APPLICATION OF NEW HOMOTOPY ANALYSIS METHOD AND OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR SOLVING FUZZY FRACTIONAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstrak

Fenomena fizikal yang kompleks dengan sifat keturunan serta ketidakpastian diiktiraf untuk dihuraikan dengan baik menggunakan persamaan pembeza biasa pecahan kabur (PPBPK). Pendekatan analitik untuk menyelesaikan PPBPK bertujuan untuk memberikan penyelesaian bentuk tertutup yang dianggap sebagai penyelesaian tepat. Walau bagaimanapun, bagi kebanyakan PPBPK, penyelesaian analitik tidak mudah diperolehi. Selain itu, kebanyakan fenomena fizikal yang kompleks cenderung kepada ketiadaan penyelesaian analitikal. Pendekatan penganggaran boleh menangani kelemahan ini dengan menyediakan penyelesaian bentuk terbuka, dengan beberapa PPBPK dapat diselesaikan menggunakan kaedah-kaedah dalam kelas penganggaran berangka. Walau bagaimanapun, kaedah tersebut kebanyakannya digunakan untuk masalah linear atau yang dilinearkan dan tidak dapat menyelesaikan PPBPK bertertib tinggi secara langsung. Sementara itu, kaedah kelas anggaran-analitik di bawah pendekatan penganggaran bukan sahaja terpakai untuk PPBPK tak linear tanpa memerlukan pelinearan atau pendiskretan tetapi juga mempunyai keupayaan untuk menentukan ketepatan penyelesaian tanpa memerlukan penyelesaian tepat untuk perbandingan. Walau bagaimanapun, kaedah-kaedah anggaran-analitik sedia ada tidak dapat memastikan penumpuan penyelesaian. Namun begitu, untuk menyelesaikan persamaan pembeza biasa pecahan bukan kabur, wujud kaedah berasaskan gangguan: kaedah analisis homotopi pecahan (KAH-P) dan kaedah asimptotik homotopi optimum pecahan (KAHO-P), yang memiliki keupayaan kawalan penumpuan. Oleh itu, penyelidikan ini bertujuan untuk membangunkan kaedah anggaran-analitik baru yang berpenumpuan terkawal: KAH-P kabur (KAH-PK) dan KAHO-P kabur (KAHO-PK), untuk menyelesaikan masalah nilai awal biasa pecahan kabur tertib pertama dan kedua serta masalah nilai sempadan biasa pecahan kabur. Dalam pembangunan teori, pemantapan penumpuan penyelesaian dibangunkan berdasarkan parameter kawalan penumpuan. Dalam kerja eksperimen, penumpuan penyelesaian ditentukan dengan menggunakan sifat nombor kabur. KAH-PK dan KAHO-PK bukan sahaja dapat menyelesaikan masalah tak linear yang sukar bahkan juga mampu menyelesaikan masalah bertertib tinggi secara langsung tanpa menurunkannya ke sistem tertib pertama. Kajian perbandingan menunjukkan prestasi cemerlang bagi kaedah yang dibangunkan berbanding dengan kaedah lain, dengan KAH-PK dan KAHO-PK secara individunya unggul dari segi ketepatan.

Kata kunci: Persamaan pembeza biasa pecahan kabur, Kaedah analisis homotopi (KAH), Kaedah asimptotik homotopi optimum (KAHO), Kaedah penganggaran, Kaedah penganggar-analitik.

Abstract

Physical phenomena that are complex and have hereditary features as well as uncertainty are recognized to be well-described using fuzzy fractional ordinary differential equations (FFODEs). The analytical approach for solving FFODEs aims to give closed-form solutions that are considered exact solutions. However, for most FFODEs, the analytical solutions are not easily derived. Moreover, most complex physical phenomena tend to lack analytical solutions. The approximation approach can handle this drawback by providing open-form solutions where several FFODEs are solvable using the approximate-numerical class of methods. However, those methods are mostly employed for linear or linearized problems, and they cannot directly solve FFODES of high order. Meanwhile, the approximate-analytic class of methods under the approximation approach are not only applicable to nonlinear FFODEs without the need for linearization or discretization, but also can determine solution accuracy without requiring the exact solution for comparison. However, existing approximate-analytical methods cannot ensure convergence of the solution. Nevertheless, to solve non-fuzzy fractional ordinary differential equations, there exist perturbation-based methods: the fractional homotopy analysis method (F-HAM) and the optimal homotopy asymptotic method (F-OHAM), that possess convergence-control ability. Therefore, this research aims to develop new convergence-controlled approximate-analytical methods, fuzzy F-HAM (FF-HAM) and fuzzy F-OHAM (FF-OHAM), for solving first-order and second-order fuzzy fractional ordinary initial value problems and fuzzy fractional ordinary boundary value problems. In the theoretical development, the establishment of the convergence of the solutions is done based on the convergence-control parameters. In the experimental work, the convergence of solutions is determined using properties of fuzzy numbers. FF-HAM and FF-OHAM are not only able to solve difficult nonlinear problems but are also able to solve high-order problems directly without reducing them into first-order systems. The developed methods demonstrate the excellent performance of the developed methods in comparison to other methods, where FF-HAM and FF-OHAM are individually superior in terms of accuracy.

Keywords: Fuzzy fractional ordinary differential equations, Homotopy analysis method (HAM), Optimal homotopy asymptotic method (OHAM), Approximation methods, Approximate-analytical methods.

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List of Abbreviations

ODE	Ordinary Differential Equation
IVP	Initial Value Problem
BVP	Boundary Value Problem
FODE	Fractional Ordinary Differential Equation
FOIVP	Fractional Ordinary Initial Value Problem
FOBVP	Fractional Ordinary Boundary Value Problem
FFOIVP	Fuzzy Fractional Ordinary Initial Value Problem
FFODE	Fuzzy Fractional Ordinary Differential Equation
FFOBVP	Fuzzy Fractional Ordinary Boundary Value Problem
HAM	Homotopy Analysis Method
F-HAM	Fractional Homotopy Analysis Method
FF-HAM	Fuzzy Fractional Homotopy Analysis Method
OHAM	Optimal Homotopy Asymptotic Method
F-OHAM	Fractional Optimal Homotopy Asymptotic Method
FF-OHAM	Fuzzy Fractional Optimal Homotopy Asymptotic Method
RKHSM	Reproducing Kernel Hilbert Space Method
FRPSM	Fractional Residual Power Series Method
SCM	The spectral collocation method

CHAPTER ONE
INTRODUCTION

1.1 Background of the Study

Classical calculus provides a powerful tool in the modelling of dynamic processes. However, there are many complex systems with anomalous dynamics in nature, possessing hereditary properties of various materials and processes (Cui et al., 2018). For such systems, classical models are often not enough to describe their features. Fractional-order models are more accurate than integer-order models since there are more degrees of freedom in the fractional-order models. The fractional calculus apparently captures some of the hereditary properties in the system (Failla & Zingales, 2020). Fractional calculus is not modern; it is a generalization of the traditional calculus theory which deals with the integer order (Machado et al., 2014). In fractional calculus, the derivative and integral found in classical calculus are generalized to arbitrary real or complex order, that is, to non-integer order (Dalir & Bashour, 2010). The beginning of the theory of fractional calculus dated back to the seventeenth century when Leibniz wrote to L'Hôpital in the year 1695 to tell him about the derivative $\frac{d^{(\beta)}}{dx^{(\beta)}}$ of order $\beta = 0.5$. This letter marked the first appearance of fractional calculus (Dalir & Bashour, 2010).

Whilst classical calculus has unique definitions and clear physical as well as geometrical interpretations for the integer order derivatives and integrals, definitions for the derivative and integral of factional order are not unique where several definitions have been proposed since 1695 (Li & Deng, 2007). The definitions include Riemann-Liouville (Li et al., 2011), Caputo (Li et al., 2011), Riesz (Çelik & Duman,

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