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CAN A FINITE DEGENERATE 'STRING' HEAR ITSELF? THE EXACT SOLUTION TO A SIMPLIFIED IBVP

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Abstract. The separation of variables based solution to a simplified (compared to that published earlier in JODEA, **28** (1) (2020), 1 – 42) initial boundary value problem for a 1D linear degenerate wave equation, posed in a space-time rectangle, has been presented in a fully complete form. Degeneracy of the equation is due to vanishing its coefficient in an interior point of the spatial segment being the side of the rectangle. For the sake of convenience, the solution is interpreted as a vibrating 'string'. The solution obtained in the case of weak degeneracy is smooth and bounded, whereas that in the case of strong degeneracy is piece-wise smooth, piece-wise continuous and unbounded in a neighborhood of the point of degeneracy, nevertheless being satisfied some regularity conditions, including square-integrability. In both cases the travelling waves pass through the point of degeneracy, and this phenomenon is referred to as an ability of the 'string' to hear itself. The total energy of the 'string' is shown to conserve in both cases of degeneracy, provided the ends of the 'string' are fixed, though the above vibrating 'string' analogy fails in the case of strong degeneracy. The total energy conservation implies the uniqueness of the solution to the problem in both cases of degeneracy.

Key words: degenerate wave equation, vibrating string, separation of variables, series solution, transmission condition, travelling wave, conservation of energy.

2010 Mathematics Subject Classification: 35C10, 35L05, 35L10, 35L20, 35L80.

1. Introduction and the problem formulation

The current study is a sequel to [2,4] and deals with the following 1-parameter simplified initial boundary value problem (IBVP) for the degenerate wave equation, posed in the space-time rectangle $[0,T]\times[-1,+1]\subset\mathbb{R}^+_t\times\mathbb{R}_x$ wrt $u(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial u}{\partial x}\right)=0\,, & (t,|x|)\in(0,T]\times(0,1)\,,\\ \left.\begin{aligned}\dfrac{\partial u(0,x;\alpha)}{\partial t}&=\overset{**}{u}(x;\alpha)\\ u(0,x;\alpha)&=\overset{*}{u}(x;\alpha)\end{aligned}\right\}, & x\in[-1,+1]\,,\\ \left.\begin{aligned}u(t,-1;\alpha)&=h_1(t;\alpha)\\ u(t,+1;\alpha)&=h_2(t;\alpha)\end{aligned}\right\}, & t\in[0,T]\,,\end{cases}\tag{1.1}$$

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where known control functions $h_{1,2}(t;\alpha)\in\mathscr{C}^1[0,T]\bigcap\mathscr{C}^2(0,T]$ obey the compatibility conditions: $h_1(0;\alpha)=\overset{*}{u}(-1;\alpha)$, $h_1'(0;\alpha)=\overset{**}{u}(-1;\alpha)$, $h_2(0;\alpha)=\overset{*}{u}(+1;\alpha)$, $h_2'(0;\alpha)=\overset{**}{u}(+1;\alpha)$, and the 1-parameter family of coefficient functions is defined as follows

$$a(x;\alpha)=|x|^\alpha, \qquad x\in[-1,+1]\,, \tag{1.2}$$

the parameter of degeneracy $\alpha\in(0,2)$, and all the variables are nondimensional. The point $x=0$, where the coefficient (1.2) vanishes, is referred below to as the degeneracy point, whereas $[0,T]\times[-1,+1]\supset[0,T]\times\{0\}$ is referred to as the degeneracy segment, or the dividing segment of the space-time rectangle. Dealing with (1.1), (1.2), we distinguish between the cases of: 1) weak degeneracy, $\alpha\in(0,1)$, 2) strong degeneracy, $\alpha\in(1,2)$, and 3) non-degeneracy, $\alpha=0$ (the limiting case).

For the solution to the problem to exist and to be unique, some matching conditions must be imposed on the required solution at the degeneracy segment, which will be discussed below.

The above problem is simplified compared to that of [2, 3] due to extending the original power law for the coefficient function

$$a(x;\alpha)=\begin{cases} a_*|x|^\alpha, & 0\leqslant|x|\leqslant c\,,\\ \quad 1\,, & c\leqslant|x|\leqslant 1\,,\end{cases} \tag{1.3}$$

where $a_*c^\alpha=1$, to the segment $[-1,+1]$, as in (1.2). We will further refer to (1.1), (1.2) as the IBVPS, for short.

The transformation of the independent variables based on the characteristics variables [2] (refer to Fig. 1.1)

$$\begin{cases} \tau=t\,,\\ \xi=\mp|x|^{\frac{\theta}{2}}, \end{cases} \qquad (t,|x|)\in[0,T]\times[0,1]\,, \tag{1.4}$$

where the upper and lower signs refer to $x<0$ and $x>0$, respectively, yields to the following formulation of the IBVPS, posed in the space-time rectangle $[0,T]\times[-1,+1]\subset\mathbb{R}^+_\tau\times\mathbb{R}_\xi$ wrt $U(\tau,\xi;\alpha)$

$$\begin{cases} \dfrac{\partial^2U}{\partial\tau^2}-\left(\dfrac{\theta}{2}\right)^2\dfrac{\partial^2U}{\partial\xi^2}-\dfrac{\theta}{2}\,\dfrac{\alpha}{2\xi}\,\dfrac{\partial U}{\partial\xi}=0\,, & (\tau,|\xi|)\in(0,T]\times(0,1)\,,\\ \left.\begin{aligned}\dfrac{\partial U(0,\xi;\alpha)}{\partial\tau}&=\overset{**}{U}(\xi;\alpha)\\ U(0,\xi;\alpha)&=\overset{*}{U}(\xi;\alpha)\end{aligned}\right\}, & \xi\in[-1,+1]\,,\\ \left.\begin{aligned}U(\tau,-1;\alpha)&=h_2(\tau;\alpha)\\ U(\tau,+1;\alpha)&=h_1(\tau;\alpha)\end{aligned}\right\}, & \tau\in[0,T]\,,\end{cases} \tag{1.5}$$

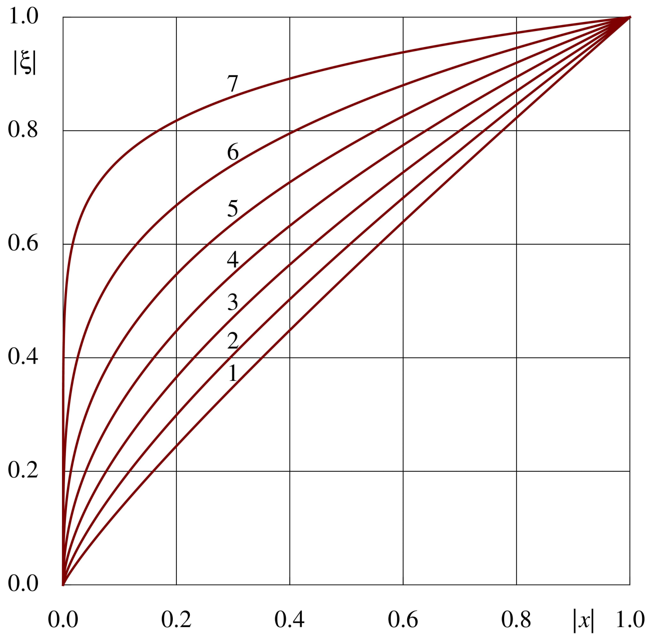

Fig. 1.1. Transformation (1.4) maps space-time rectangle $(t, |x|) \in [0, T] \times [0, 1]$ onto space-time rectangle $(\tau, |\xi|) \in [0, T] \times [0, 1]$ and 'inflates' the degeneracy of the original wave equation in (1.1) by stretching variable ξ near the degeneracy segment: curves 1 − 7 are drawn for $\alpha = 0.25\,(0.25)\,1.75$, respectively

referred to as the simplified transformed initial boundary value problem (IBVPT, for short).

Wherever it is useful and convenient, the solution to the IBVPS will be interpreted (and referred to) as the distributed over segment $[-1, +1]$ displacements of a vibrating 'string': 1) subject to known controls $h_1(t)$ and $h_2(t)$, imposed on both ends of the 'string', and 2) having the initial distributed displacements $\overset{*}{u}(x; \alpha)$ and velocities $\overset{**}{u}(x; \alpha)$.

From a physical point of view, the coefficient function $a(x; \alpha)$ of the degenerate wave equation of the IBVPS can be treated as a) the ratio of the local tension and the local density of the 'string' or b) the local 'speed of sound' (the velocity of the travelling waves) squared. Indeed, the expanded form of the degenerate wave equation of (1.1) can be presented using the differential operator, satisfying the following identity [1]

$$\frac{\partial^2}{\partial t^2} - a\,\frac{\partial^2}{\partial x^2} - a'\,\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial t} - \sqrt{a}\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \sqrt{a}\,\frac{\partial}{\partial x}\right) - \frac{a'}{2}\,\frac{\partial}{\partial x} =$$

$$= \left(\frac{\partial}{\partial t} + \sqrt{a}\,\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \sqrt{a}\,\frac{\partial}{\partial x}\right) - \frac{a'}{2}\,\frac{\partial}{\partial x}\,,$$

and then can be easily rewritten as any of two following systems

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \sqrt{a}\,\dfrac{\partial u_1}{\partial x} - \qquad u_2 \qquad = 0\,, \\ \dfrac{\partial u_2}{\partial t} - \dfrac{a'}{2}\,\dfrac{\partial u_1}{\partial x} - \sqrt{a}\,\dfrac{\partial u_2}{\partial x} = 0\,, \end{cases} \tag{1.6}$$

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \sqrt{a}\,\dfrac{\partial v_1}{\partial x} - \qquad v_2 \qquad = 0\,, \\ \dfrac{\partial v_2}{\partial t} - \dfrac{a'}{2}\,\dfrac{\partial v_1}{\partial x} + \sqrt{a}\,\dfrac{\partial v_2}{\partial x} = 0\,, \end{cases} \tag{1.7}$$

where $u_1(t,x;\alpha) = v_1(t,x;\alpha) = u(t,x;\alpha)$, whereas functions $u_2(t,x;\alpha)$, $v_2(t,x;\alpha)$ are fully determined by the first equations of systems (1.6), (1.7), respectively.

Both systems can be presented in matrix form, for example the former reads

$$\frac{\partial \mathbf{U}}{\partial t} + \mathsf{A}\,\frac{\partial \mathbf{U}}{\partial x} + \mathsf{A}_\star \mathbf{U} = \mathbf{O}\,, \tag{1.8}$$

where $\mathbf{U}(t,x;\alpha)$ is the state matrix-column, whereas $\mathsf{A}(x;\alpha)$ and $\mathsf{A}_\star$ are quadratic coefficient matrices as follows

$$\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} +\sqrt{a} & 0 \\ -\frac{1}{2}\,a' & -\sqrt{a} \end{pmatrix}, \quad \mathsf{A}_\star = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

Matrix A has 1) two real and distinct eigenvalues $\mp\sqrt{a}$ being the velocities of the travelling waves and 2) a complete set of right (in columns of matrix R) and left (in rows of matrix L) eigenvectors

$$\mathsf{R} = \mathsf{L}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2}\,(\sqrt{a}\,)' \end{pmatrix}, \quad \mathsf{L} = \mathsf{R}^{-1} = \begin{pmatrix} +\frac{1}{2}\,(\sqrt{a}\,)' & 1 \\ 1 & 0 \end{pmatrix},$$

hence, system (1.8) is strictly hyperbolic outside the degeneracy segment, where it degenerates.

The fact that $a(x;\alpha)$ vanishes at $x = 0$ makes it possible to assume that the 'string' *a*) acts like a swivel or is 'too heavy' in a close vicinity to this point, and even *b*) can prevent the travelling waves from passing through this point.

One question among those, raised in [2], was that concerning the travelling waves on one part of a degenerate 'string', say, on the left one, initially excited due to a choice of $\overset{*}{u}(x;\alpha)$, $\overset{**}{u}(x;\alpha)$, $x \in [-1,0)$, or being excited due to a choice of $h_1(t;\alpha)$, $t \in [0,T]$ (provided that $h_2(t;\alpha) \equiv 0$, $t \in [0,T]$). Can such waves enter the right part of the 'string' being at rest and come back to the left part (violating the above assumption *b*)? Using power series solutions to the degenerate wave equation valid in some close vicinity of the degeneracy point, it was succeeded in [2] to prove that some necessary conditions for passing the travelling waves through the point of degeneracy hold. Being impressed by the title [7], we refer to the question concerning passing the travelling waves through the degeneracy point (from the left part of the ‘string’ being disturbed to the right one being undisturbed) in the way the current study is entitled, realizing that our local implication of verb ‘to hear’ and that used in [7] are quite different.

From a mathematical point of view, formulation of any correct IBVP for the vibrating ‘string’ implies imposing the proper matching conditions at both sides of the degeneracy point (more precisely, at both sides of the degeneracy segment of the space-time rectangle): 1) the continuity of the displacement $u(t,x;\alpha)$ and 2) the continuity of the tension, or the flux $f(t,x;\alpha)$, being equal to the product of $a(x;\alpha)$ and the local slope of the displacement (the latter condition usually is referred to as the transmission one). Therefore the matching conditions for the IBVPS read as the continuity of $u(t,x;\alpha)$ and $f(t,x;\alpha)$ on the degeneracy segment

$$\begin{cases} u(t,x;\alpha)\big|_{x=0-0} = u(t,x;\alpha)\big|_{x=0+0}\,, \\ f(t,x;\alpha)\big|_{x=0-0} = f(t,x;\alpha)\big|_{x=0+0}\,, \end{cases} \qquad t\in[0,T]\,. \tag{1.9}$$

whereas for the IBVPT they read as continuity of $U(\tau,\xi;\alpha)$ and the flux $F(\tau,\xi;\alpha)$

$$\begin{cases} U(\tau,\xi;\alpha)\big|_{\xi=0-0} = U(\tau,\xi;\alpha)\big|_{\xi=0+0}\,, \\ F(\tau,\xi;\alpha)\big|_{\xi=0-0} = F(\tau,\xi;\alpha)\big|_{\xi=0+0}\,, \end{cases} \qquad \tau\in[0,T]\,. \tag{1.10}$$

$F(\tau,\xi;\alpha)$ being equal to the product of $\dfrac{\theta}{2}\,|\xi|^{\frac{\alpha}{\theta}}$ and the local slope of $U(\tau,\xi;\alpha)$.

We explain our approaches to solve the IBVPS and the IBVPT using separation of variables (SV) and imposing the matching conditions by referring to the IBVP for vibrating string with a piecewise constant density (a particular non-degenerate case). Let the IBVP be posed in the space-time rectangle $[0,T]\times[0,l]$, then the coefficient reads: $a(x;\alpha)\equiv a_1^2$, $x\in[0,x_0)$; $a(x;\alpha)\equiv a_2^2$, $x\in(x_0,l]$; x_0 being the point of discontinuity at which the matching conditions are imposed.

The first approach follows an algorithm given, for example, in the collection of problems [5] (for example, problems 164 – 166 on p. 37; problem 57 on p. 128), supplementing the textbook [12]. The algorithm utilizes the core idea of formulating ‘the global’ (or the composite) Sturm – Liouville boundary value problem (BVP) for the segment $[0,l]$, rather than formulating two ‘local’ Sturm – Liouville BVPs for the subsegments $[0,x_0]$, $[x_0,l]$. The governing equation of the global BVP has the above discontinuity in its coefficient at $x=x_0$, nevertheless there exist the complete countable sets of the ‘global’ eigenvalues and eigenfunctions (modes), the latter being smooth over the segment $[0,l]$. This means that the resulting solution to the IBVP, based on those sets due to SV, obey the matching conditions.

The second approach to the IBVP is based on: 1) splitting $[0,T]\times[0,l]$ into two space-time subrectangles $[0,T]\times[0,x_0]$ and $[0,T]\times[x_0,l]$; 2) reformulating the IBVP into two ‘local’ IBVPs in the above space-time subrectangles; 3) formulating and solving two associated ‘local’ incomplete BVPs (due to not imposing the matching conditions at x_0) as two sets of the 'local' eigenvalues and eigenfunctions (modes) ; 4) solving the 'local' IBVPs; 5) imposing matching conditions on the solutions to the 'local' IBVPs for obtaining the solution to the IBVP.

Previously, in [4], it was attempted to develop the second SV based approach to the IBVPS, nevertheless the first SV based approach was presented as well.

The goal of the current study is:

1) to develop SV [4] for solving the IBVPS and IBVPT, utilizing the first approach, in a fully complete form, supplemented by illustrative test cases;

2) to demonstrate that multiple passing the travelling waves through the point of degeneracy occurs in the cases of weak and strong degeneracy at any type of exciting one part of the 'string', the other part being at rest.

The current study is arranged as follows.

In Sect. 2 we: 1) define the complete sets of the eigenvalues and the eigenfunctions for the composite BVPs, associated with the IBVPS $(\lambda_{k,\mu}(\alpha),\ X_{k,\mu}(x;\alpha))$ and the IBVPT $(\lambda_{k,\mu}(\alpha),\ \Phi_{k,\mu}(\xi;\alpha))$; 2) discuss the properties of the above sets to build solutions $u(t,x;\alpha)$ to the IBVPS and $U(\tau,\xi;\alpha)$ to the IBVPT, both bounded and unbounded, respectively in the cases of weak and strong degeneracy.

In Sect. 3 we apply the second SV based approach to solve exactly the IBVPS and IBVPT in a fully complete form.

In Sect. 4 we introduce proper function spaces H^1_a and H^1_ξ for the SV based solutions $u(t,x;\alpha)$ and $U(\tau,\xi;\alpha)$ of Sect. 3, respectively to the IBVPS and IBVPT, and prove the uniqueness of the above solutions, 1) using the energy method, and 2) accounting that $X_{k,\mu}(x;\alpha) \subset H^1_a$ and $\Phi_{k,\mu}(\xi;\alpha) \subset H^1_\xi$.

In Sect. 5 we discuss a proper choice of the blending functions $\phi_1(x;\alpha)$ and $\phi_2(x;\alpha)$, used in SV of Sect. 3 to replace the IBVPS and IBVPT with the associated IBVPs.

In Sect. 6 we give some comments on ways used to suppress the Gibbs phenomenon, that arises when expanding: 1) the initial functions $\overset{*}{u}(x;\alpha)$, $\overset{**}{u}(x;\alpha)$, $\overset{*}{U}(\xi;\alpha)$, and $\overset{**}{U}(\xi;\alpha)$, and 2) the blending functions $\phi_1(x;\alpha)$, $\phi_2(x;\alpha)$ in general Fourier series wrt the eigenfunctions $X_{k,\mu}(x;\alpha)$ and $\Phi_{k,\mu}(\xi;\alpha)$ of Sect. 2.

In Sect. 7 we apply the obtained exact solutions of Sect. 3 to two test cases of the IBVPS and IBVPT and discuss in detail the observable properties of the solutions to the test cases.

In Sect. 8 we summarize the results obtained and some observations on the procedures applied in the current study.

Before completing this introductory Sect., we would like to point out other studies on the subject [8–10], carried out purely by methods of functional analysis.

2. Preliminaries to separation of variables

Implementing SV to the IBVPS and IBVPT of Sect. 1 is essentially based on the assertions being formulated below in Prop. 2.1 and Prop. 2.2 relating to the IBVPS and in Prop. 2.3 relating to the IBVPT. Note, that Props. 2.1, 2.2 are given with some modifications compared to those presented in [4].

The Bessel functions $\mathsf{J}_{\mp\varrho}(s)$ of the first kind and orders $\mp\varrho$ [14], being referenced to in Props. 2.1 − 2.3, satisfy the ordinary differential equation

$$s^2\,\mathsf{J}''_{\mp\varrho}(s) + s\,\mathsf{J}'_{\mp\varrho}(s) + \left(s^2-\varrho^2\right)\mathsf{J}_{\mp\varrho}(s) = 0$$

and have the following power series representations

$$\mathsf{J}_{\mp\varrho}(s) = \left(\frac{s}{2}\right)^{\mp\varrho} \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!\,\Gamma(1\mp\varrho+\gamma)} \left(\frac{s}{2}\right)^{2\gamma}. \tag{2.1}$$

Proposition 2.1. *Let the following incomplete 1-parameter boundary value problems be given*

$$\begin{cases} \left[a(x;\alpha)\,Z'_j(x;\alpha)\right]' + \lambda_j(\alpha)\,Z_j(x;\alpha) = 0\,, \qquad 0<|x|<1\,, \\ Z_j(\mp 1;\alpha) = 0\,, \end{cases} \tag{2.2}$$

where $j\in\{1,2\}$ *relates to* $[-1,0]$ *and* $[0,+1]$ *respectively, then*: 1) *the eigenvalues* $\lambda_{j,k,\mu}(\alpha)\equiv\lambda_{k,\mu}(\alpha)$ *and the eigenfunctions* $Z_{j,k,\mu}(x;\alpha)\equiv Z_{k,\mu}(x;\alpha)$ *of the problems* (2.2) *of the two kinds* (*marked with* $k\in\{1,2\}$) *are defined as follows*

$$\begin{cases} \lambda_{1,\mu}(\alpha) = \left(\dfrac{\theta}{2}\,s_{1,\mu}\right)^2 \equiv \sigma^2_{1,\mu}\,, \qquad Z_{1,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\,\mathsf{J}_{-\varrho}\left(s_{1,\mu}\,|x|^{\frac{\theta}{2}}\right), \\ \lambda_{2,\mu}(\alpha) = \left(\dfrac{\theta}{2}\,s_{2,\mu}\right)^2 \equiv \sigma^2_{2,\mu}\,, \qquad Z_{2,\mu}(x;\alpha) = |x|^{\frac{\nu}{2}}\,\mathsf{J}_{+\varrho}\left(s_{2,\mu}\,|x|^{\frac{\theta}{2}}\right), \end{cases} \tag{2.3}$$

where ν,θ,ϱ *are the* α*-dependent quantities*

$$\nu(\alpha) = 1-\alpha\,, \qquad \theta(\alpha) = 2-\alpha\,, \qquad \varrho(\alpha) = \frac{\nu}{\theta} = \frac{1-\alpha}{2-\alpha}\,; \tag{2.4}$$

$\mathsf{J}_{\mp\varrho}(s)$, $\varrho\notin\mathbb{Z}$, *are the linearly independent Bessel functions of the first kind and orders* $\mp\varrho$ [14]; $\{s_{k,\mu}\}_{\mu=1}^{\infty}$ *are the unbounded monotonically increasing sequences of the zeros of functions* $\mathsf{J}_{\mp\varrho}(s)$; 2) *the eigenfunctions of each kind are orthogonal in* $L_2[-1,0]$ *and* $L_2[0,+1]$ *respectively, that is*

$$\mp\int_0^{\mp 1} Z_{k,\mu}(x;\alpha)\,Z_{k,\gamma}(x;\alpha)\,\mathrm{d}x = \frac{1}{\theta}\,\mathsf{J}^2_{\mp\varrho+1}(s_{k,\mu})\,\delta_{\mu,\gamma} \equiv \|Z_{k,\mu}\|_1^2\,\delta_{\mu,\gamma}\,, \tag{2.5}$$

where $\mu,\gamma\in\mathbb{N}$, *and* $\delta_{\mu,\gamma}$ *is the Kronecker delta.*

Proposition 2.2. *Let the following composite 1-parameter boundary value problem be given*

$$\begin{cases} D'(x;\alpha) + \lambda(\alpha)\,X(x;\alpha) = 0\,, \qquad 0<|x|<1\,, \\ a)\ X(\mp 1;\alpha) = 0\,, \qquad b)\ X(x;\alpha)\big|_{x=0-0} = X(x;\alpha)\big|_{x=0+0}\,, \\ \qquad\qquad\qquad\qquad\quad c)\ D(x;\alpha)\big|_{x=0-0} = D(x;\alpha)\big|_{x=0+0}\,, \end{cases} \tag{2.6}$$

where $D(x;\alpha)=a(x;\alpha)\,X'(x;\alpha)$ *is the flux of* $X(x;\alpha)$*, then in the case of weak degeneracy:* 1) *the eigenvalues* $\lambda_{k,\mu}(\alpha)$ *and the eigenfunctions* $X_{k,\mu}(x;\alpha)$ *of the problem* (2.6) *of the two kinds* (*marked with* $k\in\{1,2\}$) *are defined as follows*

$$\begin{cases} \lambda_{1,\mu}(\alpha) = \sigma^2_{1,\mu}\,, & X_{1,\mu}(x;\alpha) = \qquad Z_{1,\mu}(x;\alpha)\,, \\ \lambda_{2,\mu}(\alpha) = \sigma^2_{2,\mu}\,, & X_{2,\mu}(x;\alpha) = \operatorname{sgn} x\; Z_{2,\mu}(x;\alpha)\,, \end{cases} \tag{2.7}$$

where $\sigma^2_{k,\mu}(\alpha)$ *and* $Z_{k,\mu}(x;\alpha)$ *are given in* (2.3) *of Prop. 2.1;* 2) *the eigenfunctions of both kinds are orthogonal in* $\mathscr{L}_2[-1,+1]$*, that is*

$$\begin{cases} \displaystyle\int_{-1}^{+1} X_{k,\mu}(x;\alpha)\,X_{k,\gamma}(x;\alpha)\,\mathrm{d}x = 2\,\|Z_{k,\mu}\|_1^2\;\delta_{\mu,\gamma} \equiv \|X_{k,\mu}\|_2^2\;\delta_{\mu,\gamma}\,, \\ \displaystyle\int_{-1}^{+1} X_{1,\mu}(x;\alpha)\,X_{2,\gamma}(x;\alpha)\,\mathrm{d}x = 0\,. \end{cases} \tag{2.8}$$

The eigenfunctions $X_{k,\mu}(x;\alpha)$ and their fluxes $D_{k,\mu}(x;\alpha)=a(x;\alpha)\,X'_{k,\mu}(x;\alpha)$ (refer to Fig. 2.2), due to (2.1) and (2.7), have the following series representations

$$\begin{cases} X_{1,\mu}(x;\alpha) = \qquad\qquad \Big(\dfrac{s_{1,\mu}}{2}\Big)^{-\varrho} \displaystyle\sum_{\gamma=0}^{\infty} \dfrac{(-1)^\gamma\,|x|^{\gamma\theta}}{\gamma!\,\Gamma(1-\varrho+\gamma)}\Big(\dfrac{s_{1,\mu}}{2}\Big)^{2\gamma}\,, \\ X_{2,\mu}(x;\alpha) = \operatorname{sgn} x\;|x|^{\nu}\Big(\dfrac{s_{2,\mu}}{2}\Big)^{+\varrho} \displaystyle\sum_{\gamma=0}^{\infty} \dfrac{(-1)^\gamma\,|x|^{\gamma\theta}}{\gamma!\,\Gamma(1+\varrho+\gamma)}\Big(\dfrac{s_{2,\mu}}{2}\Big)^{2\gamma}\,, \end{cases} \tag{2.9}$$

$$\begin{cases} D_{1,\mu}(x;\alpha) = \Big(\dfrac{s_{1,\mu}}{2}\Big)^{-\varrho}\theta\,x \displaystyle\sum_{\gamma=1}^{\infty} \dfrac{(-1)^\gamma\gamma\,|x|^{(\gamma-1)\theta}}{\gamma!\,\Gamma(1-\varrho+\gamma)}\Big(\dfrac{s_{1,\mu}}{2}\Big)^{2\gamma}\,, \\ D_{2,\mu}(x;\alpha) = \Big(\dfrac{s_{2,\mu}}{2}\Big)^{+\varrho} \displaystyle\sum_{\gamma=0}^{\infty} \dfrac{(-1)^\gamma\,[\nu+\gamma\theta]\,|x|^{\gamma\theta}}{\gamma!\,\Gamma(1+\varrho+\gamma)}\Big(\dfrac{s_{2,\mu}}{2}\Big)^{2\gamma}\,. \end{cases} \tag{2.10}$$

The resulting series (2.9), (2.10) give the following values on the degeneracy segment

$$\begin{cases} X_{1,\mu}(0;\alpha) = \Big(\dfrac{s_{1,\mu}}{2}\Big)^{-\varrho}\dfrac{1}{\Gamma(1-\varrho)}\,, & \alpha\in(0,2)\,, \\ X_{2,\mu}(0;\alpha) = \qquad\qquad 0\,, & \alpha\in(0,1)\,, \\ \operatorname{sgn} x\,\displaystyle\lim_{x\to 0}|x|^{-\nu}X_{2,\mu}(x;\alpha) = \Big(\dfrac{s_{2,\mu}}{2}\Big)^{+\varrho}\dfrac{1}{\Gamma(1+\varrho)}\,, & \alpha\in[1,2)\,, \end{cases} \tag{2.11}$$

$$\begin{cases} D_{1,\mu}(0;\alpha) = \qquad\qquad 0\,, \\ D_{2,\mu}(0;\alpha) = \Big(\dfrac{s_{2,\mu}}{2}\Big)^{\varrho}\dfrac{\nu}{\Gamma(1+\varrho)} \neq 0\,, \end{cases} \qquad \alpha\in(0,2)\,. \tag{2.12}$$

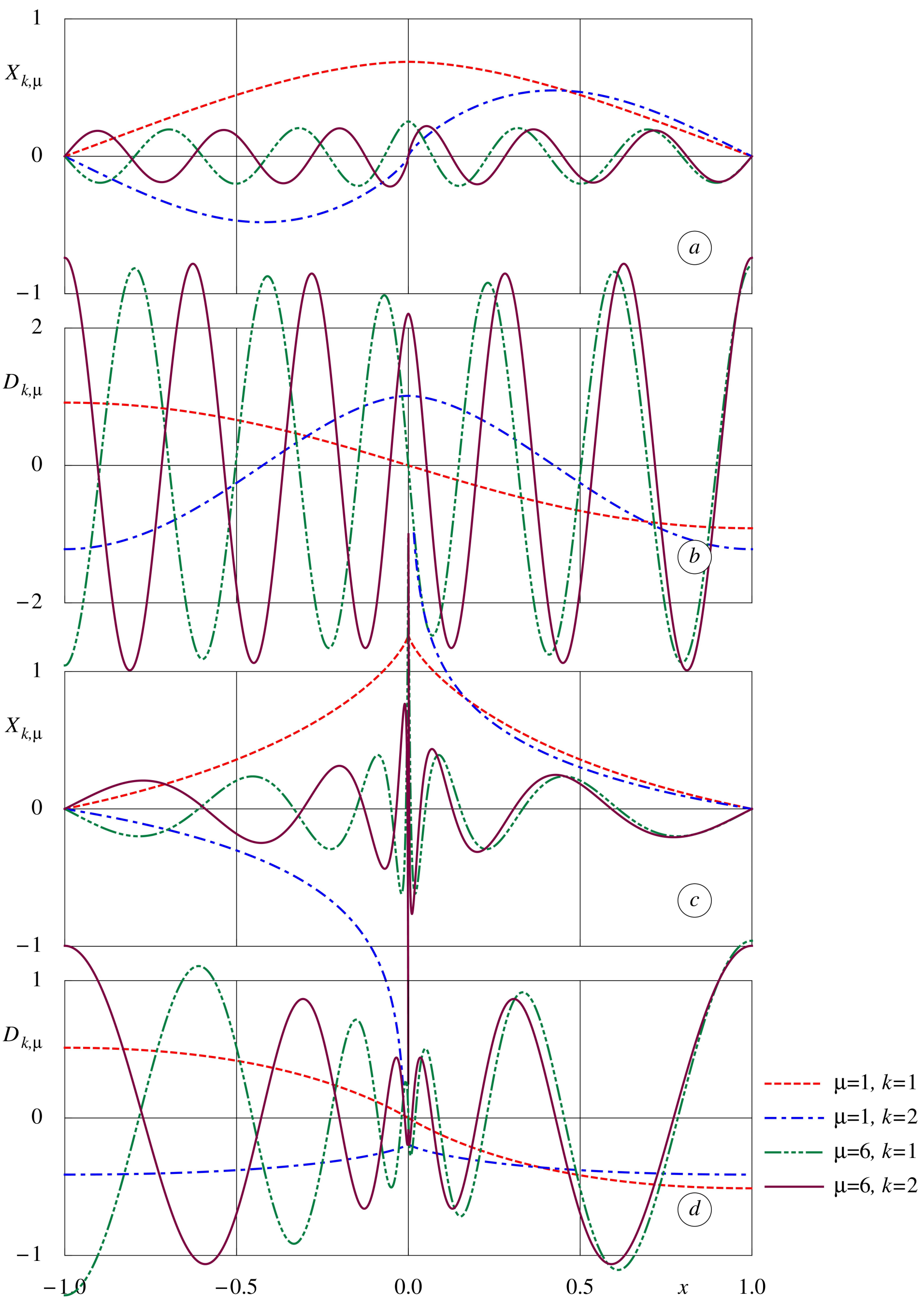

Fig. 2.2. Eigenvalues $X_{k,\mu}(x;\alpha)$ (2.7) (a,c) of the composite boundary-value problem (2.6) and their fluxes $D_{k,\mu}(x;\alpha) = a(x;\alpha)\,X'_{k,\mu}(x;\alpha)$ (b,d) for $\alpha = 0.25$ (a,b) and $\alpha = 1.25$ (c,d)

1) In the case of weak degeneracy the above values are in agree with conditions *b*) and *c*) of Prop. 2.2.

2) In the case of strong degeneracy, the eigenfunctions $X_{1,\mu}(x;\alpha)$ still obey conditions *b*) and *c*), whereas the eigenfunctions $X_{2,\mu}(x;\alpha)$, being unbounded and discontinuous at $x=0$, do not obey condition *b*), nevertheless do obey condition *c*). We shall try to build formally the solution to the IBVPS for $\alpha\in(1,2)$, keeping in mind this property of $X_{2,\mu}(x;\alpha)$.

3) In the case of non-degeneracy ($\nu=1$, $\theta=2$), the Bessel functions of the first kind and orders $\mp\varrho=\mp\frac{1}{2}$ are known [13, 14] to simplify to

$$\begin{cases} \mathtt{J}_{-\varrho}(s)=\mathtt{J}_{-\frac{1}{2}}(s)=\sqrt{\dfrac{2}{\pi s}}\,\cos s\,, \\ \mathtt{J}_{+\varrho}(s)=\mathtt{J}_{+\frac{1}{2}}(s)=\sqrt{\dfrac{2}{\pi s}}\,\sin s\,, \end{cases} \qquad s>0\,, \tag{2.13}$$

then the eigenfunctions read

$$\begin{cases} X_{1,\mu}(x;0)= \qquad |x|^{\frac{1}{2}}\,\mathtt{J}_{-\varrho}\big(s_{1,\mu}\,|x|\big)=\sqrt{\dfrac{2}{\pi\,s_{1,\mu}}}\,\cos\big(s_{1,\mu}\,x\big)\,, \\ X_{2,\mu}(x;0)=\operatorname{sign}x\,|x|^{\frac{1}{2}}\,\mathtt{J}_{+\varrho}\big(s_{2,\mu}\,|x|\big)=\sqrt{\dfrac{2}{\pi\,s_{2,\mu}}}\,\sin\big(s_{2,\mu}\,x\big)\,, \end{cases} \qquad x\in[-1,+1]\,,$$

where the eigenvalues are as follows

$$\begin{cases} s_{1,\mu}(0)=\sigma_{1,\mu}(0)=(2\mu-1)\,\dfrac{\pi}{2}\equiv s_{1,\mu}\,, \\ s_{2,\mu}(0)=\sigma_{2,\mu}(0)=(2\mu-0)\,\dfrac{\pi}{2}\equiv s_{2,\mu}\,, \end{cases} \qquad \mu\in\mathbb{N}\,. \tag{2.14}$$

Taking the above eigenfunctions, for the sake of convenience, as

$$\begin{cases} X_{1,\mu}(x;0)=\cos\big(\sigma_{1,\mu}\,x\big)\,, \\ X_{2,\mu}(x;0)=\sin\big(\sigma_{2,\mu}\,x\big)\,, \end{cases} \qquad x\in[-1,+1]\,, \tag{2.15}$$

we obtain the well-known orthonormal system of the eigenfunctions of the Sturm-Liouville BPV [12]

$$\begin{cases} X''(x;0)+\lambda\,X(x;0)=0\,, \qquad x\in[-1,+1]\,, \\ X(\mp1;0)=0\,, \end{cases} \tag{2.16}$$

satisfying conditions *b*) and *c*).

The next proposition directly follows from Prop. 2.2, transformation (1.4) and its inverse, though it can be easily proved independently.

Proposition 2.3. *Let the following composite 1-parameter boundary value problem be given*

$$\begin{cases} \dfrac{\theta}{2}\,|\xi|^{-\frac{\alpha}{\theta}}\,\Psi'(\xi;\alpha)+\lambda\,\Phi(\xi;\alpha)=0\,, \qquad 0<|\xi|<1\,, \\ a)\ \Phi(\mp 1;\alpha)=0\,, \qquad b)\ \Phi(\xi;\alpha)\big|_{\xi=0-0}=\Phi(\xi;\alpha)\big|_{\xi=0+0}\,, \\ \qquad\qquad\qquad\qquad\quad c)\ \Psi(\xi;\alpha)\big|_{\xi=0-0}=\Psi(\xi;\alpha)\big|_{\xi=0+0}\,, \end{cases} \tag{2.17}$$

where $\Psi(\xi;\alpha)=\dfrac{\theta}{2}\,|\xi|^{\frac{\alpha}{\theta}}\,\Phi'(\xi;\alpha)$ *is the flux of* $\Phi(\xi;\alpha)$*, then in the case of weak degeneracy:* 1) *the eigenvalues* $\lambda_{k,\mu}(\alpha)$ *and the eigenfunctions* $\Phi_{k,\mu}(\xi;\alpha)$ *of the problem* (2.17) *of the two kinds* (*marked with* $k\in\{1,2\}$) *are defined as follows*

$$\begin{cases} \lambda_{1,\mu}(\alpha)=\sigma^2_{1,\mu}\,, \qquad \Phi_{1,\mu}(\xi;\alpha)= |\xi|^{\varrho}\,\mathrm{J}_{-\varrho}\Big(s_{1,\mu}\,|\xi|\Big)\,, \\ \lambda_{2,\mu}(\alpha)=\sigma^2_{2,\mu}\,, \qquad \Phi_{2,\mu}(\xi;\alpha)=\operatorname{sgn}\xi\ |\xi|^{\varrho}\,\mathrm{J}_{+\varrho}\Big(s_{2,\mu}\,|\xi|\Big)\,, \end{cases} \tag{2.18}$$

where $\sigma^2_{k,\mu}$ *are given in* (2.3) *of Prop. 2.1;* 2) *the eigenfunctions of both kinds are orthogonal in* $L_2[-1,+1]$ *with the weight* $|\xi|^{\frac{\alpha}{\theta}}$ (*for detail refer to Sect. 4*)*, that is*

$$\begin{cases} \displaystyle\int_{-1}^{+1}|\xi|^{\frac{\alpha}{\theta}}\,\Phi_{k,\mu}(\xi;\alpha)\,\Phi_{k,\gamma}(\xi;\alpha)\,\mathrm{d}\xi=\|\Phi_{k,\mu}\|_3^2\,\delta_{\mu,\gamma}=\frac{\theta}{2}\,\|X_{k,\mu}\|_2^2\,\delta_{\mu,\gamma}\,, \\ \displaystyle\int_{-1}^{+1}|\xi|^{\frac{\alpha}{\theta}}\,\Phi_{1,\mu}(\xi;\alpha)\,\Phi_{2,\gamma}(\xi;\alpha)\,\mathrm{d}\xi=0\,. \end{cases} \tag{2.19}$$

The eigenfunctions $\Phi_{k,\mu}(\xi;\alpha)$ and their fluxes $\Psi_{k,\mu}(\xi;\alpha)=\dfrac{\theta}{2}\,|\xi|^{\frac{\alpha}{\theta}}\,\Phi'_{k,\mu}(\xi;\alpha)$ (refer to Fig. 2.3), due to (2.1), (2.18), have the following series representations

$$\begin{cases} \Phi_{1,\mu}(\xi;\alpha)= \displaystyle\Big(\frac{s_{1,\mu}}{2}\Big)^{-\varrho}\sum_{\gamma=0}^{\infty}\frac{(-1)^\gamma}{\gamma!\,\Gamma(1-\varrho+\gamma)}\,\Big(\frac{s_{1,\mu}|\xi|}{2}\Big)^{2\gamma}\,, \\ \Phi_{2,\mu}(\xi;\alpha)=\operatorname{sgn}\xi\ |\xi|^{2\rho}\displaystyle\Big(\frac{s_{2,\mu}}{2}\Big)^{+\varrho}\sum_{\gamma=0}^{\infty}\frac{(-1)^\gamma}{\gamma!\,\Gamma(1+\varrho+\gamma)}\,\Big(\frac{s_{2,\mu}|\xi|}{2}\Big)^{2\gamma}\,, \end{cases} \tag{2.20}$$

$$\begin{cases} \Psi_{1,\mu}(x;\alpha)=\theta\ \operatorname{sgn}\xi\,|\xi|^{\frac{2}{\theta}}\displaystyle\Big(\frac{s_{1,\mu}}{2}\Big)^{-\varrho}\sum_{\gamma=1}^{\infty}\frac{(-1)^\gamma\gamma\,|\xi|^{2(\gamma-1)}}{\gamma!\,\Gamma(1-\varrho+\gamma)}\,\Big(\frac{s_{1,\mu}}{2}\Big)^{2\gamma}\,, \\ \Psi_{2,\mu}(x;\alpha)= \displaystyle\Big(\frac{s_{2,\mu}}{2}\Big)^{+\varrho}\sum_{\gamma=0}^{\infty}\frac{(-1)^\gamma\,[\nu+\gamma\theta]\,|\xi|^{2\gamma}}{\gamma!\,\Gamma(1+\varrho+\gamma)}\,\Big(\frac{s_{2,\mu}}{2}\Big)^{2\gamma}\,, \end{cases} \tag{2.21}$$

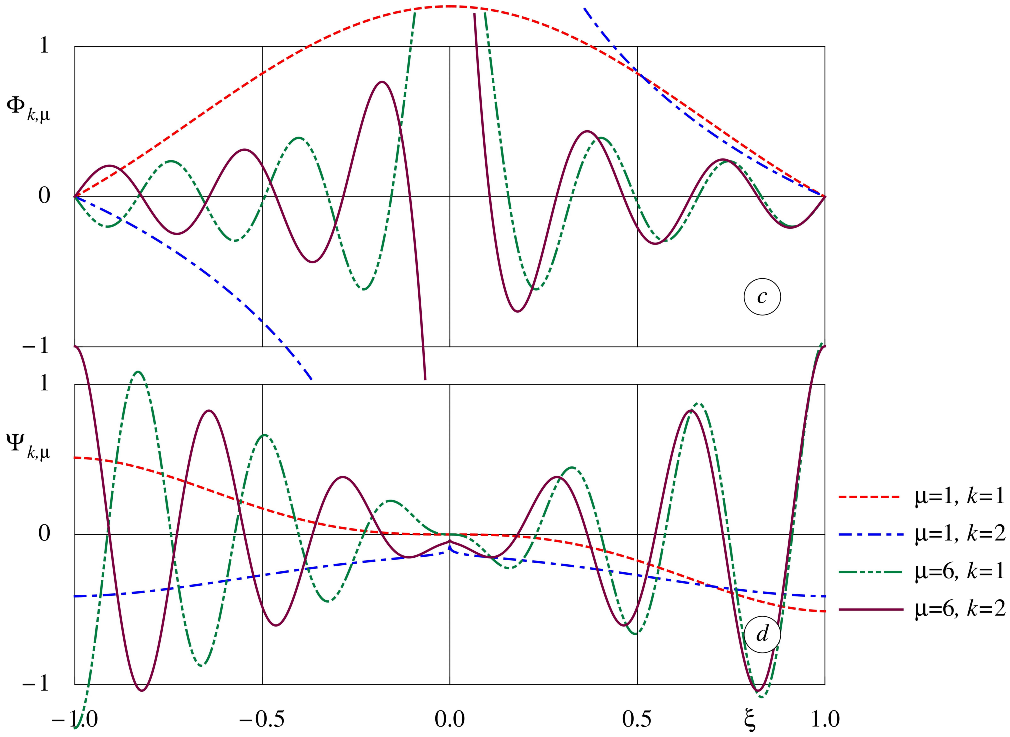

Fig. 2.3. Eigenvalues $\Phi_{k,\mu}(\xi;\alpha)$ (2.18) (c) of the composite boundary-value problem (2.17) and their fluxes $\Psi_{k,\mu}(\xi;\alpha) = \Phi'_{k,\mu}(x;\alpha)$ (d) for $\alpha = 1.25$ (c,d) (cf. Fig. 2.2, c,d)

and take exactly the same values on the degeneracy segment as those (2.11), (2.12) for the eigenfunctions $X_{k,\mu}(\xi;\alpha)$ and their fluxes $D_{k,\mu}(\xi;\alpha)$.

3. Applying separation of variables to the problem

The solution to the IBVPS is assumed to have the following representation

$$u(t,x;\alpha) = v(t,x;\alpha) + w(t,x;\alpha)\,, \tag{3.1}$$

where: a) $v(t,x;\alpha)$ is the required function; b) the univariate interpolation function $w(t,x;\alpha)$ is the result of blending the control functions and is given as follows

$$w(t,x;\alpha) = \phi_1(x;\alpha)\,h_1(t;\alpha) + \phi_2(x;\alpha)\,h_2(t;\alpha)\,; \tag{3.2}$$

c) the smooth blending functions $\phi_1(x;\alpha)$, $\phi_2(x;\alpha)$ obey the boundary and regularity conditions, respectively

$$\begin{cases} \phi_1(-1;\alpha) = 1\,, & \phi_1(+1;\alpha) = 0\,, \\ \phi_2(-1;\alpha) = 0\,, & \phi_2(+1;\alpha) = 1\,; \end{cases} \tag{3.3}$$

$$\begin{cases} \psi_1(x;\alpha) \equiv \varphi_1'(x;\alpha) = \big[a(x;\alpha)\,\phi_1'(x;\alpha)\big]' \in \mathscr{C}[-1,+1]\,, \\ \psi_2(x;\alpha) \equiv \varphi_2'(x;\alpha) = \big[a(x;\alpha)\,\phi_2'(x;\alpha)\big]' \in \mathscr{C}[-1,+1]\,. \end{cases} \tag{3.4}$$

Combining (3.1) – (3.3) we obtain: *a*) the initial conditions for $v(t,x;\alpha)$

$$\begin{cases} v(0,x;\alpha) = u(0,x;\alpha) - w(0,x;\alpha) \equiv \overset{*}{v}(x;\alpha)\,, \\ \dfrac{\partial v(0,x;\alpha)}{\partial t} = \dfrac{\partial u(0,x;\alpha)}{\partial t} - \dfrac{\partial w(0,x;\alpha)}{\partial t} \equiv \overset{**}{v}(x;\alpha)\,, \end{cases} \tag{3.5}$$

and *b*) the reformulated IBVPS wrt $v(t,x;\alpha)$

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial}{\partial x}\left(a\,\dfrac{\partial v}{\partial x}\right) = g\,, & (t,x) \in (0,T]\times(-1,+1)\,, \\ \left.\begin{matrix} v(t,-1;\alpha) = 0 \\ v(t,+1;\alpha) = 0 \end{matrix}\right\}, & t \in [0,T]\,, \\ \left.\begin{matrix} \dfrac{\partial v(0,x;\alpha)}{\partial t} = \overset{**}{v}(x;\alpha) \\ v(0,x;\alpha) = \overset{*}{v}(x;\alpha) \end{matrix}\right\}, & x \in [-1,+1]\,, \end{cases} \tag{3.6}$$

where the right-hand side of the degenerate wave equation reads

$$\begin{aligned} -g(t,x;\alpha) &= \frac{\partial^2 w(t,x;\alpha)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x;\alpha)\,\frac{\partial w(t,x;\alpha)}{\partial x}\right) \\ &= \phi_1(x;\alpha)\,h_1''(t;\alpha) - \psi_1(x;\alpha)\,h_1(t;\alpha) \\ &\quad + \phi_2(x;\alpha)\,h_2''(t;\alpha) - \psi_2(x;\alpha)\,h_2(t;\alpha) \end{aligned} \tag{3.7}$$

Applying SV to (3.6) implies that the initial functions (3.5) and the right-hand side (3.7) are expanded into the series wrt $X_{k,\mu}(x;\alpha)$

$$\begin{cases} \overset{*}{v}(x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} \overset{*}{v}_{1,\mu}(\alpha)\,X_{1,\mu}(x;\alpha) + \sum_{\mu=1}^{\infty} \overset{*}{v}_{2,\mu}(\alpha)\,X_{2,\mu}(x;\alpha)\,, \\ \overset{**}{v}(x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} \overset{**}{v}_{1,\mu}(\alpha)\,X_{1,\mu}(x;\alpha) + \sum_{\mu=1}^{\infty} \overset{**}{v}_{2,\mu}(\alpha)\,X_{2,\mu}(x;\alpha)\,, \end{cases} \tag{3.8}$$

$$g(t,x;\alpha) = \sum_{\mu=1}^{\infty} g_{1,\mu}(t;\alpha)\,X_{1,\mu}(x;\alpha) + \sum_{\mu=1}^{\infty} g_{2,\mu}(t)(t;\alpha)\,X_{2,\mu}(x;\alpha)\,, \tag{3.9}$$

where the coefficients are calculated by direct integration

$$\begin{cases} \overset{*}{v}_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \overset{*}{v}(x;\alpha)\, X_{k,\mu}(x;\alpha)\, \mathrm{d}x\,, \\ \overset{**}{v}_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \overset{**}{v}(x;\alpha)\, X_{k,\mu}(x;\alpha)\, \mathrm{d}x\,, \\ g_{k,\mu}(t;\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} g(t,x;\alpha)\, X_{k,\mu}(x;\alpha)\, \mathrm{d}x\,. \end{cases} \tag{3.10}$$

Accounting for (3.9) the latter coefficients can be presented as follows

$$\begin{aligned} g_{k,\mu}(t;\alpha) &= c_{k,\mu}(\alpha)\, h_1(t;\alpha) - a_{k,\mu}(\alpha)\, h_1''(t;\alpha) \\ &\quad + d_{k,\mu}(\alpha)\, h_2(t;\alpha) - b_{k,\mu}(\alpha)\, h_2''(t;\alpha)\,, \end{aligned} \tag{3.11}$$

where

$$\begin{cases} a_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \phi_1(x;\alpha)\, X_{k,\mu}(x)\, \mathrm{d}x\,, \\ b_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \phi_2(x;\alpha)\, X_{k,\mu}(x)\, \mathrm{d}x\,, \\ c_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \psi_1(x;\alpha)\, X_{k,\mu}(x)\, \mathrm{d}x\,, \\ d_{k,\mu}(\alpha) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{+1} \psi_2(x;\alpha)\, X_{k,\mu}(x)\, \mathrm{d}x\,. \end{cases} \tag{3.12}$$

Then the standard SV procedure yields to the solution to the reformulated IBVPS (3.6), (3.7)

$$v(t,x;\alpha) = \sum_{\mu=1}^{\infty} O_{1,\mu}(t;\alpha)\, X_{1,\mu}(x;\alpha) + \sum_{\mu=1}^{\infty} O_{2,\mu}(t;\alpha)\, X_{2,\mu}(x;\alpha)\,, \tag{3.13}$$

where the time dependent coefficient functions read

$$\begin{aligned} O_{k,\mu}(t;\alpha) &= \overset{*}{v}_{k,\mu}(\alpha)\, \cos\big(\sigma_{k,\mu} t\big) + \sigma_{k,\mu}^{-1}\, \overset{**}{v}_{k,\mu}(\alpha)\, \sin\big(\sigma_{k,\mu} t\big) \\ &\quad + \sigma_{k,\mu}^{-1} \int_0^t g_{k,\mu}(\tau;\alpha)\, \sin\big[\sigma_{k,\mu}(t-\tau)\big]\, \mathrm{d}\tau\,. \end{aligned} \tag{3.14}$$

Finally, (3.13), (3.14) and representation (3.1) give the required solution to the original IBVPS (1.1), (1.2)

$$\begin{cases} u(t,x;\alpha) = \displaystyle\sum_{\mu=1}^{\infty} O_{1,\mu}(t;\alpha)\, X_{1,\mu}(x;\alpha) + \phi_1(x;\alpha)\, h_1(t;\alpha) \\ \qquad\qquad + \displaystyle\sum_{\mu=1}^{\infty} O_{2,\mu}(t;\alpha)\, X_{2,\mu}(x;\alpha) + \phi_2(x;\alpha)\, h_2(t;\alpha)\,, \end{cases} \tag{3.15}$$

with the flux

$$\begin{cases} f(t,x;\alpha) = \sum_{\mu=1}^{\infty} O_{1,\mu}(t;\alpha)\, D_{1,\mu}(x;\alpha) + \varphi_1(x;\alpha)\, h_1(t;\alpha) \\ \qquad\qquad + \sum_{\mu=1}^{\infty} O_{2,\mu}(t;\alpha)\, D_{2,\mu}(x;\alpha) + \varphi_2(x;\alpha)\, h_2(t;\alpha)\,, \end{cases} \tag{3.16}$$

being continuous on the space-time rectangle $[0,T] \times [-1,+1] \subset \mathbb{R}_t^+ \times \mathbb{R}_x$, due to: *a*) Prop. 2.2 and *b*) the regularity conditions (3.4). From this it immediately follows that the matching conditions (1.9) for the displacement (3.15) and the flux (3.16), imposed at the degeneracy segment, are satisfied in the case of weak degeneracy. In the case of strong degeneracy the displacement (3.15) is unbounded on the degeneracy segment, nevertheless the flux (3.15) is still continuous, therefore the first of the matching conditions fails, whereas the second one holds.

The solution $U(\tau,\xi;\alpha)$ to the IBVPT, with the flux $F(\tau,\xi;\alpha)$ being continuous on the space-time rectangle $[0,T] \times [-1,+1] \subset \mathbb{R}_t^+ \times \mathbb{R}_\xi$ due to Prop. 2.3, can be obtained in exactly the same way as the above solution to the IBVPS.

4. Existence and uniquiness of the solution to the problem

Now we refer to proving the uniqueness of the obtained in Sect. 3 solutions to the IBVPS and IBVPT, using a quite standard procedure [12], based on the equations governing the total energy wrt time for both problems. It is known [12], that multiplying the wave equation for a finite vibrating string by the partial derivative of the solution wrt time and integrating the product over the spatial segment yields, after some transformations of the integrand, to the required equation. This procedure is applicable for the degenerate wave equation of the IBVPS as well and yields to the following total energy equation

$$E(t;\alpha) = \frac{1}{2}\int_{-1}^{+1} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\sqrt{a}\,\frac{\partial u}{\partial x}\right)^2 \right] \mathrm{d}x = \text{const}\,, \qquad t \in [0,T]\,. \tag{4.1}$$

Applying the above procedure for the degenerate wave equation of the IBVPT involves sophisticated calculations, therefore we perform changing the variable of integration in (4.1) by referring to transformation (1.4) and its inverse, to obtain the following total energy equation

$$\Theta(\tau;\alpha) = \frac{1}{2}\int_{-1}^{+1} |\xi|^{\frac{\alpha}{\theta}} \left[\frac{2}{\theta}\left(\frac{\partial U}{\partial \tau}\right)^2 + \frac{\theta}{2}\left(\frac{\partial U}{\partial \xi}\right)^2 \right] \mathrm{d}\xi = \text{const}\,, \quad \tau \in [0,T]\,. \tag{4.2}$$

In (4.1), (4.2) the first and second terms in the brackets are responsible for the kinetic and potential energy, respectively.

Then we introduce a function space, associated with the IBVPS and denoted as $H^1_a\left([0,T]\times[-1,+1]\right)\equiv H^1_a$, of all functions satisfying the following conditions

$$\begin{cases} a) \qquad u(t,|x|;\alpha) \quad \in PC^{(2,2)}\left([0,T]\times(0,1]\right), \\ \left.\begin{array}{l} b) \qquad u(t,x;\alpha) \\ c) \ \sqrt{a(x;\alpha)}\ \dfrac{\partial u(t,x;\alpha)}{\partial x} \end{array}\right\} \in L_2[-1,+1], \qquad t\in[0,T]. \end{cases} \tag{4.3}$$

The above space is of Hilbert type wrt the scalar product

$$(u,v)_4=\int_{-1}^{+1}\left[uv+a\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)\right]\mathrm{d}x \tag{4.4}$$

and the associated norm

$$\|u\|_4^2=\int_{-1}^{+1}\left[u^2+a\left(\frac{\partial u}{\partial x}\right)^2\right]\mathrm{d}x\,, \tag{4.5}$$

determined for any $u(t,x;\alpha), v(t,x;\alpha)\in H^1_a$.

Doing in the same way, we also introduce a function space, associated with the IBVPT and denoted as $H^1_\xi\left([0,T]\times[-1,+1]\right)\equiv H^1_\xi$, of all functions satisfying the following conditions

$$\begin{cases} a) \qquad U(\tau,|\xi|;\alpha) \quad \in PC^{(2,2)}\left([0,T]\times(0,1]\right), \\ \left.\begin{array}{l} b) \quad |\xi|^{\frac{1}{2}\frac{\alpha}{\theta}}\ U(\tau,\xi;\alpha) \\ c) \ |\xi|^{\frac{1}{2}\frac{\alpha}{\theta}}\ \dfrac{\partial U(\tau,\xi;\alpha)}{\partial \xi} \end{array}\right\} \in L_2[-1,+1], \qquad \tau\in[0,T]. \end{cases} \tag{4.6}$$

The space again is of Hilbert type wrt the scalar product

$$(U,V)_5=\int_{-1}^{+1}|\xi|^{\frac{\alpha}{\theta}}\left[UV+a\left(\frac{\partial U}{\partial \xi}\right)\left(\frac{\partial V}{\partial \xi}\right)\right]\mathrm{d}\xi \tag{4.7}$$

and the associated norm

$$\|U\|_5^2=\int_{-1}^{+1}|\xi|^{\frac{\alpha}{\theta}}\left[U^2+\left(\frac{\partial U}{\partial \xi}\right)^2\right]\mathrm{d}\xi\,, \tag{4.8}$$

determined for any $U(\tau,\xi;\alpha), V(\tau,\xi;\alpha)\in H^1_\xi$.

Two following Props. 4.1, 4.2 can be readily proved: 1) using the total energy equations(4.1), (4.2); 2) accounting that $X_{k,\mu}(x;\alpha)\subset H^1_a$, $\Phi_{k,\mu}(\xi;\alpha)\subset H^1_\xi$; and 3) following any textbook on partial differential equations, for example [12].

Proposition 4.1. *Let the IBVPS be posed for functions $u(t,x;\alpha)\in H^1_a$, then there exists a solution to the IBVPS (for example, obtained by SV) and this solution is unique.*

Proposition 4.2. *Let the IBVPT be posed for functions $U(\tau, \xi; \alpha) \in H^1_\xi$, then there exists a solution to the IBVPT (for example, obtained by SV) and this solution is unique.*

5. Calculating the coefficients a, b, c, and d

We take for blending in (3.2) the following power functions

$$\begin{cases} \phi_1(x;\alpha) = \dfrac{1 - \operatorname{sgn} x}{2}\,|x|^{\omega_1}, \\ \phi_2(x;\alpha) = \dfrac{1 + \operatorname{sgn} x}{2}\,|x|^{\omega_2}, \end{cases} \tag{5.1}$$

where the exponents $\omega_j(\alpha)$ are to be determined. To impose the proper constraint on the exponents, we calculate: 1) the 'fluxes' $\varphi_j(x;\alpha) = a(x;\alpha)\,\phi_j'(x;\alpha)$

$$\begin{cases} \varphi_1(x;\alpha) = \dfrac{\operatorname{sgn} x - 1}{2}\,\omega_1\,|x|^{\omega_1 - \theta + 1}, \\ \varphi_2(x;\alpha) = \dfrac{1 + \operatorname{sgn} x}{2}\,\omega_2\,|x|^{\omega_2 - \theta + 1}, \end{cases}$$

and 2) their derivatives $\psi_j(x;\alpha) = \varphi_j'(x;\alpha) = \big[a(x;\alpha)\,\phi_j'(x;\alpha)\big]'$

$$\begin{cases} \psi_1(x;\alpha) = \dfrac{1 - \operatorname{sgn} x}{2}\,\omega_1\,[\omega_1 - \theta + 1]\,|x|^{\omega_1 - \theta}, \\ \psi_2(x;\alpha) = \dfrac{1 + \operatorname{sgn} x}{2}\,\omega_2\,[\omega_2 - \theta + 1]\,|x|^{\omega_2 - \theta}, \end{cases} \tag{5.2}$$

and assume that the latter vanish at $x = 0$ smoothly: $\psi_j(0;\alpha) = 0$, $\psi_j'(0;\alpha) = 0$, therefore $\omega_j - \theta = 1 + \epsilon_j$, where the free parameter $\epsilon_j > 0$. Taking $\omega_j \equiv \omega$ and substituting the functions $\phi_j(x;\alpha,\epsilon)$ (5.1), $\psi_j(x;\alpha,\epsilon)$ (5.2) into (3.12) yields to

$$\begin{cases} a_{k,\mu}(\alpha,\epsilon) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{0} \phi_1(x;\alpha,\epsilon)\,X_{k,\mu}(x;\alpha)\,\mathrm{d}x\,, \\ c_{k,\mu}(\alpha,\epsilon) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{-1}^{0} \psi_1(x;\alpha,\epsilon)\,X_{k,\mu}(x;\alpha)\,\mathrm{d}x\,, \end{cases}$$

$$\begin{cases} b_{k,\mu}(\alpha,\epsilon) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{0}^{1} \phi_2(x;\alpha,\epsilon)\,X_{k,\mu}(x;\alpha)\,\mathrm{d}x\,, \\ d_{k,\mu}(\alpha,\epsilon) = \dfrac{1}{\|X_{k,\mu}\|_2^2} \displaystyle\int_{0}^{1} \psi_2(x;\alpha,\epsilon)\,X_{k,\mu}(x;\alpha)\,\mathrm{d}x\,. \end{cases}$$

Accounting for the definition of $X_{k,\mu}(x;\alpha)$ (refer to Prop. 2.2 on p. 95), the above coefficients can be rewritten as

$$\begin{cases} a_{k,\mu}(\alpha,\epsilon) = \mp\dfrac{1}{2}\,\dfrac{Q_{k,\mu}(\alpha,\epsilon)}{\|Z_{k,\mu}\|_1^2}\,, & c_{k,\mu}(\alpha,\epsilon) = \pm\dfrac{\vartheta}{2}\,\dfrac{P_{k,\mu}(\alpha,\epsilon)}{\|Z_{k,\mu}\|_1^2}\,, \\ b_{k,\mu}(\alpha,\epsilon) = -\dfrac{1}{2}\,\dfrac{Q_{k,\mu}(\alpha,\epsilon)}{\|Z_{k,\mu}\|_1^2}\,, & d_{k,\mu}(\alpha,\epsilon) = +\dfrac{\vartheta}{2}\,\dfrac{P_{k,\mu}(\alpha,\epsilon)}{\|Z_{k,\mu}\|_1^2}\,, \end{cases} \tag{5.3}$$

where $\vartheta = \omega\,[\omega-\theta+1]$, $\theta\,\|Z_{k,\mu}\|^2 = \mathtt{J}^2_{\mp\varrho+1}(s_{k,\mu})$ (2.5), and the quantities $P_{k,\mu}(\alpha,\epsilon)$, $Q_{k,\mu}(\alpha,\epsilon)$ can be readily presented as follows

$$\begin{cases} P_{k,\mu}(\alpha,\epsilon) = \displaystyle\int_0^1 x^{\omega-\theta}\,Z_{k,\mu}(x;\alpha)\,\mathrm{d}x = \int_0^1 x^{\epsilon+1}\,x^{\frac{\nu}{2}}\,\mathtt{J}_{\mp\varrho}\Big(s_{k,\mu}\,|x|^{\frac{\theta}{2}}\Big)\,\mathrm{d}x \\ \qquad\qquad\; = \dfrac{2}{\theta}\left(\dfrac{1}{s_{k,\mu}}\right)^{\upsilon+1} P^*_{k,\mu}(\alpha,\epsilon)\,, \\ Q_{k,\mu}(\alpha,\epsilon) = \displaystyle\int_0^1 x^{\omega}\,Z_{k,\mu}(x;\alpha)\,\mathrm{d}x = \int_0^1 x^{\theta+\epsilon+1}\,x^{\frac{\nu}{2}}\,\mathtt{J}_{\mp\varrho}\Big(s_{k,\mu}\,|x|^{\frac{\theta}{2}}\Big)\,\mathrm{d}x \\ \qquad\qquad\; = \dfrac{2}{\theta}\left(\dfrac{1}{s_{k,\mu}}\right)^{\upsilon+3} Q^*_{k,\mu}(\alpha,\epsilon)\,, \end{cases} \tag{5.4}$$

applying the variable transformation $s = s_{k,\mu}\,x^{\frac{\theta}{2}}$ and notation $\upsilon\theta = 2\epsilon+3$.

The definite integrals in (5.4)

$$P^*_{k,\mu}(\alpha,\epsilon) = \int_0^{s_{k,\mu}} s^{\upsilon}\,\mathtt{J}_{\mp\varrho}(s)\,\mathrm{d}s\,, \qquad Q^*_{k,\mu}(\alpha,\epsilon) = \int_0^{s_{k,\mu}} s^{\upsilon+2}\,\mathtt{J}_{\mp\varrho}(s)\,\mathrm{d}s\,, \tag{5.5}$$

in turn, can be calculated exactly by parts, if the positive values of those, produced by the formulas: 1) $\epsilon = -1 + n\,\theta$, $n \in \mathbb{N}$, and 2) $\epsilon = -2 + m\,\theta$, $m \in \mathbb{N}$, are used for the free parameter ϵ, respectively, in cases: 1) $k=1$ and the upper sign in $\mathtt{J}_{\mp\varrho}(s)$, and 2) $k=2$ and the lower sign in $\mathtt{J}_{\mp\varrho}(s)$ [4]. Unfortunately, simultaneous (for the unique choice of ϵ in both cases) exact calculation of the above integrals is impossible. Subtracting one formula from the other gives the equality $1 = k\theta$, $k \in \mathbb{Z}$, which does not hold for any $\alpha \in [0,2]$, except for $\alpha = 1$. Therefore, numerical approaches are used to calculate (5.5).

6. Suppressing the Gibbs phenomenon

Expanding the initial functions $\overset{*}{u}(x;\alpha)$, $\overset{**}{u}(x;\alpha)$, $\overset{*}{U}(\xi;\alpha)$, $\overset{**}{U}(\xi;\alpha)$ of Sect. 1, and the blending functions $\phi_1(x;\alpha)$, $\phi_2(x;\alpha)$ of Sect. 5 is the only source of the spurious oscillations, or the Gibbs phenomenon, in the series solutions of Sect. 3. We tested some approaches [6], being different filters, to suppress the Gibbs phenomenon, especially near the degeneracy segment. Note, that method [11] can be easily adjusted to the eigenfunctions of Props. 2.2, 2.3. To avoid the Gibbs phenomenon, produced by the initial functions, we used mollifiers to smooth piecewise constant discontinuous initial functions (7.1) of test case A of Sect. 7. Therefore test case A is free from any sources of the Gibbs phenomenon, whereas test case B is sensitive to the Gibbs phenomenon, due to expanding the blending functions, therefore their expansions were properly filtered.

7. Test cases of the problem

We chose for the IBVPS and the IBVPT two test cases differing in the way the 'string' is excited: 1) by disturbing the initial shape of its left part, the 'string' being at rest, and 2) by moving its left end periodically, the right end being fixed.

Five different values, presented in Table. 1, were assigned to exponent α in simplified power law (1.2).

Table 1. α- and N-dependent quantities for the test cases

α	ν	θ	ϱ	$\Delta\xi$	$(\Delta x)_{min}$
0	1	2	$+\frac{1}{2}$	$1.000{\cdot}10^{-3}$	$1.000 \cdot 10^{-3}$
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{7}{4}$	$+\frac{3}{7}$	$1.000{\cdot}10^{-3}$	$3.378 \cdot 10^{-4}$
$\frac{2}{4}$	$\frac{2}{4}$	$\frac{6}{4}$	$+\frac{1}{3}$	$1.000{\cdot}10^{-3}$	$7.942 \cdot 10^{-5}$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{5}{4}$	$+\frac{1}{5}$	$1.000{\cdot}10^{-3}$	$1.047 \cdot 10^{-5}$
$\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{3}$	$1.000{\cdot}10^{-3}$	$3.154 \cdot 10^{-9}$

It is clear that there is no necessity to solve directly the IBVPT, using the explicit formulas similar to those (3.15), (3.16), to study the solution $u(t, x; \alpha)$ to the IBVPS in a close vicinity of the degeneracy segment, but it is quite enough to invoke the inverse of transformation (1.4) and apply it as a 'magnifying glass' to 'inflate' the known solution $u(t, x; \alpha)$ to the required solution $U(\tau, \xi; \alpha)$.

To this end, we first introduce the uniform grid on segment $[-1, +1] \subset \mathbb{R}_\xi$ with spacing $\Delta\xi$ between the nodes and the coordinates ξ_i of the formers, where

$$(N-1)\,\Delta\xi = 2\,, \qquad \xi_i = -1 + (i-1)\,\Delta\xi\,, \qquad i = 1, \dots, N\,,$$

N being the number of the nodes. In the case $\mathrm{mod}\,(N, 2) = 0$ two central nodes are biased wrt the degeneracy point $\xi = 0$ by half of $\Delta\xi$.

Second, we apply the inverse of transformation (1.4) to calculate the coordinates x_i of the grid nodes on segment $[-1,+1] \subset \mathbb{R}_x$, for the nodes to cluster near the degeneracy point $x = 0$.

All the results presented below were obtained at $N = 2000$, the corresponding values of $\Delta\xi$ and $(\Delta x)_{min}$ are presented in Table 1, to evaluate clustering the grid nodes x_i.

7.1. Test case A

In test case A: 1) the initially ($t = 0$) disturbed 'string' is at rest

$$\overset{**}{u}(x;\alpha) \equiv \overset{**}{u}_0 = 0\,, \quad \overset{*}{u}(x;\alpha) = \begin{cases} 0\,, & |x - x_0| > \delta\,, \\ \overset{*}{u}_0\,, & |x - x_0| \leqslant \delta\,, \end{cases} \qquad x \in [-1,+1]\,; \quad (7.1)$$

2) both ends of the 'string' are fixed

$$u(-1,t;\alpha) = u(+1,t;\alpha) = 0\,, \qquad t \in [0,T]\,, \tag{7.2}$$

i. e., both controls are absent: $h_1(t;\alpha) = h_2(t;\alpha) \equiv 0$.

The coefficient functions (3.14) then reduce to

$$O_{k,\mu}(t;\alpha) = \overset{*}{v}_{k,\mu}(\alpha)\,\cos\left(\sigma_{k,\mu} t\right), \tag{7.3}$$

where $\overset{*}{v}_{k,\mu}$ are given in (3.10) and $x_0 = -0.50$, $\delta = 0.10$, $\overset{*}{u}_0 = 0.50$. Recall, referring to Sect. 6, that the piece-wise constant discontinuous initial conditions (7.1) are smoothed before substituting into (3.10).

Below we present the solutions $u(t,x;\alpha)$ and $U(\tau,\xi;\alpha)$ to the IBVPS and to the IBVPT as the shapes or the displacements of the 'string', respectively, in Fig. 7.4, 7.5 and Fig. 7.6–7.9, at some instants of $t, \tau \in [0,9]$.

The initial shape (7.1) ($t = 0.0$) immediately breaks into two travelling waves moving into the opposite directions ($t = 0.4$). In the non-degenerate case the velocity of both travelling waves is known to be constant. In the case of weak degeneracy the travelling waves behave themselves similarly to those in the non-degenerate case, but that moving to the right slows down and increases its displacement near the degeneracy point ($t = 0.8$). In the case of strong degeneracy the 'string' loses the continuity of its shape, and the displacements become unbounded when the wave approaches the point of degeneracy ($t = 1.6$), they remain such both just after the wave passes the point ($t = 2.4$) and then during its further motion.

Reaching the right end of the 'string' the wave reflects with overturning and runs towards the left end, again passing through the degeneracy point. The same does the wave starting to move to the left from the beginning. From this we conclude that only a part of the total energy is spent to drive the degeneracy point in vertical direction, whereas the other part of the total energy is continuously redistributed between the kinetic and potential energy to provide motion of the waves. Note, that multiple passing through the degeneracy point leads to essential distortion of the initial shape of the travelling waves.

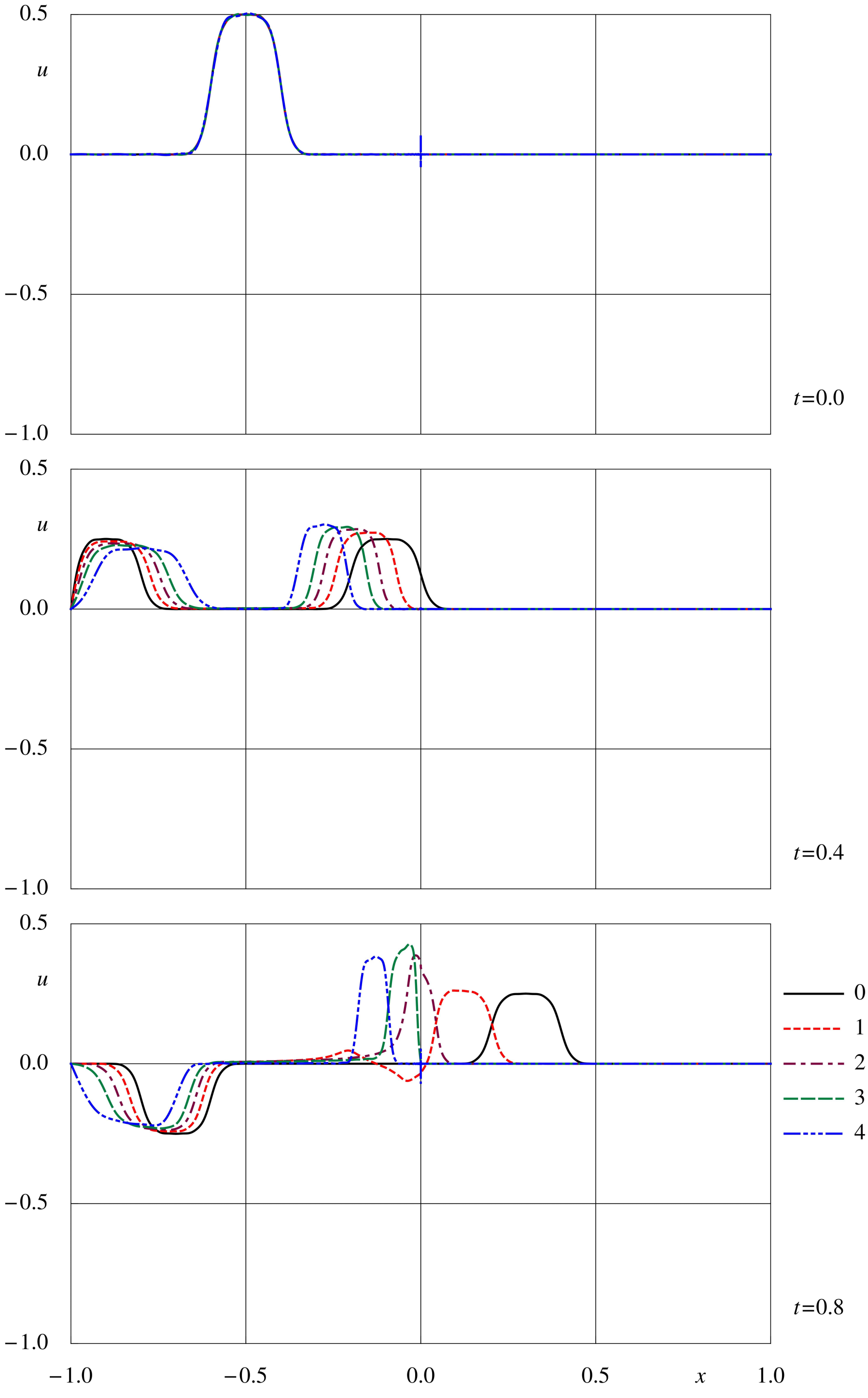

Fig. 7.4. Test case A: solution $u(t, x; \alpha)$ to the IBVPS (1.1), (1.2)

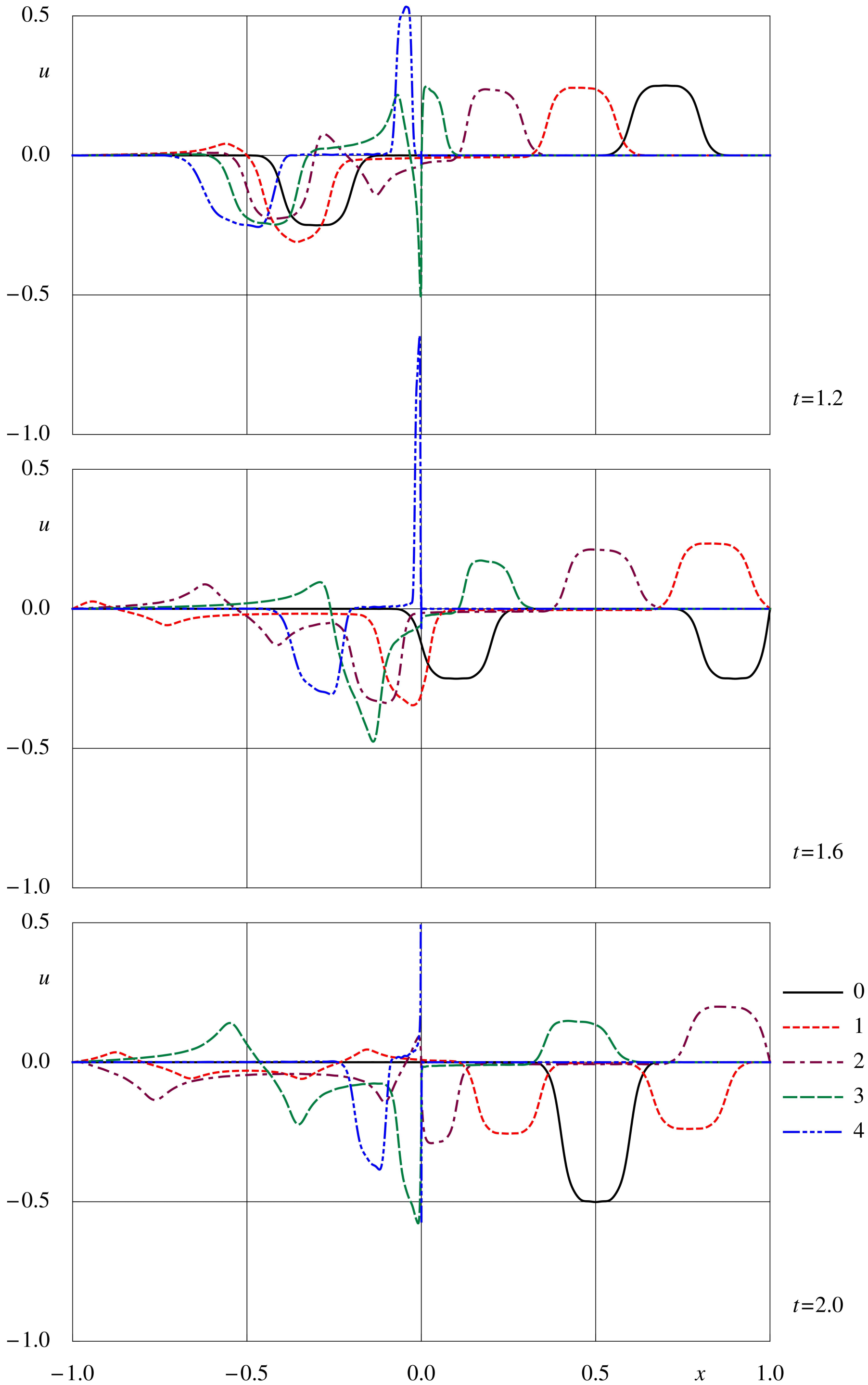

Fig. 7.5. Test case A: solution $u(t, x; \alpha)$ to the IBVPS (1.1), (1.2) (continued from Fig. 7.4)

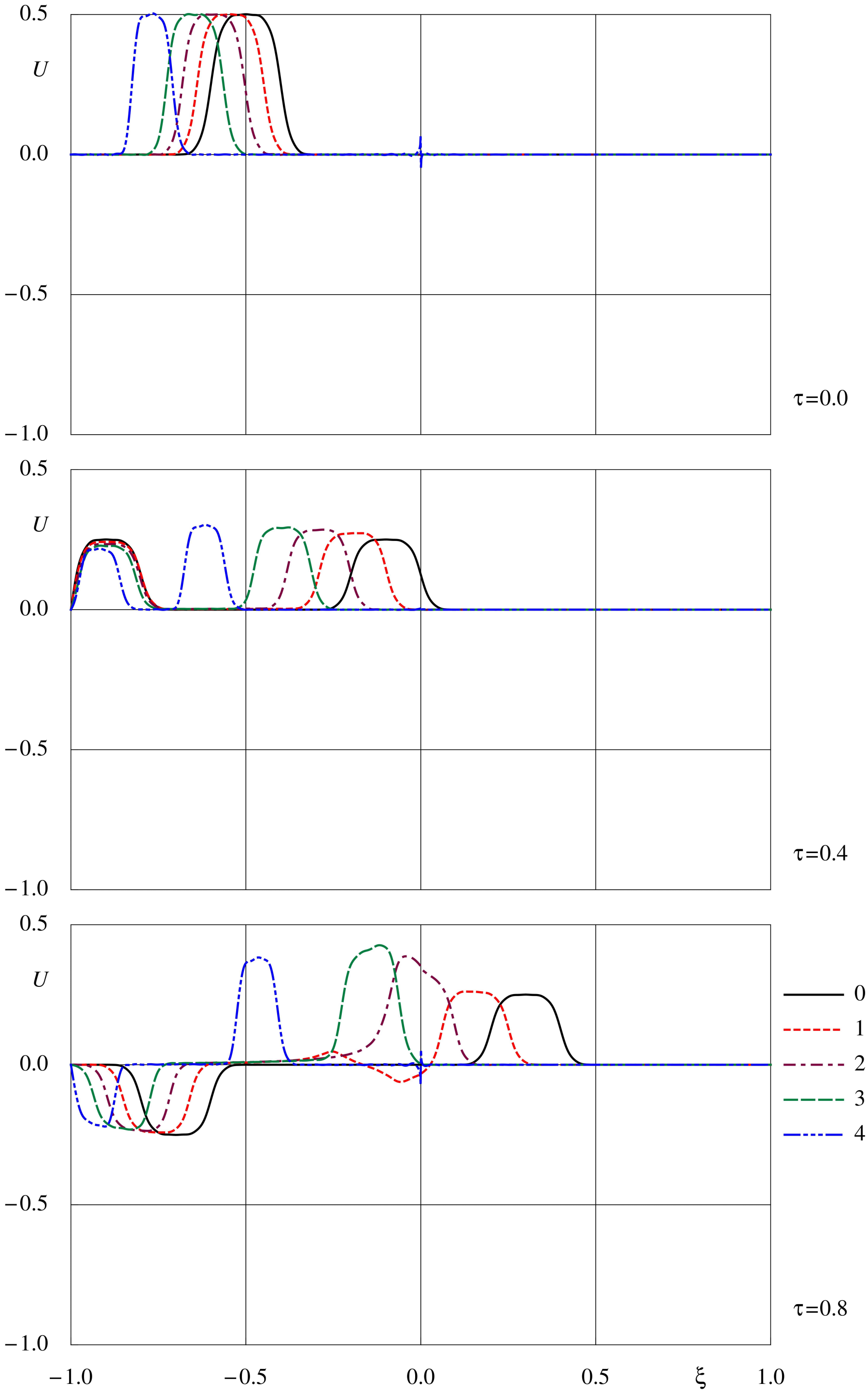

Fig. 7.6. Test case A: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5)

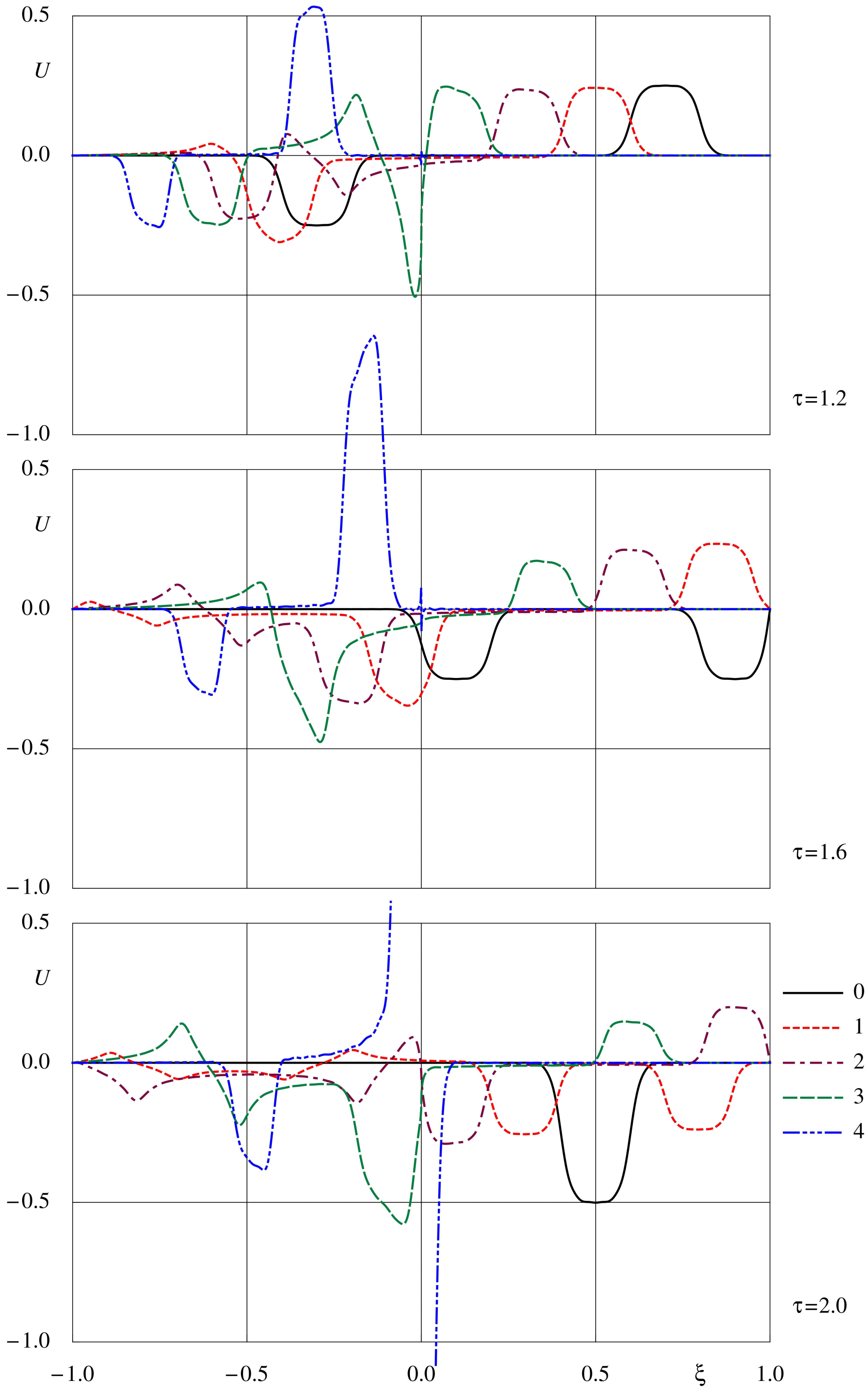

Fig. 7.7. Test case A: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5) (continued from Fig. 7.6)

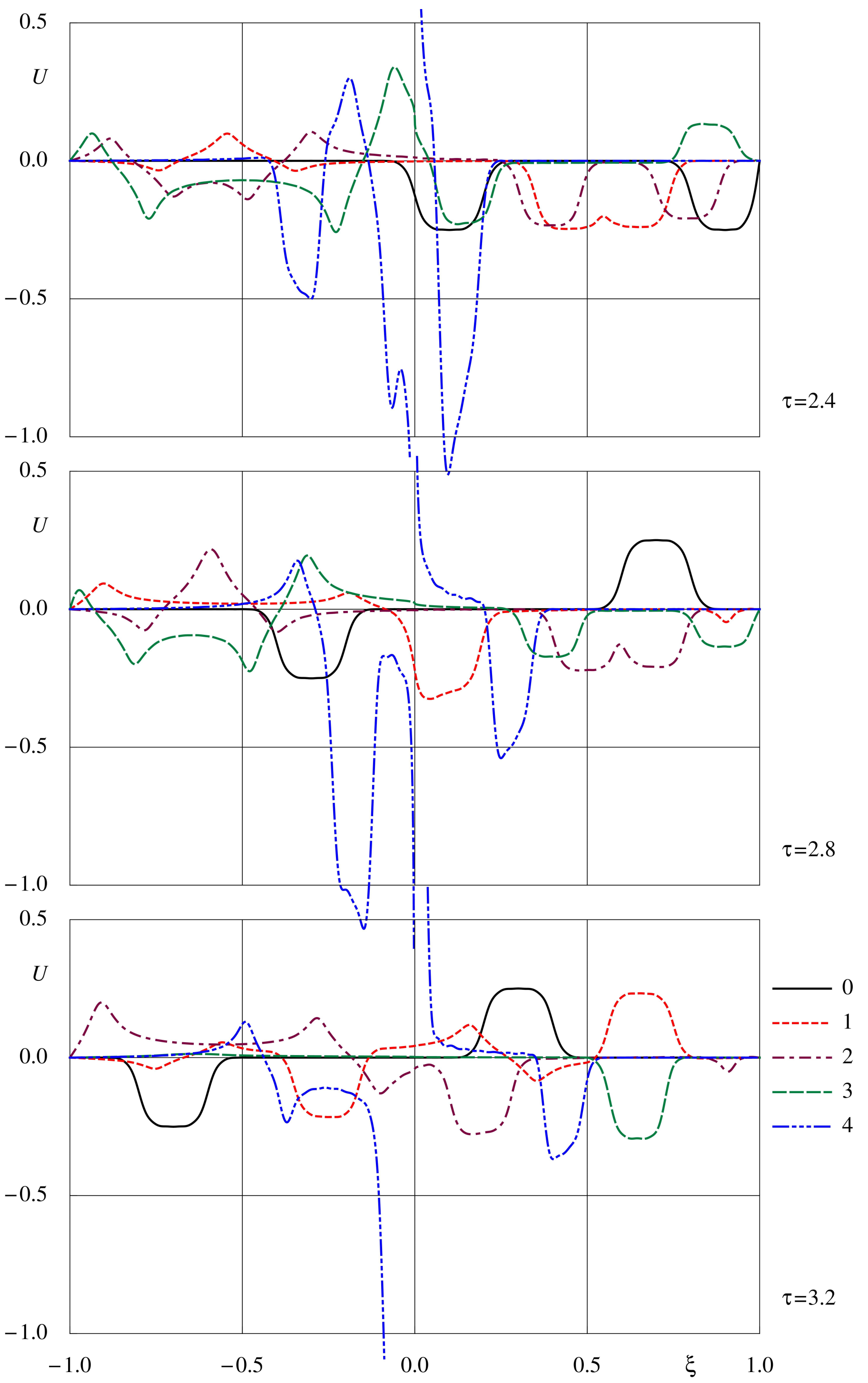

Fig. 7.8. Test case A: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5) (continued from Fig. 7.7)

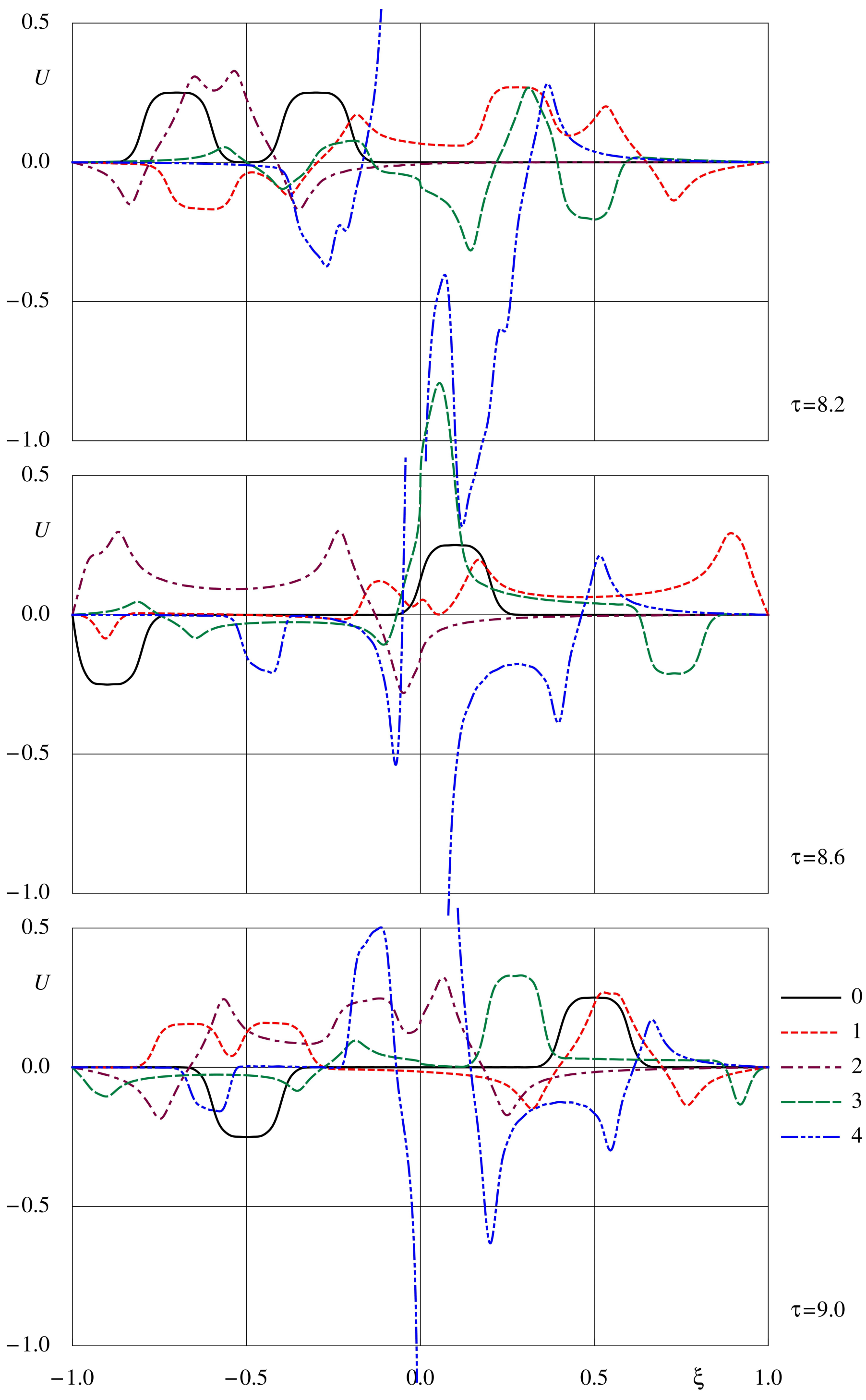

Fig. 7.9. Test case A: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5) (continued from Fig. 7.8)

7.2. Test case B

In test case B: 1) the initially ($t = 0$) undisturbed 'string' ($\overset{*}{u}(x;\alpha) \equiv 0$) is at rest ($\overset{**}{u}(x;\alpha) \equiv 0$); 2) the right control is absent ($h_2(t;\alpha) \equiv 0$, $t \in [0,T]$).

The coefficient functions (3.14) then reduce to

$$O_{k,\mu}(t;\alpha) = \sigma_{k,\mu}^{-1} \int_0^t g_{k,\mu}(\tau;\alpha)\ \sin\left[\sigma_{k,\mu}(t-\tau)\right]\,\mathrm{d}\tau\,, \tag{7.4}$$

$g_{k,\mu}(t;\alpha)$ (3.11) simplify to

$$g_{k,\mu}(t;\alpha) = c_{k,\mu}\,h_1(t;\alpha) - a_{k,\mu}\,h_1''(t;\alpha)\,, \tag{7.5}$$

and $a_{k,\mu}, c_{k,\mu}$ are given in (3.12), therefore (7.4) read

$$\begin{aligned} O_{k,\mu}(t;\alpha) &= \sigma_{k,\mu}^{-1}\,c_{k,\mu} \underbrace{\int_0^t h_1(\tau;\alpha)\,\sin\left[\sigma_{k,\mu}(t-\tau)\right]\,\mathrm{d}\tau}_{I_{k,\mu}(t;\alpha)} \\ &\quad - \sigma_{k,\mu}^{-1}\,a_{k,\mu} \underbrace{\int_0^t h_1''(\tau;\alpha)\,\sin\left[\sigma_{k,\mu}(t-\tau)\right]\,\mathrm{d}\tau}_{J_{k,\mu}(t;\alpha)}\,. \end{aligned} \tag{7.6}$$

Calculation of the second integral in (7.6) is performed accounting for the compatibility conditions $h_1(0;\alpha) = \overset{*}{u}(-1;\alpha)$, $h_1'(0;\alpha) = \overset{**}{u}(-1;\alpha)$, introduced in Sect. 1 for the left control $h_1(t;\alpha)$, and applying integration by parts, as follows

$$\begin{aligned} J_{k,\mu}(t;\alpha) &= \underbrace{h_1'(\tau;\alpha)\,\sin\left[\sigma_{k,\mu}(t-\tau)\right]\Big|_0^t}_{0} + \sigma_{k,\mu} \int_0^t h_1'(\tau;\alpha)\,\cos\left[\sigma_{k,\mu}(t-\tau)\right]\,\mathrm{d}\tau \\ &= \sigma_{k,\mu} \underbrace{h_1(\tau;\alpha)\,\cos\left[\sigma_{k,\mu}(t-\tau)\right]\Big|_0^t}_{h_1(t;\alpha)} - \sigma_{k,\mu}^2 \int_0^t h_1(\tau;\alpha)\,\sin\left[\sigma_{k,\mu}(t-\tau)\right]\,\mathrm{d}\tau\,, \end{aligned}$$

and finally

$$J_{k,\mu}(t;\alpha) = \sigma_{k,\mu}\,h_1(t;\alpha) - \sigma_{k,\mu}^2\,I_{k,\mu}(t;\alpha)\,, \tag{7.7}$$

wherefrom (7.6) yields to

$$\begin{aligned} O_{k,\mu}(t;\alpha) &= \sigma_{k,\mu}^{-1}\,c_{k,\mu}\,I_{k,\mu}(t;\alpha) - \sigma_{k,\mu}^{-1}\,a_{k,\mu}\,J_{k,\mu}(t;\alpha) \\ &= \sigma_{k,\mu}^{-1}\left(c_{k,\mu} + a_{k,\mu}\sigma_{k,\mu}^2\right) I_{k,\mu}(t;\alpha) - a_{k,\mu}\,h_1(t;\alpha)\,. \end{aligned} \tag{7.8}$$

We chose the left control in the form (refer to Fig. 7.10)

$$h_1(t;\alpha) = \frac{(-1)^{n(t)}}{2}\,A_1(\alpha)\left[1 - \cos\left(\omega_1(\alpha)\,t\right)\right] \equiv C_1(t;\alpha)\left[1 - \cos\left(\omega_1 t\right)\right], \tag{7.9}$$

where the time-dependent exponent reads

$$n(t) = \text{entier}\left(\frac{\omega_1 t}{2\pi}\right) = \text{entier}\left(\frac{t}{T_1}\right), \tag{7.10}$$

$2T_1$ being the period of (7.9), wherefrom we find

$$h_1'(t;\alpha) = C_1(t;\alpha)\,\omega_1 \sin(\omega_1 t)\,, \qquad h_1''(t;\alpha) = C_1(t;\alpha)\,\omega_1^2 \cos(\omega_1 t)\,, \tag{7.11}$$

and the above 'left' compatibility conditions are evidently to meet.

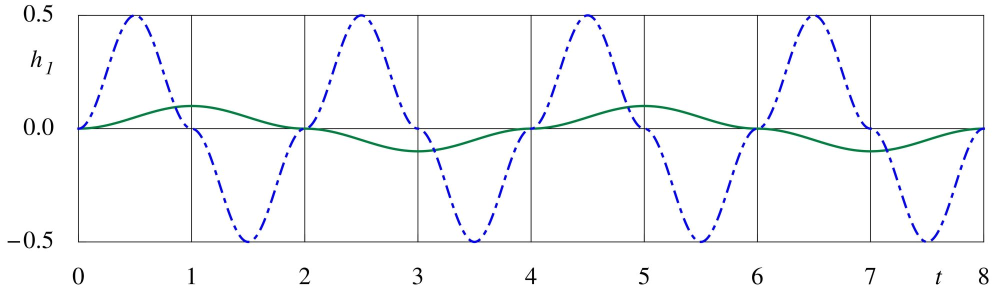

Fig. 7.10. Left control function $h_1(t;\alpha)$ (7.9), satisfying the compatibility conditions: $A_1 = 0.10$, $T_1 = 2$ (solid green curve), $A_1 = 0.25$, $T_1 = 1$ (dashed blue curve)

Below the solution to the IBVPT is presented in Fig. 7.11 – 7.14, as the shapes or the displacements of the 'string', at some instants of $\tau \in [0, 4.4]$, $A_1 = 0.10$, $T_1 = 2$. It is clear that the way the 'string' is excited does not significantly affect the behavior of the traveling waves, including its passing the degeneracy point.

8. Conclusions

1. The SV based methods of solving the IBVPS and IBVPT have been presented in the fully completed form.

2. The solutions to the above problems have been proven using the energy method to be unique in properly introduced function spaces.

3. In the case of weak degeneracy the travelling waves pass through the point of degeneracy, the displacements of the 'string' being smooth and bounded, that is the 'string' retains its integrity and can hear itself.

4. In the case of strong degeneracy the travelling waves, passing the degeneracy point, lead to local unboundness and violating the continuity of the displacements of the 'string', the displacements being piece-wise smooth, that is the 'string' does not retain its integrity, nevertheless it still can hear itself.

5. The way the degenerate 'string' is excited does not affect significantly passing the traveling waves through the point of degeneracy, the total energy of the 'string' being conserved.

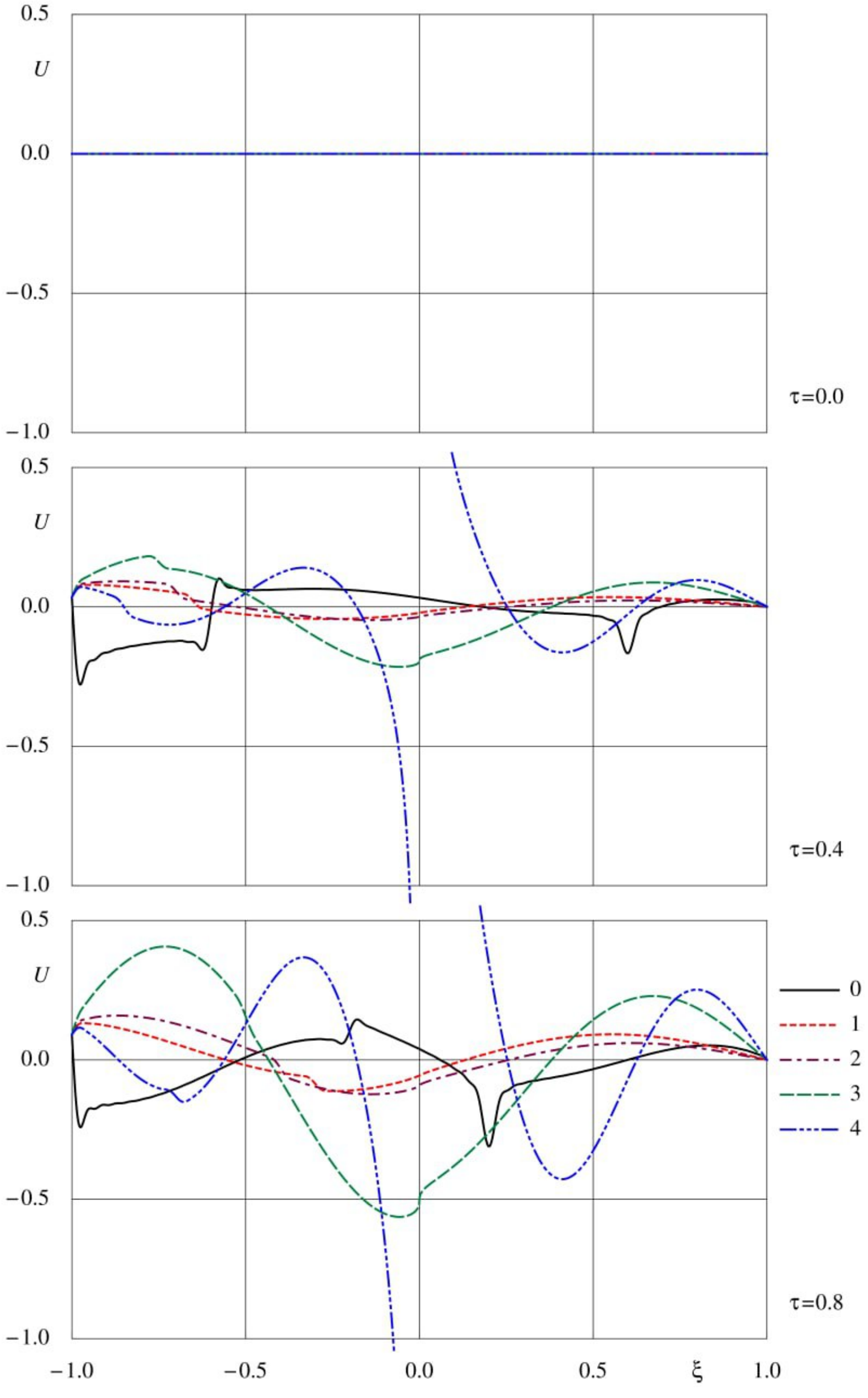

Fig. 7.11. Test case B: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5)

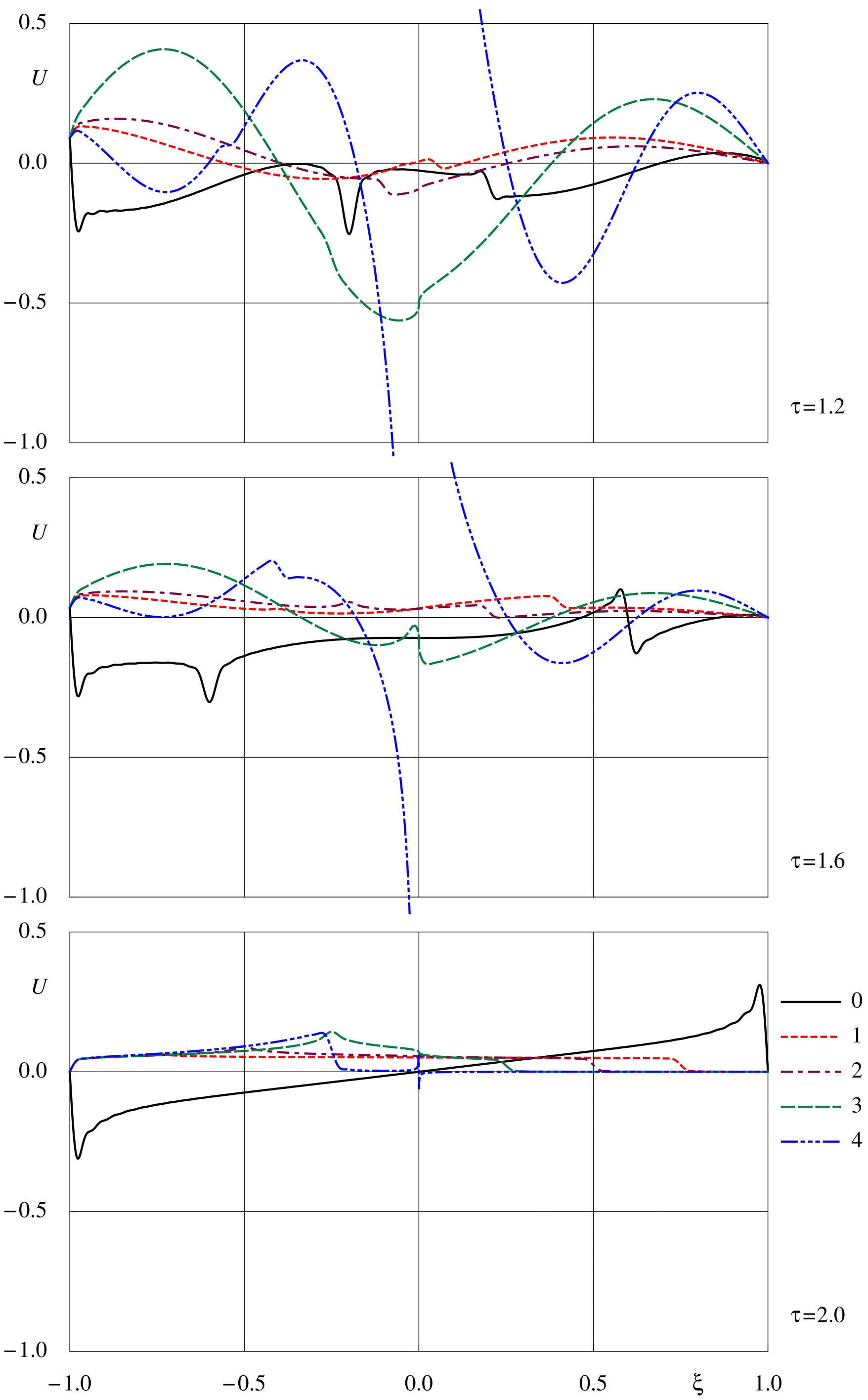

Fig. 7.12. Test case B: solution $U(\tau,\xi;\alpha)$ to the IBVPT (1.5) (continued from Fig. 7.11)

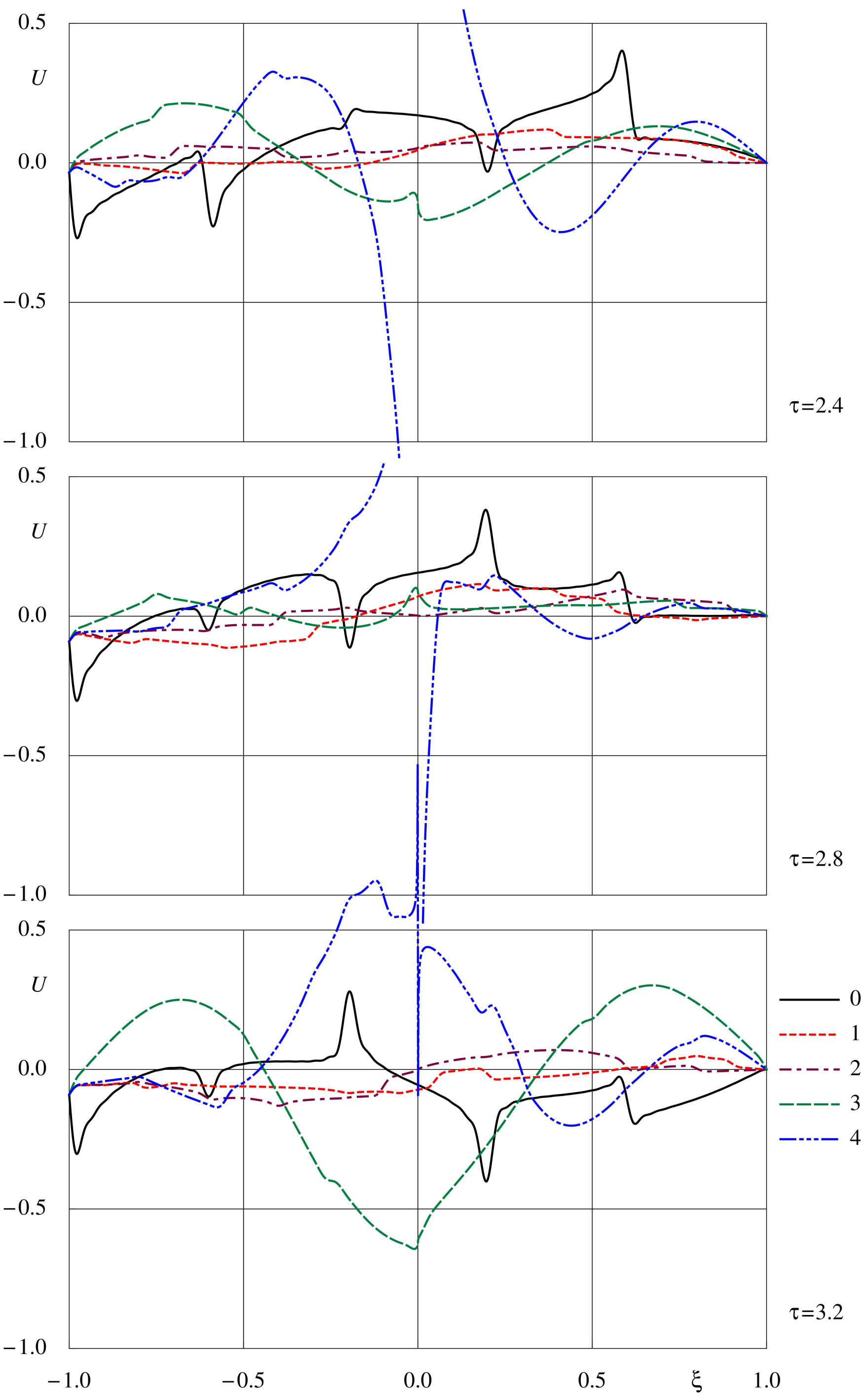

Fig. 7.13. Test case B: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5) (continued from Fig. 7.12)

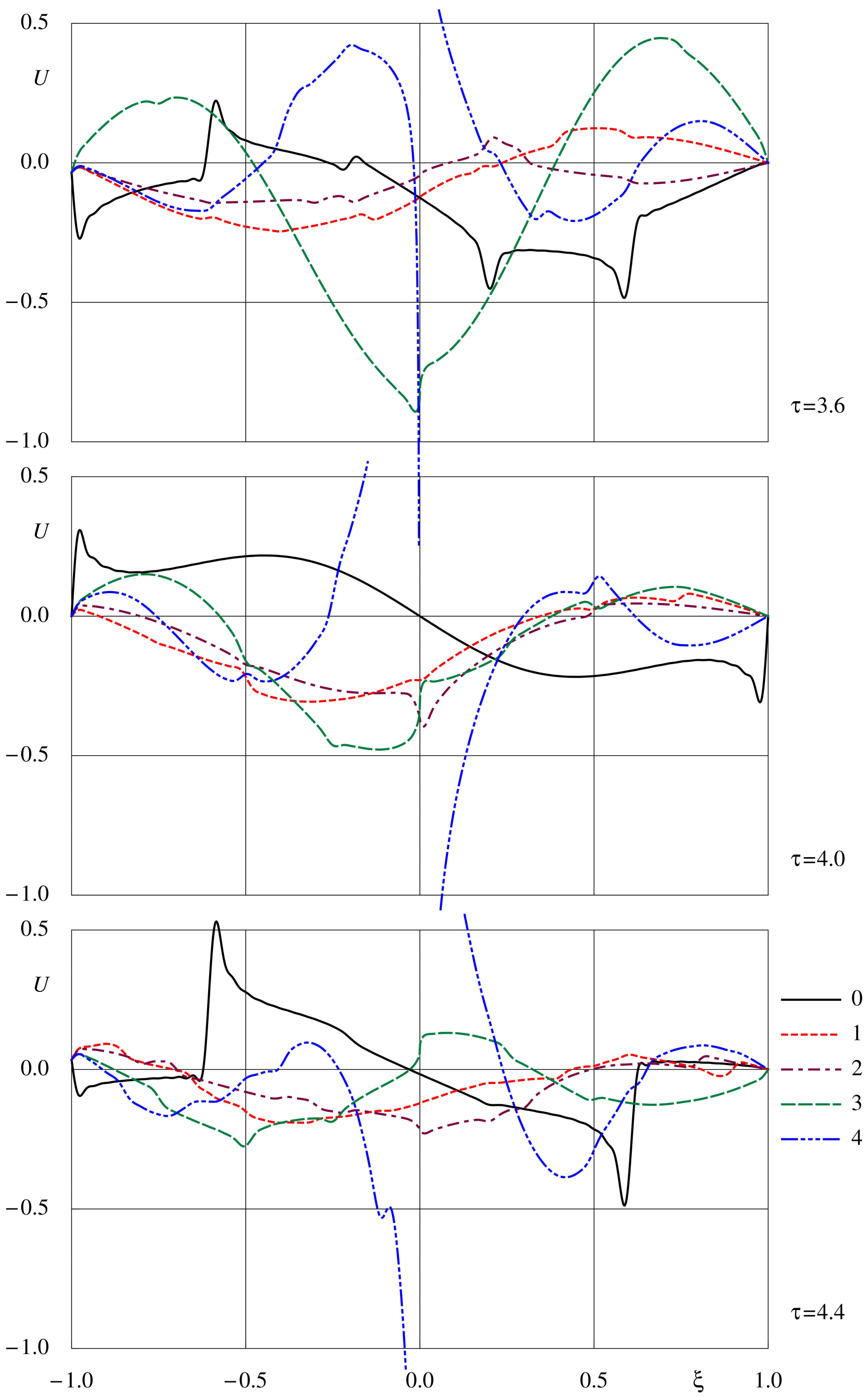

Fig. 7.14. Test case B: solution $U(\tau, \xi; \alpha)$ to the IBVPT (1.5) (continued from Fig. 7.13)

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