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Error estimates of anti-Gaussian quadrature formulae ☆

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This work is dedicated to the memory of Professor Franz Peherstorfer.

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ABSTRACT

Anti-Gauss quadrature formulae associated with four classical Chebyshev weight functions are considered. Complex-variable methods are used to obtain expansions of the error in anti-Gaussian quadrature formulae over the interval $[-1, 1]$. The kernel of the remainder term in anti-Gaussian quadrature formulae is analyzed. The location on the elliptic contours where the modulus of the kernel attains its maximum value is investigated. This leads to effective L^∞-error bounds of anti-Gauss quadratures. Moreover, the effective L^1-error estimates are also derived. The results obtained here are an analogue of some results of Gautschi and Varga (1983) [11], Gautschi et al. (1990) [9] and Hunter (1995) [10] concerning Gaussian quadratures.

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1. Introduction

Let w be a given nonnegative and integrable weight function on the interval $[-1, 1]$. Let us denote by p_k the monic polynomial of degree k, which is orthogonal to $\mathbb{P}_{k-1}$ ($\mathbb{P}_k$ denotes the set of polynomials of degree at most k) with respect to w, i.e.

$$\int_{-1}^{1} x^j p_k(x) w(x) \mathrm{d}x = 0, \quad j = 0, 1, \ldots, k-1,$$

and let us recall that (p_k) satisfies a three-term recurrence relation of the form

$$p_{k+1}(x) = (x - a_k)p_k(x) - b_k p_{k-1}(x), \quad k = 0, 1, \ldots, \tag{1.1}$$

where $p_{-1}(x) = 0$, $p_0(x) = 1$ and the b_k's have the property of being positive.

The unique interpolatory quadrature formula with n nodes and the highest possible degree of exactness $2n - 1$ is the Gaussian formula with respect to the weight w,

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x = G_n[f] + E_n(f), \quad G_n[f] = \sum_{j=1}^{n} \lambda_j^G f\left(x_j^G\right) \ (n \in \mathbb{N}). \tag{1.2}$$

In [1], Laurie introduced quadrature rules that he referred to as anti-Gauss associated with the weight w,

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x = A_{n+1}[f] + R_{n+1}(f), \quad A_{n+1}[f] = \sum_{j=1}^{n+1} \lambda_j^A f\left(x_j^A\right) \ (n \in \mathbb{N}). \tag{1.3}$$

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This is an $(n+1)$-point interpolatory formula of degree $2n-1$ which integrates polynomials of degree up to $2n+1$ with an error equal in magnitude but opposite sign to that of the n-point Gaussian formula (1.2). Its intended application is to estimate the error incurred in Gaussian integration by halving the difference between the results obtained from the two formulae. Concerning with this and related problematic there appeared several papers in the last time, see [2–8]. Laurie [1] showed that an anti-Gaussian quadrature formula has positive weights and that its nodes are in the integration interval (except that for some weight functions, at most two of the nodes may be outside the integration interval) and are interlaced by those of the corresponding Gaussian formula. The anti-Gaussian formula is as easy to compute as the $(n+1)$-point Gaussian formula. Finally, the anti-Gaussian quadrature formula (1.3) is based on the zeros of polynomial

$$\pi_{n+1} = p_{n+1} - b_n p_{n-1}, \tag{1.4}$$

which is orthogonal subject to the linear functional $2\int[\cdot]w(x)\mathrm{d}x - G_n[\cdot]$.

In this paper w represents one of four classical Chebyshev weight functions:

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \qquad w_2(t) = \sqrt{1-t^2}, \qquad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \qquad w_4(t) = \sqrt{\frac{1-t}{1+t}}.$$

In these cases all nodes of the anti-Gauss quadrature formula (1.3), i.e., all zeros of the corresponding polynomial π_{n+1}, belong to the interval $[-1, 1]$. They are, in the same time, the Kronrod nodes (see [1]).

2. On the remainder term of anti-Gauss quadrature formulae for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and let D be its interior. If integrand f is analytic on D and continuous on $\overline{D}$, and if all nodes of anti-Gauss quadrature formula belong to the interval $[-1, 1]$, then the remainder term $R_{n+1}(f)$ in (1.3) admits the contour integral representation

$$R_{n+1}(f) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma K_{n+1}(z) f(z)\mathrm{d}z. \tag{2.1}$$

The *kernel* is given by

$$K_{n+1}(z) = \frac{\varrho_{n+1}(z)}{\pi_{n+1}(z)}, \quad z \notin [-1, 1], \tag{2.2}$$

where

$$\varrho_{n+1}(z) = \int_{-1}^{1} \frac{\pi_{n+1}(x)}{z-x} w(x)\mathrm{d}x.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_{n+1}(\overline{z})| = |K_{n+}(z)|$. If the weight function w is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_{n+1}(-\overline{z})| = |K_{n+1}(z)|$ (see [9]).

In many papers error bounds of $|E_n(f)|$, i.e., of the modulus of the remainder term in Gauss quadrature formula (1.2), where f is an analytic function, are considered. Two choices of the contour Γ have been widely used:

- a circle C_r with a center at the origin and a radius $r\ (>1)$, i.e., $C_r = \{z \mid |z| = r\}$, $r > 1$, and
- an ellipse $\mathcal{E}_\rho$ with foci at the points ∓ 1 and a sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \mid z = \frac{1}{2}\left(\xi + \xi^{-1}\right),\ \xi = \rho\, \mathrm{e}^{\mathrm{i}\theta},\ 0 \le \theta \le 2\pi \right\}. \tag{2.3}$$

When $\rho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1. In this paper we take Γ to be the ellipse $\mathcal{E}_\rho$.

The integral representation (2.1), for the remainder term in the anti-Gauss quadrature formula (1.3), leads to a general error estimate, by using Hölder's inequality,

$$\begin{aligned} |R_{n+1}(f)| &= \frac{1}{2\pi} \left| \oint_{\mathcal{E}_\rho} K_{n+1}(z) f(z)\mathrm{d}z \right| \\ &\le \frac{1}{2\pi} \left(\oint_{\mathcal{E}_\rho} |K_{n+1}(z)|^r |\mathrm{d}z| \right)^{1/r} \left(\oint_{\mathcal{E}_\rho} |f(z)|^{r'} |\mathrm{d}z| \right)^{1/r'}, \end{aligned}$$

i.e.,

$$|R_{n+1}(f)| \le \frac{1}{2\pi} \|K_{n+1}\|_r \|f\|_{r'},$$

where $1 \le r \le +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} \left(\oint_{\mathcal{E}_\rho} |f(z)|^r |dz| \right)^{1/r}, & 1 \le r < +\infty, \\ \max_{z \in \mathcal{E}_\rho} |f(z)|, & r = +\infty. \end{cases}$$

The case $r = +\infty$ ($r' = 1$) gives

$$|R_{n+1}(f)| \le \frac{1}{2\pi} \left(\max_{z \in \mathcal{E}_\rho} |K_{n+1}(z)| \right) \|f\|_1,$$

i.e.

$$|R_{n+1}(f)| \le \frac{\ell\left(\mathcal{E}_\rho\right)}{2\pi} \left(\max_{z \in \mathcal{E}_\rho} |K_{n+1}(z)| \right) \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right), \tag{2.4}$$

where $\ell\left(\mathcal{E}_\rho\right)$ is the length of the ellipse $\mathcal{E}_\rho$, whereas for $r = 1$ ($r' = +\infty$) we have

$$|R_{n+1}(f)| \le \frac{1}{2\pi} \left(\oint_{\mathcal{E}_\rho} |K_{n+1}(z)||dz| \right) \|f\|_\infty. \tag{2.5}$$

When w is one of the 4 classical Chebyshev weight functions, the bounds of $|E_n(f)|$, i.e., of the modulus of the remainder term in the Gauss quadrature formula (1.2), of the type (2.5) have been considered in detail in [10]. In that paper and some other error bounds, based on the an expansion $E_n(f)$ in series, are considered. These error estimates of $|R_{n+1}(f)|$ for the anti-Gauss quadrature formula (1.3) are considered in Section 3, whereas the error estimates of the type (2.5) are considered in Section 4.

When w is one of the 4 classical Chebyshev weight functions, the bounds of $|E_n(f)|$, i.e., of the modulus of the remainder term in the Gauss quadrature formula (1.2), of the type (2.4) have considered in detail in [11,9]. The error estimates of $|R_{n+1}(f)|$ of the type (2.4) for the anti-Gauss quadrature formula (1.3) are considered in Section 5.

3. Error estimates of $|R_{n+1}(f)|$ based on the expansion $R_{n+1}(f)$ in series

Following [10], for $z \notin [-1, 1]$ and $\pi_{n+1}(z) = \prod_{j=1}^{n+1} \left(z - x_j^A\right)$ in (1.4), we first have

$$\left(z - x_j^A\right)^{-1} = 2 \sum_{k=0}^{\infty} U_k\left(x_j^A\right) \xi^{-k-1}, \quad j = 1, 2, \ldots, n+1,$$

where U_k is a Chebyshev polynomial of the second kind (cf. [11, Section 5]), ξ is defined in (2.3), and then, on multiplying these expansions for $j = 1, 2, \ldots, n+1$, we get

$$\frac{1}{\pi_{n+1}(z)} = \sum_{k=0}^{\infty} \beta_k \xi^{-n-1-k},$$

with

$$\beta_k = 2^{n+1} \sum_{(\ell)} \prod_{j=1}^{n+1} U_{\ell j}\left(x_j^A\right),$$

the summation being over all $(n+1)$-tuples $(\ell) = (l_1, l_2, \ldots, l_{n+1})$ of non-negative integers for which $l_1 + l_2 + \cdots + l_{n+1} = k$.

β_k is a homogeneous symmetric function of degree k in $x_1^A, x_2^A, \ldots, x_{n+1}^A$. If w is an even function, these points are symmetrically distributed about 0. So $\beta_k = 0$ if k is odd.

Further,

$$\begin{aligned} \varrho_{n+1}(z) &= \int_{-1}^{1} \frac{\pi_{n+1}(x)}{z - x} w(x) dx \\ &= 2\xi^{-1} \int_{-1}^{1} \sum_{k=0}^{\infty} w(x)\pi_{n+1}(x)U_k(x)\xi^{-k} dx \\ &= \sum_{k=0}^{\infty} \gamma_k \xi^{-n-k}, \end{aligned}$$

where

$$\gamma_k = 2\int_{-1}^{1} w(x)\pi_{n+1}(x)U_{k+n-1}(x)\mathrm{d}x, \quad k = 0, 1, 2, \ldots.$$

If w is an even function and k is odd, then $\gamma_k = 0$.

Now,

$$\frac{\varrho_{n+1}(z)}{\pi_{n+1}(z)} = \left(\sum_{k=0}^{\infty}\beta_k\xi^{-n-1-k}\right)\left(\sum_{j=0}^{\infty}\gamma_j\xi^{-n-j}\right) = \sum_{k=0}^{\infty}\omega_k\xi^{-k-2n-1},$$

where

$$\omega_k = \sum_{j=0}^{k}\beta_j\gamma_{k-j}.$$

As f is analytic in the interior of $\mathcal{E}_\rho$, then it has the expansion

$$f(z) = {\sum_{k=0}^{\infty}}'\alpha_k T_k(z),$$

where

$$\alpha_k = \frac{1}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2}f(x)T_k(x)\mathrm{d}x,$$

which converges for all z in the interior of $\mathcal{E}_\rho$. Here, T_k is a Chebyshev polynomial of the first kind, which is given by the equation $T_k(z) = \frac{1}{2}(\xi^k + \xi^{-k})$. The prime on the summation indicates that the first term is halved.

In general, the Chebyshev coefficients α_k are unknown. However, Elliott [12] describes a number of ways of estimating or bounding them. In particular, under our assumptions,

$$|\alpha_k| \le \frac{2\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)}{\rho^k}. \tag{3.1}$$

Finally, (2.1) is reduced to

$$R_{n+1}(f) = \frac{1}{2\pi \mathrm{i}}\sum_{k=0}^{\infty}{\sum_{j=0}^{\infty}}'\alpha_j\omega_k\oint_{\mathcal{E}_\rho}T_j(z)\xi^{-2n-k-1}\mathrm{d}z,$$

i.e., by applying [10, Lemma 5],

$$R_{n+1}(f) = \sum_{k=0}^{\infty}\alpha_{2n+k}\varepsilon_k, \tag{3.2}$$

where

$$\varepsilon_0 = \frac{1}{4}\omega_0$$
$$\varepsilon_1 = \frac{1}{4}\omega_1$$
$$\varepsilon_k = \frac{1}{4}(\omega_k - \omega_{k-2}), \quad k \ge 2.$$

If w is an even function and k is odd it follows that $\omega_k = 0$ and hence $\varepsilon_k = 0$.

The quantities ε_k can be determined and by

$$\varepsilon_k = R_{n+1}(T_{2n+k}) = \sigma_{2n+k} - \sum_{j=1}^{n+1}\lambda_j^A T_{2n+k}\left(x_j^A\right),$$

where

$$\sigma_k = \int_{-1}^{1} w(x)T_k(x)\mathrm{d}x, \quad k = 0, 1, 2, \ldots.$$

Putting $k = 0, 1$ in the last formula, on the basis [1, Eq. (5)], it follows $\varepsilon_k = -\overline{\varepsilon}_k$ $(k = 0, 1)$, and therefore $\omega_k = -\overline{\omega}_k$ $(k = 0, 1)$, where $\overline{\varepsilon}_k, \overline{\omega}_k$ $(k = 0, 1)$ are the corresponding values for the Gauss quadrature formula (1.2).

3.1. The case $w(x) = w_1(x) = (1-x^2)^{-1/2}$

Here in fact, $\pi_{n+1}(z) = (T_{n+1}(z) - T_{n-1}(z))/2^n$, where $T_n(z) = \frac{1}{2}(\xi^n + \xi^{-n})$. From here on, we take $\pi_{n+1} := \kappa_n \pi_{n+1}$ ($\kappa_n > 0$), where κ_n is a suitable chosen coefficient, in order to simplify our computation, since the obtained results performed above remain the same (cf. (2.2)). Definitely, we take $\pi_{n+1}(z) = T_{n+1}(z) - T_{n-1}(z)$.

First, we have

$$\frac{1}{\pi_{n+1}(z)} = \frac{1}{T_{n+1}(z) - T_{n-1}(z)} = \frac{2}{(\xi^n - \xi^{-n})(\xi - \xi^{-1})}$$
$$= \frac{2}{\xi^{n+1}} \frac{1}{1-\xi^{-2n}} \frac{1}{1-\xi^{-2}} = \frac{2}{\xi^{n+1}} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{\xi^{2(nk+j)}},$$

from which we conclude

$$\beta_{2j} = 2(k+1), \quad nk \le j \le n(k+1) - 1, \; k = 0, 1, 2, \ldots,$$
$$\beta_{2j+1} = 0, \quad j = 0, 1, 2, \ldots.$$

Using (see [13, Eq. 3.613.1], or [11, p. 1176])

$$\int_{-1}^{1} \frac{T_n(x)}{z-x} (1-x^2)^{-1/2} \mathrm{d}x = \frac{2\pi}{\xi^n(\xi - \xi^{-1})},$$

we have

$$\varrho_{n+1}(z) = \int_{-1}^{1} \frac{T_{n+1}(x) - T_{n-1}(x)}{z-x} \frac{\mathrm{d}x}{\sqrt{1-x^2}}$$
$$= \frac{2\pi}{\xi^{n+1}(\xi - \xi^{-1})} - \frac{2\pi}{\xi^{n-1}(\xi - \xi^{-1})} = -2\pi\xi^{-n}.$$

Therefore,

$$\gamma_k = \begin{cases} -2\pi, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The kernel in this case is given by

$$K_{n+1}^{(1)}(z) = \frac{-4\pi}{\xi^n(\xi - \xi^{-1})(\xi^n - \xi^{-n})}. \tag{3.3}$$

Further, $\omega_{2j} = -2\pi\beta_{2j}$, and

$$\varepsilon_0 = \varepsilon_{2j} = -\pi \quad (j = \mathrm{l}n, \; l = 1, 2, \ldots).$$

Now, (3.2) obtains the form

$$R_{n+1}(f) = -\pi \sum_{k=0}^{\infty} \alpha_{2n(k+1)},$$

from which, on the basis of (3.1), we obtain the error bound

$$|R_{n+1}(f)| \le \sum_{k=0}^{\infty} \frac{2\left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right)}{\rho^{2nk+2n}},$$

i.e.,

$$|R_{n+1}(f)| \le \frac{2\pi\left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right)}{\rho^{2n} - 1},$$

which is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (4.4)]).

3.2. The case $w(x) = w_2(x) = (1-x^2)^{1/2}$

Here $\pi_{n+1}(z) = U_{n+1}(z) - U_{n-1}(z)$, where $U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1})$.

First, we have

$$\frac{1}{\pi_{n+1}(z)} = \frac{1}{U_{n+1}(z) - U_{n-1}(z)} = \frac{1}{\frac{\xi^{n+2}-\xi^{-n-2}}{\xi-\xi^{-1}} - \frac{\xi^{n}-\xi^{-n}}{\xi-\xi^{-1}}}$$
$$= \frac{2}{\xi^{n+1}(1+\xi^{-2n-2})} = \sum_{k=0}^{\infty}(-1)^k \xi^{-(2k+1)(n+1)},$$

from which we conclude

$$\beta_k = \begin{cases} (-1)^j, & \text{if } k = 2j(n+1), \\ 0, & \text{otherwise.} \end{cases}$$

Using (see [13, Eq. 3.613.3], or [11, p. 1177])

$$\int_{-1}^{1} \frac{U_n(x)}{z-x}(1-x^2)^{1/2}\mathrm{d}x = \frac{\pi}{\xi^{n+1}},$$

we have

$$\varrho_{n+1}(z) = \int_{-1}^{1} \frac{U_{n+1}(x) - U_{n-1}(x)}{z-x}\sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{\xi^{n+2}} - \frac{\pi}{\xi^{n}}.$$

Therefore,

$$\gamma_k = \begin{cases} -\pi, & \text{if } k = 0, \\ \pi, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The kernel in this case is given by

$$K_{n+1}^{(2)}(z) = \frac{-\pi(\xi-\xi^{-1})}{\xi^{n+1}(\xi^{n+1}+\xi^{-(n+1)})}. \tag{3.4}$$

Further, in the case $n \geq 2$, i.e., $n+1 \geq 3$, we obtain

$$\omega_{2j(n+1)} = (-1)^{j+1}\pi, \qquad \omega_{2j(n+1)+2} = (-1)^j\pi \quad (j = 0, 1, 2, \ldots),$$

and

$$\varepsilon_{2j(n+1)} = \varepsilon_{2j(n+1)+4} = \frac{(-1)^{j+1}\pi}{4}, \qquad \varepsilon_{2j(n+1)+2} = \frac{(-1)^j\pi}{2} \quad (j = 0, 1, 2, \ldots).$$

Now, (3.2) obtains the form

$$R_{n+1}(f) = \sum_{k=0}^{\infty} \alpha_{2n+k}\varepsilon_k,$$
$$= \sum_{k=0}^{\infty} \left(\alpha_{2n+2j(n+1)}\varepsilon_{2j(n+1)} + \alpha_{2n+2j(n+1)+2}\varepsilon_{2j(n+1)+2} + \alpha_{2n+2j(n+1)+4}\varepsilon_{2j(n+1)+4}\right),$$

from which, on the basis of (3.1), we obtain the error bound

$$|R_{n+1}(f)| \leq \frac{\pi\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)}{2} \sum_{j=0}^{\infty}\left(\frac{1}{\rho^{2n+2j(n+1)}} + \frac{2}{\rho^{2n+2j(n+1)+2}} + \frac{1}{\rho^{2n+2j(n+1)+4}}\right),$$

i.e.,

$$|R_{n+1}(f)| \leq \frac{\pi\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)(\rho+\rho^{-1})^2}{2(\rho^{2n+2}-1)}, \tag{3.5}$$

which is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (4.8)]).

In the case $n = 1$, i.e., $n+1 = 2$, the obtained results

$$\varepsilon_k = \begin{cases} -\dfrac{\pi}{4}, & \text{if } k = 0, \\ (-1)^j\dfrac{\pi}{2}, & \text{if } k = 2j+2 \quad (0, 1, 2, \ldots), \\ 0, & \text{otherwise,} \end{cases}$$

leadto the error bound (3.5) for $n = 1$.

3.3. The case $w(x) = w_3(x) = \sqrt{(1+x)/(1-x)}$

Let

$$A = A_n = \frac{n^2+n}{n^2-1/4}. \tag{3.6}$$

Here $\pi_{n+1}(z) = A_n p_{n+1}(z) - p_{n-1}(z)$, where (see [11, p. 1178]) $p_n(z) = (\xi^{n+1}+\xi^{-n})/(\xi+1)$.

Since $A_n \to 1$, when $n \to \infty$, we shall in this and Section 4 consider error estimates by putting $A = A_n = 1$ which are useful for larger values of n. In the general case (3.6) these two kinds of error bounds are very complicated to perform.

First, we have

$$\begin{aligned}\frac{1}{\pi_{n+1}(z)} &\cong \frac{1}{p_{n+1}(z)-p_{n-1}(z)} = \frac{1}{\frac{\xi^{n+2}+\xi^{-n-1}}{\xi+1} - \frac{\xi^{n}+\xi^{-n+1}}{\xi+1}} \\ &= \frac{1}{(\xi^{1/2}+\xi^{-1/2})(\xi^{n+1/2}-\xi^{-n-1/2})} \\ &= \left(\frac{1}{\xi^{n+1}}+\frac{1}{\xi^{n+1}}\right)\cdot\frac{1}{1-\xi^{-2}}\cdot\frac{1}{1-\xi^{-2n-1}},\end{aligned}$$

from which, by expanding the last express, we conclude

$$\beta_j = k+1, \quad k(2n+1) \le j \le (k+1)(2n+1)-1 \ (k=0,1,2,\ldots).$$

Using ([11, p. 1178])

$$\int_{-1}^{1} \frac{p_n(x)}{z-x}\sqrt{\frac{1+x}{1-x}}\mathrm{d}x = \frac{2\pi(\xi+1)}{\xi^{n+1}(\xi-\xi^{-1})},$$

we have

$$\begin{aligned}\varrho_{n+1}(z) &\cong \int_{-1}^{1}\frac{p_{n+1}(x)-p_{n-1}(x)}{z-x}\sqrt{\frac{1+x}{1-x}}\mathrm{d}x \\ &= \frac{2\pi(\xi+1)}{\xi^{n+2}(\xi-\xi^{-1})} - \frac{2\pi(\xi+1)}{\xi^{n}(\xi-\xi^{-1})} = -2\pi(\xi+1)\xi^{-n-1}.\end{aligned}$$

Therefore,

$$\gamma_k = \begin{cases}-2\pi, & \text{if } k=0,1,\\ 0, & \text{otherwise.}\end{cases}$$

The kernel in this case ($A = 1$) is given by

$$\overline{K}^{(3)}_{n+1}(z) = \frac{-2\pi(\xi+1)^2}{\xi^{n+1}(\xi^{n+1}-\xi^{-n})(\xi-\xi^{-1})} \quad \left(\cong K^{(3)}_{n+1}(z),\ \text{for large } n\right), \tag{3.7}$$

whereas in the general case (3.6) the kernel is given by

$$\begin{aligned}K^{(3)}_{n+1}(z) &= \frac{2\pi(\xi+1)^2(A\xi^{-1}-\xi)}{\xi^{n+1}(\xi-\xi^{-1})\left[\xi^{n+1}(A\xi-\xi^{-1})+\xi^{-n}(A\xi^{-1}-\xi)\right]} \\ &= \frac{2\pi(\xi^{1/2}+\xi^{-1/2})}{\xi^{n}(\xi^{1/2}-\xi^{-1/2})}\cdot\frac{A\xi^{-1}-\xi}{\xi^{n+1}(A\xi-\xi^{-1})+\xi^{-n}(A\xi^{-1}-\xi)}.\end{aligned} \tag{3.8}$$

Further,

$$\begin{aligned}&\omega_{k(2n+1)} = -2\pi(2k+1)\pi,\\ &\omega_j = -4\pi(k+1), \quad (2n+1)k < j \le (k+1)(2n+1)-1,\end{aligned}$$

for $k = 0, 1, 2, \ldots,$ and

$$\varepsilon_{j(2n+1)} = \varepsilon_{j(2n+1)+2} = -\frac{\pi}{2}, \qquad \varepsilon_{j(2n+1)+1} = -\pi \quad (j=0,1,2,\ldots).$$

Now, on the basis of (3.2), we obtain the error estimate

$$\begin{aligned}|R_{n+1}(f)| &\cong \left|\sum_{k=0}^{\infty}\alpha_{2n+k}\varepsilon_k\right| \\ &\le \pi\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right)\sum_{j=0}^{\infty}\left(\frac{1}{\rho^{2n+j(2n+1)}}+\frac{2}{\rho^{2n+j(2n+1)+1}}+\frac{1}{\rho^{2n+j(2n+1)+2}}\right),\end{aligned}$$

i.e.,

$$|R_{n+1}(f)| \cong \left|\sum_{k=0}^{\infty} \alpha_{2n+k}\varepsilon_k\right| \le \pi \left(\max_{z\in\mathcal{E}_\rho} |f(z)|\right) \frac{(\rho^{1/2}+\rho^{-1/2})^2}{\rho^{2n+1}-1},$$

which is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (4.13)]).

4. Error estimates of the type (2.5)

From here on, we use the usual notation (see for example [11])

$$a_j = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N}.$$

4.1. The case $w(x) = w_1(x) = (1-x^2)^{-1/2}$

Using

$$|\xi^k \mp \xi^{-k}| = \sqrt{2}(a_{2k} \mp \cos 2n\theta) \quad (k \in \mathbb{R})$$

and (2.2), we obtain

$$\left|K_{n+1}^{(1)}(z)\right| = \frac{2\pi}{\rho^n (a_2 - \cos 2\theta)^{1/2}(a_{2n} - \cos 2n\theta)^{1/2}}. \tag{4.1}$$

By using this formula and $|\mathrm{d}z| = 2\sqrt{a_2 - \cos 2\theta}\,\mathrm{d}\theta$, (2.5) obtains the form

$$|R_{n+1}(f)| \le \frac{1}{4}\rho^{-n}\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a_{2n} - \cos 2n\theta)^{1/2}},$$

i.e.

$$|R_{n+1}(f)| \le \frac{4}{\rho^{2n}+1} K(a_n^{-1}), \tag{4.2}$$

where

$$K(\kappa) = \int_0^{\pi/2} (1-\kappa^2\sin^2\theta)^{-1/2}\mathrm{d}\theta \quad (|\kappa| < 1)$$

is the complete elliptic integral of the first kind.

The error bound (4.2) is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (5.7)]).

4.2. The case $w(x) = w_2(x) = (1-x^2)^{1/2}$

We have

$$\left|K_{n+1}^{(2)}(z)\right| = \frac{\pi}{\rho^{n+1}} \frac{(a_2 - \cos 2\theta)^{1/2}}{(a_{2n+2} + \cos 2(n+1)\theta)^{1/2}}. \tag{4.3}$$

(2.5) obtains the form

$$|R_{n+1}(f)| \le \frac{1}{8}\rho^{-n-1} a_2 \int_0^{2\pi} \frac{1}{(a_{n+1}^2 - \sin^2(n+1)\theta)^{1/2}}\mathrm{d}\theta - \frac{1}{8}\rho^{-n-1}\int_0^{2\pi} \frac{\cos 2\theta}{(a_{n+1}^2 - \sin^2(n+1)\theta)^{1/2}}\mathrm{d}\theta,$$

i.e., since the second integral is equal to zero (see below),

$$|R_{n+1}(f)| \le \frac{2a_2}{\rho^{2n+2}+1} K(a_{n+1}^{-1}). \tag{4.4}$$

The error bound (4.4) is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (5.8)]).

We finish this subsection by proving that

$$\int_0^{2\pi} \frac{\cos 2\theta}{(a_{n+1}^2 - \sin^2(n+1)\theta)^{1/2}}\mathrm{d}\theta = 0.$$

The last integral is equal to $a_{n+1}^2 I$, where

$$I = \int_0^{2\pi} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta = \int_0^{\pi} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta + \int_{\pi}^{2\pi} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta,$$

with $m = n+1$ (≥ 2) and $\kappa = 2/a_{n+1}$. Substituting $\theta := \pi + \theta$ in the last integral, we obtain $I = 2I_1$, where

$$I_1 = \int_0^{\pi} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta = \int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta + \int_{\pi/2}^{\pi} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta.$$

In a similar way, we have $I_1 = 2I_2$, where

$$I_2 = \int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta = \int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 (1-\cos 2m\theta)/2)^{1/2}} \mathrm{d}\theta.$$

Substituting $\theta := \pi/2 - \theta$ in the last integral, we obtain

$$I_2 = -\int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 \sin^2 m\theta)^{1/2}} \mathrm{d}\theta = \int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 (1-(-1)^m \cos 2m\theta)/2)^{1/2}} \mathrm{d}\theta.$$

If m is even, then $I_2 = -I_2$, i.e., $I_2 = 0$, whereas if m is odd, then (cf. [10, p. 78])

$$I_2 = -\int_0^{\pi/2} \frac{\cos 2\theta}{(1-\kappa^2 \cos^2 m\theta)^{1/2}} \mathrm{d}\theta = -\frac{1}{2}\int_0^{\pi} \frac{\cos 2\theta}{(1-\kappa^2 \cos^2 m\theta)^{1/2}} \mathrm{d}\theta = 0.$$

The assertion follows.

4.3. *The case* $w(x) = w_3(x) = \sqrt{(1+x)/(1-x)}$

On the basis of (3.7), we have

$$\left|\overline{K}_{n+1}^{(3)}(z)\right| = \frac{2\pi}{\rho^{n+1/2}(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} - \cos(2n+1)\theta)^{1/2}}, \tag{4.5}$$

whereas on the basis of (3.8), we have

$$\begin{aligned}\left|K_{n+1}^{(3)}(z)\right| = {} & \frac{2\pi}{\rho^n} \cdot \frac{(a_1+\cos\theta)^{1/2}}{(a_1-\cos\theta)^{1/2}} \cdot (A^2\rho^{-2} + \rho^2 - 2A\cos 2\theta)^{1/2} \\ & \times [A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n})\cos 2\theta \\ & + 2A^2\rho\cos(2n+3)\theta + 2\rho\cos(2n-1)\theta - 2A(\rho^3+\rho^{-1})\cos(2n+1)\theta]^{-1/2}. \end{aligned} \tag{4.6}$$

Let

$$\overline{R}_{n+1}(f) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \overline{K}_{n+1}^{(3)}(z) f(z) \mathrm{d}z,$$

then (2.5) obtains the form

$$\begin{aligned}|R_{n+1}(f)| \cong \left|\overline{R}_{n+1}(f)\right| \leq {} & \frac{1}{16}\rho^{-n-1/2} a_1 \int_0^{2\pi} \frac{1}{(a_{n+1/2}^2 - \cos^2(n+1/2)\theta)^{1/2}} \mathrm{d}\theta, \\ & + \frac{1}{16}\rho^{-n-1} \int_0^{2\pi} \frac{\cos\theta}{(a_{n+1/2}^2 - \cos^2(n+1/2)\theta)^{1/2}} \mathrm{d}\theta,\end{aligned}$$

i.e., since the second integral is equal to zero (it can be proved in a similar way as in the previous subsection),

$$|R_{n+1}(f)| \cong \left|\overline{R}_{n+1}(f)\right| \leq \frac{4a_1}{\rho^{2n+1}+1} K(a_{n+1/2}^{-1}). \tag{4.7}$$

The error estimate (4.7) is the same as the error bound of this type for the corresponding Gauss quadrature formula (1.2) (cf. [10, Eq. (5.9)]).

5. Error bounds of the type (2.4)

The location on the elliptic contours where the modulus of the kernel attains its maximum value is investigated. This leads to effective error bounds of the corresponding anti-Gauss quadratures.

The derivation of adequate bounds for $|R_{n+1}(f)|$ on the basis of (2.4) is possible only if good estimates for $\max_{z\in\mathcal{E}_\rho}|K_{n+1}(z)|$ are available, especially if we know the location of the extremal point $\eta \in \mathcal{E}_\rho$ at which $|K_{n+1}|$ attains its maximum. In such a case, instead of looking for upper bounds for $\max_{z\in\mathcal{E}_\rho}|K_{n+1}(z)|$ one can simply try to calculate $|K_{n+1}(\eta, w)|$. In general, this may not be an easy task, but in the case of the Gauss-type quadrature formula (1.2) there exist effective algorithms for calculation of the kernel at any point z outside $[-1, 1]$ (see [11]).

So far, this approach (cf. (2.4)) was discussed for Gaussian quadrature rules (1.2) with respect to the Chebyshev weight functions $w_i,\ i = 1, 2, 3, 4$ (see [11,9]), and later has been extended by Schira to symmetric weight functions under restriction of monotonicity type (either $w(t)\sqrt{1-t^2}$ is increasing on $(0, 1)$ or $w(t)/\sqrt{1-t^2}$ is decreasing on $(0, 1)$), including certain Gegenbauer weight functions (see [14]).

5.1. The case $w(x) = w_1(x) = (1-x^2)^{-1/2}$

Theorem 5.1. *For the anti-Gauss quadrature formula (1.3), $n \in \mathbb{N}$, with the weight function $w_1(x)$ on $(-1, 1)$, the modulus of the kernel $\left|K^{(1)}_{n+1}(z)\right|$ attains its maximum value on the real axis ($\theta = 0$), i.e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K^{(1)}_{n+1}(z)\right| = -K^{(1)}_{n+1}\left(\frac{1}{2}(\rho+\rho^{-1})\right).$$

Proof. The modulus of the kernel $\left|K^{(1)}_{n+1}(z)\right|$ is given by (4.1). It is obvious that

$$\frac{1}{(a_2-\cos 2\theta)(a_{2n}-\cos 2n\theta)} \le \frac{1}{(a_2-1)(a_{2n}-1)},$$

for all $\theta \in [0, \pi/2]$ (w_1 is even weight), all n, with equality holding when $\theta = 0$. With $z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho$, this gives the desired result. □

5.2. The case $w(x) = w_2(x) = (1-x^2)^{1/2}$

Theorem 5.2. *For the anti-Gauss quadrature formula (1.3), $n \in \mathbb{N}$, n is even, with the weight function $w_2(x)$ on $(-1, 1)$, the modulus of the kernel $\left|K^{(2)}_{n+1}(z)\right|$ attains its maximum value on the imaginary axis ($\theta = \pi/2$), i.e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K^{(2)}_{n+1}(z)\right| = \left|K^{(2)}_{n+1}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

Proof. The modulus of the kernel $\left|K^{(2)}_{n+1}(z)\right|$ is given by (4.3). It is obvious that

$$\frac{a_2-\cos 2\theta}{a_{2n+2}+\cos(2n+2)\theta} \le \frac{a_2+1}{a_{2n+2}-1},$$

for all $\theta \in [0, \pi/2]$ (w_2 is even weight), all n, with equality holding when $\theta = \pi/2$. With $z = \frac{1}{2}(\rho e^{i\theta} + \rho^{-1}e^{-i\theta}) \in \mathcal{E}_\rho$, this gives the desired result. □

The value on the right-hand side of the sign in the last inequality represents and the maximum of the modulus of the kernel of n-point (n is odd) Gauss quadrature formula (1.2) with respect to the weight w_2 (see [11, Th. 5.2]).

Theorem 5.3. *For each positive integer k (≥ 2) let $\rho_k > 1$ be the unique root of*

$$\frac{a_1(\rho)}{a_k(\rho)} = \frac{1}{k} \quad (\rho > 1). \tag{5.1}$$

Then, if $n \ge 1$ is odd, we have

$$\max_{z\in\mathcal{E}_\rho}\left|K^{(2)}_{n+1}(z)\right| = \left|K^{(2)}_{n+1}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| \quad \text{if } \rho \ge \rho_{n+1}, \tag{5.2}$$

i.e., the maximum of $\left|K^{(2)}_{n+1}(z)\right|$, for $z \in \mathcal{E}_\rho$ and $\rho \ge \rho_{n+1}$, is attained on the imaginary axis. If $1 < \rho < \rho_{n+1}$, then the maximum in (5.2) is attained at some $z = z^* = \frac{1}{2}(\rho e^{i\theta^*} + \rho^{-1}e^{-i\theta^*}) \in \mathcal{E}_\rho$ with $(n\pi)/(2(n+1)) < \theta^* < \pi/2$.

The proof can be performed by using the same arguments as in [9, Th. 1], having in mind that here n is odd, and $\kappa_n(\theta) = (a_2 - \cos 2\theta)/(a_{2n+2} + \cos(2n+2)\theta)$, $\theta \in [0, \pi/2]$. The roots $\rho_k > 1$ of (5.1) satisfy $(2k)^{1/k} < \rho_k < \mu_k$, for all $k \ge 2$, where μ_k (> 1) is the unique root of $\mu^{k+1} - k(\mu^2 + 1) = 0$.

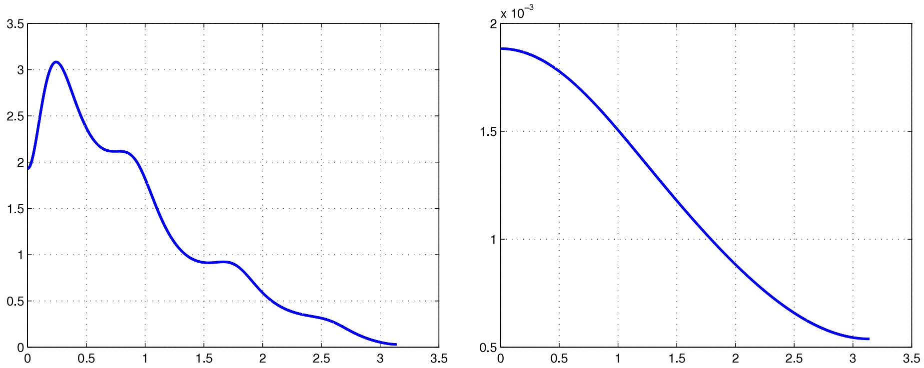

Fig. 5.1. The function $\theta \mapsto |K_4^{(3)}(z)|$, $z \in \mathcal{E}_\rho$, when $\rho = 1.3$ (left) and $\rho = 3.3$ (right).

Table 5.1 (cf. [9, p. 220]) displays the roots $\rho_k > 1$ of (5.1), and now every second value in it definitely obtains the meaning. Therefore, $\rho_2 = 2.2966302629$ represents the corresponding value of the anti-Gauss quadrature formula (1.3) with two nodes, whereas $\rho_3 = 1.9318516526$ is the corresponding value of the Gauss quadrature formula (1.2) with two nodes, etc.

5.3. The case $w(x) = w_3(x) = \sqrt{(1+x)/(1-x)}$

We consider the general case, where $A = A_n$ is given by (3.6) and $\left|K_{n+1}^{(3)}(z)\right|$ by (4.6).

Numerical results show that $|K_{n+1}^{(3)}(z)|,\ z \in \mathcal{E}_\rho$, attains its maximum on the real axis for ρ enough large (n is fixed). The graphs $\theta \mapsto |K_{n+1}^{(3)}(z)|,\ z \in \mathcal{E}_\varrho$, i.e., $\theta \in [0, \pi]$, for $n = 3$, $\rho = 1.3$ (left) and $\rho = 3.3$ (right), are displayed in Fig. 5.1.

Theorem 5.4. *For the anti-Gauss quadrature formula (1.2), $n \in \mathbb{N}$, with the weight function $w_3(x)$, there exists a $\rho^* \in (1, +\infty)$ ($\rho^* = \rho_n^*$) such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_{n+1}^{(3)}(z)\right|$ attains its maximum value on the real axis ($\theta = 0$), i.e.,*

$$\max_{z \in \mathcal{E}_\rho} \left|K_{n+1}^{(3)}(z)\right| = \left|K_{n+1}^{(3)}\left(\frac{1}{2}(\rho + \rho^{-1})\right)\right|.$$

Proof. Using (4.6), it is sufficient to prove the following inequality for $\theta \in [0, \pi]$:

$$\begin{aligned} &\frac{a_1 + \cos\theta}{a_1 - \cos\theta} \cdot (A^2\rho^{-2} + \rho^2 - 2A\cos 2\theta) \\ &\quad \times [A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n})\cos 2\theta \\ &\quad + 2A^2\rho\cos(2n+3)\theta + 2\rho\cos(2n-1)\theta - 2A(\rho^3 + \rho^{-1})\cos(2n+1)\theta]^{-1} \\ &\quad \leq \frac{a_1 + 1}{a_1 - 1} \cdot (\rho - A\rho^{-1})^2[A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} \\ &\quad - 2A(\rho^{2n+2} + \rho^{-2n}) + 2A^2\rho + 2\rho - 2A(\rho^3 + \rho^{-1})]^{-1}. \end{aligned}$$

This condition, by putting

$$\begin{aligned} &\alpha + \alpha_1 = a_1 + \cos\theta, \\ &\beta + \beta_1 = a_1 - \cos\theta, \\ &N + N_1 = A^2\rho^{-2} + \rho^2 - 2A\cos 2\theta, \\ &D + D_1 = A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n})\cos 2\theta \\ &\qquad + 2A^2\rho\cos(2n+3)\theta + 2\rho\cos(2n-1)\theta - 2A(\rho^3 + \rho^{-1})\cos(2n+1)\theta, \end{aligned}$$

where

$$\begin{aligned} &\alpha = a_1 + 1\ (>0), \qquad \alpha_1 = -2\sin^2\frac{\theta}{2}, \\ &\beta = a_1 - 1\ (>0), \qquad \beta_1 = 2\sin^2\frac{\theta}{2}, \end{aligned}$$

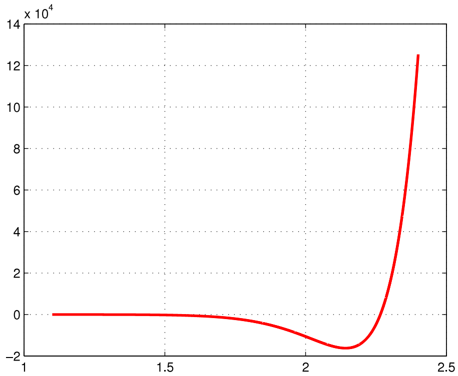

Fig. 5.2. Typical graph of the function $F_5(\rho)$ on the interval [1, 2.4].

$$N = (\rho - A\rho^{-1})^2 \ (\geq 0), \quad N_1 = 4A\sin^2\theta,$$
$$D = A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n}) + 2A^2\rho + 2\rho - 2A(\rho^3 + \rho^{-1}) \ (\geq 0),$$
$$D_1 = 4A(\rho^{2n+2} + \rho^{-2n})\sin^2\theta - 4A^2\rho\sin^2\frac{(2n+3)\theta}{2} - 4\rho\sin^2\frac{(2n-1)\theta}{2} + 4A(\rho^3 + \rho^{-1})\sin^2\frac{(2n+1)\theta}{2},$$

has the form

$$\frac{(\alpha + \alpha_1)(N + N_1)}{(\beta + \beta_1)(D + D_1)} \leq \frac{\alpha N}{\beta D}.$$

The last inequality holds, if

$$\alpha N(\beta + \beta_1)(D + D_1) - \beta D(\alpha + \alpha_1)(N + N_1) \geq 0,$$

i.e., if

$$\alpha N(\beta D_1 + \beta_1 D + \beta_1 D_1) - \beta D(\alpha N_1 + \alpha_1 N + \alpha_1 N_1) \geq 0.$$

The last inequality holds for $\theta = 0$.

Let $\theta \in (0, \pi]$. The last inequality holds, after dividing it by $2\sin^2(\theta/2)$ and using the well-known inequality $\sin^2(\zeta\theta)/\sin^2\theta \leq \zeta^2$ $(\zeta \in \mathbb{R})$, if

$$F_n(\rho) \geq 0, \tag{5.3}$$

where

$$\begin{aligned} F_n(\rho) = {} & [A^2\rho^{2n+4} + \rho^{2n} + A^2\rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n}) \\ & + 2A^2\rho + 2\rho - 2A(\rho^3 + \rho^{-1})] \cdot \left[a_1(\rho - A\rho^{-1})^2 - 4A(a_1^2 - 1)\right] \\ & - \rho(a_1 + 1)(\rho - A\rho^{-1})^2 \left[2 + 2A^2 + (a_1 - 1)\left((2n-1)^2 + A^2(2n+3)^2\right)\right], \end{aligned}$$

i.e.

$$F_n(\rho) = \frac{1}{2}A^2\rho^{2n+7} + O(\rho^{2n+6}) \quad (\rho \to \infty).$$

On the basis of the last equality, the assertion of the theorem follows. □

Typical graph of the function $F_n(\rho)$ (here for $n = 5$ on the interval [1, 2.4]) is displayed on Fig. 5.2.

The proof of Theorem 5.4 is of practical importance. Namely, we can determine the intervals $[\rho^*, +\infty)$ on which the modulus of the kernel $K_{n+1}^{(3)}$ attains its maximum value on the positive real axis. For some values of n the values of ρ_n^* are displayed in Table 5.1. Observe that the results become very satisfactory when n increases.

We end this subsection with an example.

Remainder terms for quadrature formulas are traditionally expressed in terms of some high-order derivative of the involved function. This is a serious disadvantage, if such derivatives are not known, do not exist or are too complicated to be handled.

Let us consider numerical calculation of the integral

$$I(f) = \int_{-1}^{1} f(x)\sqrt{\frac{1+x}{1-x}}\mathrm{d}x, \tag{5.4}$$

Table 5.1
The values of ρ^* for some $n \in \mathbb{N}$.

n	ρ^*	n	ρ^*	n	ρ^*
1	5.813	2	3.37	3	2.756
4	2.452	5	2.266	6	2.138
7	2.044	8	1.972	9	1.913
10	1.866	15	1.71	20	1.622
25	1.563	30	1.52	35	1.487
40	1.46	45	1.438	50	1.419
60	1.389	70	1.366	80	1.347
100	1.318	200	1.243	500	1.173

with

$$f(x) = \frac{e^{e^x}}{(a+x)^k (b+x)^\ell (c+x)^m},$$

where $c \le b \le a < -1$; $k \in \mathbb{N}$, $\ell, m \in \mathbb{N}_0$.

Under the assumption that f is analytic inside $\mathcal{E}_{\rho_{\max}}$, from (2.4) we obtain the error bound

$$|R_{n+1}(f)| \le \tilde{r}_{n+1}(f), \tag{5.5}$$

where

$$\tilde{r}_{n+1}(f) = \inf_{\rho_n^* < \rho < \rho_{\max}} \left[\frac{\ell(\mathcal{E}_\rho)}{2\pi} \left(\max_{z \in \mathcal{E}_\rho} \left| K_{n+1}^{(3)}(z) \right| \right) \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right) \right],$$

and ρ_n^* is defined by Theorem 5.4. In the case under consideration $|a| = \frac{1}{2}(\rho_{\max} + \rho_{\max}^{-1})$.

The length of the ellipse $\mathcal{E}_\rho$ can be estimated by (see [15, Eq. (2.2)])

$$\ell(\mathcal{E}_\rho) \le 2\pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right),$$

where $a_1 = (\rho + \rho^{-1})/2$.

It can be proved (see [16]) that

$$\max_{z \in \mathcal{E}_\rho} \left| \frac{e^{e^z}}{(a+z)^k (b+z)^\ell (c+z)^m} \right| = \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m},$$

where the maximum is attained at $\theta = 0$. Now, $r_{n+1}(f)$ $(\ge \tilde{r}_{n+1}(f))$ has the form

$$\begin{aligned} r_{n+1}(f) = \inf_{\rho_n^* < \rho < \rho_{\max}} \Bigg\{ & 2\pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right) \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m} \\ & \times \frac{1}{\rho^n} \frac{(a_1+1)^{1/2}}{(a_1-1)^{1/2}} \cdot (\rho - A\rho^{-1}) \\ & \times \left[A^2 \rho^{2n+4} + \rho^{2n} + A^2 \rho^{-2n-2} + \rho^{-2n+2} - 2A(\rho^{2n+2} + \rho^{-2n}) + 2A^2\rho + 2\rho - 2A(\rho^3 + \rho^{-1}) \right]^{-1/2} \Bigg\}. \end{aligned}$$

Let $c \le b \le a < -1$. This condition means that the function f is analytic inside the elliptical contour $\mathcal{E}_{\rho_{\max}}$, where $\rho_{\max} = 1 + \sqrt{2|a|}$. The classical error bound in this case is difficult to determine, since the derivatives of higher order of the function f are too complicated to be handled. However, we can use the error bound (5.5) based on the results of Theorem 5.4.

The error bound (5.5) is valid for integrands analytic on a neighborhood of the interval of integration and we compare it with the same error bounds for the corresponding Gauss quadrature formulae (1.2), i.e., $e_n(f)$ $(\ge \tilde{e}_n(f) \ge |E_n(f)|)$ intended for the same class of integrands. Derivation of $e_n(f)$ can be done in a similar way, by using [11, Th. 5.3 with Eq. (5.11)].

Let the integrand f be specialized by $k = 1$, $\ell = 5$, $m = 10$, and

$$a = -1.202083333333333(+01), \qquad b = -1.751428571428572(+01),$$
$$c = -2.301086956521739(+01),$$

which means that $\rho_{\max} = 24$.

We have calculated the values of $r_{n+1}(f)$, $e_n(f)$ for the corresponding integral $I(f)$ given by (5.4). For some values of n, the obtained results are displayed in Table 5.2. (Numbers in parentheses indicate decimal exponents.)

Table 5.2
The values of $e_n(f)$, $r_{n+1}(f)$ for some n.

n	$e_n(f)$	$r_{n+1}(f)$	n	$e_n(f)$	$r_{n+1}(f)$
1	2.395(−19)	3.478(−13)	20	3.636(−42)	3.448(−42)
2	5.371(−20)	7.601(−20)	30	4.573(−57)	4.415(−57)
3	8.02(−21)	5.31(−21)	40	9.546(−73)	9.3(−73)
5	9.11(−23)	7.232(−23)	50	5.67(−89)	5.554(−89)
10	1.395(−28)	1.25(−28)	60	1.293(−105)	1.271(−105)

Table 5.3
The values of $e_n(\bar{f})$, $r_{n+1}(\bar{f})$ for some n.

n	$e_n(\bar{f})$	$r_{n+1}(\bar{f})$	n	$e_n(\bar{f})$	$r_{n+1}(\bar{f})$
1	5.112(+00)	2.095(+02)	10	9.256(−12)	8.338(−12)
2	5.044(−01)	2.897(−01)	20	2.04(−27)	1.937(−27)
3	3.599(−02)	2.509(−02)	30	3.936(−44)	3.801(−44)
5	1.068(−04)	8.642(−05)	40	1.741(−61)	1.697(−61)

At the end, let us consider numerical calculation of the integral (5.4), with

$$f(t) = \bar{f}(t) = \mathrm{e}^{\cos t}.$$

The function $\bar{f}(z) = \mathrm{e}^{\cos z}$ is entire, and it is easy to see that

$$\max_{z \in \mathcal{E}_\rho} \left| \mathrm{e}^{\cos z} \right| = \mathrm{e}^{\cosh(b_1)},$$

where $b_1 = \frac{1}{2}(\rho - \rho^{-1})$.

And for this case, we have calculated the values of $r_{n+1}(\bar{f})$, $e_n(\bar{f})$ for the corresponding integral $I(\bar{f})$ given by (5.4). For some values of n, the obtained results are displayed in Table 5.3.

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