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On the Space Restitution of the Laguerre's Points Associated to the Perspective Elliptical Involuted Ranges

This paper discloses the proof, based on the method of the space restitution, that geometrical loci of Laguerre's points of perspective elliptical involuted ranges represent two circles. The exposed theorem is a contribution to the theory of Projective geometry; moreover it makes the constructive methods and computational algorithms of an object axial rotations in central projection more effective.

Keywords: *elliptical, Lagguerre, restitution, hyperboloid, involuted.*

1. INTRODUCTION

Every elliptical involuted range possesses a pair of Laguerre's points, which represent supports of the elliptic pencils, whose involution is circular. This paper analyses the space restitution and geometrical loci of Laguerre's points of perspective elliptical involuted ranges.

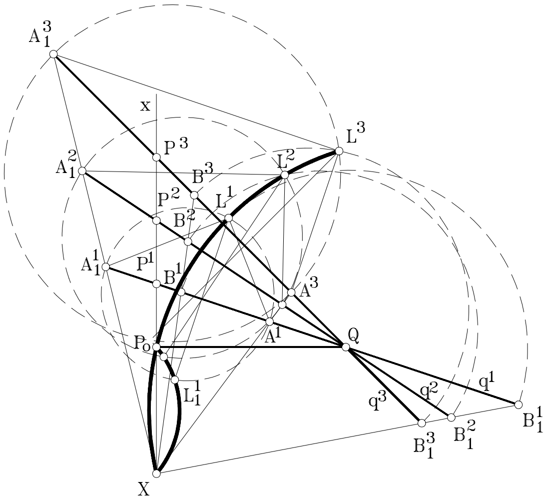

Figure 1. Perspective elliptical involuted ranges

As is shown in Fig. 1, the perspective elliptical involuted ranges are obtained by projecting the elliptical range (q) to all other straight lines of pencil (Q) from any other point X which does not belong to the straight line q. Since the point Q is common for all elliptical ranges, it will be involutively associated with the corresponding points P^i ($i = 0,1,2,3...$), which belong to the straight lines q^i in the pencil (Q). From the fact that the elliptical ranges (q^i) are perspective, we can conclude that the points P^i belong to the ray x which possesses the center of perspectivity X. Every elliptical range (q^i) has a pair of Laguerre's points L^i, L_1^i ($i = 0,1,2,3...$) and their geometrical loci represent two continual curves. For further analysis, it is very important to note that the pencil (Q) possesses the straight line q^0 which is orthogonal to the ray x and that the point P^0 of their intersection is involutively associated with the point Q. It is also important to emphasize that the trajectories of Laguerre's points of perspective elliptical involuted ranges will possess the center of perspectivity X and the point P^0.

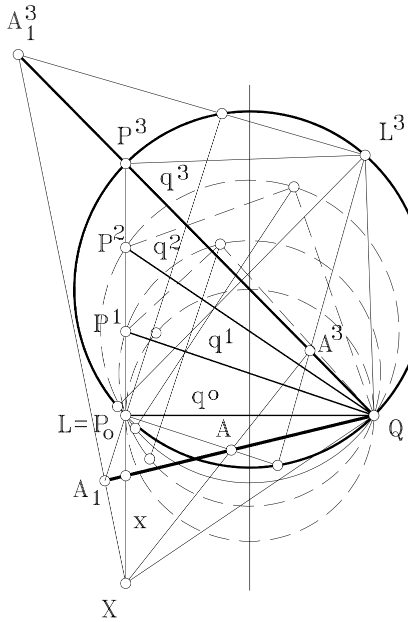

Figure 2. Elliptical pencil of circles which contains all Laguerre's points

2. A PLANIMETRIC ANALZSIS

Laguerre's point $L = P^0$ is the support of the circular pencil which is projectively associated with the circular pencils of all other Laguerre's points. Therefore, the product of the projectively associated pairs of

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pencils (L)-(L^1), (L)-(L^2), etc. will represent one elliptical pencil of circles, whose base points are P^0 and Q. This elliptical pencil of circles contains all Laguerre's points of perspective elliptical involuted ranges (q^i), as shown in Fig. 2.

The same conclusion can be drawn from a different consideration. Fig. 3. shows the unspecified straight line q^i from the pencil (Q) and the straight line q^0 which is orthogonal to the ray x and intersects it at the point P^0. The straight line q^i is the support of the elliptical range, one Laguerre's point of which is L^i. As the points Q and P^i are involutively associated into the elliptical range (q^i), the straight lines L^iP^i is orthogonal to the straight line L^iQ. It can be concluded from this that the points L^i, P^i, P^0 and Q belong to the same circle, i.e., that all Laguerre's points of elliptical involuted ranges (q^i) ($i = 0,1,2,3...$) belong to the elliptical pencil of circles whose base points are P^0 and Q.

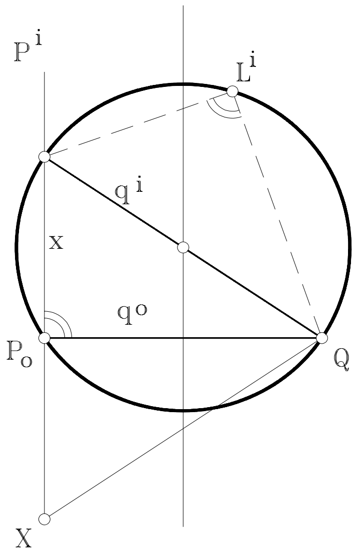

Figure 3. Circle which contains Laguerre's point L^i

In the preceding paragraphs it was emphasized that the geometrical loci of Laguerre's points of perspective elliptical involuted ranges have the point P^0. For this reason there is a possibility of forming two parabolic pencils of circles in such a way that the first one possesses the Laguerre's points L^i, and the second one the Laguerre's points L_1^i of perspective elliptical involuted ranges q^i ($i = 0,1,2,3...$). Both of the parabolic pencils have the base point P^0, and the centers of their circles are collinear to the ray x. As is shown in Fig. 4., each center of the circle in the elliptical pencil is associated exactly to the one center of the circle in the parabolic pencil in such a way that their centers in the infinity overlap. From this, it can be concluded that the centers of the above-mentioned pencils of circles represent two perspectively similar ranges of points.

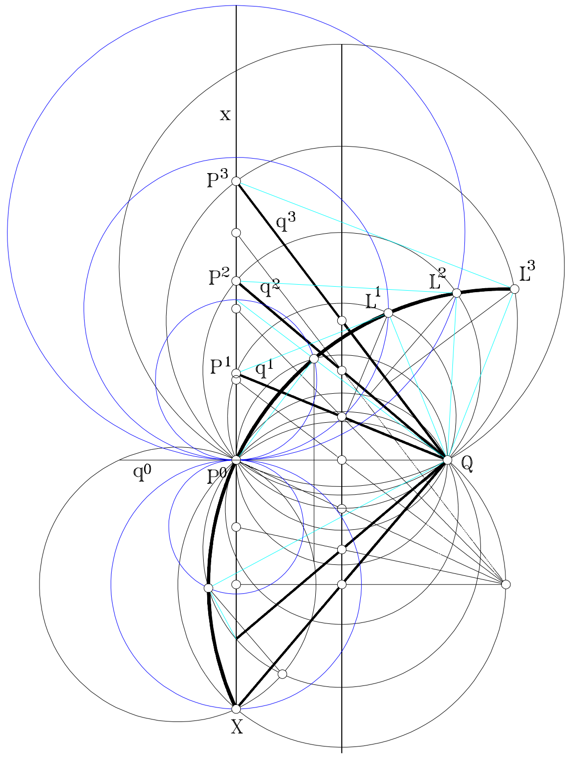

Figure 4. Elliptical and parabolic pencil of circles whose centers are perspectively similar ranges of points

3. METHOD OF THE SPACE RESTITUTION

Every elliptical pencil of circles represents the orthogonal projection of a one sheet hyperboloid which possesses the system of circle intersections whose planes are parallel to the plane of that elliptical pencil. In Fig. 5, the space restitution of the elliptical pencil of circles is accomplished into the one sheet hyperboloid by translation of the circles from the elliptical pencil in such a way that their centers become collinear to the straight line a. The base points P^0 and Q of this elliptical pencil are ray projections of the parallel generatrices p and q which belong to the one sheet hyperboloid restituted.

Every parabolic pencil of circles represents an orthogonal projection of the cone which possesses a system of circle intersections whose planes are parallel to the plane of that parabolic pencil. In Fig. 6. the space restitution of the parabolic pencil of circles is accomplished into a cone by the translation of the circles from the elliptical pencil in such a way that their centers become collinear to the straight line c. The base point P_0 of this parabolic pencil is a ray projection of the generatrix p which belongs to the cone restituted.

The method of space restitution of the elliptical and the parabolic pencils of circles, exposed in the preceding paragraphs, will be applied to the determination of the geometrical loci of the Laguerre's points of elliptical involuted ranges. In the preceding paragraphs has also been shown that the geometrical

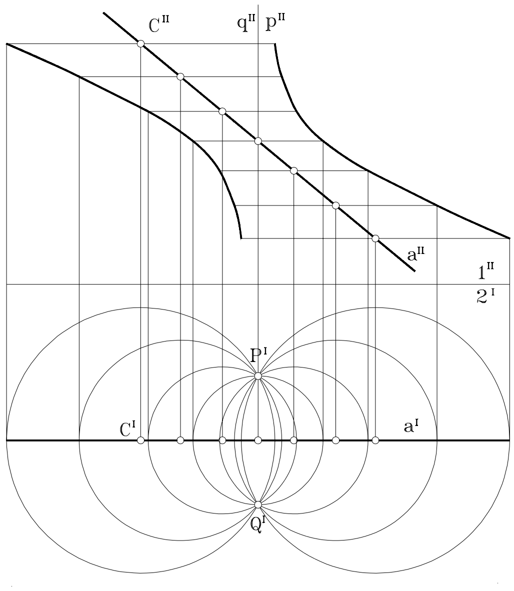

Figure 5. Restitution of the elliptical pencil of circles into the one sheet hyperboloid

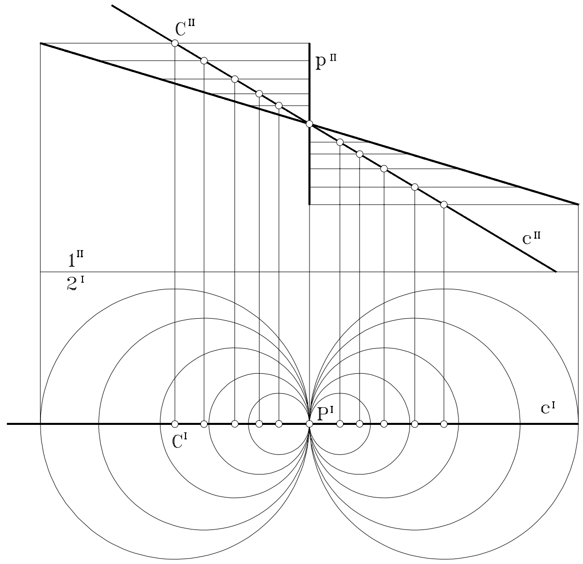

Figure 6. Restitution of the parabolical pencil of circles into the cone

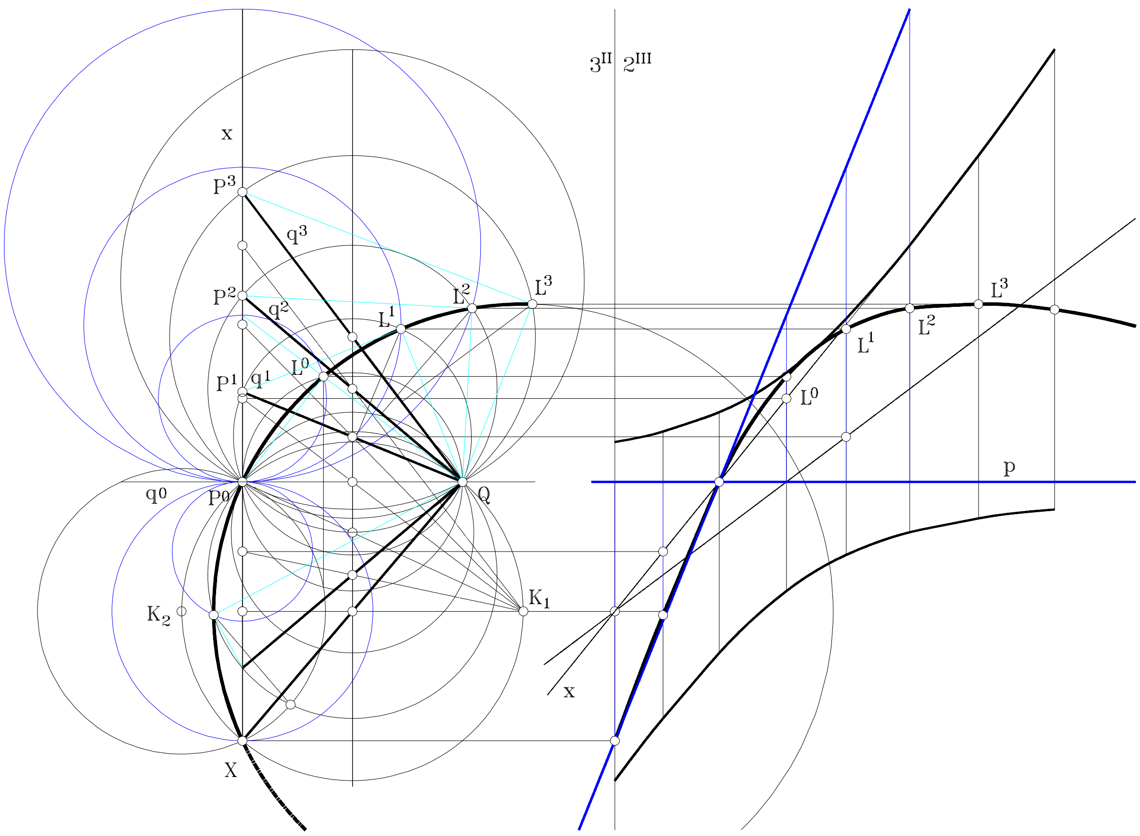

Figure 7. Intersection curve of one sheet hyperboloid and a cone obtained by the space restitution of elliptical and parabolic pencil of circles

loci of Laguerre's points of perspective elliptical involuted ranges belong to one elliptical and two parabolic pencils of circles, the radical axes of which are common, in such a way that the parabolic pencil base point overlaps with one base point of the elliptical pencil. As is shown in Fig. 7., the space restitution of that elliptical and the one of the two above-mentioned parabolic pencils of circles form, respectively, one sheet hyperboloid and a cone. It is essential to note that every two projectively associated circles from the elliptical and parabolic pencils of circles, after the space restitution, belong to the same plane which is parallel to the plane of these elliptical and parabolic pencils. These surfaces of the second order intersect in one common generatrix p and a spatial curve of the third order. This spatial curve contains a pair of absolute points, since it belongs to the circular intersections of one sheet hyperboloid and a cone. The orthogonal projection of this curve of the third order, from its point of infinity, represents a curve of the second order, which also contains a pair of absolute points, and is, consequently, a circle. From the above, one can draw the conclusion that the geometrical loci of Laguerre's points L^i and L_1^i ($i = 0,1,2,3...$) of perspective elliptical involuted ranges represent two circles. These circles intersect in the center of perspectivity X and in the base point P^0, and their centers K_1 and K_2 are the perspectivity centers of perspectively associated ranges of centers of elliptical and parabolic pencils of circles.

4. CONCLUSION

This theorem of Laguerre's points circle trajectories can be applied to constructive methods and corresponding computational algorithms of an axially rotating object in the central projection, based on the theory of general collineation. If the axially rotating object is geometrically assigned by the system of parallel intersections, the central projections of their planes will represent the pairs of general collinear fields. These fields possess two pencils of vanishing lines, and the absolute involutions mapped, whose supports they are, represent elliptical involuted ranges. The axial rotation of an object can be obtained in the central projection by using absolute involuted ranges of Laguerre's points, by a direct mapping of the central projections of its parallel intersections. Since absolute involutions mapped represent perspective elliptical involuted ranges, the exposed theorem of geometrical loci of Laguerre's points can be directly applied, which will make the constructive methods and corresponding algorithms of an object axial rotation in central projection more effective.

REFERENCES

[1] Dovniković, L.: Descriptive Geometrical Methods of Space Restitution, Proc. 3rd ICECGDG, Vienna, Volume 1, pp. 109-116, 1988.

[2] Popkonstantinović, B.: General Collinear Fields Obtained by Axially Rotating Central Space Projection, M.S. Dissertation, Belgrade, 1994.

[3] Obradović, M.: *General Collinear Fields with Conjugate Imaginary Double Elements*, M.S. Dissertation, Belgrade, 1995.

[4] Bennett, M.K.: *Affine and Projective Geometry* Wiley-Interscience , USA, 1995 .

[5] Hilbert, D., Cohn-Vossen, S.: *The Second-Order Surfaces*, New York: Chelsea, pp. 12-19, 1999.

[6] Coxeter, H.S.M.: *Projective Geometry*, Spriger-Verlag, New York Inc., USA, 2003.

[7] Coxeter, H.S.M.: *Non –Euclidean Geometry*, The Matemarical Association of America., USA, 2003.

ПРОСТОРНА РЕСТИТУЦИЈА ЛАГЕРОВИХ ТАЧАКА ПРИДРУЖЕНИХ ПЕРСПЕКТИВНО ЕЛИПТИЧКИМ ИНВОЛУТОРНИМ НИЗОВИМА

Драган Петровић,
Бранислав Попконстантиновић

Овај рад излаже доказ, заснован на методи просторне реституције, да геометријско место Лагерових тачака перспективних елиптичних инволуторних низова представља два круга. Осим што даје допринос теорији Пројективне геометрије, изложена теорема омогућава већу ефикасност конструктивних поступака и компјутерских алгоритама осне ротације објеката у централној пројекцији.