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SHAPE, VELOCITY, AND EXACT CONTROLLABILITY FOR THE WAVE EQUATION

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Abstract. A new method to prove exact controllability for the wave equation is demonstrated and discussed on several examples. The method of proof first uses a dynamical argument to prove shape controllability and velocity controllability, thereby solving their associated moment problems. This enables one to solve the moment problem associated to exact controllability.

Key words: Exact controllability, wave equation, shape controllability, velocity controllability, moment problem.

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1. Introduction

Controllability properties of the wave equation is a central topic of the control theory of partial differential equations. A large number of papers describe many powerful methods, which prove controllability of the wave equation in various spatial domains under the action of various types of controls (see, e.g. [25], [24], [23], [26] and references therein). In this paper we describe an approach that is based on the relationship between exact controllability, on one hand, and shape and velocity controllability on the other hand. This relationship was used in [4] for a vector wave equation, in [5], [6] for a string with attached point masses, and in [15] for the wave equation on a metric tree graph. In the present paper we consider three control problems for the wave equation: on an interval with one and two boundary controls, and on a graph with cycle — a ring with two attached edges.

The purpose of the first two examples is partly methodological. We demonstrate how the known (we can say classical) results in PDE control theory can be obtained in a much more simple way using a new approach. The third example contains a new result in control theory for PDEs on metric graphs. Control problems for the wave equation on graphs have important applications in science and engineering and were studied in many papers (see the monographs [10], [20], [22];

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the surveys [2], [27]; and references therein). They also have a deep connection with inverse problems on graphs, see, e.g. [17], [13], [1], [7], [21]. In this paper we consider exact controllability for the wave equation of the form

$$u_{tt} - u_{xx} + q(x)u = 0.$$

There is a growing body of work in the case where the graph is a tree, i.e. a graph without cycles, and the controls are assumed to act on the boundary. Typically, the so-called Kirchhoff-Neumann (KN) conditions are assumed at all interior vertices. This problem was studied, e.g. in [18], [20], [22], [15] (in those papers the problem was stated in slightly different forms). It was proved that the system is exactly controllable if the control functions act at all or at all but one of the boundary vertices.

In the case of graphs with cycles, there are only few results concerning exact controllability [8, 9, 16]. In these papers it was assumed that the controls and solutions are pretty regular. In the present paper we consider the classical for PDE control theory case of L^2 Dirichlet type controls.

To reach the exact controllability of systems on graphs with cycles we need to use not only boundary but also interior controls, as was proposed in [3], for the graph, denoted Ω, consisting of a ring with two attached edges, see Figure 1.1. First we prove the shape and velocity controllability using the dynamical method — we reduce these problems to the Volterra integral equations of the second kind. Then we prove exact controllability using the spectral approach — the method of moments and properties of exponential families.

Let $l, T > 0$. We consider the following initial boundary value problem (IBVP):

$$u_{tt} - u_{xx} + q(x)u = 0, \ (x,t) \in (0,l) \times (0,T), \tag{1.1}$$

$$u(x,0) = u_t(x,0) = 0, \ x \in (0,l), \tag{1.2}$$

$$u(0,t) = f(t), \ u(l,t) = 0, \ t \in (0,T). \tag{1.3}$$

Here q is a continuous real valued function on $[0,l]$, $f \in H^1_*(0,T) := \{h \in H^1(0,T) : h(0) = 0\}$. In what follows, we will refer to f as a Dirichlet boundary control. For any given f, this IBVP has a unique solution $u = u^f(x,t)$ such that

$$u \in C(0,T; H^1(0,l)), \ u_t \in C(0,T; L^2(0,l)).$$

The following theorem concerning controllability of the system (1.1)–(1.3) is known, see, e.g. [11], however, its proof is not elementary and requires using delicate results of nonharmonic Fourier series.

Theorem 1.1. *Let $(\phi, \psi) \in H^1_0(0,l) \times L^2(0,l)$ and $T \geq 2l$. There exists a control $f \in H^1_0(0,T)$ such that the solution u^f to the IBVP (1.1)–(1.3) satisfies the equalities*

$$u^f(\cdot, T) = \phi, \ u^f_t(\cdot, T) = \psi. \tag{1.4}$$

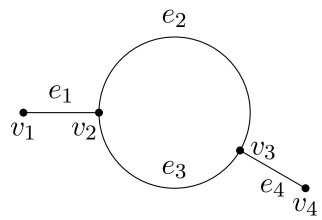

Fig. 1.1. A ring with two attached edges

We will present an elementary proof of this theorem and demonstrate that our approach can be extended to much more general situations. A key ingredient in the proof of Theorems 1 are shape and velocity control results.

Theorem 1.2.

a) (Shape control) Let $T \geq l$ and $\phi \in H_0^1(0,l)$. There exists a control $f \in H_0^1(0,T)$ such that $u^f(\cdot,T) = \phi$.

b) (Velocity control). Let $T \geq l$ and $\psi \in L^2(0,l)$. There exists a control $f \in H_*^1(0,T)$ such that $u_t^f(\cdot,T) = \psi$.

The second problem we choose to demonstrate our approach is the IBVP described by equations (1.1), (1.2) with two boundary controls, Dirichlet and Neumann:

$$u(0,t) = f(t),\ u_x(l,t) = g(t),\ t \in (0,T),\ f \in H_*^1(0,T),\ g \in L^2(0,T). \tag{1.5}$$

Theorem 1.3.

a) Let $\phi \in H_*^1(0,l)$ and $T > l/2$. There exist controls $f \in H_0^1(0,T)$, $g \in L^2(0,T)$ such that the solution $u^{f,g}$ to the IBVP (1.1), (1.2), (1.5) satisfies the equality $u^{f,g}(\cdot,T) = \phi$.

b) Let $\psi \in L^2(0,l)$ and $T \geq l/2$. There exist controls $f \in H_*^1(0,T)$, $g \in L^2(0,T)$ such that $u^{f,g}(\cdot,T) = \psi$.

c) Let $(\phi,\psi) \in H_*^1(0,l) \times L^2(0,l)$ and $T > l$. There exist controls $f \in H_0^1(0,T)$, $g \in L^2(0,T)$ such that $u^{f,g}(\cdot,T) = \phi$, $u_t^{f,g}(\cdot,T) = \psi$.

A similar problem with two Dirichlet controls was studied in [10, Sec. VII.4]. The statement c) of Theorem 1.3 can be proved following the methods developed there methods based on the theory of vector exponential functions [10]. In the present paper we demonstrate a much more simple proof of this theorem.

In the last example we consider the wave equation on the graph shown in Figure 1.1. Our graph $\Omega = \{V, E\}$ consists of four vertices, $V = \{v_j,\ j = 1,\dots,4\}$, and four edges, $E = \{e_j,\, j = 1, \dots, 4\}$. Denote the length of e_j by l_j. In what follows, we denote by ϕ_j the restriction of function ϕ to the edge e_j. Assume q_j is a uniformly continuous, real valued function for each j. We denote the space $L^2(\Omega)$ by $\mathcal{H}$. Let $\mathcal{H}^1$ be the space of continuous functions on Ω whose restriction to each edge e is in $H^1(e)$. Define $\mathcal{H}^1_0$ as the set of $\phi \in \mathcal{H}^1$ such that

$$\phi_1(v_1) = \phi_4(v_4) = 0.$$

We set its norm by $\|\phi\|^2_{\mathcal{H}^1} = \sum_{j=1}^4 \|\phi'_j\|^2_{L^2(e_j)} + \|\phi_j\|^2_{L^2(e_j)}$. Let $\mathcal{H}^{-1}$ be the dual space of $\mathcal{H}^1_0$. In what follows, we denote by $\partial\phi_j(v_i)$ the derivative of ϕ_j at v_i in direction away from v_i. We define $\mathcal{H}^2$ as the subset of $\mathcal{H}^1$ such that

$$\phi \in \mathcal{H}^2 \text{ if and only if } \phi_j \in H^2(e_j) \text{ and } \sum_{j=1}^{3} \partial\phi_j(v_2) = \sum_{j=2}^{4} \partial\phi_j(v_3) = 0.$$

The self-adjoint operator L in x associated to our wave equation has operator domain $\mathcal{H}^2 \cap \mathcal{H}^1_0$ and acts on functions by the rule

$$(L\phi)(x) = -\phi''(x) + q(x)\phi(x),\ x \in \Omega \setminus V.$$

To motivate our control problem for the wave equation on this graph it is convenient to start with the observation problem. We consider the following IBVP on $[\Omega \setminus V] \times [0, T]$

$$w_{tt} - w_{xx} + q(x)w = 0, \tag{1.6}$$

$$w|_{t=0} = w^0 \in \mathcal{H}^1_0,\ w_t|_{t=0} = w^1 \in \mathcal{H}, \tag{1.7}$$

$$w_1(v_2, t) = w_2(v_2, t) = w_3(v_2, t),\ \sum_{j=1}^{3} \partial w_j(v_2, t) = 0. \tag{1.8}$$

$$w_2(v_3, t) = w_3(v_3, t) = w_4(v_3, t),\ \sum_{j=2}^{4} \partial w_j(v_3, t) = 0. \tag{1.9}$$

Conditions (1.8) and (1.9) are called the Kirchhoff–Neumann (KN) conditions. Using the Fourier method one can show, following the approach presented in [14], that this IBVP has a unique generalized solution such that for any $i, j = 1, 2, 3, 4$,

$$\begin{gathered} w \in C([0, T]; \mathcal{H}^1_0),\ w_t \in C([0, T]; \mathcal{H}), \\ w(v_i, \cdot) \in H^1(0, T),\ \partial w_j(v_i, \cdot) \in L^2(0, T). \end{gathered} \tag{1.10}$$

We introduce two observations: $\partial w_1(v_1, t), \partial w_2(v_2, t),\ t \in [0, T]$. We say that the system (1.6)–(1.9) is observable in time T if there is a positive constant C, independent of w^0, w^1, such that

$$\|\partial w_1(v_1, \cdot)\|^2_{L^2(0,T)} + \|\partial w_2(v_2, \cdot)\|^2_{L^2(0,T)} \geq C \left\{ \|w^0\|^2_{\mathcal{H}^1} + \|w^1\|^2_{\mathcal{H}} \right\} \tag{1.11}$$

for every $w^0 \in \mathcal{H}^1,\ w^1 \in \mathcal{H}$. We emphasize that our observations are feasible and nondestructive, and hence the corresponding dual control problem is natural.

To state the exact controllability result that is equivalent by duality to the observability inequality (1.11), we consider the following system:

$$u_{tt} - u_{xx} + q(x)u = 0, \tag{1.12}$$

$$u(\cdot, 0) = u_t(\cdot, 0) = 0, \tag{1.13}$$

$$u_1(v_1, t) = f_1(t), \quad u_4(v_4, t) = 0, \tag{1.14}$$

$$u_2(v_2, t) = u_1(v_2, t) + f_2(t),\ u_3(v_2, t) = u_1(v_2, t);\ \sum_{j=1}^{3} \partial u_j(v_2, t) = 0; \tag{1.15}$$

$$u_j(v_3, t) = u_k(v_3, t),\ j \neq k; \quad \sum_{j=2}^{4} \partial u_j(v_3, t) = 0. \tag{1.16}$$

Here $f_1,\ f_2 \in L^2(0, T)$. Evidently, the control f_2 should have the effect of breaking the symmetries, which obstruct controllability, that are caused by the cycle.

Well-posedness of this system is proven in [14], where it is shown that for any $T > 0$, the solution u satisfies

$$u \in C(0, T; \mathcal{H}) \cap C^1(0, T; \mathcal{H}^{-1}).$$

Without loss of generality, in what follows we will assume that $l_2 \geq l_3$.

Theorem 1.4. *Assume the operator kernel of L is trivial. Let*

$$T_* = \max\{\, l_1 + l_2,\, l_3 + l_4 \,\}, \quad T \geq 2T_*,$$

and $(\phi, \psi) \in \mathcal{H} \times \mathcal{H}^{-1}$. *There exist controls*

$$\mathbf{f} := (f_1, f_2) \in L^2(0, T) \times L^2(0, T)$$

such that the solution $u^{\mathbf{f}}$ *to System (1.12)-(1.16) satisfies*

$$u^{\mathbf{f}}(\cdot, T) = \phi,\ u_t^{\mathbf{f}}(\cdot, T) = \psi, \tag{1.17}$$

and there exists a constant C *that depends only on* $q, l_j, j = 1, \ldots, 4,$ *such that*

$$\|f_1\|_{L^2(0,T)} + \|f_2\|_{L^2(0,T)} \leq C\, (\|\phi\|_{\mathcal{H}} + \|\psi\|_{\mathcal{H}^{-1}}). \tag{1.18}$$

Key ingredients in the proof of Theorem 1.4 are shape and velocity control results that can be formulated as follows.

Theorem 1.5. *a) (Shape control) Let* $T \geq T_*$ *and* $\phi \in \mathcal{H}$. *There exist controls* $\mathbf{f} := (f_1, f_2) \in L^2(0, T) \times L^2(0, T)$ *such that*

$$u^{\mathbf{f}}(\cdot, T) = \phi,$$

and there exists a constant C *that depends only on* $q, l_j, j = 1, \ldots, 4,$ *such that*

$$\|f_1\|_{L^2(0,T)} + \|f_2\|_{L^2(0,T)} \leq C\, \|\phi\|_{\mathcal{H}}. \tag{1.19}$$

b) *(Velocity control). Assume the operator L has trivial kernel. Let $T \geq T_*$ and $\psi \in \mathcal{H}^{-1}$. There exist controls* $\mathbf{f} := (f_1, f_2) \in L^2(0,T) \times L^2(0,T)$ *such that*

$$u_t^{\mathbf{f}}(\cdot, T) = \psi,$$

and there exists a constant C that depends only on $q, l_j, j = 1, \ldots, 4$, such that

$$\|f_1\|_{L^2(0,T)} + \|f_2\|_{L^2(0,T)} \leq C\, \|\psi\|_{\mathcal{H}^{-1}}. \tag{1.20}$$

2. Proof of Main Results

2.1. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.2. When $T \leqslant l$ and $f \in L_2(0,T)$, it is well known that the IBVP (1.1)–(1.3) has a unique generalized solution u^f presented in the form

$$u^f(x,t) = \begin{cases} 0, & 0 < t < x \\ f(t-x) + \displaystyle\int_x^t k(x,s) f(t-s)\, ds, & x \leqslant t. \end{cases} \tag{2.1}$$

Here $k(x,t)$ is a solution to the Goursat problem

$$\begin{cases} k_{tt} - k_{xx} + q(x)k = 0, & 0 < x < t, \\ k(0,t) = 0, k(x,x) = -\dfrac{1}{2}\displaystyle\int_0^x q(s)\, ds \end{cases} \tag{2.2}$$

that can be found by a standard iteration method (see, e.g., [12] for $q \in L^1(0,l)$).

Then the equation $u^f(x,l) = \phi(x)$ can be written as a Volterra integral equation of the second kind (VESK):

$$f(l-x) + \int_x^l k(x,s) f(l-s)\, ds = \phi(x). \tag{2.3}$$

For any $\phi \in H_0^1(0,l)$, this equation has a unique solution $f \in H_0^1(0,l)$, which proves the shape controllability for $T = l$. Then the system is clearly shape controllable for any $T \geq l$.

Using equation (2.1), the velocity control problem for $T = l$ is reduced to solvability of the VESK

$$f'(l-x) + \int_x^l k(x,s) f'(l-s)\, ds = \psi(x). \tag{2.4}$$

For any $\psi \in L^2(0,l)$, this equation has a unique solution $f' \in L^2(0,l)$. That gives a unique $f \in H_*^1(0,l) : f(t) = \int_0^t f'(s)\, ds$. It proves the velocity controllability for $T = l$, and hence for all $T \geq l$.

Proof of Theorem 1.3 (ab). For t, l, the solution $u^{f,g}$ to the IBVP (1.1), (1.2), (1.5) can be presented in the form

$$u^{f,g}(x,t) = f(t-x) + \int_x^t k(x,s) f(t-s)\, ds \\ +G(t-l+x) + \int_{l-x}^t r(l-x,s)\, G(t-s)\, ds, \tag{2.5}$$

where $G(t) = \int_0^t g$, and r is a solution to the Goursat problem similar to (2.2) with the Neumann boundary condition $r_x(0,s) = 0$. Then from (2.5),

$$u_t^{f,g}(x,t) = f'(t-x) + \int_x^t k(x,s) f'(t-s)\, ds \\ +g(t-l+x) + \int_{l-x}^t r(l-x,s)\, g(t-s)\, ds, \tag{2.6}$$

The proofs of each a) and b) are thus reduced to solving a VESK.

Proof of Theorem 1.1. Now we prove the exact controllability in time $2T$, for $T \geq T_*$, using the shape and velocity controllability in time T. Let $\{\omega_n^2\}_1^\infty$ and $\{\varphi_n\}_1^\infty$ be the eigenvalues and corresponding normalized eigenfunctions of the classical Sturm–Liouville problem

$$-\varphi_n'' + q(x)\varphi_n = \omega_n^2 \varphi_n, \ 0 < x < l, \tag{2.7}$$

$$\varphi_n(0) = \varphi_n(l) = 0. \tag{2.8}$$

We represent the solution to the IBVP (1.1)–(1.3) in the form $u^f(x,t) = \sum_{n=1}^\infty a_n(t)\, \phi_n(x)$. It is well known (see, e.g. [10, Ch. III]) that

$$a_n(T) = \int_0^T f(t)\, \varphi_n'(0)\, \frac{\sin \omega_n (T-t)}{\omega_n}\, dt. \tag{2.9}$$

Therefore, the shape controllability result can be formulated as solvability of the moment problem

$$\omega_n a_n = \int_0^T f_0(t)\, \varphi_n'(0) \sin \omega_n (T-t)\, dt, \ \forall n, \tag{2.10}$$

for arbitrary sequence $\{a_n \omega_n\} \in \ell^2$ by the function $f_0 \in H_0^1(0,T)$.

Differentiating (2.9) with respect to T, we reformulate the velocity controllability result as solvability of the moment problem

$$b_n = \int_0^T f_1(t)\, \varphi_n'(0)\, \cos \omega_n (T-t)]\, dt, \ \forall n, \tag{2.11}$$

for arbitrary sequences $\{b_n\} \in \ell^2$ by the function $f_1 \in H_*^1(0,T)$.

Now we extend the function f_0 in the odd way with respect to $t = T$ from the interval $[0, T]$ to $[0, 2T]$, and the function f_1 — in the even way. We define the functions

$$f = \frac{f_0 + f_1}{2} \in H_0^1(0, 2T)$$

and observe that this function solves the moment problems

$$a_n \omega_n = \int_0^{2T} f(t)\, \varphi_n'(0) \sin \omega_n (T - t)\, dt, \tag{2.12}$$

$$b_n = \int_0^{2T} f(t)\, \varphi_n'(0)\, \cos \omega_n (T - t)]\, dt, \tag{2.13}$$

for arbitrary sequences $\{a_n \omega_n\},\ \{b_n\} \in \ell^2$. By rewriting sine and cosine in the last two equations in terms of complex exponentials, it becomes an easy algebra exercise to check that solvability of the moment problem (2.12), (2.13) is equivalent to solvability of the moment problem

$$\alpha_n = \int_0^{2T} f(t)\, \varphi_n'(0)\, \sin \omega_n (2T - t)\, dt \tag{2.14}$$

$$\beta_n = \int_0^{2T} f(t)\, \varphi_n'(0)\, \cos \omega_n (2T - t)\, dt, \tag{2.15}$$

for arbitrary sequences $\{\alpha_n\}, \{\beta_n\} \in \ell^2$. It's not hard to see that having solved these moment problems is equivalent to proving exact controllability in time $2T$. The proof of Theorem 1.1 is complete.□

Proof of Theorem 1.3 (c) is similar to the proof of Theorem 1.1. We consider the eigenvalue spectral problem

$$-\varphi_n'' + q(x)\varphi_n = \omega_n^2 \varphi_n,\ 0 < x < l, \tag{2.16}$$

$$\varphi_n(0) = \varphi_n'(l) = 0. \tag{2.17}$$

Presenting the solution to the IBVP (1.1), (1.2), (1.5) as a series with respect to eigenfunctions of this problem, $u^{f,g} = \sum_{n=1}^{\infty} a_n(t)\, \phi_n(x)$, we get

$$a_n(T) = \int_0^T [f(t)\, \varphi_n'(0) + g(t)\, \varphi_n(l)]\, \frac{\sin \omega_n (T - t)}{\omega_n}\, dt. \tag{2.18}$$

Therefore, the shape controllability result can be formulated as solvability of the moment problem

$$\omega_n a_n = \int_0^T [f_0(t)\, \varphi_n'(0) + g_0(t)\, \varphi_n(l)] \sin \omega_n (T - t)\, dt,\ \forall n, \tag{2.19}$$

for arbitrary sequence $\{a_n \omega_n\} \in \ell^2$ by the functions $f_0 \in H_0^1(0, T)$, $g_0 \in L^2(0, T)$.

Differentiating (2.18) with respect to T, we reformulate the velocity controllability result as solvability of the moment problem

$$b_n = \int_0^T [f_1(t)\,\varphi_n'(0) + g_1(t)\,\varphi_n(l)]\,\cos\omega_n(T-t)]\,dt,\ \forall n, \tag{2.20}$$

for arbitrary sequences $\{b_n\} \in \ell^2$ by the functions $f_1 \in H_*^1(0,T)$, $g_1 \in L^2(0,T)$.

Now we extend the functions f_0, g_0 in the odd way with respect to $t = T$ from the interval $[0,T]$ to $[0,2T]$, and the functions f_1, g_1 — in the even way. We define the functions

$$f = \frac{f_0 + f_1}{2} \in H_0^1(0,2T),\ g = \frac{g_0 + g_1}{2} \in L^2(0,2T),$$

and observe that these functions solve the moment problems

$$a_n\omega_n = \int_0^{2T} [f(t)\,\varphi_n'(0) + g(t)\,\varphi_n(l)]\sin\omega_n(T-t)\,dt, \tag{2.21}$$

$$b_n = \int_0^{2T} [f(t)\,\varphi_n'(0) + g(t)\,\varphi_n(l)]\,\cos\omega_n(T-t)]\,dt \tag{2.22}$$

for arbitrary sequences $\{a_n\omega_n\},\ \{b_n\} \in \ell^2$. By rewriting sine and cosine in the last two equations in terms of complex exponentials, it becomes an easy algebra exercice to check that solvability of the moment problem (2.21), (2.22) is equivalent to solvability of the moment problem

$$\alpha_n = \int_0^{2T} [f(t)\,\varphi_n'(0) + g(t)\,\varphi_n(l)]\,\sin\omega_n(2T-t)\,dt \tag{2.23}$$

$$\beta_n = \int_0^{2T} [f(t)\,\varphi_n'(0) + g(t)\,\varphi_n(l)]\,\cos\omega_n(2T-t)\,dt, \tag{2.24}$$

for arbitrary sequences $\{\alpha_n\}, \{\beta_n\} \in \ell^2$. Having solved these moment problems is equivalent to proving exact controllability in time $2T$. The proof of Theorem 1.3 is complete.□

2.2. Forward Problem for the Interval

A key ingredient for our proof of Theorem 1.5 is a representation of the solution of the wave equation on the interval which accounts for reflections off the boundary points, found in [15], generalizing d'Alembert's original representation to a finite interval with $q \neq 0$. We consider the IBVP:

$$v_{tt} - v_{xx} + q(x)\,v = 0,\ 0 < x < l,\ t \in (0,T), \tag{2.25}$$

$$v(x,0) = v_t(x,0) = 0,\ 0 < x < l, \tag{2.26}$$

$$v(0,t) = f(t), \tag{2.27}$$

$$v(l,t) = g(t),\ t > 0. \tag{2.28}$$

Throughout this section, we assume that $f, g \in L^2(0,T)$.

We begin by solving

$$\begin{cases} u_{tt} - u_{xx} + q(x)\,u = 0, \quad 0 < x < l,\ 0 < t < T \\ u|_{t\leqslant 0} = 0, \quad u(0,t) = f(t), \quad u(l,t) = 0. \end{cases} \tag{2.29}$$

For $f \in H^1_*(0,T)$, the system (2.29) has a unique solution $u^{f,0} \in C([0,T]; H^1(0,l))$. This solution can be presented by the so-called "folding ruler formula", see [15]:

$$\begin{aligned} u^{f,0}(x,t) = \quad & f(t-x) + \int_x^t k(x,s) f(t-s)\,ds \\ & - f(t-2l+x) - \int_{2l-x}^t k(2l-x,s) f(t-s)\,ds \\ & + f(t-2l-x) + \int_{2l+x}^t k(2l+x,s) f(t-s)\,ds \\ & - f(t-4l+x) - \int_{4l-x}^t k(4l-x,s) f(t-s)\,ds + \dots \\ = \quad & \sum_{n=0}^{\frac{t-x}{2l}} \left(f(t-2nl-x) + \int_{2nl+x}^t k(2nl+x,s) f(t-s)\,ds \right) \\ & - \sum_{n=1}^{\frac{t+x}{2l}} \left(f(t-2nl+x) + \int_{2nl-x}^t k(2nl-x,s) f(t-s)\,ds \right), \end{aligned} \tag{2.30}$$

where $\lfloor \cdot \rfloor$ is the floor function; $k(x,t)$ is a solution to the Goursat problem (2.2) in which the potential $q(x)$ is extended to the semi-axis $x > 0$ by the rule $q(2nl \pm x) = q(x)$ for all $n \in \mathbb{N}$. Setting $f(t) = 0$ for $t < 0$ guarantees that the sums above are finite.

The folding ruler formula gives convenient presentations of the solution with various boundary conditions and will be used in many constructions of this paper. When the Dirichlet control function $g \in H^1_*(0,T)$ is applied at $x = l$, the IBVP

$$\begin{cases} u_{tt} - u_{xx} + q(x)\,u = 0, \quad 0 < x < l, t > 0 \\ u|_{t\leqslant 0} = 0, \quad u(0,t) = 0, \quad u(l,t) = g(t). \end{cases} \tag{2.31}$$

can be solved by changing of variables in (2.29). We put $p(x) = q(l-x)$ and extend p by letting $p(2nl \pm x) = p(x)$. Letting $\overline{k}(x,t)$ be the solution to the Goursat problem (2.2), where $q(x)$ is replaced with $p(x)$, then

$$\begin{aligned} u^{0,g}(x,t) \quad = \quad & g(t-l+x) + \int_{l-x}^t \overline{k}(l-x,s) g(t-s)\,ds \\ & - g(t-l-x) - \int_{l+x}^t \overline{k}(l+x,s) g(t-s)\,ds + \dots \end{aligned} \tag{2.32}$$

By linearity, the general solution $v^{f,g}$ to (2.25)-(2.28) is thus given by

$$v^{f,g}(x,t) = v^{f,0}(x,t) + v^{0,g}(x,t).$$

2.3. Velocity Control for Graph with Cycle

In this section we prove part b) of Theorem 1.5. The proof for part a) is similar the proof of Proposition 1 below, and is left to the reader.

One technical challenge for proving velocity control is that it is unclear how to restrict a generalized function in $\mathcal{H}^{-1}$ to proper subsets. We avoid this difficulty by first proving velocity control in a more regular space, and then using a functional analytic argument to conclude velocity controllability for $\mathcal{H}^{-1}$.

Proposition 1. Let $T \geq T_*$ and $\psi \in \mathcal{H}_0^1$. There exist controls $\mathbf{f} := (f_1, f_2) \in H_0^2(0,T) \times H_0^2(0,T)$ such that

$$u_t^{\mathbf{f}}(\cdot, T) = \psi,$$

and there exists a constant C that depends only on $q, l_j, j = 1, \ldots, 4$, such that

$$\|f_1\|_{H^2(0,T)} + \|f_2\|_{H^2(0,T)} \leq C\, \|\psi\|_{\mathcal{H}^1}. \tag{2.33}$$

We construct our controls in several steps.

Step 1. We begin by rewriting our interior vertex conditions, (1.15) and (1.16), in a convenient way. Let u be the solution to the system (1.12)-(1.16). We adopt the following notation:

$$g_j(t) = u_j(v_2,t),\ j = 1,2,3, \text{ and } h(t) = u_k(v_3,t). \tag{2.34}$$

We first express (1.15) in terms of $\{f_1, g_j, h\}$. We identify e_j with $(0, l_j)$ with $x = 0$ corresponding to v_2. Then for each j, we can use (2.30) and (2.32) to represent u_j. In particular,

$$u_1(x,t) = v^{g_1,f_1}(x,t);\ u_j(x,t) = v^{g_j,h}(x,t) \text{ for } j = 2,3, \tag{2.35}$$

with the associated integral kernels labelled k_j or $\overline{k}_j$. By (1.15), we have

$$0 = \sum_1^3 \partial u_j(0,t) = \frac{\partial}{\partial x}\big(v^{g_1,f_1}(x,t) + v^{g_2,h}(x,t) + v^{g_3,h}(x,t)\big)|_{x=0},$$

and hence:

$$
\begin{aligned}
0 = &-\sum_{j=1}^{3}\big(g_j'(t) - 2\sum_{n\geq 1} g_j'(t-2nl_j)\big) - 2\sum_{j=1}^{3}\sum_{n\geq 1} k_j(2nl_j, 2nl_j)g_j(t-2nl_j)\\
&+\sum_{j=1}^{3}\big(\int_0^t \partial k_j(0,s)g_j(t-s)ds + 2\sum_{n\geq 1}\int_{2nl_j}^t \partial k_j(2nl_j,s)g_j(t-s)ds\big)\\
&+2\sum_{n\geq 1}\big(f_1'(t-(2n-1)l_1) + \overline{k}_1((2n-1)l_1,(2n-1)l_1)f_1(t-(2n-1)l_1)\\
&-\int_{(2n-1)l_1}^t \partial\overline{k}_1((2n-1)l_1,s)f_1(t-s)ds\big)\\
&+2\sum_{j=2}^{3}\Big[\sum_{n\geq 1}\big(h'(t-(2n-1)l_j) + \overline{k}_j((2n-1)l_j,(2n-1)l_j)h(t-(2n-1)l_j)\\
&-\int_{(2n-1)l_j}^t \partial\overline{k}_j((2n-1)l_j,s)h(t-s)ds\big)\Big]. \qquad (2.36)
\end{aligned}
$$

Also, by (1.15), we get the equations:

$$g_1(t) + f_2(t) = g_2(t), \qquad (2.37)$$

$$g_1(t) = g_3(t). \qquad (2.38)$$

We now rewrite (1.16) in terms of $\{g_j, h\}$. To this end, for this paragraph we identify e_j for $j = 2,3,4$ with $(0, l_j)$ with $x = 0$ identified with v_3. Let κ_j, resp. $\overline{\kappa}_j$ be the integral kernels associated to (2.30), resp. (2.32), for $j = 2,3,4$. Thus $u_4(x,t) = v^{h,0}(x,t)$. A similar argument to the derivation of (2.36) at v_3 gives

$$
\begin{aligned}
0 = &-\sum_{j=2}^{4}\big(h'(t) - 2\sum_{n\geq 1} h'(t-2nl_j)\big) - 2\sum_{j=2}^{4}\sum_{n\geq 1} \kappa_j(2nl_j, 2nl_j)h(t-2nl_j)\\
&+\sum_{j=2}^{4}\big(\int_0^t \partial \kappa_j(0,s)h(t-s)ds + 2\sum_{n\geq 1}\int_{2nl_j}^t \partial \kappa_j(2nl_j,s)h(t-s)ds\big)\\
&+2\sum_{j=2}^{3}\Big[\sum_{n\geq 1}\big(g_j'(t-(2n-1)l_j) + \overline{\kappa}_j((2n-1)l_j,(2n-1)l_j)g_j(t-(2n-1)l_j)\\
&-\int_{(2n-1)l_j}^t \partial\overline{\kappa}_j((2n-1)l_j,s)g_j(t-s)ds\big)\Big]. \qquad (2.39)
\end{aligned}
$$

Step 2. Let $\psi \in \mathcal{H}_0^1$ be our target velocity. To construct our velocity control, we will use f_2 alone to attain the desired shape on e_4, while f_1, f_2 will be used jointly to attain control on the other edges. Recall $T_* = \max(l_1 + l_2, l_3 + l_4)$. In the construction that follows, f_2 will be supported in the interval $(T_* - \max(l_3 + l_4, l_2), T_*))$ while f_1 will be supported in $(T_* - l_2 - l_1, T_*)$.

In this step, we solve for h. Recall we identify e_4 with $(0, l_4)$ with $x = 0$ identified with v_3. First, we will solve $v^{h,0}(x, T_*) = \phi_4(x)$. We set $h(t) = 0$ for $t < T_* - l_4$ in (2.3), and $\tilde{h}(t) = h(t - (T_* - l_4))$. Then

$$\psi_4(x) = \tilde{h}'(l_4 - x) + \int_x^{l_4} \kappa_4(x, s)\tilde{h}'(l_4 - s)ds,\ x \in (0, l_4).$$

This is a VESK, and thus has a unique solution $\tilde{h}' \in H^1(0, l_4)$. Furthermore, the equation above gives we $\tilde{h}'(0) = \psi_4(l_4) = 0$. Translating again in time, $h' \in H_0^1(T_* - l_4, T_*)$. We set $h'(t) = 0$ for $t < T_* - l_4$, and integrate using $h(0) = 0$. We get $h \in H_*^2(0, T_*)$.

Step 3 In this step, we compute

$$g_2 \in H_*^2(T_* - l_2, T_*) \quad \text{and} \quad g_3 \in H_*^2(T_* - l_4 - l_3, T_* - l_3),$$

from which we will fully determine f_2 and partly determine f_1.

Let $\Delta = \min(l_1, l_3)$ if $l_2 = l_3$, and $\Delta = \min(l_1, l_3, l_2 - l_3)$ if $l_2 > l_3$. Define $t_n^* = n\Delta$ for $n \geq 0$. We will use (2.39) to solve for g_3 on successive intervals $[t_n^*, t_{n+1}^*]$ in terms of the known function h; this process will stop when $t < T_* - l_3$.. As n increases, the number of non-zero terms in (2.39) increases. In what follows, we will denote by $\alpha(t)$ various functions, which change from line to line, that are known for $t < t_j^*$. Also, $\beta(t)$ are functions, that varying from line to line, that are determined by h, and hence are known for all t in the argument below.

In this paragraph, we use (2.39) to solve for $g_3(t)$ for $t < T_* - l_3$. Assume first that $t < t_1^*$. We begin For $n \geq 1$, we have $t - 2nl_j < 0$ for $j = 1, 2, 3$ by the definition of t_1^*, and hence the terms in such as $g_j(t - 2nl_j)$ all vanish. In addition, since $g_2(t)$ will vanish for $t < T_* - l_2$, the wave generated by g_2 will not interact with the vertex v_3. Thus (2.39) simplifies to

$$\alpha(t) = 2g_3'(t - l_2) + \overline{\kappa}_3(l_3, l_3)g_3(t - l_3) - \int_{l_3}^{t} \partial\overline{\kappa}_3(l_3, s)g_3(t - s)ds,\ t < t_1^*.$$

Integrating, we get

$$\alpha(t) = 2g_3(t) + \int_0^t \overline{K}_3(t, r)g_3(r)dr,\ t < t_1^*.$$

where

$$\overline{K}_3(t - l_3, y) = \overline{\kappa}_3(l_3, l_3) + \int_{r=0}^{t-l_3} \overline{\kappa}_3(0, r + l_3 - y)dy.$$

This is a VESK, and thus $g_3(t)$ is uniquely solvable for $t < t_1^*$. By the regularity of $\overline{K}_3$, we have $g_3 \in H^2$, and one can also verify that $g_3(0) = g_3'(0) = 0$. Now assume $g_3(t)$ has been determined for $t < t_k^*$. In this case, $t < t_{k+1}^*$ implies that for $n > 0$ and $j = 1, 2, 3$, we have $(t - 2nl_j) < t_k^*$, and so the function $g_3(t - 2nl_j)$ have been previously calculated. Thus (2.39) can be rewritten

$$\alpha(t) = 2g_3(t) + \int_{t_k^*}^{t} \overline{K}_3(t, r)g_3(r)dr,\ t \in [t_k^*, t_{k+1}^*].$$

Thus we solve this VESK for $g_3(t)$, $t < t^*_{k+1}$. Iterating the argument above, until $t^*_{k+1} \geq T_* - l_3$, we determine g_3 for $t \in (0, T_* - l_3)$. We remark in passing that by definition, we have $g_3(t) = 0$ until $t^*_k > T_* - l_4 - l_3$. Also, by construction g_3, g_3' are continuous, hence

$$g_3(T_* - l_4 - l_3) = g_3'(T_* - l_4 - l_3) = 0.$$

We now solve for $g_3(t)$ for $t > T_* - l_3$. By (2.30), (2.32),

$$\begin{aligned}\psi_3(x) = g_3'(T_* - l_3 + x) + \int_{l_3 - x}^{T_*} \overline{k}_3(l_3 - x, s) g_3'(T_3 - s) - \ldots \\ + h'(T_* - x) + \int_x^{T_*} k_3(x, s) h'(T_* - s)\, ds \\ - h'(T_* - 2l_3 + x) - \int_{2l_3 - x}^{T_*} \overline{k}_3(2l_3 - x, s) h'(T_* - s)\, ds \ + \ldots,\ x \in (0, l_3).\end{aligned}$$

Observe that the only unknown terms in this last equation are the first two on the right hand side. Thus we solve this VESK to determine $g_3'(t)$ for $t > T_* - l_3$. Then $g_3'(t)$ is continuous at $t = T_* - l_3$ by the following argument. Letting $x \to l^-_{m(i)}$ in the equation above, one gets $g_3'((T - l_3)^-) = g_3'((T - l_3)^+)$, so that $g_3'(t)$ extends to a continuous function at $x = T - l_3$, and so $g_3' \in H^1_*[T_* - l_4 - l_3, T_*]$. Integrating, we get $g_3 \in H^2_*(0, T_*)$. We now compute $g_2 \in H^2_*(T_* - l_2, T_2)$. By (2.30), (2.32), we have

$$\begin{aligned}\psi_2(x) = h'(T_* - x) + \int_x^{T_*} k_2(x, s) h'(T_* - s) ds - h'(T_* + x - 2l_2) - \ldots \\ + g_2'(T_* - l_2 + x) + \int_{l_2 - x}^{T_*} \overline{k}(l_2 - x, s) g_2'(T_* - s),\ x \in (0, l_2).\end{aligned}$$

Only the last two terms in this equation are unknown, so we solve this VESK to obtain g_2'. Letting $x \to 0^+$ in this equation, we can show $g_2'(T_* - l_2) = 0$. We set $g_2'(t) = 0$ for $t < T_* - l_2$, and then solve for g_2 by integration. Thus $g_2 \in H^2_*(0, T_*)$.

We now apply (2.37) and (2.38) to solve for f_2, g_1. Since $g_3, g_3 \in H^2_*(0, T_*)$, the same is true for f_2. We also have $f_2(T_*) = 0$ by (2.37), (2.38), and the continuity of u at v_2. Also, $f_2'(T) = 0$ by (2.37), (2.38), and the continuity of u_t at v_2. Thus $f_2 \in H^2_0(0, T_*)$.

Step 4. In this step, we solve for f_1. First, we will apply (2.36) to solve for $f_1(t)$ for $t \in (T_* - l_2 - l_1, T_* - l_2)$. We have already determined h and g_j in that equation, so it simplifies to

$$\begin{aligned}\beta(t) = 2 \sum_{n \geq 1} \big(f_1'(t - (2n-1)l_1) + \overline{k}_1((2n-1)l_1, (2n-1)l_1) f_1(t - (2n-1)l_1) \\ - \int_{(2n-1)l_1}^{t} \partial \overline{k}_1((2n-1)l_1, s) f_1(t - s) ds.\end{aligned}$$

We can use an iterative argument as in Step 2 to solve for $f_1(t)$ with $t < T_* - l_1$, with $f_1(0) = f_1'(0) = 0$; the details are left to the reader.

For $t < T_* - l_1$, we identify e_1 with the interval $(0, l_1)$ with $x = 0$ identified with v_2. Using (2.30), (2.32), we get

$$\psi_1(x) = g_1'(T_* - x) + \int_{l_1 - x}^{T_*} k_1(x,s) g_1'(T_* - s) ds - ...$$
$$+ f_1'(T_* - l_1 + x) + \int_{l_1 - x}^{T_*} \overline{k}_1(l_1 - x, s) f_1'(T_* - s) - ..., \ x \in (0, l_1).$$

Solving this VESK, it is straightforward to verify that $f_1'(t)$ is then continuous at $T_* - l_1$, and that $f_1'(T_* - l_1 - l_2) = 0$. If $T_* - l_1 - l_2 > 0$, then we extend f_1' trivially to zero. Integrating, we get $f_1 \in H_*^2(0, T_*)$. We can then prove $f_1'(T_*) = 0$ by noting $f_1'(T_*) = u_t(v_1, T_*) = \psi(v_1) = 0$. That $f_1(T_*) = 0$ follows from $\psi(v_1) = 0$. Thus $f_1 \in H_0^2(T_* - l_2 - l_2, T_*)$.

The proof of Proposition 1 is complete.

Remark 2.1. Recalling that $T_* = \max\{ l_1 + l_2, \ l_3 + l_4 \}$, we summarize the supports of the controls f_1, f_2.

1) if $l_1 + l_2 + \mu = l_3 + l_4, \ \mu \geq 0$, then $T_* = l_3 + l_4$, supp $\ f_1 \subset [\mu, T_*]$ and supp $f_2 \subset [0, T_*]$;

2) if $l_1 + l_2 = l_3 + l_4 + \mu, \ \mu \geq 0$, then $T_* = l_1 + l_2$, supp $\ f_1 \subset [0, T_*]$ and supp $f_2 \subset [\tau, T_*]$, where $\tau = \min\{\mu, l_1\}$.

The proof of the following result, which holds trivially in the classical setting, is left to the reader.

Lemma 2.1. *Let* $\mathbf{f} \in H_0^2(0,T) \times H_0^2(0,T)$. *Then*

$$\partial_t^2 u_t^{\mathbf{f}}(*, t) = u_t^{\mathbf{f}''}(*, t), \tag{2.40}$$

this equation holding in $C(0, T; \mathcal{H}^{-1})$.

The following completes our proof of Theorem 1.5 b).

Corollary 2.1. *Assume 0 is not an eigenvalue of L. Let* $T \geq T_*$ *Let* $\psi \in \mathcal{H}^{-1}$. *Then there exist* $f_1, f_2 \in L^2(0,T)$, *such that*

$$u_t^{\mathbf{f}}(*, T) = \psi(*).$$

Proof. Because zero is not an eigenvalue, L

$$L : \mathcal{H}_0^1 \mapsto \mathcal{H}^{-1}$$

is an isometry. Thus, using (2.40), there exist $\tilde{\psi} \in \mathcal{H}_0^1$ and $\tilde{\mathbf{f}} \in H_0^2(0,T) \times H_0^2(0,T)$ such that

$$\begin{aligned} \psi &= L\tilde{\psi}, \\ &= L u_t^{\tilde{\mathbf{f}}}(*, T), \\ &= \partial_t^2 u_t^{\tilde{\mathbf{f}}}(*, T) = u_t^{\tilde{\mathbf{f}}''}(*, T), \end{aligned}$$

so it suffices to set $\mathbf{f} = \tilde{\mathbf{f}}''$. □

2.4. Proof of Theorem 1.4

Now we prove the exact controllability in time $2T$, for $T \geq T_*$, using the shape and velocity controllability in time T. Let $\{\omega_n^2\}_1^\infty$ and $\{\varphi_n\}_1^\infty$ be the eigenvalues and corresponding normalized eigenfunctions of the spectral problem associated to System (1.12)-(1.16):

$$-\varphi_n'' + q(x)\varphi_n = \omega_n^2 \varphi_n,\ x \in e_j, j = 1, ..., 4, \tag{2.41}$$

$$(\varphi_n)_1(v_1) = (\varphi_n)_4(v_4) = 0, \tag{2.42}$$

$$(\varphi_n)_1(v_2) = (\varphi_n)_2(v_2) = (\varphi_n)_3(v_2),\ \sum_{j=1}^{3} \partial(\varphi_n)_j(v_2) = 0, \tag{2.43}$$

$$(\varphi_n)_2(v_3) = (\varphi_n)_3(v_3) = (\varphi_n)_4(v_3),\ \sum_{j=2}^{4} \partial(\varphi_n)_j(v_3) = 0. \tag{2.44}$$

Here and in what follows, we denote the restriction of φ_n to edge e_j by $(\varphi_n)_j$.

The following estimates can be extracted from [14], [19], and [21]:

$$|\partial(\varphi_n)_1(v_1)| \prec n,\ |(\varphi_n)_2(v_2)| \prec 1,\ |\omega_n| \asymp n.$$

We write $u(x,t) = \sum_1^\infty a_n(t)\varphi_n(x)$. To find the coefficients a_n we integrate by parts in x the identity

$$0 = \int_0^T \int_\Omega [\partial_{tt} u - \partial_{xx} u + q(x)u]\, \varphi_n(x)\mu(t)\, dxdt$$

with arbitrary $\mu \in C_0^2[0,T]$. We obtain the initial value problem

$$a_n''(t) + \omega_n^2 a_n(t) = f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2),\ a_n(0) = a_n'(0) = 0.$$

By variation of parameters, we get

$$a_n(T) = \int_0^T [f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2)]\, \frac{\sin\omega_n(T-t)}{\omega_n}\, dt. \tag{2.45}$$

In what follows, the controls solving the shape control problem will be denoted by $f_j = f_{j0}$, and the controls solving the velocity control problem will be denoted by $f_j = f_{j1}$, for $j = 1, 2$. Setting $a_n = a_n(T)$, the shape controllability result can be formulated as solvability of the moment problem

$$a_n = \int_0^T [f_{10}(t)(\partial\varphi_n)_1(v_1) + f_{20}(t)(\partial\varphi_n)_2(v_2)]\, \frac{\sin\omega_n(T-t)}{\omega_n}\, dt,\ \forall n, \tag{2.46}$$

for arbitrary sequence $\{a_n\} \in \ell^2$ by the functions $f_{10},\ f_{20} \in L^2(0,T)$.

Differentiating (2.45) with respect to T, we reformulate the velocity controllability result as solvability of the moment problem

$$b_n = \int_0^T [f_{11}(t)(\partial\varphi_n)_1(v_1) + f_{21}(t)(\partial\varphi_n)_2(v_2)] \frac{\cos\omega_n(T-t)}{\omega_n}\, dt,\ \forall n, \qquad (2.47)$$

for arbitrary sequences $\{b_n\} \in \ell^2$ by the functions $f_{11},\ f_{21} \in L^2(0,T)$.

Now we extend the functions f_{10}, f_{20} in the odd way with respect to $t = T$ from the interval $[0,T]$ to $[0,2T]$, and the functions f_{11}, f_{21} — in the even way. We define the functions

$$f_1 = \frac{f_{10}+f_{11}}{2} \in L^2(0,2T),\ f_2 = \frac{f_{20}+f_{21}}{2} \in L^2(0,2T),$$

and observe that these functions solve the moment problems

$$a_n = \int_0^{2T} [f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2)] \frac{\sin\omega_n(T-t)}{\omega_n}\, dt, \qquad (2.48)$$

$$b_n = \int_0^{2T} [f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2)] \frac{\cos\omega_n(T-t)}{\omega_n}\, dt \qquad (2.49)$$

for arbitrary sequences $\{a_n\},\ \{b_n\} \in \ell^2$. By rewriting sine and cosine in the last two equations in terms of complex exponentials, it becomes an easy algebra exercice to check that solvability of the moment problem (2.48), (2.49) is equivalent to solvability of the moment problem

$$\alpha_n = \int_0^{2T} [f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2)] \frac{\sin\omega_n(T-t)}{\omega_n}\, dt \qquad (2.50)$$

$$\beta_n = \int_0^{2T} [f_1(t)(\partial\varphi_n)_1(v_1) + f_2(t)(\partial\varphi_n)_2(v_2)] \frac{\cos\omega_n(T-t)}{\omega_n}\, dt, \qquad (2.51)$$

for arbitrary sequences $\{\alpha_n\}, \{\beta_n\} \in \ell^2$. Having solved these moment problems is equivalent to proving exact controllability in time $2T$. The proof of Theorem 1.4 is complete.$\square$

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