NAVIER-STOKES-FOURIER ANALYTIC SOLUTIONS FOR NON-ISOTHERMAL COUETTE SLIP GAS FLOW

by

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The explicit and reliable analytical solutions for steady plane compressible non-isothermal Couette gas flow are presented. These solutions for velocity and temperature are developed by macroscopic approach from Navier-Stokes-Fourier system of continuum equations and the velocity slip and the temperature jump first order boundary conditions. Variability of the viscosity and thermal conductivity with temperature is involved in the model. The known result for the gas flow with constant and equal temperatures of the walls (isothermal walls) is verified and a new solution for the case of different temperature of the walls is obtained. Evan though the solution for isothermal walls correspond to the gas flow of the Knudsen number (Kn ≤ 0.1), i. e. to the slip and continuum flow, it is shown that the gas velocity and related shear stress are also valid for the whole range of the Knudsen number. The deviation from numerical results for the same system is less than 1%. The reliability of the solution is confirmed by comparing with results of other authors which are obtained numerically by microscopic approach. The advantage of the presented solution compared to previous is in a very simple applicability along with high accuracy.

Key words: *micro-Couette gas flow, non-isothermal, slip flow, analytical solution*

Introduction

Micro-Couette flow is often a part of different micro-electro-mechanical-systems, such as microcomb mechanisms, microgears, microbearings, and micromotors [1, 2]. Heaters installed in these systems can lead to the non-isothermal regime. For instance, heaters are embedded near the trailing edge in the thermal nanoactuated sliders. The temperature difference, between head and disk varies from 20-30 °C for the electrical heaters to 300-400 °C for the optical heaters. In this case the gas flow in the head-disk interface can be described by the non-isothermal micro-Couette gas flow. The slip flow regime (Kn < 0.1) prevails in microsystems, hence the results that describe that conditions are practical especially if the thermal effects are taken into account.

Isothermal Couette flow is solved using microscopic approach mostly, either by solving the kinetic Boltzmann equation [3-7] or by the direct simulation Monte-Carlo (DSMC) [3, 4, 7-12]. Also there are results with macroscopic approach by solving Navier-Stokes-Fourier (NSF) [3] or Burnett equations [8] numerically.

There are several analytical solutions for non-isothermal Couette flow with isothermal walls. An analytical solution by continuum theory for the constant viscosity and the

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thermal conductivity were compared with DSMC results in [9]. The analytical solutions of non-linear lattice Boltzmann (LB) kinetic equation for D2Q9 and D2Q16 models were presented in [13]. Couette flow with isothermal walls is solved analytically by regularized 13-moment equations in [5]. The analytical solutions by perturbation method were derived for micro-Couette non-isothermal [14] as well as microchannel [15] and microbearing isothermal [16] and non-isothermal [17] gas flows. Besides the perturbation method, the exact analytical solutions for micro-Couette gas flow with isothermal walls were also derived [14]. Now, these are presented and analyzed again to show that it is applicable not only for continuum and slip gas flow [14], but for all values of Knudsen numbers. It is confirmed here that velocity solution and shear stress are in a good agreement with velocity and shear stress obtained by DSMC [10] for the whole range of the Knudsen number, for different gases and also regardless of the values of the accommodation coefficient, σ_v.

The velocity and temperature fields for flow with isothermal as well as with constant but different walls temperatures by non-linear Bathangar-Gross-Krook equation in the whole range of Knudsen numbers were given in [6]. Also, the velocity and temperature fields for different walls temperatures obtained by DSMC are shown in [11, 12].

In this paper, the analytical solutions for micro-Couette slip gas flow are found by macroscopic approach. In the previous paper [14] the solutions for the same problem were obtained by perturbation method and were given in the form of complex and very long expressions with long terms defined in *Appendix*, while in this paper the system of governing equations are solved directly and the obtained analytical solutions are more concise and easy to use. Moreover, the previous solutions obtained by perturbation method [14] are approximate, while the solutions presented here are exact. Furthermore, the explicit analytical solutions for temperature and velocity fields for the flow with constant but different walls temperatures are presented here for the first time and also are verified by DSMC solutions of other authors [11, 12]. The problems are solved by continuum governing equations (continuity, NSF momentum and energy) accompanied by the Maxwell-Smoluchowski first-order velocity slip and temperature jump boundary conditions. Second order boundary condition does not contribute to the solution accuracy [18]. In this paper the influences of the viscosity and thermal conductivity dependence on temperature are taken into account. The presented solutions also agree with the numerical solutions of NSF governing equations [14], with deviation less than 1%.

As opposed to the molecular approach that is prevalent in the literature, solutions presented in this paper have been obtained completely analytically, by applying the theory of the continuum. Therefore, these solutions have a much simpler form that allows their easier application. Despite their simplicity and conciseness, comparing with solutions obtained by DSMC method and with numerical solutions of NSF equations for Couette slip gas flow, it has been shown in this paper that the obtained solutions are highly reliable. To the best of our knowledge, the solutions for velocity and temperature for micro-Couette slip gas flow for different temperatures of the walls are the first of this kind in the open literature.

Problem formulation and governing equation

A compressible non-isothermal Couette slip gas flow between two parallel plates at different but constant wall temperatures $\tilde{T}_\mathrm{r} \mp \Delta\tilde{T}/2$, , which are moving with constant velocities $\pm\tilde{u}_\mathrm{w}$ is considered, fig. 1. The temperature of the upper plate is $\tilde{T}_\mathrm{r} - \Delta\tilde{T}/2$ and it is moving to the right, while the temperature of the lower plate is $\tilde{T}_\mathrm{r} + \Delta\tilde{T}/2$ and it is moving to the left. The Couette flow as the shear-driven flow is fully developed in the stream-wise direction $\tilde{x}$, so all flow parameters (pressure, temperature, and velocity) do not change in the stream-wise direction. Furthermore, from the continuity equation and the fact that the cross-wise velocity at the parallel walls is zero $(\tilde{v}_\mathrm{w} = 0)$ follow that the cross-wise velocity is $\tilde{v} = 0$ in the whole flow domain. In order to transform the momentum and energy equations into a dimensionless form all variables are scaled in the following way: $y = \tilde{y}/\tilde{h}$, $u = \tilde{u}/\tilde{u}_\mathrm{w}$, $p = \tilde{p}/\tilde{p}_\mathrm{r}$, $T = \tilde{T}/\tilde{T}_\mathrm{r}$, $\rho = \tilde{\rho}/\tilde{\rho}_\mathrm{r}$, $\mu = \tilde{\mu}/\tilde{\mu}_\mathrm{r}$, and $k = \tilde{k}/\tilde{k}_\mathrm{r}$, where $\tilde{h}$ is the distance between the plates, $\tilde{p}_\mathrm{r}$ – the pressure at $\tilde{y} = 0$, $\tilde{T}_\mathrm{r}$ – the average temperature for the lower and the upper plate, and $\tilde{\rho}_\mathrm{r} = \tilde{p}_\mathrm{r}/\mathrm{R}\tilde{T}_\mathrm{r}$ (R is the gas constant). The reference dynamics viscosity, $\tilde{\mu}_\mathrm{r}$, and thermal conductivity, $\tilde{k}_\mathrm{r}$, correspond to the reference temperature, $\tilde{T}_\mathrm{r}$. All dimensional quantities are marked with a tilde, while the dimensionless are without it.

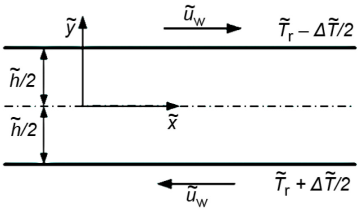

Figure 1. Overview of micro-Couette flow with different wall temperatures

Hence, with all previous assumptions, the momentum equations in the stream-wise and cross-wise directions and NSF energy equation in the non-dimensional form are:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mu \frac{\mathrm{d}u}{\mathrm{d}y}\right) = 0 \tag{1}$$

$$\frac{\mathrm{d}p}{\mathrm{d}y} = 0 \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(k \frac{\mathrm{d}T}{\mathrm{d}y}\right) + (\kappa - 1)\mathrm{Ma}_\mathrm{r}^2 \mathrm{Pr}\mu \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 = 0 \tag{3}$$

where $\mathrm{Ma}_\mathrm{r} = \tilde{u}_\mathrm{w}/(\kappa \mathrm{R}\tilde{T}_\mathrm{r})^{1/2}$ is the reference Mach number, κ – the ratio of the specific heats, $\mathrm{Pr} = c_\mathrm{P}\tilde{\mu}/\tilde{k}$ – the Prandtl number, c_p – the specific heat at constant pressure, $\tilde{\mu}$ – the dynamic viscosity, and $\tilde{k}$ – the thermal conductivity. The dynamic viscosity and the thermal conductivity vary with temperature as $\mu = k = \tilde{T}^a/\tilde{T}_\mathrm{r}^a = T^a$. For the hard sphere molecule model the value of the viscosity-temperature index is in the range $a \in (0.5,\ 1)$ [19]. The value a = 0.5 corresponds to the elastic sphere molecule model, while a = 1 to the Maxwellian molecules. If the viscosity and conductivity dependence on the temperature is neglected, the viscosity-temperature index is a = 0. Since the pressure does not change in the x-direction, the eq. (2) leads to p = 1. Hence, the equation of the state in the non-dimensional form is:

$$\rho = \frac{1}{T} \tag{4}$$

The gas velocity and the temperature at the walls in the non-dimensional form are Maxwell-Smoluchowski first-order slip boundary conditions:

$$y = \pm 0.5, \qquad u = \pm 1 \mp \frac{2 - \sigma_v}{\sigma_v} \mathrm{Kn}_\mathrm{r} T^{a+0.5} \left.\frac{\partial u}{\partial y}\right|_\mathrm{w}, \qquad v = 0 \tag{5}$$

$$y = \pm 0.5, \qquad T = 1 \mp \theta \mp \frac{2 - \sigma_T}{\sigma_T} \frac{2\kappa}{\kappa + 1} \frac{\mathrm{Kn}_\mathrm{r}}{\mathrm{Pr}} T^{a+0.5} \left.\frac{\partial T}{\partial y}\right|_\mathrm{w} \tag{6}$$

where σ_v and σ_T are the momentum and thermal accommodation coefficients and $\theta = \Delta T/2$. The local Knudsen number is $\mathrm{Kn} = \tilde{\lambda}/\tilde{h}$ where the local molecular mean-free path is defined as $\tilde{\lambda} = \tilde{\mu}[(\pi R\tilde{T}/2)^{1/2}]/\tilde{p}$. Since the reference Knudsen number is defined as $\mathrm{Kn_r} = \tilde{\lambda}_\mathrm{r}/\tilde{h}$ where $\tilde{\lambda}_\mathrm{r} = \tilde{\mu}_\mathrm{r}[(\pi R\tilde{T}_\mathrm{r}/2)^{1/2}]/\tilde{p}$, the local and reference Knudsen numbers relates as $\mathrm{Kn} = \mathrm{Kn_r} T^{a+0.5}$.

Solution for the isothermal walls

In the case of Couette flow with constant and equal wall temperatures ($\theta = 0$), although the temperature field is not isothermal, it was shown that the dynamic viscosity and thermal conductivity dependence on the temperature is negligible [14]. Accordingly, solving the previous system of governing eqs. (1) and (3) with boundary conditions, eqs. (5) and (6), and relying on the assumption about anti-symmetric velocity profile and symmetric temperature profile, it is possible to get the exact analytical solution for the velocity and temperature fields. For the constant dynamics viscosity and thermal conductivity ($\mu = k = 1$, $a = 0$), the solutions [14] are:

$$u = C_0 y \tag{7}$$

$$T = -C(C_0 y)^2 + 2CC_0^2 \left(\frac{2-\sigma_T}{\sigma_T} \frac{\kappa}{\kappa+1} \frac{\mathrm{Kn_r}}{\mathrm{Pr}} + \frac{1}{8} \right) + 1 \tag{8}$$

where constants C_0 and C are:

$$C_0 = \frac{2}{1 + 2\dfrac{2-\sigma_v}{\sigma_v}\mathrm{Kn_r}}, \qquad C = \frac{\kappa-1}{2}\mathrm{Ma_r^2}\,\mathrm{Pr} \tag{9}$$

From eqs. (7) and (8), the shear stress and normal flux are derived:

$$\tau = C_0 \tag{10}$$

$$q_y = 2CC_0^2 y \tag{11}$$

The results from (7) and (8) are explicit and reliable solutions of the continuum governing NSF equations for the continuum and slip Couette gas flow. They differ from the numerical solutions of NSF equations [4, 14] much less than one percent.

The velocity profile (7) could be reduced to the same form as the first term of LB hierarchy *i. e.* D2Q9 model which was given in [13], where was shown that beside the agreement in the continuum and slip domain, the solutions for the velocity and hence the shear stress are valid for the whole range of the Knudsen number. Moreover, the solutions (7) and (10) for the $\sigma_v = 1$ and mono-atomic gas ($\kappa = 5/3$, Pr = 5/3) are compared with DSMC [10]. Evan thought the solutions (7)-(11) are the consequence of the NSF continuum equations and are appropriate for the slip and continuum regimes, this comparison also shows that the solutions are applicable for all Knudsen number values. It is validated for different gases (argon, helium, nitrogen, and air) and for different momentum accommodation coefficients. In fig. 2 the velocity and shear stress are presented only for argon. In the continuum and slip regime complete agreement with DSMC [10] is achieved, as it is expected. The velocity results obtained by presented model (7) in the transition domain (p = 1 Pa *i. e.* Kn = 6.3, p = 10 Pa *i. e.* Kn = 0.63) deviate from DSMC, which can be seen from the velocity diagram in the fig. 2(a). Transition domain is known in the literature as an unstable zone so it is the most difficult to obtain the reliable results in that domain. Taking this into account the presented model gives results with acceptable accuracy. For higher values of Knudsen numbers (p = 0.1 Pa *i. e.* Kn = 63), that correspond to the fully diluted gas, presented solution for velocity (7) is completely consistent with DSMC. The NSF continuum equations could not achieve the s-shape for velocity profile that is obtained by DSMC. In that sense, the solutions obtained by DSMC method are more reliable in the vicinity of the walls for all regimes.

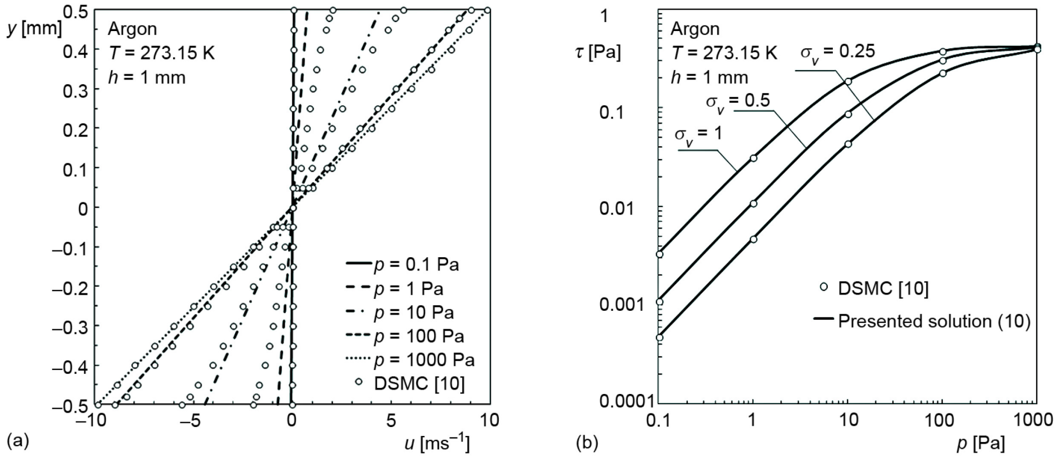

Figure 2. Comparison of presented velocity (7) (a) and shear stress (10), (b) with DSMC [10], for $\tilde{u}_w = \pm 10$ m/s and different pressures *i. e.* Knudsen numbers

In fig. 2(b) the shear stress dependence on the pressure for the three different momentum accommodation coefficients is presented also. The results correspond to the wide range of pressure values ($0.1 \le p \le 1000$), *i. e.* Knudsen numbers ($63 \ge \text{Kn} \ge 0.0063$). It is shown that the agreement of the presented model for the shear stress, eq. (10) with DSMC [10] is good for all values of Knudsen numbers.

Solution for the different temperature walls

The other case, Couette slip gas flow with different but constant temperatures of the walls, is considered also. The several next steps show that it is possible to get an analytical solution for the Couette slip gas flow and continuum gas flow and that it is completely consistent to the numerical result obtained by Runge-Kutta method, presented in [14]. The procedure for solving the system of NSF governing eqs. (1) and (3), and boundary conditions (5) and (6) is the same for the slip and continuum regimes.

Double integration of the energy eq. (3) using the momentum eq. (1) the temperature solution as the function of velocity field:

$$T = -Cu^2 + \frac{C_1}{C_3}u + C_2 \tag{12}$$

where C is defined by eq. (9). The constants of integration C_1, C_2, and C_3 must be defined by boundary conditions. The first integration of energy eq. (3) gives the exact expression for normal heat flux:

$$q_y = -k\frac{\mathrm{d}T}{\mathrm{d}y} = 2CC_3u - C_1 \tag{13}$$

Integration of the momentum eq. (1) gives the equation that separates the variables:

$$\int\left(-Cu^2+\frac{C_1}{C_3}u+C_2\right)^a \mathrm{d}u = C_3 y + C_4 \tag{14}$$

A good agreement with the numerical solutions for the velocity and temperature flow fields is achieved after expanding the previous integrand into the series on u and keeping only the first two terms. The velocity follows from the quadratic equation implicitly:

$$C_2^a u + \frac{aC_1}{2C_3} C_2^{a-1} u^2 = C_3 y + C_4 \tag{15}$$

and explicitly:

$$u_{1,2} = \frac{C_2 C_3}{aC_1}\left[-1 \pm \sqrt{1+2aC_1 \frac{\left(\frac{C_4}{C_3}+y\right)}{C_2^{a+1}}}\right] \tag{16}$$

The acceptable is only solution with sign +, because the velocity is positive on the lower and negative on the upper wall. Thus, the velocity solution is:

$$u = \frac{C_2 C_3}{aC_1}\left[-1 + \sqrt{1+2aC_1 \frac{\left(\frac{C_4}{C_3}+y\right)}{C_2^{a+1}}}\right] \tag{17}$$

Putting the solution for velocity (17) and temperature (12) into the velocity (5) and temperature (6) boundary conditions is:

$$1+\theta+\frac{2-\sigma_T}{\sigma_T}\frac{2\kappa}{\kappa+1}\frac{\mathrm{Kn_r}}{\mathrm{Pr}}(-2C\,C_3 u\big|_{y=-0,5}+C_1)\sqrt{T\big|_{y=-0,5}} = T\big|_{y=-0,5} \tag{18}$$

$$1-\theta-\frac{2-\sigma_T}{\sigma_T}\frac{2\kappa}{\kappa+1}\frac{\mathrm{Kn_r}}{\mathrm{Pr}}(-2C\,C_3 u\big|_{y=0,5}+C_1)\sqrt{T\big|_{y=0,5}} = T\big|_{y=0,5} \tag{19}$$

$$-1+\frac{2-\sigma_v}{\sigma_v}C_3\mathrm{Kn_r}\sqrt{T\big|_{y=-0,5}} = u\big|_{y=-0,5} \tag{20}$$

$$1-\frac{2-\sigma_v}{\sigma_v}C_3\mathrm{Kn_r}\sqrt{T\big|_{y=0,5}} = u\big|_{y=0,5} \tag{21}$$

The constants C_1, C_2, C_3, and C_4 are simply found from the system (18)-(21) numerically (the authors used a package MATHEMATICA), using the solutions for velocity (17) and temperature (12) at both walls ($y = \pm 0.5$). In the case of continuum, appropriate constants C_{c1}, C_{c2}, C_{c3}, and C_{c4} can be found by putting in the system (18)-(21) the appropriate boundary conditions at the walls:

$$u\big|_{y=\pm 0.5} = \pm 1, \qquad T\big|_{y=\pm 0.5} = 1 \mp \theta \tag{22}$$

Those constants are then:

$$C_{c1} = -2\theta(1+C)^a, \quad C_{c2} = 1+C, \quad C_{c3} = 2(1+C)^a, \quad C_{c4} = -\frac{a\theta}{2}(1+C)^{(a-1)} \tag{23}$$

Putting the constants (23) into the velocity eq. (17) and temperature eq. (12) the solutions for velocity and temperature for continuum Couette gas flow in the case of different temperatures of the walls read, respectively:

$$u = C_c - \sqrt{C_c(C_c - 4y) + 1} \tag{24}$$

$$T = -C\left[C_c - \sqrt{C_c(C_c - 4y)+1}\right]^2 - \theta\left[C_c - \sqrt{C_c(C_c-4y)+1}\right] + C + 1 \tag{25}$$

where C_c is the constant defined as $C_c = (1+C)/(a\theta)$.

In fig. 3 the influence of temperature on the transport properties *e. g.* on the temperature (12) and velocity (17) profiles is presented taking into account the three values of the viscosity-temperature index $a = 0$ (constant dynamic viscosity and thermal conductivity), $a = 0.5$ (the elastic sphere molecule model) and $a = 1$ (Maxwellian molecules). The presented analytical solutions for the temperature (12) and velocity (17) are compared with numerical solutions of NSF system of equations for this Couette slip gas flow explained in [14] and with the appropriate DSMC solutions from [11] and [12]. The momentum and thermal accommodation coefficients are taken as $\sigma_v = 1$ and $\sigma_T = 1$. The temperature and velocity difference of the walls are $\Delta\tilde{T} = 70$ K and $\Delta\tilde{u} = 100$ m/s, respectively. The referent temperature and pressure are $\tilde{T}_r = 273.15$ K and $\tilde{p}_r = 266.644$ Pa, respectively. Furthermore, the lower wall is colder and is moving to the left ($\tilde{u}_w = 50$ m/s), whereas the upper wall is warmer and moving to the right with the same velocity. The all considered conditions are the same to the ones provided by [11]. The results are compared for Maxwellian molecules $a = 1$ and a very good agreement is achieved.

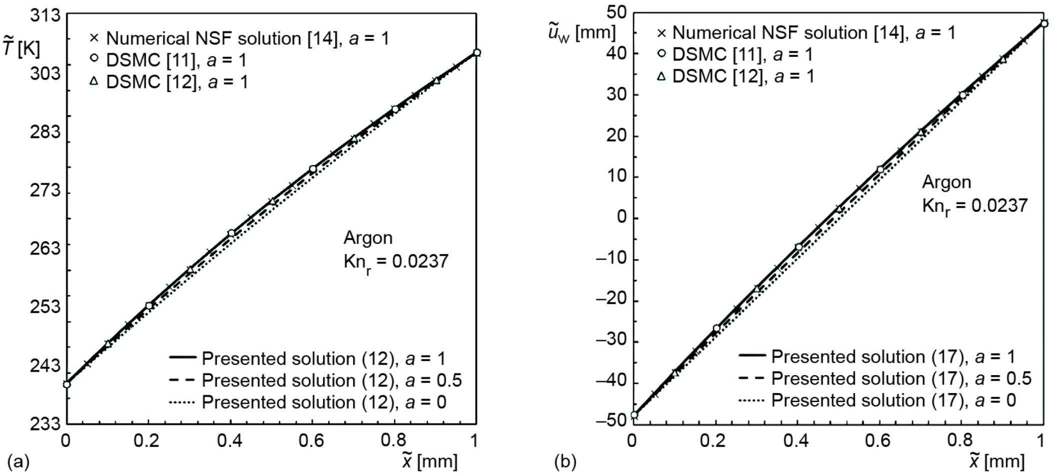

Figure 3. Comparison of presented temperature (12) (a) and velocity (17) (b) solutions with DSMC [11] and [12]

The results for different temperatures of the walls for the negligible temperature difference tend to the solutions for the isothermal walls. In that case the solutions for the velocity (7) and temperature (8) that correspond to the micro-Couette gas flow between walls with equal temperatures are reliable.

Solution for the one adiabatic wall

The obtained solutions for temperature (12) and velocity (17) can cover not only the previously solved case, but a wider range of cases. With appropriate boundary conditions they can be used for solving similar, but different problems. As the variety of micro gauges often contains micro-Couette gas flow, an interesting example is the case when one of the walls is adiabatically insulated.

For example, we can consider the Couette microflow with adiabatic bottom wall. The problem is solved with the same procedure as the previous one, where the walls were on different temperature. The only difference when the bottom wall is adiabatic is in the temperature boundary conditions (6). There is no temperature jump at the bottom wall, and if the temperature is scaled by the top wall temperature, the boundary conditions for the bottom and the top wall are respectively:

$$\frac{2-\sigma_T}{\sigma_T}\frac{2\kappa}{\kappa+1}\frac{\mathrm{Kn_r}}{\mathrm{Pr}}(-2C\,C_3u\big|_{y=-0,5}+C_1)\sqrt{T\big|_{y=-0,5}}=0 \tag{26}$$

$$1-\frac{2-\sigma_T}{\sigma_T}\frac{2\kappa}{\kappa+1}\frac{\mathrm{Kn_r}}{\mathrm{Pr}}(-2C\,C_3u\big|_{y=0,5}+C_1)\sqrt{T\big|_{y=0,5}}=T\big|_{y=0,5} \tag{27}$$

The velocity boundary conditions are given by the same equations (20) and (21) as in the previous case where the walls were on different temperature. Then, the appropriate constants C_1, C_2, C_3, and C_4 are simply found from the system (20), (21), (26), and (27) numerically. The solutions for temperature (12) and velocity (17) with these constants are compared with the corresponding results that are given by Karniadakis *et al.* [2] and a good agreement is achieved.

Conclusions

Based on the obtained solutions, it is possible to derive the net mass flow in both considered cases. The obtained analytical solutions imply that in the case of Couette gas flow between isothermal walls, although the temperature of the walls is the same and constant, and the distance between walls is of micron scale, the obtained gas temperature profile is non--uniform. The velocity profile is asymmetric and the temperature profile is symmetric. Hence, it follows that the net mass flow is zero. On the other hand, in the second case, unequal walls temperatures lead to the non-zero net mass flow. Even though the walls are moving with the same velocities in opposite directions, the velocity profile is not asymmetric due to the non--isothermal flow. The velocity solution shows that the slip is always larger at the warmer wall than at the colder one. However, whether the temperature jump is larger at the warmer or at the colder wall depends on the walls velocity.

Microchannel gas flows are encountered in a wide variety of contemporary applications. Hence, analytical solutions for slip gas flows are very useful [20], as they can serve as accuracy verification of experimental results and numerical calculations. In this paper the solutions for velocity and temperature fields take into account the influence of temperature on the transport properties. The obtained analytical solutions for the non-isothermal Couette slip gas flow with equal and different walls temperatures might serve as the benchmark test for research in this field, because they are in a good agreement with other authors' results, but however they are simple and explicit. Also, differently from the results found in the open literature for the non--isothermal micro-Couette flow, our results are obtained analytically, by a macroscopic approach.

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References

[1] Gad-el-Hak, M., *The MEMS Handbook,* CRC Press, Taylor and Franciz, N. Y., USA, 2002
[2] Karniadakis, G. E., *et al.*, *Microflows and Nanoflows. Fundamentals and Simulation,* Springer, Berlin, 2005
[3] Gu, X. J., Emerson, D. R., A Computational Strategy for the Regularized 13 Moment Equations with Enhanced Wall-Boundary Conditions, *J. Comput. Phys., 225* (2007), 1, pp. 263-283
[4] Gu, X. J., Emerson, D. R., A High-Order Moment Approach for Capturing Non-Equilibrium Phenomena in the Transition Regime, *J. Fluid Mech., 636* (2009), Oct., pp. 177-216
[5] Taheri, P., *et al.*, Couette and Poiseuille Microflows: Analytical Solutions for Regularized 13-Moment Equations, *Phys. Fluids, 21* (2009), ID 017102
[6] Misdanitis, S., Valougeorgis, D., Couette Flow with Heat Transfer in the whole Range of the Knudsen Number, *Proceedings*, 6th ICNMM (ASME), Darmstadt, Germany, 2008
[7] Xue, H., *et al.*, Prediction of Flow and Heat Transfer Characteristics in Micro-Couette Flow, *Microscale Thermophysical Engineering, 7* (2003), 1, pp. 51-68
[8] Lockerby, D. A., Reese, J. M., High-Resolution Burnett Simulations of Micro Couette Flow and Heat Transfer, *J. Comput. Physics, 18* (2003), 2, pp. 333-347
[9] Marques, W. Jr., *et al.*, Couette Flow with Slip and Jump Boundary Conditions, *Continuum Mech. Thermodyn., 12* (2000), 6, pp. 379-386
[10] Torczynski, J. R., Gallis, M. A., DSMC-Based Shear-Stress/Velocity-Slip Boundary Condition for Navier-Stokes Couette-Flow Simulations, *Proceedings,* 27th International Symposium on Rarefied Gas Dynamics, AIP Conference 1333*,* Pacific Grove, Cal., USA, 2010, pp. 802-807
[11] Gallis, M. A., *et al.*, Normal Solutions of the Boltzmann Equation for Highly Nonequilibrium Fourier Flow and Couette Flow, *Phys. Fluids, 18* (2006), ID 017104
[12] Zhou, W. D., *et al.*, Rarefied-Gas Heat Transfer in Micro- and Nanoscale Couette Flows, *Phys. Rev. E, 81* (2010), ID 011204
[13] Ansumali, S., *et al.*, Hydrodynamics Beyond Navier-Stokes: Exact Solution to the Lattice Boltzmann Hierarchy, *Phys. Rev. Lett., 98* (2007), ID 124502
[14] Milicev, S. S., Stevanovic, D. N., A Non-Isothermal Couette Slip Gas Flow, *Sci. China-Phys. Mech. Astron., 56* (2013), 9, pp. 1782-1797
[15] Stevanovic, D. N., A New Analytical Solution of Microchannel Gas Flow, *J. Micromech. Microeng., 17* (2007), 8, pp. 1695-1702
[16] Stevanovic, D. N., Analytical Solution of Gas Lubricated Slider Microbearing, *Microfluid. Nanofluid., 7* (2009), 1, pp. 97-105
[17] Milićev, S. S., Stevanović, D. N., A Microbearing Gas Flow with Different Walls' Temperatures, *Thermal Science, 16* (2012), 1, pp. 119-132
[18] Hamdan, M. A., *et al.*, Effect of Second Order Velocity-Slip/Temperature-Jump on Basic Gaseous Fluctuating Micro-Flows, *ASME J. Fluids Eng., 132* (2010), 7, pp. 0745031-0745036
[19] Vincenti, W. G., Kruger, C. H., *Introduction to Physical Gas Dynamics,* John Wiley & Sons, N. Y., USA, 1965
[20] Wang, C. Y., Brief Review of Exact Solutions for Slip-Flow in Ducts and Channels, *ASME J. Fluids Eng., 134* (2012), 9, ID 094501

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