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**APPLICATION OF NEW HOMOTOPY ANALYSIS METHOD
AND OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR
SOLVING FUZZY FRACTIONAL ORDINARY DIFFERENTIAL
EQUATIONS**



**DOCTOR OF PHILOSOPHY
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of Arts And Sciences

Universiti Utara Malaysia

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Abstrak

Fenomena fizikal yang kompleks dengan sifat keturunan serta ketidakpastian diiktiraf untuk dihuraikan dengan baik menggunakan persamaan pembeza biasa pecahan kabur (PPBPK). Pendekatan analitik untuk menyelesaikan PPBPK bertujuan untuk memberikan penyelesaian bentuk tertutup yang dianggap sebagai penyelesaian tepat. Walau bagaimanapun, bagi kebanyakan PPBPK, penyelesaian analitik tidak mudah diperolehi. Selain itu, kebanyakan fenomena fizikal yang kompleks cenderung kepada ketiadaan penyelesaian analitikal. Pendekatan penganggaran boleh menangani kelemahan ini dengan menyediakan penyelesaian bentuk terbuka, dengan beberapa PPBPK dapat diselesaikan menggunakan kaedah-kaedah dalam kelas penganggaran berangka. Walau bagaimanapun, kaedah tersebut kebanyakannya digunakan untuk masalah linear atau yang dilinearakan dan tidak dapat menyelesaikan PPBPK bertertib tinggi secara langsung. Sementara itu, kaedah kelas anggaran-analitik di bawah pendekatan penganggaran bukan sahaja terpakai untuk PPBPK tak linear tanpa memerlukan pelinearan atau pendiskretan tetapi juga mempunyai keupayaan untuk menentukan ketepatan penyelesaian tanpa memerlukan penyelesaian tepat untuk perbandingan. Walau bagaimanapun, kaedah-kaedah anggaran-analitik sedia ada tidak dapat memastikan penumpuan penyelesaian. Namun begitu, untuk menyelesaikan persamaan pembeza biasa pecahan bukan kabur, wujud kaedah berasaskan gangguan: kaedah analisis homotopi pecahan (KAH-P) dan kaedah asimptotik homotopi optimum pecahan (KAHO-P), yang memiliki keupayaan kawalan penumpuan. Oleh itu, penyelidikan ini bertujuan untuk membangunkan kaedah anggaran-analitik baru yang berpenumpuan terkawal: KAH-P kabur (KAH-PK) dan KAH-P kabur (KAHO-PK), untuk menyelesaikan masalah nilai awal biasa pecahan kabur tertib pertama dan kedua serta masalah nilai sempadan biasa pecahan kabur. Dalam pembangunan teori, pemantapan penumpuan penyelesaian dibangunkan berdasarkan parameter kawalan penumpuan. Dalam kerja eksperimen, penumpuan penyelesaian ditentukan dengan menggunakan sifat nombor kabur. KAH-PK dan KAH-PK bukan sahaja dapat menyelesaikan masalah tak linear yang sukar bahkan juga mampu menyelesaikan masalah bertertib tinggi secara langsung tanpa menurunkannya ke sistem tertib pertama. Kajian perbandingan menunjukkan prestasi cemerlang bagi kaedah yang dibangunkan berbanding dengan kaedah lain, dengan KAH-PK dan KAH-PK secara individunya unggul dari segi ketepatan.

Kata kunci: Persamaan pembeza biasa pecahan kabur, Kaedah analisis homotopi (KAH), Kaedah asimptotik homotopi optimum (KAHO), Kaedah penganggaran, Kaedah penganggaran-analitik.

Abstract

Physical phenomena that are complex and have hereditary features as well as uncertainty are recognized to be well-described using fuzzy fractional ordinary differential equations (FFODEs). The analytical approach for solving FFODEs aims to give closed-form solutions that are considered exact solutions. However, for most FFODEs, the analytical solutions are not easily derived. Moreover, most complex physical phenomena tend to lack analytical solutions. The approximation approach can handle this drawback by providing open-form solutions where several FFODEs are solvable using the approximate-numerical class of methods. However, those methods are mostly employed for linear or linearized problems, and they cannot directly solve FFODEs of high order. Meanwhile, the approximate-analytic class of methods under the approximation approach are not only applicable to nonlinear FFODEs without the need for linearization or discretization, but also can determine solution accuracy without requiring the exact solution for comparison. However, existing approximate-analytical methods cannot ensure convergence of the solution. Nevertheless, to solve non-fuzzy fractional ordinary differential equations, there exist perturbation-based methods: the fractional homotopy analysis method (F-HAM) and the optimal homotopy asymptotic method (F-OHAM), that possess convergence-control ability. Therefore, this research aims to develop new convergence-controlled approximate-analytical methods, fuzzy F-HAM (FF-HAM) and fuzzy F-OHAM (FF-OHAM), for solving first-order and second-order fuzzy fractional ordinary initial value problems and fuzzy fractional ordinary boundary value problems. In the theoretical development, the establishment of the convergence of the solutions is done based on the convergence-control parameters. In the experimental work, the convergence of solutions is determined using properties of fuzzy numbers. FF-HAM and FF-OHAM are not only able to solve difficult nonlinear problems but are also able to solve high-order problems directly without reducing them into first-order systems. The developed methods demonstrate the excellent performance of the developed methods in comparison to other methods, where FF-HAM and FF-OHAM are individually superior in terms of accuracy.

Keywords: Fuzzy fractional ordinary differential equations, Homotopy analysis method (HAM), Optimal homotopy asymptotic method (OHAM), Approximation methods, Approximate-analytical methods.

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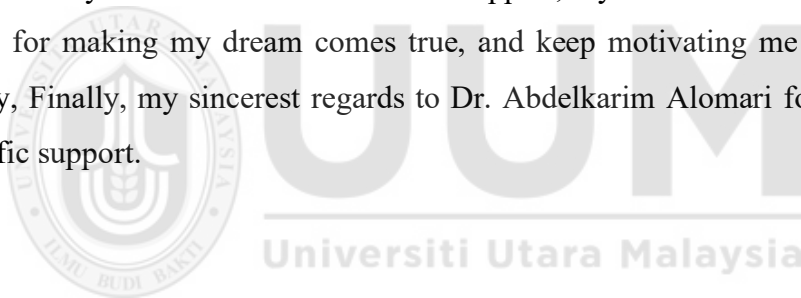


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List of Abbreviations

ODE	Ordinary Differential Equation
IVP	Initial Value Problem
BVP	Boundary Value Problem
FODE	Fractional Ordinary Differential Equation
FOIVP	Fractional Ordinary Initial Value Problem
FOBVP	Fractional Ordinary Boundary Value Problem
FFOIVP	Fuzzy Fractional Ordinary Initial Value Problem
FFODE	Fuzzy Fractional Ordinary Differential Equation
FFOBVP	Fuzzy Fractional Ordinary Boundary Value Problem
HAM	Homotopy Analysis Method
F-HAM	Fractional Homotopy Analysis Method
FF-HAM	Fuzzy Fractional Homotopy Analysis Method
OHAM	Optimal Homotopy Asymptotic Method
F-OHAM	Fractional Optimal Homotopy Asymptotic Method
FF-OHAM	Fuzzy Fractional Optimal Homotopy Asymptotic Method
RKHSM	Reproducing Kernel Hilbert Space Method
FRPSM	Fractional Residual Power Series Method
SCM	The spectral collocation method

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Classical calculus provides a powerful tool in the modelling of dynamic processes. However, there are many complex systems with anomalous dynamics in nature, possessing hereditary properties of various materials and processes (Cui et al., 2018). For such systems, classical models are often not enough to describe their features. Fractional-order models are more accurate than integer-order models since there are more degrees of freedom in the fractional-order models. The fractional calculus apparently captures some of the hereditary properties in the system (Failla & Zingales, 2020). Fractional calculus is not modern; it is a generalization of the traditional calculus theory which deals with the integer order (Machado et al., 2014). In fractional calculus, the derivative and integral found in classical calculus are generalized to arbitrary real or complex order, that is, to non-integer order (Dalir & Bashour, 2010). The beginning of the theory of fractional calculus dated back to the seventeenth century when Leibniz wrote to L'Hôpital in the year 1695 to tell him about the derivative $\frac{d^{(\beta)}}{dx^{(\beta)}}$ of order $\beta = 0.5$. This letter marked the first appearance of fractional calculus (Dalir & Bashour, 2010).

Whilst classical calculus has unique definitions and clear physical as well as geometrical interpretations for the integer order derivatives and integrals, definitions for the derivative and integral of fractional order are not unique where several definitions have been proposed since 1695 (Li & Deng, 2007). The definitions include Riemann-Liouville (Li et al., 2011), Caputo (Li et al., 2011), Riesz (Çelik & Duman,

REFERENCES

- Abdel Aal, M., Abu-Darwish, N., Abu Arqub, O., Al-Smadi, M., & Momani, S. (2019). Analytical solutions of fuzzy fractional boundary value problem of order 2α by using RKHS algorithm. *Applied Mathematics and Information Sciences*, 13(4), 523–533. <https://doi.org/10.18576/amis/130402>
- Abdelkawy, M. A., Zaky, M. A., Bhrawy, A. H., & Baleanu, D. (2015). Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model. *Romanian Reports in Physics*, 67(3), 773–791.
- Abdollahi, R., Farshbaf Moghimi, M., Khastan, A., & Hooshmandasl, M. R. (2019). Linear fractional fuzzy differential equations with Caputo derivative. *Computational Methods for Differential Equations*, 7(2), 252–265.
- Abdul Rahman, N. A., & Ahmad, M. Z. (2017). Solving fuzzy fractional differential equations using fuzzy Sumudu transform. *The Journal of Nonlinear Sciences and Applications*, 10(05), 2620–2632. <https://doi.org/10.22436/jnsa.010.05.28>
- Agarwal, R. P., Baleanu, D., Nieto, J. J., Torres, D. F. M., & Zhou, Y. (2018). A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *Journal of Computational and Applied Mathematics*, 339, 3–29. <https://doi.org/10.1016/j.cam.2017.09.039>
- Agarwal, R. P., Lakshmikantham, V., & Nieto, J. J. (2010). On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Analysis: Theory, Methods & Applications*, 72(6), 2859–2862. <https://doi.org/10.1016/j.na.2009.11.029>

- Ahmad, M. Z., Hasan, M. K., & Abbasbandy, S. (2013). Solving fuzzy fractional differential equations using Zadeh's extension principle. *The Scientific World Journal*, 2013(1), 1–11.
- Ahmad, M. Z., Hasan, M. K., & De Baets, B. (2013). Analytical and numerical solutions of fuzzy differential equations. *Information Sciences*, 236, 156–167. <https://doi.org/10.1016/j.ins.2013.02.026>
- Ahmadian, A., Ismail, F., Salahshour, S., Baleanu, D., & Ghaemi, F. (2017). Uncertain viscoelastic models with fractional order: A new spectral tau method to study the numerical simulations of the solution. *Communications in Nonlinear Science and Numerical Simulation*, 53, 44–64. <https://doi.org/10.1016/j.cnsns.2017.03.012>
- Ahmadian, A., Salahshour, S., Baleanu, D., Amirkhani, H., & Yunus, R. (2015). Tau method for the numerical solution of a fuzzy fractional kinetic model and its application to the oil palm frond as a promising source of xylose. *Journal of Computational Physics*, 294, 562–584. <https://doi.org/10.1016/j.jcp.2015.03.011>
- Ahmadian, Ali, Suleiman, M., Salahshour, S., & Baleanu, D. (2013). A Jacobi operational matrix for solving a fuzzy linear fractional differential equation. *Advances in Difference Equations*, 2013(1), 1–29. <https://doi.org/10.1186/1687-1847-2013-104>
- Alaroud, M., Saadeh, R., Al-smadi, M., Ahmad, R. R., Din, ummul K. S., & Arqub, O. abu. (2019). Solving nonlinear fuzzy fractional IVPs using fractional residual power series algorithm. *IACM*, 170–175.
- Alderremy, A. A., Gómez-Aguilar, J. F., Aly, S., & Saad, K. M. (2021). A fuzzy fractional model of coronavirus (COVID-19) and its study with Legendre spectral

method. *Results in Physics*, 21, 103773.

<https://doi.org/10.1016/j.rinp.2020.103773>

Allahviranloo, T., Salahshour, S., & Abbasbandy, S. (2012). Explicit solutions of fractional differential equations with uncertainty. *Soft Computing*, 16(2), 297–302. <https://doi.org/10.1007/s00500-011-0743-y>

Allahviranloo, T., Abbasbandy, S., Shahryari, M. R. B., Salahshour, S., & Baleanu, D. (2013). On solutions of linear fractional differential equations with uncertainty. *Abstract and Applied Analysis*, 2013, 1–13. <https://doi.org/10.1155/2013/178378>

Allahviranloo, Tofigh, Kiani, N. A., & Motamedi, N. (2009). Solving fuzzy differential equations by differential transformation method. *Information Sciences*, 179(7), 956–966.

Alshorman, M. A., Zamri, N., Ali, M., & Albzeirat, A. K. (2018). New implementation of residual power series for solving fuzzy fractional Riccati equation. *Journal of Modeling and Optimization*, 10(2), 81–87.

Askari, S., Allahviranloo, T., & Abbasbandy, S. (2019). Solving fuzzy fractional differential equations by adomian decomposition method used in optimal control theory. *International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies*, 10(12), 1–10. <https://doi.org/10.6084/m9.figshare.11110514>

Atangana, A., & Secer, A. (2013). A note on fractional order derivatives and table of fractional derivatives of some special functions. *Abstract and Applied Analysis*, 2013(1), 1–8. <https://doi.org/10.1155/2013/279681>

Bahia, G., Ouannas, A., Batiha, I. M., & Odibat, Z. (2021). The optimal homotopy

analysis method applied on nonlinear time-fractional hyperbolic partial differential equations. *Numerical Methods for Partial Differential Equations*, 37(3), 2008–2022.

Băleanu, D., & Mustafa, O. G. (2010). On the global existence of solutions to a class of fractional differential equations. *Computers & Mathematics with Applications*, 59(5), 1835–1841. <https://doi.org/10.1016/j.camwa.2009.08.028>

Bencsik, A. L., Bede, B., Tar, ozsef K., & Fodor, J. (2006). Fuzzy differential equations in modeling of hydraulic differential servo cylinders. In *Third Romanian-Hungarian Joint Symposium on Applied Computational Intelligence (SACI)*.

Bodjanova, S. (2006). Median alpha-levels of a fuzzy number. *Fuzzy Sets and Systems*, 157(7), 879–891. <https://doi.org/10.1016/j.fss.2005.10.015>

Bonyah, E., Atangana, A., & Chand, M. (2019). Analysis of 3D IS-LM macroeconomic system model within the scope of fractional calculus. *Chaos, Solitons & Fractals: X*, 2. <https://doi.org/10.1016/j.csf.2019.100007>

Buckley, J.J., & Yan, A. (2000). Fuzzy functional analysis (I): Basic concepts. *Fuzzy Sets and Systems*, 115(3), 393–402. [https://doi.org/10.1016/S0165-0114\(98\)00161-4](https://doi.org/10.1016/S0165-0114(98)00161-4)

Buckley, James J., & Feuring, T. (2001). Fuzzy initial value problem for Nth-order linear differential equations. *Fuzzy Sets and Systems*, 121(2), 247–255. [https://doi.org/10.1016/S0165-0114\(00\)00028-2](https://doi.org/10.1016/S0165-0114(00)00028-2)

Bulut, H., Baskonus, H. M., & Belgacem, F. B. M. (2013). The analytical solution of some fractional ordinary differential equations by the Sumudu transform method.

Abstract and Applied Analysis, 2013.

- Çelik, C., & Duman, M. (2012). Crank–Nicolson method for the fractional diffusion equation with the Riesz fractional derivative. *Journal of Computational Physics*, 231(4), 1743–1750. <https://doi.org/10.1016/j.jcp.2011.11.008>
- Chang, S. S. L., & Zadeh, L. A. (1972). On fuzzy mapping and control. *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-2(1), 30–34. <https://doi.org/10.1109/TSMC.1972.5408553>
- Chang, S. S. L., & Zadeh, L. A. (1996). On fuzzy mapping and control. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* (pp. 180–184). World Scientific.
- Cui, Y., Ma, W., Sun, Q., & Su, X. (2018). New uniqueness results for boundary value problem of fractional differential equation. *Nonlinear Analysis: Modelling and Control*, 23(1), 31–39. <https://doi.org/10.15388/NA.2018.1.3>
- Dalir, M., & Bashour, M. (2010). Applications of fractional calculus. *Applied Mathematical Sciences*, 4(21), 1021–1032.
- Das, A. K., & Roy, T. K. (2017). Exact solution of some linear fuzzy fractional differential equation using Laplace transform method. *Global Journal of Pure and Applied Mathematics*, 13(9), 5427–5435.
- Das, S., Pan, I., & Das, S. (2013). Fractional order fuzzy control of nuclear reactor power with thermal-hydraulic effects in the presence of random network induced delay and sensor noise having long range dependence. *Energy Conversion and Management*, 68, 200–218. <https://doi.org/10.1016/j.enconman.2013.01.003>

- Dehghan, M., Manafian, J., & Saadatmandi, A. (2011). Analytical treatment of some partial differential equations arising in mathematical physics by using the Exp-function method. *International Journal of Modern Physics B*, 25(22), 2965–2981.
- Demirci, E., & Ozalp, N. (2012). A method for solving differential equations of fractional order. *Journal of Computational and Applied Mathematics*, 236(11), 2754–2762. <https://doi.org/10.1016/j.cam.2012.01.005>
- Deng, Y. (2019). Fractional-order fuzzy adaptive controller design for uncertain robotic manipulators. *International Journal of Advanced Robotic Systems*, 16(2), 172988141984022. <https://doi.org/10.1177/1729881419840223>
- Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer Science & Business Media. <https://doi.org/10.1007/978-3-642-14574-2>
- Dubois, D., & Prade, H. (1982). Towards fuzzy differential calculus part3: Differentiation. *Fuzzy Sets and Systems*, 8(1982), 225–233.
- Efe, M. O. (2008). Fractional Fuzzy Adaptive Sliding-Mode Control of a 2-DOF Direct-Drive Robot Arm. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 38(6), 1561–1570. <https://doi.org/10.1109/TSMCB.2008.928227>
- Esmailbeigi, M., Paripour, M., & Garmanjani, G. (2018). Approximate solution of the fuzzy fractional Bagley-Torvik equation by the RBF collocation method. *Computational Methods for Differential Equations*, 6(2), 186–214.
- Failla, G., & Zingales, M. (2020). Advanced materials modelling via fractional calculus: Challenges and perspectives. *Philosophical Transactions of the Royal*

Society A: Mathematical, Physical and Engineering Sciences, 378, 1–13.
<https://doi.org/10.1098/rsta.2020.0050>

Fard, O. S. (2009). An iterative scheme for the solution of generalized system of linear fuzzy differential equations. *World Applied Sciences Journal*, 7(12), 1597–1604.

Farooq, M., Khan, A., Nawaz, R., Islam, S., Ayaz, M., & Chu, Y. M. (2021). Comparative study of generalized couette flow of couple stress fluid using optimal homotopy asymptotic method and new iterative method. *Scientific Reports*, 11(1), 1–20. <https://doi.org/10.1038/s41598-021-82746-8>

Frolov, A. L., Frolova, O. A., Sumina, R. S., & Sviridova, E. N. (2020). Mathematical modeling of axisymmetric flow of granular materials. *Journal of Physics: Conference Series*, 1479(1). <https://doi.org/10.1088/1742-6596/1479/1/012115>

Garrappa, R. (2018). Numerical solution of fractional differential equations: A survey and a software tutorial. *Mathematics*, 6(2), 1–23.
<https://doi.org/10.3390/math6020016>

Ghanbari, B., & Akgul, A. (2021). Abundant new analytical and approximate solutions to the generalized schamel equation. *Materials and Design*, 11(20), 1–32.

Ghazanfari, B., & Veisi, F. (2011). Homotopy analysis method for the fractional nonlinear equations. *Journal of King Saud University - Science*, 23(4), 389–393.
<https://doi.org/10.1016/j.jksus.2010.07.019>

Ghoreishi, M., Ismail, A. I. B. M., & Alomari, A. K. (2011). Application of the homotopy analysis method for solving a model for HIV infection of CD4+ T-cells. *Mathematical and Computer Modelling*, 54(11–12), 3007–3015.
<https://doi.org/10.1016/j.mcm.2011.07.029>

- Ghoreishi, M., Ismail, A. I. B. M., Alomari, A. K., & Sami Bataineh, A. (2012). The comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population models. *Communications in Nonlinear Science and Numerical Simulation*, 17(3), 1163–1177. <https://doi.org/10.1016/j.cnsns.2011.08.003>
- Grover, M., & Tomer, A. (2011). Comparison of optimal homotopy asymptotic method with homotopy perturbation method of twelfth order boundary value problems. *International Journal on Computer Science and Engineering*, 3(7), 2739–2747.
- Guang-Quan, Z. (1991). Fuzzy continuous function and its properties. *Fuzzy Sets and Systems*, 43(2), 159–171. [https://doi.org/10.1016/0165-0114\(91\)90074-Z](https://doi.org/10.1016/0165-0114(91)90074-Z)
- Hamarshah, M., Ismail, A., & Odibat, Z. (2015). Optimal homotopy asymptotic method for solving fractional relaxation-oscillation equation. *Journal of Interpolation and Approximation in Scientific Computing*, 2015(2), 98–111. <https://doi.org/10.5899/2015/jiasc-00081>
- Hasan, S., Alawneh, A., Al-Momani, M., & Momani, S. (2017). Second order fuzzy fractional differential equations under Caputo's H-differentiability. *Applied Mathematics & Information Sciences*, 11(6), 1–12. <https://doi.org/10.18576/amis/110606>
- Hashim, I., Abdulaziz, O., & Momani, S. (2009). Homotopy analysis method for fractional IVPs. *Communications in Nonlinear Science and Numerical Simulation*, 14(3), 674–684. <https://doi.org/10.1016/j.cnsns.2007.09.014>
- He, J.-H. (2004). Comparison of homotopy perturbation method and homotopy

analysis method. *Applied Mathematics and Computation*, 156(2), 527–539.

Hoang, N. Van, Vu, H., & Duc, T. M. (2019). Fuzzy fractional differential equations under Caputo–Katugampola fractional derivative approach. *Fuzzy Sets and Systems*, 375, 70–99. <https://doi.org/10.1016/j.fss.2018.08.001>

Ilie, M., Biazar, J., & Ayati, Z. (2019). Analytic solution for second-order fractional differential equations via OHAM. *Journal of Fractional Calculus and Applications*, 10(1), 105–119.

Ismail, M., Saeed, U., Alzabut, J., & ur Rehman, M. (2019). Approximate solutions for fractional boundary value problems via green-CAS wavelet method. *Mathematics*, 7(12), 1–20. <https://doi.org/10.3390/MATH7121164>

Jafari, H., & Tajadodi, H. (2010). He's variational iteration method for solving fractional Riccati differential equation. *International Journal of Differential Equations*, 1–8. <https://doi.org/10.1155/2010/764738>

Jain, P., Kumbhakar, M., & Ghoshal, K. (2021). Application of homotopy analysis method to the determination of vertical sediment concentration distribution with shear-induced diffusivity. *Engineering with Computers*, 1–20.

Jameel, A. F., Saaban, A., Altaie, S. A., Anakira, N. R., Alomari, A. K., & Ahmad, N. (2018). Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. *International Journal of Pure and Applied Mathematics*, 118(1), 49–64. <https://doi.org/10.12732/ijpam.v118i1.5>

Jena, R. M., Chakraverty, S., & Jena, S. K. (2019). Dynamic response analysis of fractionally damped beams subjected to external loads using homotopy analysis method. *Journal of Applied and Computational Mechanics*, 5(2), 355–366.

<https://doi.org/10.22055/jacm.2019.27592.1419>

Kaleva, O. (2006). A note on fuzzy differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(5), 895–900.

<https://doi.org/10.1016/j.na.2005.01.003>

Kandel, A., & Byatt, W. J. (1980). Fuzzy processes. *Fuzzy Sets and Systems*, 4(2), 117–152. [https://doi.org/10.1016/0165-0114\(80\)90032-9](https://doi.org/10.1016/0165-0114(80)90032-9)

Kaur, D., Agarwal, P., Rakshit, M., & Chand, M. (2020). Fractional calculus involving (p, q)-mathieu type series. *Applied Mathematics and Nonlinear Sciences*, 5(2), 15–34. <https://doi.org/10.2478/amns.2020.2.00011>

Khan, N. A., Riaz, F., & Razzaq, O. A. (2014). A comparison between numerical methods for solving fuzzy fractional differential equations. *Nonlinear Engineering*, 3(3), 155–162. <https://doi.org/10.1515/nleng-2013-0029>

Khodadadi, E., & Çelik, E. (2013). The variational iteration method for fuzzy fractional differential equations with uncertainty. *Fixed Point Theory and Applications*, 2013(1), 1–7. <https://doi.org/10.1186/1687-1812-2013-13>

Kim, T., & Kim, D. S. (2020). Note on the degenerate gamma function. *Russian Journal of Mathematical Physics*, 27(3), 352–358. <https://doi.org/10.1134/S1061920820030061>

Kudryashov, N. A. (2020). Method for finding highly dispersive optical solitons of nonlinear differential equations. *Optik*, 206, 163550.

Kumar, A., & Lata, S. (2012). A note on fuzzy initial value problem for Nth-order fuzzy linear differential equations. *Journal of Fuzzy Set Valued Analysis*, 2012,

1–3. <https://doi.org/10.5899/2012/jfsva-00103>

Kumar, S., Kumar, R., Singh, J., Nisar, K. S., & Kumar, D. (2020). An efficient numerical scheme for fractional model of HIV-1 infection of CD4+ T-cells with the effect of antiviral drug therapy. *Alexandria Engineering Journal*, 59(4), 2053–2064.

Lee, M. O., Kumaresan, N., & Ratnavelu, K. (2016). Solution of fuzzy fractional differential equations using homotopy analysis method. *Applied Mathematical Modelling*, 32(2), 113–119. <https://doi.org/10.1016/j.apm.2009.09.011>

Li, C., & Chen, A. (2018). Numerical methods for fractional partial differential equations. *International Journal of Computer Mathematics*, 95(6–7), 1048–1099. <https://doi.org/10.1080/00207160.2017.1343941>

Li, C., & Deng, W. (2007). Remarks on fractional derivatives. *Applied Mathematics and Computation*, 187(2), 777–784. <https://doi.org/10.1016/j.amc.2006.08.163>

Li, C., Qian, D., & Chen, Y. (2011). On Riemann-Liouville and Caputo derivatives. *Discrete Dynamics in Nature and Society*, 2011, 1–15. <https://doi.org/10.1155/2011/562494>

Li, C., & Zeng, F. (2013). The finite difference methods for fractional ordinary differential equations. *Numerical Functional Analysis and Optimization*, 34(2), 149–179. <https://doi.org/10.1080/01630563.2012.706673>

Liao, S. (2004). On the homotopy analysis method for nonlinear problems. *Applied Mathematics and Computation*, 147, 499–513. [https://doi.org/10.1016/S0096-3003\(02\)00790-7](https://doi.org/10.1016/S0096-3003(02)00790-7)

- Liao, S. (2005). Comparison between the homotopy analysis method and homotopy perturbation method. *Applied Mathematics and Computation*, 169(2), 1186–1194. <https://doi.org/10.1016/j.amc.2004.10.058>
- Liao, S. (2009). Notes on the homotopy analysis method: Some definitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, 14(4), 983–997. <https://doi.org/10.1016/j.cnsns.2008.04.013>
- Liao, S. J. (1999). An explicit, totally analytic approximate solution for Blasius' viscous flow problems. *International Journal of Non-Linear Mechanics*, 34(4), 759–778. [https://doi.org/10.1016/S0020-7462\(98\)00056-0](https://doi.org/10.1016/S0020-7462(98)00056-0)
- Liu, Z.-J., Adamu, M. Y., Suleiman, E., & He, J.-H. (2017). Hybridization of homotopy perturbation method and Laplace transformation for the partial differential equations. *Thermal Science*, 21(4), 1843–1846.
- Mabood, F., Ismail, A. I. M., & Hashim, I. (2013). Application of optimal homotopy asymptotic method for the approximate solution of Riccati equation. *Sains Malaysiana*, 42(6), 863–867.
- Machado, J. A. T., Galhano, A. M. S. F., & Trujillo, J. J. (2014). On development of fractional calculus during the last fifty years. *Scientometrics*, 98(1), 577–582. <https://doi.org/10.1007/s11192-013-1032-6>
- Maier, C., Mattke, J., Pflügner, K., & Weitzel, T. (2020). Smartphone use while driving: A fuzzy-set qualitative comparative analysis of personality profiles influencing frequent high-risk smartphone use while driving in Germany. *International Journal of Information Management*, 55, 1–12. <https://doi.org/10.1016/j.ijinfomgt.2020.102207>

- Manafian, J., & Teymuri sindi, C. (2018). An optimal homotopy asymptotic method applied to the nonlinear thin film flow problems. *International Journal of Numerical Methods for Heat and Fluid Flow*, 28(12), 2816–2841. <https://doi.org/10.1108/HFF-08-2017-0300>
- Mansour, N., Cherif, M. S., & Abdelfattah, W. (2019). Multi-objective imprecise programming for financial portfolio selection with fuzzy returns. *Expert Systems with Applications*, 138. <https://doi.org/10.1016/j.eswa.2019.07.027>
- Mansouri, S. S., & Ahmady, N. (2012). A numerical method for solving Nth-order fuzzy differential equation by using characterization theorem. *Communications in Numerical Analysis*, 2012, 1–12. <https://doi.org/10.5899/2012/cna-00054>
- Marinca, V., Herișanu, N., & Nemeș, I. (2008). Optimal homotopy asymptotic method with application to thin film flow. *Open Physics*, 6(3), 648–653. <https://doi.org/10.2478/s11534-008-0061-x>
- Matar, M. M. (2018). Solution of sequential Hadamard fractional differential equations by variation of parameter technique. *Abstract and Applied Analysis*, 2018(3), 1–7. <https://doi.org/10.1155/2018/9605353>
- Matlob, M. A., & Jamali, Y. (2019). The concepts and applications of fractional order differential calculus in modeling of viscoelastic systems: A primer. *Critical Reviews in Biomedical Engineering*, 47(4), 249–276. <https://doi.org/10.1615/CritRevBiomedEng.2018028368>
- Mazandarani, M., & Kamyad, A. V. (2013). Modified fractional Euler method for solving fuzzy fractional initial value problem. *Communications in Nonlinear Science and Numerical Simulation*, 18(1), 12–21.

<https://doi.org/10.1016/j.cnsns.2012.06.008>

- Mizukoshi, M. T., Barros, L. C., Chalco-Cano, Y., Román-Flores, H., & Bassanezi, R. C. (2007). Fuzzy differential equations and the extension principle. *Information Sciences*, 177(17), 3627–3635. <https://doi.org/10.1016/j.ins.2007.02.039>
- Mohammed, O. H., & Ahmed, S. A. (2013). Solving fuzzy fractional boundary value problems using fractional differential transform method. *Journal of Al-Nahrain University Science*, 16(4), 225–232. <https://doi.org/10.22401/JNUS.16.4.28>
- Momani, S., & Shawagfeh, N. (2006). Decomposition method for solving fractional Riccati differential equations. *Applied Mathematics and Computation*, 182(2), 1083–1092. <https://doi.org/10.1016/j.amc.2006.05.008>
- Moore, T. J., & Ertürk, V. S. (2020). Comparison of the method of variation of parameters to semi-analytical methods for solving nonlinear boundary value problems in engineering. *Nonlinear Engineering*, 9(1), 1–13. <https://doi.org/10.1515/nleng-2018-0148>
- Morales, O. S., & Mendez, J. J. S. (2012). Partition of a nonempty fuzzy set in nonempty convex fuzzy subsets. *Applied Mathematical Sciences*, 6(59), 2917–2921.
- Mosleh, M., & Otadi, M. (2015). Approximate solution of fuzzy differential equations under generalized differentiability. *Applied Mathematical Modelling*, 39(10–11), 3003–3015. <https://doi.org/10.1016/j.apm.2014.11.035>
- Najariyan, M., & Zhao, Y. (2018). Fuzzy Fractional Quadratic Regulator Problem Under Granular Fuzzy Fractional Derivatives. *IEEE Transactions on Fuzzy Systems*, 26(4), 2273–2288. <https://doi.org/10.1109/TFUZZ.2017.2783895>

- Nawaz, R., Islam, S., & Yasin, S. (2010). Solution of tenth order boundary value problems using optimal homotopy asymptotic method (OHAM). *Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering & Medicin*, 1(2), 37–54.
- Ngan, R. T., Son, L. H., Ali, M., Tamir, D. E., Rishe, N. D., & Kandel, A. (2020). Representing complex intuitionistic fuzzy set by quaternion numbers and applications to decision making. *Applied Soft Computing Journal*, 87, 105961. <https://doi.org/10.1016/j.asoc.2019.105961>
- Omana, R. W. (2009). Lower and upper solutions and existence of $W_{1,1}$ -solutions of fuzzy differential equations. *Southern Africa Journal of Pure and Applied Mathematics*, 4(2009), 29–42.
- Otadi, M., & Mosleh, M. (2016). Solution of fuzzy differential equations. *International Journal of Industrial Mathematics*, 8(1), 73–80.
- Pagnini, G. (2012). Erdélyi-Kober fractional diffusion. *Fractional Calculus and Applied Analysis*, 15(1), 117–127. <https://doi.org/10.2478/s13540-012-0008-1>
- Pakdaman, M., Ahmadian, A., Effati, S., Salahshour, S., & Baleanu, D. (2017). Solving differential equations of fractional order using an optimization technique based on training artificial neural network. *Applied Mathematics and Computation*, 293(2017), 81–95. <https://doi.org/10.1016/j.amc.2016.07.021>
- Panahi, A. (2017). Approximate solution of fuzzy fractional differential equations. *International Journal of Industrial Mathematics*, 9(2), 111–118.
- Patrício, M. F. S., Ramos, H., & Patrício, M. (2019). Solving initial and boundary value problems of fractional ordinary differential equations by using collocation

and fractional powers. *Journal of Computational and Applied Mathematics*, 354, 348–359. <https://doi.org/10.1016/j.cam.2018.07.034>

Picozzi, S., & West, B. J. (2002). Fractional langevin model of memory in financial markets. *Physical Review E* - 66, 66(4), 12. <https://doi.org/10.1103/PhysRevE.66.046118>

PIEGAT, A. (2005). A new definition of the fuzzy set. *Int. J. Appl. Math. Comput. Sci*, 15(1), 125–140.

Prakash, P., Nieto, J. J., Senthilvelavan, S., & Sudha Priya, G. (2015). Fuzzy fractional initial value problem. *Journal of Intelligent and Fuzzy Systems*, 28(6), 2691–2704. <https://doi.org/10.3233/IFS-151547>

Raj, S. R., & Saradha, M. (2015). Solving hybrid fuzzy fractional differential equations by Adam-Bashforth method. *Applied Mathematical Sciences*, 9(29), 1429–1432. <https://doi.org/10.12988/ams.2015.4121047>

Rana, J., & Liao, S. (2019a). A general analytical approach to study solute dispersion in non-Newtonian fluid flow. *European Journal of Mechanics, B/Fluids*, 77, 183–200. <https://doi.org/10.1016/j.euromechflu.2019.04.013>

Rana, J., & Liao, S. (2019b). On time independent Schrödinger equations in quantum mechanics by the homotopy analysis method. *Theoretical and Applied Mechanics Letters*, 9(6), 376–381. <https://doi.org/10.1016/j.taml.2019.05.006>

Rashid, S., Ashraf, R., & Bayones, F. S. (2021). A Novel Treatment of Fuzzy Fractional Swift–Hohenberg Equation for a Hybrid Transform within the Fractional Derivative Operator. *Fractal and Fractional*, 5(4), 209. <https://doi.org/10.3390/fractalfract5040209>

- Rivaz, A., Fard, O. S., & Bidgoli, T. A. (2016). Solving fuzzy fractional differential equations by a generalized differential transform method. *SeMA Journal*, 73(2), 149–170. <https://doi.org/10.1007/s40324-015-0061-x>
- Roszkowska, E., & Kacprzak, D. (2016). The fuzzy saw and fuzzy topsis procedures based on ordered fuzzy numbers. *Information Sciences*, 369, 564–584. <https://doi.org/10.1016/j.ins.2016.07.044>
- Salahshour, S. (2011). Nth-order fuzzy differential equations under generalized differentiability. *Journal of Fuzzy Set Valued Analysis*, 2011, 1–14. <https://doi.org/10.5899/2011/jfsva-00043>
- Salahshour, S, Allahviranloo, T., & Abbasbandy, S. (2012). Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Communications in Nonlinear Science and Numerical Simulation*, 17(3), 1372–1381. <https://doi.org/10.1016/j.cnsns.2011.07.005>
- Salahshour, S, Allahviranloo, T., Abbasbandy, S., & Baleanu, D. (2012). Existence and uniqueness results for fractional differential equations with uncertainty. *Advanced in Difference Equations*, 2012(1), 1–12.
- Salahshour, Soheil, Ahmadian, A., Senu, N., Baleanu, D., & Agarwal, P. (2015). On Analytical Solutions of the Fractional Differential Equation with Uncertainty: Application to the Basset Problem. *Entropy*, 17(2), 885–902. <https://doi.org/10.3390/e17020885>
- Scalas, E., Gorenflo, R., & Mainardi, F. (2000). Fractional calculus and continuous-time finance. *Physica A: Statistical Mechanics and Its Applications*, 284(2000), 376–384. [https://doi.org/10.1016/S0378-4371\(00\)00255-7](https://doi.org/10.1016/S0378-4371(00)00255-7)

- Shah, N. A., Ahmad, I., Bazighifan, O., Abouelregal, A. E., & Ahmad, H. (2020). Multistage optimal homotopy asymptotic method for the nonlinear Riccati ordinary differential equation in nonlinear physics. *Applied Mathematics & Information Sciences*, 14(6), 1009–1016. <https://doi.org/10.18576/amis/140608>
- Shahidi, M., & Khastan, A. (2018). Solving fuzzy fractional differential equations by power series expansion method. *2018 6th Iranian Joint Congress on Fuzzy and Intelligent Systems (CFIS)*, 2018, 37–39. <https://doi.org/10.1109/CFIS.2018.8336621>
- Sin, K., Chen, M., Choi, H., & Ri, K. (2017). Fractional Jacobi operational matrix for solving fuzzy fractional differential equation1. *Journal of Intelligent & Fuzzy Systems*, 33(2), 1041–1052. <https://doi.org/10.3233/JIFS-162374>
- Sin, K., Chen, M., Wu, C., Ri, K., & Choi, H. (2018). Application of a spectral method to Fractional Differential Equations under uncertainty1. *Journal of Intelligent & Fuzzy Systems*, 35(4), 4821–4835. <https://doi.org/10.3233/JIFS-18732>
- Somathilake, L. W. (2020). An efficient numerical method for fractional ordinary differential equations - based on exponentially decreasing random memory on uniform meshes. *Journal of the National Science Foundation of Sri Lanka*, 48(2), 163–174. <https://doi.org/10.4038/jnsfsr.v48i2.9026>
- Sotonwa, O., & Obabiyi, O. (2019). The convergence of homotopy analysis method for solving Onchocerciasis (Riverblindness). *IOSR Journal of Mathematics*, 15(5), 79–93. <https://doi.org/10.9790/5728-1505047993>
- Stefanini, L. (2009). A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems*, 161, 1564–1584.

<https://doi.org/10.1016/j.fss.2009.06.009>

Stefanini, L., Sorini, L., & Guerra, M. L. (2006). Parametric representation of fuzzy numbers and application to fuzzy calculus. *Fuzzy Sets and Systems*, 157(18), 2423–2455. <https://doi.org/10.1016/j.fss.2006.02.002>

Sun, H. G., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. Q. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*, 64, 213–231. <https://doi.org/10.1016/j.cnsns.2018.04.019>

Sun, H., Song, X., & Chen, Y. (2010). A class of fractional dynamic systems with fuzzy order. *2010 8th World Congress on Intelligent Control and Automation*, 197–201.

Takači, D., Takači, A., & Takači, A. (2014). On the solutions of fuzzy fractional differential equations. *Fractional Calculus and Applied Analysis*, 4(1), 98–103.

Tan, Y., & Abbasbandy, S. (2008). Homotopy analysis method for quadratic Riccati differential equation. *Communications in Nonlinear Science and Numerical Simulation*, 13(3), 539–546. <https://doi.org/10.1016/j.cnsns.2006.06.006>

Tropanevsky, M. I., Seminara, S. A., & Fabio, M. A. (2019). A review on fractional differential equations and a numerical method to solve some boundary value problems. In *Nonlinear Systems -Theoretical Aspects and Recent Applications* (pp. 1–19). IntechOpen. <https://doi.org/10.5772/intechopen.86273>

Turkyilmazoglu, M. (2011). Some issues on HPM and HAM methods: A convergence scheme. *Mathematical and Computer Modelling*, 53(2011), 1929–1936. <https://doi.org/10.1016/j.mcm.2011.01.022>

- Uehara, K., & Fujise, M. (1993). Fuzzy inference based on families of α -level sets. *IEEE Transactions on Fuzzy Systems*, 1(2), 111–124.
- Ullah, A., Ullah, A., Ahmad, S., Ahmad, I., & Akgül, A. (2020). On solutions of fuzzy fractional order complex population dynamical model. *Numerical Methods for Partial Differential Equations*, num.22654. <https://doi.org/10.1002/num.22654>
- Veerasha, P., Prakasha, D. G., & Baskonus, H. M. (2019). Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Mathematical Sciences*, 13(2), 115–128.
- Verma, P., & Kumar, M. (2020). An analytical solution of linear/nonlinear fractional-order partial differential equations and with new existence and uniqueness conditions. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, 1–9.
- Vu, H., An, T. V., & Van Hoa, N. (2019). On the initial value problem for random fuzzy differential equations with Riemann-Liouville fractional derivative: Existence theory and analytical solution. *Journal of Intelligent and Fuzzy Systems*, 36(6), 6503–6520. <https://doi.org/10.3233/JIFS-182876>
- Wasques, V., Lariate, B., Santo Pedro, F., Esmi, E., & de Barros, L. C. (2020). *Interactive Fuzzy Fractional Differential Equation: Application on HIV Dynamics* (pp. 198–211). https://doi.org/10.1007/978-3-030-50153-2_15
- Wu, H.-C. (2000). The fuzzy Riemann integral and its numerical integration. *Fuzzy Sets and Systems*, 110, 1–25. [https://doi.org/10.1016/S0020-0255\(98\)00016-4](https://doi.org/10.1016/S0020-0255(98)00016-4)
- Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1), 3–28.

Zhu, Y. (2015). Existence and uniqueness of the solution to uncertain fractional differential equation. *Journal of Uncertainty Analysis and Applications*, 3(1), 1–11. <https://doi.org/10.1186/s40467-015-0028-6>

