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Medium Domination Decomposition of Graphs

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Abstract

A set of vertices S in a graph G dominates G if every vertex in G is either in S or adjacent to a vertex in S. The size of any smallest dominating set is called domination number of G. The concept of Medium Domination Number was introduced by Vargor and Dunder which finds the total number of vertices that dominate all pairs of vertices and evaluate the average of this value. The Medium domination Number is a notation which uses neighbourhood of each pair of vertices. For $G = (V, E)$ and $\forall u, v \in V$ *if* u, v are adjacent they dominate each other, then atleast *dom (u, v) = 1*. The total number of vertices that dominate every pair of vertices is defined as TDV(G)=$\sum$ dom (u, v), for every u, v$\in$ V(G). For any connected, undirected, loopless graph G of order p, the Medium Domination Number MD(G) = $\frac{TDV(G)}{pC_2}$. In this paper we have introduced the new concept Medium Domination Decomposition. A decomposition $(G_1, G_2, \dots, G_n)$ of a graph G is said to be Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i - 1, i = 1,2,\dots,n$.

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1. Introduction

Graph is a mathematical representation of a network and it describes the relationship between vertices and edges. Let $G = (V, E)$ be a simple, connected, undirected, loopless graph with p vertices and q edges and G_i be the subgraph of G with p_i vertices and q_i edges, where $1 \leq i \leq n$, n is the number of subgraphs of G. The length of a shortest $u - v$ path in a connected graph G is called the distance from a vertex u to a vertex v. $d(u, v)$ denotes the distance between v and u. Two $u - v$ paths are internally disjoint if they have no vertices in common, other than u and v. The degree of a vertex v in a graph G is the number of edges of incident with v and denoted by $\deg(v)$. The minimum degree among the vertices of a graph G is denoted by $\delta(G)$. The maximum degree among vertices of a graph G is denoted by $\Delta(G)$. [1] The concept of Medium Domination Number was introduced by Vargor and Dunder which finds the total number of vertices that dominate all pairs of vertices and evaluate the average of this value.[5] T.I. Joel and E.E. Merly introduced the concept of Geodetic Decomposition of Graphs. Motivated by the above we have introduced the new concept of Medium Domination Decomposition of Graphs. For basic terminologies in graph theorem, we refer [2], [3] and [4]. The following are the basic definitions and results needed for the main section.

Definition 1.1. [1] For $G = (V, E)$ and $\forall\, u, v \in V$, if u and v are adjacent they dominate each other, then atleast $dom\ (u, v) = 1$.

Definition 1.2. [1] For $G = (V, E)$ and $\forall\, u, v \in V$, the total number of vertices that dominate every pair of vertices is defined as $TDV(G) = \Sigma_{\forall u, v \in V(G)} dom\ (u, v)$.

Definition 1.3. [1] For any connected, undirected, loopless graph G of order p the Medium Domination Number of G is defined as $MD(G) = \frac{TDV(G)}{\binom{P}{2}}$.

2. Medium Domination Decomposition of Graphs

Definition. 2.1 Let G be a simple connected (p, q) graph. A decomposition $(G_1, G_2, \ldots, G_n)$ of a graph G is said to be a Medium Domination Decomposition (MDD) if $\lfloor MD(G_i) \rfloor = i - 1, i = 1,2,3, \ldots., n$.

Example. 2.2

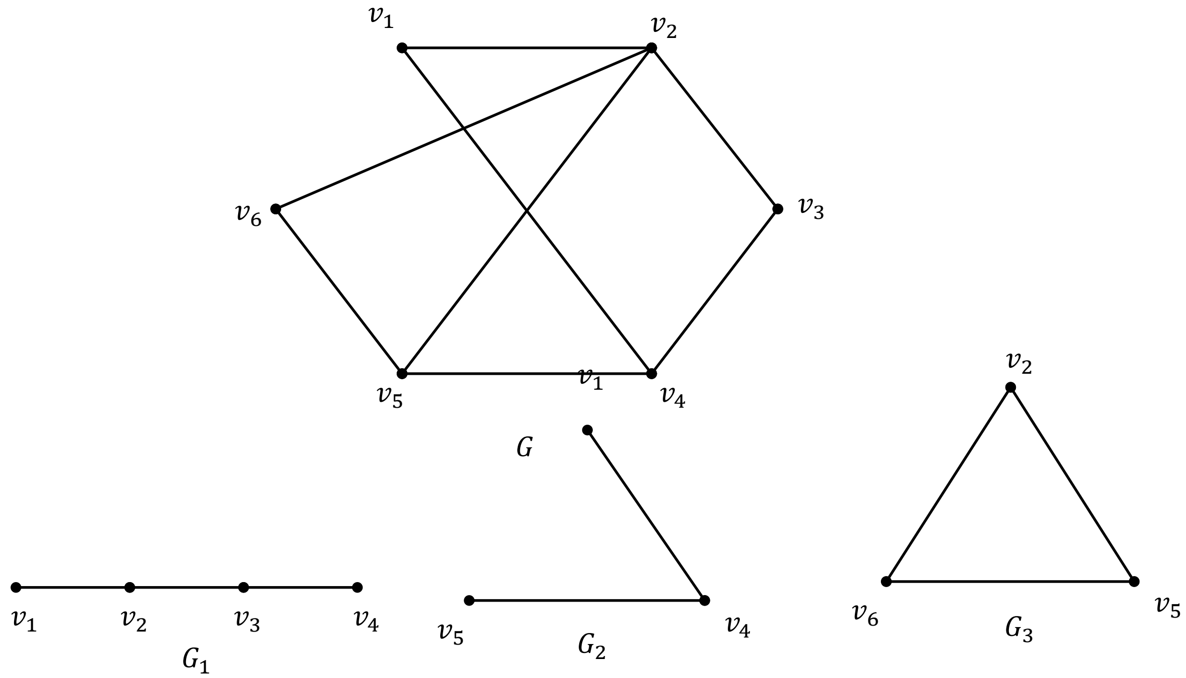

Figure 2.3

Here $MD(G_1) = 0.8, MD(G_2) = 1$ and $MD(G_3) = 2$
That is $\lfloor MD(G_1) \rfloor = 0, MD(G_2) = 1$ and $MD(G_3) = 2$

Remark. 2.4

i) Star Graph does not admit MDD
ii) $K_p, p \leq 4$, does not admit MDD

Theorem. 2.5 If a graph G admits $MDD(G_1, G_2, \ldots, G_n)$, then $p \geq 4$ and $q \geq 3$.
Proof.

Since the Medium Domination Number is one, when $p \leq 3$ and $q \leq 2$ and the Medium Domination Number is two, when p and q is 3, we can't get any subgraph with $\lfloor MD(G_i) \rfloor = 0$.

Note: 2.6 The converse of the above theorem is need not be true. For example, the complete graph K_4.

Theorem: 2.7

Let G be a graph and G admits $MDD(G_1, G_2, \ldots, G_n)$. Then
(i) $MD(G_2) = p_2 - \Delta(G_2)$ if and only if G_2 is $K_{1,m}$ for any m.
(ii) $MD(G_i) = \Delta(G_i)$ if and only if G_i is a complete graph, where $i = 2,3, \ldots., n$
Proof:

Suppose G admits $MDD(G_1, G_2, \ldots, G_n)$.

(i) Let G_2 be $K_{1,m}$ for any m.
$\Leftrightarrow G_2$ has p_2 vertices and $q_2 (= p_2 - 1)$ edges and the maximum degree of $G_2 = p_2 - 1$
$\Leftrightarrow MD(G_2) = \frac{(p_2-1)+\binom{p_2-1}{2}}{\binom{p_2}{2}}$
$\Leftrightarrow MDD(G_2) = 1$
$\Leftrightarrow MD(G_2) = p_2 - \Delta(G_2)$
(ii) Let G_i be a complete graph, $i = 1,2,3, \ldots, n$
$\Leftrightarrow G_i$ has p_i vertices and $q_i = \frac{p_i(p_i-1)}{2}$ edges and the maximum degree of $G_i = p_i - 1$.
$\Leftrightarrow MD(G_i) = \frac{\frac{p_i(p_i-1)}{2}+p_i[p_i-1)C_2]}{\binom{p_i}{2}}$
$\Leftrightarrow MD(G_i) = p_i - 1$
$\Leftrightarrow MD(G_i) = \Delta(G_i), i = 2,3, \ldots, n$
Hence the proof.

Theorem: 2.8 Let G be a graph and G admits $MDD(G_1, G_2, \ldots, G_n)$. Then $\sum_{i=1}^{n} \lfloor MD(G_i) \rfloor < \frac{n(n+1)}{2} \lceil MD(G) \rceil$, where n is the number of decompositions of G.
Proof:
We prove this theorem by induction on n.
When $n = 1$,
Then $\lfloor MD(G_1) \rfloor < \lceil MD(G) \rceil$,
Therefore, the result is true for $n = 1$
Assume that the theorem is true for $n - 1$
That is, $\sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor < \frac{n(n-1)}{2} \lceil MD(G) \rceil$
To prove: the theorem is true for n.
That is, $\sum_{i=1}^{n} \lfloor MD(G_i) \rfloor < \frac{n(n+1)}{2} \lceil MD(G) \rceil$.
Now,
$\sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor = \lfloor MD(G_1) \rfloor + \lfloor MD(G_2) \rfloor + \cdots + \lfloor MD(G_n) \rfloor$
$\Rightarrow \sum_{i=1}^{n-1} \lfloor MD(G_i) \rfloor + \lfloor MD(G_n) \rfloor < \frac{n(n+1)}{2} + \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$\Rightarrow \sum_{i=1}^{n} \lfloor MD(G_n) \rfloor < \frac{n^2-n}{2} + \lfloor MD(G) \rfloor + \lceil MD(G_n) \rceil$
$= \left(\frac{n^2 - n + n - n}{2} \right) \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$= \left(\frac{(n^2 + n)}{2} - \frac{2n}{2} \right) \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$= \left(\frac{n^2 + n}{2} \right) \lceil MD(G) \rceil - n \lceil MD(G) \rceil + \lfloor MD(G_n) \rfloor$
$< \left(\frac{n^2+n}{2} \right) \lceil MD(G) \rceil$, since the value of $-n \lceil MD(G) \rceil + \lfloor MD(G) \rfloor$ is negative
$\sum_{i=1}^{n} \lfloor MD(G_n) \rfloor < \left(\frac{n(n+1)}{2} \right) \lfloor MD(G) \rfloor$

The result is true for n. Hence the proof.

Theorem: 2.8 If G admits $MDD(G_1, G_2, \dots, G_n)$ and $\Delta(G) = p - 1$ then
i) $\gamma(G) < \lceil MD(G) \rceil$
ii) $\gamma(G) \leq \gamma(G_i)$
iii) $\gamma(G) \leq \lceil MD(G_i) \rceil$
Proof:
Let G be a graph with p vertices and q edges and $\Delta(G) = p - 1$.
Since $\Delta(G) = p - 1, \gamma(G) = 1$. But the minimum value of $\lceil MD(G) \rceil = 1$.
Therefore $\gamma(G) < \lceil MD(G) \rceil$.
The proof of (ii) and (iii) is obvious.

Remark: 2.9 The equality holds in theorem 2.8, (ii) and (iii) when G_i is star.

Theorem: 2.10 If G admits $MDD(G_1, G_2, \dots, G_n)$ then
i) $\lceil MD(G) \rceil < \gamma(G) + \Delta(G)$
ii) $\gamma(G_i) < \gamma(G) + \Delta(G)$
The Proof is obvious.

Theorem: 2.11 If G admits $MDD(G_1, G_2, \dots, G_n)$ and H is a Spanning subgraph of a graph G, then
(i) $\gamma(G) \leq \gamma(H)$
(ii) $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$
Proof:
Let G be a simple connected graph and G admits MDD.
Case (i): $n = 1$
Then $\gamma(G) = \gamma(H)$ and $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$.Hence the result is obvious.
Case (ii): $n > 1$
Then, let $G_1, G_2, \dots G_i, \dots, G_n$ be the decomposition of G.
Let $H = G_i$ be the spanning subgraph of G.
Since the number of decompositions is more than one, $|E(H)| < |E(G)|$
Also $\Delta(G) \geq \Delta(H)$ and $\delta(G) \geq \delta(H)$.
Therefore $\gamma(H) \geq \gamma(G)$.
Obviously, since H is a spanning subgraph of G, the Medium Domination Number of H is less than the Medium Domination Number of G.
Therefore $\lceil MD(G) \rceil > \lfloor MD(H) \rfloor$.
Hence the proof.

3. Conclusion

In this paper, we calculated the number of vertices that are capable of dominating both of u and v. The total number of vertices that dominate every pair of vertices is examined and the average of this value is calculated which is called "the medium domination number" of graph. Some theorems and results on the Medium Domination Decomposition of a graph and basic graph classes are given. Further this concept can be extended to some family of graphs.

References

[1] Duygu Vargor, Pinar Dundar, "The Medium Domination Number of a Graph", *International Journal of pure and applied mathematics volume of No.3, 2011 297-306.*

[2] Fairouz Beggas, "Decomposition and Domination of Some Graphs" Data Structures and Algorithms [cs.DS]. University Claude Bernard Lyon 1,2017.

[3] S. Arumugan and S. Ramachandran, "Invitation to Graph Theory", SciTech publications (India) PVT. LTD. (2003).

[4] Teresa W. Haynes, Stephen T. Hedetnimi and Peter J. Slater, "Fundamentals of Domination in Graphs" Marcel Dekkar, Inc., New York, 1998.

[5] T.I. Joel and E.E.R. Merly, "Geodetic Decomposition of Graphs", *Journal of Computer and Mathematical Sciences, 9(7) (2018), 829 - 833.*

[6] Sr Little Femilin Jana. D., Jaya. R., Arokia Ranjithkumar, M., Krishnakumar. S., "Resolving Sets and Dimension in Special Graphs", *Advances And Applications In Mathematical Sciences* 21 (7) (2022), 3709-3717.