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EXISTENCE AND UNIQUENESS SOLUTIONS OF FUZZY FRACTIONAL INTEGRAL EQUATION OF VOLTERRA-STIELTJES TYPE

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Abstract. In this paper, we establish the existence and uniqueness results to the Cauchy problem posed for a fuzzy fractional Volterra-Stieltjes integrodifferential equation. The method of successive approximations is used to prove the existence, whereas the contraction theory is applied to prove the uniqueness of the solution to the problem.

Key words: Fuzzy theory, Existence and uniqueness theorem, fractional Volterra-Stieltjes equation.

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1. Introduction

The theory of integral operators of all kinds generates an important branch of regenerative nonlinear functional analysis. The theories of linear integral operators (Friedholm, Volterra) and the theories of different types of nonlinear integral operators (Hammerstein, Orison) have many applications in mathematics, physics, chemistry, biology and engineering. Many of the above integral operators can be treated as special cases of Stieltjes integral operators with the kernel dependent on two variables [6, 7, 9].
The idea of fuzzy group was created by Zadeh [25] as a mathematical solution to represent fuzziness in daily life. In 1965, Zadeh put forward the concept of fuzzy groups. Subsequently, the application of this fuzzy group in modelling gradually appeared. Thus, the development and application of fuzzy differential equations (FDEs) has increased rapidly in the past 50 years [4]. In order to use the Zadeh concept in topology and functional analysis, many scholars have defined fuzzy metric spaces in several ways [2, 10, 18]. Georg and Ferramani [12] modified the concept of fuzzy metric space introduced by Kramusel and Michalck [18] and were also successful in defining the Hausdorff topology in such a fuzzy metric space that it is often used in current research nowadays and is used in this research work as well. The power of fuzzy mathematics lies in its clear and fruitful applications in practical life. Many interesting examples of fuzzy scales in the sense can be read in [8, 13] given by George and Viramani and such fuzzy scales have also been used to process color images.

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In recent years the theory of fuzzy fractional integral equations has developed rapidly. The concept of fuzzy integral equation was studied in 2011 by Allahviranloo [4]. They considered two new uniqueness results for fuzzy fractional equation with two type of conditions Naqumo and Krasnoselskii-Krein by using the methods of successive approximation and contraction principle. In 2013 Allahviranloo [5] have used the fuzzy Caputo derivatives under generalized Hukuhara difference to introduce fuzzy fractional Volterra-Fredholm integral equations and they proved the existence and uniqueness of the solutions for this class of equations using the methods of successive approximation and contraction principle respectively. In the same year Robab and Fariba [3] investigate the solutions to the same kind of equations employing the method of upper and lower solutions. Nagarajan and Radhakrishnan [20] in 2022 introduced the uniqueness of solution for Sobolev fuzzy integral equation with nonlocal condition also by using the idea of contraction theory.
In this paper, we prove the existence and uniqueness theorems of solutions to the fuzzy fractional Volterra- Stieltjes integrodifferential equation:

$$\begin{gathered} {}^{c}D_{0^+}^{\beta,g(t,s)}y(t)=Q(t,y(t)), \\ y(0)=y_0=\widetilde{\mathbb{O}}. \end{gathered} \tag{1.1}$$

Where $y(t)\in\mathcal{J}\subset\mathbb{E}^n,\mathcal{J}$ is closed and bounded domain, $\mathbb{E}^n=\{u:\mathcal{R}^n\to[0,1]\}$, $Q(t,y(t)):[0,T]\times\mathbb{E}^n\to\mathbb{E}^n$ is levelwise continuous, defined on the domain $\mathcal{J}_1=\{(t,y),t\in[0,T],y\in\mathbb{E}^n\}$, $s,t\in[0,T],1<\beta\le 2,g(t,s):[0,T]\times[0,T]\to\mathcal{R}$ is continuous and differentiable given function such that for all $t,s\in[0,T],|t-s|\le h$, and $g'(t,s)\le\mathcal{K}\in\mathcal{R}^+,\widetilde{\mathbb{O}}\in\mathbb{E}^n$, such that $\widetilde{\mathbb{O}}(t)=1$ for $t=0,0$ otherwise , and $\mathfrak{D}(Q,\widetilde{\mathbb{O}})\le\eta$ for any $(t,y)\in\mathcal{J}_1$ and $\eta\in\mathcal{R}^+$, $\{\mathfrak{D}$ is defined below$\}$.

2. Preliminaries

Let $\mathcal{F}_{\mathcal{C}}(\mathcal{R}^n)$ denote the family of all nonempty compact convex subsets of $\mathcal{R}^n$ and define the addition and scalar multiplication in $\mathcal{F}_{\mathcal{C}}(\mathcal{R}^n)$ as usual. Let $\mathcal{M}$ and $\mathcal{N}$ be two nonempty bounded subsets of $\mathcal{R}^n$. The distance between $\mathcal{M}$ and $\mathcal{N}$ is defined by the Hausdorff metric

$$d(\mathcal{M},\mathcal{N})=\max\left\{\sup_{m\in\mathcal{M}}\inf_{n\in\mathcal{N}}\|m-n\|,\sup_{n\in\mathcal{N}}\inf_{m\in\mathcal{M}}\|m-n\|\right\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathcal{R}^n$. Then it is clear that $(\mathcal{F}_{\mathcal{C}}(\mathcal{R}^n),d)$ becomes a metric space [1].

Let $\mathcal{T}=[c,d]\subset\mathcal{R}$ be a compact interval and denote $\mathbb{E}^n=\{u:\mathcal{R}^n\to[0,1]\mid u$ satisfies $(1-4)$ below $\}$
1. u is normal, i.e., there exists an $x_0\in\mathcal{R}^n$ such that $u(x_0)=1$,
2. u is fuzzy convex,
3. u is upper semicontinuous,

4. $[u]^0 = \text{cl}\{x \in \mathcal{R}^n : u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote $[u]^\alpha = \{x \in \mathcal{R}^n : u(x) \ge \alpha\}$, then from (1-4), it follows that the α - level set $[u]^\alpha \in \mathcal{F}_\mathcal{C}(\mathcal{R}^n)$ for all $0 < \alpha \le 1$.

Define $\mathfrak{D} : \mathbb{E}^n \times \mathbb{E}^n \to \mathcal{R}^+ \cup \{0\}$ by the equation

$$\mathfrak{D}(u, v) = \sup_{\alpha \in [0,1]} d\left([u]^\alpha, [v]^\alpha\right),$$

where d is the Hausdorff metric define in $\mathcal{F}_\mathcal{C}(\mathcal{R}^n)$. We list the following properties of $\mathfrak{D}(u, v)$:
1. $\mathfrak{D}(u + w, v + w) = \mathfrak{D}(u, v)$, and $\mathfrak{D}(u, v) = \mathfrak{D}(v, u)$,
2. $\mathfrak{D}(\lambda u, \lambda v) = \lambda \mathfrak{D}(u, v)$,
3. $\mathfrak{D}(u, v) \le \mathfrak{D}(u, w) + \mathfrak{D}(w, v)$,
for all $u, v, w \in \mathbb{E}^n$ and $\lambda \in \mathcal{R}$ [19].

Definition 2.1. [16]. A mapping $f : T \times \mathbb{E}^n \to \mathbb{E}^n$ is called levelwise continuous at point $(t_0, x_0) \in T \times \mathbb{E}^n$ provided, for any fixed $\alpha \in [0, 1]$. and arbitrary $\epsilon > 0$, there exists a $\zeta(\epsilon, \alpha)$ such that:

$$d\left([f(t; x)]^\alpha, [f(t_0; x_0)]^\alpha\right) < \epsilon,$$

whenever $|t - t_0| < \zeta(\epsilon, \alpha)$ *and* $([f(t; x)]^\alpha, [f(t_0; x_0)]^\alpha) < \zeta(\epsilon, \alpha)$ for all $t \in T$, $x \in \mathbb{E}^n$.

Definition 2.2. [22]. A mapping $\mathbb{F} : T \to \mathbb{E}^n$ is called levelwise continuous at $t_0 \in T$ if the set valued mapping $\mathbb{F}_\alpha(t) = [\mathbb{F}(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0, 1]$.

A mapping $\mathbb{F} : T \to \mathbb{E}^n$ is called integrally bounded if there exists an integrable function $h(t)$ such that $\|x\| \le h(t)$ for all $x \in \mathbb{F}_0(t)$.

Definition 2.3. [23]. A mapping $\mathbb{F} : T \to \mathbb{E}^n$ is called fuzzy Hukuhara differentiable at $t_0 \in T$ if, for any $\alpha \in [0, 1]$, the set valued mapping $\mathbb{F}_\alpha(t) = [\mathbb{F}(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $D_H\mathbb{F}_\alpha(t_0)$ and the family $\{D_H\mathbb{F}_\alpha(t_0) |\ \alpha \in [0, 1]\}$ define a fuzzy number $\mathbb{F}(t_0) \in \mathbb{E}^n$.
If $\mathbb{F} : T \to \mathbb{E}^n$ is differentiable at $t_0 \in T$, then we say that $\mathbb{F}(t_0)$ is the fuzzy derivative of $\mathbb{F}(t)$ at the point t_0 .

Definition 2.4. [14]. The Riemann-Liouville fractional integral of order q is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)\, ds,$$

$q > 0$, provided the integral exists.

Definition 2.5 ([15]). The Riemann-Liouville fractional derivative of order q is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\, ds,$$

$n-1 < q \leq n, q > 0$, Provided the right-hand side is pointwise defined on $(0, +\infty)$.

Theorem 2.1 ([24]). *The metric space* $(\mathcal{F}_{\mathcal{C}}(\mathcal{R}^n), d)$ *is complete and separable.*

Theorem 2.2 ([17]). *If* $f(x)$ *is continuous and* $\alpha'(x)$ *is Riemann integrable over the specified interval, then*

$$\int_a^b f(x) d\alpha(x) = \int_a^b f(x)\alpha'(x) dx.$$

Theorem 2.3 ([22]). *Let* $F, \mathbb{G} : T \to \mathbb{E}^n$ *be integrable, and* $\lambda \in \mathcal{R}$. *Then*

1. $\displaystyle\int_T (\mathbb{F}(t) + \mathbb{G}(t))\, dt = \int_T \mathbb{F}(t)\, dt + \int_T \mathbb{G}(t)\, dt,$
2. $\displaystyle\int_T \lambda \mathbb{F}(t)\, dt = \lambda \int_T \mathbb{F}(t)\, dt,$
3. $\mathfrak{D}(\mathbb{F}, \mathbb{G})$ *is integrable,*
4. $\displaystyle\mathfrak{D}\left(\int_T \mathbb{F}(t)\, dt, \int_T \mathbb{G}(t)\, dt\right) \leq \int_T \mathfrak{D}(\mathbb{F}(t), \mathbb{G}(t))\, dt.$

Theorem 2.4 ([21]). *If* $u \in \mathbb{E}^n$, *then*

1. $[u]^\alpha \in \mathcal{F}_{\mathcal{C}}(\mathcal{R}^n)$ *for all* $0 \leq \alpha \leq 1$,
2. $[u]^{\alpha_1} \subset [u]^{\alpha_2}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,
3. *if* $\{\alpha_c\} \subset [0, 1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then*

$$[u]^\alpha = \bigcap_{c \geq 1} [u]^{\alpha_c},$$

Conversely, if $\{\mathcal{A}^\alpha : 0 \leq \alpha \leq 1\}$ *is a family of subsets of* $\mathcal{R}^n$ *satisfying* $(1-3)$, *then there exists* $u \in \mathbb{E}^n$ *such that*

$$[u]^\alpha = \mathcal{A}^\alpha \quad \text{for } 0 \leq \alpha < 1,$$

$$[u]^0 = \overline{\bigcup_{0 \leq \alpha < 1} \mathcal{A}^\alpha} \subset \mathcal{A}^0.$$

3. Main Result

Consider the fuzzy fractional integral equation (1.1) where $y_0 \in \mathbb{E}^n$, we define the nonempty set $\mathcal{J}_Q = \mathcal{J} - \frac{\eta \mathcal{K} T^\beta}{\Gamma(\beta+1)}$. A mapping $y : \mathcal{J} \to \mathbb{E}^n$ is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation

$$y(t) = y_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q\left(s, y(s)\right) d_s g(t,s).$$

According to the method of successive approximation, let us consider the sequence $\{y_n(t)\}$ such that

$$y_n(t) = y_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q\left(s, y_{n-1}(s)\right) d_s g(t,s), \quad n = 1, 2, ... \tag{3.1}$$

Theorem 3.1. *Assume that $Q : [0,T] \times \mathbb{E}^n \to \mathbb{E}^n$ is levelwise continuous, and satisfies Lipschitz condition, such that for any pair $(t, y_1), (t, y_2) \in \mathcal{J}_1$*

$$\mathfrak{D}\left(Q\left(t, y_1\right), Q\left(t, y_2\right)\right) \leq \mathbb{L} \mathfrak{D}\left(y_1, y_2\right). \tag{3.2}$$

where $\mathbb{L} > 0$. Then there exists a unique solution $y(t)$ of (1.1) defined on the domain $\mathcal{J}$. Moreover, there exists a fuzzy set-valued mapping $y(t) : \mathcal{J} \to \mathbb{E}^n$ such that $\mathfrak{D}\left(y_n(t), y(t)\right) \to 0$ on $\mathcal{J}$ as $n \to \infty$, provided that $\mathcal{K}^{n+1} = \frac{1}{\mathbb{L}^n T^{(n+1)\beta}}$.

Proof. Let $t \in [0,T]$, from (3.1), it follows that, for $n = 1$,

$$y_1(t) = y_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q\left(s, y_0(s)\right) \, d_s g(t,s).$$

which proves that $y_1(t)$ is levelwise continuous on $\mathcal{J}$. Moreover, using theorem (2.3), for any $\alpha \in [0,1]$, we have

$$\begin{aligned} d\left([y_1(t)]^\alpha, [y_0]^\alpha\right) &= d\left(\left[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_0(s)\right) \, d_s g(t,s)\right]^\alpha, \widetilde{\mathbb{O}}\right), \\ &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} d\left(\left[Q\left(s, y_0(s)\right)\right]^\alpha, \widetilde{\mathbb{O}}\right) \, d_s g(t,s), \end{aligned}$$

$$\begin{aligned} \mathfrak{D}\left(y_1(t), y_0\right) &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q\left(s, y_0(s)\right), \widetilde{\mathbb{O}}\right) \, d_s g(t,s), \\ &\leq \eta \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} d_s g(t,s) \leq \eta \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g'(t,s) \, ds \leq \frac{\eta \mathcal{K} T^\beta}{\Gamma(\beta+1)}. \end{aligned}$$

That is $y_1(t) \in \mathcal{J}_Q$, from (3.1), we deduce that $y_n(t)$ is levelwise continuous and

$$d\left([y_2(t)]^\alpha, [y_0]^\alpha\right) = d\left(\left[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_1(s)\right)\, d_s g(t,s)\right]^\alpha, \widetilde{\mathbb{O}}\right),$$

$$\mathfrak{D}\left(y_2(t), y_0\right) \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q\left(s, y_1(s)\right), \widetilde{\mathbb{O}}\right) d_s g(t,s) \leq \frac{\eta \mathcal{K} T^\beta}{\Gamma(\beta+1)},$$

$$\mathfrak{D}\left(y_n(t), y_0\right) \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q\left(s, y_{n-1}(s)\right), \widetilde{\mathbb{O}}\right)\, d_s g(t,s) \leq \frac{\eta \mathcal{K} T^\beta}{\Gamma(\beta+1)}.$$

Consequently, we conclude that $\{y_n(t)\}$ consists of levelwise continuous mappings on $\mathcal{J}$ and $y_n(t) \in \mathcal{J}_Q, n = 1, 2, \ldots$, for all $t \in [0, h]$.

Now we prove that there exists a fuzzy set-valued mapping $y(t) : [0, T] \to \mathbb{E}^n$ such that $\mathfrak{D}\left(y_n(t), y(t)\right) \to 0$ uniformly on $\mathcal{J}$ as $n \to \infty$. For $n = 2$

$$y_2(t) = y_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q\left(s, y_1(s)\right)\, d_s g(t,s).$$

from (3.1) and (3.2) we have

$$y_1(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_0(s)\right)\, d_s g(t,s),$$

$$y_2(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_1(s)\right)\, d_s g(t,s),$$

$$d\big([y_2(t)]^\alpha, [y_1(t)]^\alpha\big) \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} d\left([Q\left(s, y_1(s)\right)]^\alpha, [Q\left(s, y_0(s)\right)]^\alpha\right)\, d_s g(t,s),$$

$$\begin{aligned}
\mathfrak{D}\left(y_2(t), y_1(t)\right) &\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q\left(s, y_1(s)\right), Q\left(s, y_0(s)\right)\right)\, d_s g(t,s) \\
&\leq \mathbb{L} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(y_1(s), y_0(s)\right)\, d_s g(t,s) \\
&\leq \frac{\eta \mathcal{K} T^\beta}{\Gamma(\beta+1)} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g'(t,s)\, ds \\
&\leq \frac{\eta \mathbb{L} \mathcal{K}^2 T^{2\beta}}{(\Gamma(\beta+1))^2}.
\end{aligned}$$

Assume that

$$\mathfrak{D}\left(y_n(t), y_{n-1}(t)\right) \leq \frac{\eta \mathbb{L}^{n-1} \mathcal{K}^n T^{n\beta}}{(\Gamma(\beta+1))^n}.$$

Indeed, from (3.1) and condition (3.2), it follows that

$$y_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_{n-1}(s)\right)\, d_s g(t,s), \tag{3.3}$$

$$y_{n+1}(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_n(s)\right)\, d_s g(t,s), \tag{3.4}$$

$$\begin{aligned} d\left([y_{n+1}(t)]^\alpha, [y_n(t)]^\alpha\right) \leq & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \\ & \times d\left([Q\left(s, y_n(s)\right)]^\alpha, [Q\left(s, y_{n-1}(s)\right)]^\alpha\right) d_s g(t,s), \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathfrak{D}\left(y_{n+1}(t), y_n(t)\right) \leq & \frac{\mathbb{L}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \\ & \times \mathfrak{D}\left(Q\left(s, y_n(s)\right), Q\left(s, y_{n-1}(s)\right)\right) d_s g(t,s), \\ \leq & \mathbb{L} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(y_n(s), y_{n-1}(s)\right) g'(t,s)\, ds \\ \leq & \frac{\eta \mathbb{L}^n \mathcal{K}^n T^{n\beta}}{(\Gamma(\beta+1))^n} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \quad g'(t,s)\, ds \\ \leq & \frac{\eta \mathbb{L}^n \mathcal{K}^{n+1} T^{(n+1)\beta}}{(\Gamma(\beta+1))^{n+1}}. \end{aligned} \tag{3.6}$$

Rewrite $\{y_n(t)\}$ as

$$y_0 \oplus \sum_{n=1}^{\infty} \left[y_n(t) - y_{n-1}(t)\right]. \tag{3.7}$$

From (3.6), according to the convergence criterion of Weierstrass, it follows that the series (3.7) having the general term $y_n(t) - y_{n-1}(t)$, so $\mathfrak{D}(y_n(t), y_{n-1}(t)) \to 0$ uniformly on $\mathcal{J}$ as $n \to \infty$, hence, there exists a fuzzy set-valued mapping $y(t) : \mathcal{J} \to \mathbb{E}^n$ such that $\mathfrak{D}(y_n(t), y(t)) \to 0$ on $\mathcal{J}$ as $n \to \infty$.

To prove that this solution is unique, let $\chi(t)$ be another solution of (1.1) on $\mathcal{J}$, that is

$$\chi(t) = \chi_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q(s, \chi(s))\, d_s g(t,s),$$
$$\chi(0) = \chi_0 = \widetilde{\mathbb{O}},$$

then setting

$$\chi_1(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, \chi_0(s)\right)\, d_s g(t,s),$$
$$y_1(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_0(s)\right)\, d_s g(t,s),$$

we obtain

$$d\left([\chi_1(t)]^\alpha, [y_1(t)]^\alpha\right) \le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} d\left([Q\left(s,\chi_0(s)\right)]^\alpha, [Q\left(s,y_0(s)\right)]^\alpha\right) d_s g(t,s),$$

$$\mathfrak{D}\left(\chi_1(t), y_1(t)\right) \le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q\left(s,\chi_0(s)\right), Q\left(s,y_0(s)\right)\right) d_s g(t,s),$$

$$\le \frac{\mathbb{L}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathfrak{D}\left(\chi_0(s), y_0(s)\right) d_s g(t,s). \tag{3.8}$$

Since $\mathfrak{D}\left(\chi_0(s), y_0(s)\right) = 0$, then the right-hand side in (3.8) tend to 0, hence $\chi_1(t) = y_1(t)$. Assume that $\chi_{n-1}(t) = y_{n-1}(t)$. Then

$$\chi_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, \chi_{n-1}(s)\right) d_s g(t,s),$$

$$y_n(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q\left(s, y_{n-1}(s)\right) d_s g(t,s),$$

$$d\left([\chi_n(t)]^\alpha, [y_n(t)]^\alpha\right) \le \frac{\mathbb{L}}{\Gamma(\beta)} \int_0^\tau (t-s)^{\beta-1} \mathfrak{D}\left(\chi_{n-1}(s), y_{n-1}(s)\right) d_s g(t,s) = 0.$$

Hence, $\chi_n(t) = y_n(t)$. This proves the uniqueness of the solution for (1.1). □

Theorem 3.2. *Let Q in* (1.1) *be a continuous function, satisfies Lipschitz condition, such that for any pair $(t,\chi), (t,y) \in \mathcal{J}_1$, $\mathbb{L} > 0$, we have*

$$\mathfrak{D}(Q(t,\chi), Q(t,y)) \le \mathbb{L}\mathfrak{D}(\chi, y).$$

Then the IVP (1.1) *has a unique solution, provided that $\frac{L\mathcal{K}T^\beta}{\Gamma(\beta+1)} < 1$.*

Proof. Define $\mathfrak{G} : C\left([0,T], \mathcal{F}_\mathcal{C}\right) \to C\left([0,T], \mathcal{F}_\mathcal{C}\right)$ as:

$$\mathfrak{G}y(t) = y_0 \oplus \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} Q(s, y(s))\, d_s g(t,s).$$

Since Q and g are continuous functions and $t \in [0,T]$ then the right-hand side generates a continuous fuzzy number valued function on $[0,T]$ and hence it's well defined. Now, consider the following

$$d\left([\mathfrak{G}\chi(t)]^\alpha, [\mathfrak{G}y(t)]^\alpha\right)$$

$$= d\left(\left[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q(s,\chi(s)) d_s g(t,s)\right]^\alpha, \left[\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} Q(s,y(s))\, d_s g(t,s)\right]^\alpha\right),$$

$$\le \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, d([Q(s,\chi(s))]^\alpha, [Q(s,y(s))]^\alpha)\, d_s g(t,s),$$

$$\mathfrak{D}(\mathfrak{G}\chi(t), \mathfrak{G}y(t)) \leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \sup_{\alpha\in[0,1]} d([Q(s,\chi(s))]^\alpha, [Q(s,y(s))]^\alpha \, d_s g(t,s)),$$
$$\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mathfrak{D}\left(Q(s,\chi(s)), Q(s,y(s)\right) d_s g(t,s)\ ,$$

$$\mathfrak{D}\left(\mathfrak{G}\chi(t), \mathfrak{G}y(t)\right) \leq \frac{\mathbb{L}\mathfrak{D}(\chi,y)}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g'(t,s)\, ds \leq \frac{\mathbb{L}\mathcal{K}T^\beta}{\Gamma(\beta+1)} \mathfrak{D}(\chi,y).$$

For $\frac{L\mathcal{K}T^\beta}{\Gamma(\beta+1)} < 1$, implies that $\mathfrak{G}$ is a contraction, and so the prove complete. □

4. Conclusion

In this paper, we have obtained under the suitable restrictions on the function Q the existence and uniqueness result of a fuzzy fractional Volterra-Stieltjes integral equation. We have used Picard's successive approximation method for the existence, and Banach contraction principle for the uniqueness.

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