $C(\overline{Q_T})$, if

$$\lim_{k\to\infty} \int_{Q_T} f_k\varphi\, dxdt = \int_{Q_T} f\varphi\, dxdt, \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^N). \tag{2.31}$$

We make use of the following result concerning the lower semicontinuity property of the variable $L^{p_k(\cdot)}$-norm with respect to the weak convergence in $L^{p_k(\cdot)}(Q_T)$ (we refer to [43, Lemma 13.3] for comparison).

Proposition 2.2. If a bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(Q_T)\right\}_{k\in\mathbb{N}}$ converges weakly in $L^\alpha(Q_T)$ to f for some $\alpha > 1$, then $f \in L^{p(\cdot)}(Q_T)$, $f_k \rightharpoonup f$ in variable $L^{p_k(\cdot)}(Q_T)$, and

$$\liminf_{k\to\infty} \int_{Q_T} |f_k(t,x)|^{p_k(t,x)}\, dxdt \geqslant \int_{Q_T} |f(t,x)|^{p(t,x)}\, dxdt. \tag{2.32}$$

Proof. In view of the fact that

$$\begin{aligned}\left|\int_{Q_T}|f_k(t,x)|^{p_k(t,x)}\,dxdt-\int_{Q_T}\frac{p(t,x)}{p_k(t,x)}|f_k(t,x)|^{p_k(t,x)}\,dxdt\right|\\ \leqslant \|p_k-p\|_{C(\overline{Q_T})}\int_{Q_T}\frac{1}{p_k(t,x)}|f_k(t,x)|^{p_k(t,x)}\,dxdt\\ \leqslant \frac{\|p_k-p\|_{C(\overline{Q_T})}}{\alpha}\int_{Q_T}|f_k(t,x)|^{p_k(t,x)}\,dx \overset{\text{by (2.30)}}{\to} 0 \text{ as } k\to\infty,\end{aligned}$$

we see that

$$\liminf_{k\to\infty}\int_{Q_T}|f_k(t,x)|^{p_k(t,x)}\,dxdt=\liminf_{k\to\infty}\int_{Q_T}\frac{p(t,x)}{p_k(t,x)}|f_k(t,x)|^{p_k(t,x)}\,dxdt. \quad (2.33)$$

Using the Young inequality $ab\leqslant |a|^p/p+|b|^{p'}/p'$, we have

$$\begin{aligned}\int_{Q_T}\frac{p(t,x)}{p_k(t,x)}|f_k(t,x)|^{p_k(t,x)}\,dxdt\\ \geqslant \int_{Q_T}p(t,x)f_k(t,x)\varphi(t,x)\,dxdt-\int_{Q_T}\frac{p(t,x)}{p_k'(t,x)}|\varphi(t,x)|^{p_k'(t,x)}\,dxdt, \quad (2.34)\end{aligned}$$

for $p_k'(t,x)=p_k(t,x)/(p_k(t,x)-1)$ and any $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^2)$.

Then passing to the limit in (2.34) as $k\to\infty$ and utilizing property (2.29) and the fact that

$$\begin{aligned}\lim\nolimits_{k\to\infty}\int_{Q_T}f_k(t,x)\varphi(t,x)\,dxdt=\int_{Q_T}f(t,x)\varphi(t,x)\,dxdt\\ \text{for all }\ \varphi\in L^{\alpha'}(Q_T),\end{aligned} \quad (2.35)$$

we obtain

$$\begin{aligned}\liminf_{k\to\infty}\int_{Q_T}|f_k(t,x)|^{p_k(t,x)}\,dxdt\\ \geqslant \int_{Q_T}p(t,x)f(t,x)\varphi(t,x)\,dxdt-\int_{Q_T}\frac{p(t,x)}{p'(t,x)}|\varphi(t,x)|^{p'(t,x)}\,dxdt. \quad (2.36)\end{aligned}$$

Since the last inequality is valid for all $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$ and $C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$ is dense subset of $L^{p'(\cdot)}(Q_T)$, it follows that this relation also holds true for $\varphi\in L^{p'(\cdot)}(Q_T)$. So, taking $\varphi=|f(t,x)|^{p(t,x)-2}f(t,x)$, we arrive at the announced inequality (2.32). So, as a consequence of (2.32) and estimate (2.24), we get: $f\in L^{p(\cdot)}(Q_T)$.

To end the proof, it remains to observe that relation (2.35) holds true for $\varphi\in C_c^\infty(\mathbb{R}\times\mathbb{R}^N)$ as well. From this the weak convergence $f_k\rightharpoonup f$ in the variable Orlicz space $L^{p_k(\cdot)}(Q_T)$ follows. □

2.5. On Weighted Energy Space with Variable Exponent

Let $D(t,x,w)$ be the diffusion tensor associated with some function $w \in C([0,T];L^2(\Omega))$ and given by the rule (2.17). For our further analysis, we define the weighted energy space $W_w(Q_T)$ as the set of all functions $u(t,x)$ such that

$$\begin{aligned} & u \in L^2(Q_T), \quad u(t,\cdot) \in W^{1,1}(\Omega) \ \text{ for a.e.} t \in [0,T], \\ & \int_{Q_T} |D(t,x,w)\nabla u|^{p_w(t,x)}\, dxdt < +\infty. \end{aligned} \tag{2.37}$$

We equip $W_w(Q_T)$ with the norm

$$\|u\|_{W_w(Q_T)} = \|u\|_{L^2(Q_T)} + \|D(\cdot,\cdot,w)\nabla u\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)}, \tag{2.38}$$

where the second term on the right-hand side is the norm of the vector-valued function $D(t,x,w)\nabla u(t,x)$ in the Orlicz space $L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)$. Since (see (2.18))

$$d_1^2|\xi|^2 \leqslant \xi^t\,[D(t,x,u)]^2\,\xi \leqslant d_2^2|\xi|^2, \quad \forall\,\xi \in \mathbb{R}^N,\ \forall\,(t,x) \in Q_T, \tag{2.39}$$

it follows that the set $W_w(Q_T)$, equiped with the norm (2.38), is a reflexive Banach space. Moreover, due to the fact that the exponent $p_w : Q_T \to \mathbb{R}$ is Lipschitz continuous, the smooth functions are dense in the weighted Sobolev-Orlicz space $W_w(Q_T)$ (see [3]). So, $W_w(Q_T)$ can be considered as the closure of the set $\left\{\varphi \in C^\infty(\overline{Q}_T)\right\}$ with respect to the norm $\|\cdot\|_{W_w(Q_T)}$.

2.6. On the weak convergence of fluxes to flux

Let us consider the following collection of parabolic equations of monotone type

$$\frac{\partial u_k}{\partial t} - div\, A_k(t,x,\nabla u_k) = f, \quad (t,x) \in Q_T, \tag{2.40}$$

where $f \in L^2(\Omega)$ and $k = 1,2,\dots$. Let u_k be a solution of (2.40) for a given $k \in \mathbb{N}$ and this solution is understood in the sense of distributions. Assume that $A_k(\cdot,\cdot,\xi) \to A(\cdot,\cdot,\xi)$ as $k \to \infty$ pointwise a.e. with respect to the first two arguments and for all $\xi \in \mathbb{R}^N$.

A typical situation arising in the study of most optimization problems and which is of fundamental importance in many others areas of nonlinear analysis, can be stated as follows: suppose it is known that a solution $u_k \in L^2(0,T;W^{1,\alpha}(\Omega))$ of (2.40) and the corresponding flow $w_k = A_k(\cdot,\cdot,\nabla u_k) \in L^{\beta'}(Q_T;\mathbb{R}^N)$ converge weakly, namely,

$$\begin{gathered} u_k \rightharpoonup u \ \text{ in } \ L^2(0,T;W^{1,\alpha}(\Omega)), \quad w_k \rightharpoonup w \ \text{ in } \ L^{\beta'}(Q_T;\mathbb{R}^N), \\ 1 < \alpha < \beta,\ \beta' = \frac{\beta}{\beta - 1}. \end{gathered}$$

The main question is whether a flow converges to a flow, i.e., whether the equality for the limit elements $A(t,x,\nabla u) = w$ holds. The situation is not trivial because the function $A(\cdot,\cdot,v)$ is nonlinear in v and the weak convergence $v_k \rightharpoonup v$ is far from sufficient to derive the limit relation $A_k(\cdot,\cdot,v_k) \rightharpoonup A(\cdot,\cdot,v)$. So, the problem is to show that $w = A(\cdot,\cdot,\nabla u)$, although the validity of this equality is by no means obvious at this stage. The conditions (first of all, on the exponents α and β) under which the answer to the above question is affirmative, have been obtained by Zhikov and Pastukhova in their celebrated paper [45].

Theorem 2.2. *Assume that the following conditions are satisfied:*

(C1) $A_k(t,x,\xi)$ *and* $A(t,x,\xi)$ *are* $\mathbb{R}^N$*-valued Carathéodory functions, that is, these functions are continuous in* $\xi \in \mathbb{R}^N$ *for a.e.* $(t,x) \in Q_T$ *and measurable with respect to* $(t,x) \in Q_T$ *for each* $\xi \in \mathbb{R}^N$*;*

(C2) $\Big(A_k(t,x,\xi) - A_k(t,x,\zeta), \xi - \zeta\Big) \geqslant 0$, $A_k(t,x,0) = 0 \quad \forall \xi,\zeta \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$*;*

(C3) $|A_k(t,x,\xi)| \leqslant c(|\xi|) < \infty$ *and* $\lim_{k\to\infty} A_k(t,x,\xi) = A(t,x,\xi)$ *for all* $\xi \in \mathbb{R}^N$ *and for a.e.* $(t,x) \in Q_T$*;*

(C4) $u_k \rightharpoonup u$ *in* $L^\alpha(0,T;W^{1,\alpha}(\Omega))$, $\alpha > 1$, *and* $\{u_k\}_{k\in\mathbb{N}}$ *are bounded in the space* $L^\infty(0,T;L^2(\Omega))$*;*

(C5) $w_k = A_k(t,x,\nabla u_k) \rightharpoonup w$ *in* $L^{\beta'}(Q_T;\mathbb{R}^N)$, $\beta > 1$*;*

(C6) $u_k \in L^\beta(0,T;W^{1,\beta}(\Omega))$ *for all* $k \in \mathbb{N}$, *and* $\sup_{k\in\mathbb{N}} \|\,(w_k,\nabla u_k)\,\|_{L^1(Q_T)} < \infty$*;*

(C7) $1 < \alpha < \beta < 2\alpha$.

Then the flow $A_k(t,x,\nabla u_k)$ *weakly converges in the Lebesgue space* $L^{\beta'}(Q_T;\mathbb{R}^N)$ *to the flow* $A(t,x,\nabla u)$.

For our further analysis, we make also use of the following well-known results.

Lemma 2.3 ([42]). *Let* Ψ *be a class of integrands* $F(t,x,\xi)$ *that are convex with respect to* $\xi \in \mathbb{R}^N$, *measurable with respect to* $(t,x) \in Q_T$, *and satisfy the estimate*

$$c_1|\xi|^\alpha \leqslant F(t,x,\xi) \leqslant c_2|\xi|^\beta, \quad 1 < \alpha \leqslant \beta < \infty,\ c_1,c_2 > 0.$$

Suppose that F_k *and* F *belong to the class* Ψ *and the following condition holds:*

$$\lim_{k\to\infty} F_k(t,x,\xi) = F(t,x,\xi) \quad \text{for a.e. } (t,x) \in Q_T \text{ and any } \xi \in \mathbb{R}^N.$$

Then the following lower semicontinuity property is valid:
if $v_k \rightharpoonup v$ *in* $L^1(Q_T;\mathbb{R}^N)$ *then*

$$\liminf_{k\to\infty} \int_{Q_T} F_k(t,x,v_k)\,dxdt \geqslant \int_{Q_T} F(t,x,v)\,dxdt. \tag{2.41}$$

Lemma 2.4 ([44]). *Let $A_k(t,x,\xi)$ and $A(t,x,\xi)$ be $\mathbb{R}^N$-valued Carathéodory functions with properties (C1)–(C3). Assume that*

$$v_k \rightharpoonup v \quad \text{and} \quad w_k = A_k(t,x,v_k) \rightharpoonup w \quad \text{in } L^1(Q_T;\mathbb{R}^N) \text{ as } k\to\infty,$$

and $(w,v)\in L^1(Q_T)$. Then

$$\liminf_{k\to\infty}\int_{Q_T}(A_k(t,x,v_k),v_k)\,dxdt \geqslant \int_{Q_T}(w,v)\,dxdt. \tag{2.42}$$

Lemma 2.5 ([3]). *Let ϵ be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. Assume that the following conditions*

$$\begin{aligned}
&(i) \quad p_\epsilon, p\in C(\overline{Q_T}), \quad p_\epsilon\to p \ \text{ in } C(\overline{Q_T}) \text{ as } \epsilon\to 0,\\
&(ii) \quad v_\epsilon\in L^1(Q_T;\mathbb{R}^N), \quad \int_{Q_T}\left[|v_\epsilon|^{p_\epsilon}+\epsilon|v_\epsilon|^\beta\right]dxdt\leqslant K<\infty \ \text{ for each } \epsilon>0,\\
&(iii) \quad |v_\epsilon|^{p_\epsilon}+\epsilon|v_\epsilon|^\beta \rightharpoonup z \ \text{ in } L^{\beta'}(Q_T), \quad \beta'=\beta/(\beta-1) \text{ as } \epsilon\to 0
\end{aligned}$$

hold true with some α and β such that $1<\alpha\leqslant p_\epsilon(t,x)\leqslant\beta<\infty$ for all $\epsilon>0$ and $(t,x)\in Q_T$. Then $z\in L^{p'(\cdot)}(Q_T)$.

3. Existence Result for a Class of Parabolic Equations with Variable Nonlinearity Exponent

Let $f\in L^2(Q_T)$ and $f_0\in L^2(\Omega)$ be given distributions. Let us consider the following initial-boundary value problem (IBVP)

$$\frac{\partial u}{\partial t} - div\, A_u(t,x,\nabla u) + u = f \quad \text{in } Q_T, \tag{3.1}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{3.2}$$

$$u(0,\cdot) = f_0 \quad \text{in } \Omega. \tag{3.3}$$

Here,

$$A_w(t,x,\nabla u) := |D(t,x,w)\nabla u|^{p_w(t,x)-2}D(t,x,w)\nabla u, \tag{3.4}$$

the exponent $p_w : Q_T\to(1,2]$ is given by the rule (1.4), the symmetric matrix $D(t,x,w)$ is defined in (2.17), and ∂_ν stands for the outward normal derivative.

As follows from (3.4), (2.17), and Lemma 2.1, for each fixed function $w\in C([0,T];L^2(\Omega))$, the mapping $(t,x,\xi)\mapsto A_w(t,x,\xi)$ is a Carathéodory vector function; that is, $A_w(t,x,\xi)$ is continuous in $\xi\in\mathbb{R}^N$ and is measurable with respect to (t,x) for each $\xi\in\mathbb{R}^N$. Moreover, the following monotonicity, coerciveness and boundedness conditions hold for a.e. $(t,x) \in Q_T$ [43]:

$$\Big(A_w(t,x,\xi) - A_w(t,x,\zeta), \xi - \zeta\Big) \geqslant 0, \quad \forall\, \xi, \zeta \in \mathbb{R}^N, \tag{3.5}$$

$$\begin{aligned} (A_w(t,x,\xi),\xi) &= |D(t,x,w)\xi|^{p_w(t,x)-2}\left(D(t,x,w)\xi, D^{-1}(t,x,w)D(t,x,w)\xi\right) \\ &\overset{\text{by } (2.39)}{\geqslant} d_2^{-1} d_1^{p_w(t,x)} |\xi|^{p_w(t,x)} \geqslant d_2^{-1} d_1^2 |\xi|^{p_w(t,x)}, \quad \forall\, \xi \in \mathbb{R}^N, \end{aligned} \tag{3.6}$$

$$|A_w(t,x,\xi)|^{p_w'(t,x)} \leqslant d_2^{p_w(t,x)} |\xi|^{p_w(t,x)} \leqslant d_2^2 |\xi|^{p_w(t,x)}, \quad \forall\, \xi \in \mathbb{R}^N. \tag{3.7}$$

In general, the principle operator $-\,div\;\; A_u(t,x,\nabla u) + u$ provides an example of a strongly non-linear, non-monotone, and non-coercive operator in divergence form. Hence, the existence result for the IBVP (3.1)–(3.3) and uniqueness of its solution remain, apparently, open questions for nowadays [25, Chapter III].

Definition 3.1. We say that, for given $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$, a function u is a weak solution to the problem (3.1)–(3.3) if $u \in W_u(Q_T)$, i.e.,

$$\begin{gathered} u \in L^2(Q_T),\ u(t,\cdot) \in W^{1,1}(\Omega) \ \text{ for a.e. } t \in [0,T], \\ \int_{Q_T} |D(t,x,u)\nabla u|^{p_u(t,x)}\, dxdt < +\infty, \end{gathered} \tag{3.8}$$

and the integral identity

$$\begin{aligned} \int_{Q_T} \left(-u\frac{\partial \varphi}{\partial t} + (A_u(t,x,\nabla u), \nabla\varphi) + u\varphi\right) dxdt& \\ = \int_{Q_T} f\varphi\, dxdt + \int_\Omega f_0 \varphi|_{t=0}\, dx& \end{aligned} \tag{3.9}$$

holds true for any function $\varphi \in \Phi$, where $\Phi = \left\{\varphi \in C^\infty(\overline{Q}_T) \;:\; \varphi|_{t=T} = 0\right\}$.

In order to find out in what sense the weak solution takes the initial value $u(0,\cdot) = f_0$, we give the following result.

Proposition 3.1. Let $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ be given distributions. Let $u \in W_u(Q_T)$ be a weak solution to the problem (3.1)–(3.3). Then, for any $\eta \in C^\infty(\overline{\Omega})$, the scalar function $h(t) = \displaystyle\int_\Omega u(t,x)\eta(x)\, dx$ belongs to $W^{1,1}(0,T)$ and $h(0) = \displaystyle\int_\Omega f_0(x)\eta(x)\, dx$.

Proof. We set $\varphi(t,x) = \eta(x)\zeta(t)$, where $\zeta(\cdot)$ is a smooth function on $[0,T]$ and $\zeta(T) = 0$. Then it is clear that $\varphi \in \Phi$ and, therefore, the integral identity (3.9) yields the equality

$$\int_0^T \left[-h(t)\zeta'(t)\right]\, dt + \int_0^T \underbrace{\left(\int_\Omega (A_u(t,x,\nabla u), \nabla \eta)\, dx + \int_\Omega \eta u\, dx - \int_\Omega f\eta\, dx\right)}_{H(t)} \zeta(t)\, dt = \left(\int_\Omega f_0 \eta\, dx\right)\zeta(0). \quad (3.10)$$

Since $h \in L^1(0,T)$, it follows from (2.6), (3.7), and (3.10) that $H \in L^1(0,T)$. Hence, $h \in W^{1,1}(0,T)$, i.e., the function $h(t)$ is absolutely continuous on $[0,T]$. Moreover, from (3.10) we deduce that $h(0) = \int_\Omega f_0 \eta\, dx$. □

Our main intention in this section is to show the existence of a weak solution to the problem (3.1)–(3.3). To this end, we make use of the perturbation technique and the classical fixed point theorem of Schauder [35] (we refer to [16,23,27,29,31] where the similar technique has been used).

We begin with the following auxiliary result.

Theorem 3.1. *Let $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ be given distributions. Then, for each positive value $\epsilon > 0$, the Cauchy-Neumann problem*

$$\frac{\partial u}{\partial t} - \epsilon \Delta u - div\, A_u(t,x,\nabla u) + u = f \quad in \;\; Q_T := (0,T)\times\Omega, \quad (3.11)$$

$$\partial_\nu u = 0 \quad on \;\; (0,T)\times\partial\Omega, \quad (3.12)$$

$$u(0,\cdot) = f_0 \quad in \;\; \Omega, \quad (3.13)$$

has a weak solution $u_\epsilon \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$ verifying (3.11)–(3.13) *in the sense of distributions.*

Proof. We introduce the space

$$W(0,T) = \left\{ w \in L^2(0,T; W^{1,2}(\Omega)),\ \frac{dw}{dt} \in L^2(0,T; \left[W^{1,2}(\Omega)\right]')\right\}.$$

This space is a Hilbert space with respect to the graph norm. Let us fix an arbitrary function $w \in W(0,T) \cap L^\infty(0,T; L^2(\Omega))$ such that

$$\left.\begin{aligned} \|w\|_{L^2(0,T;W^{1,2}(\Omega))} &\leqslant C_1, \\ \|w\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sqrt{2}C_1, \\ \|\tfrac{\partial w}{\partial t}\|_{L^2(0,T;(W^{1,2}(\Omega))')} &\leqslant C_3, \\ w(0,\cdot) &= f_0 \;\text{ in } \;\Omega, \end{aligned}\right\} \quad (3.14)$$

with

$$
\begin{aligned}
C_1 &= \sqrt{\|f\|^2_{L^2(Q_T)} + 2\|f_0\|^2_{L^2(\Omega)}}\ ,\\
C_2 &= \left(\frac{d_2}{d_1^2}C_1^2 + 1\right)^{1/\alpha},\\
C_3 &= 2C_1 + \|f\|_{L^2(Q_T)} + \left(1 + d_2^2\left(1 + C_2^2\right)\right)^{1/2}\left(1 + T|\Omega|\right)^{1/2}.
\end{aligned}
$$

For the convenience, we divide the rest proof onto several steps.

Step 1. We associate with w the following variational problem: Find $u = U_\epsilon(w) \in W(0,T)$ satisfying

$$
\begin{aligned}
\left\langle \frac{\partial u(t)}{\partial t}, v\right\rangle &+ \int_\Omega \left[\epsilon\left(\nabla u(t), \nabla v\right) + \left(A_w(t,x,\nabla u(t)), \nabla v\right) + u(t)v\right]\, dx\\
&= \int_\Omega f(t)v\, dx, \quad \forall\, v \in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], && (3.15)\\
u(0) &= f_0. && (3.16)
\end{aligned}
$$

Since the condition $w \in W(0,T)$ implies $w \in C([0,T]; L^2(\Omega))$ (see [21, Chapter XVIII]), it follows from Lemma2.1 that the corresponding exponent

$$
p_w := 1 + g\left(\frac{1}{h}\int_{t-h}^{t} \left|(\nabla G_\sigma * \widetilde{w}(\tau,\cdot))\right|^2\, d\tau\right)
$$

is such that $p_w \in C^{0,1}(Q_T)$ and $1 < \alpha \leqslant q(\cdot,\cdot) \leqslant \beta$ in $\overline{Q_T}$, with $\beta = 2$ and $\alpha = 1 + \delta = 1 + ah\left[ah + \|G_\sigma\|^2_{C^1(\overline{\Omega - \Omega})}|\Omega| \sup_{k\in\mathbb{N}} \|w\|^2_{L^2(0,T;L^2(\Omega))}\right]^{-1}$.

Moreover, we see that the structure tensor $J_\rho(w_\sigma)$ is C^∞ and, as follows from (2.17), $D(\cdot,\cdot,w) \in L^\infty(0,T; C^\infty(\Omega))$, where the anisotropic diffusion tensor $D(t,x,w)$ satisfies the two-side inequality (2.18). Hence, it is easy now to deduce that, for a given value $\epsilon > 0$, the principle operator $B : L^2(0,T; W^{1,2}(\Omega)) \to L^2(0,T; \left(W^{1,2}(\Omega)\right)')$, defined by the rule

$$
\langle Bu, v\rangle = \int_{Q_T} \left(\epsilon\nabla u + A_w(t,x,\nabla u), \nabla v\right)\, dxdt + \int_{Q_T} uv\, dxdt,
$$

is coercive, monotone, and hemicontinuous, where the hemicontinuity means the continuity of the scalar function

$$
\begin{aligned}
z(\lambda) = \langle B(u + \lambda v), \varphi\rangle = &\int_{Q_T} \left(\epsilon(\nabla u + \lambda\nabla v) + A_w(t,x,\nabla u + \lambda\nabla v), \nabla\varphi\right)\, dxdt\\
&+ \int_{Q_T} (u + \lambda v)v\varphi\, dxdt, \quad \forall\, u, v, \varphi \in L^2(0,T; W^{1,2}(\Omega))
\end{aligned}
$$

at the point $\lambda = 0$. Since A_w is a Carathéodory functions, this property can be easily derived with the use of the Lebesgue theorem and the following estimate

$$\begin{aligned}
&|A_w(t,x,\nabla u{+}\lambda\nabla v)||\nabla\varphi| \\
&\qquad \leqslant \frac{1}{p'_w(t,x)}|A_w(t,x,\nabla u+\lambda\nabla v)|^{p'_w(t,x)} + \frac{1}{p_w(t,x)}|\nabla\varphi|^{p_w(t,x)} \\
&\qquad \overset{\text{by (2.5), (3.7)}}{\leqslant} \frac{d_2^2}{2}|\nabla u+\lambda\nabla v|^{p_w(t,x)} + \frac{1}{\alpha}|\nabla\varphi|^{p_w(t,x)} \\
&\qquad \leqslant c_1\left(|\nabla u|^{p_w(t,x)} + |\nabla v|^{p_w(t,x)}\right) + \frac{1}{\alpha}|\nabla\varphi|^{p_w(t,x)} \in L^1(Q_T). \qquad (3.17)
\end{aligned}$$

Hence, by the classical results on parabolic equations [32] (see also results of Alkhutov and Zhikov [3, 4]), we deduce that the problem (3.15)–(3.16) has a unique weak solution $U_\epsilon(w) \in W(0,T)$ in the sense of distributions. Since the integral identity (3.15) is valid for all test functions $v = v(t,x)$ which are stepwise with respect to variable t, it follows that this identity remains true for all $v \in L^2(0,T;W^{1,2}(\Omega))$, and hence for all $v \in W^{1,2}(Q_T)$ such that $v(T,\cdot) = 0$. So, utilizing the integration by parts formula, one can easily deduces from (3.15) that the solution $U_\epsilon(w)$ satisfies both the integral identity

$$\begin{aligned}
&\int_{Q_T}\left(-U_\epsilon(w)\frac{\partial\varphi}{\partial t} + (\epsilon\nabla U_\epsilon(w) + A_w(t,x,\nabla U_\epsilon(w)),\nabla\varphi) + U_\epsilon(w)\varphi\right)dxdt \\
&\qquad\qquad = \int_{Q_T} f\varphi\,dxdt + \int_\Omega f_0\varphi|_{t=0}\,dx \quad \forall\,\varphi\in\Phi \qquad (3.18)
\end{aligned}$$

and the energy equality

$$\begin{aligned}
&\frac12\int_\Omega U_\epsilon^2(w)\,dx \\
&\quad + \int_0^t\int_\Omega\left(\epsilon|\nabla U_\epsilon(w)|^2 + (A_w(t,x,\nabla U_\epsilon(w)),\nabla U_\epsilon(w)) + U_\epsilon^2(w)\right)dxdt \\
&\quad = \int_{Q_T} fU_\epsilon(w)\,dxdt + \int_\Omega f_0^2\,dx, \quad \forall\,t\in[0,T]. \qquad (3.19)
\end{aligned}$$

Step 2. Using (3.19), we derive the following estimates:

$$\begin{aligned}
&\|U_\epsilon(w)\|^2_{L^2(Q_T)} \leqslant \|f\|^2_{L^2(Q_T)} + 2\|f_0\|^2_{L^2(\Omega)} =: C_1^2, \qquad (3.20)\\
&\|\nabla U_\epsilon(w)\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)} \overset{\text{by (2.24)}}{\leqslant} \left(\int_{Q_T}|\nabla U_\epsilon(w)|^{p_w(t,x)}\,dxdt + 1\right)^{1/\alpha} \\
&\quad \overset{\text{by (3.19)}}{\leqslant} \left(\frac{d_2}{d_1^2}\left(\|f_0\|^2_{L^2(\Omega)} + \frac12\|f\|^2_{L^2(Q_T)} + \frac12\|U_\epsilon(w)\|^2_{L^2(Q_T)}\right) + 1\right)^{1/\alpha} \\
&\quad \overset{\text{by (3.20)}}{\leqslant} \left(\frac{d_2}{d_1^2}C_1^2 + 1\right)^{1/\alpha} =: C_2, \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}\|U_\epsilon(w)\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sqrt{2\left(\|f_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(Q_T)} + \frac{1}{2}\|U_\epsilon(w)\|^2_{L^2(Q_T)}\right)}\\ &\leqslant \sqrt{2}C_1, \qquad (3.22)\\ \|\nabla U_\epsilon(w)\|_{L^2(Q_T;\mathbb{R}^N)} &\leqslant \frac{1}{\sqrt{\epsilon}}\left(\|f_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(Q_T)} + \frac{1}{2}\|U_\epsilon(w)\|^2_{L^2(Q_T)}\right)\\ &\leqslant \frac{1}{\sqrt{\epsilon}}C_1. \qquad (3.23)\end{aligned}$$

We also notice that

$$\begin{aligned}&\left|\left\langle \frac{\partial U_\epsilon(w)}{\partial t}, v\right\rangle\right| \overset{\text{by (3.15), (2.28)}}{\leqslant} \sqrt{\epsilon}\|\nabla U_\epsilon(w)\|_{L^2(Q_T;\mathbb{R}^N)}\|\nabla v\|_{L^2(Q_T;\mathbb{R}^N)}\\ &\quad + 2\|A_w(t,x,\nabla U_\epsilon(w))\|_{L^{p'_w(\cdot)}(Q_T;\mathbb{R}^N)}\|\nabla v\|_{L^{p_w(\cdot)}(Q_T;\mathbb{R}^N)}\\ &\quad + \|U_\epsilon\|_{L^2(Q_T)}\|v\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)}\|v\|_{L^2(Q_T)}\\ &\overset{\text{by (2.27)}}{\leqslant} \left[\sqrt{\epsilon}\|\nabla U_\epsilon(w)\|_{L^2(Q_T;\mathbb{R}^N)} + \|U_\epsilon\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)}\right]\|v\|_{L^2(0,T;W^{1,2}(\Omega))}\\ &\quad + \left(1 + \int_{Q_T} |A_w(t,x,\nabla U_\epsilon(w))|^{p'_w(t,x)}\,dxdt\right)^{1/2}(1+T|\Omega|)^{1/2}\|v\|_{L^2(Q_T)}\\ &\overset{\text{by (3.20)–(3.23)}}{\leqslant} C_3\|v\|_{L^2(0,T;W^{1,2}(\Omega))}, \quad \forall\, v \in L^2(0,T;W^{1,2}(\Omega)),\end{aligned}$$

where

$$C_3 := 2C_1 + \|f\|_{L^2(Q_T)} + \left(1 + d_2^2\left(1+C_2^2\right)\right)^{1/2}(1+T|\Omega|)^{1/2}.$$

Hence,

$$\left\|\frac{\partial U_\epsilon(w)}{\partial t}\right\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leqslant C_3. \qquad (3.24)$$

Taking into account these estimates, we introduce the following subset W_0 of the space $W(0,T)$

$$W_0 = \left\{ z \in W(0,T) \;\middle|\; \begin{array}{c} \|z\|_{L^2(0,T;W^{1,2}(\Omega))} \leqslant \left(1+\frac{1}{\sqrt{\epsilon}}\right)C_1,\\ \|z\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \sqrt{2}C_1,\\ \|\frac{\partial z}{\partial t}\|_{L^2(0,T;(W^{1,2}(\Omega))')} \leqslant C_3,\\ z(0,\cdot) = f_0 \end{array}\right\}$$

In view of estimates (3.20)–(3.24) and condition (3.14), it is clear that $w \in W_0$ and, hence, U_ϵ can be interpreted as a mapping from W_0 into W_0. Moreover, we see that W_0 is a nonempty, convex, and weakly compact subset of $W(0,T)$. So, in order to apply the Schauder fixed-point theorem, we have to show that the mapping U_ϵ is weakly continuous from W_0 into W_0. Since the embedding of $W^{1,2}(\Omega)$ in $L^2(\Omega)$ is compact, a refinement of Aubin's lemma (see, e.g. [40, Section 8, Corollary 4] ensures that any bounded subset of $W(0,T)$ is relatively compact in $L^2(Q_T)$. As a result, the Schauder fixed-point theorem will provide the existence of element u_ϵ in W_0 such that $u_\epsilon = U_\epsilon(u_\epsilon)$.

Step 3. Let $\{w_j\}_{j\in\mathbb{R}}$ be a sequence in W_0 converging weakly in W_0 to some $w \in W_0$. Setting $u_{\epsilon,j} = U_\epsilon(w_j)$ and utilizing the weak compactness of the set W_0, we see that $\{u_{\epsilon,j}\}_{j\in\mathbb{R}}$ contains a subsequence such that

$$u_{\epsilon,j} \rightharpoonup u_\epsilon \quad \text{weakly in } \ L^2(0,T;W^{1,2}(\Omega)), \tag{3.25}$$

$$\frac{\partial u_{\epsilon,j}}{\partial t} \rightharpoonup \frac{\partial u_\epsilon}{\partial t} \quad \text{weakly in } \ L^2(0,T;\left(W^{1,2}(\Omega)\right)'), \tag{3.26}$$

$$u_{\epsilon,j} \to u_\epsilon \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)) \text{ and a.e. in } \ Q_T, \tag{3.27}$$

$$\frac{\partial u_{\epsilon,j}}{\partial x_i} \rightharpoonup \frac{\partial u_\epsilon}{\partial x_i} \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)), \tag{3.28}$$

$$w_j \to w \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)). \tag{3.29}$$

Then Lemmas 2.1 and 2.2 imply that

$$\begin{gathered} D(t,x,w_j) \to D(t,x,w) \text{ and } p_{w_j}(t,x) \to p_w(t,x) \\ \text{uniformly in } \overline{Q_T} \text{ as } j \to \infty. \end{gathered} \tag{3.30}$$

Moreover, taking into account that

$$\begin{aligned} \|A_{w_j}(t,x,\nabla u_{\epsilon,j})\|^{\beta'}_{L^{\beta'}(Q_T;\mathbb{R}^N)} &\overset{\text{by (2.26)}}{\leqslant} (1+T|\Omega|)\, \|A_{w_j}(t,x,\nabla u_{\epsilon,j})\|^{\beta'}_{L^{p'_{w_j}(\cdot)}(Q_T;\mathbb{R}^N)} \\ &\overset{\text{by (2.24)}}{\leqslant} (1+T|\Omega|) \left(1 + \int_{Q_T} |A_{w_j}(t,x,\nabla u_{\epsilon,j})|^{p'_{w_j}(t,x)}\, dxdt\right) \\ &\overset{\text{by (3.7)}}{\leqslant} (1+T|\Omega|) \left(1 + d_2^2 \int_{Q_T} |\nabla u_{\epsilon,j}|^{p_{w_j}}\, dxdt\right) \overset{\text{by (3.21)}}{<} \infty, \end{aligned} \tag{3.31}$$

we deduce from (3.23) and (2.22) that the sequence $\left\{\epsilon \nabla u_{\epsilon,j} + A_{w_j}(t,x,\nabla u_{\epsilon,j})\right\}_{j\in\mathbb{R}}$ is bounded in $L^{\beta'}(Q_T;\mathbb{R}^N)$. Hence, we can suppose that there exists an element $z \in L^{\beta'}(Q_T;\mathbb{R}^N)$ such that

$$\epsilon \nabla u_{\epsilon,j} + A_{w_j}(t,x,\nabla u_{\epsilon,j}) \rightharpoonup z \quad \text{weakly in } L^{\beta'}(Q_T;\mathbb{R}^N) \text{ as } j \to \infty. \tag{3.32}$$

Utilizing this fact together with the properties

$$\begin{aligned} u_{\epsilon,j} \rightharpoonup u_\epsilon \ \text{ in } L^\alpha(0,T;W^{1,\alpha}(\Omega)) \text{ with } \alpha = 1+\delta \text{ (by (3.25))}, \\ \{u_{\epsilon,j}\}_{j\in\mathbb{N}} \ \text{ are bounded in } L^\infty(0,T;L^2(\Omega)) \text{ (by (3.22))}, \\ u_{\epsilon,j} \in L^\beta(0,T;W^{1,\beta}(\Omega))\ \forall\, j \in \mathbb{N} \ \text{ by (3.23)}, \\ \sup_{j\in\mathbb{N}} \|\left(A_{w_j}(t,x,\nabla u_{\epsilon,j}), \nabla u_{\epsilon,j}\right)\|_{L^1(Q_T)} < \infty \text{ (by (3.17))}, \end{aligned} \tag{3.33}$$

and taking into account that $1 < 1+\delta = \alpha < \beta = 2 < 2\alpha$, we deduce from Theorem 2.2 that the flow $A_{w_j}(t,x,\nabla u_{\epsilon,j})$ weakly converges in $L^{\beta'}(Q_T;\mathbb{R}^N)$ to the flow $A_w(t,x,\nabla u_\epsilon)$, i.e., $z = A_w(t,x,\nabla u_\epsilon)$.

Then we can pass to the limit in relations (3.15)–(3.16) with $u = u_{\epsilon,j}$ and $w = w_j$ as $j \to \infty$. This yields

$$\left\langle \frac{\partial u_\epsilon(t)}{\partial t}, v \right\rangle + \int_\Omega [\epsilon (\nabla u_\epsilon(t), \nabla v) + (A_w(t,x,\nabla u_\epsilon(t)), \nabla v) + u_\epsilon(t) v] \, dx$$

$$= \int_\Omega f(t) v \, dx, \quad \forall\, v \in W^{1,2}(\Omega) \quad \text{a.e. in } [0,T], \tag{3.34}$$

$$u_\epsilon(0) = f_0, \tag{3.35}$$

i.e., $u_\epsilon = U_\epsilon(w)$. Moreover, since variational problem (3.34)–(3.35) has a unique solution, it follows that the whole sequence $\{u_{\epsilon,j}\}_{j\in\mathbb{R}}$ converges weakly in $W(0,T)$ to $u_\epsilon = U_\epsilon(w)$.

Thus, the mapping $U_\epsilon : W_0 \mapsto W_0$ is weakly continuous and, hence, by the Schauder fixed point theorem, u_ϵ is a weak solution of the perturbed problem (3.11)–(3.13). □

Remark 3.1. Since the uniqueness of weak solutions of the perturbed problem (3.11)–(3.13) seems to be an open question, we call the weak solution u_ϵ, given by Theorem 3.1, just W_0-attainable.

Corollary 3.1. *For any $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and each positive value $\epsilon > 0$, the Cauchy-Neumann problem* (3.11)–(3.13) *admits at least one W_0-attainable weak solution.*

As a direct consequence of the previous results, we can indicate the following additional properties of W_0-attainable solutions.

Corollary 3.2. *Let $f \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, and $\epsilon > 0$ be given. Let $u_\epsilon \in W(0,T)$ be a W_0-attainable weak solution of* (3.11)–(3.13). *Then u_ϵ is a weak solution to this problem in the sense of Definition 3.1, and the following energy equality holds*

$$\frac{1}{2}\int_\Omega u_\epsilon^2 \, dx + \int_0^t \int_\Omega \left(\epsilon |\nabla u_\epsilon|^2 + (A_{u_\epsilon}(t,x,\nabla u_\epsilon), \nabla u_\epsilon) + u_\epsilon^2 \right) dxdt$$

$$= \int_{Q_T} f u_\epsilon \, dxdt + \int_\Omega f_0^2 \, dx \quad \forall\, t \in [0,T]. \tag{3.36}$$

Proof. As follows from (3.18), if u_ϵ is a weak solution of (3.11)–(3.13) in the distributional sense, then u_ϵ satisfies the integral identity

$$\int_{Q_T} \left(-u_\epsilon \frac{\partial \varphi}{\partial t} + \epsilon (\nabla u_\epsilon, \nabla \varphi) + (A_{u_\epsilon}(t,x,\nabla u_\epsilon), \nabla \varphi) + u_\epsilon \varphi \right) dxdt$$

$$= \int_{Q_T} f \varphi \, dxdt + \int_\Omega f_0 \varphi|_{t=0} \, dx \quad \forall\, \varphi \in \Phi. \tag{3.37}$$

Therefore, in order to show that u_ϵ is a weak solution, it's enough to establish the inclusion $u_\epsilon \in W_{u_\epsilon}(Q_T)$. Taking into account the definition of the space $W_{u_\epsilon}(Q_T)$ (see (3.8)), let us show that

$$\int_{Q_T} |D(t,x,u_\epsilon)\nabla u_\epsilon|^{p_{u_\epsilon}(t,x)}\,dxdt < +\infty \quad \text{for a.e.} t \in [0,T].$$

Indeed, since u_ϵ is a W_0-attainable solution of (3.11)–(3.13), it follows that there exists a sequence $\{u_{\epsilon,j}\}_{j\in\mathbb{R}} \in W_0$ with properties (3.25)–(3.28) and such that $u_{\epsilon,j} = U_\epsilon(u_{\epsilon,j-1})$, for $j = 2,3,\dots$. Then this sequence possesses the properties (3.32)–(3.33). Hence, $\nabla u_{\epsilon,j} \in L^1(Q_T;\mathbb{R}^N)$ by (3.25), and

$$\begin{aligned}
&\int_{Q_T} \left[|\nabla u_{\epsilon,j}|^{p_{u_{\epsilon,j}}} + \epsilon|\nabla u_{\epsilon,j}|^\beta\right] dxdt \\
&\quad\leqslant \max\left\{1, \frac{d_2}{d_1^2}\right\} \int_{Q_T} \left[\frac{d_1^2}{d_2}|\nabla u_{\epsilon,j}|^{p_{u_{\epsilon,j}}} + \epsilon|\nabla u_{\epsilon,j}|^\beta\right] dxdt \\
&\quad\leqslant \max\left\{1, \frac{d_2}{d_1^2}\right\} \sup_{j\in\mathbb{N}} \int_{Q_T} \left[|\left(A_{u_{\epsilon,j-1}}(t,x,\nabla u_{\epsilon,j}), \nabla u_{\epsilon,j}\right)| + \epsilon|\nabla u_{\epsilon,j}|^2\right] dxdt \\
&\quad\overset{\text{by (3.6),(3.33)}}{<} \infty, \qquad\qquad (3.38)
\end{aligned}$$

$$|\nabla u_{\epsilon,j}|^{p_{u_{\epsilon,j}}} + \epsilon|\nabla u_{\epsilon,j}|^\beta \overset{\text{by (3.32)}}{\rightharpoonup} z \quad \text{in} \quad L^{\beta'}(Q_T). \qquad (3.39)$$

As a result, Lemma 2.5 implies that $z \in L^{p_{u_\epsilon}(\cdot)}(Q_T)$. Therefore,

$$\begin{aligned}
\int_{Q_T} |D(t,x,u_\epsilon)\nabla u_\epsilon|^{p_{u_\epsilon}(t,x)}\,dxdt &\leqslant d_2^2 \int_{Q_T} |\nabla u_\epsilon|^{p_{u_\epsilon}(t,x)}\,dxdt \\
&\overset{\text{by (2.24)}}{\leqslant} d_2^2\left(\|\nabla u_\epsilon\|^\beta_{L^{p_{u_\epsilon}(\cdot)}(Q_T)} + 1\right) \\
&\overset{\text{by (3.21)}}{\leqslant} d_2^2\,(C_2+1) < +\infty. \qquad (3.40)
\end{aligned}$$

Thus, $u_\epsilon \in W_{u_\epsilon}(Q_T)$ and, therefore, in view of Definition 3.1, u_ϵ is a weak solution to the problem (3.11)–(3.13).

It remains to prove the energy equality (3.36). Since u_ϵ is in $W(0,T)$ and the set of test functions $C^\infty([0,T];C_c^\infty(\mathbb{R}^N))$ is dense in $L^2(0,T;W^{1,2}(\Omega))$, it follows that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C^\infty([0,T];C_c^\infty(\mathbb{R}^N))$ such that

$$\varphi_j \to u_\epsilon \quad \text{in } L^2(0,T;W^{1,2}(\Omega)) \quad \text{as } j\to\infty. \qquad (3.41)$$

Taking into account that, for each $j \in \mathbb{N}$, the integral identity

$$\begin{aligned}
&\int_0^t \int_\Omega \left[\epsilon\left(\nabla u_\epsilon(t), \nabla\varphi_j(t)\right) + \left(A_{u_\epsilon}(t,x,\nabla u_\epsilon(t)), \nabla\varphi_j(t)\right) + u_\epsilon(t)\varphi_j(t)\right] dxdt \\
&+ \int_0^t \left\langle \frac{\partial u_\epsilon(t)}{\partial t}, \varphi_j(t)\right\rangle dt = \int_0^t\int_\Omega f(t)\varphi_j(t)\,dxdt, \quad \forall\, j\in\mathbb{N},\ \forall\, t\in[0,T] \qquad (3.42)
\end{aligned}$$

holds true, we can pass to the limit in (3.42) as $j \to \infty$. To do so, we notice that

$$\int_0^t \int_\Omega (A_{u_\epsilon}(t,x,\nabla u_\epsilon(t)), \nabla \varphi_j(t))\, dxdt = \int_0^t \int_\Omega (A_{u_\epsilon}(t,x,\nabla u_\epsilon(t)), \nabla u_\epsilon(t))\, dxdt$$
$$+ \int_0^t \int_\Omega (A_{u_\epsilon}(t,x,\nabla u_\epsilon(t)), \nabla \varphi_j(t) - \nabla u_\epsilon(t))\, dxdt,$$

where $\nabla\varphi_j - \nabla u_\epsilon \to 0$ a.e. in Q_T by (3.41), and

$$|(A_{u_\epsilon}(t,x,\nabla u_\epsilon), \nabla\varphi_j - \nabla u_\epsilon)| \leqslant c_1 |\nabla u_\epsilon|^{p_{u_\epsilon}} + \frac{1}{\alpha}|\nabla\varphi_j - \nabla u_\epsilon|^{p_{u_\epsilon}} \in L^1(Q_T)$$

by (3.17). Hence, utilizing the Lebesgue dominated theorem, the limit passage in (3.42) leads to the equality

$$\int_0^t \int_\Omega \left[\epsilon\,(\nabla u_\epsilon(t), \nabla u_\epsilon(t)) + (A_w(t,x,\nabla u_\epsilon(t)), \nabla u_\epsilon(t)) + u_\epsilon^2(t)\right] dxdt$$
$$+ \int_0^t \left\langle \frac{\partial u_\epsilon(t)}{\partial t}, u_\epsilon(t)\right\rangle dt = \int_0^t \int_\Omega f(t) u_\epsilon(t)\, dxdt, \quad \forall\, t \in [0,T]. \quad (3.43)$$

Thus, to obtain the energy equality (3.36), it remains to use the integration by parts formula. □

Before proceeding further, we make use of the following result.

Lemma 3.1. *Let $\{v_\epsilon\}_{\epsilon\to 0} \subset W(0,T)$ be a sequence such that*

$$\sup_{\epsilon\to 0}\left(\epsilon \int_{Q_T} |\nabla u_\epsilon|^2\, dxdt\right) < +\infty. \quad (3.44)$$

Then $\epsilon\nabla u_\epsilon \rightharpoonup 0$ in $L^2(Q_T;\mathbb{R}^N)$.

Proof. Let $\varphi \in C_0^\infty(Q_T)$ be an arbitrary vector-function. Then

$$\left|\int_{Q_T} (\epsilon\nabla u_\epsilon, \varphi)\, dxdt\right| \leqslant \sqrt{\epsilon}\left(\int_{Q_T} \epsilon|\nabla u_\epsilon|^2\, dxdt\right)^{1/2}\left(\int_{Q_T} |\varphi|^2\, dxdt\right)^{1/2}.$$

Hence, the sequence $\{\epsilon\nabla u_\epsilon\}_{\epsilon\to 0}$ is bounded in $L^2(Q_T;\mathbb{R}^N)$. As a result, we have

$$\left|\int_{Q_T} (\epsilon\nabla u_\epsilon, \varphi)\, dxdt\right| \overset{\text{by (3.44)}}{\leqslant} C\sqrt{\epsilon}\left(\int_{Q_T} \epsilon|\nabla u_\epsilon|^2\, dxdt\right)^{1/2} \leqslant \widehat{C}\sqrt{\epsilon} \to 0.$$

The proof is complete. □

We are now in a position to prove the main result of this section.

Theorem 3.2. *Let $f \in L^2(Q_T)$ and $f_0 \in L^2(\Omega)$ be given distributions. Then initial-boundary value problem* (3.1)–(3.3) *admits at least one weak solution $u \in W_u(Q_T)$.*

Proof. Let ϵ be a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. Let $\{u_\epsilon \in W(0,T)\}_{\epsilon\to 0}$ be a sequence of W_0-attainable weak solutions to the approximating problem (3.11)–(3.13). Then, for each $\epsilon > 0$, u_ϵ satisfies the integral identity (3.37) and the energy equality (3.36). Hence, we can deduce from (3.36) the following estimates

$$\sup_{\varepsilon>0} \|u_\epsilon\|^2_{L^2(Q_T)} \leqslant C_1^2 = \|f\|^2_{L^2(Q_T)} + 2\|f_0\|^2_{L^2(\Omega)}, \tag{3.45}$$

$$\begin{aligned}
\sup_{\varepsilon>0} \|\nabla u_\epsilon\|_{L^{p_{u_\epsilon}(\cdot)}(Q_T;\mathbb{R}^N)} &\overset{\text{by (2.24)}}{\leqslant} \sup_{\varepsilon>0}\left(\int_{Q_T} |\nabla u_\epsilon|^{p_{u_\epsilon}(t,x)}\,dxdt + 1\right)^{1/\alpha}\\
&\overset{\text{by (3.36)}}{\leqslant} \sup_{\varepsilon>0}\left(\frac{d_2}{d_1^2}\left(\|f_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(Q_T)} + \frac{1}{2}\|u_\epsilon\|^2_{L^2(Q_T)}\right) + 1\right)^{1/\alpha}\\
&\overset{\text{by (3.45)}}{\leqslant} C_2 = \left(\frac{d_2}{d_1^2}C_1^2 + 1\right)^{1/\alpha},
\end{aligned} \tag{3.46}$$

$$\begin{aligned}
\sup_{\varepsilon>0} \|\nabla u_\epsilon\|_{L^{\alpha}(Q_T;\mathbb{R}^N)} &\overset{\text{by (2.26)}}{\leqslant} \sup_{\varepsilon>0} (1 + T|\Omega|)^{1/\alpha} \|\nabla u_\epsilon\|_{L^{p_{u_\epsilon}(\cdot)}(Q_T;\mathbb{R}^N)}\\
&\overset{\text{by (3.46)}}{\leqslant} (1 + T|\Omega|)^{1/\alpha} C_2,
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
\sup_{\varepsilon>0} \|u_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} &\leqslant \sup_{\varepsilon>0} \sqrt{2\left(\|f_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(Q_T)} + \frac{1}{2}\|u_\epsilon\|^2_{L^2(Q_T)}\right)}\\
&\leqslant \sqrt{2}C_1,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\|\nabla u_\epsilon\|_{L^2(Q_T;\mathbb{R}^N)} &\leqslant \frac{1}{\sqrt{\epsilon}}\left(\|f_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|f\|^2_{L^2(Q_T)} + \frac{1}{2}\|u_\epsilon\|^2_{L^2(Q_T)}\right)\\
&\leqslant \frac{1}{\sqrt{\epsilon}} C_1^2.
\end{aligned} \tag{3.49}$$

Before proceeding further, it is worth to emphasize that the exponent α in (3.47) is given by the rule

$$\begin{aligned}
\alpha &\overset{\text{by (2.5)}}{=} 1 + \delta\\
&\overset{\text{by (2.8)}}{=} ah\left[ah + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega| \sup_{\epsilon>0} \|u_\epsilon\|^2_{L^2(0,T;L^2(\Omega))}\right]^{-1}\\
&\overset{\text{by (3.45)}}{\geqslant} ah\left[ah + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}|\Omega| \left(\|f\|^2_{L^2(Q_T)} + 2\|f_0\|^2_{L^2(\Omega)}\right)\right]^{-1}
\end{aligned}$$

Taking into account these estimates, we see that the sequence $\{u_\epsilon\}_{\epsilon\to 0}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and in $L^\alpha(0,T;W^{1,\alpha}(\Omega))$. Therefore, there exists an element

$$u \in L^\alpha(0,T;W^{1,\alpha}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \tag{3.50}$$

such that, up to a subsequence, $u_\epsilon \rightharpoonup u$ in $L^\alpha(0,T;W^{1,\alpha}(\Omega))$ as $\epsilon \to 0$. Moreover, the uniform boundedness of fluxes $\{A_{u_\epsilon}(t,x,\nabla u_\epsilon)\}_{\epsilon\to 0}$ in $L^{\beta'}(Q_T;\mathbb{R}^N)$ with respect to $\epsilon > 0$ implies that this sequence is sequentially weakly compact in $L^{\beta'}(Q_T;\mathbb{R}^N)$ (for arguments see (3.31)). Hence, we may suppose the existence of some vector-function w such that $w_\epsilon = A_{u_\epsilon}(t,x,\nabla u_\epsilon) \rightharpoonup w$ in $L^{\beta'}(Q_T;\mathbb{R}^N)$ as $\epsilon \to 0$. Arguing as in the proof of Theorem 3.1, it can be shown that $\sup_{\epsilon\to 0} \|\,(w_\epsilon, \nabla u_\epsilon)\,\|_{L^1(Q_T)} < \infty$. As a result, Theorem 2.2 implies that the flow $A_{u_\epsilon}(t,x,\nabla u_\epsilon)$ weakly converges in $L^{\beta'}(Q_T;\mathbb{R}^N)$ to the flow $w = A_u(t,x,\nabla u)$. Then, utilizing Lemma 3.1, we see that the passage to the limit in the integral identity (3.37) leads to a similar identity for equation (3.1). It remains to take into account Lemma 2.5 and relations (3.38)–(3.40) in order to deduce that $\displaystyle\int_{Q_T} |D(t,x,u)\nabla u|^{p_u(t,x)}\,dxdt < +\infty$. Thus, u is an element of the space $W_u(Q_T)$ and, as a consequence, u is a weak solution to the problem (3.1)–(3.3). □

We can supplement the result on Theorem 3.2 with the following assertions.

Corollary 3.3. *Let $u \in W_u(Q_T)$ be a weak solution to the problem* (3.1)–(3.3) *that has been obtained as a cluster point of W_0-attainable solutions $\{u_\epsilon \in W(0,T)\}$. Then the following energy inequality*

$$\frac{1}{2}\int_\Omega u^2\,dx + \int_0^t\int_\Omega \big((A_u(t,x,\nabla u),\nabla u) + u^2\big)\,dxdt \leqslant \int_{Q_T} fu\,dxdt + \int_\Omega f_0^2\,dx \quad (3.51)$$

holds true for all $t \in [0,T]$.

Proof. To deduce this inequality it is enough to pass to the limit in relation (3.36) as $\epsilon \to 0$ using the weak convergence $u_\epsilon \rightharpoonup u$ in $L^\alpha(0,T;W^{1,\alpha}(\Omega))$ and utilize Lemma 2.4 and the weak convergence of fluxes to flux (Theorem 2.2). □

Corollary 3.4. *Let $N = 2$ and let $u \in W_u(Q_T)$ be a weak solution to the problem* (3.1)–(3.3) *that has been obtained as a cluster point of W_0-attainable solutions $\{u_\epsilon \in W(0,T)\}_{\epsilon\to 0}$. Then the following higher integrability property is valid:*

$$u \in L^{2\alpha}(Q_T) \quad \text{with estimate} \quad \|u\|_{L^{2\alpha}(Q_T)} \leqslant \left(\sqrt{2}C_1C_2\,(1+T|\Omega|)^{\frac{1}{\alpha}}\right)^{\frac{1}{2}}. \quad (3.52)$$

where the exponent $\alpha > 1$ is given by (2.5)*, and the constants C_1 and C_2 are defined in* (3.45) *and* (3.46)*, respectively.*

Proof. As follows from a priori estimates (3.47)–(3.48), the given weak solution satisfies inclusion (3.50). Hence, in the case $N = 2$ the announced embedding result together with estimate (3.52) is a direct consequence of Corollary 2.1. □

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