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Abstract metric spaces and Hardy–Rogers-type theorems

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ABSTRACT

The purpose of the present paper is to establish coincidence point theorem for two mappings and fixed point theorem for one mapping in abstract metric space which satisfy contractive conditions of Hardy–Rogers type. Our results generalize fixed point theorems of Nemytzki [V.V. Nemytzki, Fixed point method in analysis, Uspekhi Mat. Nauk 1 (1936) 141–174], Edelstein [M. Edelstein, On fixed and periodic point under contractive mappings, J. Lond. Math. Soc. 37 (1962) 74–79] and Huang, Zhang [L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2) (2007) 1468–1476] from abstract metric spaces to symmetric spaces (Theorem 2.1) and to metric spaces (Theorem 2.4, Corollaries 2.6–2.8). Two examples are given to illustrate the usability of our results.

1. Introduction and preliminaries

Abstract (cone) metric spaces were introduced in [1], in which authors described convergence in abstract metric spaces and introduced completeness. Then, they proved some fixed point theorems of contractive mappings on cone metric spaces. Also, in [2–10], some common fixed point theorems were proved for maps on cone metric spaces.

Consistent with Huang and Zhang [1] and Deimling [11], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a cone if and only if

(a) P is closed, nonempty and $P \neq \{\theta\}$;
(b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(c) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P). A cone $P \subset E$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number satisfying the above inequality is called the normal constant of P. It is clear that $K \geq 1$. Most of ordered Banach spaces used in applications posses a cone with the normal constant $K = 1$, and if this is the case, proofs of the corresponding results are much alike as in the metric setting. If $K > 1$, this is not the case. From [11], we know that there exists ordered Banach space E with cone P which is not normal but with $\operatorname{int} P \neq \emptyset$.

The cone P is called regular if every increasing sequence in E which is bounded from above is convergent. That is, if $(x_n)_{n\geq 1}$ is a sequence in E such that $x_1 \preceq x_2 \preceq \cdots \preceq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. Every regular cone is normal [11], but the converse is not true.

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Definition 1.1 (*[1]*). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.
For basic properties of cone metric spaces, we refer to [1].

Definition 1.2 (*[1]*). Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(e) a Cauchy sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
(f) a convergent sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. In the case of normal cone, it is known [1] that $\{x_n\}$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

For the given cone metric space (X, d), one can construct a symmetric space (X, D) where "symmetric" (in sense of [12–14]) $D : X \times X \to \mathbb{R}$ is given by $D(x, y) = \|d(x, y)\|$ (for details see [15]). In the case when (X, d) is a cone metric space with a normal cone P, then (1.1) implies

$$D(x, y) \le K(D(x, z) + D(z, y)),$$

$x, y, z \in X$, K is a normal constant of P.
We find it convenient to introduce the following definition.

Definition 1.3 (*See [4]*). Let (X, d) be a cone metric space and P a cone with nonempty interior. Suppose that the mappings $f, g : X \to X$ are such that the range of g contains the range of f, and $f(X)$ or $g(X)$ is a complete subspace of X. In this case we will say that the pair (f, g) is Abbas and Jungck's pair, or shortly AJ's pair.

Definition 1.4 (*See [2]*). Let f and g be self-maps of a set X (i.e., $f, g : X \to X$). If $\omega = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and ω is called a point of coincidence of f and g. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for $x \in X$, then $fgx = gfx$.

Proposition 1.5 (*See [2]*). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $\omega = fx = gx$, then ω is the unique common fixed point of f and g.*

In the following, we always suppose that E is a Banach space, P is a normal cone in E with $\operatorname{int} P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.
The next example shows that the fixed point problem cannot be solved in symmetric spaces as in the metric setting.

Example 1.6. Let $X = (0, \infty)$ and $d(x, y) = (x - y)^2$. Obviously, (X, d) is a symmetric space. The mapping $fx = \frac{1}{2}x$, $x \in X$ is a contraction in the Banach sense with $\frac{1}{4} \le \lambda < 1$, because

$$d(fx, fy) = (fx - fy)^2 = \frac{1}{4}(x - y)^2 = \frac{1}{4}d(x, y) \le \lambda d(x, y),$$

for $\lambda \in [\frac{1}{4}, 1)$. However, f has no fixed points.

However, symmetric space (X, D) obtained as associated with cone metric space (X, d) with a normal cone P has a property that any contractive mapping possesses a fixed point (see [9]).

2. Main results

Theorem 2.1. *Let (X, d) be a complete cone metric space, P a normal cone with normal constant K. Suppose that (f, g) is AJ's pair, and that there exist nonnegative constants a_i, $i = \overline{1, 5}$ satisfying $\sum_{i=1}^{3} a_i + K(a_4 + a_5) < 1$ such that, for every $x, y \in X$,*

$$D(fx, fy) \le a_1 D(gx, gy) + a_2 D(gx, fx) + a_3 D(gy, fy) + a_4 D(gx, fy) + a_5 D(gy, fx). \tag{2.1}$$

Then, f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and let $x_1 \in X$ be chosen such that $y_0 = f(x_0) = g(x_1)$. This can be done, since $f(X) \subseteq g(X)$. Let $x_2 \in X$ be such that $y_1 = f(x_1) = g(x_2)$. Continuing this process, having chosen $x_n \in X$, we choose x_{n+1} in X such that

$$y_n = f(x_n) = g(x_{n+1}).$$

We have to show that

$$D(y_n, y_{n+1}) \le \lambda D(y_{n-1}, y_n) \quad \text{for some } \lambda \in [0,1),\ n \ge 1. \tag{2.2}$$

From

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) \le a_1 D(gx_n, gx_{n+1}) + a_2 D(gx_n, fx_n) \\ &\quad + a_3 D(gx_{n+1}, fx_{n+1}) + a_4 D(gx_n, fx_{n+1}) + a_5 D(gx_{n+1}, fx_n) \\ &= a_1 D(y_{n-1}, y_n) + a_2 D(y_{n-1}, y_n) + a_3 D(y_n, y_{n+1}) + a_4 D(y_{n-1}, y_{n+1}) + a_5 D(y_n, y_n) \\ &\le (a_1 + a_2 + Ka_4) D(y_{n-1}, y_n) + (a_3 + Ka_4) D(y_n, y_{n+1}), \end{aligned}$$

and from

$$\begin{aligned} D(y_{n+1}, y_n) &= D(fx_{n+1}, fx_n) \le a_1 D(gx_{n+1}, gx_n) + a_2 D(gx_{n+1}, fx_{n+1}) \\ &\quad + a_3 D(gx_n, fx_n) + a_4 D(gx_{n+1}, fx_n) + a_5 D(gx_n, fx_{n+1}) \\ &= a_1 D(y_n, y_{n-1}) + a_2 D(y_n, y_{n+1}) + a_3 D(y_{n-1}, y_n) + a_4 D(y_n, y_n) + a_5 D(y_{n-1}, y_{n+1}) \\ &\le (a_1 + a_3 + Ka_5) D(y_{n-1}, y_n) + (a_2 + Ka_5) D(y_n, y_{n+1}), \end{aligned}$$

we obtain

$$2D(y_{n+1}, y_n) \le (2a_1 + a_2 + a_3 + K(a_4 + a_5)) D(y_n, y_{n-1}) + (a_2 + a_3 + K(a_4 + a_5)) D(y_{n+1}, y_n),$$

that is,

$$D(y_{n+1}, y_n) \le \lambda D(y_n, y_{n-1}), \quad \lambda = \frac{2a_1 + a_2 + a_3 + K(a_4 + a_5)}{2 - (a_2 + a_3 + K(a_4 + a_5))} < 1,\ n = 1, 2, \ldots.$$

Further, (2.2) implies that

$$D(y_n, y_{n-1}) \le \lambda D(y_{n-1}, y_{n-2}) \le \cdots \le \lambda^{n-1} D(y_1, y_0). \tag{2.3}$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence. By the triangle inequality, for $n > m$ we have

$$d(y_n, y_m) \preceq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m).$$

Hence, as P is a normal cone, we have

$$\begin{aligned} D(y_n, y_m) = \|d(y_n, y_m)\| &\le K(\|d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m)\|) \\ &\le K(\|d(y_n, y_{n-1})\| + \|d(y_{n-1}, y_{n-2})\| + \cdots + \|d(y_{m+1}, y_m)\|) \\ &= KD(y_n, y_{n-1}) + KD(y_{n-1}, y_{n-2}) + \cdots + KD(y_{m+1}, y_m). \end{aligned}$$

Now by (2.3), it follows that

$$\begin{aligned} D(y_n, y_m) &\le K(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) D(y_1, y_0) \\ &\le \frac{K\lambda^m}{1-\lambda} D(y_1, y_0) \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

From [1, Lemma 4] follows that $\{y_n\} = \{fx_n\} = \{gx_{n+1}\}$ is a Cauchy sequence. Since $g(X)$ is complete, there exists a q in $g(X)$ such that $y_n \to q$ as $n \to \infty$. Consequently, we can find p in X such that $g(p) = q$. We shall show that $f(p) = q$. Substituting $x = p$, $y = x_n$ in (2.1), we get

$$D(fp, fx_n) \le a_1 D(gp, gx_n) + a_2 D(gp, fp) + a_3 D(gx_n, fx_n) + a_4 D(gp, fx_n) + a_5 D(gx_n, fp).$$

According to [1, Lemma 5], it follows

$$\begin{aligned} D(fp, q) &\le a_1 D(q, q) + a_2 D(q, fp) + a_3 D(q, q) + a_4 D(q, q) + a_5 D(q, fp) \\ &= (a_2 + a_5) D(fp, q) < D(fp, q), \end{aligned}$$

because $a_2 + a_5 \le \sum_{i=1}^{3} a_i + K(a_4 + a_5) < 1$. Now, if we suppose that $fp \ne q$, then we have a contradiction. Hence, $gp = fp = q$. We shall show that f and g have a unique point of coincidence. For this, assume that there exists another point of coincidence $q_1 \ne q$ in X such that $fp_1 = gp_1 = q_1$. Now,

$$\begin{aligned} D(q, q_1) &= D(fp, fp_1) \le a_1 D(gp, gp_1) + a_2 D(gp, fp) + a_3 D(gp_1, fp_1) + a_4 D(gp, fp_1) + a_5 D(gp_1, fp) \\ &= a_1 D(q, q_1) + a_2 D(q, q) + a_3 D(q_1, q_1) + a_4 D(q, q_1) + a_5 D(q_1, q) \\ &= (a_1 + a_4 + a_5) D(q_1, q) < D(q_1, q). \end{aligned}$$

As $a_1 + a_4 + a_5 \leq \sum_{i=1}^{3} a_i + K(a_4 + a_5) < 1$, we get $D(q, q_1) = 0$, that is, $q = q_1$. From the Proposition 1.5, it follows that f and g have a unique common fixed point. □

Corollary 2.2. *In Theorem 2.1. by setting $E = \mathbb{R}, P = [0, +\infty), \|x\| = |x|, x \in E, g = I_X$, we get the main theorem of Hardy–Rogers from* [16].

Corollary 2.3. *Putting $a_1 = \lambda, a_2 = a_3 = a_4 = a_5 = 0$ (respect. $a_2 = a_3 = \lambda, a_1 = a_4 = a_5 = 0$ that is, $a_1 = a_2 = a_3 = 0, a_4 = a_5 = \lambda$) in Theorem 2.1, we obtain Theorem 2.3 from* [9].

Since the four points $\{x, y, fx, fy\}$ determine six distances in X, the condition (2.4) in the following theorem to say that the image distance $d(fx, fy)$ never exceeds a fixed convex combination of the remaining five distances. Geometrically, this type of condition is quite natural.

Theorem 2.4. *Let (X, d) be a sequentially compact cone metric space, P a regular cone and $f : X \to X$ a continuous mapping such that*

$$d(fx, fy) \prec a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 d(x, fy) + a_5 d(y, fx) \tag{2.4}$$

for all $x, y \in X, x \neq y$ where $a_i \in [0, 1),\ i = \overline{1, 5}$ and $\sum_{i=1}^{5} a_i = 1$. Then, f has a unique fixed point.

In order to prove Theorem 2.4., we shall need the following lemma.

Lemma 2.5. *Let (X, d) be a cone metric space, $f : X \to X$ a mapping satisfying (2.4) for all $x, y \in X, x \neq y$ where $a_i \in [0, 1),\ i = \overline{1, 5}$ and $\sum_{i=1}^{5} a_i = 1$. Then*

$$d(f^2x, fx) \prec d(x, fx), \quad \text{for each } x \in X \text{ with } x \neq fx.$$

Proof. Putting $y = fx$ in (2.4), we have

$$\begin{aligned} d(fx, f^2x) &\prec a_1 d(x, fx) + a_2 d(x, fx) + a_3 d(fx, f^2x) + a_4 d(x, f^2x) + a_5 d(fx, fx) \\ &\preceq (a_1 + a_2 + a_4) d(x, fx) + (a_3 + a_4) d(fx, f^2x), \end{aligned}$$

that is,

$$d(fx, f^2x) \prec \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} d(x, fx). \tag{2.5}$$

By symmetry in (2.4), we have

$$d(fy, fx) \prec a_1 d(y, x) + a_2 d(y, fy) + a_3 d(x, fx) + a_4 d(y, fx) + a_5 d(x, fy), \tag{2.6}$$

i.e., putting $y = fx$ in (2.6), we obtain

$$\begin{aligned} d(f^2x, fx) &\prec a_1 d(fx, x) + a_2 d(fx, f^2x) + a_3 d(x, fx) + a_4 d(fx, fx) + a_5 d(x, f^2x) \\ &\preceq (a_1 + a_3 + a_5)\, d(x, fx) + (a_2 + a_5) d(fx, f^2x), \end{aligned}$$

that is,

$$d(f^2x, fx) \prec \frac{a_1 + a_3 + a_5}{1 - a_2 - a_5} d(x, fx). \tag{2.7}$$

If $k = \min\left\{\frac{a_1+a_2+a_4}{1-a_3-a_4}, \frac{a_1+a_3+a_5}{1-a_2-a_5}\right\} \in [0, 1)$, then

$$d(fx, f^2x) \preceq k d(x, fx) \prec d(x, fx). \quad \square \tag{2.8}$$

Proof of the Theorem 2.4. First, if u and v are two different fixed points of f, according to (2.4), we have

$$\begin{aligned} d(u, v) &= d(fu, fv) \prec a_1 d(u, v) + a_2 d(u, fu) + a_3 d(v, fv) + a_4 d(u, fv) + a_5 d(v, fu) \\ &= a_1 d(u, v) + a_4 d(u, v) + a_5 d(u, v) = (a_1 + a_4 + a_5) d(u, v) \prec d(u, v). \end{aligned}$$

This is a contradiction.

Let $x_0 \in X$. We define the sequence $x_n = f^n x_0, n = 0, 1, 2, \ldots$. If $x_{n+1} = x_n$ for some n, then x_n is a fixed point of f. Suppose that $x_{n+1} \neq x_n$ for each n. Then, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f^n x_0, f^{n+1} x_0) = d(ff^{n-1} x_0, ff^n x_0) \\ &\prec a_1 d(f^{n-1} x_0, f^n x_0) + a_2 d(f^{n-1} x_0, f^n x_0) + a_3 d(f^n x_0, f^{n+1} x_0) \\ &\quad + a_4 d(f^{n-1} x_0, f^{n+1} x_0) + a_5 d(f^n x_0, f^n x_0) \\ &\preceq a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, x_n) + a_3 d(x_n, x_{n+1}) + a_4 d(x_{n-1}, x_n) + a_4 d(x_n, x_{n+1}), \end{aligned}$$

that is,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f^{n+1}x_0, f^n x_0) = d(ff^n x_0, ff^{n-1}x_0) \\ &\prec a_1 d(f^n x_0, f^{n-1}x_0) + a_2 d(f^n x_0, f^{n+1}x_0) + a_3 d(f^{n-1}x_0, f^n x_0) \\ &\quad + a_4 d(f^n x_0, f^n x_0) + a_5 d(f^{n-1}x_0, f^{n+1}x_0) \\ &\preceq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 d(x_{n-1}, x_n) + a_5 d(x_{n-1}, x_n) + a_5 d(x_n, x_{n+1}). \end{aligned}$$

Now, we obtain

$$2d(x_n, x_{n+1}) \prec (2a_1 + a_2 + a_3 + a_4 + a_5)d(x_{n-1}, x_n) + (a_2 + a_3 + a_4 + a_5)d(x_n, x_{n+1}),$$

i.e.,

$$d(x_n, x_{n+1}) \prec \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} d(x_{n-1}, x_n) = \frac{a_1 + 1}{a_1 + 1} d(x_{n-1}, x_n) = d(x_{n-1}, x_n).$$

Hence, the sequence $d_n = d(x_n, x_{n+1})$ is strictly decreasing bounded below by θ. Since P is regular, there is $d^* \in E$ such that $d_n \to d^*(n \to \infty)$. From the sequence compactness of (X, d), there are subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x^* \in X$ such that $x_{n_i} \to x^*(i \to \infty)$. Since mappings f and f^2 are continuous, we have

$$fx_{n_i} \to fx^* \quad \text{and} \quad f^2 x_{n_i} \to f^2 x^*.$$

By using [1, Lemma 5], we have

$$d(fx_{n_i}, x_{n_i}) \to d(fx^*, x^*) \quad (i \to \infty) \quad \text{and} \quad d(f^2 x_{n_i}, fx_{n_i}) \to d(f^2 x^*, fx^*) \quad (i \to \infty).$$

It is obvious that

$$\begin{aligned} &d(fx_{n_i}, x_{n_i}) = d_{n_i} \to d^* = d(fx^*, x^*) \quad (i \to \infty) \quad \text{and} \\ &d(f^2 x_{n_i}, fx_{n_i}) = d_{n_i+1} \to d^* = d(f^2 x^*, fx^*) \quad (i \to \infty). \end{aligned} \tag{2.9}$$

Now we shall prove that $fx^* = x^*$. If $fx^* \neq x^*$, then $d^* \neq 0$. From (2.9) and according to Lemma 2.5, it follows

$$d^* = \lim_{i\to\infty} d_{n_i+1} = \lim_{i\to\infty} d(f^2 x_{n_i}, fx_{n_i}) = d(f^2 x^*, fx^*) \prec d(fx^*, x^*) = d^*.$$

We have a contradiction, so $fx^* = x^*$. That is, x^* is a fixed point of f. This completes the proof of Theorem 2.4. □

We now list some corollaries of Theorem 2.4.

Corollary 2.6 ([17]). *In Theorem 2.4 by setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|x\| = |x|$, $x \in E$, $a_1 = 1$, $a_i = 0$, $i = \overline{2, 5}$, we get the well-known Nemytzki's result for contractive mappings on compact metric spaces.*

Corollary 2.7 ([18]). *In Theorem 2.4 by setting $E = \mathbb{R}$, $P = [0, +\infty)$, $\|x\| = |x|$, $x \in E$, $a_1 = 1$, $a_i = 0$, $i = \overline{2, 5}$, we get the well-known Edelstein's result for contractive mappings on compact metric spaces.*

Corollary 2.8. *Putting $a_1 = 1$, $a_i = 0$, $i = \overline{2, 5}$ in Theorem 2.4, we get the result from* [1, *Theorem 2*].

Remark 2.9. For contractive conditions (2.1) and (2.4) Hardy–Rogers type see also [5,16].

If the space (X, d) is not sequentially compact, condition (2.4) is not sufficient for the existence of a fixed point of the mapping f.

Example 2.10. Let $X = [1, +\infty)$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\} \subset \mathbb{R}^2$ and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Consider the mapping $f : X \to X$ defined as $fx = x + \frac{1}{x}$. For arbitrary $x, y \in X$, we have that

$$\begin{aligned} d(fx, fy) &= (|fx - fy|, \alpha|fx - fy|) = \left(|x-y|\left(1 - \frac{1}{xy}\right), \alpha|x-y|\left(1 - \frac{1}{xy}\right)\right) \\ &\prec (|x-y|, \alpha|x-y|) = d(x, y). \end{aligned}$$

Obviously, the condition (2.4) holds, but f has not fixed points. The space (X, d) is a regular complete cone metric space which is not sequentially compact.

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