JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **30**, Issue **2**, December **2022**, pp. 62–78, DOI 10.15421/142210

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

DISCRETE PROCESSES AND CHAOS IN SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Vasiliy Ye. Belozyorov,* Svetlana A. Volkova†

Abstract. A method for constructing a one-dimensional discrete mapping describing a certain periodic process in a general system of ordinary autonomous differential equations is proposed. The resulting discrete mapping is then used to prove the existence of chaos in the original system of differential equations.

Key words: system of ordinary autonomous differential equations, limit cycle, chaotic attractor, $1D$ exponential discrete map.

2010 Mathematics Subject Classification: 34A34, 34C28, 37D45.

Communicated by Prof. V. Ye. Kapustyan

1. Introduction

An extensive scientific literature is devoted to the issue of the appearance of chaos in nonlinear dynamic systems. A good presentation of the reasons for the appearance of chaotic behavior in dynamical systems (the birth of a chaotic attractor) is presented, for example, in the book [1]. It should be said that there are two main methods for searching for chaotic attractors for nonlinear systems. These methods are based either on the fact the existence of a homoclinic (or heteroclinic) orbit for a given system (a1) or on constructing a discrete mapping for the same system and proving its state of chaos (a2).

(a1) Consider the following $3D$ system of real ordinary autonomous differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p}) \in \mathbb{R}^3;\ \mathbf{x} = (x, y, z)^T,\ \mathbf{f}(\mathbf{0}, \mathbf{p}) = \mathbf{0}. \tag{1.1}$$

Here $\mathbf{f}(\mathbf{x}, \mathbf{p}) : \mathbb{R}^3 \times \mathbb{R}^m \to \mathbb{R}^3$ is a continuously differentiable function; $\mathbf{p} \in \mathbb{R}^m$ is a parameter vector.

Let the point $\mathbf{0}$ be a saddle focus of system (1.1). Assume also that at some parameter vector $\mathbf{p} = \mathbf{p}_0$ in system (1.1) there exists a limit cycle $L = L(\mathbf{p}_0)$ of period $T = T(\mathbf{p}_0)$.

In world scientific literature dedicated to the problems of chaotic dynamics a few scenarios of transition to the chaos in system (1.1) are considered. For a dissipative system (1.1) one of these scenarios is offered in paper [2].

*Department of Applied Mathematics, Oles Honchar Dnipro National University, Gagarin's Avenue, 72, 49010, Dnipro, Ukraine, belozvye2017@gmail.com

†Department of Computer Information Technologies, Ukrainian State University of Science and Technologies, Academican Lazaryan's Street, 2, 49010, Dnipro, Ukraine, svolkovav2017@gmail.com

According to one of statements of the mentioned work, the transition to chaos happens through the infinite cascade of period-doubling bifurcations according to the Feigenbaum's scenario. This scenario generates the cascade of the period-doubling bifurcations of limit cycles $L \to L(\mathbf{p}_0)$: $T \to 2^k T$; $k = 1, 2, \ldots$. Further, Feigenbaum's scenario continues by the subharmonic cascade of bifurcations of limit cycles $L \to L(\mathbf{p})$, the periods $T \to m(k)T$ of which are defined by Sharkovsky's ordering $m(k)$, where $m(k)$ is an integer-valued sequence; $k = 1, 2, \ldots$ (The subharmonic cascade of bifurcations is finished by the cycle of period 3.) Finally, the subharmonic cascade is ended by the homoclinic cascade of bifurcations of stable cycles, which converges to the homoclinic orbit connected at $\mathbf{0}$. The existence conditions of homoclinic (heteroclinic) orbit for system (1.1) are given by the known Shilnikov Homoclinic (Heteroclinic) Theorem. (Note that in the Shilnikov Homoclinic (Heteroclinic) Theorem the existence of homoclinic (heteroclinic) orbit is a key condition for appearance of chaotic dynamics in system (1.1). Due to the existence of homoclinic (heteroclinic) orbit the chaotic behavior of the known Lorenz system at the suitable parameter vector $\mathbf{p}$ was proved.)

(a2) Let $A(x^*, y^*, z^*) \in L(\mathbf{p}_0)$ be a point on the limit cycle $L = L(\mathbf{p}_0)$. Consider the folowing $1D$ discrete process

$$x_{n+1} = h(x_n, \mathbf{p}); \; n = 0, 1, 2, \ldots, \tag{1.2}$$

which is generated by system (1.1) [3, 4]. Here $h(x, \mathbf{p}) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuously differentiable function.

It is clear that if $\mathbf{p} = \mathbf{p}_0$, then the point x^* is a fixed point of the function $h(x, \mathbf{p})$: $x^* = h(x^*, \mathbf{p}_0)$. (It means that there exists the limit cycle in system (1.1).)

Suppose that at $\mathbf{p} = \mathbf{p}_c$ the discrete process (1.2) demonstrates the chaotic behavior. In this case system (1.1) also demonstrates the chaotic behavior [5, 6].

Now assume that for some $\mathbf{p} = \mathbf{p}_c$ in system (1.1) the conditions of the Shilnikov Homoclinic (Heteroclinic) Theorem is fulfilled. Then at $\mathbf{p} = \mathbf{p}_c$ process (1.2) is chaotic.

The converse statement is incorrect. If process (1.2) is chaotic, then system (1.1) is also chaotic, but in this system the homoclinic (or heteroclinic) orbit can not exist. Consequently, in such cases the Shilnikov Homoclinic (Heteroclinic) Theorem is inapplicable.

Let $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ be a dynamical system. In the present article, an attempt to find the conditions for the appearance of chaos in this system was made. (Here it is important to note that the existence of equilibrium points, homoclinic and heteroclinic orbits in studied dynamical systems, was not supposed.)

Denote by $\mathbb{R}^n$ a real space of dimension n. Let $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ be an unknown vector whose coordinates are functions of time t. Let also $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^n$ be a real vector function of variable $\mathbf{x}$.

Consider the autonomous real dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T, \tag{1.3}$$

where functions $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ are real continuous.

The function $\mathbf{f}(\mathbf{x})$ is said to be continuously differentiable at a point $\mathbf{x}_0$ if the partial derivatives $\partial f_i(x_1, \ldots, x_n)/\partial x_j$ exist and are continuous at $\mathbf{x}_0$ for $i, j = 1, \ldots, n$. For the continuously differentiable function $\mathbf{f}(\mathbf{x})$, the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ is an $n \times n$ matrix whose element in the ith row and jth column is $\partial f_i(\mathbf{x})/\partial x_j$ [7].

In future we will assume that only one of two following situations takes place:

(b1) the functions $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ are independent. Thus, $\mathtt{rank} D\mathbf{f}(\mathbf{x}) = n$ and there doesn't exist nonzero a real continuous function $\Phi(\xi_1, \ldots, \xi_n)$ depending on n variables $\xi_1, \ldots, \xi_n$ such that $\forall \mathbf{x} \in \mathbb{R}^n \ \Phi(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \equiv 0$;

(b2) $\mathtt{rank} D\mathbf{f}(\mathbf{x}) = n - 1$. It means that there exists the nonzero real function $\Phi(\xi_1, \ldots, \xi_n)$ depending on n variables $\xi_1, \ldots, \xi_n$ such that $\forall \mathbf{x} \in \mathbb{R}^n \ \Phi(f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \equiv 0$.

Consider the autonomous real system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{a} + B\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)), \mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n. \tag{1.4}$$

Here $\mathbf{a} = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$, $B = (b_{ij}) \in \mathbb{R}^{n \times n}$, $\mathbf{f}(\mathbf{x}) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))^T \in \mathbb{R}^n$, and $f_1(x_1, \ldots, x_n) \not\equiv 0$.

Note that with a suitable change of variables $\mathbf{x} \to S\mathbf{x}, \det S \neq 0$, the vector $\mathbf{a}$ can be represented as $\mathbf{a} = (0, a_2, \ldots, a_n)^T$. Moreover, in matrix $B \neq 0$, we can always fulfill the condition $b_{11} \neq 0$. Therefore, we will assume that in system (1.4), we have $a_1 = 0$ and $b_{11} \neq 0$.

Assume that the $f_1(\ldots), \ldots, f_n(\ldots)$ are continuously differentiable functions such that

$$\begin{aligned} f_i(0, \ldots, 0) = 0, &\quad i = 1, \ldots, n; \\ \frac{\partial f_i(x_1, \ldots, x_n)}{\partial x_j}(0, \ldots, 0) = 0; &\quad i = 1, \ldots, n; \quad j = 1, \ldots, n. \end{aligned}$$

(A situation such that some of the functions $f_1(\ldots), \ldots, f_n(\ldots)$ identically equal to the zero, is not excepted.)

Assume that all functions $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ are polynomials. In this case it is clear that if the function $f_i(x_1, \ldots, x_n) \not\equiv 0$, then we will have $\deg f_i(x_1, \ldots, x_n) \geq 2; i \in \{1, \ldots, n\}$.

Let $b_{11} \neq 0$. Change the variable x_1 by the same variable x_1 on the formula: $x_1 \to x_1 - a_1/b_{11}$. Then in these variables system (1.4) can be rewritten in the following form:

$$\begin{cases} \dot{x}_1(t) = a_1 + \sum\limits_{j=1}^{n} b_{1j}x_j(t) + f_1(x_1(t),\dots,x_n(t)) \equiv g_1(x_1(t),\dots,x_n(t)), \\ \dot{x}_2(t) = a_2 + \sum\limits_{j=1}^{n} b_{2j}x_j(t) + f_2(x_1(t),x_2(t),\dots,x_n(t)) \equiv g_2(x_1(t),\dots,x_n(t)), \\ \dots\dots\dots\dots\dots\dots , \\ \dot{x}_n(t) = a_n + \sum\limits_{j=1}^{n} b_{nj}x_j(t) + f_n(x_1(t),x_2(t),\dots,x_n(t)) \equiv g_n(x_1(t),\dots,x_n(t)). \end{cases} \tag{1.5}$$

where $a_1 = 0$. (For simplicity we have left the former designations of variables $x_1,\dots,x_n$, functions $g_1(\dots),\dots,\ g_n(\dots)$, and corresponding coefficients.)

Let $\mathbf{g}(\mathbf{x}) \equiv (g_1(x_1,\dots,x_n),\dots,g_n(x_1,\dots,x_n))^T$. By $\mathbf{x}^T(0) = \mathbf{x}_0^T = (x_{10},\dots, x_{n0})$ denote a vector of initial values for system (1.5).

Definition 1.1. [7]. A set $\mathbb{L} \subset \mathbb{R}^n$ is said to be a positively invariant set of system (1.5) if from the condition $\mathbf{x}_0 \subset \mathbb{L}$ it follows that $\mathbf{x}(t,\mathbf{x}_0) \subseteq \mathbb{L}, \forall t \geq 0$.

Definition 1.2. System (1.5) is called either degenerate, if $\forall \mathbf{x} \in \mathbb{R}^n\ \mathtt{rank} D\mathbf{g}(\mathbf{x}) < n$ or regular, if there exists a point $\mathbf{x}_0 \in \mathbb{R}^n$ such that $\mathtt{rank} D\mathbf{g}(\mathbf{x}_0) = n$.

Now by $\mathbf{x}_0$ denote an equilibrium point of system (1.5). In future existence (or absence) of equilibrium points at system (1.3) is not assumed. Nevertheless, if these equilibriium points exist, they must possess next properties.

Definition 1.3. [1]. The equilibrium $\mathbf{x}_0$ is called a saddle if the matrix $D\mathbf{g}(\mathbf{x}_0)$ has at least one eigenvalue with a positive real part and one with a negative real part.

The equilibrium $\mathbf{x}_0$ is called a center if all of eigenvalues of the matrix $D\mathbf{g}(\mathbf{x}_0)$ have zero real parts with distinct eigenvalues.

2. Construction of $1D$ exponential discrete map generating chaos in system (1.5)

Let condition (b1) be satisfied for system (1.5). In this case, the functions $g_2(x_1,\dots,x_n),\dots,g_n(x_1,\ \dots,x_n)$ in this system are independent.

Now if condition (b1) is false, then let condition (b2) be valid. A situation, which is possible in this case: the functions $g_2(x_1,\dots,x_n),\dots,g_n(x_1,\dots,x_n)$ are dependent. Therefore, it is necessary to do linear nonsingular replacements of variables $x_1 \to x_1$, $x_2 \to \sum \alpha_{1j}x_j$,..., $x_n \to \sum \alpha_{nj}x_j$ such that in new variables new functions $\overline{g_2}(\alpha_{ij},x_1,\dots,x_n),\dots,\ \overline{g_n}(\alpha_{ij},x_1,\dots,x_n)$ (these are linear combinations of former functions $g_1(\dots),g_2(\dots),\dots,g_n(\dots)$) will be independent.

2.1. Case when at (1.5) $f_1(x_1,\dots,x_n) \equiv f_1(0,x_2,\dots,x_n)$

Let function $f_1(x_1,\dots,x_n)$ be independent of variable x_1: $f_1 \equiv f_1(x_2,\dots,x_n)$.

Introduce in system (1.5) (or (1.4)) new real variables $\rho > 0, x, \phi_1, \dots, \phi_{n-1}$, which are given by the following formulas: $x_1 = x,\ x_2 = \rho\cos\phi_1, \dots,\ x_n = \rho\cos\phi_{n-1}$, where $\cos^2\phi_1 + \dots + \cos^2\phi_{n-1} \equiv 1$. Then, after replacement of variables and multiplication of the second, third,..., and the last equations of system (1.5) on the corresponding coordinates of row-vector $(\cos\phi_1, \dots, \cos\phi_{n-1})$ and summation, we get the first and second equations of system (1.5) in such aspect:

$$\begin{cases} \dot{x}(t) = b_{11}x(t) + \rho(t)h_1(\rho(t), \phi_1(t), \dots, \phi_{n-1}(t)) \equiv b_{11}x(t) + \rho(t)\omega_1(\rho(t), t), \\ \dot{\rho}(t) = a_2\cos\phi_1(t) + \dots + a_n\cos\phi_{n-1}(t) + h_2(x(t), \rho(t), \phi_1(t), \dots, \phi_{n-1}(t)) \\ \qquad \equiv a_0(t) + \omega_2(x(t), \rho(t), t). \end{cases} \tag{2.1}$$

Here $a_0(t) \equiv a_2\cos\phi_1(t) + \dots + a_n\cos\phi_{n-1}(t), h_1(\dots), \omega_1(\dots), h_2(\dots), \omega_2(\dots)$ are known functions of their arguments. (The equations $\dot{\phi}_1(t) = v_1(\dots), \dots, \dot{\phi}_{n-1}(t) = v_{n-1}(\dots)$ derived from equations (1.5) in future aren't used.)

Remind that $b_{11} \neq 0$. Then from the first equation of system (2.1), we have

$$x(t) = -(\rho(t)\omega_1(\rho(t), t))/b_{11} + \dot{x}(t)/b_{11}.$$

From here and the second equation of system (2.1), we also have

$$\dot{\rho}(t) = a_0(t) + \rho(t)\Phi(\rho(t), t) + \dot{x}(t)\Psi(\rho(t), \dot{x}(t), t), \tag{2.2}$$

where $a_0(t), \Phi(\rho(t), t)$, and $\Psi(\rho(t), \dot{x}(t), t)$ are continuous functions of variables $t, \rho(t), \dot{x}(t)$ with periodic coefficients depending on t.

To solve the last equation, we use the well-known Lagrange method of variation of an arbitrary constant. Then, the solution of this equation can be written in the following form:

$$\rho(t) = \rho(t_0)\exp\left(\int_{t_0}^{t} \Phi(\rho(\tau), \tau)d\tau\right) \\ + \int_{t_0}^{t} \Big[a_0(\tau) + \dot{x}(\tau)\Psi(\rho(\tau), \dot{x}(\tau), \tau)\Big] \exp\left(\int_{t_0}^{\tau} \Phi(\rho(\tau - \nu), \tau - \nu)d\nu\right)d\tau,\ \tau > \nu. \tag{2.3}$$

We can suppose that for system (2.1) the following variant takes place: the increasing sequence $t_k < t_{k+1} < t_{k+2} < \dots$ is a join of sequences of maximums $t_k < t_{k+2} < t_{k+4} < \dots$ and minimums $t_{k+1} < t_{k+3} < t_{k+5} < \dots$ of the function $u(t)$, where $\dot{u}(t) = a_0(t) + \dot{x}(t)\Psi(\rho(t), \dot{x}(t), t)$.

We also assume that there exists a positive number T such that $t_{2k+2} - t_{2k} = t_{2k+1} - t_{2k-1} = T$. Since we have $t_{2k+2} - t_{2k+1} = t_{k+1} - t_k$, then for the sequence $\rho_k = \rho(t_k), \rho_{k+1} = \rho(t_{k+1}),\ \dots$ formula (2.3) can be represented in the following aspect:

$$\rho(t_{k+1}) = \rho(t_k)\exp\left(\int\limits_{t_k}^{t_{k+1}} \Phi(\rho(\tau),\tau)d\tau\right)$$
$$+\int\limits_{t_k}^{t_{k+1}} \Big[a_0(\tau) + \dot{x}(\tau)\Psi(\rho(\tau),\dot{x}(\tau),\tau)\Big]\exp\left(\int\limits_{t_k}^{\tau} \Phi(\rho(\tau-\nu),\tau-\nu)d\nu\right)d\tau, \quad (2.4)$$

where $\tau > \nu$. (Compact notation is $\rho_{k+1} = w(\rho_k), k = 0, 1, \dots,$ where $w(\rho) \geq 0$ is a $1D$ continuous real map defined by (2.4)).

3. Conditions for the existence of chaos in system (1.5)

There are a few determinations of chaotic dynamics in the discrete dynamical systems [8]. We will adhere to one of them.

Definition 3.1. [8]. The function $w(\rho) : [0,\infty) \to [0,\infty)$ is called chaotic if: 1) w is transitive and 2) a set of periodic points of w is dense in $[0,\infty)$.

Definition 3.2. System (1.5) is called chaotic if the function $w(\rho)$ is chaotic.

Let $\mathbb{Y} \subset \mathbb{R}^n$ be a linear subspace in $\mathbb{R}^n$ of dimension p (here $0 < p < n$).

Denote by $\mathbf{P} = \mathbf{g}|_{\mathbb{Y}}$ the restriction of $\mathbf{g}$ to $\mathbb{Y}$. Assume that $\forall \mathbf{y} \in \mathbb{Y}$, we have $\mathbf{P}(\mathbf{y}) \in \mathbb{Y}$.

Definition 3.3. The system $\dot{\mathbf{y}}(t) = \mathbf{P}(\mathbf{y}(t)), \mathbf{y}(t) \in \mathbb{Y}$, is called a nontrivial subsystem of system (1.5).

Let $\phi(x_1) = g_1(x_1, 0, \dots, 0) \equiv x_1(b_{11} + \theta(x_1))$ be a function differentiable on the interval $(-\infty,\infty)$.

Theorem 3.1. *Assume that for system (1.5) the following conditions are fulfilled:*

(c1) there doesn't exist nontrivial subsystems in system (1.5);

(c2) $\int\limits_{-\infty}^{\infty} (b_{11} + \theta(x_1(t))dt < 0;$

(c3) there exists an open, connected, positively invariant set $\mathbb{X}_0 \subset \mathbb{R}^n$ (without stable equilibrium points) such that for an arbitrary vector of initial values $(x_{10},\dots,x_{n0})^T \in \mathbb{X}_0$ the solutions $x_1(x_{10},\dots,x_{n0},t),\dots,x_n(x_{10}, \dots,x_{n0},t)$ of system (1.5) are bounded, and these solutions satisfy to equality

$$\liminf_{t\to\infty} \rho(t) \equiv \liminf_{t\to\infty} \sqrt{x_2^2(x_{10},\dots,x_{n0},t) + \cdots + x_n^2(x_{10},\dots,x_{n0},t)} = 0.$$

Then in system (1.5) there exists a chaotic dynamics.

3.1. Proof of Theorem 3.1 in the case when at (1.5) $f_1 :\equiv f_1(x_2, \dots, x_n)$

We will look at the iterative process (2.4) as a discrete dynamical system.

We formally construct for system (1.5) the discrete process (2.4).

Further, if system (1.5) has no equilibria, then $\mathtt{rank} D\mathbf{g}(\mathbf{x}) = n$ or $\mathtt{rank} D\mathbf{g}(\mathbf{x}) = n-1$. In this case by linear replacements of variables it is always possible to obtain that in system (1.5) the functions $g_2(x_1, \dots, x_n), \dots, g_n(x_1, \dots, x_n)$ will be independent. (It is always possible for the system with equilibrium points.) Therefore, we can consider that such independence takes place.

Now we suppose that condition (c1) of Theorem 3.1 is not valid. In this case there exists a basis of space $\mathbb{R}^n$, in which system (1.5) may be represented in the following form:

$$\begin{cases} \dot{y}_1(t) = P_1(y_1(t), \dots, y_n(t)), \\ \dot{y}_2(t) = P_2(y_2(t), \dots, y_n(t)), \\ \dots\dots\dots\dots\dots, \\ \dot{y}_n(t) = P_n(y_2(t), \dots, y_n(t)), \end{cases}$$

where $y_1, \dots, y_n$ are new variables and $P_1(\dots), \dots, P_n(\dots)$ are continuous functions.

From here it follows that in system (1.5) there exists $(n-1)D$ nontrivial subsystem. In this case, if $n > 3$, then in system (1.5) can exists a chaotic attractor embedded in $(n-1)D$ subspace in $\mathbb{R}^n$. (It attractor can arise up in the subsystem $\dot{y}_2(t) = P_2(y_2(t), \dots, y_n(t)), \dots, \dot{y}_n(t) = P_n(y_2(t), \dots, y_n(t))$.) Condition (c1) excepts such situation. (Condition (c1) guarantees that system (1.5) cannot be reduced to the so-called "triangular form" by suitable changes of variables.)

Further, we assume that for system (1.5) condition (c3) of Theorem 3.1 is fulfilled. It means that the condition $\liminf\limits_{k\to\infty} \rho_k = 0$ is valid. From here it follows that $\liminf\limits_{j\to\infty} \rho_{2j+1} = 0$. In this case, for the sequence $t_1, t_3, \dots$ of minimums of function $x(t)$, we can rewrite formula (2.4) in such aspect:

$$\begin{aligned} \rho(t_{2k+1}) = {} & \rho(t_{2k-1}) \exp\left(\int\limits_{t_{2k-1}}^{t_{2k+1}} \Phi(\rho(\tau), \tau) d\tau \right) \\ & + \int\limits_{t_{2k-1}}^{t_{2k+1}} \Big[a_0(\tau) + \dot{x}(\tau)\Psi(\rho(\tau), \dot{x}(\tau), \tau) \Big] \\ & \times \exp\left(\int\limits_{t_{2k-1}}^{\tau} \Phi(\rho(\tau-\nu), \tau-\nu) d\nu \right) d\tau, \ \tau > \nu, \quad (3.1) \end{aligned}$$

where $k > 0$.

Fix a small enough number $\epsilon > 0$. Further, we choose from the sequence $t_1, t_3, \dots, t_{2k-1}, \dots$ a subsequence $s_1 = t_{i_1}, s_2 = t_{i_2}, \dots, s_p = t_{i_p}, \dots$ such that $\rho(s_1) < \epsilon$, $\rho(s_2) < \epsilon, \dots, \rho(s_p) < \epsilon, \dots$

Since $\rho(t) \geq 0$, then for large enough integers p_i, p_{i+1}, we can consider that $\rho(s_{p_i}) \approx \rho(s_{p_{i+1}}) \approx 0$. Let's change variables $t_{2k-1} \to s_{p_i}$ and $t_{2k+1} \to s_{p_{i+1}}$. Then, from here and from formula (3.1), it follows that

$$\liminf_{i\to\infty} \int\limits_{s_{p_i}}^{s_{p_{i+1}}} \Big[a_0(\tau) + \dot{x}(\tau)\Psi(\rho(\tau), \dot{x}(\tau), \tau)\Big] \times \exp\left(\int\limits_{s_{p_i}}^{\tau} \Phi(\rho(\tau-\nu), \tau-\nu)d\nu\right)d\tau = 0. \quad (3.2)$$

It is clear that in order that equality (3.2) took place, it is necessary that the function $u(t)$ was with alternating signs on the interval $(s_{p_i}, s_{p_{i+1}})$ (the function $\dot{u}(t) = a_0(t) + \dot{x}(t)\Psi(\rho(t), \dot{x}(t), t)$ has a root on $(s_{p_i}, s_{p_{i+1}})$.)

It means that for $i \to \infty$ formula (3.1) can be represented as

$$\liminf_{i\to\infty} \rho_{s_{p_{i+1}}} = \liminf_{i\to\infty} \rho_{s_{p_i}} \exp\left(\int\limits_{s_{p_i}}^{s_{p_{i+1}}} \Phi(\rho(\tau), \tau)d\tau\right) = 0. \quad (3.3)$$

(Note that if $\liminf\limits_{k\to\infty} \rho_k = 0$, then $\lim\limits_{t\to\infty} T = \infty$ and the function $\rho(t)$ at some values of parameters of system (2.1) becomes unperiodic.)

Further, under the condition $\liminf\limits_{k\to\infty} \rho_k = 0$ and the boundedness of $\rho(t)$, we can pass from system (2.1) to the system

$$\begin{cases} \liminf\limits_{k\to\infty} \dot{x}(t_k) = \liminf\limits_{k\to\infty} b_{11}x(t_k), \\ \liminf\limits_{k\to\infty} \dot{\rho}(t_k) = \liminf\limits_{k\to\infty} (a_0(t_k) + \omega_2(x(t_k), \rho(t_k), t_k)). \end{cases}$$

Thus, in order that the solution $x(t)$ of first equation of system (2.1) was bounded, it is necessary that in this system we have $b_{11} < 0$. (It is condition (c2) of Theorem 3.1.)

Thus, if $\liminf\limits_{k\to\infty} \rho_k = 0$ the iterated process (2.4) for system (1.5) is described by the formula

$$\rho(t_{k+1}) = \rho(t_k) \exp\left(\int\limits_{t_k}^{t_{k+1}} \Phi(\rho(\tau), \tau)d\tau\right). \quad (3.4)$$

If the sequence of values $\rho(t_1), \rho(t_3), \ldots$ of the function $\rho(t)$ is built, then formula (3.4) can be rewritten in the following form:

$$\rho(t_{2i+1}) = \rho(t_{2i-1}) \exp\left(\int\limits_{t_{2i-1}}^{t_{2i-1}+T} \Phi(\rho(\tau), \tau)d\tau\right); \quad i = 1, 2, \ldots, \infty;\ T > 0. \quad (3.5)$$

Using the Taylor series expansion, the smooth function $\Phi(\rho(\tau),\tau)$ near point $\rho(t_{2i-1})$ can be described as follows:

$$\Phi(\rho,\tau)=\sum_{j=0}^{\infty}\frac{\Phi^{(j)}(\rho(t_{2i-1}),t_{2i-1})}{j!}(\rho(\tau)-\rho(t_{2i-1}))^j;\Phi^{(j)}(\rho,\tau)=\frac{d^j\Phi(\rho,\tau)}{d\rho^j}. \quad (3.6)$$

Without loss of generality it is possible to assume that series (3.6) converges uniformly to $\Phi(\rho,\tau)$ on the interval $[\rho(t_{2i-1}),\rho(t_{2i+1})]$. In this case, we can consider that

$$\int\limits_{t_{2i-1}}^{t_{2i-1}+T}\Phi(\rho(\tau),\tau)d\tau=\int\limits_{t_{2i-1}}^{t_{2i-1}+T}\sum_{j=0}^{\infty}h_j(\tau)(\rho(\tau)-\rho(t_{2i-1}))^j d\tau, \quad (3.7)$$

where $h_j(\tau)$ are bounded functions on the interval $[t_{2i-1},t_{2i-1}+T]$.

Now using known First Theorem About Mean Value from (3.7) we get such presentation:

$$\int\limits_{t_{2i-1}}^{t_{2i-1}+T}\Phi(\rho(\tau),\tau)d\tau$$
$$=\int\limits_{t_{2i-1}}^{t_{2i-1}+T}\sum_{j=0}^{\infty}h_j(\tau)(\rho-\rho(t_{2i-1}))^j d\tau=\sum_{j=0}^{\infty}w_j(\xi)(\rho(t_{2i+1})-\rho(t_{2i-1}))^j,$$

where $w_j(\xi)=\int\limits_{t_{2i-1}}^{t_{2i-1}+T}h_j(\tau)d\tau$, $t_{2i-1}<\xi<t_{2i+1}$.

Since $\liminf\limits_{k\to\infty}\rho_k=0$, then we can choose from the sequence of minimums $t_1,t_3,\dots,t_{2i+1},\dots$ of function $a_0(t)+\dot{x}(t)\Psi(\rho(t),\dot{x}(t),t)$ (see (2.2)) the subsequence $s_1=t_{p_1},s_3=t_{p_3},\dots,s_{2i+1}=t_{p_{2i+1}},\dots$ such that $\rho(s_{2i+1})=o(\rho(s_{2i-1}))$ and the sequence $\rho(s_1),\rho(s_3),\dots,\rho(s_{2i-1}),\dots$ is convergent ($\lim_{i\to\infty}\rho(s_{2i-1})=h>0$, where h is small enough). In this case formula (3.5) can be rewritten as

$$\rho(s_{2i+1})=\rho(s_{2i-1})\exp\left(\sum_{j=0}^{\infty}(-1)^j w_j(s_{2i-1})\rho^j(s_{2i-1})\right);\quad i=1,2,\dots,\infty. \quad (3.8)$$

Now we rewrite formula (3.8) in the following form:

$$\rho(s_{2i+1})=\rho(s_{2i-1})\exp(\lambda(s_{2i-1})-\rho(s_{2i-1})\mu(\rho(s_{2i-1})));\quad i=1,2,\dots,\infty. \quad (3.9)$$

Here $\lambda(s_{2i-1})=w_0(s_{2i-1}),\mu(\rho(s_{2i-1}))=\sum\limits_{j=1}^{\infty}(-1)^j w_j(s_{2i-1})\rho^j(s_{2i-1})/\rho(s_{2i-1})$.

Further, from (3.9) it follows that

$$\rho(s_{2i+1})\mu(\rho(s_{2i-1})) = \rho(s_{2i-1})\mu(\rho(s_{2i-1})) \\ \times \exp(\lambda(s_{2i-1}) - \rho(s_{2i-1})\mu(\rho(s_{2i-1}))) = 0; \quad i = 1, 2, \dots, \infty. \quad (3.10)$$

Now we take into consideration condition (c3) in following aspect:

$$\liminf_{i\to\infty} \rho(s_{2i+1}) = \liminf_{i\to\infty} \rho(s_{2i-1}).$$

Consequently, we have

$$\liminf_{i\to\infty} \rho(s_{2i+1})\mu(\rho(s_{2i+1})) \\ = \liminf_{i\to\infty} \rho(s_{2i-1})\mu(\rho(s_{2i-1})) = \liminf_{i\to\infty} \rho(s_{2i+1})\mu(\rho(s_{2i-1})).$$

In this case equality (3.10) as $i \to \infty$ can be rewritten in such form:

$$v(s_{2i+1}) = v(s_{2i-1}) \exp(\lambda(s_{2i-1}) - v(s_{2i-1})); \quad i = 1, 2, \dots, \infty. \quad (3.11)$$

Here $v(s_{2i-1}) = \rho(s_{2i-1})\mu(\rho(s_{2i-1})), \lambda(s_{2i-1}) > 0$.

Process (3.11) is generated by the $1D$ map $\delta(v) = v \exp(\lambda - v)$ (it is Ricker's map [6]). It is known that for some λ Ricker's map is chaotic (this fact was proved in [4, 5]). (It should also be noted that process (3.11) is non-autonomous.)

Note that if the functions $f_1(\dots), \dots, f_n(\dots)$ in system (1.5) are polynomials, then at some values of parameters sequence (3.8) will be chaotic [4] – [6]. (In this case in formula (3.8) under the sign $\exp(\dots)$ there will be a polynomial instead of series.)

Above it was shown, as process (3.8) can be reduced to process (3.11). At some values of parameters process (3.11) (and (3.8)) demonstrates the chaotic behavior. Consequently, there must be the values of parameters at which the solutions of system (1.5) also will demonstrate the chaotic behavior. □

3.2. Proof of Theorem 3.1 when at (1.5) $f_1 :\equiv f_1(x_1, \dots, x_n)$

In this case, the proof of Theorem 3.1 almost completely repeats the proof of Subsection 3.1. There are only a few differences to which you should pay attention.

1. The first equation of system (2.1) will have the following form:

$$\dot{x}(t) = x(t)(b_{11} + \theta(x(t))) + \rho(t)h_1(x(t), \rho(t), \phi_1(t), \dots, \phi_{n-1}(t)). \quad (3.12)$$

2. Let condition (c3) of Theorem 3.1 be satisfied. Then, for $t \to \infty$, Equation (3.12) can be represented as

$$\dot{x}(t) = x(t)(b_{11} + \theta(x(t))) + O(\rho(t)).$$

The solution to the last equation is this:

$$x(t) = x(t_0)\exp\left(\int_{t_0}^{t}(b_{11}+\theta(x(\tau)))d\tau\right)$$
$$+\int_{t_0}^{t}\Big[O(\rho(\tau))\Big]\cdot\exp\left(\int_{t_0}^{\tau}(b_{11}+\theta(x(\tau-\nu)))d\nu\right)d\tau. \quad (3.13)$$

Obviously, if condition (c2) of Theorem 3.1 is satisfied, then the solution $x(t)$ is bounded.

3. Let $t_c > 0$ be a point such that $\dot{x}|_{t=t_c} = 0$. Then Equation (3.12) can be written as

$$\Delta(x,\rho) = 0,$$

where $\Delta(0,0) = 0$.

Now we use the well-known Implicit Function Theorem [7]. According to this theorem, there exists a neighborhood S of the point $\rho = 0$ such that in this neighborhood $x = \gamma(\rho)$, where γ is continuously differentiable at $\rho = 0$.

Further, we use relation the $x = \gamma(\rho)$ to obtain an equation of type (2.2), from the second equation of system (2.1). Now it remains to repeat the procedure for proving Theorem 3.1 presented in Subsection 3.1. □

Let us calculate Lyapunov's exponent $\Lambda[f]$ for a real function $f(t)$ [3,4]:

$$\Lambda[f] = \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\left|\frac{f(t)}{f(t_0)}\right|, f(t_0) \neq 0. \quad (3.14)$$

Let $\Lambda[x_1], \Lambda[x_2], \dots \Lambda[x_n]$ be all n Lyapunov's exponents of system (1.5).

Corollary (necessary conditions for the existence of chaos). *Under the conditions of Theorem 3.1, among all Lyapunov's exponents of system (1.5), there exist at least one positive, one negative, and one zero exponent.*

Proof of Corollary. From the first equation of system (2.1) it follows that

$$x(t) = \exp(b_{11}t)x_0 + \int_0^t \exp(b_{11}(t-\tau))\rho(\tau)\omega_1(\rho(\tau),\tau)d\tau.$$

Then, from here and (3.14), we have

$$\Lambda[x] = b_{11} + \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\frac{1}{|x_0|}\left|x_0 + \int_0^t \exp(-b_{11}\tau))\rho(\tau)\omega_1(\rho(\tau),\tau)d\tau\right| = b_{11} < 0.$$

Further, we recall that $x_i(t) = \rho(t)\cos\phi(t), \quad i = 2,\dots,n$. In addition, one of properties of Lyapunov's exponents is the following: $\Lambda[x_i] = \Lambda[\rho] + \Lambda[\cos\phi_i]$, where $\Lambda[\cos\phi_i] \leq 0$; $\quad i = 2, \ldots, n$. As $\Lambda[x_1] < 0$, then the following variants are possible:

(d1) if $\Lambda[\rho] < 0$, then $\Lambda[x_i] < 0$, $\quad i = 2, \ldots, n$. Thus, in $\mathbb{X}_0$ there is a stable equilibrium.

(d2) if $\Lambda[\rho] = 0$, then $\Lambda[x_i] = 0$, $\quad i = 2, \ldots, p$, and $\Lambda[x_i] < 0$, $\quad i = p + 1, \ldots, n$, where $p \geq 2$. Thus, in $\mathbb{X}_0$ there is a limit cycle (or center). In this case the situation $\liminf\limits_{t\to\infty} \rho(t) = 0$ is impossible.

(d31) if $\Lambda[\rho] > 0$, then $\Lambda[x_i] > 0$, $\quad i = 2, \ldots, p$, and $\Lambda[x_i] < 0$, $\quad i = p + 1, \ldots, n$, where $p \geq 2$. Thus, in $\mathbb{X}_0$ there is a saddle point. In this case the situation $\liminf\limits_{t\to\infty} \rho(t) = 0$ is also impossible.

(d32) if $\Lambda[\rho] > 0$, then $\Lambda[x_i] > 0$, $\quad i = 2, \ldots, p$; $\Lambda[x_i] = 0$, $\quad i = p+1, \ldots, p+q$, and $\Lambda[x_i] < 0$, $\quad i = p + q + 1, \ldots, n$. Here $p \geq 2$ and $q \geq p + 1$.

Note that variants (d1), (d2), and (d31) contradict condition (c3) of Theorem 3.1. However, there is condition (d32), which satisfies the condition (c3). This remark completes the proof of Corollary. □

It should be said that the verification of condition (c1) is trivial. (For example, it is fulfilled for almost all models, which describe real technical systems.) The verification of condition (c2) is also almost trivial. If this condition is not fulfilled, then by linear replacements of variables it is necessary to obtain that in the first equation of the again got system (1.5) condition (c2) was satisfied.

Thus, it is really difficult to check up only condition (c3) of Theorem 3.1. (It means the boundedness of solutions of system (1.5).)

We take advantage of known results about of structure of a central manifold (Theorem 4.2 [7]) and stability of perturbed systems (Theorem 5.3 [7]).

Without loss of generality it is possible to consider that $\texttt{rank}D\mathbf{g}(\mathbf{x}) = n - 1$.

Let $\mathbf{a} = 0$. Then we have that the point $(0, \ldots, 0)^T \in \mathbb{R}^n$ is an equilibrium of system (1.5). We will consider that the matrix B has one zero eigenvalue and $(n-1)$ negative eigenvalues. Introduce the new real variables $u \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^{n-1}$ by the formula

$$\mathbf{x}_d = T \cdot \begin{pmatrix} u \\ \mathbf{v} \end{pmatrix}, T \in \mathbb{R}^{n\times n},$$

where $\det T \neq 0$. In this case the matrix T can be chosen so that system (1.5) in new variables will take the following form:

$$\dot{u}(t) = d_0 u^p(t) + F(u(t), \mathbf{v}(t)), \tag{3.15}$$

$$\dot{\mathbf{v}}(t) = Q\mathbf{v}(t) + \mathbf{P}(u(t), \mathbf{v}(t)). \tag{3.16}$$

Here $Q \in \mathbb{R}^{(n-1)\times(n-1)}$, the map $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $d_0 \neq 0$, and integer $p > 1$.

Consider the vector partial differential equation

$$\frac{\partial \mathbf{H}(u)}{\partial u} \cdot (d_0 u^p + F(u, \mathbf{H}(u)) - Q\mathbf{H}(u) - \mathbf{P}(u, \mathbf{H}(u)) = \mathbf{0}, \tag{3.17}$$

with boundary conditions

$$\mathbf{H}(0) = \mathbf{0}, \ \frac{\partial \mathbf{H}(u)}{\partial u}(0) = \mathbf{0}, \tag{3.18}$$

where $\mathbf{H}(u) = (h_1(u), \dots, h_{n-1}(u))^T$ is unknown vector function.

Definition 3.4. [7]. The set $\mathbf{H}(u)$ of all solutions of the equations (3.17) and (3.18) is called a center manifold.

In order to study conditions (3.18) it is necessary that the vector $\mathbf{H}(u)$ had the form

$$\mathbf{H}(u) = \sum_{i=p}^{\infty} \mathbf{d}_{i-p} u^i, \ \mathbf{d}_i \in \mathbb{R}^{n-1}.$$

In this case, from (3.17) it follows that

$$\mathbf{H}(u) = -Q^{-1} \cdot \mathbf{P}(u, \mathbf{H}(u)); \mathbf{0} = (0, \dots, 0)^T \in \mathbb{R}^{n-1}.$$

Thus, on the central manifold equation (3.15) has the following aspect:

$$\dot{u}(t) = d_0 u^p - Q^{-1} \cdot \mathbf{P}(u, \mathbf{H}(u)) \cdot u^p + O(\|u\|^{p+1}) \equiv s_0 u^p + O(\|u\|^{p+1}).$$

If p is odd and $s_0 = d_0 - Q^{-1} \cdot \mathbf{P}(u, \mathbf{H}(u)) < 0$, then the conclusion of Theorem 4.2 [7] is valid. In this case the origin of system (1.5) is asymptotically stable.

Let in system (1.5) be $\mathbf{a} \neq 0$. We assume that the vector $\mathbf{a} = \mathbf{a}^*$ is the root of the equation $\mathbf{a} + B\mathbf{a} + \mathbf{f}(\mathbf{a}) = \mathbf{0}$.

Let's introduce a new variable $\mathbf{y} = \mathbf{x} - \mathbf{a}^*$. Then using Taylor's series expansion, the right side of system (1.5) near point $\mathbf{a}^*$ can be described as follows:

$$\dot{\mathbf{y}}(t) = (B + D\mathbf{f}(\mathbf{a}^*))\mathbf{y}(t) + \dots$$

Assume that the origin of the last system is asymptotically stable. In this case the solutions of system (1.5) are bounded (Theorem 5.3 [7]). In the turn the last assertion is also necessary to fulfill the condition (c3) of Theorem 3.1.

Now let $\mathtt{rank} D\mathbf{g}(\mathbf{x}) = n$. Then for verification of boundedness of solutions of system (1.5) it is enough to use only Theorem 5.3 [7].

4. Examples

Consider the following system without equilibrium points [9]:

$$\begin{cases} \dot{x}(t) = a(y(t) - x(t)), \\ \dot{y}(t) = -by(t) + nx(t)z(t) + cw(t), \\ \dot{z}(t) = d - \exp(x(t)y(t)), \\ \dot{w}(t) = -my(t), \end{cases} \tag{4.1}$$

where $a > 0, b > 0, c > 0, d > 1, m > 0, n > 0$.

The last two columns of the Jacobian matrix of system (4.1) are $(0, nx, 0, 0)^T$ and $(0, c, 0, 0)^T$. In this case system (4.1) is degenerate.

For a proof of boundedness of solutiions of system (4.1) it is possible again to take advantage of Theorems 4.2 and 5.3 [7]. However simpler it is to apply Theorem 3.1.

Since here we can put $b_{11} = b > 0$, then all conditions of Theorem 3.1 (with the exception of (c3)) are fulfilled. The verification of condition (c3) is represented in Figure 4.1. (In this case $\rho^2 = x^2 + z^2 + w^2$.)

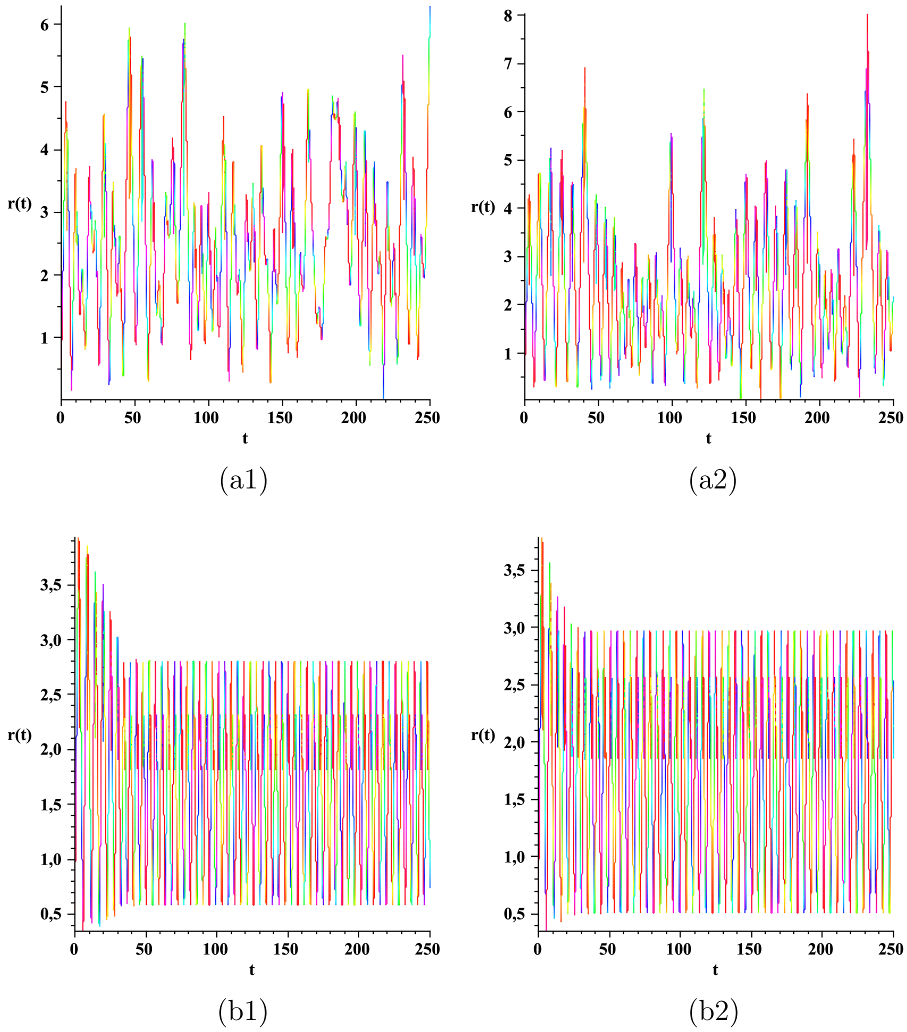
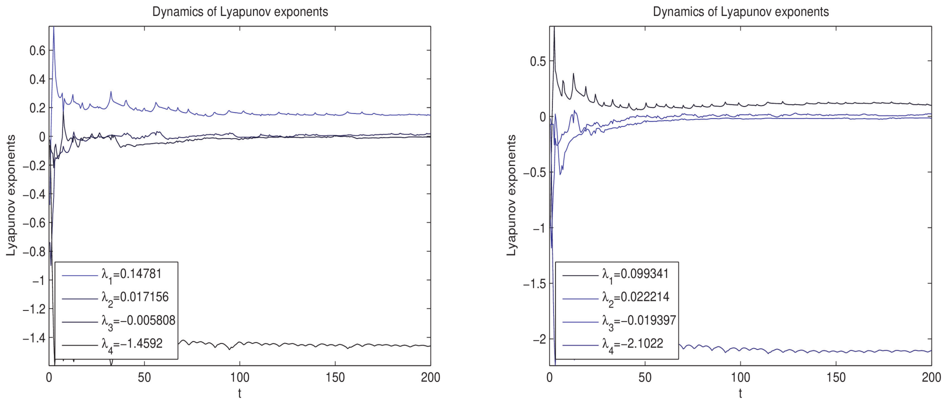


Fig. 4.1. The behavior of function $\rho(t)$ for system (4.1) at $b = m = 0.5$, $c = 0.2$, $n = 1$, $d = 2.5$, and also at $a = 0.8$ (a1), $a = 1.5$ (a2), $a = 2.8$ (b1), $a = 3.8$ (b2)

It is possible to consider condition (c3) of Theorem 3.1 ($\liminf_{t\to\infty} \rho(t) = 0$) for cases (a1), (a2) of Figure 4.1 was achieved. It means that at the indicated values of parameters the solutions of system (4.1) are chaotic. In cases (b1), (b2) of



(a1) (a2)

Fig. 4.2. The Lyapunov exponents $\lambda_1, ..., \lambda_4$ for system (4.1) at $b = m = 0.5$, $c = 0.2$, $n = 1$, $d = 2.5$, and also at $a = 0.8$ (a1), $a = 1.5$ (a2)

Figure 4.1 the solutions of system (4.1) are periodic. A proof of chaotic behavior of solutions of system (4.1) confirms Figure 4.2. Here a dynamics of Lyapunov exponents is shown. There are two positive Lyapunov exponents. It means an existence of hyperchaotic dynamics.

In Figure 4.3 we show the fulfillment of condition (c3) of Theorem 3.1 for point $(x, z, w) = (0, 0, 0) \in \mathbb{R}^3$, which can be achieved at $t^* \approx 219$. (In this case $x(219) = z(219) = w(219) = 0, y(219) = 0.4$.) However, this condition is not satisfied for the origin $(x, y, z, w) = (0, 0, 0, 0) \in \mathbb{R}^4$:

$$\liminf_{t\to\infty} \rho(t) = \liminf_{t\to\infty} \sqrt{x^2(x_0, y_0, z_0, w_0, t) + \cdots + w^2(x_0, y_0, z_0, w_0, t)} \approx \rho(219)$$

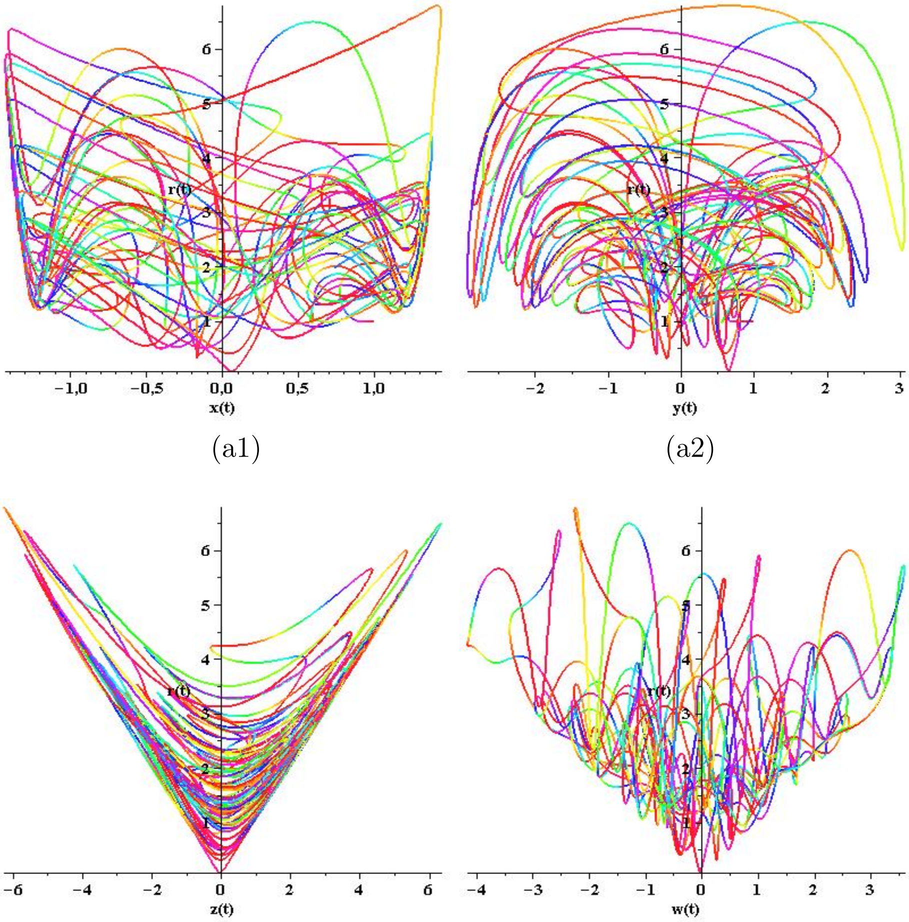
$$= \sqrt{0^2 + (0.4)^2 + 0^2 + 0^2} = 0.4 \neq 0.$$

This result is a consequence of the fact that system (4.1) has no equilibrium points. Therefore, the criteria for the existence of chaos proposed in [10, 11] are unsuitable for such systems.

In conclusion, we note that as the coefficient b_{11} from Theorem 3.1, we can take the coefficient $a = 0.8 > 0$. In this case we have $\rho^2 = y^2 + z^2 + w^2$ and the results will be similar to the previous ones.

5. Conclusion

It is known that the chaos in the autonomous nonlinear dynamical system begins with bifurcations of the limit cycle. Consequently, the existence of limit cycle in this system is a necessary condition for the existence of chaos. In turn, the minimum dimension of an autonomous dynamical system, in which a limit cycle can exist, is equal to 2. Exactly this circumstance dictated the introduction of system (2.1).



(a1) (a2)

(a3) (a4)

Fig. 4.3. The behavior of function $\rho(t) = h_1(x(t))(a1)$, $\rho(t) = h_2(y(t))(a2)$, $\rho(t) = h_3(z(t))(a3)$, and $\rho(t) = h_4(w(t))(a4)$ for system (4.1) at $b = m = 0.5$, $c = 0.2$, $n = 1$, $d = 2.5$, and $a = 0.8$

The present article is a continuation of work [10]. In this paper questions of existence of chaotic dynamics in the autonomous polynomial dynamic systems are extended to continuous systems, the right-hand sides of which are continuously differentiable functions.

Thus, the idea of reducing the problem of the existence of chaos in system (1.4) to the study of the problem of the existence of chaos in the $2D$ system (2.1) got further development.

Basic results got in the present work are such:

1. The sufficient conditions of existence of chaotic dynamics in continuous dynamic systems, the right-hand sides of which are continuously differentiable functions, are found.

2. These conditions do not suppose of the presence in system (1.4) either homoclinic or heteroclinic orbits. (The existence of such orbits almost always

results to the beginning of chaotic processes.)

3. The existence of equilibrium points in the same system (1.4) is not also assumed.

4. The results represented in papers [10, 11] are got under the condition $\mathbb{X}_0 = \mathbb{R}^n$. (This circumstance diminishes the class of systems for which the quoted results can be applied.) Theorem 3.1 of the present article does not suppose that the condition $\mathbb{X}_0 = \mathbb{R}^n$ takes place. (Here we have the condition $\mathbb{X}_0 \subset \mathbb{R}^n$ only.) Thus, the class of chaotic systems that satisfy the conditions of Theorem 3.1 can be much wider than the class of chaotic systems described in [10–12].

Nevertheless, the advantage of the criteria for the existence of chaos proposed in [12] is that the right-hand sides of system (1.4) can only be continuous (differentiability is not assumed).

References

1. A. C. J. Luo, *Regularity and Complexity in Dynamical Systems*, (Springer/ New York, Dordrecht, Heidelberg, London), 2012.
2. N. A. Magnitskii, *Universal theory of dynamical chaos in nonlinear dissipative systems of differential equations*, Communications in Nonlinear Science and Numerical Simulation, **13**, (2008), 416-433.
3. V. Ye. Belozyorov, *General method of construction of implicit discrete maps generating chaos in 3D quadratic systems of differential equations*, International Journal of Bifurcation and Chaos, **24**, (2014), ID 1450025-1-23.
4. V. Ye. Belozyorov, *Exponential-algebraic maps and chaos in 3D autonomous quadratic systems*, International Journal of Bifurcation and Chaos, **25**, (2015), ID 1550048-1-24.
5. V. Ye. Belozyorov, *Research of chaotic dynamics of 3D autonomous quadraic systems by their reduction to special 2D quadratic systems*, Mathematical Problems in Engineering, **2015**, (2015), ID 271637-1-15.
6. V. Ye. Belozyorov, S. A. Volkova, *Role of logistic and Ricker's maps in appearance of chaos in autonomous quadratic dynamical systems*, Nonlinear Dynamics, **83**, (2016), 719-729.
7. H. K. Khalil, *Nonlinear Systems – 2nd Edition*, (Prentice Hall/New-Jersy), 1996.
8. R. M. Crownover, *Introduction to Fractals and Chaos*, (Jones and Bartlett Publishers/ Boston, London), 1995.
9. D. V. Hoang, S. T. Kingni, V. T. Pham, *A no-equilibrium hyperchaotic system and its fractional-order form*, Mathematical Problems in Engineering, **2017**, (2017), ID 3927184-1-11.
10. V. Ye. Belozyorov, *On novel conditions of chaotic attractors existence in autonomous polynomial dynamical systems*, Nonlinear Dynamics, **91**, (2018), 2435–2452.
11. V. Ye. Belozyorov, D. V. Dantsev, *Modeling of chaotic processes by means of antisymmetric neural ODEs*, Journal of Optimization, Differential Equations and Their Applications (JODEA), **30**(1), (2022), 1–41.
12. V. Ye. Belozyorov, S. A. Volkova, *Odd and even functions in the design problem of new chaotic attractors*, International Journal of Bifurcation and Chaos, **32**, (2022), ID 2250218-1-26.

Received 28.08.2022