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Computation of Input Disturbance Sets for Constrained Output Reachability

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***Abstract*—Linear models with additive unknown-but-bounded input disturbances are extensively used to model uncertainty in robust control systems design. Typically, the disturbance set is either assumed to be known a priori or estimated from data through set-membership identification. However, the problem of computing a suitable input disturbance set in case the set of possible output values is assigned a priori has received relatively little attention. This problem arises in many contexts, such as in supervisory control, actuator design, decentralized control, and others. In this paper, we propose a method to compute input disturbance sets (and the corresponding set of states) such that the resulting set of outputs matches as closely as possible a given set of outputs, while additionally satisfying strict (inner or outer) inclusion constraints. We formulate the problem as an optimization problem by relying on the concept of robust invariance. The effectiveness of the approach is demonstrated in numerical examples that illustrate how to solve safe reference set and input-constraint set computation problems.**

***Index Terms*—Disturbance sets, Constrained linear systems, Invariant sets.**

I. Introduction

The theory of set invariance plays a key role in the analysis of uncertain dynamical systems, as it provides the tools for the synthesis of robust controllers that can satisfy constraints in the presence of disturbances [1]. Of particular interest are Robust Positive Invariant (RPI) sets [2], the characterization and computation of which has been a very active area of research [3]–[5]. RPI sets are used to provide robust stability and constraint satisfaction guarantees of various robust Model Predictive Control (MPC) and Reference Governor (RG) schemes [6]–[8]. These guarantees are usually established using the maximal robust positive invariant (MRPI) set [5], which is the largest RPI set included in the constraint set. The minimal RPI (mRPI) [1] set, which is the smallest RPI set for a given disturbance set [5], is used to design trajectory tubes in robust MPC [9], and to analyze the existence of MRPI sets. It was shown that an output-admissible RPI set exists for the system if and only if the mRPI set is included in the constraint set [10]. In order to enforce this inclusion, several methods were proposed in the literature to design feedback controllers that sufficiently attenuate the effects of disturbances [11], [12]. On the other hand, in applications such as fault-tolerant control [13], RPI sets that include a given set are computed and used for sensor fault isolation. All the aforementioned applications were developed under the assumption that *the disturbance set is known a priori*.

In many practical cases, however, while the set of admissible states can be estimated from sensor measurements or pre-specified from given constraints to be satisfied, *the disturbance set is unknown*, leaving the designer the task of suitably defining it, especially in case one must satisfy a given set of constraints on the system, e.g., encoding known physical limitations, or undesired states. For example, in a decentralized MPC (DeMPC) application such as [12], [14], the dynamic coupling between subsystems is modeled as an additive disturbance. Then, the disturbance set for a given subsystem represents the state-constraint sets of the neighboring subsystems. Another example is presented in [15], in which the tracking references are modeled as disturbances acting on a system, such that a feasible disturbance set is the set of all feedforward tracking references guaranteeing constraint satisfaction. In both these cases, it is desirable to compute the *largest* feasible disturbance sets. In particular, a large disturbance set in the DeMPC case ensures that the region of attraction of the DeMPC scheme in which recursive feasibility and stability is guaranteed is maximized. Similarly, in the reference tracking case, a large disturbance set ensures that the operating range of the tracking control system is maximized. On the other hand, the reachability properties of a system under the application of a set of inputs can be exploited for actuator design as follows. Given an input-constraint set, a corresponding subset of the output space is reachable by the plant. Then, if a pre-specified output set is required to be reachable, the input-constraint set must be chosen such that the pre-specified output set is included in the output reachable set. Hence, the actuator design task involves computing an input-constraint set such that the output inclusion holds, and it is desirable to compute the *smallest* input-constraint set satisfying the output inclusion to avoid oversizing the actuators. Since the same reachability properties hold for input-constraint sets and disturbance sets, this problem can be posed as computing the smallest feasible disturbance set. Moreover, in disturbance identification techniques such as [16], [17], one is interested in obtaining a small disturbance set that can explain the data. In this paper, we propose a method to tackle disturbance set computation problems such as those described above. In particular, we compute a set of disturbances such that the corresponding output-reachable set approximately matches an assigned one. This method is centered on the formulation of an optimization problem, with the input disturbance set being the unknown and the approximation error between the obtained and assigned output sets being the objective function to minimize.

We propose the formulation of the optimization problem for stable linear systems and polytopic sets: since the construction of the output set requires the computation of an RPI set, we adopt the notions of [18], [19] to encode the computation of a parametrized RPI set within the problem. Then, we propose to use a penalty-function [20] to solve the resulting bilevel linear program. Finally, we show the effectiveness of the approach through numerical examples related to safe reference set and input-constraint set computation problems.

The paper is organized as follows. We introduce some notation and recall basic definitions regarding set operations in Section II. Then, we introduce the problems we solve, along with a discussion on their (non)-convexity aspects, and the advantages and limitations of the methods we propose. Then, in Section IV, we present the main results, using which we formulate tractable optimization problems to compute polytopic disturbance sets. Finally, in Section V we present some numerical results along with some application demonstrations.

II. Notation and Set Operations

Consider the sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, and vectors $a \in \mathbb{R}^{n_a}$ and $b \in \mathbb{R}^{n_b}$. Given a matrix $L \in \mathbb{R}^{n \times m}$, we denote by $L\mathcal{X}$ the image $\{y \in \mathbb{R}^m : y = Lx, x \in \mathcal{X}\}$ of $\mathcal{X}$ under the linear transformation induced by L. We denote the i-th row of matrix L by L_i, an element in row i and column j by L_{ij}, the rank of L by $\text{rank}(L)$, the

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image-space of L by $\mathrm{Im}(L)$, and the null-space of L by $\mathrm{null}(L)$. Given a square matrix $L \in \mathbb{R}^{n\times n}$, $\rho(L)$ denotes its spectral radius. The set $\mathcal{B}_p^n := \{x : \|x\|_p \leq 1\}$ is the unit p-norm ball in $\mathbb{R}^n$. A polyhedron is the intersection of a finite number of half-spaces, and a polytope is a compact polyhedron. Given two matrices L, $M \in \mathbb{R}^{n\times m}$, $L \leq M$ denotes element-wise inequality. The symbols $\mathbf{1}$, $\mathbf{0}$, and $\mathbf{I}$ denote all-ones, all-zeros and identity matrix respectively, with dimensions specified if the context is ambiguous. The set of natural numbers between two integers m and n, $m \leq n$, is denoted by $\mathbb{I}_m^n := \{m, \ldots, n\}$. Given a vector $x \in \mathbb{R}^n$, $\mathrm{diag}(x) \in \mathbb{R}^{n\times n}$ is a matrix with diagonal values x_i. The Minkowski set addition is defined as $\mathcal{X} \oplus \mathcal{Y} := \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$. Given some vector $z \in \mathbb{R}^n$ and polytopes $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, $z \in \mathcal{X} \oplus \mathcal{Y}$ holds if and only if there exists an $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $z = x + y$. If $\mathcal{X} \subset \mathbb{R}^{n+m}$, the projection operation onto the first n coordinates is defined as $\Pi_x\mathcal{X} := \{x : \exists y \in \mathbb{R}^m : [x^\top \ y^\top]^\top \in \mathcal{X}\}$. The support function of a compact set $\mathcal{X} \subset \mathbb{R}^n$ at some $y \in \mathbb{R}^n$ is $h_\mathcal{X}(y) := \max_{x\in\mathcal{X}} y^\top x$. For polytopes $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$, support functions are positively homogeneous, i.e., $h_{\alpha\mathcal{X}}(y) = \alpha h_\mathcal{X}(y)$ for any scalar $\alpha \geq 0$. For any vector $y \in \mathbb{R}^n$, $h_{\mathcal{X}\oplus\mathcal{Y}}(y) = h_\mathcal{X}(y) + h_\mathcal{Y}(y)$ holds. The inclusion $\mathcal{X} \subseteq \mathcal{Y}$ holds if and only if $h_\mathcal{X}(z) \leq h_\mathcal{Y}(z)$ for all $z \in \mathcal{B}_p^n$. Suppose $\mathcal{Y} := \{x : Mx \leq b\}$ with $M \in \mathbb{R}^{n_b\times n}$, $b \in \mathbb{R}^{n_b}$, then $\mathcal{X} \subseteq \mathcal{Y}$ holds if and only if $h_\mathcal{X}(M_i^\top) \leq h_\mathcal{Y}(M_i^\top) \leq b_i$ for all $i \in \mathbb{I}_1^{n_b}$, with $h_\mathcal{Y}(M_i^\top) = b_i$ if M, b define a minimal hyperplane representation of $\mathcal{Y}$. We use the Hausdorff distance between polytopes $\mathcal{X}$ and $\mathcal{Y}$ defined as $d_\mathrm{H}(\mathcal{X}, \mathcal{Y}) := \max_{z\in\mathcal{B}_p^n} |h_\mathcal{X}(z) - h_\mathcal{Y}(z)|$. If $\mathcal{X} \subseteq \mathcal{Y}$, then $d_\mathrm{H}(\mathcal{X}, \mathcal{Y}) = \min\{\epsilon : \mathcal{Y} \subseteq \mathcal{X} \oplus \epsilon\mathcal{B}_p^n\} \geq 0$.

III. Problem Definition and Approximations

Consider the linear time-invariant discrete-time system

$$x(t + 1) = Ax(t) + Bw(t), \tag{1a}$$
$$y(t) = Cx(t) + Dw(t), \tag{1b}$$

with state $x \in \mathbb{R}^{n_x}$, output $y \in \mathbb{R}^{n_y}$ and disturbance $w \in \mathbb{R}^{n_w}$. Given a polytopic set $\mathcal{Y} := \{y : Gy \leq g\}$, $g \in \mathbb{R}^{m_Y}$, of outputs, our goal is to compute a disturbance set $\mathcal{W}$ such that $\mathcal{Y}$ is "reachable" by the output y, in a sense which we will define precisely later. We refer to w as a "disturbance", as it is customary in the literature of uncertain systems. Depending on the application, however, it could also represent a set of command inputs, as we will show through application examples. We work under the following assumption.

Assumption 1: Matrix A is strictly stable, i.e., $\rho(A) < 1$. □

In this paper, we focus on the computation of a disturbance set $\mathcal{W}$ parametrized as the polytope $\mathbb{W}(\epsilon^w) := \{w : Fw \leq \epsilon^w\}$ with $\epsilon^w \in \mathbb{R}^{m_W}$. We assume that the row vectors $F_i^\top \in \mathbb{R}^{n_w}$ of F are given a priori, and restrict our attention to computing ϵ^w. For simplicity, we also enforce that $\mathbf{0} \in \mathcal{W}$, which is equivalent to $\epsilon^w \geq \mathbf{0}$. In the next section, we present a method to relax this restriction, i.e., permit the computation of a disturbance set $\mathcal{W}$ that does not contain the origin.

Given a disturbance set $\mathbb{W}(\epsilon^w)$, the *forward computation* problem, which is typically tackled in the literature [5], [21], entails computing a suitable RPI set $\mathcal{X} := \{x : Ax + Bw \in \mathcal{X}, \forall\ w \in \mathbb{W}(\epsilon^w)\}$. Of particular interest is the *minimal* RPI (mRPI) set $\mathcal{X}_\mathrm{m}(\epsilon^w)$, given by

$$\mathcal{X}_\mathrm{m}(\epsilon^w) = \oplus_{t=0}^{\infty} A^t B\mathbb{W}(\epsilon^w), \tag{2}$$

and is the smallest RPI set contained in every RPI set. If $\mathbf{0} \in \mathbb{W}(\epsilon^w)$, and Assumption 1 holds, then a compact, convex and unique mRPI set $\mathcal{X}_\mathrm{m}(\epsilon^w)$ exists with $\mathbf{0} \in \mathcal{X}_\mathrm{m}(\epsilon^w)$ [5]. Moreover, it is the limit of all state trajectories of (1a) under persistent disturbances $w \in \mathbb{W}(\epsilon^w)$ [1]. Then, the corresponding limit set of output trajectories is $\mathcal{Y}_\mathrm{m}(\epsilon^w) := C\mathcal{X}_\mathrm{m}(\epsilon^w) \oplus D\mathbb{W}(\epsilon^w)$ as per (1b). The set $\mathcal{Y}_\mathrm{m}(\epsilon^w)$ exists, is compact and convex with $\mathbf{0} \in \mathcal{Y}_\mathrm{m}(\epsilon^w)$ if $\epsilon^w \geq \mathbf{0}$.

In this paper, we tackle the *reverse computation* problem, i.e., given an output polytope $\mathcal{Y}$, compute the vector ϵ^w such that $\mathcal{Y}_\mathrm{m}(\epsilon^w) = \mathcal{Y}$. This problem, however, might not have a solution, i.e., there might not exist any ϵ^w satisfying the output-set equality because of either of the following two reasons. (1) The mRPI set $\mathcal{X}_\mathrm{m}(\epsilon^w)$ is not finitely determined, except in a few special cases, e.g., nilpotent systems [5]. Then, depending on the set $\mathcal{X}_\mathrm{m}(\epsilon^w)$ and the structure of matrix C, the set $\mathcal{Y}_\mathrm{m}(\epsilon^w)$ might also not be finitely determined. In this case, enforcing $\mathcal{Y}_\mathrm{m}(\epsilon^w) = \mathcal{Y}$, with $\mathcal{Y}$ defined using a finite number of hyperplanes is not possible. (2) Even if the set $\mathcal{X}_\mathrm{m}(\epsilon^w)$ and matrix C are such that $\mathcal{Y}_\mathrm{m}(\epsilon^w)$ is finitely determined, its shape is in general not arbitrary, but is a function of the dynamics of system (1a) and parametrization of set $\mathbb{W}(\epsilon^w)$. This implies enforcing $\mathcal{Y}_\mathrm{m}(\epsilon^w) = \mathcal{Y}$, with $\mathcal{Y}$ being a user-specified arbitrarily shaped polytope, might not be possible. Hence, we instead tackle the problem

$$\min_{\epsilon^w \geq \mathbf{0}} \ d_\mathrm{H}(\mathcal{Y}_\mathrm{m}(\epsilon^w), \mathcal{Y}). \tag{3}$$

This formulation includes the case $\mathcal{Y}_\mathrm{m}(\epsilon^w) = \mathcal{Y}$, which holds if and only if $d_\mathrm{H}(\mathcal{Y}_\mathrm{m}(\epsilon^w), \mathcal{Y}) = 0$. Moreover, we consider *inner* and *outer* approximation settings of Problem (3), since many applications typically enforce stronger inclusion requirements as $\mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y}$ or $\mathcal{Y} \subseteq \mathcal{Y}_\mathrm{m}(\epsilon^w)$. We formulate the *inner-approximation* setting by appending the constraint $\mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y}$ to Problem (3). For simplicity, we use the ∞-norm induced Hausdorff distance $d_\mathrm{H}(\mathcal{Y}_\mathrm{m}(\epsilon^w), \mathcal{Y}) = \min\{\epsilon : \mathcal{Y} \subseteq \mathcal{Y}_\mathrm{m}(\epsilon^w) \oplus \epsilon\mathcal{B}_\infty^{n_y}\}$ to model the objective of Problem (3), resulting in the following optimization problem:

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \ \epsilon \ \text{ s.t. } \ \mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y}, \ \mathcal{Y} \subseteq \mathcal{Y}_\mathrm{m}(\epsilon^w) \oplus \epsilon\mathcal{B}_\infty^{n_y}. \tag{4}$$

Similarly, we formulate the *outer-approximation* setting by appending the constraint $\mathcal{Y} \subseteq \mathcal{Y}_\mathrm{m}(\epsilon^w)$ to Problem (3). We again model the objective of Problem (3) using the ∞-norm induced Hausdorff distance $d_\mathrm{H}(\mathcal{Y}_\mathrm{m}(\epsilon^w), \mathcal{Y}) = \min\{\epsilon : \mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y} \oplus \epsilon\mathcal{B}_\infty^{n_y}\}$, resulting in the following optimization problem:

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \ \epsilon \ \text{ s.t. } \ \mathcal{Y} \subseteq \mathcal{Y}_\mathrm{m}(\epsilon^w), \ \mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y} \oplus \epsilon\mathcal{B}_\infty^{n_y}. \tag{5}$$

Unfortunately, Problems (4)-(5) cannot be solved exactly since they are formulated using the mRPI set $\mathcal{X}_\mathrm{m}(\epsilon^w)$. Hence, in the rest of this paper, we develop methods to approximately solve Problems (4)-(5) by using a parametrized RPI set to approximate the mRPI set.

Before we proceed, a brief discussion regarding the convexity of Problems (4)-(5) is due. We present the discussion for Problem (4), since a similar analysis follows for Problem (5). We define the set of feasible ϵ^w as $\mathcal{O}_\mathcal{Y}(F) := \{\epsilon^w \geq \mathbf{0} : \mathcal{Y}_\mathrm{m}(\epsilon^w) \subseteq \mathcal{Y}\}$. By properties of support functions, $\epsilon^w \in \mathcal{O}_\mathcal{Y}(F)$ if and only if $\forall\ j \in \mathbb{I}_1^{m_Y}$,

$$\textstyle\sum_{t=0}^{\infty} h_{\mathbb{W}(\epsilon^w)}((G_jCA^tB)^\top) + h_{\mathbb{W}(\epsilon^w)}((G_jD)^\top) \leq g_j. \tag{6}$$

Considering two feasible vectors $\epsilon^{w,1}, \epsilon^{w,2} \in \mathcal{O}_\mathcal{Y}(F)$, a scalar $\zeta \in [0, 1]$, and the convex combination $\tilde{\epsilon}^w := \zeta\epsilon^{w,1} + (1 - \zeta)\epsilon^{w,2}$, the inclusion $\mathbb{W}(\zeta\epsilon^{w,1}) \oplus \mathbb{W}((1 - \zeta)\epsilon^{w,2}) \subseteq \mathbb{W}(\tilde{\epsilon}^w)$ holds for a general disturbance set parameterizing matrix F (from duality in Linear Programing (LP), or [22, Proposition 1]). This implies that for any $r \in \mathbb{R}^{n_w}$, $h_{\mathbb{W}(\zeta\epsilon^{w,1})}(r) + h_{\mathbb{W}((1-\zeta)\epsilon^{w,2})}(r) \leq h_{\mathbb{W}(\tilde{\epsilon}^w)}(r)$ holds, such that $\tilde{\epsilon}^w$ does not necessarily satisfy (6). Hence, $\tilde{\epsilon}^w$ does not necessarily belong to $\mathcal{O}_\mathcal{Y}(F)$, and *Problem* (4) *is in general nonconvex*. However, if the matrix F is such that $\mathbb{W}(\epsilon^w)$ is in a minimal representation for all feasible ϵ^w, Problem (4) is convex, since $\mathbb{W}(\zeta\epsilon^{w,1}) \oplus \mathbb{W}((1 - \zeta)\epsilon^{w,2}) = \mathbb{W}(\tilde{\epsilon}^w)$ then holds. For example, $F = [\tilde{F}^\top \ -\hat{F}^\top]^\top$, $\tilde{F} \in \mathbb{R}^{n_w\times n_w}$ with a well-defined $\hat{F} := (\tilde{F}\tilde{F}^\top)^{-1}\tilde{F}$ satisfies this condition ([23, Theorem 1]). However, the investigation of uniqueness of the optimizer is more involved and is a subject of future study.

In this paper, we focus on general polytopic parametrizations of $\mathbb{W}(\epsilon^w)$, for which Problems (4)-(5) are generally nonconvex. This is motivated primarily by the fact that parametrizations of $\mathbb{W}(\epsilon^w)$ that ensure convexity of Problems (4)-(5) might be excessively conservative in certain applications, e.g. decentralized MPC [14], in which the disturbance sets represent state-constraint sets of dynamically coupled subsystems. We use a polytopic RPI set parametrized with a priori fixed hyperplanes to approximate the mRPI set. The main advantage of this approach is that the RPI set complexity is fixed a priori, thus allowing for a tradeoff between computational complexity and conservativeness. The main limitation, however, is that the distance between the parametrized RPI set and the mRPI set is not quantified a priori, which implies that the suboptimality with respect to Problems (4)-(5) cannot be specified a priori. The development of methods that exploit convexity of Problems (4)-(5) under the aforementioned assumption on F, and guarantee an a priori upper bound on the optimality gap, are subjects of future research.

IV. Polytopic RPI set parametrization

In this section, we use a polytope $\mathbb{X}(\epsilon^x) := \{x : Ex \leq \epsilon^x\} \subset \mathbb{R}^{n_x}$ with $\epsilon^x \in \mathbb{R}^{m_X}$ and matrix E given a priori to approximate the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ in Problems (4)-(5) as $\mathcal{X}_{\mathrm{m}}(\epsilon^w) \subseteq \mathbb{X}(\epsilon^x)$, since $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ is included in all RPI sets corresponding to $\mathbb{W}(\epsilon^w)$. This polytopic parametrization of the RPI set with fixed hyperplanes was originally proposed in [24], and later used in [18], in the more general setting of a linear non-autonomous system controlled by a positively homogeneous state-feedback control law. As done in [19], we specialize this setting to the case of an autonomous LTI system satisfying Assumption 1 (or a non-autonomous LTI system with a stabilizing linear feedback controller). Using the parametrized RPI set $\mathbb{X}(\epsilon^x)$ to approximate $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$, we initially approximate Problems (4)-(5) as bilevel problems containing $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ in the lower-level problems. Then, we present results to eliminate $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$, and obtain equivalent single-level reformulations that can be numerically solved.

Using $\mathbb{X}(\epsilon^x)$ to approximate $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ and defining the output-set $\mathbb{Y}(\epsilon^x, \epsilon^w) := C\mathbb{X}(\epsilon^x) \oplus D\mathbb{W}(\epsilon^w)$, we approximate Problem (3) as

$$\min_{\epsilon^w \geq \mathbf{0}} \; d_{\mathrm{H}}(\mathbb{Y}(\epsilon^x, \epsilon^w), \mathcal{Y}) \tag{7a}$$

$$\begin{aligned} \text{s.t. } \epsilon^x = & \arg\min_{\underline{\epsilon}^x} \; d_{\mathrm{H}}(\mathbb{X}(\underline{\epsilon}^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w)) \\ & \text{s.t. } \; A\mathbb{X}(\underline{\epsilon}^x) \oplus B\mathbb{W}(\epsilon^w) \subseteq \mathbb{X}(\underline{\epsilon}^x), \end{aligned} \tag{7b}$$

where (7b) enforces the RPI inclusion $A\mathbb{X}(\epsilon^x) \oplus B\mathbb{W}(\epsilon^w) \subseteq \mathbb{X}(\epsilon^x)$, and $\mathbb{Y}(\epsilon^x, \epsilon^w) \supseteq \mathcal{Y}_{\mathrm{m}}(\epsilon^w)$ since $\mathbb{X}(\epsilon^x) \supseteq \mathcal{X}_{\mathrm{m}}(\epsilon^w)$. This problem is formulated using the following rationale: for any $\epsilon^w \geq \mathbf{0}$, $d_{\mathrm{H}}(\mathcal{Y}_{\mathrm{m}}(\epsilon^w), \mathcal{Y}) \leq d_{\mathrm{H}}(\mathcal{Y}_{\mathrm{m}}(\epsilon^w), \mathbb{Y}(\epsilon^x, \epsilon^w)) + d_{\mathrm{H}}(\mathbb{Y}(\epsilon^x, \epsilon^w), \mathcal{Y})$ follows from the triangle inequality, the second part of which is minimized in (7a). For the first part, let $d_{\mathrm{H}}^{\mathrm{x}} := d_{\mathrm{H}}(\mathbb{X}(\epsilon^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w))$, such that $\mathbb{X}(\epsilon^x) \subseteq \mathcal{X}_{\mathrm{m}}(\epsilon^w) \oplus d_{\mathrm{H}}^{\mathrm{x}} \mathcal{B}_p^{n_x}$. Then, $\mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y}_{\mathrm{m}}(\epsilon^w) \oplus d_{\mathrm{H}}^{\mathrm{x}} C\mathcal{B}_p^{n_x}$ follows, which by definition of the Hausdorff distance implies $d_{\mathrm{H}}(\mathbb{Y}(\epsilon^x, \epsilon^w), \mathcal{Y}_{\mathrm{m}}(\epsilon^w)) \leq d_{\mathrm{H}}^{\mathrm{x}} ||h_{C\mathcal{B}_p^{n_x}}(r)||_\infty, \forall\, r \in \mathcal{B}_p^{n_y}$. Since (7b) minimizes $d_{\mathrm{H}}^{\mathrm{x}}$, (7) minimizes an upper bound to (3) as

$$d_{\mathrm{H}}(\mathcal{Y}_{\mathrm{m}}(\epsilon^w), \mathcal{Y}) \leq d_{\mathrm{H}}^{\mathrm{x}} ||h_{C\mathcal{B}_\infty^{n_x}}(r)||_\infty + d_{\mathrm{H}}(\mathbb{Y}(\epsilon^x, \epsilon^w), \mathcal{Y}). \tag{8}$$

From this bound, we see that the suboptimality introduced due to RPI set parametrization is reduced if tight RPI approximations of the mRPI set that result in small $d_{\mathrm{H}}^{\mathrm{x}}$ are considered. However, we do not explicitly characterize the optimality gap between Problem (7) and Problem (3). We remark that this is a standard drawback of many state-of-the-art RPI set computation approaches formulated using an optimization framework, in which an explicit computation of a parametrized polytopic RPI set is sought, e.g., [25].

We will now formulate the inner and outer approximation settings of Problem (7), i.e., approximate Problems (4)-(5) using $\mathbb{Y}(\epsilon^x, \epsilon^w)$ instead of $\mathcal{Y}_{\mathrm{m}}(\epsilon^w)$. For the inner-approximation problem, we append the constraint $\mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y}$ to enforce the inclusion $\mathcal{Y}_{\mathrm{m}}(\epsilon^w) \subseteq \mathcal{Y}$, leading to the following approximation of Problem (4):

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \; \epsilon \quad \text{s.t.} \quad \text{(7b)}, \; \mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y}, \; \mathcal{Y} \subseteq \mathbb{Y}(\epsilon^x, \epsilon^w) \oplus \epsilon \mathcal{B}_\infty^{n_y}. \tag{9}$$

For the outer-approximation problem, for some user-defined integer $\bar{s} > 0$, we define $\mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w) := \oplus_{t=0}^{\bar{s}-1} CA^t B\mathbb{W}(\epsilon^w) \oplus D\mathbb{W}(\epsilon^w)$, and append the constraint $\mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w)$. Then, the desired inclusion $\mathcal{Y} \subseteq \mathcal{Y}_{\mathrm{m}}(\epsilon^w)$ holds since $\mathcal{Y}(\bar{s}, \epsilon^w) \subseteq \mathcal{Y}_{\mathrm{m}}(\epsilon^w)$ for any $\bar{s} > 0$. Using this set, we approximate Problem (5) as

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \; \epsilon \quad \text{s.t.} \quad \text{(7b)}, \; \mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w), \; \mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y} \oplus \epsilon \mathcal{B}_\infty^{n_y}. \tag{10}$$

In this section, we develop methods to solve Problems (9)-(10). We will first derive the requirements that matrix E must satisfy such that $\mathbb{X}(\epsilon^x)$ is an RPI set. Then, we show that the constraint (7b) involving the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ can be replaced by an equivalent constraint formulated directly in terms of ϵ^x and ϵ^w. Finally, we describe a numerical optimization method to solve Problems (9)-(10).

1) Existence conditions for a polytopic RPI set: The set $\mathbb{X}(\epsilon^x)$ is RPI for a given disturbance set $\mathbb{W}(\epsilon^w)$ if and only if it satisfies the inclusion $A\mathbb{X}(\epsilon^x) \oplus B\mathbb{W}(\epsilon^w) \subseteq \mathbb{X}(\epsilon^x)$. However, since ϵ^w is an optimization variable in Problems (9)-(10), we must ensure that there always exists some ϵ^x satisfying the inclusion for every $\epsilon^w \geq \mathbf{0}$. We pose this condition as requirements on matrix E to ensure that the constraint-set of Problem (7b) is nonempty for every $\epsilon^w \geq \mathbf{0}$. To this end, we define the support functions

$$\forall\; i \in \mathbb{I}_1^{m_X}, \begin{cases} \boldsymbol{c}_i(\epsilon^x) := h_{A\mathbb{X}(\epsilon^x)}(E_i^\top), \; \boldsymbol{d}_i(\epsilon^w) := h_{B\mathbb{W}(\epsilon^w)}(E_i^\top), \\ \boldsymbol{b}_i(\epsilon^x) := h_{\mathbb{X}(\epsilon^x)}(E_i^\top), \end{cases}$$

such that $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) \leq \boldsymbol{b}(\epsilon^x)$ is equivalent to the RPI inclusion. Then, we define the set of all ϵ^x satisfying the RPI condition for a given ϵ^w as $\mathcal{E}(\boldsymbol{d}(\epsilon^w)) := \{\epsilon^x \geq \mathbf{0} : \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) \leq \boldsymbol{b}(\epsilon^x)\}$. Using this set, we write Problem (7b) equivalently as

$$\epsilon^x = \arg\min_{\underline{\epsilon}^x} \; d_{\mathrm{H}}(\mathbb{X}(\underline{\epsilon}^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w)) \quad \text{s.t.} \quad \underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d}(\epsilon^w)). \tag{11}$$

We now formulate the requirements that matrix E must satisfy to ensure that $\mathcal{E}(\boldsymbol{d}(\epsilon^w))$ is nonempty for every $\epsilon^w \geq \mathbf{0}$.

Assumption 2: Matrix E is chosen such $\boldsymbol{b}(\mathbf{1}) = \mathbf{1}$, and there exists an $\hat{\epsilon}^x \geq \mathbf{0}$ satisfying the inequality $\boldsymbol{c}(\hat{\epsilon}^x) + \mathbf{1} \leq \boldsymbol{b}(\hat{\epsilon}^x)$. □

Assumption 2 implies that there exists an RPI set $\mathbb{X}(\hat{\epsilon}^x)$ for the system $x(t+1) = Ax(t) + \tilde{w}(t)$ with $\tilde{w} \in \mathbb{X}(\mathbf{1})$. In the following result, we show that there always exists an RPI set $\mathbb{X}(\epsilon^x)$ for system (1a) with the disturbance set $\mathbb{W}(\epsilon^w)$ under Assumption 2.

Proposition 1: If Assumption 2 holds, then there always exists an $\epsilon^x \geq \mathbf{0}$ satisfying $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) \leq \boldsymbol{b}(\epsilon^x)$ for all $\epsilon^w \geq \mathbf{0}$. □

Proof: Under Assumption 2, there exist nonnegative multipliers $\hat{\Lambda}_{\mathbf{c}}, \hat{\Lambda}_{\mathbf{b}} \in \mathbb{R}^{m_X \times m_X}$ satisfying $\hat{\Lambda}_{\mathbf{c}}^\top \hat{\epsilon}_x + \mathbf{1} \leq \hat{\Lambda}_{\mathbf{b}}^\top \hat{\epsilon}_x$, $\hat{\Lambda}_{\mathbf{c}}^\top E = EA$, $\hat{\Lambda}_{\mathbf{b}}^\top E = E$ by LP duality and Farkas' lemma [1]. There exists an $\epsilon^x \geq \mathbf{0}$ satisfying $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) \leq \boldsymbol{b}(\epsilon^x)$ for any $\epsilon^w \geq \mathbf{0}$ if and only if there exist nonnegative multipliers $\Lambda_{\mathbf{c}}, \Lambda_{\mathbf{b}} \in \mathbb{R}^{m_X \times m_X}$ satisfying $\Lambda_{\mathbf{c}}^\top \epsilon_x + \boldsymbol{d}(\epsilon^w) \leq \Lambda_{\mathbf{b}}^\top \epsilon_x$, $\Lambda_{\mathbf{c}}^\top E = EA$, $\Lambda_{\mathbf{b}}^\top E = E$. The proof is concluded by noting that $\epsilon^x = ||\boldsymbol{d}(\epsilon^w)||_\infty \hat{\epsilon}^x$, $\Lambda_{\mathbf{c}} = \hat{\Lambda}_{\mathbf{c}}$, and $\Lambda_{\mathbf{b}} = \hat{\Lambda}_{\mathbf{b}}$ satisfy these conditions. ■

Remark 1: Assumption 2 can be verified by checking the boundedness of LP (8) in [19]. An iterative procedure to obtain a matrix E that verifies Assumption 2 was presented in [26]. □

Remark 2: The choice of polytopic parametrizations with fixed hyperplanes for the disturbance and RPI sets is motivated primarily by their computational convenience. In particular, the choice of matrix

F is completely independent of system (1), while matrix E must satisfy Assumption 2, which depends on system (1). Conservativeness introduced by this parametrization can potentially be reduced by also optimizing over the hyperplane matrices E and F. The results presented in the rest of this paper hold in the presence of these additional variables. Alternative RPI parameterizations such as zonotopic sets [1], [27] can also be considered. However, efficient methods to embed them in an optimization framework requires further research. □

2) Elimination of $\mathcal{X}_{\mathrm{m}}(\epsilon^{\mathrm{w}})$ from (11): Having established that $\mathcal{E}(\boldsymbol{d}(\epsilon^w))$ is nonempty for any $\epsilon^w \geq \mathbf{0}$ under Assumption 2, we will now eliminate the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ from Problem (11). To this end, we recall the following results from [18] (specialized to the case of an autonomous stable LTI system), which state that the solution of Problem (11) can be obtained using fixed-point iterations for a given $\epsilon^w \geq \mathbf{0}$. We denote $\boldsymbol{d}(\epsilon^w)$ by $\boldsymbol{d}$ for ease of notation.

Lemma 1: [18, Theorems 1 and 2, Corollary 1] Suppose Assumption 2 holds, and $\mathcal{H}(\boldsymbol{d}) := \{\epsilon^x : \mathbf{0} \leq \epsilon^x \leq \|\boldsymbol{d}\|_\infty \hat{\epsilon}^x\}$. Then, for any $\epsilon^w \geq \mathbf{0}$, the following results hold: (1) For all $\underline{\epsilon}^x \in \mathcal{H}(\boldsymbol{d})$, it holds that $\boldsymbol{c}(\underline{\epsilon}^x) + \boldsymbol{d} \in \mathcal{H}(\boldsymbol{d})$, and there exists atleast one solution $\epsilon^x_*(\boldsymbol{d}) \in \mathcal{H}(\boldsymbol{d})$ for the fixed-point equations $\boldsymbol{c}(\epsilon^x_*(\boldsymbol{d}))+\boldsymbol{d} = \boldsymbol{b}(\epsilon^x_*(\boldsymbol{d}))$ and $\boldsymbol{b}(\epsilon^x_*(\boldsymbol{d})) = \epsilon^x_*(\boldsymbol{d})$. Hence, the set of all fixed-point solutions $\mathcal{R}(\boldsymbol{d}) := \{\epsilon^x \in \mathcal{H}(\boldsymbol{d}) : \boldsymbol{c}(\epsilon^x) + \boldsymbol{d} = \boldsymbol{b}(\epsilon^x), \ \boldsymbol{b}(\epsilon^x) = \epsilon^x\}$ is nonempty; (2) Starting from the initial-condition $\epsilon^x_{[0]} = \mathbf{0}$, the sequence generated by the iterations $\epsilon^x_{[k+1]} := \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}$ converges to a fixed-point solution $\lim_{k\to\infty} \epsilon^x_{[k]} := \epsilon^x_*(\mathbf{0}, \boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$. Moreover, $\epsilon^x_*(\mathbf{0}, \boldsymbol{d})$ is the minimal fixed-point, i.e., $\epsilon^x_*(\mathbf{0}, \boldsymbol{d}) \leq \underline{\epsilon}^x$ for all $\underline{\epsilon}^x \in \mathcal{R}(\boldsymbol{d}) \subseteq \mathcal{E}(\boldsymbol{d})$. Consequently, the set $\mathbb{X}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d}))$ satisfies satisfies $\mathcal{X}_{\mathrm{m}}(\epsilon^w) \subseteq \mathbb{X}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d})) = \bigcap_{\underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d})} \mathbb{X}(\underline{\epsilon}^x)$, and hence is the minimal parametrized RPI set. □

From Lemma 1.2, we see that $\epsilon^x_*(\mathbf{0}, \boldsymbol{d}(\epsilon^w))$ is the solution of the Problem (11), since the RPI set $\mathbb{X}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d}(\epsilon^w)))$ satisfies the inequality $d_{\mathrm{H}}(\mathbb{X}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d}(\epsilon^w))), \mathcal{X}_{\mathrm{m}}(\epsilon^w)) \leq d_{\mathrm{H}}(\mathbb{X}(\underline{\epsilon}^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w))$ over all $\underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d}(\epsilon^w))$. Since this solution also satisfies $\epsilon^x_*(\mathbf{0}, \boldsymbol{d}(\epsilon^w)) \leq \underline{\epsilon}^x$ over all $\underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d}(\epsilon^w))$, it has the smallest 1-norm value over all $\underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d}(\epsilon^w))$. Hence, we write Problem (11) equivalently as

$$\epsilon^x = \arg\min_{\underline{\epsilon}^x} \ \left\|\underline{\epsilon}^x\right\|_1 \ \text{ s.t. } \underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d}(\epsilon^w)). \tag{12}$$

Thus, in the rest of this section, we tackle Problems (9)-(10) formulated with constraint (7b) replaced by (12) that is independent of the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$. This results in Problem (9) being equivalent to

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \ \epsilon \ \text{ s.t. } \ (12), \ \mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y}, \ \mathcal{Y} \subseteq \mathbb{Y}(\epsilon^x, \epsilon^w) \oplus \epsilon \mathcal{B}^{n_y}_\infty, \tag{13}$$

and Problem (10) being equivalent to

$$\min_{\epsilon, \epsilon^w \geq \mathbf{0}} \ \epsilon \ \text{ s.t. } \ (12), \ \mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w), \ \mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y} \oplus \epsilon \mathcal{B}^{n_y}_\infty. \tag{14}$$

In the sequel, we transform Problems (13)-(14) into implementable forms under the following feasibility assumptions on output-set $\mathcal{Y}$.

Assumption 3: (*Inner*): The origin belongs to the output-set, i.e., $\{\mathbf{0}\} \in \mathcal{Y} = \{y : Gy \leq g\}$; (*Outer*): $\mathcal{Y}$ belongs to the output controllable subspace, i.e., $\mathcal{Y} \subset \mathrm{Im}([CB \ CAB \ \cdots CA^{n_x - 1}B \ D])$. □

Under Assumption 3-*Inner*, $g \geq \mathbf{0}$, and $(\epsilon^x, \epsilon^w) = \mathbf{0}$ are feasible solutions of Problem (13). Under Assumption 3-*Outer*, all $y \in \mathcal{Y}$ can be reached from the origin with inputs $w \in \mathbb{W}(\epsilon^w)$. Then, Problem (14) is feasible for all $\bar{s} \geq n_x$. Moreover, if $\mathrm{rank}([CB \ CAB \ \cdots CA^{n_x - 1}B \ D]) = n_y$, then (14) is feasible for every $\mathcal{Y} \neq \emptyset$ and $\bar{s} \geq n_x$ since system (1) is then output-controllable.

We now present a brief discussion on the suboptimality of Problems (13)-(14) with respect to Problems (4)-(5), resulting from approximating the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ by the RPI set $\mathbb{X}(\epsilon^x)$ solving (12). We note from (8) that $d^{\mathrm{x}}_{\mathrm{H}} = d_{\mathrm{H}}(\mathbb{X}(\epsilon^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w))$ characterizes this suboptimality. Then, we recall from [21] that $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ is the limit (in Hausdorff distance) of the sets $\mathcal{X}_k := \oplus_{t=0}^k A^t B \mathbb{W}(\epsilon^w)$ as $k \to \infty$. Similarly, from Lemma 1.2 we know that $\mathbb{X}(\epsilon^x)$ is the limit of the sets $\mathbb{X}(\epsilon^x_{[k]})$ obtained using the iteration $\epsilon^x_{[k+1]} = \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}(\epsilon^w)$ and $\epsilon^x_{[0]} = \mathbf{0}$. Hence, $d^{\mathrm{x}}_{\mathrm{H}}$ is the limit of the distance between the sets $\mathcal{X}_k$ and $\mathbb{X}(\epsilon^x_{[k]})$. In order to examine the distance between $\mathcal{X}_k$ and $\mathbb{X}(\epsilon^x_{[k]})$, we note that $\mathcal{X}_k \subseteq \mathbb{X}(\epsilon^x_{[k]})$ for all $k \geq 0$: The inclusion $B\mathbb{W}(\epsilon^w) \subseteq \mathbb{X}(\boldsymbol{d}(\epsilon^w))$ holds by definition of $\boldsymbol{d}(\cdot)$, which implies $\mathcal{X}_0 \subseteq \mathbb{X}(\epsilon^x_{[0]})$. Similarly, for any $\epsilon^x \geq 0$, the inclusion $A\mathbb{X}(\epsilon^x) \subseteq \mathbb{X}(\boldsymbol{c}(\epsilon^x))$ holds by definition of $\boldsymbol{c}(\cdot)$. Finally, for any $k > 0$, $\mathcal{X}_{k+1} = A\mathcal{X}_k \oplus B\mathbb{W}(\epsilon^w)$ follows from basic properties of Minkowski sums. Then, if $\mathcal{X}_k \subseteq \mathbb{X}(\epsilon^x_{[k]})$ holds for some $k > 0$,

$$\begin{aligned}\mathcal{X}_{k+1} = A\mathcal{X}_k \oplus B\mathbb{W}(\epsilon^w) &\subseteq A\mathbb{X}(\epsilon^x_{[k]}) \oplus B\mathbb{W}(\epsilon^w) \\ &\subseteq \mathbb{X}(\boldsymbol{c}(\epsilon^x_{[k]})) \oplus \mathbb{X}(\boldsymbol{d}(\epsilon^w)) = \mathbb{X}(\epsilon^x_{[k+1]})\end{aligned} \tag{15}$$

follows (The last equality holds since $\boldsymbol{b}(\boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}(\epsilon^w)) = \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}(\epsilon^w)$ from [18, Proposition 1]). By induction, this implies $\mathbb{X}(\epsilon^x_{[k]})$ is an outer-approximation of the k-step reachable set $\mathcal{X}_k$ at every $k \geq 0$. The error in this approximation accumulates over the iterations k, as seen in the second inclusion in (15). This accumulation is referred to as the *wrapping effect* [28], and a reduction in this effect can be obtained by a selecting a matrix E that ensures that the distance between $A\mathbb{X}(\epsilon^x)$ and $\mathbb{X}(\boldsymbol{c}(\epsilon^x))$, and $B\mathbb{W}(\epsilon^w)$ and $\mathbb{X}(\boldsymbol{d}(\epsilon^w))$ is not too large for all reachable $\epsilon^x \geq \mathbf{0}$ and permitted $\epsilon^w \geq \mathbf{0}$. Derivation of an upper-bound to this error as a function of E is a subject of future research. We note, however, that an a posteriori upper-bound of $d^{\mathrm{x}}_{\mathrm{H}}$ can be computed by using μ-RPI approximations $\mathcal{X}_\mu(\epsilon^w)$ of the mRPI set, i.e, an RPI set satisfying $d_{\mathrm{H}}(\mathcal{X}_\mu(\epsilon^w), \mathcal{X}_{\mathrm{m}}(\epsilon^w)) \leq \mu$, for a given ϵ^w. Such sets can be computed for arbitrarily small $\mu > 0$ using the methods in [29], and the upper-bound can be derived using the triangle inequality as $d^{\mathrm{x}}_{\mathrm{H}} \leq d_{\mathrm{H}}(\mathbb{X}(\epsilon^x), \mathcal{X}_\mu(\epsilon^w)) + \mu$.

3) Characterization of RPI Constraints: In this subsection, we show that the minimal parametrized RPI constraint (12) in Problems (13)-(14) can be replaced by the equality $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) = \epsilon^x$, i.e.,

$$(12) \iff \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) = \epsilon^x \tag{16}$$

holds, thus obtaining equivalent single-level reformulations of Problems (13)-(14). For simplicity, we denote $\boldsymbol{d}(\epsilon^w)$ by $\boldsymbol{d}$ in the sequel, since the results are presented for a fixed $\epsilon^w \geq \mathbf{0}$. We recall that the fixed-point solution $\epsilon^x_*(\mathbf{0}, \boldsymbol{d}) = \arg\min_{\epsilon^x} \{\|\underline{\epsilon}^x\|_1 \ \text{ s.t. } \underline{\epsilon}^x \in \mathcal{E}(\boldsymbol{d})\}$ exists, and satisfies $\boldsymbol{c}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d})) + \boldsymbol{d} = \boldsymbol{b}(\epsilon^x_*(\mathbf{0}, \boldsymbol{d})) = \epsilon^x_*(\mathbf{0}, \boldsymbol{d})$ from Lemma 1. We recall further that $\mathcal{R}(\boldsymbol{d}) \subseteq \mathcal{E}(\boldsymbol{d})$ is the set of all fixed-points, i.e., all $\epsilon^x \in \mathcal{E}(\boldsymbol{d})$ that satisfy $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d} = \boldsymbol{b}(\epsilon^x) = \epsilon^x$. Then, if there exists a unique fixed-point $\epsilon^x_\#(\boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$, we have $\epsilon^x_*(\mathbf{0}, \boldsymbol{d}) = \epsilon^x_\#(\boldsymbol{d})$. Moreover, since $\boldsymbol{b}(\boldsymbol{c}(\epsilon^x) + \boldsymbol{d}) = \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}$ for every $\epsilon^x \in \mathcal{E}(\boldsymbol{d})$ from [18, Proposition 1], every $\epsilon^x \in \mathcal{E}(\boldsymbol{d})$ that satisfies $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d} = \epsilon^x$ satisfies $\boldsymbol{b}(\epsilon^x) = \epsilon^x$. Hence, the existence of a unique fixed-point $\epsilon^x_\#(\boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$ implies that we can replace constraint (12) by $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d} = \epsilon^x$. In the following result, uniqueness of $\epsilon^x_\#(\boldsymbol{d})$ was shown under a slightly more restrictive assumption.

Lemma 2: [19, Theorem 3] Suppose Assumption 2 holds and $\boldsymbol{d} > \mathbf{0}$, then there exists a unique fixed-point $\epsilon^x_\#(\boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$. □

We now present a brief discussion regarding the restrictions imposed by the assumption $\boldsymbol{d} > \mathbf{0}$: recalling the definition of the support function $\boldsymbol{d}_i = \max_w \{E_i B w \text{ s.t. } Fw \leq \epsilon^w\}$, we see that $\boldsymbol{d}_i > 0$ for all $i \in \mathbb{I}_1^{m_X}$ only if $E_i B \neq \mathbf{0}$ for each $i \in \mathbb{I}_1^{m_X}$ and $\epsilon^w > \mathbf{0}$. While the condition $\epsilon^w > \mathbf{0}$ can be enforced easily through a linear constraint in Problems (13)-(14), the former condition holds only if the additional assumption $E_i^\top \notin \mathrm{null}(B^\top)$ (or the stronger assumption $\mathrm{rank}(B) = n_x$) is satisfied: these assumptions restrict the class of systems and RPI set parametrizations that are often encountered. Moreover, they lead to excessively conservative RPI set parametrizations. For example, an uncontrollable system would

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require an RPI set that always includes the origin within its interior. We prove next that there exists a unique fixed-point $\epsilon^x_{\#}(\boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$ if $\boldsymbol{d} \geq \mathbf{0}$, thus overcoming the limitations with $\boldsymbol{d} > \mathbf{0}$. We first characterize the fixed-points using the following LP, similarly to [19]:

$$\max_{\boldsymbol{c},\mathbf{x}:=\{\mathbf{x}^i, i\in\mathbb{I}_1^{m_X}\}} \quad \textstyle\sum_{i=1}^{m_X} \boldsymbol{c}_i \tag{17a}$$

$$\text{s.t.} \quad \boldsymbol{c}_i - E_i A\mathbf{x}^i \leq 0, \quad i \in \mathbb{I}_1^{m_X}, \tag{17b}$$

$$E\mathbf{x}^i \leq \boldsymbol{c} + \boldsymbol{d}, \qquad i \in \mathbb{I}_1^{m_X}, \tag{17c}$$

and we denote the set of all optimizers $(\boldsymbol{c}^*, \mathbf{x}^*)$ of LP (17) as $\mathcal{S}$.

Proposition 2: Suppose Assumption 2 holds. Then if $\bar{\epsilon}^x \in \mathcal{R}(\boldsymbol{d})$ there exists a $(\bar{\boldsymbol{c}}, \bar{\mathbf{x}}) \in \mathcal{S}$ such that $\bar{\boldsymbol{c}}_i = E_i A\bar{\mathbf{x}}^i$ and $\bar{\epsilon}^x = \bar{\boldsymbol{c}} + \boldsymbol{d}$. □

Proof: If Assumption 2 holds, Lemma 1.1 entails that $\mathcal{R}(\boldsymbol{d})$ is nonempty for every $\boldsymbol{d} \geq \mathbf{0}$. At every fixed-point solution $\bar{\epsilon}^x \in \mathcal{R}(\boldsymbol{d})$, $\bar{\epsilon}^x = \boldsymbol{c}(\bar{\epsilon}^x) + \boldsymbol{d}$ holds. Define $\bar{\mathbf{x}}^i := \arg\max_{\mathbf{x}^i}\{E_i A\mathbf{x}^i \text{ s.t. } E\mathbf{x}^i \leq \bar{\epsilon}^x\}$ and $\bar{\boldsymbol{c}}_i := E_i A\bar{\mathbf{x}}^i$; by definition of $\boldsymbol{c}(\cdot)$ we have $\bar{\epsilon}^x_i = \bar{\boldsymbol{c}}_i + \boldsymbol{d}_i$ for each $i \in \mathbb{I}_1^{m_X}$. We combine the LPs defining $\bar{\mathbf{x}}^i$ into a single LP by defining $\bar{\mathbf{x}} := \{\bar{\mathbf{x}}^i, i \in \mathbb{I}_1^{m_X}\}$, and adopting an epigraph form [30] by introducing variables $\boldsymbol{c}_i$ to obtain

$$\max_{\boldsymbol{c},\mathbf{x}} \textstyle\sum_{i=1}^{m_X} \mathbf{c}_i \quad \text{s.t.} \quad \mathbf{c}_i \leq E_i A\mathbf{x}^i, \; E\mathbf{x}^i \leq \bar{\boldsymbol{c}} + \boldsymbol{d}, \; \forall \; i \in \mathbb{I}_1^{m_X}, \tag{18}$$

in which we write $\bar{\epsilon}^x = \bar{\boldsymbol{c}} + \boldsymbol{d}$. Since $(\boldsymbol{c}, \mathbf{x}) = (\bar{\boldsymbol{c}}, \bar{\mathbf{x}})$ is feasible for LP (18), and the optimal value is $\sum_{i=1}^{m_X} \bar{\boldsymbol{c}}_i$, we can replace $\bar{\mathbf{c}}$ by $\mathbf{c}$ to obtain LP (17), and $(\bar{\boldsymbol{c}}, \bar{\mathbf{x}})$ will be one of the optimizers. ■

Proposition 2 entails that every fixed-point $\bar{\epsilon}^x \in \mathcal{R}(\boldsymbol{d})$ can be expressed as $\bar{\epsilon}^x = \bar{\boldsymbol{c}} + \boldsymbol{d}$ for some $\bar{\boldsymbol{c}} \in \Pi_{\boldsymbol{c}^*}\mathcal{S}$ (Note that, for now, $\Pi_{\boldsymbol{c}^*}\mathcal{S}$ need not be singleton). In Theorem 1, we exploit this property to show that the fixed-point is unique. To this end, we first present the following general result that we use later to establish uniqueness.

Lemma 3: Let $M \in \mathbb{R}^{p\times p}$ be a matrix with $M_{ij} \geq 0, \forall\, i, j \in \mathbb{I}_1^p$, and $N := M(\mathbf{I}+\text{diag}(M\mathbf{1}))^{-1}$, i.e., $N_{ij} = M_{ij}/(1+\sum_{k=1}^p M_{jk})$. Then, (*a*) $Z := \mathbf{I} - N^\top$ is invertible; (*b*) $\rho(N^\top) < 1$. □

Proof: (*a*) Matrix Z is invertible if and only if $Z^\top$ is invertible. Suppose there exists some $q \in \mathbb{R}^p$ satisfying

$$Nq + \mathbf{1} \leq q, \qquad\qquad q \geq \mathbf{0}, \tag{19}$$

such that $Nq < q$ holds. Then, $(\mathbf{I} - N)q > \mathbf{0}$ and $q \geq \mathbf{0}$ follow, which, by [31, Theorems 4.1,4.6], implies that $\mathbf{I} - N$ is invertible (Since $N_{ii} \leq 1$, by construction $Z_{ii} \geq 0$, $Z_{ij} \leq 0, \forall\, i, j \in \mathbb{I}_1^p$ such that Z is a $\mathcal{Z}$-matrix [31, Definition 1]). We show next that indeed $\exists\, q \in \mathbb{R}^p$ satisfying (19). We introduce a slack variable $s \in \mathbb{R}^p$ in the LP formulation, and write (19) equivalently as

$$[N - \mathbf{I} \;\; \mathbf{I}][q^\top \;\; s^\top]^\top = -\mathbf{1}, \qquad [q^\top \;\; s^\top]^\top \geq \mathbf{0}. \tag{20}$$

By Farkas' lemma [32, Corollary 7.1d], there exist $[q^\top \; s^\top]^\top$ satisfying (20) if and only if for every $\zeta \in \mathcal{T} := \{\zeta : N^\top\zeta \geq \zeta, \zeta \geq \mathbf{0}\}$, the inequality $\zeta^\top\mathbf{1} \leq \mathbf{0}$ holds. Since $\zeta \geq \mathbf{0}$ for every $\zeta \in \mathcal{T}$, $\zeta^\top\mathbf{1} \leq \mathbf{0}$ holds if and only if $\zeta = \mathbf{0}$, i.e., $\mathcal{T} = \{\mathbf{0}\}$. To show $\mathcal{T} = \{\mathbf{0}\}$, we rewrite $N^\top\zeta \geq \zeta$ as $(\mathbf{I} + \text{diag}(M\mathbf{1}))^{-\top}M^\top\zeta \geq \zeta$ (using the definition of N), and multiply both sides by the positive diagonal matrix $(\mathbf{I} + \text{diag}(M\mathbf{1}))$ to obtain

$$M^\top\zeta \geq \zeta + \text{diag}(M\mathbf{1})\zeta = \begin{Bmatrix} \sum_{i=1}^p M_{i1}\zeta_i \geq \zeta_1 + \sum_{k=1}^p M_{1k}\zeta_1, \\ \vdots \\ \sum_{i=1}^p M_{ip}\zeta_i \geq \zeta_p + \sum_{k=1}^p M_{pk}\zeta_p \end{Bmatrix}.$$

We further manipulate these inequalities as

$$\begin{aligned} &\textstyle\sum_{i=1}^p M_{i1}\zeta_i \geq \zeta_1 + M_{11}\zeta_1 + M_{12}\zeta_1 + \cdots + M_{1p}\zeta_1 && \to \text{Row 1} \\ &M_{12}\zeta_1 \geq \zeta_2 + \textstyle\sum_{k=1}^p M_{2k}\zeta_2 - \sum_{i=2}^p M_{i2}\zeta_i && \to \text{Row 2} \\ &\vdots \\ &M_{1p}\zeta_1 \geq \zeta_p + \textstyle\sum_{k=1}^p M_{pk}\zeta_p - \sum_{i=2}^p M_{ip}\zeta_i && \to \text{Row } p \end{aligned}$$

Substituting Rows 2-p in Row 1 to replace $M_{1i}\zeta_1$ terms, we obtain

$$\sum_{i=1}^p M_{i1}\zeta_i \geq \sum_{l=1}^p \zeta_l + \sum_{j=1}^p M_{j1}\zeta_j + \sum_{j=2}^p\sum_{k=2}^p M_{jk}\zeta_j - \sum_{j=2}^p\sum_{i=2}^p M_{ij}\zeta_i,$$

which, after elementary operations, yields $\sum_{l=1}^p \zeta_l \leq 0$. Hence, the set $\mathcal{T} = \{\zeta : \sum_{l=1}^p \zeta_l \leq 0, \zeta \geq \mathbf{0}\} = \{\mathbf{0}\}$, such that $\zeta^\top\mathbf{1} \leq \mathbf{0}$ for all $\zeta \in \mathcal{T}$. Thus, $\exists\, q \in \mathbb{R}^p$ satisfying (19), concluding the proof of the first claim. (*b*) Since $(\mathbf{I} - N^\top)^{-1} = \sum_{k=0}^\infty (N^\top)^k$ is well-defined, it implies $\lim_{k\to\infty}(N^\top)^k = \mathbf{0}$, or, equivalently, $\rho(N^\top) < 1$. ■

Theorem 1: Suppose that Assumption 2 holds and $\boldsymbol{d} \geq \mathbf{0}$, then there exists a unique fixed-point $\epsilon^x_{\#}(\boldsymbol{d}) \in \mathcal{R}(\boldsymbol{d})$. □

Proof: By Assumption 2, Lemma 1 entails $\mathcal{R}(\boldsymbol{d}) \neq \emptyset$, and the fixed-point $\bar{\epsilon}^x = \epsilon^x_*(\mathbf{0}, \boldsymbol{d})$ reached from $\epsilon^x_{[0]} = \mathbf{0}$ with the iterations $\epsilon^x_{[k+1]} = \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}$ is the minimal fixed-point, i.e.,

$$\bar{\epsilon}^x \leq \tilde{\epsilon}^x, \qquad\qquad \forall\; \tilde{\epsilon}^x \in \mathcal{R}(\boldsymbol{d}). \tag{21}$$

In order to show uniqueness of this fixed-point, we show that the iterations $\epsilon^x_{[k+1]} = \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}$ starting from any initial-condition $\epsilon^x_{[0]} \geq \bar{\epsilon}^x$ converge to $\bar{\epsilon}^x$. To this end, we observe that Proposition 2 entails that there exists some optimizer $\bar{\boldsymbol{c}} \in \Pi_{\boldsymbol{c}^*}\mathcal{S}$ of LP (17) such that $\bar{\epsilon}^x = \bar{\boldsymbol{c}} + \boldsymbol{d}$. Then, we write the dual LP of LP (17) as

$$\min_{(\boldsymbol{\lambda},\boldsymbol{\eta}):=\{\lambda_i\geq 0, \eta^i\geq\mathbf{0}_{m_X}, i\in\mathbb{I}_1^{m_X}\}} \quad \textstyle\sum_{i=1}^{m_X} {\eta^i}^\top \boldsymbol{d} \tag{22a}$$

$$\text{s.t.} \quad \lambda_i = 1 + \textstyle\sum_{j=1}^{m_X} \eta_i^j, \quad i \in \mathbb{I}_1^{m_X}, \tag{22b}$$

$${\eta^i}^\top E = \lambda_i E_i A, \qquad i \in \mathbb{I}_1^{m_X}, \tag{22c}$$

where λ_i and η^i are the dual variables associated to constraints (17b) and (17c) respectively. We denote the optimal dual variables corresponding to $\bar{\boldsymbol{c}}$ as λ_i^* and η^{i*}, and define matrix Θ^* with rows $\Theta_i^* := {\eta^{i*}}^\top/\lambda_i^*$, where $\lambda_i^* \geq 1$ by (22b). We recall that $\bar{\boldsymbol{c}} = \Theta^*(\bar{\boldsymbol{c}} + \boldsymbol{d}) = \Theta^*\bar{\epsilon}^x$, since $\bar{\boldsymbol{c}}$ optimizes LP (17) ([19, Theorem 4]). Then we apply Lemma 3 with $M = [\eta^{1*} \cdots \eta^{m_X*}]$, such that $N = \Theta^*$. Hence, $\rho(\Theta^*) < 1$ from Lemma 3(*b*). For any $\epsilon^x \in \mathcal{H}(\boldsymbol{d})$,

$$\boldsymbol{c}_i(\epsilon^x) = \begin{Bmatrix} \max_x E_i Ax \\ \text{s.t. } Ex \leq \epsilon^x \end{Bmatrix} = \begin{Bmatrix} \min_\gamma \gamma^\top\epsilon^x \\ \text{s.t. } \gamma^\top E = E_i A, \gamma \geq \mathbf{0} \end{Bmatrix} \leq \Theta_i^*\epsilon^x,$$

holds, where the second equality follows from strong duality for LPs, and the inequality follows since $\gamma^\top = \Theta_i^*$ is feasible for the dual LP. This implies $\boldsymbol{c}(\epsilon^x) \leq \Theta^*\epsilon^x$ holds for all $\epsilon^x \in \mathcal{H}(\boldsymbol{d})$. Hence, for the iterations $\epsilon^x_{[k+1]} = \boldsymbol{c}(\epsilon^x_{[k]}) + \boldsymbol{d}$ from any $\epsilon^x_{[0]} \in \mathcal{H}(\boldsymbol{d})$, we obtain $\epsilon^x_{[k+1]} \leq \Theta^*\epsilon^x_{[k]} + \boldsymbol{d}$. Subtracting by $\bar{\epsilon}^x = \Theta^*\bar{\epsilon}^x + \boldsymbol{d}$, the inequality $\epsilon^x_{[k+1]} - \bar{\epsilon}^x \leq \Theta^*(\epsilon^x_{[k]} - \bar{\epsilon}^x)$ follows. Applying recursively, the inequality $\epsilon^x_{[k]} - \bar{\epsilon}^x \leq (\Theta^*)^k(\epsilon^x_{[0]} - \bar{\epsilon}^x)$ holds. If $\bar{\epsilon}^x \leq \epsilon^x_{[0]}$, then $\bar{\epsilon}^x \leq \epsilon^x_{[k]}, \forall\; k \geq 0$ by monotonicity of $\boldsymbol{c}(\cdot)$, and definition of $\bar{\epsilon}^x$. Then, $\rho(\Theta^*) < 1$ implies $(\Theta^*)^k \to \mathbf{0}$ as $k \to \infty$, such that

$$\forall\; \delta > 0, \; \exists\; k < \infty \;:\; \epsilon^x_{[k]} - \bar{\epsilon} \leq \delta\mathbf{1}. \tag{23}$$

If the initial condition $\epsilon^x_{[0]} = \tilde{\epsilon}^x \in \mathcal{R}(\boldsymbol{d})\backslash\{\bar{\epsilon}^x\}$, then $\epsilon^x_{[k]} = \tilde{\epsilon}^x$ for all $k \geq 1$, since $\tilde{\epsilon}^x$ is a fixed-point. From (23), this implies $\tilde{\epsilon}^x \leq \bar{\epsilon}^x + \delta\mathbf{1}$ for every $\delta > 0$. From (21), we know that $\bar{\epsilon}^x \leq \tilde{\epsilon}^x$. Suppose there exist some index $i \in \mathbb{I}_1^{m_X}$ such that $\bar{\epsilon}^x_i < \tilde{\epsilon}^x_i$. Then, for every scalar $\beta \in (0, \tilde{\epsilon}^x_i - \bar{\epsilon}^x_i)$, $\tilde{\epsilon}^x \nleq \bar{\epsilon}^x + \beta\mathbf{1}$ holds, which contradicts (23) with $\epsilon^x_{[k]} = \tilde{\epsilon}^x$. Hence, $\epsilon^x_{\#}(\boldsymbol{d}) = \bar{\epsilon}^x = \tilde{\epsilon}^x$, which concludes the proof. ■

Remark 3: We note that $\rho(\Theta^*) \in [\rho(A), 1)$: Let (α, κ_α) be an eigenpair of A, such that $A\kappa_\alpha = \alpha\kappa_\alpha$. Multiplying by E, we obtain $\Theta^*(E\kappa_\alpha) = \alpha(E\kappa_\alpha)$ since $\Theta^*E = EA$ from (22b)-(22c). Hence, the eigenvalues of A are a subset of the eigenvalues of Θ^*. □

This theorem validates (16) and allows us to replace constraint (12) by the equivalent condition $\boldsymbol{c}(\epsilon^x) + \boldsymbol{d} = \epsilon^x$ in Problems (13)-(14).

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Remark 4: While we assume that $\mathbf{0} \in \mathbb{W}(\epsilon^w)$, there exist cases where this is not known a priori. Such cases can be accommodated in Problems (13)-(14) by considering the disturbance set parametrization $\{\bar{w}\} \oplus \mathbb{W}(\epsilon^w)$, where $\mathbf{0} \in \mathbb{W}(\epsilon^w)$ if $\epsilon^w \geq \mathbf{0}$, and $\bar{w}$ represents the origin offset. Then, an RPI set parametrized as $\{\bar{x}\} \oplus \mathbb{X}(\epsilon^x)$ satisfies $\{A\bar{x} + B\bar{w}\} \oplus A\mathbb{X}(\epsilon^x) \oplus B\mathbb{W}(\epsilon^w) \subseteq \{\bar{x}\} \oplus \mathbb{X}(\epsilon^x)$, or equivalently $EA\bar{x} + EB\bar{w} - E\bar{x} + \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) \leq \boldsymbol{b}(\epsilon^x)$, the first part of which can be eliminated by using the state offset $\bar{x} = (\mathbf{I} - A)^{-1}B\bar{w}$. □

4) Encoding inclusion constraints: We now encode the inclusion constraints in Problems (13)-(14), assuming that the vertices $\{\mathrm{y}_{[p]}, p \in \mathbb{I}_1^{v_\mathcal{Y}}\}$ of the output-set $\mathcal{Y} = \{y : Gy \leq g\}$ are known.

(a) Inner-approximation Problem (13): We encode the inclusion constraint $\mathcal{Y} \subseteq \mathbb{Y}(\epsilon^x, \epsilon^w) \oplus \epsilon\mathcal{B}_\infty^{n_y}$ as $(\epsilon^x, \epsilon^w, \epsilon) \in \Xi^{\mathrm{I}}$, where

$$\Xi^{\mathrm{I}} := \{(\epsilon^x, \epsilon^w, \epsilon) : \mathrm{y}_{[p]} \in \mathbb{Y}(\epsilon^x, \epsilon^w) \oplus \epsilon\mathcal{B}_\infty^{n_y}, \ \forall\, p \in \mathbb{I}_1^{v_\mathcal{Y}}\} \quad (24)$$

is a set of linear constraints. To encode $\mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y}$, we define

$$\forall\, k \in \mathbb{I}_1^{m_Y}, \left\{ \boldsymbol{l}_k(\epsilon^x) := h_{C\mathbb{X}(\epsilon^x)}(G_k^\top), \ \boldsymbol{m}_k(\epsilon^w) := h_{D\mathbb{W}(\epsilon^w)}(G_k^\top), \right.$$

using support functions, and enforce $\boldsymbol{l}(\epsilon^x) + \boldsymbol{m}(\epsilon^w) \leq g$. Hence, using the proposed inclusion encodings and the RPI set equivalence in (16), we write Problem (13) as

$$\min_{\epsilon, \epsilon^w, \epsilon^x} \left\{ \epsilon \ \text{ s.t. } \begin{array}{c} \epsilon^w \geq \mathbf{0}, \quad \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) = \epsilon^x, \\ \boldsymbol{l}(\epsilon^x) + \boldsymbol{m}(\epsilon^w) \leq g, \ (\epsilon^x, \epsilon^w, \epsilon) \in \Xi^{\mathrm{I}}. \end{array} \right\} \quad (25)$$

(b) Outer-approximation Problem(14): We encode the inclusion constraint $\mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w)$ as $\epsilon^w \in \Xi^{\mathrm{O}}$, where

$$\Xi^{\mathrm{O}} := \{\epsilon^w : \mathrm{y}_{[p]} \in \mathcal{Y}(\bar{s}, \epsilon^w), \ \forall\, p \in \mathbb{I}_1^{v_\mathcal{Y}}\} \quad (26)$$

is a set of linear constraints. In order to encode the inclusion $\mathbb{Y}(\epsilon^x, \epsilon^w) \subseteq \mathcal{Y} \oplus \epsilon\mathcal{B}_\infty^{n_y}$, we consider the set $\mathbb{B} := \{y : Hy \leq \mathbf{1}\}$ with $H \in \mathbb{R}^{m_B \times n_y}$, and the vectors $H_j^\top$ are sampled from $\mathcal{B}_\infty^{n_y}$. We then approximately encode the inclusion using support functions

$$\forall\, j \in \mathbb{I}_1^{m_B}, \begin{cases} \boldsymbol{n}_j(\epsilon^x) := h_{C\mathbb{X}(\epsilon^x)}(H_j^\top), \ \boldsymbol{p}_j(\epsilon^w) := h_{D\mathbb{W}(\epsilon^w)}(H_j^\top), \\ \boldsymbol{g}_j^{\mathrm{O}} := h_{\mathcal{Y}}(H_j^\top), \end{cases}$$

as $\boldsymbol{n}(\epsilon^x) + \boldsymbol{p}(\epsilon^w) - \epsilon\mathbf{1} \leq \boldsymbol{g}^{\mathrm{O}}$. Since this condition is only necessary for the inclusion, we compute a lower-bound to the actual Hausdorff distance. Hence, using the proposed inclusion encodings and the RPI set equivalence in (16), we write Problem (14) as

$$\min_{\epsilon, \epsilon^w, \epsilon^x} \left\{ \epsilon \ \text{ s.t. } \begin{array}{c} \epsilon^w \geq \mathbf{0}, \quad \boldsymbol{c}(\epsilon^x) + \boldsymbol{d}(\epsilon^w) = \epsilon^x, \\ \boldsymbol{n}(\epsilon^x) + \boldsymbol{p}(\epsilon^w) - \epsilon\mathbf{1} \leq \boldsymbol{g}^{\mathrm{O}}, \ \epsilon^w \in \Xi^{\mathrm{O}}. \end{array} \right\} \quad (27)$$

Remark 5: If the vertices $\{\mathrm{y}_{[p]}, p \in \mathbb{I}_1^{v_\mathcal{Y}}\}$ of the output-set $\mathcal{Y}$ are not known, then $\mathcal{Y} \subseteq \mathbb{Y}(\epsilon^x, \epsilon^w) \oplus \epsilon\mathcal{B}_\infty^{n_y}$ and $\mathcal{Y} \subseteq \mathcal{Y}(\bar{s}, \epsilon^w)$ can be encoded as a set of linear constraints directly in terms of the hyperplane notation of $\mathcal{Y}$ using results from [33, Theorem 1]. □

5) Numerical optimization: In this subsection, we adopt a penalty function approach to solve Problems (25)-(27). To this end, we first note that ϵ^w might be unbounded above in case of a nonminimal representation of $\mathbb{W}(\epsilon^w)$. We tackle this issue using the support function $\boldsymbol{q}_t(\epsilon^w) := h_{\mathbb{W}(\epsilon^w)}(F_t^\top), \ \forall\, t \in \mathbb{I}_1^{m_W}$, such that $\boldsymbol{q}(\epsilon^w) = \epsilon^w$ if and only if $\mathbb{W}(\epsilon^w)$ is in minimal representation. Then, we modify the objective of Problems (25)-(27) as

$$\epsilon + \sigma \textstyle\sum_{t=1}^{m_W} (\epsilon_t^w - \boldsymbol{q}_t(\epsilon^w)), \quad (28)$$

where $\sigma > 0$ is some scalar tuning parameter. This modification ensures that (a) the solution ϵ^w is such that $\mathbb{W}(\epsilon^w)$ is in a minimal representation; (b) the solution is not perturbed, since $\mathbb{W}(\boldsymbol{q}(\epsilon^w)) = \mathbb{W}(\epsilon^w)$. This modification is not required for ϵ^x, since the RPI constraint enforces uniqueness of ϵ^x for a given value of ϵ^w.

We use the penalty function approach from [20] to solve Problems (25)-(27) modified with objective (28). We present the approach for Problem (25), since a very similar method follows for Problem (27). Considering Problem (25) with objective (28), the LPs formulating the support functions satisfy strong duality [34] since they are feasible and bounded for every bounded $\epsilon^x \geq \mathbf{0}$ and $\epsilon^w \geq \mathbf{0}$. This property is exploited in the penalty function algorithm to compute local optima. Introducing the optimal primal and dual variables

Support function LP	$\boldsymbol{c}_i(\epsilon^x)$	$\boldsymbol{d}_i(\epsilon^w)$	$\boldsymbol{l}_k(\epsilon^x)$	$\boldsymbol{m}_k(\epsilon^w)$	$\boldsymbol{q}_t(\epsilon^w)$
Primal Variables	$\mathrm{z}^{\boldsymbol{c}_i}$	$\mathrm{z}^{\boldsymbol{d}_i}$	$\mathrm{z}^{\boldsymbol{l}_k}$	$\mathrm{z}^{\boldsymbol{m}_k}$	$\mathrm{z}^{\boldsymbol{q}_t}$
Dual Variables	$\lambda^{\boldsymbol{c}_i}$	$\lambda^{\boldsymbol{d}_i}$	$\lambda^{\boldsymbol{l}_k}$	$\lambda^{\boldsymbol{m}_k}$	$\lambda^{\boldsymbol{q}_t}$

strong duality of the LPs implies that these values are primal and dual feasible, and have a zero duality gap. For the support function $\boldsymbol{c}_i(\epsilon^x)$, these conditions are $E\mathrm{z}^{\boldsymbol{c}_i} \leq \epsilon^x$, $E^\top \lambda^{\boldsymbol{c}_i} = A^\top E_i^\top$, $\lambda^{\boldsymbol{c}_i} \geq \mathbf{0}_{m_X}$ and $\lambda^{\boldsymbol{c}_i\,\top} \epsilon^x = E_i A\mathrm{z}^{\boldsymbol{c}_i}$. Introducing these variables along with the optimality conditions, Problem (25) is reformulated to a single-level problem using a penalty function to penalize the duality gap as

$$\begin{aligned} \min_{\epsilon, \epsilon^w, \epsilon^x, \mathrm{z}^*, \lambda^*} \ & \epsilon + \sigma \textstyle\sum_{t=1}^{m_W} (\epsilon_t^w - F_t\mathrm{z}^{\boldsymbol{q}_t}) + \mathbf{K}\mathcal{P}(\epsilon^x, \epsilon^w, \mathrm{z}^*, \lambda^*) \quad (29) \\ \text{s.t. } & E_iA\mathrm{z}^{\boldsymbol{c}_i} + E_iB\mathrm{z}^{\boldsymbol{d}_i} = \epsilon_i^x, \quad G_kC\mathrm{z}^{\boldsymbol{l}_k} + G_kD\mathrm{z}^{\boldsymbol{m}_k} \leq g_k, \\ & \mathrm{z}^{\boldsymbol{c}_i}, \mathrm{z}^{\boldsymbol{l}_k} \in \mathbb{X}(\epsilon^x), \quad \mathrm{z}^{\boldsymbol{d}_i}, \mathrm{z}^{\boldsymbol{m}_k}, \mathrm{z}^{\boldsymbol{q}_t} \in \mathbb{W}(\epsilon^w), \\ & E^\top\lambda^{\boldsymbol{c}_i} = A^\top E_i^\top, \ F^\top\lambda^{\boldsymbol{d}_i} = B^\top E_i^\top, \ E^\top\lambda^{\boldsymbol{l}_k} = C^\top G_k^\top, \\ & F^\top\lambda^{\boldsymbol{m}_k} = D^\top G_k^\top, \ F^\top\lambda^{\boldsymbol{q}_t} = F_t^\top, \\ & (\epsilon^x, \epsilon^w, \epsilon) \in \Xi^{\mathrm{I}}, \ \epsilon^w \geq \mathbf{0}, \ \lambda^* \geq \mathbf{0}, \\ & \forall\, i \in \mathbb{I}_1^{m_X}, \forall\, k \in \mathbb{I}_1^{m_Y}, \forall\, t \in \mathbb{I}_1^{m_W}, \end{aligned}$$

where $z^* \in \mathbb{R}^{m_X(n_x+n_w)+m_Y(n_x+n_w)+m_Wn_w}$ denotes the optimal primal variables, $\lambda^* \in \mathbb{R}^{m_X(m_X+m_W)+m_Y(m_X+m_W)+m_W^2}$ denotes the optimal dual variables, and the penalty function $\mathcal{P}(\epsilon^x, \epsilon^w, \mathrm{z}^*, \lambda^*) := \sum_{t=1}^{m_W}(\lambda^{\boldsymbol{q}_t\top}\epsilon^w - F_t\mathrm{z}^{\boldsymbol{q}_t}) + \sum_{i=1}^{m_X}(\lambda^{\boldsymbol{c}_i\top}\epsilon^x - E_iA\mathrm{z}^{\boldsymbol{c}_i} + \lambda^{\boldsymbol{d}_i\top}\epsilon^w - E_iB\mathrm{z}^{\boldsymbol{d}_i}) + \sum_{k=1}^{m_Y}(\lambda^{\boldsymbol{l}_k\top}\epsilon^x - G_kC\mathrm{z}^{\boldsymbol{l}_k} + \lambda^{\boldsymbol{m}_k\top}\epsilon^w - G_kD\mathrm{z}^{\boldsymbol{m}_k})$ weighted by some constant $\mathbf{K} > 0$ penalizes the duality gap. We denote Problem (29) as $\mathcal{F}(\epsilon^x, \epsilon^w, \epsilon, \mathrm{z}^*, \lambda^*, \mathbf{K})$. The main idea behind the penalty function approach is that there exists a parameter $\mathbf{K}^*$ that, if Problem (29) is solved with $\mathbf{K} > \mathbf{K}^*$, then $\mathcal{P}(\epsilon^x, \epsilon^w, \mathrm{z}^*, \lambda^*) = 0$ at the solution which also solves Problem (25) with objective (28). An iterative method is proposed to solve (29), with each iteration composed of two LPs. Denoting an iteration by the subscript $\{l\}$, the first LP solved is $\mathcal{F}(\epsilon^x, \epsilon^w, \epsilon, \mathrm{z}^*, \lambda^*_{\{l-1\}}, \mathbf{K}_{\{l-1\}})$. Using the solution $(\epsilon^x_{\{l\}}, \epsilon^w_{\{l\}}, \epsilon_{\{l\}}, \mathrm{z}^*_{\{l\}})$, the next step consists of solving LP $\mathcal{F}(\epsilon^x_{\{l\}}, \epsilon^w_{\{l\}}, \epsilon_{\{l\}}, \mathrm{z}^*_{\{l\}}, \lambda^*, \mathbf{K}_{\{l-1\}})$ for $\lambda^*_{\{l\}}$. If the obtained values solve Problem (29) with zero duality gap, the algorithm is terminated. Else, it is repeated with $\mathbf{K}_{\{l\}} \geq \mathbf{K}_{\{l-1\}}$. This algorithm was shown to converge to a local solution of (25) in [20].

Remark 6: We propose to initialize the optimization algorithm using the scaling $\boldsymbol{\zeta} \geq 0$ as $\epsilon^w_{\{0\}} = \boldsymbol{\zeta}\mathbf{1}$ and $\epsilon^x_{\{0\}} = \boldsymbol{\zeta}\hat{\epsilon}^x_{\{0\}}$ satisfying $\boldsymbol{c}(\hat{\epsilon}^x_{\{0\}}) + \boldsymbol{d}(\mathbf{1}) = \hat{\epsilon}^x_{\{0\}}$. This value can be computed using the procedure in [19], and $\boldsymbol{\zeta}$ can be selected by solving an LP that enforces desired inclusions with respect to $\mathcal{Y}$. □

Remark 7: Alternative procedures to compute disturbance sets can be derived by formulating the optimization problems in [23], [25], [35] with ϵ^w as a variable and a gain K. While the formulations in [23], [35] involves solving LPs, the *reduced-complexity* polytopes can be excessively conservative. The formulation in [25] uses *full-complexity* polytopes. However, the solution procedure involves computationally expensive iterative Semidefinite Programing (SDP). □

V. Numerical examples

A. Computation of safe reference-sets for supervisory control

We consider the system $z(t+1) = \begin{bmatrix} 1.1 & 0.2 \\ -0.3 & 0.4 \end{bmatrix} z(t) + \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix} u(t)$ with input-constraints $u \in \hat{\mathbf{U}} := \{u : |u| \leq [4\ 3]^\top\}$, and equipped with

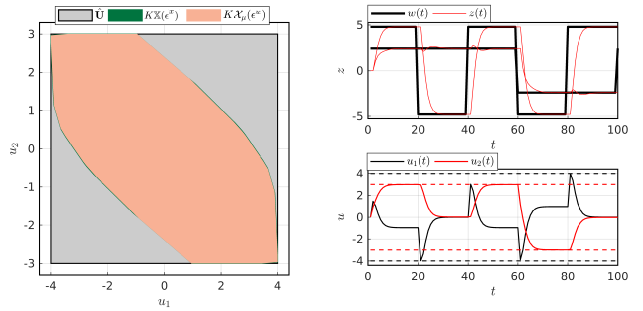

Fig. 1: Results of solving the *inner-approximation* problem. Tight RPI set $\mathcal{X}_\mu(\epsilon^w)$ is computed with $\mu = 10^{-6}$. Top-right plot shows the tracking performance with w sampled from the vertices of $\mathbb{W}(\epsilon^w)$. Bottom-right plot shows resulting closed-loop inputs.

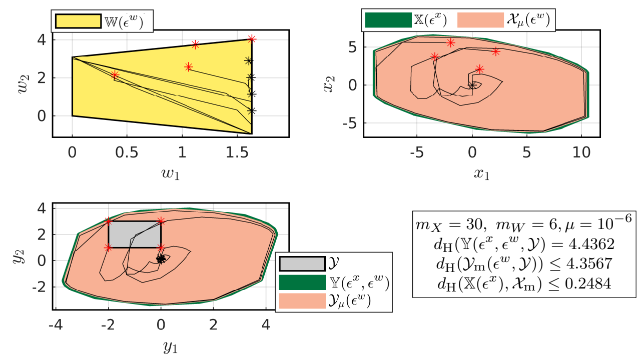

Fig. 2: Results of solving the *outer-approximation* problem. Input, state and output trajectories are plotted with $w(0), x(0), y(0)$ denoted by black $*$, $w(100), x(100), y(100)$ denoted by red $*$. Observe that the vertices of $\mathcal{Y}$ are reachable from $x(0) = \mathbf{0}$ with $w \in \mathbb{W}(\epsilon^w)$.

an LQI-tracking controller such that z tracks a reference signal w: an integral-action state q with dynamics $q(t+1) = q(t) + z(t) - w(t)$ is appended, and the state $x = [z^\top \ q^\top]^\top$ is introduced. Then, an LQI feedback gain $K = \begin{bmatrix} -1.19 & -0.1439 & -0.3154 & 0.0213 \\ 0.2777 & -0.6497 & -0.0037 & -0.3724 \end{bmatrix}$ is computed corresponding to matrices $Q = \mathrm{diag}(\mathbf{I}, 0.5\mathbf{I})$ and $R = \mathbf{I}$. The resulting closed-loop system with $u = Kx$ has the dynamics

$$x(t+1) = \begin{bmatrix} -0.09 & 0.0561 & -0.3154 & 0.0213 \\ -0.1413 & -0.2641 & -0.0353 & -0.3702 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} w(t).$$

We aim to design a supervisory controller for this system that saturates the references as $w \in \mathbb{W}(\epsilon^w) = \{w : |w| \leq \bar{\epsilon}^w\}$ such that $u \in \hat{\mathbf{U}}$ always holds. We assume that the supervisory controller cannot access the state $x(t)$ of the system, such that $\mathbb{W}(\epsilon^w)$ should guarantee input-constraint satisfaction for all reachable x. Since the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ is the set of set of all reachable x, the constraint $u \in \hat{\mathbf{U}}$ is equivalent to $K\mathcal{X}_{\mathrm{m}}(\epsilon^w) \subseteq \hat{\mathbf{U}}$. Hence, we solve the inner-approximation Problem (25) with the output equation (1b) formulated using $C = K, D = 0$, output-constraint set $\mathcal{Y} = \hat{\mathbf{U}}$, and the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ approximated using the RPI set $\mathbb{X}(\epsilon^x) = \{x : Ex \leq \epsilon^x\}$, where the matrix E is composed of hyperplanes defining the set $\oplus_{t=0}^{5} A^t B \mathbb{W}(\mathbf{1})$ (A,B denote the matrices of the closed-loop system). This choice results in $m_X = 240$. The result of solving this problem using the methods presented in this paper is shown in Figure 1 (Plotted using the MPT-toolbox [36]). The computed saturation bounds are $\bar{w}_1 = 4.7864$, $\bar{w}_2 = 2.4282$. The penalty function algorithm converges in a single iteration, with iteration time 4.3989s using the MOSEK [37] LP solver. The quick convergence is potentially related to the fact that the box-parametrization of set $\mathbb{W}(\epsilon^w)$ results in Problem (4) being convex. Further investigation of this effect is a subject of future research. We also plot the set $K\mathcal{X}_\mu(\epsilon^w)$, where $\mathcal{X}_\mu(\epsilon^w)$ is an RPI set satisfying $\mathcal{X}_\mu(\epsilon^w) \subseteq \mathcal{X}_{\mathrm{m}}(\epsilon^w) \oplus \mu\mathcal{B}_\infty^{n_x}$. This set is computed using the method in [29] for $\mu = 10^{-6}$. Using this set and the triangle inequality, we compute $d_{\mathrm{H}}(\mathbb{X}(\epsilon^x), \mathcal{X}_{\mathrm{m}}(\epsilon^w)) \leq 0.0728$, indicating that $\mathbb{X}(\epsilon^x)$ is a reasonably tight approximation of the mRPI set. We also report that at convergence, we obtain $d_{\mathrm{H}}(\mathbb{Y}(\epsilon^x, \epsilon^w), \mathcal{Y}) = 1.9917$, and using $\mathcal{X}_\mu(\epsilon^w)$, we compute $d_{\mathrm{H}}(\mathcal{Y}_{\mathrm{m}}(\epsilon^w), \mathcal{Y}) \leq 1.9970$. Closed-loop trajectories are plotted with references w sampled from the vertices of $\mathbb{W}(\epsilon^w)$, for which the input response satisfies the input-constraints. Hence, if $x(0) \in \mathbb{X}(\epsilon^x)$, the supervisory controller can command any reference $w \in \mathbb{W}(\epsilon^w)$ with guaranteed input-constraint satisfaction. An alternative approach to compute a suitable $\mathbb{W}(\epsilon^w)$ can be derived using the methods presented in [25], [38], in which a nonlinear SDP is solved with a sequential convexification technique. Using the same RPI set parametrization $\mathbb{X}(\epsilon^x)$ and inital conditions, the SDP procedure to maximize the size of $\mathbb{W}(\epsilon^w)$ converges in 18 iterations, with each iteration being computed in an average of 64.37s using the MOSEK [37] SDP solver. The disturbance bounds at convergence are $\bar{w}_1 = 4.7038$ and $\bar{w}_2 = 2.4678$. Clearly, the penalty function approach computationally outperforms the SDP approach, which implies that the proposed method can be used in practical applications in which RPI sets parametrized as polytopes are sought.

Remark 8: The mRPI set is suitable to formulate the problem in Example A since we do not have access to the state $x(t)$. If this limitation is overcome, then a reference governor scheme [8] is more suitable to design the supervisory controller, which uses control invariant sets to guarantee constraint satisfaction. □

B. Computation of input-constraint sets for output reachability

We consider system (1) with initial-state $x(0) = \mathbf{0}$, for which we compute the smallest input-constraint set $\mathbb{W}(\epsilon^w) = \{w : Fw \leq \epsilon^w\}$ with rows $F_i = [\sin(2\pi(i-1)/m_W) \ \cos(2\pi(i-1)/m_W)]$ for each $i \in \mathbb{I}_1^{m_W}$, such that all $y \in \mathcal{Y}$ can be reached with control inputs $w \in \mathbb{W}(\epsilon^w)$. To this end, we use $\boldsymbol{f}(\epsilon^w)$ as a measure of the set $\mathbb{W}(\epsilon^w)$, and formulate the optimization problem $\mathbb{P}^N$ defined as $\epsilon^{w,N} := \arg \min_{\epsilon^w \geq \mathbf{0}} \{ \boldsymbol{f}(\epsilon^w) \text{ s.t. } \mathcal{Y} \subseteq \oplus_{t=0}^{N-1} CA^t B\mathbb{W}(\epsilon^w) \oplus D\mathbb{W}(\epsilon^w)\}$ such that $\mathbb{W}(\epsilon^{w,N})$ is the smallest input-constraint set in which there exist inputs driving the output of system (1) to all $y \in \mathcal{Y}$ from the origin in N-steps. If Assumption 3-*Outer* holds, then $\mathbb{P}^N$ is feasible for all $N \geq n_x$. It can then be shown that the sequence of optimal values $\{\boldsymbol{f}(\epsilon^{w,N})\}_N$ is non-increasing, and converges to the optimal value of the problem $\mathbb{P}^*$ written as $\epsilon_*^w := \arg\min_{\epsilon^w \geq \mathbf{0}} \{\boldsymbol{f}(\epsilon^w) \text{ s.t. } \mathcal{Y} \subseteq C\mathcal{X}_{\mathrm{m}}(\epsilon^w) \oplus D\mathbb{W}(\epsilon^w)\}$, where $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ is the mRPI set corresponding to $\mathbb{W}(\epsilon^w)$. This follows from the idea that the mRPI set is the closure of the largest 0-reachable set [1]. Hence, computing the smallest input-constraint set entails solving Problem $\mathbb{P}^*$. We choose $\boldsymbol{f}(\epsilon^w) = d_{\mathrm{H}}(\mathcal{Y}_{\mathrm{m}}(\epsilon^w), \mathcal{Y})$, such that Problem $\mathbb{P}^*$ is equivalent to Problem (10). This choice ensures that we compute an input-constraint set $\mathbb{W}(\epsilon^w)$ whose 0-reachable set in the output space tightly includes the target output-set $\mathcal{Y}$. We approximately solve Problem $\mathbb{P}^*$ based on the outer-approximation formulation in Problem (27): we approximate the mRPI set using the polytopic RPI set $\mathbb{X}(\epsilon^x) = \{x : Ex \leq \epsilon^x\}$ with rows $E_i = [\sin(2\pi(i-1)/m_X) \ \cos(2\pi(i-1)/m_X)]$ for each $i \in \mathbb{I}_1^{m_X}$. The results of solving this problem using the methods presented in this paper are shown in Figure 2. We consider system (1) with matrices $A = \begin{bmatrix} 0.8966 & 0.8822 \\ -0.2068 & 0.3244 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & 0.6 \end{bmatrix}$,

$D = \begin{bmatrix} 0.01 & -0.01 \\ 0.03 & 0.05 \end{bmatrix}$ and the output-set $\mathcal{Y} = \{[-1\ \ 2]^\top\} \oplus \mathcal{B}_\infty^2$. This system is the closed-loop form of the standard double-integrator with feedback gain $K = [0.2068\ 0.6756]$. We choose $\bar{s} = 100$ in the formulation of Problem (27). We see that the computed set $\mathbb{W}(\epsilon^w)$ is such all $y \in \mathcal{Y}$ are reachable from the origin. We also plot tight approximation RPI set $\mathcal{X}_\mu(\epsilon^w)$ of the mRPI set $\mathcal{X}_{\mathrm{m}}(\epsilon^w)$ using the methods presented in [29], in a manner similar to the previous example. We observe through the set $\mathcal{Y}_\mu(\epsilon^w) := C\mathcal{X}_\mu(\epsilon^w) \oplus D\mathbb{W}(\epsilon^w)$ that $\mathcal{Y} \subseteq \mathcal{Y}_{\mathrm{m}}(\epsilon^w) \subseteq \mathcal{Y}_\mu(\epsilon^w) \subseteq \mathbb{Y}(\epsilon^x, \epsilon^w)$ holds, thus ensuring the desired reachability. In conclusion, one can design feedback controllers to select inputs w from the input-constraint set $\mathbb{W}(\epsilon^w)$, with the guarantee that for any $x(0) \in \mathcal{X}_{\mathrm{m}}(\epsilon^w)$, there always exist feasible inputs to reach every target output $y \in \mathcal{Y}_{\mathrm{m}}(\epsilon^w) \supset \mathcal{Y}$. In Figure 2, we also plot state, input and output trajectories with $x(0) = \mathbf{0}$ and $y(100) \in \mathcal{Y}$ to demonstrate the reachability.

VI. Conclusions

We have presented a method for computing an input disturbance set for discrete-time LTI systems such that the reachable set of outputs approximates an assigned set. To that end, we formulated an optimization problem in order to minimize the approximation error. We presented some numerical results to demonstrate the feasibility of the approach and two possible practical applications. Future research will further develop the solution algorithm by considering: (a) alternative solution methods such as, e.g., value function approaches [34]; (b) optimizing over matrices E and F. Extensions to feedback gain synthesis and system identification problems will be investigated.

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