Irreducibility of the Gorenstein loci of Hilbert schemes via ray families

Gianfranco Casnati, Joachim Jelisiejew, Roberto Notari*

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Abstract

We analyse the Gorenstein locus of the Hilbert scheme of d points on $\mathbb{P}^n$ i.e. the open subscheme parameterising zero-dimensional Gorenstein subschemes of $\mathbb{P}^n$ of degree d. We give new sufficient criteria for smoothability and smoothness of points of the Gorenstein locus. In particular we prove that this locus is irreducible when $d \leq 13$ and find its components when $d = 14$.

The proof is relatively self-contained and it does not rely on a computer algebra system. As a by–product, we give equations of the fourth secant variety to the d-th Veronese reembedding of $\mathbb{P}^n$ for $d \geq 4$.

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1 Introduction and notation

Let k be an algebraically closed field of characteristic neither 2 nor 3 and denote by $\mathcal{H}ilb_{p(t)}\mathbb{P}^N$ the Hilbert scheme parameterising closed subschemes in $\mathbb{P}^N$ with fixed Hilbert polynomial $p(t) \in \mathbb{Q}[t]$. Since A. Grothendieck proved the existence of such a parameter space in 1966 (see [Gro95]), the problem of dealing with $\mathcal{H}ilb_{p(t)}\mathbb{P}^N$ and its subloci has been a fruitful field attracting the interest of many researchers in algebraic geometry.

Only to quickly mention some of the classical results which deserve, in our opinion, a particular attention, we recall Hartshorne's proof of the connectedness of $\mathcal{H}ilb_{p(t)}\mathbb{P}^N$ (see [Har66]), the description of the locus of codimension 2 arithmetically Cohen–Macaulay subschemes due to G. Ellingsrud and J. Fogarty (see [Fog68] for the dimension zero case and [Ell75] for larger dimension) and of the study of the locus of codimension 3 arithmetically Gorenstein subschemes due to J. Kleppe and R.M. Miró–Roig (see [MR92] and [KMR98]).

If we restrict our attention to the case of zero–dimensional subschemes of degree d, i.e. subschemes with Hilbert polynomial $p(t) = d$, then the first significant results are due to J. Fogarty (see [Fog68]) and to A. Iarrobino (see [Iar72]).

In [Fog68], the author proves that $\mathcal{H}ilb_d\mathbb{P}^2$ is smooth, hence irreducible thanks to Hartshorne's connectedness result (the same result holds, when one substitutes $\mathbb{P}^2$ by any smooth surface).

On the other hand in [Iar72], A. Iarrobino deals with the reducibility when $N \geq 3$ and d is large with respect to N. In order to better understand the result, recall that the locus of reduced schemes $\mathcal{R} \subseteq \mathcal{H}ilb_d \mathbb{P}^N$ is birational to a suitable open subset of the d-th symmetric product of $\mathbb{P}^N$, thus it is irreducible of dimension dN. We will denote by $\mathcal{H}ilb_d^{gen} \mathbb{P}^N$ its closure in $\mathcal{H}ilb_d \mathbb{P}^N$. It is a well–known and easy fact that $\mathcal{H}ilb_d^{gen} \mathbb{P}^N$ is an irreducible component of dimension dN, by construction. In [Iar72], the author proves that $\mathcal{H}ilb_d \mathbb{P}^N$ is never irreducible when $d \gg N \geq 3$, showing that there is a family of schemes of dimension greater than dN. Such a family is thus necessarily contained in a component different from $\mathcal{H}ilb_d^{gen} \mathbb{P}^N$.

D.A. Cartwright, D. Erman, M. Velasco, B. Viray proved that already for $d = 8$ and $N \geq 4$, the scheme $\mathcal{H}ilb_d \mathbb{P}^N$ is reducible (see [CEVV09]).

In view of these earlier works it seems reasonable to consider the irreducibility and smoothness of open loci in $\mathcal{H}ilb_d \mathbb{P}^N$ defined by particular algebraic and geometric properties. In the present paper we are interested in the locus $\mathcal{H}ilb_d^G \mathbb{P}^N$ of points in $\mathcal{H}ilb_d \mathbb{P}^N$ representing schemes which are Gorenstein. This is an important locus: e.g. it has an irreducible component $\mathcal{H}ilb_d^{G,gen} \mathbb{P}^N := \mathcal{H}ilb_d^{gen} \mathbb{P}^N \cap \mathcal{H}ilb_d^G \mathbb{P}^N$ of dimension dN containing all the points representing reduced schemes. Moreover it is open, but in general not dense, inside $\mathcal{H}ilb_d \mathbb{P}^N$. Recently, interesting interactions between $\mathcal{H}ilb_d^G \mathbb{P}^N$ and the geometry of secant varieties and general topology have been found (see for example [BB14], [BJJM]).

Some results about $\mathcal{H}ilb_d^G \mathbb{P}^N$ are known. The irreducibility and smoothness of $\mathcal{H}ilb_d^G \mathbb{P}^N$ when $N \leq 3$ is part of the folklore (see [CN09, Cor 2.6] for more precise references). When $N \geq 4$, the properties of $\mathcal{H}ilb_d^G \mathbb{P}^N$ have been object of an intensive study in recent years.

E.g., it is classically known that $\mathcal{H}ilb_d^G \mathbb{P}^N$ is never irreducible for $d \geq 14$ and $N \geq 6$, at least when the characteristic of k is zero (see [IE78] and [IK99]: see also [CN11]). Also for $N = 4$ and $d \geq 140$ or $N = 5$ and $d \geq 42$ the scheme $\mathcal{H}ilb_d^G \mathbb{P}^N$ is reducible, see [BB14, Section 6, p. 81]. For fixed $N \in \{4, 5\}$ the minimal value of d, for which this scheme is reducible, is not known.

As reflected by the quoted papers, it is natural to ask if $\mathcal{H}ilb_d^G \mathbb{P}^N$ is irreducible when $d \leq 13$ and N is arbitrary. There is some evidence of an affirmative answer to this question. Indeed the first and third authors studied the locus $\mathcal{H}ilb_d^G \mathbb{P}^N$ when $d \leq 11$ and N is arbitrary, proving its irreducibility and dealing in detail with its singular locus in a series of papers [CN09, CN11, CN14, CN13].

A key point in the study of a zero–dimensional scheme $X \subseteq \mathbb{P}^N$ is that it is abstractly isomorphic to $\operatorname{Spec} A$ where A is an Artin k-algebra with $\dim_k(A) = d$. Moreover the irreducible components of such an X correspond bijectively to those direct summands of A, which are local. Thus, in order to deal with $\mathcal{H}ilb_d \mathbb{P}^N$, it suffices to deal with the irreducible schemes in $\mathcal{H}ilb_{d'} \mathbb{P}^N$ for each $d' \leq d$.

In all of the aforementioned papers, the methods used in the study of $\mathcal{H}ilb_d^G \mathbb{P}^N$ rely on an almost explicit classification of the possible structure of local, Artin, Gorenstein k-algebras of length d. Once such a classification is obtained, the authors prove that all the corresponding irreducible schemes are smoothable, i.e. actually lie in $\mathcal{H}ilb_d^{G,gen} \mathbb{P}^N$. To this purpose they explicitly construct a projective family flatly deforming the scheme they are interested in (or, equivalently, the underlying algebra) to reducible schemes that they know to be in $\mathcal{H}ilb_d^{G,gen} \mathbb{P}^N$ because their components have lower degree.

Though such an approach sometimes seems to be too heavy in terms of calculations, only thanks to such a partial classification it is possible to state precise results about the singularities of $\mathcal{H}ilb_d^G \mathbb{P}^N$.

However, in the papers [CN11, CN14], there are families H_d of schemes of degree d, where $d = 10, 11$, for which an explicit algebraic description in the above sense cannot be obtained (see Section 3 of [CN11] for the case $d = 10$, Section 4 of [CN14] for $d = 11$). Nevertheless, using an alternative approach the authors are still able to prove the irreducibility of $\mathcal{H}ilb_d^G \mathbb{P}^N$ and study its singular locus. Indeed, using Macaulay's theory of inverse systems, the authors check the

irreducibility of the aforementioned loci H_d inside $\mathcal{H}ilb_d^G\mathbb{P}^N$. Then they show the existence of a smooth point in $H_d \cap \mathcal{H}ilb_d^{G,gen}\mathbb{P}^N$. Hence, it follows that $H_d \subseteq \mathcal{H}ilb_d^{G,gen}\mathbb{P}^N$.

The aim of the present paper is to refine and generalise this method. First, we avoid a case by case approach by analysing large classes of algebras. Second, in [CN11, CN14] a direct check (e.g. using a computer algebra program) is required to compute the dimension of tangent space to the Hilbert scheme at some specific points to conclude that they are smooth. We avoid the need of such computations by exhibiting classes of points which are smooth, making the paper self–contained.

Using this method, we finally prove the following two statements.

Theorem A. *If the characteristic of k is neither 2 nor 3, then $\mathcal{H}ilb_d^G\mathbb{P}^N$ is irreducible of dimension dN for each $d \leq 13$ and for $d = 14$ and $N \leq 5$.*

Theorem B. *If the characteristic of k is 0 and $N \geq 6$, then $\mathcal{H}ilb_{14}^G\mathbb{P}^N$ is connected and it has exactly two irreducible components, which are generically smooth.*

Theorem A has an interesting consequence regarding secant varieties of Veronese embeddings. In [Ger99] Geramita conjectures that the ideal of the 2nd secant variety (the variety of secant lines) of the dth Veronese embedding of $\mathbb{P}^n$ is generated by the 3×3 minors of the ith catalecticant matrix for $2 \leq i \leq d - 2$. Such a conjecture was confirmed in [Rai12]. As pointed out in [BB14, Section 8.1], the above Theorem A allows to extend the above result as follows: if $r \leq 13$, $2r \leq d$ and, then for every $r \leq i \leq d - r$ the set–theoretic equations of the rth secant variety of the dth Veronese embedding of $\mathbb{P}^n$ are given by the $(r + 1) \times (r + 1)$ minors of the ith catalecticant matrix.

The proofs of Theorem A and Theorem B are highly interlaced and they follow from a long series of partial results. In order to better explain the ideas and methods behind their proofs we will describe in the following lines the structure of the paper.

In our analysis we incorporate several tools. In Section 2 we recall the classical ones, most notably Macaulay's correspondence for local, Artinian, Gorenstein algebras and Macaulay's Growth Theorem. Moreover we also list some criteria for checking the flatness of a family of algebras which will be repeatedly used throughout the whole paper.

In Section 3 we analyse Artin Gorenstein quotients of a power series ring and exploit the rich automorphism group of this ring to put the quotient into suitable *standard* form, deepening a result by A. Iarrobino.

In Section 4 we further analyse the quotients, especially their dual socle generators. We also construct several irreducible subloci of the Hilbert scheme using the theory of secant varieties. We give a small contribution to this theory, showing that the fourth secant variety to a Veronese reembedding of $\mathbb{P}^n$ is defined by minors of a suitable catalecticant matrix.

Section 5 introduces a central object in our study: a class of families, called ray families, for which we have relatively good control of the flatness and, in special cases, fibers. Most notably, Subsection 5.2 gives a class of *tangent preserving* flat families, which enable us to construct smooth points on the Hilbert scheme of points without the necessity of heavy computations.

Finally, in Section 6, we give the proofs of Theorem A and B. It is worth mentioning that these results are rather easy consequences of the introduced machinery. In this section we also prove the following general smoothability result (see Thm 6.14), which has no restriction on the length of the algebra and generalises the smoothability results from [Sal79], [CN13] and [EV11].

Theorem C. *Let k be an algebraically closed field of characteristic neither 2 nor 3. Let A be a local Artin Gorenstein k-algebra with maximal ideal $\mathfrak{m}$.*

If $\dim_k(\mathfrak{m}^2/\mathfrak{m}^3) \leq 5$ and $\dim_k(\mathfrak{m}^3/\mathfrak{m}^4) \leq 2$, then $\operatorname{Spec} A$ is smoothable.

Notation

All symbols appearing below are defined in Section 2.

k	an algebraically closed field of characteristic $\neq 2, 3$.
$P = k[x_1, \ldots, x_n]$	a polynomial ring in n variables and fixed basis.
$S = k[[\alpha_1, \ldots, \alpha_n]]$	a power series ring dual (see Subsection 2.2) to P, with a fixed (dual) basis.
$\mathfrak{m}_S$	the maximal ideal of S.
$S_{poly} = k[\alpha_1, \ldots, \alpha_n]$	a polynomial subring of S defined by the choice of the basis.
H_A	the Hilbert function of a local Artin algebra A.
$\Delta_{A,i}, \Delta_i$	the i-th row of the symmetric decomposition of the Hilbert function of a local Artin Gorenstein algebra A as in Theorem 2.3.
$e(a)$	the a-th "embedding dimension", equal to $\sum_{t=0}^{a} \Delta_t(1)$, as in Definition 3.1.
$\mathrm{ann}_S(f)$	the annihilator of $f \in P$ with respect to the action of S.
$\mathrm{Apolar}(f)$	the apolar algebra of $f \in P$, equal to $S/\mathrm{ann}_S(f)$.

2 Preliminaries

Let n be a natural number. By $(S, \mathfrak{m}_S, k)$ we denote the power series ring $k[[\alpha_1, \ldots, \alpha_n]]$ of dimension n with a fixed basis $\alpha_1, \ldots, \alpha_n$. The chosen basis determines a polynomial ring $S_{poly} = k[\alpha_1, \ldots, \alpha_n] \subseteq S$. By P we denote the polynomial ring $k[x_1, \ldots, x_n]$. We will later define a duality between S and P, see Subsection 2.2. We usually think of n being large enough, so that the considered local Artin algebras are quotients of S.

For an element $f \in P$, we say that f *does not contain* x_i if $f \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$; similarly for $\sigma \in S$ or $\sigma \in S_{poly}$. For $f \in P$, by f_d we denote the degree d part of f, with respect to the total degree; similarly for $\sigma \in S$.

By P_m and $P_{\leq m}$ we denote the space of homogeneous polynomials of degree m and (not necessarily homogeneous) polynomials of degree at most m respectively. These spaces are naturally affine spaces over k, which equips them with a scheme structure.

Recall that S has a rich automorphism group: for every choice of elements $\sigma_1, \ldots, \sigma_n \in \mathfrak{m}_S$ linearly independent in $\mathfrak{m}_S/\mathfrak{m}_S^2$ there is a unique automorphism φ of S such that $\varphi(\alpha_i) = \sigma_i$ for $i = 1, 2, \ldots, n$. The existence of such automorphisms is employed in Section 4 to put the considered Artin Gorenstein algebras in a better form. See e.g. [ER15, Section 2] for details and examples of this method.

Remark 2.1. *For the reader's convenience we introduce numerous examples, which illustrate the possible applications. In all these examples k may have arbitrary characteristic $\neq 2, 3$ unless otherwise stated.*

2.1 Artin Gorenstein schemes and algebras

In this section we recall the basic facts about Artin Gorenstein algebras. For a more thorough treatment we refer to [IK99], [Eis95], [CN09] and [Jel13].

Finite type zero-dimensional schemes correspond to Artin algebras. Every such algebra A splits as a finite product of its localisations at maximal ideals, which corresponds to the fact that the support of $\mathrm{Spec}\, A$ is finite and totally disconnected. Therefore, we will focus our interest on *local* Artin k-algebras. Since k is algebraically closed, such algebras have residue field k.

An important invariant of a local algebra $(A, \mathfrak{m}, k)$ is its Hilbert function H_A defined by $H_A(l) = \dim_k \mathfrak{m}^l/\mathfrak{m}^{l+1}$. Since $H_A(l) = 0$ for $l \gg 0$ it is usual to write H_A as the vector of its

non-zero values. The *socle degree* of A is the largest l such that $H_A(l) \neq 0$. Such an algebra is *Gorenstein* if the annihilator of $\mathfrak{m}$ is a one-dimensional vector space over k, see [Eis95, Chap 21].

We recall for reader's benefit that a finite not necessarily local algebra A is Gorenstein if and only if all its localisations at maximal ideals are Gorenstein (in particular it is meaningful to discuss the irreducibility of the Gorenstein locus in the Hilbert scheme by reducing to the study of deformations of local Gorenstein algebras: see Section 2.4).

Since k is algebraically closed, we may write each Artin local algebra $(A, \mathfrak{m}, k)$ as a quotient of the power series ring $S = k[[\alpha_1, \ldots, \alpha_n]]$ when n is large enough, in fact $n \geq H_A(1)$ is sufficient. Since $\dim_k A$ is finite, such a presentation gives a presentation $A = S_{poly}/I$, i.e. a point $[\operatorname{Spec} A]$ of the Hilbert scheme of $\mathbb{A}^n = \operatorname{Spec} S_{poly}$.

2.2 Contraction map and apolar algebras

In this section we introduce the contraction mapping, which is closely related to Macaulay's inverse systems. We refer to [Iar94] and [Eis95, Chap 21] for details and proofs.

Recall that $P = k[x_1, \ldots, x_n]$ is a polynomial ring and $S = k[[\alpha_1, \ldots, \alpha_n]]$ is a power series ring. The k-algebra S acts on P by *contraction* (see [IK99, Def 1.1]). This action is denoted by $(\cdot) \lrcorner (\cdot) : S \times P \to P$ and defined as follows. Let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n} \in P$ and $\alpha^{\mathbf{b}} = \alpha_1^{b_1} \ldots \alpha_n^{b_n} \in S$ be monomials. We write $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ for all $1 \leq i \leq n$. Then

$$\alpha^{\mathbf{b}} \lrcorner \mathbf{x}^{\mathbf{a}} := \begin{cases} \mathbf{x}^{\mathbf{a}-\mathbf{b}} & \text{if} \quad \mathbf{a} \geq \mathbf{b} \\ 0 & \text{otherwise.} \end{cases}$$

This action extends to $S \times P \to P$ by k-linearity on P and countable k-linearity on S.

The contraction action induces a perfect pairing between $S/\mathfrak{m}_S^{s+1}$ and $P_{\leq s}$, which restricts to a perfect pairing between the degree s polynomials in S_{poly} and P. These pairings are compatible for different choices of s.

If $f \in P$ then a *derivative* of f is an element of the S-module Sf, i.e. an element of the form $\partial \lrcorner f$ for $\partial \in S$. By definition, these elements form an S-submodule of P, in particular a k-linear subspace.

Let $A = S/I$ be an Artin quotient of S, then A is local. The contraction action associates to A an S-submodule $M \subseteq P$ consisting of elements annihilated by I, so that A and M are dual. If A is Gorenstein, then the S-module M is cyclic, generated by a polynomial f of degree s equal to the socle degree of A. We call every such f a *dual socle generator of the Artin Gorenstein algebra A*. Unlike M, the polynomial f is *not determined uniquely by the choice of the presentation* $A = S/I$, however if f and g are two dual socle generators, then $g = \partial \lrcorner f$, where $\partial \in S$ is invertible.

Conversely, let $f \in P$ be a polynomial of degree s. We can associate it the ideal $I := \operatorname{ann}_S(f)$ such that $A := S/I$ is a local Artin Gorenstein algebra of socle degree s. We call I the *apolar ideal* of f and A the *apolar algebra* of f, which we denote as

$$A = \operatorname{Apolar}(f).$$

From the discussion above it follows that every local Artin Gorenstein algebra is an apolar algebra of some polynomial.

Remark 2.2. *Recall that we may think of $S/\mathfrak{m}_S^{s+1}$ as the linear space dual to $P_{\leq s}$. An automorphism ψ of S or $S/\mathfrak{m}_S^{s+1}$ induces an automorphism ψ^* of the k-linear space $P_{\leq s}$. If $f \in P_{\leq s}$ and I is the apolar ideal of f, then $\psi(I)$ is the apolar ideal of $\psi^*(f)$. Moreover, f and $\psi^*(f)$ have the same degree.*

2.3   Iarrobino's symmetric decomposition of Hilbert function

One of the most important invariants possessed by a local Artin Gorenstein algebra is the symmetric decomposition of its Hilbert function, due to Iarrobino [Iar94]. To state the theorem it is convenient to define addition of vectors of different lengths position-wise: if $a = (a_0, \ldots, a_n)$ and $b = (b_0, \ldots, b_m)$ are vectors, then $a + b = (a_0 + b_0, \ldots, a_{\max(m,n)} + b_{\max(m,n)})$, where $a_i = 0$ for $i > n$ and $b_i = 0$ for $i > m$. In the following, all vectors are indexed starting from zero.

Let $(A, \mathfrak{m}, k)$ be a local Artin Gorenstein algebra. By $(0 : \mathfrak{m}^l)$ we denote the annihilator of $\mathfrak{m}^l$ in A. The chain $0 = (0 : \mathfrak{m}^0) \subseteq (0 : \mathfrak{m}^1) \subseteq \ldots$ defines a filtration on A. In general, it is different from the usual filtration $0 = \mathfrak{m}^{s+1} \subseteq \mathfrak{m}^s \subseteq \mathfrak{m}^{s-1} \subseteq \ldots$. The analysis of mutual position of these filtrations is the content of Theorem 2.3 below.

Theorem 2.3 (Iarrobino's symmetric decomposition of Hilbert function). *Let $(A, \mathfrak{m}, k)$ be a local Artin Gorenstein algebra of socle degree s and Hilbert function H_A. Let*

$$\Delta_i(t) := \dim_k \frac{(0 : \mathfrak{m}^{s+1-i-t}) \cap \mathfrak{m}^t}{(0 : \mathfrak{m}^{s-i-t}) \cap \mathfrak{m}^t + (0 : \mathfrak{m}^{s+1-i-t}) \cap \mathfrak{m}^{t+1}} \quad for \quad t = 0, 1, \ldots, s - i.$$

The vectors $\Delta_0, \Delta_1, \ldots, \Delta_s$ have the following properties:

1. *the vector Δ_i has length $s + 1 - i$ and satisfies $\Delta_i(t) = \Delta_i(s - i - t)$ for all integers $t \in [0, s - i]$.*

2. *the Hilbert function H_A is equal to the sum $\sum_{i=0}^{s} \Delta_i$.*

3. *the vector Δ_0 is equal to the Hilbert function of a local Artin Gorenstein graded algebra of socle degree s.*

Let $(A, \mathfrak{m}, k)$ be a local Artin Gorenstein algebra. There are a few important remarks to do.

1. Since Δ_0 is the Hilbert function of an algebra, we have $\Delta_0(0) = 1 = H_A(0)$. Thus for every $i > 0$ we have $\Delta_i(0) = 0$. From symmetry it follows that $\Delta_i(s + 1 - i) = 0$. In particular $\Delta_s = (0)$ and $\Delta_{s-1} = (0, 0)$, so we may ignore these vectors. On the other hand $\Delta_{s-2} = (0, q, 0)$ is in general non-zero and its importance is illustrated by Proposition 4.5.

2. Suppose that $H_A = (1, n, 1, 1)$ for some $n > 0$. Then we have $\Delta_0 = (1, *, *, 1)$ and $\Delta_1 = (0, *, 0)$, thus $\Delta_0 = (1, *, 1, 1)$, so that $\Delta_0 = (1, 1, 1, 1)$ because of its symmetry. Then $\Delta_1 = (0, n - 1, 0)$. Similarly, if $H_A = (1, n, e, 1)$ is the Hilbert function of a local Artin Gorenstein algebra, then $n \geq e$. This is a basic example on how Theorem 2.3 imposes restrictions on the Hilbert function of A.

3. If A is graded, then $\Delta_0 = H_A$ and all other $\Delta_\bullet$ are zero vectors, see [Iar94, Prop 1.7].

4. For every $a \leq s$ the partial sum $\sum_{i=0}^{a} \Delta_i$ is the Hilbert function of a local Artin graded algebra, see [Iar94, Def 1.3, Thm 1.5], see also [Iar94, Subsection 1.F]. In particular it satisfies Macaulay's Growth Theorem, see Subsection 2.5. Thus e.g. there is no local Artin Gorenstein algebra with Hilbert function decomposition satisfying $\Delta_0 = (1, 1, 1, 1, 1, 1)$ and $\Delta_1 = (0, 0, 1, 0, 0)$, because then $(\Delta_0 + \Delta_1)(1) = 1$ and $(\Delta_0 + \Delta_1)(2) = 2$.

Let us now analyse the case when $A = \text{Apolar}(f) = S/\text{ann}_S(f)$ is the apolar algebra of a polynomial $f \in P$, where $f = \sum_{i=0}^{s} f_i$ for some $f_i \in P_i$. Each local Artin Gorenstein algebra is isomorphic to such algebra, see Subsection 2.2. For the proofs of the following remarks, see [Iar94].

1. Vector Δ_0 is equal to the Hilbert function of $\text{Apolar}(f_s)$, the apolar algebra of the leading form of f.

6

2. If A is graded, then $\mathrm{ann}_S(f) = \mathrm{ann}_S(f_s)$, so that we may always assume that $f = f_s$. Moreover, in this case $H_A(m)$ is equal to $(Sf_s)_m$, the number of degree m derivatives of f_s.

3. Let f_1, f_2 be polynomials of degree s such that $f_1 - f_2$ is a polynomial of degree $d < s$. Let $A_i = \mathrm{Apolar}(f_i)$ and let $\Delta_{A_i,m}$ be the symmetric decomposition of the Hilbert function H_{A_i} of A_i for $i = 1, 2$. Then $\Delta_{A_1,m} = \Delta_{A_2,m}$ for all $m < s - d$, see [Iar94, Lem 1.10].

2.4 Smoothability and unobstructedness

An Artin algebra A is called *smoothable* if it is a (finite flat) limit of smooth algebras, i.e. if there exists a finite flat family over an irreducible base with a special fiber isomorphic to $\mathrm{Spec}\,A$ and general fiber smooth. Recall that $A \simeq A_{\mathfrak{m}_1} \times \ldots A_{\mathfrak{m}_r}$, where $\mathfrak{m}_i$ are maximal ideals of A. The algebra A is smoothable if all localisations $A_\mathfrak{m}$ at its maximal ideals are smoothable. The converse also holds, i.e. if an algebra $A \simeq B_1 \times B_2$ is smoothable, then the algebras B_1 and B_2 are also smoothable, a complete and characteristic free proof of this fact will appear shortly in [BJ]. We say that a zero-dimensional scheme $Z = \mathrm{Spec}\,A$ is *smoothable* if the algebra A is smoothable.

It is crucial that every local Artin Gorenstein algebra A with $H_A(1) \leq 3$ is smoothable, see [CN09, Prop 2.5], which follows from the Buchsbaum-Eisenbud classification of resolutions, see [BE77]. Also complete intersections are smoothable. A complete intersection $Z \subseteq \mathbb{P}^n$ is smoothable by Bertini's Theorem (see [Har10, Example 29.0.1], but note that Hartshorne uses a slightly weaker definition of smoothability, without finiteness assumption). If $Z = \mathrm{Spec}\,A$ is a complete intersection in $\mathbb{A}^n$, then Z is a union of connected components of a complete intersection $Z' = \mathrm{Spec}\,B$ in $\mathbb{P}^n$, so that $B \simeq A \times C$ for some algebra C. The algebra B is smoothable since Z' is. Thus also the algebra A is smoothable, i.e. Z is smoothable.

Definition 2.4. *A smoothable Artin algebra A of length d, corresponding to $\mathrm{Spec}\,A \subseteq \mathbb{P}^n$, is* unobstructed *if the tangent space to $\mathcal{H}ilb_d(\mathbb{P}^n)$ at the k-point $[\mathrm{Spec}\,A] =: p$ has dimension nd. If A is unobstructed, then p is a smooth point of the Hilbert scheme.*

The unobstructedness is independent of n and the chosen embedding of $\mathrm{Spec}\,A$ into $\mathbb{P}^n$, see discussion before [CN09, Lem 2.3]. The argument above shows that algebras corresponding to complete intersections in $\mathbb{A}^n$ and $\mathbb{P}^n$ are unobstructed. Every local Artin Gorenstein algebra A with $H_A(1) \leq 3$ is unobstructed, see [CN09, Prop 2.5]. Moreover, every local Artin Gorenstein algebra A with $H_A(1) \leq 2$ is a complete intersection in $\mathbb{A}^2$ by the Hilbert-Burch theorem.

Definition 2.5. *An Artin algebra A is* limit-reducible *if there exists a flat family (over an irreducible base) whose special fiber is A and general fiber is reducible. An Artin algebra A is* strongly non-smoothable *if it is not limit-reducible.*

Clearly, strongly non-smoothable algebras (other than $A = k$) are non-smoothable. The definition of strong non-smoothability is useful, because to show that there is no non-smoothable algebra of length less than d it is enough to show that there is no strongly non-smoothable algebra of length less than d.

2.5 Macaulay's Growth Theorem

We will recall Macaulay's Growth Theorem and Gotzmann's Persistence Theorem, which provide strong restrictions on the possible Hilbert functions of graded algebras. Fix $n \geq 1$. Let m be any natural number, then m may be uniquely written in the form

$$m = \binom{m_n}{n} + \binom{m_{n-1}}{n-1} + \ldots + \binom{m_1}{1},$$

7

where $m_n > m_{n-1} > \ldots > m_1$. We define

$$m^{\langle i \rangle} := \binom{m_n + 1}{n + 1} + \binom{m_{n-1} + 1}{n} + \ldots + \binom{m_1 + 1}{2}.$$

It is useful to compute some initial values of the above defined function, i.e. $1^{\langle n \rangle} = 1$ for all n, $3^{\langle 2 \rangle} = 4$, $4^{\langle 2 \rangle} = 5$, $6^{\langle 2 \rangle} = 10$ or $4^{\langle 3 \rangle} = 5$.

Theorem 2.6 (Macaulay's Growth Theorem). *If A is a graded quotient of a polynomial ring over k, then the Hilbert function H_A of A satisfies $H_A(m + 1) \leq H_A(m)^{\langle m \rangle}$ for all m.*

Proof. See [BH93, Thm 4.2.10]. $\square$

Note that the assumptions of Theorem 2.6 are satisfied for every local Artin k-algebra $(A, \mathfrak{m}, k)$, since its Hilbert function is by definition equal to the Hilbert function of the associated graded algebra.

Remark 2.7. *We will frequently use the following easy consequence of Theorem 2.6.*
Let A be a graded quotient of a polynomial ring over k. Suppose that $H_A(l) \leq l$ for some l. Then $H_A(l) = \binom{l}{l} + \binom{l-1}{l-1} + \ldots$ and $H_A(l)^{\langle l \rangle} = \binom{l+1}{l+1} + \binom{l}{l} + \ldots = H_A(l)$, thus $H_A(l+1) \leq H_A(l)$. It follows that the Hilbert function of H_A satisfies $H_A(l) \geq H_A(l+1) \geq H_A(l+2) \geq \ldots$. In particular $H_A(m) \leq l$ for all $m \geq l$.

Theorem 2.8 (Gotzmann's persistence Theorem). *Let $A = S_{poly}/I$ be a graded quotient of a polynomial ring S_{poly} over k and suppose that for some l we have $H_A(l + 1) = H_A(l)^{\langle l \rangle}$ and I is generated by elements of degree at most l. Then $H_A(m + 1) = H_A(m)^{\langle m \rangle}$ for all $m \geq l$.*

Proof. See [BH93, Thm 4.3.3]. $\square$

In the following we will mostly use the following consequence of Theorem 2.8, for which we introduce some (non-standard) notation. Let $I \subseteq S_{poly} = k[\alpha_1, \ldots, \alpha_n]$ be a graded ideal in a polynomial ring and $m \geq 0$. We say that I is *m-saturated* if for all $l \leq m$ and $\sigma \in (S_{poly})_l$ the condition $\sigma \cdot (\alpha_1, \ldots, \alpha_n)^{m-l} \subseteq I$ implies $\sigma \in I$.

Lemma 2.9. *Let $S_{poly} = k[\alpha_1, \ldots, \alpha_n]$ be a polynomial ring with maximal ideal $\mathfrak{n} = (\alpha_1, \ldots, \alpha_n)$. Let $I \subseteq S_{poly}$ be a graded ideal and $A = S_{poly}/I$. Suppose that I is m-saturated for some $m \geq 2$. Then*

1. *if $H_A(m) = m + 1$ and $H_A(m + 1) = m + 2$, then $H_A(l) = l + 1$ for all $l \leq m$, in particular $H_A(1) = 2$.*

2. *if $H_A(m) = m + 2$ and $H_A(m + 1) = m + 3$, then $H_A(l) = l + 2$ for all $l \leq m$, in particular $H_A(1) = 3$.*

Proof. 1. First, if $H_A(l) \leq l$ for some $l < m$, then by Macaulay's Growth Theorem $H_A(m) \leq l < m + 1$, a contradiction. So it suffices to prove that $H_A(l) \leq l + 1$ for all $l < m$.

Let J be the ideal generated by elements of degree at most m in I. We will prove that the graded ideal J of S_{poly} defines a $\mathbb{P}^1$ linearly embedded into $\mathbb{P}^{n-1}$.

Let $B = S_{poly}/J$. Then $H_B(m) = m + 1$ and $H_B(m + 1) \geq m + 2$. Since $H_B(m) = m + 1 = \binom{m+1}{m}$, we have $H_B(m)^{\langle m \rangle} = \binom{m+2}{m+1} = m + 2$ and by Theorem 2.6 we get $H_B(m + 1) \leq m + 2$, thus $H_B(m+1) = m+2$. Then by Gotzmann's Persistence Theorem $H_B(l) = l+1$ for all $l > m$. This implies that the Hilbert polynomial of $\operatorname{Proj} B \subseteq \mathbb{P}^{n-1}$ is $h_B(t) = t + 1$, so that $\operatorname{Proj} B \subseteq \mathbb{P}^{n-1}$ is a linearly embedded $\mathbb{P}^1$. In particular the Hilbert function and Hilbert polynomial of $\operatorname{Proj} B$ are equal for all arguments. By assumption, we have $J_l = J_l^{sat}$ for all $l < m$. Then $H_A(l) = H_{S_{poly}/J}(l) = H_{S_{poly}/J^{sat}}(l) = l + 1$ for all $l < m$ and the claim of the lemma follows.

2. The proof is similar to the above one; we mention only the points, where it changes. Let J be the ideal generated by elements of degree at most m in I and $B = S_{poly}/J$. Then $H_B(m) = m + 2 = \binom{m+1}{m} + \binom{m-1}{m-1}$, thus $H_B(m+1) \le \binom{m+2}{m+1} + \binom{m}{m} = m + 3$ and B defines a closed subscheme of $\mathbb{P}^{n-1}$ with Hilbert polynomial $h_B(t) = t + 2$. There are two isomorphism types of such subschemes: $\mathbb{P}^1$ union a point and $\mathbb{P}^1$ with an embedded double point. One checks that for these schemes the Hilbert polynomial is equal to the Hilbert function for all arguments and then proceeds as in the proof of Point 1. $\square$

Remark 2.10. *If $A = S_{poly}/I$ is a graded Artin Gorenstein algebra of socle degree s, then it is m-saturated for every $m \le s$. Indeed, we may assume that $A = \mathrm{Apolar}\,(F)$ for some homogeneous $F \in P$ of degree s, then $I = \mathrm{ann}_S\,(F)$. Let $\mathfrak{n} = (\alpha_1,\ldots,\alpha_n) \subseteq k[\alpha_1,\ldots,\alpha_n] = S_{poly}$. Take $\sigma \in (S_{poly})_l$, then $\sigma \in I$ if and only if $\sigma \lrcorner F = 0$. Similarly, $\sigma \mathfrak{n}^{m-l} \subseteq I$ if and only if every element of $\mathfrak{n}^{m-l}$ annihilates $\sigma \lrcorner F$. Since $\sigma \lrcorner F$ is either a homogeneous polynomial of degree $s - l \ge m - l$ or it is zero, both conditions are equivalent.*

Remark 2.11. *Clearly, if two graded ideals I and J of S_{poly} agree up to degree m and I is m-saturated, then also J is m-saturated.*

2.6 Flatness over $\mathrm{Spec}\,k[t]$

For further reference we explicitly state a purely elementary flatness criterion. Its formulation is a bit complicated, but this is precisely the form which is needed for the proofs. This criterion relies on the easy observation that the torsion-free modules over $k[t]$ are flat.

Proposition 2.12. *Suppose S is a k-module and $I \subseteq S[t]$ is a $k[t]$-submodule. Let $I_0 := I \cap S$. If for every $\lambda \in k$ we have*

$$(t - \lambda) \cap I \subseteq (t - \lambda)I + I_0[t],$$

then $S[t]/I$ is a flat $k[t]$-module.

Proof. The ring $k[t]$ is a principal ideal domain, thus a $k[t]$-module is flat if and only if it is torsion-free, see [Eis95, Cor 6.3]. Since every polynomial in $k[t]$ decomposes into linear factors, to prove that $M = S[t]/I$ is torsion-free it is enough to show that $t - \lambda$ are non–zerodivisors on M, i.e. that $(t - \lambda)x \in I$ implies $x \in I$ for all $x \in S[t]$, $\lambda \in k$.

Fix $\lambda \in k$ and suppose that $x \in S[t]$ is such that $(t - \lambda)x \in I$. Then by assumption $(t - \lambda)x \in (t - \lambda)I + I_0[t]$, so that $(t - \lambda)(x - i) \in I_0[t]$ for some $i \in I$. Since $S[t]/I_0[t] \simeq S/I_0[t]$ is a free $k[t]$-module, we have $x - i \in I_0[t] \subseteq I$ and so $x \in I$. $\square$

Remark 2.13. *Let $i_1,\ldots,i_r$ be the generators of I. To check the inclusion which is the assumption of Proposition 2.12, it is enough to check that $s \in (t - \lambda) \cap I$ implies $s \in (t - \lambda)I + I_0[t]$ for all $s = s_1 i_1 + \ldots + s_r i_r$, where $s_i \in S$.*

Indeed, take an arbitrary element $s \in I$ and write $s = t_1 i_1 + \ldots + t_r i_r$, where $t_1,\ldots,t_r \in S[t]$. Dividing t_i by $t - \lambda$ we obtain $s = s_1 i_1 + \ldots + s_r i_r + (t - \lambda)i$, where $i \in I$ and $s_i \in S$. Denote $s' = s_1 i_1 + \ldots + s_r i_r$, then $s \in (t - \lambda) \cap I$ if and only if $s' \in (t - \lambda) \cap I$ and $s \in (t - \lambda)I + I_0[t]$ if and only if $s' \in (t - \lambda)I + I_0[t]$.

Example 2.14. *Consider $S = k[x,y]$ and $I = xyS[t] + (x^3 - tx)S[t] \subseteq S[t]$. Take an element $s_1 xy + s_2(x^3 - tx) \in I$ and suppose $s_1 xy + s_2(x^3 - tx) \in (t - \lambda)S[t]$. We want to prove that this element lies in $I_0[t] + (t - \lambda)I$. As in Remark 2.13, by subtracting an element of $I(t - \lambda)$ we may assume that s_1, s_2 lie in S. Then $s_1 xy + s_2(x^3 - tx) \in (t - \lambda)S[t]$ if and only if $s_1 xy + s_2(x^3 - \lambda x) = 0$. In particular we have $s_2 \in yS$ so that $s_2(x^3 - tx) \in xyS[t]$, then $s_1 xy + s_2(x^3 - tx) \in xyS[t] \subseteq I_0[t]$.*

Similarly as in Example 2.14, in the following we will frequently use the following easy observation, which we state in Lemma 2.15.

Lemma 2.15. *Consider a ring $R = B[\alpha]$ graded by the degree of α. Let d be a natural number and $I \subseteq R$ be a homogeneous ideal generated in degrees less or equal to d.*

Let $q \in B[\alpha]$ be an element of α-degree strictly less than d and such that for every $b \in B$ satisfying $b\alpha^d \in I$, we have $bq \in I$. Then for every $r \in R$ the condition

$$r(\alpha^d - q) \in I \quad implies \quad r\alpha^d \in I \quad and \quad rq \in I.$$

Proof. We apply induction with respect to α-degree of r, the base case being $r = 0$. Write

$$r = \sum_{i=0}^m r_i \alpha^i, \quad \text{where} \quad r_i \in B.$$

The leading form of $r(\alpha^d - q)$ is $r_m \alpha^{m+d}$ and it lies in I. Since I is homogeneous and generated in degree at most d, we have $r_m \alpha^d \in I$. Then $r_m q \in I$ by assumption, so that $\hat{r} := r - r_m \alpha^m$ satisfies $\hat{r}(\alpha^d - q) \in I$. By induction we have $\hat{r}\alpha^d, \hat{r}q \in I$, then also $r\alpha^d, rq \in I$. $\qquad\square$

3 Standard form of the dual generator

Definition 3.1. *Let $f \in P = k[x_1, \ldots, x_n]$ be a polynomial of degree s. Let $I = \mathrm{ann}_S(f)$ and $A = S/I = \mathrm{Apolar}(f)$. By $\Delta_\bullet$ we denote the decomposition of the Hilbert function of A and we set $e(a) := \sum_{t=0}^a \Delta_t(1)$.*

We say that f is in the standard form if

$$f = f_0 + f_1 + f_2 + f_3 + \cdots + f_s, \quad \text{where} \quad f_i \in P_i \cap k[x_1, \ldots, x_{e(s-i)}] \text{ for all } i.$$

Note that if f is in the standard form and $\partial \in \mathfrak{m}_S$ then $f + \partial \lrcorner f$ is also in the standard form. We say that an Artin Gorenstein algebra S/I is in the standard form if any (or every) dual socle generator of S/I is in the standard form, see Proposition 3.5 below.

Example 3.2. *If $f = x_1^6 + x_2^5 + x_3^3$, then f is in the standard form. Indeed, $e(0) = 1$, $e(1) = 2$, $e(2) = 2$, $e(3) = 3$ so that we should check that $x_1^6 \in k[x_1]$, $x_2^5 \in k[x_1, x_2]$, $x_3^3 \in k[x_1, x_2, x_3]$, which is true. On the contrary, $g = x_3^6 + x_2^5 + x_1^3$ is not in the standard form, but may be put in the standard form via a change of variables.*

The change of variables procedure of Example 3.2 may be generalised to prove that every local Artin Gorenstein algebra can be put in a standard form, as the following Proposition 3.3 explains.

Proposition 3.3. *For every Artin Gorenstein algebra S/I there is an automorphism $\varphi : S \to S$ such that $S/\varphi(I)$ is in the standard form.*

Proof. See [Iar94, Thm 5.3AB], the proof is rewritten in [Jel13, Thm 4.38]. $\qquad\square$

The idea of the proof of Proposition 3.3 is to "linearise" some elements of S. This is quite technical and perhaps it can be best seen on the following example.

Example 3.4. *On this example we exhibit the proof of Proposition 3.3. Let $f = x_1^6 + x_1^4 x_2$. The annihilator of f in S is $(\alpha_2^2, \alpha_1^5 - \alpha_1^3 \alpha_2)$, the Hilbert function of $\mathrm{Apolar}(f)$ is $(1, 2, 2, 2, 1, 1, 1)$ and the symmetric decomposition is*

$$\Delta_0 = (1, 1, 1, 1, 1, 1, 1), \quad \Delta_1 = (0, 0, 0, 0, 0, 0), \quad \Delta_2 = (0, 1, 1, 1, 0).$$

10

This shows that $e(0) = 1$, $e(1) = 1$, $e(2) = 2$. If f is in the standard form we should have $f_5 = x_1^4 x_2 \in k[x_1, \ldots, x_{e(1)}] = k[x_1]$. This means that f is not in the standard form. The "reason" for $e(1) = 1$ is the fact that $\alpha_1^3(\alpha_2 - \alpha_1^2)$ annihilates f, and the "reason" for $f_5 \notin k[x_1]$ is that $\alpha_2 - \alpha_1^2$ is not a linear form. Thus we make $\alpha_2 - \alpha_1^2$ a linear form by twisting by a suitable automorphism of S.

We define an automorphism $\psi : S \to S$ by $\psi(\alpha_1) = \alpha_1$ and $\psi(\alpha_2) = \alpha_2 + \alpha_1^2$, so that we have $\psi(\alpha_2 - \alpha_1^2) = \alpha_2$. The automorphism maps the annihilator of f to the ideal $I := ((\alpha_2 + \alpha_1^2)^2, \alpha_1^3 \alpha_2)$. We will see that the algebra S/I is in the standard form and also find a particular dual generator obtained from f.

As mentioned in Remark 2.2, the automorphism ψ induces an automorphism ψ^* of the k-linear space $P_{\leq 6}$. This automorphism maps f to a dual socle generator $\psi^* f$ of S/I.

The element $F := \psi^* x_1^6$ is the only element of P such that $\psi(\alpha_1^7) \lrcorner F = \psi(\alpha_2) \lrcorner F = 0$, $\psi(\alpha_1^6)(F) = 1$ and $\psi(\alpha_1^l)(F) = 0$ for $l \leq 5$. Caution: in the last line we use evaluation on the functional and not the induced action (see Remark 2.2). One can compute that $\psi^* x_1^6 = x_1^6 - x_1^4 x_2 + x_1^2 x_2^2 - x_2^3$ and similarly $\psi^* x_1^4 x_2 = x_1^4 x_2 - 2x_1^2 x_2^2 + 3x_2^3$ so that $\psi^* f = x_1^6 - x_1^2 x_2^2 + 2x_2^3$. Now indeed $x_1^6 \in k[x_1]$, $x_1^2 x_2^2 \in k[x_1, x_2]$ and $2x_2^3 \in k[x_1, x_2]$ so the dual socle generator is in the standard form.

We note the following equivalent conditions for a dual socle generator to be in the standard form.

Proposition 3.5. *In the notation of Definition 3.1, the following conditions are equivalent for a polynomial $f \in P$:*

1. *the polynomial f is in the standard form,*

2. *for all r and i such that $r > e(s - i)$ we have $\mathfrak{m}_S^{i-1} \alpha_r \subseteq I = (f)^\perp$. Equivalently, for all r and i such that $r > e(i)$ we have $\mathfrak{m}_S^{s-i-1} \alpha_r \subseteq I = (f)^\perp$.*

Proof. Straightforward. $\square$

Corollary 3.6. *Let $f \in P$ be such that the algebra S/I is in the standard form, where $I = \mathrm{ann}_S(f)$. Let φ be an automorphism of S given by*

$$\varphi(\alpha_i) = \kappa_i \alpha_i + q_i \text{ where } q_i \text{ is such that } \deg(q_i \lrcorner f) \leq \deg(\alpha_i \lrcorner f) \text{ and } \kappa_i \in k \setminus \{0\}.$$

Then the algebra $S/\varphi^{-1}(I)$ is also in the standard form.

Proof. The algebras S/I and $S/\varphi^{-1}(I)$ are isomorphic, in particular they have equal functions $e(\cdot)$. By Proposition 3.5 it suffices to prove that if for some r, i we have $\mathfrak{m}_S^r \alpha_i \subseteq I$, then $\mathfrak{m}_S^r \alpha_i \subseteq \varphi^{-1}(I)$. The latter condition is equivalent to $\mathfrak{m}_S^r \varphi(\alpha_i) \subseteq I$. If $\mathfrak{m}_S^r \alpha_i \lrcorner f = 0$ then $\deg(\alpha_i \lrcorner f) < r$ so, by assumption, $\deg(q_i \lrcorner f) < r$ thus $\mathfrak{m}_S^r q_i \lrcorner f = 0$ and $\mathfrak{m}_S^r \varphi(\alpha_i) = \mathfrak{m}_S^r(\kappa_i \alpha_i + q_i) \lrcorner f = 0$. $\square$

Corollary 3.7. *Suppose that $q \in \mathfrak{m}_S^2$ does not contain α_i and let $\varphi : S \to S$ be an automorphism given by*

$$\varphi(\alpha_j) = \alpha_j \text{ for all } j \neq i \text{ and } \varphi(\alpha_i) = \kappa_i \alpha_i + q, \text{ where } \kappa_i \in k \setminus \{0\}.$$

Suppose that S/I is in the standard form, where $I = \mathrm{ann}_S(f)$ and that $\deg(q \lrcorner f) \leq \deg(\alpha_i \lrcorner f)$. Then the algebras $S/\varphi(I)$ and $S/\varphi^{-1}(I)$ are also in the standard form.

Proof. Note that $\psi : S \to S$ given by $\psi(\alpha_j) = \alpha_j$ for $j \neq i$ and $\psi(\alpha_i) = \kappa_i^{-1}(\alpha_i - q)$ is an automorphism of S and furthermore $\psi(\kappa_i \alpha_i + q) = \alpha_i - q + q = \alpha_i$ so that $\psi = \varphi^{-1}$. Both φ and ψ satisfy assumptions of Corollary 3.6 so both $S/\varphi^{-1}(I)$ and $S/\psi^{-1}(I) = S/\varphi(I)$ are in the standard form. $\square$

Remark 3.8. *The assumption $q \in \mathfrak{m}_S^2$ of Corollary 3.7 is needed only to ensure that φ is an automorphism of S. On the other hand the fact that q does not contain α_i is important, because it allows us to control φ^{-1} and in particular prove that $S/\varphi(I)$ is in the standard form.*

The following Corollary 3.9 is a straightforward generalisation of Corollary 3.7, but the notation is difficult. We first choose a set $\mathcal{K}$ of variables. The automorphism sends each variable from $\mathcal{K}$ to (a multiple of) itself plus a suitable polynomial in variables not appearing in $\mathcal{K}$.

Corollary 3.9. *Take $\mathcal{K} \subseteq \{1, 2, \ldots, n\}$ and $q_i \in \mathfrak{m}_S^2$ for $i \in \mathcal{K}$ which do not contain any variables from the set $\{\alpha_i\}_{i \in \mathcal{K}}$. Define $\varphi : S \to S$ by*

$$\varphi(\alpha_i) = \begin{cases} \alpha_i & \text{if } i \notin \mathcal{K} \\ \kappa_i \alpha_i + q_i, \text{ where } \kappa_i \in k \setminus \{0\} & \text{if } i \in \mathcal{K}. \end{cases}$$

Suppose that S/I is in the standard form, where $I = \mathrm{ann}_S(f)$ and that $\deg(q_i \lrcorner f) \leq \deg(\alpha_i \lrcorner f)$ for all $i \in \mathcal{K}$. Then the algebras $S/\varphi(I)$ and $S/\varphi^{-1}(I)$ are also in the standard form. $\square$

4 Special forms of dual socle generators

Recall that k is an algebraically closed field of characteristic neither 2 nor 3.

In the previous section we mentioned that for every local Artin Gorenstein algebra there exists a dual socle generator in the standard form, see Definition 3.1. In this section we will see that in most cases we can say more about this generator. Our main aim is to put the generator in the form $x^s + f$, where f contain no monomial divisible by a "high" power of x. We will use it to prove that families arising from certain ray decompositions (see Definition 5.2) are flat.

We begin with an easy observation.

Remark 4.1. *Suppose that a polynomial $f \in P$ is such that $H_{\mathrm{Apolar}(f)}(1)$ equals the number of variables in P. Then any linear form in P is a derivative of f. If $\deg f > 1$ then the S-submodules Sf and $S(f - f_1 - f_0)$ are equal, so analysing this modules we may assume $f_1 = f_0 = 0$, i.e. the linear part of f is zero.*

Later we use this remark implicitly.

The following Lemma 4.2 provides a method to slightly improve the given dual socle generator. This improvement is the building block of all other results in this section.

Lemma 4.2. *Let $f \in P$ be a polynomial of degree s and A be the apolar algebra of f. Suppose that $\alpha_1^s \lrcorner f \neq 0$. For every i let $d_i := \deg(\alpha_1 \alpha_i \lrcorner f) + 2$.*

Then A is isomorphic to the apolar algebra of a polynomial $\hat{f}$ of degree s, such that $\alpha_1^s \lrcorner \hat{f} = 1$ and $\alpha_1^{d_i - 1} \alpha_i \lrcorner \hat{f} = 0$ for all $i \neq 1$. Moreover, the leading forms of f and $\hat{f}$ are equal up to a non-zero constant. If f is in the standard form, then $\hat{f}$ is also in the standard form.

Proof. By multiplying f by a non-zero constant we may assume that $\alpha_1^s \lrcorner f = 1$. Denote $I := \mathrm{ann}_S(f)$. Since $\deg(\alpha_1 \alpha_i \lrcorner f) = d_i - 2$, the polynomial $\alpha_1^{d_i - 1} \alpha_i \lrcorner f = \alpha_1^{d_i - 2}(\alpha_1 \alpha_i \lrcorner f)$ is constant; we denote it by λ_i. Then

$$\left(\alpha_1^{d_i - 1} \alpha_i - \lambda_i \alpha_1^s\right) \lrcorner f = 0, \text{ so that } \alpha_1^{d_i - 1}\left(\alpha_i - \lambda_i \alpha_1^{s - d_i + 1}\right) \in I.$$

Define an automorphism $\varphi : S \to S$ by

$$\varphi(\alpha_i) = \begin{cases} \alpha_1 & \text{if } i = 1 \\ \alpha_i - \lambda_i \alpha_1^{s - d_i + 1} & \text{if } i \neq 1, \end{cases}$$

12

then $\alpha_1^{d_i-1}\alpha_i \in \varphi^{-1}(I)$ for all $i > 1$. The dual socle generator $\hat{f}$ of the algebra $S/\varphi^{-1}(I)$ has the required form. We can easily check that the graded algebras of $S/\varphi^{-1}(I)$ and S/I are equal, in particular $\hat{f}$ and f have the same leading form, up to a non-zero constant.

Suppose now that f is in the standard form. Let $i \in \{1, \ldots, n\}$. Then $d_i = \deg(\alpha_1\alpha_i \lrcorner f) + 2 \leq \deg(\alpha_i \lrcorner f) + 1$, so that $\deg(\alpha_1^{s-d_i+1} \lrcorner f) \leq d_i - 1 \leq \deg(\alpha_i \lrcorner f)$. Since φ is an automorphism of S, by Remark 3.8 we may apply Corollary 3.9 to φ. Then $S/\varphi(I)$ is in the standard form, so $\hat{f}$ is in the standard form by definition. $\qquad\square$

Example 4.3. *Let $f \in k[x_1, x_2, x_3, x_4]$ be a polynomial of degree s. Suppose that the leading form f_s of f can be written as $f_s = x_1^s + g_s$ where $g_s \in k[x_2, x_3, x_4]$. Then $\deg(\alpha_1\alpha_i \lrcorner f) \leq s - 3$ for all $i > 1$. Using Lemma 4.2 we produce $\hat{f} = x_1^s + h$ such that the apolar algebras of f and $\hat{f}$ are isomorphic and $\alpha_1^{s-2}\alpha_i \lrcorner h = 0$ for all $i \neq 1$. Then $\alpha_1^{s-2} \lrcorner h = \lambda_1 x_1 + \lambda_2$, where $\lambda_i \in k$ for $i = 1, 2$. After adding a suitable derivative to $\hat{f}$, we may assume $\lambda_1 = \lambda_2 = 0$, i.e. $\alpha_1^{s-2} \lrcorner h = 0$.*

Example 4.4. *Suppose that a local Artin Gorenstein algebra A of socle degree s has Hilbert function equal to $(1, H_1, H_2, \ldots, H_c, 1, \ldots, 1)$. The standard form of the dual socle generator of A is*

$$f = x_1^s + \kappa_{s-1}x_1^{s-1} + \cdots + \kappa_{c+2}x_1^{c+2} + g,$$

where $\deg g \leq c + 1$ and $\kappa_\bullet \in k$. By adding a suitable derivative we may furthermore make all $\kappa_i = 0$ and assume that $\alpha_1^{c+1} \lrcorner g = 0$. Using Lemma 4.2 we may also assume that $\alpha_1^c \alpha_j \lrcorner g = 0$ for every $j \neq 1$ so we may assume $\alpha_1^c \lrcorner g = 0$, arguing as in Example 4.3. This gives a dual socle generator

$$f = x_1^s + g,$$

where $\deg g \leq c + 1$ and g does not contain monomials divisible by x_1^c.

The following proposition was proved in [CN13] under the assumption that k is algebraically closed of characteristic zero and in [Jel13, Thm 5.1] under the assumption that $k = \mathbb{C}$. For completeness we include the proof (with no further assumptions on k other than the ones listed at the beginning of this section).

Proposition 4.5. *Let A be Artin local Gorenstein algebra of socle degree $s \geq 2$ such that the Hilbert function decomposition from Theorem 2.3 has $\Delta_{A,s-2} = (0, q, 0)$. Then A is isomorphic to the apolar algebra of a polynomial f such that f is in the standard form and the quadric part f_2 of f is a sum of q squares of variables not appearing in $f_{\geq 3}$ and a quadric in variables appearing in $f_{\geq 3}$.*

Proof. Let us take a standard dual socle generator $f \in P := k[x_1, \ldots, x_n]$ of the algebra A. Now we will twist f to obtain the required form of f_2. We may assume that $H_{\mathrm{Apolar}(f)}(1) = n$.

If $s = 2$, then the theorem follows from the fact that the quadric f may be diagonalised. Assume $s \geq 3$. Let $e := e(s - 3) = \sum_{t=0}^{s-3} \Delta_{A,t}(1)$. We have $n = e(s - 2) = f + q$, so that $f_{\geq 3} \in k[x_1, \ldots, x_e]$ and $f_2 \in k[x_1, \ldots, x_n]$. Note that $f_{\geq 3}$ is also in the standard form, so that every linear form in $x_1, \ldots, x_e$ is a derivative of $f_{\geq 3}$, see Remark 4.1.

First, we want to assure that $\alpha_n^2 \lrcorner f \neq 0$. If $\alpha_n \lrcorner f \in k[x_1, \ldots, x_e]$ then there exists an operator $\partial \in \mathfrak{m}_S^2$ such that $(\alpha_n - \partial) \lrcorner f = 0$. This contradicts the fact that f was in the standard form (see the discussion in Example 3.4). So we get that $\alpha_n \lrcorner f$ contains some x_r for $r > e$, i.e. f contains a monomial $x_r x_n$. A change of variables involving only x_r and x_n preserves the standard form and gives $\alpha_n^2 \lrcorner f \neq 0$.

Applying Lemma 4.2 to x_n we see that f may be taken to be in the form $\hat{f} + x_n^2$, where $\hat{f}$ does not contain x_n, i.e. $\hat{f} \in k[x_1, \ldots, x_{n-1}]$. We repeat the argument for $\hat{f}$. $\qquad\square$

Example 4.6. *If A is an algebra of socle degree 3, then $H_A = (1, n, e, 1)$ for some n, e. Moreover, $n \geq e$ and the symmetric decomposition of H_A is $(1, e, e, 1) + (0, n - e, 0)$. By Proposition 4.5 we see that A is isomorphic to the apolar algebra of*

$$f + \sum_{e < i \leq n} x_i^2,$$

where $f \in k[x_1, \ldots, x_e]$. This claim was first proved by Elias and Rossi, see [ER12, Thm 4.1].

4.1 Irreducibility for fixed Hilbert function in two variables.

Below we analyse local Artin Gorenstein algebras with Hilbert function $(1, 2, 2, \ldots)$. Such algebras are (in some cases) classified up to isomorphism in [EV11], but rather than such classification we need to know the geometry of their parameter space, which is analysed (among other such spaces) in [Iar77].

We need the following Proposition 4.7, which is part of folklore. We thank J. Buczyński for explaining the proof.

Let $r \geq 1$ be a natural number. By $\mathcal{H}ilb_r \operatorname{Spec} S$ we denote the Hilbert scheme of length r subschemes of the power series ring S. It is called the *punctual Hilbert scheme* because as a set, $\mathcal{H}ilb_r \operatorname{Spec} S$ is equal to the set of length r subschemes of $\mathbb{P}^n$ supported at a single fixed point.

We recall a classical construction. Let V be a constructible subset of $P_{\leq s}$. Assume that the apolar algebra $\operatorname{Apolar}(f)$ has length r for every closed point $f \in V$. Then we may construct the incidence scheme $\{(f, \operatorname{Apolar}(f))\} \to V$ which is a finite flat family over V and thus we obtain a morphism from V to $\mathcal{H}ilb_r \operatorname{Spec} S$. See [Jel13, Prop 4.39] for details.

Proposition 4.7. *Let $\mathcal{R} \subseteq \mathcal{H}ilb_r \operatorname{Spec} S$ be a constructible subset and $V \subseteq P$ denote the set of all possible dual socle generators of elements of $\mathcal{R}$. If $\mathcal{R}$ is irreducible, then also V is irreducible.*

Proof. Below by k^* and S^* we denote the sets of invertible elements of k and S respectively.

There is an induced surjective morphism φ from V to $\mathcal{R}$ as explained above. By construction the fiber over $\varphi(f)$ is $S^* \lrcorner f$. The image $\mathcal{R}$ of φ is irreducible, so it is enough to show the existence of an open cover $\{H_i\}$ of $\mathcal{R}$ such that every $\varphi^{-1}(H_i)$ is irreducible.

Choose an element $f \in V$ and a section of $\mathfrak{m}_S / \operatorname{ann}_S(f)$ to $\mathfrak{m}_S$, that is, a linear subspace $\mathfrak{m}(f) \subseteq \mathfrak{m}_S$ such that $\mathfrak{m}(f) \to \mathfrak{m}_S / \operatorname{ann}_S(f)$ is bijective. Let $O(f) := \mathfrak{m}(f) + k \subseteq S$, then $S \lrcorner f = O(f) \lrcorner f$. Finally let $O(f)^* := k^* + \mathfrak{m}(f)$, so that $\varphi^{-1}(\varphi(f)) = O(f)^* \lrcorner f$. Consider the set

$$U_f = \{g \in V \mid O(f) \cap \operatorname{ann}_S(g) = 0\} = \{g \in V \mid O(f) \lrcorner g = S \lrcorner g\}.$$

It is an open set in V and its image $H_f = \varphi(U_f)$ is open (hence irreducible) in the Hilbert scheme. Moreover $U_f = \varphi^{-1}(H_f)$. For every $g \in U_f$ the fiber $\varphi^{-1}(\varphi(g))$ is equal to $O(f)^* \lrcorner g$.

By [Ems78, Proposition 18 and its Corollary] there is an open neighborhood $H_f' \subseteq H_f$ of $\varphi(f)$ such that the morphism $\varphi : \varphi^{-1}(H_f') \to H_f'$ has a section i. Denoting $\varphi^{-1}(H_f')$ by U_f', we have a surjective morphism $O(f)^* \times H_f' \to U_f'$ mapping (σ, h) to $\sigma \lrcorner i(h)$. Since $O(f)^*$ and H_f' are irreducible, also U_f' is irreducible. Therefore $\{H_f'\}$ form a desired cover of $\mathcal{R}$ and so V is irreducible. □

Proposition 4.8. *Let $H = (1, 2, 2, *, \ldots, *, 1)$ be a vector of length $s + 1$. The set of polynomials $f \in k[x_1, x_2]$ such that $H_{\operatorname{Apolar}(f)} = H$ constitutes an irreducible subscheme of the affine space $k[x_1, x_2]_{\leq s}$. A general member of this set has, up to an automorphism of P induced by an automorphism of S, the form $f + \partial \lrcorner f$, where $f = x_1^s + x_2^{s_2}$ for some $s_2 \leq s$.*

Proof. Let $V \subseteq k[x_1, x_2]$ denote the set of f such that $H_{\operatorname{Apolar}(f)} = H$. Then the image of V under the mapping sending f to $\operatorname{Apolar}(f)$ is irreducible by [Iar77, Thm 3.13]. By Proposition 4.7 the set V is irreducible.

In the case $H = (1, 1, 1, \ldots, 1)$ the claim (with $s_2 = 0$) follows directly from the existence of the standard form of a polynomial. Further in the proof we assume $H(1) = 2$.

Let us take a general polynomial f such that $H_{\mathrm{Apolar}(f)} = H$. Then $\mathrm{ann}_S(f) = (q_1, q_2)$ is a complete intersection, where $q_1 \in S$ has order 2, i.e. $q_1 \in \mathfrak{m}_S^2 \setminus \mathfrak{m}_S^3$. Since f is general, we may assume that the quadric part of q_1 has maximal rank, i.e. rank two, see also [Iar77, Thm 3.14]. Then after a change of variables $q_1 \equiv \alpha_1 \alpha_2 \mod \mathfrak{m}_S^3$. Since the leading form $\alpha_1 \alpha_2$ of q_1 is reducible, $q_1 = \delta_1 \delta_2$ for some $\delta_1, \delta_2 \in S$ such that $\delta_i \equiv \alpha_i \mod \mathfrak{m}_S^2$ for $i = 1, 2$, see e.g. [Kun05, Thm 16.6]. After an automorphism of S we may assume $\delta_i = \alpha_i$, then $\alpha_1 \alpha_2 = q_1$ annihilates f, so that it has the required form. $\square$

4.2 Homogeneous forms and secant varieties

It is well-known that if $F \in P_s$ is a form such that $H_{\mathrm{Apolar}(F)} = (1, 2, \ldots, 2, 1)$ then the standard form of F is either $x_1^s + x_2^s$ or $x_1^{s-1} x_2$. In particular the set of such forms in P is irreducible and in fact it is open in the so-called secant variety. This section is devoted to some generalisations of this result for the purposes of classification of leading forms of polynomials in P.

The following proposition is well-known if the base field is of characteristic zero (see [BGI11, Thm 4] or [LO13]), but we could not find a reference for the positive characteristic case, so for completeness we include the proof.

Proposition 4.9. *Suppose that $F \in k[x_1, x_2, x_3]$ is a homogeneous polynomial of degree $s \geq 4$. The following conditions are equivalent*

1. *the algebra* $\mathrm{Apolar}(F)$ *has Hilbert function H beginning with $H(1) = H(2) = H(3) = 3$, i.e. $H = (1, 3, 3, 3, \ldots)$,*

2. *after a linear change of variables F is in one of the forms*

$$x_1^s + x_2^s + x_3^s, \qquad x_1^{s-1} x_2 + x_3^s, \qquad x_1^{s-2}(x_1 x_3 + x_2^2).$$

Furthermore, the set of forms in $k[x_1, x_2, x_3]_s$ satisfying the above conditions is irreducible.

Proof. For the characteristic zero case see [LO13] and references therein.

Let $S = k[\alpha_1, \alpha_2, \alpha_3]$ be a polynomial ring dual to P. This notation is incoherent with the global notation, but it is more readable than S_{poly}.

Let $I := \mathrm{ann}_S(F)$ and $I_2 := \langle \theta_1, \theta_2, \theta_3 \rangle \subseteq S_2$ be the linear space of operators of degree 2 annihilating F. Let $A := S/I$, $J := (I_2) \subseteq S$ and $B := S/J$. Since A has length greater than $3 \cdot 3 > 2^3$, the ideal J is not a complete intersection. Let us analyse the Hilbert function of A. By symmetry of H_A, we have $H_A(s-1) = H_A(1) = 3$. By Remark 2.7 we have $3 = H_A(3) \geq H_A(4) \geq \ldots \geq H_A(s-1) = 3$, thus

$$H_A(m) = 3 \quad \text{for all} \quad m = 1, 2, \ldots, s-1.$$

We will prove that the graded ideal J is saturated and defines a zero-dimensional scheme of degree 3 in $\mathbb{P}^2 = \mathrm{Proj}\, S$. First, $3 = H_A(3) \leq H_B(3) \leq 4$ by Macaulay's Growth Theorem. If $H_B(3) = 4$ then by Lemma 2.9 and Remark 2.10 we have $H_A(1) = 2$, a contradiction. We have proved that $H_B(3) = 3$.

Now we want to prove that $H_B(4) = 3$. By Macaulay's Growth Theorem applied to $H_B(3) = 3$ we have $H_B(4) \leq 3$. If $s > 4$ then $H_A(4) = 3$, so $H_B(4) \geq 3$. Suppose $s = 4$. By Buchsbaum-Eisenbud result [BE77] we know that the minimal number of generators of I is odd. Moreover, we know that $A_n = B_n$ for $n < 4$, thus the generators of I have degree two or four. Since I_2 is not a complete intersection, there are at least two generators of degree 4, so $H_B(4) \geq H_A(4) + 2 = 3$.

15

From $H_B(3) = H_B(4) = 3$ by Gotzmann's Persistence Theorem we see that $H_B(m) = 3$ for all $m \geq 1$. Thus the scheme $\Gamma := V(J) \subseteq \mathrm{Proj}\, k[\alpha_1, \alpha_2, \alpha_3]$ is finite of degree 3 and J is saturated. In particular, the ideal $J = I(\Gamma)$ is contained in I.

We will use Γ to compute the possible forms of F, in the spirit of Apolarity Lemma, see [IK99, Lem 1.15]. There are four possibilities for Γ:

1. Γ is a union of three distinct, non-collinear points. After a change of basis $\Gamma = \{[1:0:0]\} \cup \{[0:1:0]\} \cup \{[0:0:1]\}$, then $I_2 = (\alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1)$ and $F = x_1^s + x_2^s + x_3^s$.

2. Γ is a union of a point and scheme of length two, such that $\langle \Gamma \rangle = \mathbb{P}^2$. After a change of basis $I_\Gamma = (\alpha_1^2, \alpha_1\alpha_2, \alpha_2\alpha_3)$, so that $F = x_3^{s-1}x_1 + x_2^s$.

3. Γ is irreducible with support $[1:0:0]$ and it is not a 2-fat point. Then Γ is Gorenstein and so Γ may be taken as the curvilinear scheme defined by $(\alpha_3^2, \alpha_2\alpha_3, \alpha_1\alpha_3 - \alpha_2^2)$. Then, after a linear change of variables, $F = x_1^{s-1}x_3 + x_2^2 x_1^{s-2}$.

4. Γ is a 2-fat point supported at $[1:0:0]$. Then $I_\Gamma = (\alpha_2^2, \alpha_2\alpha_3, \alpha_3^2)$, so $F = x_1^{s-1}(\lambda_2 x_2 + \lambda_3 x_3)$ for some $\lambda_2, \lambda_3 \in k$. But then there is a degree one operator in S annihilating F, a contradiction.

The set of forms F which are sums of three powers of linear forms is irreducible. To see that the forms satisfying the assumptions of the Proposition constitute an irreducible subset of P_s we observe that every Γ as above is smoothable by [CEVV09]. The flat family proving the smoothability of Γ induces a family $F_t \to F$, such that F_λ is a sum of three powers of linear forms for $\lambda \neq 0$, see [Ems78, Corollaire in Section 2]. See also [BB14] for a generalisation of this method. $\square$

Proposition 4.10. *Let $s \geq 4$. Consider the set $\mathcal{S}$ of all forms $F \in k[x_1, x_2, x_3, x_4]$ of degree s such that the apolar algebra of F has Hilbert function $(1, 4, 4, 4, \ldots, 4, 1)$. This set is irreducible and its general member has the form $\ell_1^s + \ell_2^s + \ell_3^s + \ell_4^s$, where ℓ_1, ℓ_2, ℓ_3, ℓ_4 are linearly independent linear forms.*

Proof. First, the set $\mathcal{S}_0$ of forms equal to $\ell_1^4 + \ell_2^4 + \ell_3^4 + \ell_4^4$, where ℓ_1, ℓ_2, ℓ_3, ℓ_4 are linearly independent linear forms, is irreducible and contained in $\mathcal{S}$. Then, it is enough to prove that $\mathcal{S}$ lies in the closure of $\mathcal{S}_0$.

We follow the proof of Proposition 4.9, omitting some details which can be found there. Let $S = k[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$, $I := \mathrm{ann}_S(F)$ and $J := (I_2)$. Set $A = S/I$ and $B = S/J$. Then $H_B(2) = 4$ and $H_B(3)$ is either 4 or 5. If $H_B(3) = 5$, then by Lemma 2.9 we have $H_B(1) = 3$, a contradiction. Thus $H_B(3) = 4$.

Now we would like to prove $H_B(4) = 4$. By Macaulay's Growth Theorem $H_B(4) \leq 5$. By Lemma 2.9 $H_B(4) \neq 5$, thus $H_B(4) \leq 4$. If $s > 4$ then $H_B(4) \geq H_A(4) \geq 4$, so we concentrate on the case $s = 4$. Let us write the minimal free resolution of A, which is symmetric by [Eis95, Cor 21.16]:

$$0 \to S(-8) \to S(-4)^{\oplus a} \oplus S(-6)^{\oplus 6} \to S(-3)^{\oplus b} \oplus S(-4)^{\oplus c} \oplus S(-5)^{\oplus b} \to S(-2)^{\oplus 6} \oplus S(-4)^{\oplus a} \to S.$$

Calculating $H_A(3) = 4$ from the resolution, we get $b = 8$. Calculating $H_A(4) = 1$ we obtain $6 - 2a + c = 0$. Since $1 + a = H_B(4) \leq 4$ we have $a \leq 3$, so $a = 3$, $c = 0$ and $H_B(4) = 4$.

Now we calculate $H_B(5)$. If $s > 5$ then $H_B(5) = 4$ as before. If $s = 4$ then extracting syzygies of I_2 from the above resolution we see that $H_B(5) = 4 + \gamma$, where $0 \leq \gamma \leq 8$, thus $H_B(5) = 4$ and $\gamma = 0$. If $s = 5$, then the resolution of A is

$$0 \to S(-9) \to S(-4)^{\oplus 3} \oplus S(-7)^{\oplus 6} \to S(-3)^{\oplus 8} \oplus S(-6)^{\oplus 8} \to S(-5)^{\oplus 3} \oplus S(-2)^{\oplus 6} \to S.$$

So $H_B(5) = 56 - 20 \cdot 6 + 8 = 4$. Thus, as in the previous case we see that J is the saturated ideal of a scheme Γ of degree 4. Then Γ is smoothable by [CEVV09] and its smoothing induces a family $F_t \to F$, where $F_\lambda \in \mathcal{S}_0$ for $\lambda \neq 0$. $\qed$

The following Corollary 4.11 is a consequence of Proposition 4.10. This corollary is not used in the proofs of the main results, but it is of certain interest of its own and shows another connection with secant varieties. For simplicity and to refer to some results from [LO13], we assume that $k = \mathbb{C}$, but the claim holds for all fields of characteristic either zero or large enough.

To formulate the claim we introduce catalecticant matrices. Let $\varphi_{a,s-a} : S_a \times P_s \to P_{s-a}$ be the contraction mapping applied to homogeneous polynomials of degree s. For $F \in P_s$ we obtain $\varphi_{a,s-a}(F) : S_a \to P_{s-a}$, whose matrix is called the *a-catalecticant matrix*. It is straightforward to see that $\operatorname{rk} \varphi_{a,s-a}(F) = H_{\mathrm{Apolar}(F)}(a)$.

Corollary 4.11. *Let $s \geq 4$ and $k = \mathbb{C}$. The fourth secant variety to s-th Veronese reembedding of $\mathbb{P}^n$ is a subset $\sigma_4(v_s(\mathbb{P}^n)) \subseteq \mathbb{P}(P_s)$ set-theoretically defined by the condition $\operatorname{rk} \varphi_{a,s-a} \leq 4$, where $a = \lfloor s/2 \rfloor$.*

Proof. Since $H_{\mathrm{Apolar}(F)}(a) \leq 4$ for F which is a sum of four powers of linear forms, by semicontinuity every $F \in \sigma_4(v_s(\mathbb{P}^n))$ satisfies the above condition.

Let $F \in P_s$ be a form satisfying $\operatorname{rk} \varphi_{a,s-a}(F) \leq 4$. Let $A = \mathrm{Apolar}\,(F)$ and $H = H_A$ be the Hilbert function of A. We want to reduce to the case where $H(n) = 4$ for all $0 < n < s$.

First we show that $H(n) \geq 4$ for all $0 < n < s$. If $H(1) \leq 3$, then the claim follows from [LO13, Thm 3.2.1 (2)], so we assume $H(1) \geq 4$. Suppose that for some n satisfying $4 \leq n < s$ we have $H(n) < 4$. Then by Remark 2.7 we have $H(m) \leq H(n)$ for all $m \geq n$, so that $H(1) = H(s-1) < 4$, a contradiction. Thus $H(n) \geq 4$ for all $n \geq 4$. Moreover, $H(3) \geq 4$ by Macaulay's Growth Theorem. Suppose now that $H(2) < 4$. By Theorem 2.6 the only possible case is $H(2) = 3$ and $H(3) = 4$. But then $H(1) = 2 < 4$ by Lemma 2.9, a contradiction. Thus we have proved that

$$H(n) \geq 4 \quad \text{for all} \quad 0 < n < s. \tag{1}$$

We have $H(a) = 4$. If $s \geq 8$, then $a \geq 4$, so by Remark 2.7 we have $H(n) \leq 4$ for all $n > a$. Then by the symmetry $H(n) = H(s-n)$ we have $H(n) \leq 4$ for all n. Together with $H(n) \geq 4$ for $0 < n < s$, we have $H(n) = 4$ for $0 < n < s$. Then $F \in \sigma_4(v_s(\mathbb{P}^n))$ by Proposition 4.10. If $a = 3$ (i.e. $s = 6$ or $s = 7$), then $H(4) \leq 4$ by Lemma 2.9 and we finish the proof as in the case $s \geq 8$. If $s = 5$, then $a = 2$ and the Hilbert function of A is $(1, n, 4, 4, n, 1)$. Again by Lemma 2.9, we have $n \leq 4$, thus $n = 4$ by (1) and Proposition 4.10 applies. If $s = 4$, then $H = (1, n, 4, n, 1)$. Suppose $n \geq 5$, then Lemma 2.9 gives $n \leq 3$, a contradiction. Thus $n = 4$ and Proposition 4.10 applies also to this case. $\qed$

Note that for $s \geq 8$ the Corollary 4.11 was also proved, in the case $k = \mathbb{C}$, in [BB14, Thm 1.1].

5   Ray sums, ray families and their flatness

Recall that k is an algebraically closed field of characteristic neither 2 nor 3. Since $k[[\alpha_i]]$ is a discrete valuation ring, all its ideals have the form $\alpha_i^\nu k[[\alpha_i]]$ for some $\nu \geq 0$. We use this property to construct certain decompositions of the ideals in the power series ring S.

Definition 5.1. *Let I be an ideal of finite colength in the power series ring $k[[\alpha_1, \dots, \alpha_n]]$ and $\pi_i : k[[\alpha_1, \dots, \alpha_n]] \twoheadrightarrow k[[\alpha_i]]$ be the projection defined by $\pi_i(\alpha_j) = 0$ for $j \neq i$ and $\pi_i(\alpha_i) = \alpha_i$.*
*   The i-th* ray order *of I is a non-negative integer $\nu = \mathrm{rord}_i\,(I)$ such that $\pi_i(I) = (\alpha_i^\nu)$.*

By the discussion above, the ray order is well-defined. Below by $\mathfrak{p}_i$ we denote the kernel of π_i; this is the ideal generated by all variables except for α_i.

Definition 5.2. *Let I be an ideal of finite colength in the power series ring $S = k[[\alpha_1, \ldots, \alpha_n]]$. A ray decomposition of I with respect to α_i consists of an ideal $J \subseteq S$, such that $J \subseteq I \cap \mathfrak{p}_i$, together with an element $q \in \mathfrak{p}_i$ and $\nu \in \mathbb{Z}_+$ such that*

$$I = J + (\alpha_i^\nu - q)S.$$

Note that from Definition 5.1 it follows that for every I and i a ray decomposition (with $J = I \cap \mathfrak{p}_i$) exists and that $\nu = \mathrm{rord}_i(I)$ for every ray decomposition.

Definition 5.3. *Let $S = k[[\alpha_1, \ldots, \alpha_n]]$ and $S_{poly} = k[\alpha_1, \ldots, \alpha_n] \subseteq S$. Let $I = J + (\alpha_i^\nu - q)S$ be a ray decomposition of a finite colength ideal $I \subseteq S$. Let $J_{poly} = J \cap S_{poly}$. The associated lower ray family is*

$$k[t] \to \frac{S_{poly}[t]}{J_{poly}[t] + (\alpha_i^\nu - t \cdot \alpha_i - q)S_{poly}[t]},$$

and the associated upper ray family is

$$k[t] \to \frac{S_{poly}[t]}{J_{poly}[t] + (\alpha_i^\nu - t \cdot \alpha_i^{\nu-1} - q)S_{poly}[t]}.$$

If the lower (upper) family is flat over $k[t]$ we will call it a lower (upper) ray degeneration.

Note that the lower and upper ray degenerations agree for $\nu = 2$.

Remark 5.4. *In all considered cases the quotient S_{poly}/J_{poly} will be finite over k, so that every ray family will be finite over $k[t]$. Then every ray degeneration will give a morphism to the Hilbert scheme. We leave this check to the reader.*

Remark 5.5. *In this remark for simplicity we assume that $i = 1$ in Definition 5.3. Below we write α instead of α_1. Let us look at the fibers of the upper ray family from this definition in a special case, when $\alpha \cdot q \in J$. The fiber over $t = 0$ is isomorphic to S/I. Let us take $\lambda \neq 0$ and analyse the fiber at $t = \lambda$. This fiber is supported at $(0, 0, \ldots, 0)$ and at $(0, \ldots, 0, \lambda, 0, \ldots, 0)$, where λ appears on the i-th position. In particular, this shows that the existence of an upper ray degeneration proves that the algebra S/I is limit-reducible; this is true also for the lower ray degeneration.*

Now $\alpha^{\nu+1} - \lambda\alpha^\nu$ is in the ideal defining the fiber of the upper ray family over $t = \lambda$. Now one may compute that near $(0, \ldots, 0)$ the ideal defining the fiber is $(\lambda\alpha^{\nu-1} - q) + J$. Similarly near $(0, \ldots, 0, \lambda, 0, \ldots, 0)$ it is $(\alpha - \lambda) + (q) + J$. The argument is similar (though easier) to the proof of Proposition 5.10.

Most of the families constructed in [CEVV09] and [CN09] are ray degenerations.

Definition 5.6. *For a non-zero polynomial $f \in P$ and $d \geq 2$ the d-th ray sum of f with respect to a derivation $\partial \in \mathfrak{m}_S$ is a polynomial $g \in P[x]$ given by*

$$g = f + x^d \cdot \partial \lrcorner f + x^{2d} \cdot \partial^2 \lrcorner f + x^{3d} \cdot \partial^3 \lrcorner f + \ldots .$$

The following proposition shows that a ray sum naturally induces a ray decomposition, which can be computed explicitly.

Proposition 5.7. *Let g be the d-th ray sum of f with respect to $\partial \in \mathfrak{m}_S$ such that $\partial \lrcorner f \neq 0$. Let α be an element dual to x, so that $P[x]$ and $T := S[[\alpha]]$ are dual. The annihilator of g in T is given by the formula*

$$\mathrm{ann}_T(g) = \mathrm{ann}_S(f) + \left(\sum_{i=1}^{d-1} k\alpha^i \right) \mathrm{ann}_S(\partial \lrcorner f) + (\alpha^d - \partial)T, \tag{2}$$

where the sum denotes the sum of k-vector spaces. In particular, the ideal $\mathrm{ann}_T(g) \subseteq T$ is generated by $\mathrm{ann}_S(f)$, $\alpha \, \mathrm{ann}_S(\partial \lrcorner f)$ and $\alpha^d - \partial$. The formula (2) is a ray decomposition of $\mathrm{ann}_T(g)$ with respect to α and with $J = \mathrm{ann}_S(f)T + \alpha \, \mathrm{ann}_S(\partial \lrcorner f)T$ and $q = \partial$.

Proof. It is straightforward to see that the right hand side of Equation (2) lies in $\mathrm{ann}_T(g)$. Let us take any $\partial' \in \mathrm{ann}_T(g)$. Reducing the powers of α using $\alpha^d - \partial$ we can write

$$\partial' = \sigma_0 + \sigma_1 \alpha + \cdots + \sigma_{k-1} \alpha^{d-1},$$

where $\sigma_\bullet$ do not contain α. The action of this derivation on g gives

$$0 = \sigma_0 \lrcorner f + x \sigma_{d-1} \partial \lrcorner f + x^2 \sigma_{d-2} \partial \lrcorner f + \cdots + x^{d-1} \sigma_1 \partial \lrcorner f + x^d (\dots).$$

We see that $\sigma_0 \in \mathrm{ann}_S(f)$ and $\sigma_i \in \mathrm{ann}_S(\partial \lrcorner f)$ for $i \geq 1$, so the equality is proved. It is also clear that $J \subseteq \mathfrak{m}_S T$ and $\mathrm{ann}_T(g) = J + (\alpha^d - \partial)T$, so that indeed we obtain a ray decomposition. $\square$

Remark 5.8. *It is not hard to compute the Hilbert function of the apolar algebra of a ray sum in some special cases. We mention one such case below. Let $f \in P$ be a polynomial satisfying $f_2 = f_1 = f_0 = 0$ and $\partial \in \mathfrak{m}_S^2$ be such that $\partial \lrcorner f = \ell$ is a linear form, so that $\partial^2 \lrcorner f = 0$. Let $A = \mathrm{Apolar}(f)$ and $B = \mathrm{Apolar}(f + x^2 \ell)$. The only different values of H_A and H_B are $H_B(m) = H_A(m) + 1$ for $m = 1, 2$. The $f_2 = f_1 = f_0 = 0$ assumption is needed to ensure that the degrees of $\partial \lrcorner f$ and $\partial \lrcorner (f + x^2 \ell)$ are equal for all ∂ not annihilating f.*

5.1 Flatness of ray families

Proposition 5.9. *Let g be the d-th ray sum with respect to f and ∂. Then the corresponding upper and lower ray families are flat. Recall, that these families are explicitly given as*

$$k[t] \to \frac{T_{poly}[t]}{J_{poly}[t] + (\alpha^d - t\alpha^{d-1} - \partial)T_{poly}[t]} \qquad \textit{(upper ray family),} \tag{3}$$

$$k[t] \to \frac{T_{poly}[t]}{J_{poly}[t] + (\alpha^d - t\alpha - \partial)T_{poly}[t]} \qquad \textit{(lower ray family),} \tag{4}$$

where T_{poly} is the fixed polynomial subring of T.

Proof. We start by proving the flatness of Family (4).

We want to use Proposition 2.12. To simplify notation let $J := J_{poly}$. Denote by $\mathfrak{I}$ the ideal defining the family and suppose that some $z \in \mathfrak{I}$ lies in $(t - \lambda)$ for some $\lambda \in k$. Write z as $i + i_2(\alpha^d - t\alpha - \partial)$, where $i \in J[t]$, $i_2 \in T_{poly}[t]$, and note that by Remark 2.13 we may assume $i \in J$, $i_2 \in T_{poly}$. Since $z \in (t - \lambda)$, we have that $i + i_2(\alpha^d - \lambda\alpha - \partial) = 0$, so

$$i_2(\alpha^d - \lambda\alpha - \partial) = -i \in J.$$

By Proposition 5.7 the ideal J is homogeneous with respect to grading by α. More precisely it is equal to $J_0 + J_1 \alpha$, where $J_0 = \mathrm{ann}_S(f)T$, $J_1 = \mathrm{ann}_S(\partial \lrcorner f)T$ are generated by elements not containing α, so that J is generated by elements of α-degree at most one. We now check the assumptions of Lemma 2.15. Note that $\partial J \subseteq J_0$ by definition of J. If $r \in T_{poly}$ is such that $r\alpha^d \in J$, then $r \in J_1$, so that $r(\lambda\alpha + \partial) \in \alpha J_1 + J_0 \subseteq J$. Therefore the assumptions are satisfied and the Lemma shows that $i_2 \alpha^d \in J$. Then $i_2 \alpha \in J$, thus $i_2(\alpha^d - t\alpha) \in J[t] \subseteq (\mathfrak{I} \cap T_{poly})[t]$. Since $i_2 \partial \in \mathfrak{I} \cap T_{poly}$ by definition, this implies that $i + i_2(\alpha^d - t\alpha - \partial) \in J[t] \subseteq (\mathfrak{I} \cap T_{poly})[t]$. Now the flatness follows from Proposition 2.12.

The same proof works equally well for upper ray family: one should just replace α by α^{d-1} in appropriate places of the proof. For this reason we leave the case of Family (3) to the reader. $\square$

Proposition 5.10. *Let us keep the notation of Proposition 5.9. Let $\lambda \in k \setminus \{0\}$. The fibers of the Family (3) and Family (4) over $t - \lambda$ are reducible.*

Suppose that $\partial^2 \lrcorner f = 0$ and the characteristic of k does not divide $d - 1$. The fiber of the Family (4) over $t - \lambda$ is isomorphic to

$$\mathrm{Spec}\,\mathrm{Apolar}(f) \sqcup (\mathrm{Spec}\,\mathrm{Apolar}(\partial f))^{\sqcup d-1}.$$

Proof. For both families the support of the fiber over $t - \lambda$ contains the origin. The support of the fiber of Family (3) contains furthermore a point with $\alpha = \lambda$ and other coordinates equal to zero. The support of the fiber of Family (4) contains a point with $\alpha = \omega$, where $\omega^{d-1} = \lambda$.

Now let us concentrate on Family (4) and on the case $\partial^2 \lrcorner f = 0$. The support of the fiber over $t - \lambda$ is $(0, \ldots, 0, 0)$ and $(0, \ldots, 0, \omega)$, where $\omega^{d-1} = \lambda$ are $(d-1)$-th roots of λ, which are pairwise different because of the characteristic assumption. We will analyse the support point by point. By hypothesis $\partial \in \mathrm{ann}_S (\partial \lrcorner f)$, so that $\alpha \cdot \partial \in J$, thus $\alpha^{d+1} - \lambda \cdot \alpha^2$ is in the ideal I of the fiber over $t = \lambda$.

Near $(0, 0, \ldots, 0)$ the element $\alpha^{d-1} - \lambda$ is invertible, so α^2 is in the localisation of the ideal I, thus $\alpha + \lambda^{-1} \partial$ is in the ideal. Now we check that the localisation of I is equal to $\mathrm{ann}_S (f) + (\alpha + \lambda^{-1} \partial) T_{poly}$. Explicitly, one should check that

$$\left(\mathrm{ann}_S (f) + (\alpha + \lambda^{-1} \partial) T_{poly} \right)_{(0, \ldots, 0)} = \left(\mathrm{ann}_S (f) + (\alpha^d - \lambda \alpha - \partial) T_{poly} \right)_{(0, \ldots, 0)}.$$

Then the stalk of the fiber at $(0, \ldots, 0)$ is isomorphic to $\mathrm{Spec\,Apolar}\,(f)$.

Near $(0, 0, \ldots, 0, \omega)$ the elements α and $\frac{\alpha^{k+1} - \lambda \cdot \alpha^2}{\alpha - \omega}$ are invertible, so $\mathrm{ann}_S (\partial \lrcorner f)$ and $\alpha - \omega$ are in the localisation of I. This, along with the other inclusion, proves that this localisation is generated by $\mathrm{ann}_S (\partial \lrcorner f)$ and $\alpha - \omega$ and thus the stalk of the fiber is isomorphic to $\mathrm{Spec\,Apolar}\,(\partial f)$. $\square$

We make the most important corollary explicit:

Corollary 5.11. *We keep the notation of Proposition 5.9. Suppose that* char k *does not divide* $d - 1$ *and* $\partial^2 \lrcorner f = 0$. *If both apolar algebras of* f *and* $\partial \lrcorner f$ *are smoothable then also the apolar algebra of every ray sum of* f *with respect to* ∂ *is smoothable.* $\square$

Example 5.12. *Let* $f \in k[x_1, \ldots, x_n]$ *be a dual socle generator of an algebra* A. *Then the algebra* $B = \mathrm{Apolar}\left(f + x_{n+1}^2\right)$ *is limit-reducible: it is a limit of algebras of the form* $A \times k$. *In particular, if* A *is smoothable, then* B *is also smoothable.*

Combining this with Proposition 4.5, we see that every local Gorenstein algebra A *of socle degree* s *with* $\Delta_{A, s-2} = (0, q, 0)$, *where* $q \neq 0$, *is limit-reducible.*

If $\deg f \geq 2$, *then the Hilbert functions of* $A = \mathrm{Apolar}\,(f)$ *and* $B = \mathrm{Apolar}\left(f + x_{n+1}^2\right)$ *are related by* $H_B(m) = H_A(m)$ *for* $m \neq 1$ *and* $H_B(1) = H_A(1) + 1$.

Above, we took advantage of the explicit form of ray decompositions coming from ray sums to analyse the resulting ray families in depth. In Proposition 5.13 below we prove the flatness of the upper ray family without such knowledge. The price paid for this is the fact that we get no information about the fibers of this family.

Proposition 5.13. *Let* $f = x_1^s + g \in P$ *be a polynomial of degree* s *such that* $\alpha_1^c \lrcorner g = 0$ *for some* c *satisfying* $2c \leq s$. *Then any ray decomposition* $\mathrm{ann}_S (f) = (\alpha_1^\nu - q) + J$, *where* $J = \mathrm{ann}_S (f) \cap (\alpha_2, \ldots, \alpha_n)$, *gives rise to an upper ray degeneration. In particular* $\mathrm{Apolar}\,(f)$ *is limit-reducible.*

Proof. Let $\mathfrak{I} := (\alpha_1^\nu - t\alpha_1^{\nu-1} - q) + J$ be the ideal defining the ray family and recall that $q, J \subseteq \mathfrak{p}_1$, where $\mathfrak{p}_1 = (\alpha_2, \ldots, \alpha_n)$.

Since $\alpha_1^\nu - q \in \mathrm{ann}_S (f)$, we have $q \lrcorner g = q \lrcorner f = \alpha_1^\nu \lrcorner f = x_1^{s-\nu} + \alpha_1^\nu \lrcorner g$. Then $\alpha_1^{s-\nu}(q \lrcorner g) = \alpha_1^{s-\nu} \lrcorner x_1^{s-\nu} + \alpha_1^s \lrcorner g = 1$, thus $\alpha_1^{s-\nu} \lrcorner g \neq 0$. It follows that $s - \nu \leq c - 1$, so $\nu - 1 \geq s - c \geq c$, thus $\alpha_1^{\nu-1} \lrcorner g = 0$. For all $\gamma \in \mathfrak{p}_1$, we claim that

$$\gamma \cdot (\alpha_1^\nu - t\alpha_1^{\nu-1} - q) \in J[t]. \tag{5}$$

Note that $(\alpha_1^\nu - q) \lrcorner f = 0$ and $\alpha_1^{\nu-1} \gamma \lrcorner f = \alpha_1^{\nu-1} \gamma \lrcorner g = 0$. This means that $\alpha_1^{\nu-1} \gamma \in J$. Since always $(\alpha_1^\nu - q)\gamma \in J$, we have proved (5).

Let $\mathfrak{I} \subseteq S_{poly}[t]$ be the ideal defining the upper ray family. Take any $\lambda \in k$ and an element $i \in \mathfrak{I} \cap (t - \lambda)$. We will prove that $i \in \mathfrak{I}(t - \lambda) + \mathfrak{I}_0[t]$, where $\mathfrak{I}_0 = \mathfrak{I} \cap S$, then Proposition 2.12 asserts that $S[t]/\mathfrak{I}$ is flat. Write $i = i_1 + i_2(\alpha_1^\nu - t\alpha_1^{\nu-1} - q)$. As before, we may assume $i_1 \in J$, $i_2 \in S$. Since $i \in (t - \lambda)$, we have $i_1 + i_2(\alpha_1^\nu - \lambda\alpha_1^{\nu-1} - q) = 0$. Since $i_1 \in \mathfrak{p}_1$, we also have $i_2 \in \mathfrak{p}_1$. But then by Inclusion (5) we have $i_2(\alpha_1^\nu - t\alpha_1^{\nu-1} - q) \subseteq \mathfrak{I}_0[t]$. Since clearly $i_1 \in J \subseteq \mathfrak{I}_0[t]$, the assumptions of Proposition 2.12 are satisfied, thus the upper ray family is flat.

Now, Remark 5.5 shows that a general fiber of the upper ray degeneration is reducible, thus $\mathrm{Apolar}\,(f)$ is a flat limit of reducible algebras, i.e. limit-reducible. $\qquad\square$

Example 5.14. *Let $f \in k[x_1, x_2, x_3, x_4]$ be a polynomial of degree 4. Suppose that the leading form f_4 of f can be written as $f_4 = x_1^4 + g_4$ where $g_4 \in k[x_2, x_3, x_4]$. We will prove that $\mathrm{Apolar}\,(f)$ is limit-reducible. By Example 4.3 we may assume that $f = x_1^4 + g$, where $\alpha_1^2 \lrcorner g = 0$. By Proposition 5.13 we see that $\mathrm{Apolar}\,(f)$ is limit-reducible.*

Example 5.15. *Suppose that an Artin local Gorenstein algebra A has Hilbert function $H_A = (1, H_1, \ldots, H_c, 1, \ldots, 1)$ and socle degree $s \geq 2c$. By Example 4.4 we may assume that $A \simeq \mathrm{Apolar}\,(x_1^s + g)$, where $\alpha_1^c \lrcorner g = 0$ and $\deg g \leq c + 1$. Then by Proposition 5.13 we obtain a flat degeneration*

$$k[t] \to \frac{S[t]}{(\alpha_1^\nu - t\alpha_1^{\nu-1} - q) + J}. \tag{6}$$

Thus A is limit-reducible in the sense of Definition 2.5. Let us take $\lambda \neq 0$. By Remark 5.5 the fiber over $t = \lambda$ is supported at $(0, 0, \ldots, 0)$ and at $(\lambda, 0, \ldots, 0)$ and the ideal defining this fiber near $(0, 0, \ldots, 0)$ is $I_0 = (\lambda\alpha_1^{\nu-1} - q) + J$. From the proof of 5.13 it follows that $\alpha_1^{\nu-1} \lrcorner g = 0$. Then one can check that I_0 lies in the annihilator of $\lambda^{-1}x_1^{s-1} + g$. Since $\sigma \lrcorner (x_1^s + g) = \sigma \lrcorner (\lambda^{-1}x_1^{s-1} + g)$ for every $\sigma \in (\alpha_2, \ldots, \alpha_n)$, one calculates that the apolar algebra of $\lambda^{-1}x_1^{s-1} + g$ has Hilbert function $(1, H_1, \ldots, H_c, 1, \ldots, 1)$ and socle degree $s - 1$. Then $\dim_k \mathrm{Apolar}\,(x_1^{s-1} + g) = \dim_k \mathrm{Apolar}\,(\lambda^{-1}x_1^s + g) - 1$. Thus the fiber is a union of a point and $\mathrm{Spec}\,\mathrm{Apolar}\,(\lambda^{-1}x_1^s + g)$, i.e. degeneration (6) peels one point off A.

5.2 Tangent preserving ray degenerations

A (finite) ray degeneration gives a morphism from $\mathrm{Spec}\,k[t]$ to the Hilbert scheme, i.e. a curve on the Hilbert scheme $\mathcal{H}ilb(\mathbb{P}^n)$. In this section we prove that in some cases the dimension of the tangent space to $\mathcal{H}ilb(\mathbb{P}^n)$ is constant along this curve. This enables us to prove that certain points of this scheme are smooth without the need for lengthy computations.

This section seems to be the most technical part of the paper, so we include even more examples. The most important results here are Theorem 5.18 together with Corollary 5.20; see examples below Corollary 5.20 for applications.

Recall (e.g. [Jel13, Prop 4.10] or [CN09]) that the dimension of the tangent space to $\mathcal{H}ilb(\mathbb{P}^n)$ at a k-point corresponding to a Gorenstein scheme $\mathrm{Spec}\,S/I$ is $\dim_k S/I^2 - \dim_k S/I$.

Lemma 5.16. *Let $d \geq 2$. Let g be the d-th ray sum of $f \in P$ with respect to $\partial \in S$ such that $\partial^2 \lrcorner f = 0$. Denote $I := \mathrm{ann}_S(f)$ and $J := \mathrm{ann}_S(\partial \lrcorner f)$. Take $T = S[[\alpha]]$ to be the ring dual to $P[x]$ and let*

$$\mathfrak{I} := \left(I + J\alpha + (\alpha^d - t\alpha - \partial) \right) \cdot T[t]$$

be the ideal in $T[t]$ defining the associated lower ray degeneration, see Proposition 5.9. Then the family $k[t] \to T[t]/\mathfrak{I}^2$ is flat if and only if $(I^2 : \partial) \cap I \cap J^2 \subseteq I \cdot J$.

Proof. To prove flatness we will use Proposition 2.12. Take an element $i \in \mathfrak{I}^2 \cap (t - \lambda)$. We want to prove that $i \in \mathfrak{I}^2(t - \lambda) + \mathfrak{I}_0[t]$, where $\mathfrak{I}_0[t] = \mathfrak{I}^2 \cap T$. Let $\mathcal{J} := (I + J\alpha)T$. Subtracting a suitable element of $\mathfrak{I}^2(t - \lambda)$ we may assume that

$$i = i_1 + i_2(\alpha^d - t\alpha - \partial) + i_3(\alpha^d - t\alpha - \partial)^2,$$

21

where $i_1 \in \mathcal{J}^2$, $i_2 \in \mathcal{J}$ and $i_3 \in T$. We will in fact show that $i \in \mathfrak{I}^2(t - \lambda) + \mathcal{J}^2[t]$.

To simplify notation denote $\sigma = \alpha^d - \lambda\alpha - \partial$. Note that $J\sigma \subseteq \mathcal{J}$. We have $i_1 + i_2\sigma + i_3\sigma^2 = 0$. Let $j_3 := i_3\sigma$. We want to apply Lemma 2.15, below we check its assumptions. The ideal $\mathcal{J}$ is homogeneous with respect to α, generated in degrees less than d. Let $s \in T$ be an element satisfying $s\alpha^d \in \mathcal{J}$. Then $s \in J$, which implies $s(\lambda\alpha + \partial) \in \mathcal{J}$. By Lemma 2.15 and $i_3\sigma^2 = j_3\sigma \in \mathcal{J}$ we obtain $j_3\alpha^d \in \mathcal{J}$, i.e. $i_3\sigma\alpha^d \in \mathcal{J}$. Applying the same argument to $i_3\alpha^d$ we obtain $i_3\alpha^{2d} \in \mathcal{J}$, therefore $i_3 \in JT$. Then

$$i_3(\alpha^d - t\alpha - \partial)^2 - i_3\sigma(\alpha^d - t\alpha - \partial) = i_3\alpha(t - \lambda)(\alpha^d - t\alpha - \partial) \in \mathcal{J}(t - \lambda)(\alpha^d - t\alpha - \partial) \subseteq \mathfrak{I}^2(t - \lambda).$$

Subtracting this element from i and substituting $i_2 := i_2 + i_3\sigma$ we may assume $i_3 = 0$. We obtain

$$0 = i_1 + i_2\sigma = i_1 + i_2(\alpha^d - \lambda\alpha - \partial). \tag{7}$$

Let $i_2 = j_2 + v_2\alpha$, where $j_2 \in S$, i.e. it does not contain α. Since $i_2 \in \mathcal{J}$, we have $j_2 \in I$. As before, we have $v_2\alpha((\alpha^d - t\alpha - \partial) - \sigma) = v_2\alpha^2(t - \lambda) \in \mathfrak{I}^2(t - \lambda)$, so that we may assume $v_2 = 0$.

Comparing the top α-degree terms of (7) we see that $j_2 \in J^2$. Comparing the terms of (7) not containing α, we deduce that $j_2\partial \in I^2$, thus $j_2 \in (I^2 : \partial)$. Jointly, $j_2 \in I \cap J^2 \cap (I^2 : \partial)$, thus $j_2 \in IJ$ by assumption. But then $j_2\alpha \in J^2$, thus $j_2(\alpha^d - t\alpha - \partial) \in \mathcal{J}^2[t]$ and since $i_1 \in \mathcal{J}^2$, the element i lies in $\mathcal{J}^2[t] \subseteq \mathfrak{I}_0[t]$. Thus the assumptions of Proposition 2.12 are satisfied and the family $T[t]/\mathfrak{I}^2$ is flat over $k[t]$.

The converse is easier: one takes $i_2 \in I \cap J^2 \cap (I^2 : \partial)$ such that $i_2 \notin IJ$. On one hand, the element $j := i_2(\alpha^d - \partial)$ lies in $\mathcal{J}^2$ and we get that $i_2(\alpha^d - t\alpha - \partial) - j = ti_2\alpha \in \mathfrak{I}^2$. On the other hand if $i_2\alpha \in \mathfrak{I}^2$, then $i_2\alpha \in (\mathfrak{I}^2 + (t)) \cap T = (\mathcal{J} + (\alpha^d - \partial))^2$, which is not the case. $\square$

Remark 5.17. *Let us keep the notation of Lemma 5.16. Fix $\lambda \in k \setminus \{0\}$ and suppose that the characteristic of k does not divide $d - 1$. The supports of the fibers of $S[t]/\mathfrak{I}$, $\mathfrak{I}/\mathfrak{I}^2$ and $S[t]/\mathfrak{I}^2$ over $t = \lambda$ are finite and equal. In particular from Proposition 5.10 it follows that the dimension of the fiber of $\mathfrak{I}/\mathfrak{I}^2$ over $t - \lambda$ is equal to $\tan(f) + (d - 1)\tan(\partial \lrcorner f)$, where $\tan(h) = \dim_k \operatorname{ann}_S(h) / \operatorname{ann}_S(h)^2$ is the dimension of the tangent space to the point of the Hilbert scheme corresponding to $\operatorname{Spec} S / \operatorname{ann}_S(h)$.*

Theorem 5.18. *Suppose that a polynomial $f \in P$ corresponds to a smoothable, unobstructed algebra $\operatorname{Apolar}(f)$. Let $\partial \in S$ be such that $\partial^2 \lrcorner f = 0$ and the algebra $\operatorname{Apolar}(\partial \lrcorner f)$ is smoothable and unobstructed. The following are equivalent:*

1. *the d-th ray sum of f with respect to ∂ is unobstructed for some d such that $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$).*

1a. *the d-th ray sum of f with respect to ∂ is unobstructed for all d such that $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$).*

2. *The $k[t]$-module $\mathfrak{I}/\mathfrak{I}^2$ is flat, where $\mathfrak{I}$ is the ideal defining the lower ray family of the d-th ray sum for some $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$), see Definition 5.3.*

2a. *The $k[t]$-module $\mathfrak{I}/\mathfrak{I}^2$ is flat, where $\mathfrak{I}$ is the ideal defining the lower ray family of the d-th ray sum for every $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$), see Definition 5.3.*

3. *The family $k[t] \to S[t]/\mathfrak{I}^2$ is flat, where $\mathfrak{I}$ is the ideal defining the lower ray family of the d-th ray sum for some $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$).*

3a. *The family $k[t] \to S[t]/\mathfrak{I}^2$ is flat, where $\mathfrak{I}$ is the ideal defining the lower ray family of the d-th ray sum for every $2 \leq d \leq \operatorname{char} k$ (or $2 \leq d$ if $\operatorname{char} k = 0$).*

4. The following inclusion (equivalent to equality) of ideals in S holds: $I \cap J^2 \cap (I^2 : \partial) \subseteq I \cdot J$, where $I = \mathrm{ann}_S(f)$ and $J = \mathrm{ann}_S(\partial \lrcorner f)$.

Proof. It is straightforward to check that the inclusion $I \cdot J \subseteq I \cap J^2 \cap (I^2 : \partial) \subseteq I \cdot J$ in Point 4 always holds, thus the other inclusion is equivalent to equality.

3. $\iff$ 4. $\iff$ 3a. The equivalence of Point 3 and Point 4 follows from Lemma 5.16. Since Point 4 is independent of d, the equivalence of Point 4 and Point 3a also follows.

2. $\iff$ 3. and 2a. $\iff$ 3a. We have an exact sequence of $k[t]$-modules

$$0 \to \mathfrak{I}/\mathfrak{I}^2 \to S[t]/\mathfrak{I}^2 \to S[t]/\mathfrak{I} \to 0.$$

Since $S[t]/\mathfrak{I}$ is a flat $k[t]$-module by Proposition 5.9, we see from the long exact sequence of Tor that $\mathfrak{I}/\mathfrak{I}^2$ is flat if and only if $S[t]/\mathfrak{I}^2$ is flat.

1. $\iff$ 2. and 1a. $\iff$ 2a. Let $g \in P[x]$ be the d-th ray sum of f with respect to ∂. We may consider $\mathrm{Apolar}(g)$, $\mathrm{Apolar}(f)$, $\mathrm{Apolar}(\partial \lrcorner f)$ as quotients of a polynomial ring T_{poly}, corresponding to points of the Hilbert scheme. The dimension of the tangent space at $\mathrm{Apolar}(g)$ is given by $\dim_k \mathfrak{I}/\mathfrak{I}^2 \otimes k[t]/t = \dim_k \mathfrak{I}/(\mathfrak{I}^2 + (t))$. By Remark 5.17 it is equal to the sum of the dimension of the tangent space at $\mathrm{Apolar}(f)$ and $(d-1)$ times the dimension of the tangent space to $\mathrm{Apolar}(\partial \lrcorner f)$. Since both algebras are smoothable and unobstructed we conclude that $\mathrm{Apolar}(g)$ is also unobstructed. On the other hand, if $\mathrm{Apolar}(g)$ is unobstructed, then $\mathfrak{I}/\mathfrak{I}^2$ is a finite $k[t]$-module such that the length of the fiber $\mathfrak{I}/\mathfrak{I}^2 \otimes k[t]/\mathfrak{m}$ does not depend on the choice of the maximal ideal $\mathfrak{m} \subseteq k[t]$. Then $\mathfrak{I}/\mathfrak{I}^2$ is flat by [Har77, Ex II.5.8] or [Har77, Thm III.9.9] applied to the associated sheaf. $\square$

Remark 5.19. *The condition from Point 4 of Theorem 5.18 seems very technical. It is enlightening to look at the images of $(I^2 : \partial) \cap I$ and $I \cdot J$ in I/I^2. The image of $(I^2 : \partial) \cap I$ is the annihilator of ∂ in I/I^2. This annihilator clearly contains $(I : \partial) \cdot I/I^2 = J \cdot I/I^2$. This shows that if the S/I-module I/I^2 is "nice", for example free, we should have an equality $(I^2 : \partial) \cap I = I \cdot J$. More generally this equality is connected to the syzygies of I/I^2.*

In the remainder of this subsection we will prove that in several situations the conditions of Theorem 5.18 are satisfied.

Corollary 5.20. *We keep the notation and assumptions of Theorem 5.18. Suppose further that the algebra $S/I = \mathrm{Apolar}(f)$ is a complete intersection. Then the equivalent conditions of Theorem 5.18 are satisfied.*

Proof. Since S/I is a complete intersection, the S/I-module I/I^2 is free, see e.g. [Mat86, Thm 16.2] and discussion above it or [Eis95, Ex 17.12a]. It implies that $(I^2 : \partial) \cap I = (I : \partial)I = JI$, because $J = \mathrm{ann}_S(\partial \lrcorner f) = \{s \in S \mid s\partial \lrcorner f = 0\} = (\mathrm{ann}_S(f) : \partial) = (I : \partial)$. Thus the condition from Point 4 of Theorem 5.18 is satisfied. $\square$

Example 5.21. *If $A = S/I$ is a complete intersection, then it is smoothable and unobstructed (see Subsection 2.4). The apolar algebras of monomials are complete intersections, therefore the assumptions of Theorem 5.18 are satisfied e.g. for $f = x_1^2 x_2^2 x_3$ and $\partial = \alpha_2^2$. Now Corollary 5.20 implies that the equivalent conditions of the Theorem are also satisfied, thus $x_1^2 x_2^2 x_3 + x_4^d x_1^2 x_3 = (x_2^2 x_3)(x_1^2 + x_4^d)$ is unobstructed for every $d \geq 2$ (provided $\mathrm{char}\, k = 0$ or $d \leq \mathrm{char}\, k$). Similarly, $x_1^2 x_2 x_3 + x_4^2 x_1$ is unobstructed and has Hilbert function $(1, 4, 5, 3, 1)$.*

Example 5.22. *Let $f = (x_1^2 + x_2^2)x_3$, then $\mathrm{ann}_S(f) = (\alpha_1^2 - \alpha_2^2, \alpha_1 \alpha_2, \alpha_3^2)$ is a complete intersection. Take $\partial = \alpha_1 \alpha_3$, then $\partial \lrcorner f = x_1$ and $\partial^2 \lrcorner f = 0$, thus $f + x_4^2 \partial \lrcorner f = x_1^2 x_3 + x_2^2 x_3 + x_4^2 x_1$ is unobstructed. Note that by Remark 5.8 the apolar algebra of this polynomial has Hilbert function $(1, 4, 4, 1)$.*

Proposition 5.23. *Let $f \in P$ be such that $\mathrm{Apolar}\,(f)$ is a complete intersection.*

Let d be a natural number. Suppose that $\mathrm{char}\,k = 0$ or $d \leq \mathrm{char}\,k$. Take $\partial \in S$ such that $\partial^2 \lrcorner f = 0$ and $\mathrm{Apolar}\,(\partial \lrcorner f)$ is also a complete intersection. Let $g \in P[y]$ be the d-th ray sum f with respect to ∂, i.e. $g = f + y^d \partial \lrcorner f$.

Suppose that $\deg \partial \lrcorner f > 0$. Let β be the variable dual to y and $\sigma \in S$ be such that $\sigma \lrcorner (\partial \lrcorner f) = 1$. Take $\varphi := \sigma \beta \in T = S[[\beta]]$. Let h be any ray sum of g with respect to φ, explicitly

$$h = f + y^d \partial \lrcorner f + z^m y^{d-1}$$

for some $m \geq 2$.

Then the algebra $\mathrm{Apolar}\,(h)$ is unobstructed.

Proof. First note that $\varphi \lrcorner g = y^{d-1}$ and so $\varphi^2 \lrcorner g = \sigma \lrcorner y^{d-2} = 0$, since $\sigma \in \mathfrak{m}_S$. Therefore indeed h has the presented form.

From Corollary 5.20 it follows that $\mathrm{Apolar}\,(g)$ is unobstructed. Since $\varphi \lrcorner g = y^{d-1}$, the algebra $\mathrm{Apolar}\,(\varphi \lrcorner g)$ is unobstructed as well. Now by Theorem 5.18 it remains to prove that

$$(I_g^2 : \varphi) \cap I_g \cap J_g^2 \subseteq I_g J_g, \tag{8}$$

where $I_g = \mathrm{ann}_T\,(g)$, $J_g = \mathrm{ann}_T\,(\varphi \lrcorner g)$. The rest of the proof is a technical verification of this claim. Denote $I_f := \mathrm{ann}_S\,(f)$ and $J_f := \mathrm{ann}_S\,(\partial \lrcorner f)$; note that we take annihilators in S. By Proposition 5.7 we have $I_g = I_f T + \beta J_f T + (\beta^d - \partial)T$. Consider $\gamma \in T$ lying in $(I_g^2 : \varphi) \cap I_g \cap J_g^2$. Write $\gamma = \gamma_0 + \gamma_1 \beta + \gamma_2 \beta^2 + \dots$ where $\gamma_i \in S$, so they do not contain β. We will prove that $\gamma \in I_g J_g$.

First, since $(\beta^d - \partial)^2 \in I_g J_g$ we may reduce powers of β in γ using this element and so we assume $\gamma_i = 0$ for $i \geq 2d$. Let us take $i < 2d$. Since $\gamma \in J_g^2 = \left(\mathrm{ann}_T\,(y^{d-1})\right)^2 = \left(\mathfrak{m}_S, \beta^d\right)^2$ we see that $\gamma_i \in \mathfrak{m}_S \subseteq J_g$. For $i > d$ we have $\beta^i \in I_g$, so that $\gamma_i \beta^i \in J_g I_g$ and we may assume $\gamma_i = 0$. Moreover, $\beta^d \gamma_d - \partial \gamma_d \in I_g J_g$ so we may also assume $\gamma_d = 0$, obtaining

$$\gamma = \gamma_0 + \dots + \gamma_{d-1} \beta^{d-1}.$$

From the explicit description of I_g in Proposition 5.7 it follows that $\gamma_i \in J_f$ for all i.

Let $M = I_g^2 \cap \varphi T = I_g^2 \cap J_f \beta T$. Then for γ as above we have $\gamma \varphi \in M$, so we will analyse the module M. Recall that

$$I_g^2 = I_f^2 \cdot T + \beta I_f J_f \cdot T + \beta^2 J_f^2 \cdot T + (\beta^d - \partial) I_f \cdot T + (\beta^d - \partial)\beta J_f \cdot T + (\beta^d - \partial)^2 \cdot T. \tag{9}$$

We claim that

$$M \subseteq I_f^2 \cdot T + \beta I_f J_f \cdot T + \beta^2 J_f^2 \cdot T + (\beta^d - \partial)\beta J_f \cdot T. \tag{10}$$

We have $I_g^2 \subseteq J_f \cdot T + (\beta^d - \partial)^2 \cdot T$, so if an element of I_g^2 lies in $J_f \cdot T$, then its coefficient standing next to $(\beta^d - \partial)^2$ in Presentation (9) is an element of J_f by Lemma 2.15. Since $J_f \cdot (\beta^d - \partial) \subseteq I_f + \beta J_f$, we may ignore the term $(\beta^d - \partial)^2$:

$$M \subseteq I_f^2 \cdot T + \beta I_f J_f \cdot T + \beta^2 J_f^2 \cdot T + (\beta^d - \partial) I_f \cdot T + (\beta^d - \partial)\beta J_f \cdot T. \tag{11}$$

Choose an element of M and let $i \in I_f \cdot T$ be the coefficient of this element standing next to $(\beta^d - \partial)$. Since $I_f T \cap \beta T \subseteq J_f T$ we may assume that i does not contain β, i.e. $i \in I_f$. Now, if an element of the right hand side of (11) lies in $\beta \cdot T$, then the coefficient i satisfies $i \cdot \partial \in I_f^2$, so that $i \in (I_f^2 : \partial)$. Since I_f is a complete intersection ideal the S/I_f-module I_f / I_f^2 is free, see Corollary 5.20 for references. Then we have $(I_f^2 : \partial) = (I_f : \partial) I_f$ and $i \in (I_f : \partial) I_f = I_f J_f$. Then $i \cdot (\beta^d - \partial) \subseteq I_f^2 + \beta \cdot I_f \cdot J_f$ and so the Inclusion (10) is proved. We come back to the proof of proposition.

From Lemma 2.15 applied to the ideal $J_f^2 T$ and the element $\beta(\beta^d - \partial)$ and the fact that $\beta \partial J_f^2 \subseteq I_g^2$ we compute that $M \cap \{\delta \mid \deg_\beta \delta \le d\}$ is a subset of $I_f^2 \cdot T + \beta \cdot I_f J_f \cdot T + \beta^2 J_f^2 \cdot T$. Then $\gamma\varphi = \gamma\beta\sigma$ lies in this set, so that $\gamma_0 \in (I_f J_f : \sigma)$ and $\gamma_n \in (J_f^2 : \sigma)$ for $n > 1$. Since Apolar (f) and Apolar $(\partial_\lrcorner f)$ are complete intersections, we have $\gamma_0 \in I_f \mathfrak{m}_S$ and $\gamma_i \in J_f \mathfrak{m}_S$ for $i \ge 1$. It follows that $\gamma \in I_g \mathfrak{m}_S \subseteq I_g J_g$. $\qquad \square$

Example 5.24. *Let $f \in P$ be a polynomial such that $A = \mathrm{Apolar}\,(f)$ is a complete intersection. Take ∂ such that $\partial_\lrcorner f = x_1$ and $\partial^2_\lrcorner f = 0$. Then the apolar algebra of $f + y_1^d x_1 + y_2^m y_1^{d-1}$ is unobstructed for any $d, m \ge 2$ (less or equal to char k if it is non-zero). In particular $g = f + y_1^2 x_1 + y_2^2 y_1$ is unobstructed.*

Continuing Example 5.22, if $f = x_1^2 x_3 + x_2^2 x_3$, then $x_1^2 x_3 + x_2^2 x_3 + x_4^2 x_1 + x_5^2 x_4$ is unobstructed. The apolar algebra of this polynomial has Hilbert function $(1, 5, 5, 1)$.

Let $g = x_1^2 x_3 + x_2^2 x_3 + x_4^2 x_1$, then $x_1^2 x_3 + x_2^2 x_3 + x_4^2 x_1 + x_5^2 x_4$ is a ray sum of g with respect to $\partial = \alpha_4 \alpha_1$. Let $I := \mathrm{ann}_S(g)$ and $J := (I : \partial)$. In contrast with Corollary 5.20 and Example 5.22 one may check that all three terms I, J^2 and $(I^2 : \partial)$ are necessary to obtain equality in the inclusion (8) for g and ∂, i.e. no two ideals of I, J^2, $(I^2 : \partial)$ have intersection equal to IJ.

Example 5.25. *Let $f = x_1^5 + x_2^4$. Then the annihilator of f in $k[\alpha_1, \alpha_2]$ is a complete intersection, and this is true for every $f \in k[x_1, x_2]$. Let $g = f + x_3^2 x_1^2$ be the second ray sum of f with respect to α_1^3 and $h = g + x_4^2 x_3$ be the second ray sum of g with respect to $\alpha_3 \alpha_1^2$. Then the apolar algebra of*

$$h = x_1^5 + x_2^4 + x_3^2 x_1^2 + x_4^2 x_3$$

is smoothable and not obstructed. It has Hilbert function $(1, 4, 4, 3, 1, 1)$.

Remark 5.26. *The assumption $\deg \partial_\lrcorner f > 0$ in Proposition 5.23 is necessary: the polynomial $h = x_1 x_2 x_3 + x_4^2 + x_5^2 x_4$ is obstructed, with length 12 and tangent space dimension $67 > 12 \cdot 5$ over $k = \mathbb{C}$. The polynomial g is the fourth ray sum of $x_1 x_2 x_3$ with respect to $\alpha_1 \alpha_2 \alpha_3$ and h is the second ray sum of $g = x_1 x_2 x_3 + x_4^2$ with respect to α_4, thus this example satisfies the assumptions of Proposition 5.23 except for $\deg \partial_\lrcorner f > 0$. Note that in this case $\alpha_4^2 \lrcorner g \ne 0$.*

6 Proof of Main Theorem and comments on the degree 14 case

6.1 Preliminary results

Let $r \ge 1$ be a natural number and V be a constructible subset of $P_{\le s}$. Assume that the apolar algebra Apolar (f) has length r for every closed point $f \in V$. Then we may construct the incidence scheme $\{(f, \mathrm{Apolar}\,(f))\} \to V$ which is a finite flat family over V and thus we obtain a morphism from V to the (punctual) Hilbert scheme of r points on an appropriate $\mathbb{P}^n$. See [Jel13, Prop 4.39] for details.

Consider $f \in P_{\le s}$. The apolar algebra of f has length at most r if and only if the matrix of partials $S_{\le s} f$ has rank at most r. This is a closed condition, so we obtain the following Remark 6.1.

Remark 6.1. *Let s be a positive integer and $V \subseteq P_{\le s}$ be a constructible subset. Then the set U, consisting of $f \in V$ such that the apolar algebra of f has the maximal length (among the elements of V), is open in V. In particular, if V is irreducible then U is also irreducible.*

Example 6.2. *Let $P_{\ge 4} = k[x_1, \ldots, x_n]_{\ge 4}$ be the space of polynomials that are sums of monomials of degree at least 4. Suppose that the set $V \subseteq P_{\ge 4}$ parameterising algebras with fixed Hilbert function H is irreducible. Then also the set W of polynomials $f \in P$ such that $f_{\ge 4} \in V$ is irreducible. Let $e := H(1)$ and suppose that the symmetric decomposition of H has zero rows $\Delta_{s-3} = (0, 0, 0, 0)$ and $\Delta_{s-2} = (0, 0, 0)$, where $s = \deg f$. We claim that general element of*

W corresponds to an algebra B with Hilbert function: $H_{max} = H + (0, n - e, n - e, 0)$. Indeed, since we may only vary the degree three part of the polynomial, the function H_B has the form $H + (0, a, a, 0) + (0, b, 0)$ for some a, b such that $a + b \leq n - e$. Therefore algebras with Hilbert function H_{max} are precisely the algebras of maximal possible length. Since H_{max} is attained for $f_{\geq 4} + x_{e+1}^3 + \ldots + x_n^3$, the claim follows from Remark 6.1.

6.2 Lemmas on Hilbert functions

In the following H_A denotes the Hilbert function of an algebra A.

Lemma 6.3. *Suppose that A is a local Artin Gorenstein algebra of socle degree $s \geq 3$ such that $\Delta_{A,s-2} = (0, 0, 0)$. Then $\operatorname{len} A \geq 2\left(H_A(1) + 1\right)$. Furthermore, equality occurs if and only if $s = 3$.*

Proof. Consider the symmetric decomposition $\Delta_\bullet = \Delta_{A,\bullet}$ of H_A. From symmetry we have $\sum_j \Delta_0(j) \geq 2 + 2\Delta_0(1)$ with equality only if Δ_0 has no terms between 1 and $s - 1$ i.e. when $s = 3$. Similarly $\sum_j \Delta_i(j) \geq 2\Delta_i(1)$ for all $1 \leq i < s - 2$. Summing these inequalities we obtain

$$\operatorname{len} A = \sum_{i < s-2} \sum_j \Delta_i(j) \geq 2 + \sum_{i < s-2} 2\Delta_i(1) = 2 + 2H_A(1). \qquad \square$$

Lemma 6.4. *Let A be a local Artin Gorenstein algebra of length at most 14. Suppose that $4 \leq H_A(1) \leq 5$. Then $H_A(2) \leq 5$.*

Proof. Let s be the socle degree of A. Suppose $H_A(2) \geq 6$. Then $H_A(3) + H_A(4) + \cdots \leq 3$, thus $s \in \{3, 4, 5\}$. The cases $s = 3$ and $s = 5$ immediately lead to contradiction – it is impossible to get the required symmetric decomposition. We will consider the case $s = 4$. In this case $H_A = (1, *, *, *, 1)$ and its symmetric decomposition is $(1, e, q, e, 1) + (0, m, m, 0) + (0, t, 0)$. Then $e = H_A(3) \leq 14 - 2 - 4 - 6 = 2$. Since $H_A(1) < H_A(2)$ we have $e < q$. This can only happen if $e = 2$ and $q = 3$. But then $14 \geq \operatorname{len} A = 9 + 2m + t$, thus $m \leq 2$ and $H_A(2) = m + q \leq 5$. A contradiction. $\qquad \square$

Lemma 6.5. *There does not exist a local Artin Gorenstein algebra with Hilbert function*

$$(1, 4, 3, 4, 1, \ldots, 1).$$

Proof. See [Iar94, pp. 99-100] for the proof or [CJN13, Lem 5.3] for a generalisation. We provide a sketch for completeness. Suppose such an algebra A exists and fix its dual socle generator $f \in k[x_1, \ldots, x_4]_s$ in the standard form. Let $I = \operatorname{ann}_S(f)$. The proof relies on two observations. First, the leading term of f is, up to a constant, equal to x_1^s and in fact we may take $f = x_1^s + f_{\leq 4}$. Moreover from the symmetric decomposition it follows that the Hilbert functions of $\operatorname{Apolar}(x_1^s + f_4)$ and $\operatorname{Apolar}(f)$ are equal. Second, $h(3) = 4 = 3^{\langle 2 \rangle} = h(2)^{\langle 2 \rangle}$ is the maximal growth, so arguing similarly as in Lemma 2.9 we may assume that the degree two part I_2 of the ideal of $\operatorname{gr} A$ is equal to $((\alpha_3, \alpha_4)S)_2$. Then any derivative of $\alpha_3 \lrcorner f_4$ is a derivative of x_1^s, i.e. a power of x_1. It follows that $\alpha_3 \lrcorner f_4$ itself is a power of x_1; similarly $\alpha_4 \lrcorner f_4$ is a power of x_1. It follows that $f_4 \in x_1^3 \cdot k[x_1, x_2, x_3, x_4] + k[x_1, x_2]$, but then f_4 is annihilated by a linear form, which contradicts the fact that f is in the standard form. $\qquad \square$

The following lemmas essentially deal with the limit-reducibility in the case $(1, 4, 4, 3, 1, 1)$. Here the method is straightforward, but the cost is that the proof is broken into several cases and quite long.

Lemma 6.6. *Let $f = x_1^5 + f_4$ be a polynomial such that $H_{\operatorname{Apolar}(f)}(2) < H_{\operatorname{Apolar}(f_4)}(2)$. Let $\mathcal{Q} = S_2 \cap \operatorname{ann}_S(x_1^5) \subseteq S_2$. Then $x_1^2 \in \mathcal{Q}f_4$ and $\operatorname{ann}_S(f_4)_2 \subseteq \mathcal{Q}$.*

Proof. Note that $\dim \mathcal{Q}f_4 \geq \dim S_2 f_4 - 1 = H_{\mathrm{Apolar}(f_4)}(2) - 1$. If $\mathrm{ann}_S(f_4)_2 \not\subseteq \mathcal{Q}$, then there is a $q \in \mathcal{Q}$ such that $\alpha_1^2 - q \in \mathrm{ann}_S(f_4)$. Then $\mathcal{Q}f_4 = S_2 f_4$ and we obtain a contradiction. Suppose that $x_1^2 \notin \mathcal{Q}f_4$. Then the degree two partials of f contain a direct sum of kx_1^2 and $\mathcal{Q}f_4$, thus they are at least $H_{\mathrm{Apolar}(f_4)}(2)$-dimensional, so that $H_{\mathrm{Apolar}(f)}(2) \geq H_{\mathrm{Apolar}(f_4)}(2)$, a contradiction. $\square$

Lemma 6.7. *Let $f = x_1^5 + f_4 \in P$ be a polynomial such that $H_{\mathrm{Apolar}(f)} = (1,3,3,3,1,1)$ and $H_{\mathrm{Apolar}(f_4)} = (1,3,4,3,1)$. Suppose that $\alpha_1^3 \lrcorner f_4 = 0$ and that $(\mathrm{ann}_S(f_4))_2$ defines a complete intersection. Then $\mathrm{Apolar}(f_4)$ and $\mathrm{Apolar}(f)$ are complete intersections.*

Proof. Let $I := \mathrm{ann}_S(f_4)$. First we will prove that $\mathrm{ann}_S(f_4) = (q_1, q_2, c)$, where $\langle q_1, q_2 \rangle = I_2$ and $c \in I_3$. Then of course $\mathrm{Apolar}(f_4)$ is a complete intersection. By assumption, q_1, q_2 form a regular sequence. Thus there are no syzygies of degree at most three in the minimal resolution of $\mathrm{Apolar}(f_4)$. By the symmetry of the minimal resolution, see [Eis95, Cor 21.16], there are no generators of degree at least four in the minimal generating set of I. Thus I is generated in degree two and three. But $H_{S/(q_1,q_2)}(3) = 4 = H_{S/I}(3) + 1$, thus there is a cubic c, such that $I_3 = kc \oplus (q_1, q_2)_3$, then $(q_1, q_2, c) = I$, thus $\mathrm{Apolar}(f_4) = S/I$ is a complete intersection.

Let $\mathcal{Q} := \mathrm{ann}_S(x_1^5) \cap S_2 \subseteq S_2$. By Lemma 6.6 we have $q_1, q_2 \in \mathcal{Q}$, so that $\alpha_1^3 \in I \setminus (q_1, q_2)$, then $I = (q_1, q_2, \alpha_1^3)$. Moreover, by the same Lemma, there exists $\sigma \in \mathcal{Q}$ such that $\sigma \lrcorner f_4 = x_1^2$.

Now we prove that $\mathrm{Apolar}(f)$ is a complete intersection. Let $J := (q_1, q_2, \alpha_1^3 - \sigma) \subseteq \mathrm{ann}_S(f)$. We will prove that S/J is a complete intersection. Since q_1, q_2, α_1^3 is a regular sequence, the set $S/(q_1, q_2)$ is a cone over a scheme of dimension zero and α_1^3 does not vanish identically on any of its components. Since σ has degree two, $\alpha_1^3 - \sigma$ also does not vanish identically on any of the components of $\mathrm{Spec}\, S/(q_1, q_2)$, thus $\mathrm{Spec}\, S/J$ has dimension zero, so it is a complete intersection (see also [VV78, Cor 2.4, Rmk 2.5]). Then the quotient by J has length at most $\deg(q_1)\deg(q_2)\deg(\alpha_1^3 - \sigma) = 12 = \dim_k S/\mathrm{ann}_S(f)$. Since $J \subseteq \mathrm{ann}_S(f)$, we have $\mathrm{ann}_S(f) = J$ and $\mathrm{Apolar}(f)$ is a complete intersection. $\square$

Lemma 6.8. *Let $f = x_1^5 + f_4 + g$, where $\deg g \leq 3$, be a polynomial such that $H_{\mathrm{Apolar}(f_{\geq 4})} = (1,3,3,3,1,1)$ and $H_{\mathrm{Apolar}(f_4)} = (1,3,4,3,1)$. Suppose that $\alpha_1^3 \lrcorner f_4 = 0$ and that $(\mathrm{ann}_S(f_4))_2$ does not define a complete intersection. Then $\mathrm{Apolar}(f)$ is limit-reducible.*

Proof. Let $\langle q_1, q_2 \rangle = (\mathrm{ann}_S(f_4))_2$. Since q_1, q_2 do not form a regular sequence, we have, after a linear transformation φ, two possibilities: $q_1 = \alpha_1 \alpha_2$ and $q_2 = \alpha_1 \alpha_3$ or $q_1 = \alpha_1^2$ and $q_2 = \alpha_1 \alpha_2$. Let β be the image of α_1 under φ, so that $\beta^3 \lrcorner f_4 = 0$.

Suppose first that $q_1 = \alpha_1 \alpha_2$ and $q_2 = \alpha_1 \alpha_3$. If β is up to constant equal to α_1, then $\alpha_1 \alpha_2, \alpha_1 \alpha_3, \alpha_1^3 \in \mathrm{ann}_S(f_4)$, so that α_1^2 is in the socle of $\mathrm{Apolar}(f_4)$, a contradiction. Thus we may assume, after another change of variables, that $\beta = \alpha_2$, $q_1 = \alpha_1 \alpha_2$ and $q_2 = \alpha_1 \alpha_3$. Then $f = x_2^5 + f_4 + \hat{g} = x_2^5 + x_1^4 + \hat{h} + \hat{g}$, where $\hat{h} \in k[x_1, x_3]$ and $\deg(\hat{g}) \leq 3$. Then by Lemma 4.2 we may assume that $\alpha_1^2 \lrcorner f = 0$, so $\mathrm{Apolar}(f)$ is limit-reducible by Proposition 5.13. See also Example 5.14 (the degree assumption in the Example can easily be modified).

Suppose now that $q_1 = \alpha_1^2$ and $q_2 = \alpha_1 \alpha_2$. If β is not a linear combination of α_1, α_2, then we may assume $\beta = \alpha_3$. Let m in f_4 be any monomial divisible by x_1. Since $q_1, q_2 \in \mathrm{ann}_S(f_4)$, we see that $m = \lambda x_1 x_3^3$ for some $\lambda \in k$. But since $\beta^3 \in \mathrm{ann}_S(f_4)$, we have $m = 0$. Thus f_4 does not contain x_1, so $H_{\mathrm{Apolar}(f_4)}(1) < 3$, a contradiction. Thus $\beta \in \langle \alpha_1, \alpha_2 \rangle$. Suppose $\beta = \lambda \alpha_1$ for some $\lambda \in k \setminus \{0\}$. Applying Lemma 6.6 to $f_{\geq 4}$ we see that x_1^2 is a derivative of f_4, so $\beta^2 \lrcorner f_4 \neq 0$, but $\beta^2 \lrcorner f_4 = \lambda^2 q_1 \lrcorner f_4 = 0$, a contradiction. Thus $\beta = \lambda_1 \alpha_1 + \lambda_2 \alpha_2$ and changing α_2 we may assume that $\beta = \alpha_2$. This substitution does not change $\langle \alpha_1^2, \alpha_1 \alpha_2 \rangle$. Now we directly check that $f_4 = x_3^2(\kappa_1 x_1 x_3 + \kappa_2 x_2^2 + \kappa_3 x_2 x_3 + \kappa_4 x_3^2)$, for some $\kappa_\bullet \in k$. Since x_1 is a derivative of f, we have $\kappa_1 \neq 0$. Then a non-zero element $\kappa_2 \alpha_1 \alpha_3 - \kappa_1 \alpha_2^2$ annihilates f_4. A contradiction with $H_{\mathrm{Apolar}(f_4)}(2) = 4$. $\square$

Lemma 6.9. *Let a quartic f_4 be such that $H_{\mathrm{Apolar}(f_4)} = (1,3,3,3,1)$ and $\alpha_1^3 \lrcorner f_4 = 0$. Then $H_{\mathrm{Apolar}(x_1^5 + f_4)}(2) \geq 4$.*

Proof. Let $\mathcal{Q} = \mathrm{ann}_S \left(x_1^5 \right)_2 \subseteq S_2$. Let I denote the apolar ideal of f_4. By Proposition 4.9 we see that I is minimally generated by three elements of degree two and two elements of degree four. In particular, there are no cubics in the generating set. Since $\alpha_1^3 \in I_3$, there is an element in $\sigma \in I_2$ such that $\sigma = \alpha_1^2 - q$, where $q \in \mathcal{Q}$. Therefore $\mathcal{Q} \lrcorner f_4 = S_2 \lrcorner f_4$. Moreover, σ does not annihilate x_1^2, so that x_1^2 is not a partial of f_4. We see that x_1^2 and $\mathcal{Q} \lrcorner f_4$ are leading forms of partials of $x_1^5 + f_4$, thus

$$H_{\mathrm{Apolar}(x_1^5+f_4)}(2) \geq 1 + \dim(\mathcal{Q} \lrcorner f_4) = 1 + \dim(S_2 \lrcorner f_4) = 1 + H_{\mathrm{Apolar}(f_4)}(2) = 4. \qquad \square$$

Remark 6.10. *In the setting of Lemma 6.9, it is not hard to deduce that $H_{\mathrm{Apolar}(x_1^5+f_4)} = (1,3,4,3,1,1)$ by analysing the possible symmetric decompositions. We do not need this stronger statement, so we omit the proof.*

6.3 Proofs

The following Proposition 6.11 generalises results about algebras with Hilbert function $(1,5,5,1)$, obtained in [Jel14] and [BCR12].

Proposition 6.11. *Let A be a local Artin Gorenstein algebra of socle degree three and $H_A(2) \leq 5$. Then A is smoothable.*

Proof. Suppose that the Hilbert function of A is $(1,n,e,1)$. By Proposition 4.5 the dual socle generator of A may be put in the form $f + x_{e+1}^2 + \cdots + x_n^2$, where $f \in k[x_1, \ldots, x_e]$. By repeated use of Example 5.12 we see that A is a limit of algebras of the form $\mathrm{Apolar}(f) \times k^{\oplus n-e}$. Thus it is smoothable if and only if $B = \mathrm{Apolar}(f)$ is.

Let $e := H_A(2)$, then $H_B = (1,e,e,1)$. If $H_B(1) = e \leq 3$ then B is smoothable. It remains to consider $4 \leq e \leq 5$. The set of points corresponding to algebras with Hilbert function $(1,e,e,1)$ is irreducible in $\mathcal{H}ilb_{2e+2}(\mathbb{P}^e)$ by Remark 6.1 for obvious parameterisation (as mentioned in [Iar84, Thm I, p. 350]), thus it will be enough to find a smooth point in this set which corresponds to a smoothable algebra. The cases $e = 4$ and $e = 5$ are considered in Example 5.22 and Example 5.24 respectively. $\qquad \square$

Remark 6.12. *The claim of Proposition 6.11 holds true if we replace the assumption $H_A(2) \leq 5$ by $H_A(2) = 7$, thanks to the smoothability of local Artin Gorenstein algebras with Hilbert function $(1,7,7,1)$, see [BCR12]. We will not use this result.*

Lemma 6.13. *Let A be a local Artin Gorenstein algebra with Hilbert function H_A beginning with $H_A(0) = 1$, $H_A(1) = 4$, $H_A(2) = 5$, $H_A(3) \leq 2$. Then A is smoothable.*

Proof. Let f be a dual socle generator of A in the standard form. From Macaulay's Growth Theorem it follows that $H_A(m) \leq 2$ for all $m \geq 3$, so that $H_A = (1,4,5,2,2,\ldots,2,1,\ldots,1)$. Let s be the socle degree of A.

Let $\Delta_{A,s-2} = (0,q,0)$ be the $(s-2)$-nd row of the symmetric decomposition of H_A. If $q > 0$, then by Example 5.12 we know that A is limit-reducible; it is a limit of algebras of the form $B \times k$, such that $H_B(1) = H_A(1) - 1 = 3$. Then the algebra B is smoothable (see [CN09, Prop 2.5]), so A is also smoothable. In the following we assume that $q = 0$.

We claim that $f_{\geq 4} \in k[x_1, x_2]$. Indeed, the symmetric decomposition of the Hilbert function is either $(1,1,\ldots,1) + (0,1,\ldots,1,0) + (0,0,1,0,0) + (0,2,2,0)$ or $(1,2,\ldots,2,1) + (0,0,1,0,0) + (0,2,2,0)$. In particular $e(s-3) = \sum_{i \geq 3} \Delta_i(1) = 2$, so that $f_{\geq 4} \in k[x_1, x_2]$ and $H_{\mathrm{Apolar}(f_{\geq 4})}(1) =$

2, in particular x_1 is a derivative of $f_{\geq 4}$, i.e. there exist a $\partial \in S$ such that $\partial \lrcorner f_{\geq 4} = x_1$. Then we may assume $\partial \in \mathfrak{m}_S^3$, so $\partial^2 \lrcorner f = 0$.

Let us fix $f_{\geq 4}$ and consider the set of all polynomials of the form $h = f_{\geq 4} + g$, where $g \in k[x_1, x_2, x_3, x_4]$ has degree at most three. By Example 6.2 the apolar algebra of a general such polynomial will have Hilbert function H_A. The set of polynomials h with fixed $h_{\geq 4} = f_{\geq 4}$, such that $H_{\mathrm{Apolar}(h)} = H_A$, is irreducible. This set contains $h := f_{\geq 4} + x_3^2 x_1 + x_4^2 x_3$. To finish the proof is it enough to show that h is smoothable and unobstructed. Since Apolar $(f_{\geq 4})$ is a complete intersection, this follows from Example 5.24. $\square$

The following Theorem 6.14 generalises numerous earlier smoothability results on stretched (by Sally, see [Sal79]), 2-stretched (by Casnati and Notari, see [CN13]) and almost-stretched (by Elias and Valla, see [EV11]) algebras. It is important to understand that, in contrast with the mentioned papers, we avoid a full classification of algebras. In the course of the proof we give some partial classification.

Theorem 6.14. *Let A be a local Artin Gorenstein algebra with Hilbert function H_A satisfying $H_A(2) \leq 5$ and $H_A(3) \leq 2$. Then A is smoothable.*

Proof. We proceed by induction on $\mathrm{len}\, A$, the case $\mathrm{len}\, A = 1$ being trivial. If A has socle degree three, then the result follows from Proposition 6.11. Suppose that A has socle degree $s \geq 4$.

Let f be a dual socle generator of A in the standard form. If the symmetric decomposition of H_A has a term $\Delta_{s-2} = (0, q, 0)$ with $q \neq 0$, then by Example 5.12, we have that A is a limit of algebras of the form $B \times k$, where B satisfies the assumptions $H_B(2) \leq 5$ and $H_B(2) \leq 2$ on the Hilbert function. Then B is smoothable by induction, so also A is smoothable. Further in the proof we assume that $\Delta_{A, s-2} = (0, 0, 0)$.

We would like to understand the symmetric decomposition of the Hilbert function H_A of A. Since H_A satisfies the Macaulay growth condition (see Subsection 2.5) it follows that $H_A = (1, n, m, 2, 2, \ldots, 2, 1, \ldots, 1)$, where the number of "2" is possibly zero. If follows that the possible symmetric decompositions of the Hilbert function are

1. $(1, 2, 2, \ldots, 2, 1) + (0, 0, 1, 0, 0) + (0, n-3, n-3, 0)$,

2. $(1, 1, 1 \ldots, 1, 1) + (0, 1, 1, \ldots, 1, 0) + (0, 0, 1, 0, 0) + (0, n-3, n-3, 0)$,

3. $(1, 1, 1 \ldots, 1, 1) + (0, 1, 2, 1, 0) + (0, n-3, n-3, 0)$,

4. $(1, \ldots, 1) + (0, n-1, n-1, 0)$,

5. $(1, 2, \ldots, 2, 1) + (0, n-2, n-2, 0)$,

6. $(1, \ldots, 1) + (0, 1, \ldots, 1, 0) + (0, n-2, n-2, 0)$,

and that the decomposition is uniquely determined by the Hilbert function. In all cases we have $H_A(1) \leq H_A(2) \leq 5$, so $f \in k[x_1, \ldots, x_5]$. Let us analyse the first three cases. In each of them we have $H_A(2) = H_A(1) + 1$. If $H_A(1) \leq 3$, then A is smoothable, see [CN09, Cor 2.4]. Suppose $H_A(1) \geq 4$. Since $H_A(2) \leq 5$, we have $H_A(2) = 5$ and $H_A(1) = 4$. In this case the result follows from Lemma 6.13 above.

It remains to analyse the three remaining cases. The proof is similar to the proof of Lemma 6.13, however here it essentially depends on induction. Let $f_{\geq 4}$ be the sum of homogeneous components of f which have degree at least four. Since f is in the standard form, we have $f_{\geq 4} \in k[x_1, x_2]$. The decomposition of the Hilbert function Apolar $(f_{\geq 4})$ is one of the decompositions $(1, \ldots, 1)$, $(1, 2 \ldots, 2, 1)$, $(1, \ldots, 1) + (0, 1, \ldots, 1, 0)$, depending on the decomposition of the Hilbert function of Apolar (f).

Let us fix a vector $\hat{h} = (1, 2, 2, 2, \ldots, 2, 1, 1, \ldots, 1)$ and take the sets

$$V_1 := \left\{ f \in k[x_1, x_2] \mid H_{\mathrm{Apolar}(f)} = \hat{h} \right\} \text{ and } V_2 := \{ f \in k[x_1, \ldots, x_n] \mid f_{\geq 4} \in V_1 \}.$$

By Proposition 4.8 the set V_1 is irreducible and thus V_2 is also irreducible. The Hilbert function of the apolar algebra of a general member of V_2 is, by Example 6.2, equal to H_A. It remains to show that the apolar algebra of this general member is smoothable.

Proposition 4.8 implies that the general member of V_2 has (after a nonlinear change of coordinates) the form $f + \partial_\lrcorner f$, where $f = x_1^s + x_2^{s_2} + g$ for some g of degree at most three. Using Lemma 4.2 we may assume (after another nonlinear change of coordinates) that $\alpha_1^2 \lrcorner g = 0$.

Let $B := \mathrm{Apolar}\,(x_1^s + x_2^{s_2} + g)$. We will show that B is smoothable. Since $s \geq 4 = 2 \cdot 2$ Proposition 5.13 shows that B is limit-reducible. Analysing the fibers of the resulting degeneration, as in Example 5.15, we see that they have the form $B' \times k$, where $B' = \mathrm{Apolar}\left(\hat{f}\right)$ and $\hat{f} = \lambda^{-1} x_1^{s-1} + x_2^{s_2} + g$. Then $H_{B'}(3) = H_{\mathrm{Apolar}(\hat{f}_{\geq 4})}(3) \leq 2$. Moreover, $\hat{f} \in k[x_1, \ldots, x_5]$, so that $H_{B'}(1) \leq 5$. Now analysing the possible symmetric decompositions of $H_{B'}$, which are listed above, we see that $H_{B'}(2) \leq H_{B'}(1) = 5$. It follows from induction on the length that B' is smoothable, thus $B' \times k$ and B are smoothable. $\square$

Proposition 6.15. *Let A be a local Artin Gorenstein algebra of socle degree four satisfying* $\mathrm{len}\, A \leq 14$. *Then A is smoothable.*

Proof. We proceed by induction on the length of A. Then by Proposition 6.11 (and the fact that all algebras of socle degree at most two are smoothable) we may assume that all algebras of socle degree *at most* four and length less than $\mathrm{len}\, A$ are smoothable.

If $\Delta_{A,1} = (0, q, 0)$ with $q \neq 0$, then by Example 5.12 the algebra A is a limit of algebras of the form $A' \times k$, where A' has socle degree four. Hence A is smoothable. Therefore we assume $q = 0$. Then $H_A(1) \leq 5$ by Lemma 6.3. Moreover, we may assume $H_A(1) \geq 4$ since otherwise A is smoothable by [CN09, Cor 2.4].

The symmetric decomposition of H_A is $(1, n, m, n, 1) + (0, p, p, 0)$ for some n, m, p. By the fact that $n \leq 5$ and Stanley's result [Sta96, p. 67] we have $n \leq m$, thus $n \leq 4$ and $H_A(2) \leq H_A(1) \leq 5$. Due to $\mathrm{len}\, A \leq 14$ we have four cases: $n = 1, 2, 3, 4$ and five possible shapes of Hilbert functions: $H_A = (1, *, *, 1, 1)$, $H_A = (1, *, *, 2, 1)$, $H_A = (1, 4, 4, 3, 1)$, $H_A = (1, 4, 4, 4, 1)$, $H_A = (1, 4, 5, 3, 1)$.

The conclusion in the first two cases follows from Theorem 6.14. In the remaining cases we first look for a suitable irreducible set of dual socle generators parameterising algebras with prescribed H_A. We examine the case $H_A = (1, 4, 4, 3, 1)$. We claim that the set of $f \in P = k[x_1, x_2, x_3, x_4]$ in the standard form, which are generators of algebras with Hilbert function H_A is irreducible. Since the leading form f_4 of such f has Hilbert function $(1, 3, 3, 3, 1)$, the set of possible leading forms is irreducible by Proposition 4.9. Then the irreducibility follows from Example 6.2. The irreducibility in the cases $H_A = (1, 4, 4, 4, 1)$ and $H_A = (1, 4, 5, 3, 1)$ follows similarly from Proposition 4.10 together with Example 6.2. In the first two cases we see that f_4 is a sum of powers of variables, then Example 5.14 shows that the apolar algebra A of a general f is limit-reducible. More precisely, A is limit of algebras of the form $A' \times k$, where A' has socle degree at most four (compare Example 5.15). Then A is smoothable. In the last case Example 5.21 gives an unobstructed algebra in this irreducible set. This completes the proof. $\square$

Lemma 6.16. *Let A be a local Artin Gorenstein algebra with Hilbert function $(1, 4, 4, 3, 1, 1)$. Then A is limit-reducible.*

Proof. Let $s = 5$ be the socle degree of A. If $\Delta_{A,s-2} \neq (0, 0, 0)$ then A is limit-reducible by Example 5.12, so we assume $\Delta_{A,s-2} = (0, 0, 0)$. The only possible symmetric decomposition of

30

the Hilbert function H_A with $\Delta_{A,s-2} = (0,0,0)$ is

$$(1,4,4,3,1,1) = (1,1,1,1,1,1) + (0,2,2,2,0) + (0,1,1,0). \tag{12}$$

Let us take a dual socle generator f of A. We assume that f is in the standard form: $f = x_1^5 + f_4 + g$, where $\deg g \leq 3$. Then $H_{\mathrm{Apolar}(x_1^5+f_4)} = (1,3,3,3,1,1)$. We analyse the possible Hilbert functions of $B = \mathrm{Apolar}(f_4)$. By Lemma 4.2 we may assume that $\alpha_1^3 \lrcorner f_4 = 0$. Suppose first that $H_B(1) \leq 2$. From (12) it follows that $H_{\mathrm{Apolar}(x_1^5+f_4)}(1) = 3$, so that $H_B(1) = 2$ and we may assume that $f_4 \in k[x_2, x_3]$. Then by Lemma 4.2 we may further assume $\alpha_1^2 \lrcorner (f - x_1^5) = 0$, then Proposition 5.13 asserts that $A = \mathrm{Apolar}(f)$ is limit-reducible.

Suppose now that $H_B(1) = 3$. Since x_1^5 is annihilated by a codimension one space of quadrics, we have $H_B(2) \leq H_A(2) + 1$, so there are two possibilities: $H_B = (1,3,3,3,1)$ or $H_B = (1,3,4,3,1)$. By Lemma 6.9 the case $H_B = (1,3,3,3,1)$ is not possible, so that $H_B = (1,3,4,3,1)$. Now by Lemma 6.8 we may consider only the case when $(\mathrm{ann}_S(f_4))_2$ is a complete intersection, then by Lemma 6.7 we have that $\mathrm{Apolar}(x_1^5 + f_4)$ is a complete intersection. In this case we will actually prove that A is smoothable.

By Example 6.2 the set W of polynomials f with fixed leading polynomial $f_{\geq 4}$ and Hilbert function $H_{\mathrm{Apolar}(f)} = (1,4,4,3,1,1)$ is irreducible. Consider the apolar algebra B of the polynomial $x_1^5 + f_4 + x_4^2 x_1 \in W$. By Proposition 5.10, this algebra is the limit of smoothable algebras $\mathrm{Apolar}(x_1^5 + f_4) \times \mathrm{Apolar}(x_1)$, thus it is smoothable. By Corollary 5.20 the algebra B is unobstructed. Thus apolar algebra of every element of W is smoothable; in particular A is smoothable. $\qquad\square$

Now we are ready to prove Theorem 6.17 which is the algebraic counterpart of Theorems A and B.

Theorem 6.17. *Let A be an Artin Gorenstein algebra of length at most 14. Then either A is smoothable or it is local with Hilbert function $(1,6,6,1)$. In particular, if A has length at most 13, then A is smoothable.*

Proof. By the discussion in Section 2.4 it is enough to consider local algebras. Let A be a local algebra of length at most 14 and of socle degree s. By H we denote the Hilbert function of A. As mentioned in Subsection 2.4 it is enough to prove A is limit-reducible. On the contrary, suppose that A is strongly non-smoothable in the sense of Definition 2.5. By Example 5.12 we have $\Delta_{A,s-2} = (0,0,0)$. Then by Lemma 6.3 we see that either $H = (1,6,6,1)$ or $H(1) \leq 5$. It is enough to consider $H(1) \leq 5$. If $s = 3$ then $H(2) \leq H(1) \leq 5$, so by Proposition 6.11 we may assume $s > 3$. By Proposition 6.15 it follows that we may consider only $s \geq 5$.

If $H(1) \leq 3$ then A is smoothable by [CN09, Cor 2.4], thus we may assume $H(1) \geq 4$. By Lemma 6.4 we see that $H(2) \leq 5$. Then by Theorem 6.14 we may reduce to the case $H(3) \geq 3$. By Macaulay's Growth Theorem we have $H(2) \geq 3$. Then $\sum_{i>3} H(i) \leq 14 - 11$, so we are left with several possibilities: $H = (1,4,3,3,1,1,1)$, $H = (1,4,3,3,2,1)$ or $H = (1,*,*,*,1,1)$. In the first two cases it follows from the symmetric decomposition that $\Delta_{A,s-2} \neq (0,0,0)$ which is a contradiction. We examine the last case. By Lemma 6.5 there does not exist an algebra with Hilbert function $(1,4,3,4,1,1)$. Thus the only possibilities are $(1,4,3,3,1,1)$, $(1,5,3,3,1,1)$ and $(1,4,4,3,1,1)$. Once more, it can be checked directly that in the first two cases $\Delta_{A,s-2} \neq (0,0,0)$. The last case is the content of Lemma 6.16. $\qquad\square$

Remark 6.18. *Assume* char $k = 0$. *In [IE78] Emsalem and Iarrobino analysed the tangent space to the Hilbert scheme. Iarrobino and Kanev claim that using Macaulay they are able to check that the tangent space to $\mathcal{H}ilb_6(\mathbb{P}^{14})$ has dimension 76 at a point corresponding to a general local Gorenstein algebra A with Hilbert function $(1,6,6,1)$, see [IK99, Lem 6.21], see also [CN11] for further details. Since $76 < (1+6+6+1) \cdot 6$ this shows that A is non-smoothable. Moreover, since all algebras of degree at most 13 are smoothable, A is strongly non-smoothable.*

To prove Theorem B, we need to show that the non-smoothable part of $\mathcal{H}ilb_{14}^G\mathbb{P}^n$ (for $n \geq 6$) is irreducible. The algebraic version of (a generalisation of) this statement is the following lemma.

Lemma 6.19. *Let $n \geq m$ be natural numbers and $V \subseteq P_{\leq 3} = k[x_1,\ldots,x_n]_{\leq 3}$ be the set of $f \in P$ such that $H_{\mathrm{Apolar}(f)} = (1, m, m, 1)$. Then V is constructible and irreducible.*

Proof. Let $V_{gr} = V \cap P_3$ denote the set of *graded* algebras with Hilbert function $(1, m, m, 1)$. This is a constructible subset of P_3. To an element $f_3 \in V_{gr}$ we may associate the tangent space to Apolar (f_3), which is isomorphic to $S_2 \lrcorner f_3$. We define

$$\{(f_3, [W]) \in V_{gr} \times \mathrm{Gr}(m, n) \mid W \supseteq S_2 \lrcorner f_3\},$$

which is an open subset in a vector bundle $\{(f_3, [W]) \in P_3 \times \mathrm{Gr}(m, n) \mid W \supseteq S_2 \lrcorner f_3\}$ over $\mathrm{Gr}(m, n)$, given by the condition $\dim S_2 \lrcorner f_3 \geq m$. Let $f \in V$ and write it as $f = f_3 + f_{\leq 2}$, where $\deg f_{\leq 2} \leq 2$. Then $H_{\mathrm{Apolar}(f_3)} = (1, m, m, 1)$. Therefore we obtain a morphism $\varphi : V \to V_{gr}$ sending f to f_3. We will analyse its fibers. Let $f_3 \in V_{gr}$ and $f = f_3 + f_{\leq 2} \in P_{\leq 3}$, where $\deg f_{\leq 2} \leq 2$. Then $H_{\mathrm{Apolar}(f)} = (1, M, m, 1)$ for some $M \geq m$. Moreover $M = m$ if and only if $\alpha \lrcorner f_{\leq 2}$ is a partial of f_3 for every α annihilating f_3. The fiber of φ over f is an affine subspace of $P_{\leq 2}$ defined by these conditions and the morphism

$$\{(f = f_3 + f_{\leq 2}, [W]) \in V \times \mathrm{Gr}(m, n) \mid W \supseteq S_2 \lrcorner f_3\} \to \{(f_3, [W]) \in V_{gr} \times \mathrm{Gr}(m, n) \mid W \supseteq S_2 \lrcorner f_3\}$$

is a projection from a vector bundle, which is thus irreducible. Since V admits a surjection from this bundle, it is irreducible as well. Moreover, the above shows that V is constructible. $\square$

Proof of Theorems A and B. The locus of points of the Hilbert scheme corresponding to smooth (i.e. reduced) algebras of length d is irreducible, as an image of an open subset of the d–symmetric product of $\mathbb{P}^n$, and smooth. The locus of points corresponding to smoothable algebras is the closure of the aforementioned locus, so it is also irreducible. If $d \leq 13$ or $d \leq 14$ and $n \leq 5$, this locus is the whole Hilbert scheme by Theorem 6.17 and the claim follows.

Now consider the case $d = 14$ and $n \geq 6$. Let $\mathcal{V}$ be the set of points of the Hilbert scheme corresponding to local Gorenstein algebras with Hilbert function $(1, 6, 6, 1)$. By Remark 6.18 these are the only non-smoothable algebras of length 14, thus they deform only to local algebras with the same Hilbert function. Therefore, $\mathcal{V}$ is a sum of irreducible components of the Hilbert scheme. We will prove that $\mathcal{V}$ is an irreducible set, whose general point is smooth.

Let $\mathcal{V}_p \subseteq \mathcal{V}$ denote the set consisting of schemes supported at a fixed point $p \in \mathbb{P}^n$. Then $\mathcal{V}$ is dominated by a set $\mathcal{V}_p \times \mathbb{P}^n$. Note that an irreducible scheme supported at a point p may be identified with a Gorenstein quotient of the power series ring having Hilbert function $(1, 6, 6, 1)$. These quotients are parameterised by the dual generators. More precisely, the set of V of $f \in k[x_1,\ldots,x_n]_{\leq 3}$ such that $H_{\mathrm{Apolar}(f)} = (1, 6, 6, 1)$ gives a morphism

$$V \to \mathcal{V}_p \subseteq \mathcal{H}ilb_{14}^G\mathbb{P}^n$$

which sends f to Spec Apolar (f) supported at p (see subsection 6.1). Since $V \to \mathcal{V}_p$ is surjective and V is irreducible by Lemma 6.19, we see that $\mathcal{V}_p$ is irreducible. Then $\mathcal{V}$ is irreducible as well.

Take a smooth point of $\mathcal{H}ilb_{14}^G\mathbb{P}^6$ which corresponds to an algebra A with Hilbert function $(1, 6, 6, 1)$. Then any point of $\mathcal{H}ilb_{14}^G\mathbb{P}^n$ corresponding to an embedding Spec $A \subseteq \mathbb{P}^n$ is smooth by [CN09, Lem 2.3]. This concludes the proof. $\square$

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Gianfranco Casnati,
Dipartimento di Scienze Matematiche, Politecnico di Torino,
corso Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: `gianfranco.casnati@polito.it`

Joachim Jelisiejew,
Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw,
Banacha 2, 02-097 Warsaw, Poland
`j.jelisiejew@mimuw.edu.pl`

Roberto Notari,
Dipartimento di Matematica, Politecnico di Milano,
via Bonardi 9, 20133 Milano, Italy
e-mail: `roberto.notari@polimi.it`