HARNACK'S INEQUALITY AND HÖLDER CONTINUITY FOR WEAK SOLUTIONS OF DEGENERATE QUASILINEAR EQUATIONS WITH ROUGH COEFFICIENTS

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Abstract. We continue to study regularity results for weak solutions of the large class of second order degenerate quasilinear equations of the form

$$\operatorname{div}\big(A(x,u,\nabla u)\big) = B(x,u,\nabla u) \text{ for } x \in \Omega$$

as considered in our paper [MRW]. There we proved only local boundedness of weak solutions. Here we derive a version of Harnack's inequality as well as local Hölder continuity for weak solutions. The possible degeneracy of an equation in the class is expressed in terms of a nonnegative definite quadratic form associated with its principal part. No smoothness is required of either the quadratic form or the coefficients of the equation. Our results extend ones obtained by J. Serrin [S] and N. Trudinger [T] for quasilinear equations, as well as ones for subelliptic linear equations obtained in [SW1, 2].

1. Introduction

1.1. **General Comments.** Our main goal is to prove Harnack's inequality and local Hölder continuity for weak solutions u of quasilinear equations of the form

$$\operatorname{div}\big(A(x,u,\nabla u)\big) = B(x,u,\nabla u) \tag{1.1}$$

in an open set $\Omega \subset \mathbf{R}^n$. The vector-valued function A and the scalar function B will be assumed to satisfy the same structural conditions as in our earlier paper [MRW], where we proved that weak solutions are locally bounded. The possible degeneracy of equation (1.1) is expressed in terms of a matrix $Q(x)$, that may vanish or become singular, associated with the functions A, B. More precisely, given p with $1 < p < \infty$ and an $n \times n$ nonnegative definite symmetric matrix $Q(x)$ satisfying $|Q| \in L^{p/2}_{loc}(\Omega)$, we assume the following structural conditions: For $(x,z,\xi) \in \Omega\times\mathbf{R}\times\mathbf{R}^n$, there is a vector $\tilde{A}(x,z,\xi)$ with values in $\mathbf{R}^n$ such that for a.e. $x \in \Omega$ and all $(z,\xi) \in \mathbf{R}\times\mathbf{R}^n$,

$$\begin{cases} (i) \quad A(x,z,\xi) = \sqrt{Q(x)}\,\tilde{A}(x,z,\xi), \\ (ii) \quad \xi\cdot A(x,z,\xi) \geq a^{-1}\Big|\sqrt{Q(x)}\,\xi\Big|^p - h(x)|z|^\gamma - g(x), \\ (iii) \quad \Big|\tilde{A}(x,z,\xi)\Big| \leq a\Big|\sqrt{Q(x)}\,\xi\Big|^{p-1} + b(x)|z|^{\gamma-1} + e(x), \\ (iv) \quad \Big|B(x,z,\xi)\Big| \leq c(x)\Big|\sqrt{Q(x)}\,\xi\Big|^{\psi-1} + d(x)|z|^{\delta-1} + f(x), \end{cases} \tag{1.2}$$

where $a, \gamma, \psi, \delta > 1$ are constants, and b, c, d, e, f, g, h are nonnegative measurable functions of $x \in \Omega$.

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The sizes of the exponents are restricted to the ranges

$$\gamma \in (1, \sigma(p-1)+1), \quad \psi \in (1, p+1-\sigma^{-1}), \quad \delta \in (1, p\sigma), \tag{1.3}$$

where $\sigma > 1$ is a constant that measures the gain in integrability in a naturally associated Sobolev estimate (see (2.8) below). For the classical Euclidean metric $|x-y|$, nondegenerate Q and $1 < p < n$, the Sobolev gain factor σ is $n/(n-p)$. Furthermore, the functions b, c, d, e, f, g, h will be assumed to lie in certain Lebesgue or Morrey spaces, and to satisfy the minimal integrability conditions

$$\begin{gathered} c \in L_{\text{loc}}^{\frac{\sigma p}{\sigma p - 1 - \sigma(\psi - 1)}}(\Omega), \qquad e \in L_{\text{loc}}^{p'}(\Omega), \qquad f \in L_{\text{loc}}^{(\sigma p)'}(\Omega), \\ b \in L_{\text{loc}}^{\frac{\sigma p}{\sigma(p-1)-\gamma+1}}(\Omega), \qquad d \in L_{\text{loc}}^{\frac{p\sigma}{p\sigma - \delta}}(\Omega). \end{gathered} \tag{1.4}$$

Here and elsewhere we use a prime to denote the dual exponent, for example, $1/p + 1/p' = 1$ when $1 \le p \le \infty$, with the standard convention that 1 and ∞ are dual exponents.

The quadratic form associated with $Q(x)$ will be denoted

$$Q(x, \xi) = \langle Q(x)\xi, \xi \rangle, \quad (x, \xi) \in \Omega \times \mathbf{R}^n, \tag{1.5}$$

and we note that $Q(x, \xi)$ may vanish when $\xi \neq 0$, i.e., $Q(x)$ may be singular (degenerate). As in [MRW] and following [SW1, 2], our weak solutions are pairs $(u, \nabla u)$ which belong to an appropriate Banach space $\mathcal{W}_Q^{1,p}(\Omega)$ obtained by isomorphism from the degenerate Sobolev space $W_Q^{1,p}(\Omega)$, defined as the completion with respect to the norm

$$||u||_{W_Q^{1,p}(\Omega)} = \left(\int_\Omega |u|^p \, dx + \int_\Omega Q(x, \nabla u)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \tag{1.6}$$

of the class of functions in $\text{Lip}_{\text{loc}}(\Omega)$ with finite $W_Q^{1,p}(\Omega)$ norm. Technical facts about these Banach spaces are given in [MRW], [SW1, 2] with weighted versions in [CRW], and some of them will be recalled below. For now, we mention only that when Q is degenerate, it is important to think of an element of the Banach space $\mathcal{W}_Q^{1,p}(\Omega)$ as a pair $(u, \nabla u)$ rather than as just the first component u, due to the possibility that ∇u may not be uniquely determined by u. Nonuniqueness of ∇u causes us little difficulty since our primary regularity results concern estimates of u rather than ∇u. Except for the need to consider a pair, the notions of weak solution, weak supersolution and weak subsolution that we will use are standard, namely, we say that a pair $(u, \nabla u) \in W_Q^{1,p}(\Omega)$ satisfies

$$\text{div}(A(x, u, \nabla u)) \quad = (\le, \ge) \quad B(x, u, \nabla u) \quad \text{for } x \in \Omega \tag{1.7}$$

in the weak sense if for every nonnegative test function $\varphi \in Lip_0(\Omega)$, the corresponding integral expression

$$\int_\Omega \left[\nabla\varphi \cdot A(x, u, \nabla u) + \varphi B(x, u, \nabla u) \right] dx = (\ge, \le)\, 0 \tag{1.8}$$

holds. The integrals in (1.8) converge absolutely due to (1.2)–(1.4); see [MRW, Proposition 2.5, Corollary 2.6, Proposition 2.7].

Our results and analysis are carried out in the context of a quasimetric ρ on Ω, that is, $\rho : \Omega \times \Omega \to [0, \infty)$ and satisfies the following for all $x, y, z \in \Omega$:

- $\rho(x, y) = \rho(y, x)$ (symmetry),
- $\rho(x, y) = 0 \iff x = y$ (positivity),
- (1.9) $\rho(x, y) \le \kappa[\rho(x, z) + \rho(y, z)]$ (triangle inequality),

where $\kappa \ge 1$ is independent of $x, y, z \in \Omega$. In particular, we will assume that appropriate Sobolev-Poincaré estimates hold and that Lipschitz cutoff functions exist for the class of quasimetric ρ-balls defined for $x \in \Omega$ and $r > 0$ by

$$B(x, r) \quad = \quad \{y \in \Omega : \rho(x, y) < r\}. \tag{1.10}$$

We will refer to $B(x,r)$ as the ρ-ball of radius $r>0$ and center x. All ρ-balls lie in Ω by their definition, and they are assumed to be open with respect to the usual Euclidean topology. The estimates we need are summarized in §2.

1.2. **Some Known Results.** In the standard elliptic case when $Q(x) =$ Identity and $\rho(x,y) = |x-y|$ is the ordinary Euclidean metric, regularity results including Harnack's inequality and local Hölder continuity for weak solutions of (1.1) were derived in [S] and [T] under structural conditions more restrictive than (1.2). Obtaining analogues of these results in the degenerate case is our main concern.

In the degenerate (or subelliptic) case, Harnack's inequality and Hölder continuity have been studied in [SW1, 2] for *linear* equations with rough coefficients and nonhomogeneous terms, and those results are included among the ones we derive here. Moreover, in the degenerate *quasilinear* case, and under the same structural assumptions as in (1.2), local boundedness of weak solutions is proved in [MRW]. In fact, a rich variety of local boundedness estimates is given there depending on the strength and type of condition imposed on the coefficients, but still without any assumption about their differentiability.

In order to describe a known estimate in the degenerate quasilinear case, we now record (without listing the precise technical data) a fairly typical form of the local boundedness estimates proved in [MRW] in case $\gamma=\delta=p$ and $\psi\in[p,p+1-\sigma^{-1})$: If $(u,\nabla u)\in W^{1,p}_Q(\Omega)$ is a weak solution of (1.1) in a ρ-ball $B(y,r)$, then for any $k>0$, there are positive constants τ, C, and $\bar{Z}$ such that

$$\operatorname*{ess\,sup}_{x\in B(y,\tau r)} \big(|u(x)|+k\big) \le C\bar{Z}\left(\frac{1}{|B(y,r)|}\int_{B(y,r)} \big(|u(x)|+k\big)^p\,dx\right)^{\frac{1}{p}}. \tag{1.11}$$

Here, τ and C are independent of $u, k, B(y,r), b, c, d, e, f, g$ and h, but $\bar{Z}$ generally depends on all these quantities in very specific ways described in [MRW] and later in this paper. The richness of boundedness estimates that we mentioned above results from estimating $\bar{Z}$ under various assumptions on the coefficients. In fact, the estimates in Corollaries 1.8–1.11 of [MRW] offer only a sample of those which are possible. Understanding $\bar{Z}$, removing its dependence on u and some of the other data, and generalizing the mean-value estimates which lead to (1.11) are important ingredients in deriving the regularity results in this paper, where in the broad sense we follow the Moser method.

In order to state our results carefully, including (1.11), we must describe the technical background, which is considerable. This is done in the next section.

2. Technical Background and Hypotheses

Our principal results are axiomatic in nature and based mainly on the existence of appropriate Sobolev-Poincaré inequalities and Lipschitz cutoff functions in a space of homogeneous type. In this section, we describe the setting for our work and list our main assumptions.

2.1. **Homogeneous Spaces.** Let $\Omega\subset\mathbb{R}^n$ be an open set and ρ be a quasimetric defined on Ω satisfying (1.9). We will make two a priori assumptions relating the ρ-balls defined in (1.10) and the Euclidean balls

$$D(x,r) \quad = \quad \{y\in\Omega : |x-y|<r\}.$$

Note that $D(x,r)$ is the intersection with Ω of the ordinary Euclidean ball with center x and radius r, and recall that all ρ-balls are also subsets of Ω. As we already mentioned, we will always assume that every $B(x,r)$ is an open set according to the Euclidean topology. Second, we will always assume that

$$\text{for all } x\in\Omega,\ |x-y|\to 0 \text{ if } \rho(x,y)\to 0. \tag{2.1}$$

As a consequence of (2.1), for every $x \in \Omega$ there exists $R_0(x) > 0$ such that the Euclidean closure $\overline{B(x,r)}$ of $B(x,r)$ satisfies $\overline{B(x,r)} \subset \Omega$ for all $0 < r < R_0(x)$. See Lemma 2.1 of [MRW] for this result.

Remark 2.1. *Since ρ-balls are assumed to be open sets, the converse of* (2.1) *automatically holds:*

$$\text{for all } x, y \in \Omega,\ \rho(x,y) \to 0 \text{ if } |x-y| \to 0. \tag{2.2}$$

Furthermore, since ρ-balls are open, every ρ-ball has positive Lebesgue measure.

As is well-known, the triangle inequality (1.9) implies that ρ-balls have the following swallowing property (see e.g. [CW1, Observation 2.1] for the simple proof):

Lemma 2.2. *If $x, y \in \Omega$, $0 < t \le r$ and $B(y,t) \cap B(x,r) \ne \emptyset$, then*

$$B(y,t) \subset B(x, \gamma^* r) \tag{2.3}$$

where $\gamma^* = \kappa + 2\kappa^2$ with κ as in (1.9).

Remark 2.3. *The constant γ^* in the conclusion of Lemma 2.2 can be decreased if we only require information about the center of the smaller ball. Indeed, if $x, y \in \Omega$, $0 < t \le r$, and $B(y,t) \cap B(x,r) \ne \emptyset$, then $y \in B(x, 2\kappa r)$ by* (1.9).

Definition 2.4. *We call the triple (Ω, ρ, dx) a* local homogeneous space *if Lebesgue measure is locally a doubling measure for ρ-balls, i.e., if there are constants $C_0, d_0 > 0$ and a function $R_1 : \Omega \to (0, \infty)$ such that if $x, y \in \Omega$, $0 < t \le r < R_1(x)$ and $B(y,t) \cap B(x,r) \ne \emptyset$, then*

$$|B(x,r)| \le C_0 \Big(\frac{r}{t}\Big)^{d_0} |B(y,t)|. \tag{2.4}$$

This notion generalizes that of a symmetric general homogeneous space as defined in [SW1, p. 71]. Also, due to the swallowing property, (2.4) has an equivalent form: There are constants $C_0', c' > 0$ such that if $x, y \in \Omega$, $0 < t \le r < c' R_1(x)$ and $B(y,t) \subset B(x,r)$, then

$$|B(x,r)| \le C_0' \Big(\frac{r}{t}\Big)^{d_0} |B(y,t)| \tag{2.5}$$

for the same d_0 and $R_1(x)$ as in (2.4).

Remark 2.5. *By a result of Korobenko-Maldonado-Rios (see* [KMR]*), the validity of the local doubling condition* (2.4) *for some exponent $d_0 > 0$ and function $R_1(x) > 0$ is a consequence of two conditions that will be introduced below: the local Sobolev inequality* (2.8) *and the existence of appropriate sequences of Lipschitz cutoff functions, supported in pseudometric balls with small radius and adapted to the matrix Q, as described in* (2.10).

We will usually require that $R_1(x)$, as well as similar functions we will use to restrict sizes of radii, satisfies the local comparability condition described in the next definition.

Definition 2.6. *Let $E \subset \Omega$. We say that a function $f : \Omega \to (0, \infty)$ satisfies a* local uniformity condition with respect to ρ in E *if there is a constant $A_* = A_*(f, E) \in (0,1)$ such that for all $x \in E$ and all $y \in B(x, f(x))$,*

$$A_* < \frac{f(y)}{f(x)} < \frac{1}{A_*}. \tag{2.6}$$

Condition (2.6) is automatically true in case f is bounded above on E and also has a positive lower bound on E. This condition will be helpful in our proof of the John-Nirenberg estimate using techniques related to those in [SW1]. It is not required in [SW1] since there, $R_0(x), R_1(x)$ (and $R_2(x)$ in §2.2 below) are chosen to be the same fixed multiple of the Euclidean distance $\mathrm{dist}(x, \partial\Omega)$ and so (2.6) holds with $f(x) = R_0(x) = R_1(x) = R_2(x)$ on any set E satisfying $\overline{E} \subset \Omega$. In some proofs to follow we will choose E to be a specific quasimetric ball $B(z,r)$.

2.2. **Poincaré-Sobolev Estimates and Cutoff Functions.** Let p and Q be as in (1.2), and recall that $p \in (1,\infty)$ and $|Q| \in L^{p/2}_{loc}(\Omega)$. Before we state the Sobolev and Poincaré estimates that we require, let us make a few more comments about the Sobolev space $W^{1,p}_Q(\Omega)$. A fuller discussion can be found in [MRW], [SW2], and [CRW]. Let $Lip_{Q,p}(\Omega)$ denote the class of locally Lipschitz functions with finite $W^{1,p}_Q(\Omega)$ norm; see (1.6). The space $W^{1,p}_Q(\Omega)$ is by definition the Banach space of equivalence classes of sequences in $Lip_{Q,p}(\Omega)$ which are Cauchy sequences with respect to the norm (1.6). Here two Cauchy sequences are called equivalent if they are equiconvergent in $W^{1,p}_Q(\Omega)$.

To further describe $W^{1,p}_Q(\Omega)$, we consider the form-weighted space consisting of all (Lebesgue) measurable $\mathbf{R}^n$-valued functions $\mathbf{f}(x)$ defined in Ω for which

$$(2.7)\qquad ||\mathbf{f}||_{\mathcal{L}^p(\Omega,Q)} = \left\{\int_\Omega Q(x,\mathbf{f}(x))^{\frac{p}{2}}\,dx\right\}^{\frac{1}{p}} < \infty.$$

By identifying any two measurable $\mathbf{R}^n$-valued functions $\mathbf{f}$ and $\mathbf{g}$ with $||\mathbf{f}-\mathbf{g}||_{\mathcal{L}^p(\Omega,Q)} = 0$, (2.7) defines a norm on the resulting Banach space of equivalence classes. We denote this Banach space of equivalence classes by $\mathcal{L}^p(\Omega,Q)$. If $\{w_k\} = \{w_k\}_{k=1}^\infty \in W^{1,p}_Q(\Omega)$, meaning that $\{w_k\}$ is a Cauchy sequence of $Lip_{Q,p}(\Omega)$ functions with respect to (1.6), then there is a unique pair $(w,\mathbf{v}) \in L^p(\Omega)\times\mathcal{L}^p(\Omega,Q)$ such that $w_k \to w$ in $L^p(\Omega)$ and $\nabla w_k \to \mathbf{v}$ in $\mathcal{L}^p(\Omega,Q)$. The pair $(w,\mathbf{v})$ represents the particular equivalence class in $W^{1,p}_Q(\Omega)$ containing $\{w_k\}$. The space $\mathcal{W}^{1,p}_Q(\Omega)$ is defined to be the collection of all pairs $(w,\mathbf{v})$ that represent equivalence classes in $W^{1,p}_Q(\Omega)$. Thus, $\mathcal{W}^{1,p}_Q(\Omega)$ is the image of the isomorphism $\mathcal{J} : W^{1,p}_Q(\Omega) \to L^p(\Omega)\times\mathcal{L}^p(\Omega,Q)$ defined by

$$\mathcal{J}([\{w_k\}]) = (w,\mathbf{v}),$$

where $[\{w_k\}]$ denotes the equivalence class in $W^{1,p}_Q(\Omega)$ containing $\{w_k\}$. Therefore, $\mathcal{W}^{1,p}_Q(\Omega)$ is a closed subspace of $L^p(\Omega)\times\mathcal{L}^p(\Omega,Q)$ and hence a Banach space as well. Since $\mathcal{W}^{1,p}_Q(\Omega)$ and $W^{1,p}_Q(\Omega)$ are isomorphic, we will often refer to elements $(w,\mathbf{v})$ of $\mathcal{W}^{1,p}_Q(\Omega)$ as elements of $W^{1,p}_Q(\Omega)$. Interestingly, $\mathbf{v}$ is generally not uniquely determined by w for pairs $(w,\mathbf{v})$ in $\mathcal{W}^{1,p}_Q(\Omega)$, i.e., the projection

$$P : \mathcal{W}^{1,p}(\Omega) \to L^p(\Omega)$$

defined by $P((w,\mathbf{v})) = w$ is not always an injection; see [FKS] for an example. However, we will generally abuse notation and denote pairs in $W^{1,p}_Q(\Omega)$ by $(w,\nabla w)$ instead of $(w,\mathbf{v})$.

$(W^{1,p}_Q)_0(\Omega)$ will denote the space analogous to $W^{1,p}_Q(\Omega)$ but where the completion is formed by using Lipschitz functions with compact support in Ω. A typical element of $(W^{1,p}_Q)_0(\Omega)$ may be thought of as a pair $(w,\nabla w) \in L^p(\Omega)\times\mathcal{L}^p(\Omega,Q)$ for which there is a sequence $\{w_k\} \subset Lip_{Q,p}(\Omega)\cap Lip_0(\Omega)$ such that $w_k \to w$ in $L^p(\Omega)$ and $\nabla w_k \to \nabla w$ in $\mathcal{L}^p(\Omega,Q)$. Here we again adopt the abuse of notation ∇w for the second component $\mathbf{v}$ of a pair $(w,\mathbf{v})$.

We can now state the Sobolev-Poincaré estimates that we will assume. We say that a *local Sobolev inequality* holds in Ω if there exists a function $R_2 : \Omega \to (0,\infty)$ and constants $C_1 > 0$ and $\sigma > 1$ such that for every ρ-ball $B(y,r)$ with $0 < r < R_2(y)$, the inequality

$$(2.8)\qquad \Big(\frac{1}{|B(y,r)|}\int_{B(y,r)}|w|^{p\sigma}dx\Big)^{\frac{1}{p\sigma}} \le C_1\Big[r\Big(\frac{1}{|B(y,r)|}\int_{B(y,r)}|\sqrt{Q}\nabla w|^{\frac{p}{2}}dx\Big)^{\frac{1}{p}} + \Big(\frac{1}{|B(y,r)|}\int_{B(y,r)}|w|^p dx\Big)^{\frac{1}{p}}\Big]$$

holds for all $(w,\nabla w) \in (W^{1,p}_Q)_0(B(y,r))$.

We say that a *local Poincaré inequality* holds in Ω if there are constants $C_2 > 0$ and $\mathfrak{b} \geq 1$ such that for every ρ-ball $B(y,r)$ with $0 < r < R_2(y)$, the inequality

$$\frac{1}{|B(y,r)|}\int_{B(y,r)} |w - w_{B(y,r)}|dx \leq C_2 r\Big(\frac{1}{|B(y,\mathfrak{b}r)|}\int_{B(y,\mathfrak{b}r)} |\sqrt{Q}\nabla w|^p dx\Big)^{\frac{1}{p}} \tag{2.9}$$

holds for all $(w,\nabla w) \in W^{1,p}_Q(\Omega)$, where $w_{B(y,r)} = \dfrac{1}{|B(y,r)|}\displaystyle\int_{B(y,r)} w dx$.

Remark 2.7. *It is easy to see that* (2.8) *and* (2.9) *hold as stated, that is, for all* $(w,\nabla w)$ *in* $(W^{1,p}_Q)_0(B(y,r))$ *or* $W^{1,p}_Q(\Omega)$ *respectively, provided they hold for all* w *in* $Lip_{Q,p}(\Omega) \cap Lip_0(B(y,r))$ *or* $Lip_{Q,p}(\Omega)$ *respectively.*

As in [MRW], we ask for two more structural requirements related to our collection of quasimetric ρ-balls $\{B(x,r)\}_{r>0;x\in\Omega}$. The first of these is the existence of appropriate sequences of Lipschitz cutoff functions (called "accumulating sequences of Lipschitz cutoff functions" in [SW1]). Specifically, for the function R_2 related to the Poincaré-Sobolev estimate (2.8), we assume there are positive constants s^*, C_{s^*}, τ and N, with $p\sigma' < s^* \leq \infty$ and $\tau < 1$, such that for every ρ-ball $B(y,r)$ with $0 < r < R_2(y)$, there is a collection of Lipschitz functions $\{\eta_j\}_{j=1}^{\infty}$ satisfying

$$\begin{cases} \operatorname{supp}\eta_1 \subset B(y,r) \\ 0 \leq \eta_j \leq 1 \quad \text{for all } j \\ B(y,\tau r) \subset \{x \in B(y,r) : \eta_j(x) = 1\} \quad \text{for all } j \\ \operatorname{supp}\eta_{j+1} \subset \{x \in B(y,r) : \eta_j(x) = 1\} \quad \text{for all } j \\ \left(\dfrac{1}{|B(y,r)|}\displaystyle\int_{B(y,r)} \left|\sqrt{Q}\nabla\eta_j\right|^{s^*} dx\right)^{1/s^*} \leq C_{s^*}\dfrac{N^j}{r} \quad \text{for all } j. \end{cases} \tag{2.10}$$

This condition is slightly weaker than the corresponding one in [SW1]; see [MRW, p. 149] for a fuller discussion. We note that since $s^* > p\sigma'$, there is a number $s' > \sigma'$ such that $s^* = ps'$. The exponent $s = \frac{s^*}{s^*-p}$ dual to s' satisfies $1 \leq s < \sigma$ and plays an important role in our results.

Remark 2.8. *As already mentioned in Remark 2.5, conditions* (2.8) *and* (2.10) *imply the validity of the local doubling condition* (2.4) *for some positive exponent* d_0 *(see* [KMR]*). It is important to note that the smaller the exponent* d_0 *in* (2.4) *can be chosen, the weaker the required assumptions of local integrability on the coefficients* b, c, d, e, f, g, h *in* (1.2) *will be in the theorems to follow. See the statements of Proposition 3.3, of Theorems 3.5, 3.7, 3.10, 3.11, 3.13, 3.15 and of Corollaries 3.9, 3.12, 3.16.*

Our last requirement is that the following pair of inequalities hold simultaneously: There exists $t \in [1,\infty]$ such that for every ρ-ball $B(y,r)$ with $0 < r < R_2(y)$, there is a constant $C_3 = C_3(B(y,r)) > 0$ such that

$$\left(\int_{B(y,r)} |\sqrt{Q}\nabla\eta|^{pt} dx\right)^{1/pt} < \infty \quad \text{and} \tag{2.11}$$

$$\left(\int_{B(y,r)} |f|^{pt'} dx\right)^{1/pt'} \leq C_3 \, ||f||_{W^{1,p}_Q(\Omega)} = C_3\Big(\int_\Omega |\sqrt{Q}\nabla f|^p dx + \int_\Omega |f|^p dx\Big)^{1/p} \tag{2.12}$$

for all $\eta \in \{\eta_j\}$, $\{\eta_j\}$ as in (2.10), and all $f \in Lip_{loc}(\Omega)$. As usual, t' denotes the dual exponent of t. In case t or t' is infinite, we simply replace the relevant term in (2.11) or (2.12) by an essential supremum.

Remark 2.9. *These inequalities are used in* [MRW] *to derive a product rule for elements of* $W^{1,p}_Q(\Omega)$*. They also imply that functions in* $W^{1,p}_Q(\Omega)$*, which are generally not compactly supported, have sufficiently high local integrability in case the Sobolev inequality (2.8) holds only for compactly* *supported Lipschitz functions. See* [MRW, Section 2, p. 162] *for these results. It is useful to note that* (2.11) *is automatically satisfied for every* t *with* $1 \le t \le s^*/p$ *by* (2.10)*. However,* (2.11) *may also hold for larger values of* t *independently of* (2.10)*. See* [MRW, p. 150] *for details. In fact, if* (2.11) *holds with* $t = \infty$ *then* (2.12) *(with* $t' = 1$*) is trivial due to the form of the* $W^{1,p}_Q(\Omega)$ *norm (1.6).*

In order to simplify notation when combining hypotheses, we fix a single function $r_1 : \Omega \to (0, \infty)$ satisfying

$$r_1(x) \;\le\; \min\{R_0(x), R_1(x), R_2(x), 1\}, \quad x \in \Omega, \tag{2.13}$$

where R_0 is as described below (2.1), R_1 is as in Definition 2.4 and R_2 is as in (2.8), (2.9), (2.10), (2.11), and (2.12).

3. Harnack's Inequality

We begin this section by recalling some notations of [MRW]. Given a measurable set E and a measurable function f on E, we write

$$||f||_{p,E;\overline{dx}} \;=\; \left(\frac{1}{|E|}\int_E |f(x)|^p dx\right)^{\frac{1}{p}} = \left(\fint_E |f(x)|^p dx\right)^{\frac{1}{p}}, \text{ and} \tag{3.1}$$

$$||f||_{p,E;dx} \;=\; \left(\int_E |f(x)|^p dx\right)^{\frac{1}{p}}. \tag{3.2}$$

In some cases when context is clear, the set E may be dropped from the left hand side in (3.1) and (3.2).

Given a function u and constants $k, \epsilon_1, \epsilon_2, \epsilon_3$ with $k > 0$ and $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1]$, we denote

$$\begin{array}{rclrcl} \bar{u} &=& |u| + k, & \bar{b} &=& b + k^{1-p}e, \\ \bar{h} &=& h + k^{-p}g, & \bar{d} &=& d + k^{1-p}f. \end{array} \tag{3.3}$$

Here, b, c, d, e, f, g, h denote the coefficients in (1.2). Furthermore, for each ρ-ball $B(y, r)$, define

$$\begin{aligned} \bar{Z} = \bar{Z}(B(y,r),\bar{u}) &= 1 + r^{p-1}\|\bar{b}\|_{p'\sigma',B(y,r);\overline{dx}} \\ &+ \left(r^p\|c^{\frac{p}{p+1-\psi}}\bar{u}^{\frac{p(\psi-p)}{p+1-\psi}}\|_{\frac{p\sigma'}{p-\epsilon_1},B(y,r);\overline{dx}}\right)^{\frac{1}{\epsilon_1}} + \left(r^p\|\bar{h}\|_{\frac{p\sigma'}{p-\epsilon_2},B(y,r);\overline{dx}}\right)^{\frac{1}{\epsilon_2}} + \left(r^p\|\bar{d}\|_{\frac{p\sigma'}{p-\epsilon_3},B(y,r);\overline{dx}}\right)^{\frac{1}{\epsilon_3}}, \end{aligned} \tag{3.4}$$

where the exponents p, ψ, σ are as usual; see (1.2) and (2.8). It is important to note that $\bar{Z}$ is not monotone in its first argument due to the normalized norms appearing in its definition. However, if $\bar{Z}(B, \bar{u}) < \infty$, then $\bar{Z}(B', \bar{u}) < \infty$ whenever $B' \subset B = B(y, r)$ with $r < R_0(y)$.

3.1. Standing Assumptions.

In order to state our main results efficiently, we list here several standing assumptions to remain in effect for the rest of this paper. As above, Ω will always denote a bounded domain in $\mathbb{R}^n$, ρ denotes a quasimetric on Ω, and $Q(x)$ denotes a measurable symmetric nonnegative definite matrix defined in Ω. We always assume the triple (Ω, ρ, dx) defines a local homogeneous space in the sense of Definition 2.4. Note that this ensures that the local doubling condition (2.4) is satisfied. We also assume the validity of the local Sobolev and Poincaré inequalities (2.8) and (2.9) and the existence of accumulating sequences of Lipschitz cutoff functions satisfying (2.10) for a fixed $\tau \in (0, 1)$ and $s^* > p\sigma'$. Here σ' denotes the dual exponent to the Sobolev gain factor σ of (2.8). Lastly, we assume that each of (2.11) and (2.12) holds for some $t \in [1, \infty]$. We can now state our core Harnack result. Under certain conditions, it will spawn other versions of Harnack's inequality that will lead to continuity of weak solutions.

3.2. **Main Results.**

Proposition 3.1. *Let $1 < p < \infty$ and $|Q(x)| \in L^{p/2}_{loc}(\Omega)$. Assume that the functions A, B of* (1.1) *satisfy* (1.2) *with*

$$\gamma = \delta = p, \quad \psi \in [p, p+1-\sigma^{-1}). \tag{3.5}$$

Fix $y \in \Omega$ and suppose there is a function $r_1(x)$ as in (2.13) *that satisfies a local uniformity condition in $B(y, r_1(y))$ with constant $A_* = A_*(y, r_1(y))$; see* (2.6)*. Let*

$$C_* = \frac{128\mathfrak{b}\kappa^{10}(\gamma^*)^8}{\tau A_*^3 \min\{A_*^2, (8\kappa^5)^{-1}\}}, \tag{3.6}$$

where $\mathfrak{b}$ is from (2.9). For $x_0 \in B\big(y, \frac{\tau}{5\kappa} r_1(y)\big)$ and $r \in \big(0, \frac{\tau A_*}{5\kappa C_*} r_1(y)\big)$, define

$$\mathfrak{E} = \{(x, l) \ : \ B(x, l) \Subset B(x_0, C_* r) \text{ and } 0 < l \le C_* r\}.$$

Let $(u, \nabla u) \in W^{1,p}(\Omega)$ be a weak solution of (1.1)*. Assume that $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1]$ and $k \ge 0$ are such that*

$$\sup_{(x,l)\in\mathfrak{E}} \bar{Z}\big(B(x, l), \bar{u}\big) = M < \infty, \tag{3.7}$$

where $\bar{Z}$ is defined by (3.4) *and $\bar{u} = |u| + k$. If $u \ge 0$ in $B(x_0, C_* r)$, then the Harnack inequality*

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)} \bar{u}(z) \ \ \le \ \ C_4\big[C_5 \bar{Z}(B(x_0, r), \bar{u})\big]^{C_6 M} \operatorname*{ess\,inf}_{z\in B(x_0,\tau r)} \bar{u}(z) \tag{3.8}$$

holds with

i) *C_4 depending on $p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3$,*
ii) *C_5 depending on $a, p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3$, C_1 in* (2.8)*, N, C_{s^*} in* (2.10)*, on the pseudometric ρ,*
iii) *C_6 depending on $a, p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3$, $C_2, \mathfrak{b}$ in* (2.9)*, τ, N, C_{s^*} in* (2.10)*, C_0, d_0 in* (2.4) *and on the pseudometric ρ.*

Remark 3.2. *Since under the hypotheses of Proposition 3.1 one has $(x_0, r) \in \mathfrak{E}$, we obtain*

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)} \bar{u}(z) \ \ \le \ \ C_4\big[C_5 M\big]^{C_6 M} \operatorname*{ess\,inf}_{z\in B(x_0,\tau r)} \bar{u}(z) \tag{3.9}$$

with C_4, C_5, C_6 independent of $(u, \nabla u), b, c, d, e, f, g, h, k, y, x_0, r, M$.

A proof of Proposition 3.1 is given in §6. The next proposition provides explicit integrability conditions on structural coefficients and choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and k that ensure condition (3.7) is satisfied. It also provides a decay condition on k essential for proving Hölder continuity of weak solutions to (1.1); see Theorem 3.7 and its proof.

Proposition 3.3. *Let $1 < p < \infty$, $|Q| \in L^{p/2}_{loc}(\Omega)$, and $\psi \in [p, p+1-\sigma^{-1})$. Let $(u, \nabla u) \in W^{1,p}_Q(\Omega)$, fix $y \in \Omega$ and suppose there is a function $r_1(x)$ as in* (2.13) *which satisfies a local uniformity condition in $B(y, r_1(y))$ with constant $A_* = A_*(y, r_1(y))$. Let C_* be defined as in* (3.6) *and assume that*

i) *$b, e \in L^{\mathcal{B}}_{loc}(\Omega)$ with $\mathcal{B} \ge \max\big\{p'\sigma', \frac{d_0}{p-1}\big\}$;*
ii) *$h, g \in L^{\mathcal{H}}_{loc}(\Omega)$ with $\mathcal{H} \ge \frac{d_0}{p}$, $\mathcal{H} > \sigma'$;*
iii) *$d, f \in L^{\mathcal{D}}_{loc}(\Omega)$ with $\mathcal{D} \ge \frac{d_0}{p}$, $\mathcal{D} > \sigma'$;*
iv) *$c \in L^{\mathcal{C}}_{loc}(\Omega)$ with $\mathcal{C} \ge \frac{d_0 p\sigma}{(p+1-\psi)(p\sigma+d_0)-d_0} > 0$ and $\mathcal{C} > \frac{p\sigma}{\sigma(p+1-\psi)-1}$.*

For every $x_0 \in B\big(y, \frac{\tau}{5\kappa} r_1(y)\big)$ and $r \in \big(0, \frac{\tau A_*}{5\kappa C_*} r_1(y)\big)$, define

$$\begin{aligned} k \ &= \ k(x_0, r) \ = \ k\big(B(x_0, C_* r)\big) \\ &= \ \Big[(C_* r)^{p-1}\|e\|_{\mathcal{B},B(x_0,C_* r);\overline{dx}}\Big]^{\frac{1}{p-1}} + \Big[(C_* r)^p\|f\|_{\mathcal{D},B(x_0,C_* r);\overline{dx}}\Big]^{\frac{1}{p-1}} + \Big[(C_* r)^p\|g\|_{\mathcal{H},B(x_0,C_* r);\overline{dx}}\Big]^{\frac{1}{p}}. \end{aligned}$$

Then

$$k(x_0,r)\le \Lambda r^{\lambda}, \tag{3.10}$$

where λ,Λ *are nonnegative numbers independent of* x_0, r *of the form*

$$\begin{aligned}\lambda &= \min\Big\{1-\frac{d_0}{(p-1)\mathcal{B}},\frac{1}{p-1}\Big(p-\frac{d_0}{\mathcal{D}}\Big),1-\frac{d_0}{p\mathcal{H}}\Big\}, \text{ and}\\ \Lambda &= C_0^{\frac{1}{(p-1)\mathcal{B}}}C_*^{1-\frac{d_0}{(p-1)\mathcal{B}}}r_1(y)^{1-\lambda}\|e\|^{\frac{1}{p-1}}_{\mathcal{B},B(y,r_1(y));\overline{dx}}\\ &\quad + C_0^{\frac{1}{(p-1)\mathcal{D}}}C_*^{\frac{p}{p-1}-\frac{d_0}{(p-1)\mathcal{D}}}r_1(y)^{\frac{p}{p-1}-\lambda}\|f\|^{\frac{1}{p-1}}_{\mathcal{D},B(y,r_1(y));\overline{dx}}\\ &\quad + C_0^{\frac{1}{p\mathcal{H}}}C_*^{1-\frac{d_0}{p\mathcal{H}}}r_1(y)^{1-\lambda}\|g\|^{\frac{1}{p}}_{\mathcal{H},B(y,r_1(y));\overline{dx}}.\end{aligned}$$

Moreover, with ϵ_1, ϵ_2 *and* ϵ_3 *defined by*

$$\epsilon_1=\min\Big\{1,\frac{p\sigma(p+1-\psi)-p-(p^2\sigma/\mathcal{C})}{(\sigma-1)(p+1-\psi)}\Big\},\quad \epsilon_2=\min\Big\{1,p-\frac{p\sigma'}{\mathcal{H}}\Big\},\quad \epsilon_3=\min\Big\{1,p-\frac{p\sigma'}{\mathcal{D}}\Big\},$$

(3.7) *is satisfied with*

$$\begin{aligned}M &= 1+C_0^{\frac{1}{\mathcal{B}}}\Big[1+r_1(y)^{p-1}\|b\|_{\mathcal{B},B(y,r_1(y));\overline{dx}}\Big]+C_0^{\frac{1}{\varepsilon_2\mathcal{H}}}\Big[1+r_1(y)^{p}\|h\|_{\mathcal{H},B(y,r_1(y));\overline{dx}}\Big]^{\frac{1}{\epsilon_2}}\\ &\quad + C_0^{\frac{1}{\varepsilon_3\mathcal{D}}}\Big[1+r_1(y)^{p}\|d\|_{\mathcal{D},B(y,r_1(y));\overline{dx}}\Big]^{\frac{1}{\epsilon_3}}\\ &\quad + C_0^{\frac{\psi-p+(p\sigma/\mathcal{C})}{\epsilon_1(p+1-\psi)\sigma}}\Bigg[r_1(y)^p\|c\|^{\frac{p}{p+1-\psi}}_{\mathcal{C},B(y,r_1(y));\overline{dx}}\Big(\|u\|_{p\sigma,B(y,r_1(y));\overline{dx}}+\Lambda r_1(y)^{\lambda}\Big)^{\frac{p(\psi-p)}{p+1-\psi}}\Bigg]^{\frac{1}{\epsilon_1}},\end{aligned}$$

where C_0 *is as in* (2.4).

Proposition 3.3 is proved in the appendix.

Remark 3.4. (1) *In part (iv), the assumption that* $\frac{d_0p\sigma}{(p+1-\psi)(p\sigma+d_0)-d_0}>0$ *follows from the condition* $\psi\in[p,p+1-\sigma^{-1})$ *provided* $d_0\le p\sigma'$*; also, in the classical Euclidean situation, the condition* $d_0\le p\sigma'$ *is true with equality. If it is the case that* $d_0>p\sigma'$*, this condition further restricts* $\psi\in[p,p+1-\frac{d_0}{d_0+p\sigma})\subsetneq[p+1-\sigma^{-1})$.

(2) *The constants* λ,Λ,M *in Proposition 3.3 are independent of* x_0,r. *Moreover* λ *is independent of* y. *The constant* M *depends on* u *only through* $\|u\|_{p\sigma,B(y,r_1(y));dx}$, *and it is independent of* u *when* $\psi=p$.

(3) *The strict inequalities in (ii), (iii) and (iv) guarantee that* ϵ_1, ϵ_2, $\epsilon_3>0$*; moreover* $\lambda>0$ *if all the inequalities in (i), (ii), (iii) are strict.*

Combining Propositions 3.1 and 3.3 we obtain the following theorem.

Theorem 3.5. *(Harnack's Inequality, when* $\gamma=\delta=p$ *and* $\psi\ge p$*) Let* $1<p<\infty$ *and* $|Q|\in L^{p/2}_{loc}(\Omega)$. *Let* A,B *be functions satisfying* (1.2) *with* γ,δ,ψ *restricted to*

$$\gamma=\delta=p,\quad \psi\in[p,p+1-\sigma^{-1}).$$

Fix $y\in\Omega$ *and suppose there is a function* $r_1(x)$ *as in* (2.13) *which satisfies a local uniformity condition in* $B(y,r_1(y))$ *with constant* $A_*=A_*(y,r_1(y))$. *Let* C_* *be as in* (3.6), $x_0\in B(y,\frac{\tau}{5\kappa}r_1(y))$ *and* $r\in(0,\frac{\tau A_*}{5\kappa C_*}r_1(y))$. *Assume that the structural functions* b,c,d,e,f,g,h *of* (1.2) *and* $\epsilon_1,\epsilon_2,\epsilon_3$ *and* $k=k(x_0,r)$ *are as in Proposition 3.3. If* $(u,\nabla u)\in W^{1,p}_Q(\Omega)$ *is a weak solution of* (1.1) *in* Ω *and* $u\ge 0$ *in* $B(x_0,C_*r)$, *then*

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)}\Big(u(z)+k(x_0,r)\Big)\le C\operatorname*{ess\,inf}_{z\in B(x_0,\tau r)}\Big(u(z)+k(x_0,r)\Big), \tag{3.11}$$

with $C = C_4(C_5M)^{C_6M}$*, M as in Proposition 3.3 and C_4, C_5, C_6 as in Proposition 3.1 with $\epsilon_1, \epsilon_2, \epsilon_3$ given in Proposition 3.3. The constant C depends on $||u||_{p\sigma,B(y,r_1(y));dx}$ only when $\psi > p$ and only through M.*

The proof of Theorem 3.5 follows by simply combining Propositions 3.1, 3.3 and is left to the reader. Theorem 3.5 will allow us to prove Hölder continuity of weak solutions to (1.1). First we recall the notions of Hölder continuity that we will use.

Definition 3.6. *Let $w : \Omega \to \mathbb{R}$ and $S \subset \Omega$. We say that w is:*

(1) **essentially Hölder continuous** *with respect to ρ in S if there are positive constants C, μ such that*

$$\operatorname*{ess\,sup}_{z,x\in S} \frac{|w(z) - w(x)|}{\rho(z, x)^\mu} \le C. \tag{3.12}$$

(2) **essentially locally Hölder continuous** *with respect to ρ in S if for every compact set $K \subset S$, there are positive constants C, μ such that*

$$\operatorname*{ess\,sup}_{z,x\in K} \frac{|w(z) - w(x)|}{\rho(z, x)^\mu} \le C. \tag{3.13}$$

In these definitions, the notion of Hölder continuity of a function is relative to the quasimetric ρ. Classical Hölder continuity with respect to the usual Euclidean metric then follows by imposing a Fefferman-Phong containment condition on the family of quasimetric ρ-balls. Recall that a Fefferman-Phong condition holds if there are positive constants C, ε such $D(x, r) \subset B(x, Cr^\varepsilon)$ for $x \in \Omega$ and $r > 0$ sufficiently small (in terms of x). Several references impose this condition for such a purpose; see [FP] and [SW1] for further discussion.

Our study of Hölder continuity of weak solutions begins with the case when the exponents γ, δ, ψ are restricted as in (3.5).

Theorem 3.7. *(Hölder continuity, when $\gamma = \delta = p$ and $\psi \ge p$) Let $1 < p < \infty$ and $|Q| \in L^{p/2}_{loc}(\Omega)$. Let $(u, \nabla u)$ be a weak solution of* (1.1) *in Ω where the functions $A(x, z, \xi)$ and $B(x, z, \xi)$ satisfy* (1.2) *with γ, δ, ψ as in* (3.5)*. Assume that the coefficient functions of* (1.2) *satisfy conditions (i)-(iv) of Proposition 3.3 with strict inequality. Let $y \in \Omega$ and suppose there is a function $r_1(x)$ as in* (2.13) *which satisfies a local uniformity condition in $B(y, r_1(y))$ with constant $A_* = A_*(y, r_1(y))$. Then u is essentially Hölder continuous with respect to ρ in $B(y, \frac{\tau^2}{5\kappa} r_1(y))$. The constants C and μ in* (3.13) *depend on $y, r_1(y), A_*$, κ as in* (1.9)*, the Harnack constant $C_4(C_5M)^{C_6M}$ which appears in Theorem 3.5, λ as in Proposition 3.3; C depends also on $\|u\|_{p\sigma,B(y,r_1(y));dx}$.*

Remark 3.8. *We explicitly note that μ in the previous Theorem depends on $\|u\|_{p\sigma,B(y,r_1(y));dx}$ only through M, and thus it depends on u itself only if $\psi > p$.*

Theorem 3.7 is proved in §7. The next result gives sufficient conditions for essential local Hölder continuity of solutions in Ω.

Corollary 3.9. *Let $1 < p < \infty$ and $|Q| \in L^{p/2}_{loc}(\Omega)$, and suppose* (1.2) *holds with γ, δ, ψ as in* (3.5)*. Assume also that the coefficient functions of* (1.2) *satisfy conditions (i)-(iv) of Proposition 3.3 with strict inequality. Let $r_1 : \Omega \to (0, \infty)$ be a function satisfying* (2.13) *with the property that given any compact $K \subset \Omega$ there is a positive constant s_0 such that $s_0 \le r_1(y) \le 1$ for every $y \in K$. Then if $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ is a weak solution of* (1.1) *in Ω, u is essentially locally Hölder continuous with respect to ρ in Ω.*

A brief proof of Corollary 3.9 can be found in §8.

3.3. **Some consequences.** The following results are concerned with some of the possible cases when the exponents γ, δ, ψ are allowed to vary in the ranges given in (1.3).

In particular, Theorems 3.10 and 3.11 and Corollary 3.12 are devoted to the case when $\gamma, \delta, \psi < p$. We consider the case when $\gamma, \delta, \psi > p$ and satisfy (1.3) in Theorems 3.13 and 3.15 and in Corollary 3.16. See §9 for their proofs.

Of course, similar results can be obtained for other choices of γ, δ, ψ in the ranges given in (1.3) but we won't list them here. Such results can all be derived from Theorems 3.5, 3.7 and Corollary 3.9. We leave the details to the interested reader.

Theorem 3.10. *(Harnack's Inequality, when $\gamma, \delta, \psi < p$) Let $1 < p < \infty$ and $|Q| \in L^{p/2}_{loc}(\Omega)$. Let A, B be functions satisfying (1.2) with γ, δ, ψ restricted to*

$$\gamma, \delta, \psi \in (1, p). \tag{3.14}$$

Fix $y \in \Omega$ and suppose there is a function $r_1(x)$ as in (2.13) which satisfies a local uniformity condition in $B(y, r_1(y))$ with constant $A_* = A_*(y, r_1(y))$. Let C_* be as in (3.6), $x_0 \in B(y, \frac{\tau}{5\kappa} r_1(y))$ and $r \in (0, \frac{\tau A_*}{5\kappa C_*} r_1(y))$. Assume that the structural functions b, d, e, f, g, h of (1.2) satisfy conditions (i), (ii) and (iii) in Proposition 3.3, that $c \in L^{\mathcal{C}}_{loc}(\Omega)$ with $\mathcal{C} \geq d_0$ and $\mathcal{C} > p\sigma'$. Let $\epsilon_2, \epsilon_3, \lambda$ be as in Proposition 3.3 and define

$$\begin{aligned}
\epsilon_1 &= \min\Big\{1, p - \frac{p^2\sigma'}{\mathcal{C}}\Big\},\\
k_1 &= k_1(x_0, r) = k_1\big(B(x_0, C_* r)\big)\\
&= \Big[(C_* r)^{p-1}\|b + e\|_{\mathcal{B}, B(x_0, C_* r); \overline{dx}}\Big]^{\frac{1}{p-1}} + \Big[(C_* r)^{p}\|c + d + f\|_{\mathcal{D}, B(x_0, C_* r); \overline{dx}}\Big]^{\frac{1}{p-1}}\\
&\quad + \Big[(C_* r)^{p}\|g + h\|_{\mathcal{H}, B(x_0, C_* r); \overline{dx}}\Big]^{\frac{1}{p}},\\
\Lambda_1 &= C_0^{\frac{1}{(p-1)\mathcal{B}}} C_*^{1 - \frac{d_0}{(p-1)\mathcal{B}}} r_1(y)^{1-\lambda} \|b + e\|^{\frac{1}{p-1}}_{\mathcal{B}, B(y, r_1(y)); \overline{dx}}\\
&\quad + C_0^{\frac{1}{(p-1)\mathcal{D}}} C_*^{\frac{p\mathcal{D} - d_0}{(p-1)\mathcal{D}}} r_1(y)^{\frac{p}{p-1} - \lambda} \|c + d + f\|^{\frac{1}{p-1}}_{\mathcal{D}, B(y, r_1(y)); \overline{dx}}\\
&\quad + C_0^{\frac{1}{p\mathcal{H}}} C_*^{1 - \frac{d_0}{p\mathcal{H}}} r_1(y)^{1-\lambda} \|g + h\|^{\frac{1}{p}}_{\mathcal{H}, B(y, r_1(y)); \overline{dx}},\\
M_1 &= 1 + C_0^{\frac{1}{\mathcal{B}}}\Big[1 + r_1(y)^{p-1}\|b\|_{\mathcal{B}, B(y, r_1(y)); \overline{dx}}\Big] + C_0^{\frac{1}{\varepsilon_2 \mathcal{H}}}\Big[1 + r_1(y)^{p}\|h\|_{\mathcal{H}, B(y, r_1(y)); \overline{dx}}\Big]^{\frac{1}{\varepsilon_2}}\\
&\quad + C_0^{\frac{1}{\varepsilon_3 \mathcal{D}}}\Big[1 + r_1(y)^{p}\|d\|_{\mathcal{D}, B(y, r_1(y)); \overline{dx}}\Big]^{\frac{1}{\varepsilon_3}} + C_0^{\frac{p}{\varepsilon_1 \mathcal{C}}}\Big[r_1(y)^{p}\|c\|^{p}_{\mathcal{C}, B(y, r_1(y)); \overline{dx}}\Big]^{\frac{1}{\varepsilon_1}}.
\end{aligned}$$

Then

$$k_1(x_0, r) \leq \Lambda_1 r^{\lambda}$$

and, if $(u, \nabla u) \in W^{1,p}_{Q}(\Omega)$ is a weak solution of (1.1) in Ω such that $u \geq 0$ in $B(x_0, C_* r)$,

$$\operatorname*{ess\,sup}_{z \in B(x_0, \tau r)} \Big(u(z) + k_1(x_0, r)\Big) \;\leq\; C \operatorname*{ess\,inf}_{z \in B(x_0, \tau r)} \Big(u(z) + k_1(x_0, r)\Big), \tag{3.15}$$

where $C = C_4(C_5 M_1)^{C_6 M_1}$, with C_4, C_5, C_6 as in Proposition 3.1.

Theorem 3.11. *(Hölder continuity, when $\gamma, \delta, \psi < p$) Let $1 < p < \infty$ and $|Q| \in L^{p/2}_{loc}(\Omega)$. Let $(u, \nabla u)$ be a weak solution of (1.1) in Ω where the functions $A(x, z, \xi)$ and $B(x, z, \xi)$ satisfy (1.2) with γ, δ, ψ as in (3.14). Assume that the coefficient functions of (1.2) satisfy the same conditions as in Theorem 3.10 with strict inequality. Let $y \in \Omega$ and suppose there is a function $r_1(x)$ as in (2.13) which satisfies a local uniformity condition in $B(y, r_1(y))$ with constant $A_* = A_*(y, r_1(y))$. Then u is essentially Hölder continuous with respect to ρ in $B(y, \frac{\tau^2}{5\kappa} r_1(y))$. The constants C and μ in (3.13) depend on $y, r_1(y), A_*$, κ as in (1.9), the Harnack constant $C_4(C_5 M_1)^{C_6 M_1}$ which* *appears in Theorem 3.10, λ as in Proposition 3.3; C depends also on $\|u\|_{p\sigma,B(y,r_1(y));dx}$, while μ is independent of $(u,\nabla u)$.*

Corollary 3.12. *Let $1<p<\infty$ and $|Q|\in L^{p/2}_{loc}(\Omega)$, and suppose* (1.2) *holds with γ,δ,ψ as in* (3.14)*. Assume also that the coefficient functions of* (1.2) *satisfy the same conditions as in Theorem 3.10 with strict inequality. Let $r_1:\Omega\to(0,\infty)$ be a function satisfying* (2.13) *with the property that given any compact $K\subset\Omega$ there is a positive constant s_0 such that $s_0\le r_1(y)\le 1$ for every $y\in K$. Then if $(u,\nabla u)\in W^{1,p}_Q(\Omega)$ is a weak solution of* (1.1) *in Ω, u is essentially locally Hölder continuous with respect to ρ in Ω with exponent μ that is independent of the weak solution $(u,\nabla u)$.*

Theorem 3.13. *(Harnack's Inequality, when $\gamma,\delta,\psi>p$) Let $1<p<\infty$ and $|Q|\in L^{p/2}_{loc}(\Omega)$. Let A,B be functions satisfying* (1.2) *with γ,δ,ψ satisfying* (1.3) *and restricted to*

$$\gamma,\delta,\psi>p. \tag{3.16}$$

Fix $y\in\Omega$ and suppose there is a function $r_1(x)$ as in (2.13) *which satisfies a local uniformity condition in $B(y,r_1(y))$ with constant $A_*=A_*(y,r_1(y))$. Let C_* be as in* (3.6)*, $x_0\in B(y,\frac{\tau}{5\kappa}r_1(y))$ and $r\in(0,\frac{\tau A_*}{5\kappa C_*}r_1(y))$. Assume that the structural functions b,d,e,f,g,h of* (1.2) *satisfy*

i) $b\in L^{\mathcal{B}_0}_{loc}(\Omega)$ *with* $\mathcal{B}_0\ge\frac{p\sigma}{p\sigma-\sigma-\gamma+1}$ *and* $\mathcal{B}_0\ge\frac{d_0p\sigma}{p(p-1)\sigma-d_0(\gamma-p)}>0$*;*
ii) $e\in L^{\mathcal{E}}_{loc}(\Omega)$ *with* $\mathcal{E}\ge\max\{p'\sigma',\frac{d_0}{p-1}\}$*;*
iii) $h\in L^{\mathcal{H}_0}_{loc}(\Omega)$ *with* $\mathcal{H}_0>\frac{p\sigma}{p\sigma-\gamma}$ *and* $\mathcal{H}_0\ge\frac{d_0p\sigma}{p^2\sigma-d_0(\gamma-p)}>0$*;*
iv) $g\in L^{\mathcal{G}}_{loc}(\Omega)$ *with* $\mathcal{G}\ge\frac{d_0}{p}$, $\mathcal{G}>\sigma'$*;*
v) $d\in L^{\mathcal{D}_0}_{loc}(\Omega)$ *with* $\mathcal{D}_0>\frac{p\sigma}{p\sigma-\delta}$ *and* $\mathcal{D}_0\ge\frac{d_0p\sigma}{p^2\sigma-d_0(\delta-p)}>0$*;*
vi) $f\in L^{\mathcal{F}}_{loc}(\Omega)$ *with* $\mathcal{F}\ge\frac{d_0}{p}$, $\mathcal{F}>\sigma'$*;*
vii) $c\in L^{\mathcal{C}}_{loc}(\Omega)$ *with* $\mathcal{C}>\frac{p\sigma}{\sigma(p+1-\psi)-1}$ *and* $\mathcal{C}\ge\frac{d_0p\sigma}{(p+1-\psi)(p\sigma+d_0)-d_0}>0$.

Define

$$\mathcal{B}=\min\left\{\frac{p\sigma}{\frac{p\sigma}{\mathcal{B}_0}+(\gamma-p)},\mathcal{E}\right\},\ \mathcal{H}=\min\left\{\frac{p\sigma}{\frac{p\sigma}{\mathcal{H}_0}+(\gamma-p)},\mathcal{G}\right\},\ \mathcal{D}=\min\left\{\frac{p\sigma}{\frac{p\sigma}{\mathcal{D}_0}+(\delta-p)},\mathcal{F}\right\}.$$

Let $k=k(x_0,r),\epsilon_1,\epsilon_2,\epsilon_3,\lambda,\Lambda$ be as in Proposition 3.3 and define

$$\begin{aligned}M_2 &= 1+C_0^{\frac{1}{\mathcal{B}}}\left[1+\frac{r_1(y)^{p-1}}{|B(y,r_1(y))|^{\frac{1}{\mathcal{B}}}}\|b\|_{\mathcal{B}_0,B(y,r_1(y));dx}\|u\|^{\gamma-p}_{p\sigma,B(y,r_1(y));dx}\right]\\ &\quad+C_0^{\frac{1}{\epsilon_2\mathcal{H}}}\left[1+\frac{r_1(y)^{p}}{|B(y,r_1(y))|^{\frac{1}{\mathcal{H}}}}\|h\|_{\mathcal{H}_0,B(y,r_1(y));dx}\|u\|^{\gamma-p}_{p\sigma,B(y,r_1(y));dx}\right]^{\frac{1}{\epsilon_2}}\\ &\quad+C_0^{\frac{1}{\epsilon_3\mathcal{D}}}\left[1+\frac{r_1(y)^{p}}{|B(y,r_1(y))|^{\frac{1}{\mathcal{D}}}}\|d\|_{\mathcal{D}_0,B(y,r_1(y));dx}\|u\|^{\delta-p}_{p\sigma,B(y,r_1(y));dx}\right]^{\frac{1}{\epsilon_3}}\\ &\quad+C_0^{\frac{\psi-p+(p\sigma/\mathcal{C})}{\epsilon_1(p+1-\psi)\sigma}}\left[r_1(y)^p\|c\|^{\frac{p}{p+1-\psi}}_{\mathcal{C},B(y,r_1(y));\overline{dx}}\left(\|u\|_{p\sigma,B(y,r_1(y));\overline{dx}}+\Lambda r_1(y)^{\lambda}\right)^{\frac{p(\psi-p)}{p+1-\psi}}\right]^{\frac{1}{\epsilon_1}},\end{aligned} \tag{3.17}$$

If $(u,\nabla u)\in W^{1,p}_Q(\Omega)$ is a weak solution of (1.1) *in Ω and $u\ge0$ in $B(x_0,C_*r)$, then*

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)}\Big(u(z)+k(x_0,r)\Big)\ \le\ C\operatorname*{ess\,inf}_{z\in B(x_0,\tau r)}\Big(u(z)+k(x_0,r)\Big), \tag{3.18}$$

where $k=k(x_0,r)$ satisfies (3.10) *and $C=C_4(C_5M_2)^{C_6M_2}$, with C_4,C_5,C_6 as in Proposition 3.1.*

Remark 3.14. *In parts (i), (iii), (v) and (vii) of the assumptions of Theorem 3.13, the positivity assumptions on* $\frac{d_0 p\sigma}{p(p-1)\sigma - d_0(\gamma-p)}$, $\frac{d_0 p\sigma}{p^2\sigma - d_0(\gamma-p)}$, $\frac{d_0 p\sigma}{p^2\sigma - d_0(\delta-p)}$ *and* $\frac{d_0 p\sigma}{(p+1-\psi)(p\sigma+d_0)-d_0}$ *are a consequence of conditions* (1.3) *and* (3.16) *when* $d_0 \le p\sigma'$. *It is also useful to note that* $d_0 \le p\sigma'$ *is true with equality in the classical Euclidean situation. In case* $d_0 > p\sigma'$, *the positivity conditions of items* $(i), (v)$, *and (vii) further restrict the ranges of* γ, δ, *and* ψ. *See also part (1) of Remark 3.4.*

Theorem 3.15. *(Hölder continuity, when* $\gamma, \delta, \psi > p$*) Let* $1 < p < \infty$ *and* $|Q| \in L^{p/2}_{loc}(\Omega)$. *Let* $(u, \nabla u)$ *be a weak solution of* (1.1) *in* Ω *where the functions* $A(x, z, \xi)$ *and* $B(x, z, \xi)$ *satisfy* (1.2) *with* γ, δ, ψ *as in* (3.16) *and* (1.3). *Assume that the coefficient functions of* (1.2) *satisfy the same conditions as in Theorem 3.13 with strict inequality. Let* $y \in \Omega$ *and suppose there is a function* $r_1(x)$ *as in* (2.13) *which satisfies a local uniformity condition in* $B(y, r_1(y))$ *with constant* $A_* = A_*(y, r_1(y))$. *Then* u *is essentially Hölder continuous with respect to* ρ *in* $B(y, \frac{\tau^2}{5\kappa} r_1(y))$. *The constants* C *and* μ *in* (3.13) *depend on* $y, r_1(y), A_*$, κ *as in* (1.9), *the Harnack constant* $C_4(C_5 M_2)^{C_6 M_2}$ *which appears in Theorem 3.13,* λ *as in Proposition 3.3; C depends also on* $\|u\|_{p\sigma, B(y, r_1(y)); dx}$.

Corollary 3.16. *Let* $1 < p < \infty$ *and* $|Q| \in L^{p/2}_{loc}(\Omega)$, *and suppose* (1.2) *holds with* γ, δ, ψ *as in* (3.16) *and* (1.3). *Assume also that the coefficient functions of* (1.2) *satisfy the same conditions as in Theorem 3.13 with strict inequality. Let* $r_1 : \Omega \to (0, \infty)$ *be a function satisfying* (2.13) *with the property that given any compact* $K \subset \Omega$ *there is a positive constant* s_0 *such that* $s_0 \le r_1(y) \le 1$ *for every* $y \in K$. *Then if* $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ *is a weak solution of* (1.1) *in* Ω, u *is essentially locally Hölder continuous with respect to* ρ *in* Ω.

We conclude the section with some comments concerning the rate growth of the Euclidean volume of pseudometric balls $B(x, r)$.

Definition 3.17. *Let* $\Theta \Subset \Omega$ *and* $r_1 : \Omega \to (0, \infty)$ *be a function satisfying* (2.13). *If* q^* *satisfies* $0 < q^* < \infty$ *and there are positive constants* C_7, α *such that*

$$|B(x, r)| \ge C_7 r^{q^*} \tag{3.19}$$

for all $x \in \Theta$ *and all* $r < \min\{1, \alpha r_1(x)\}$, *we will say that condition weak-*D_{q^*} *holds on* Θ.

A similar, but slightly stronger, condition called D_{q^*} was introduced in Definition 1.7 in [MRW] in order to derive some local boundedness results for weak solutions of equation (1.1); see Corollaries 1.8, 1.9 and 1.11 in [MRW].

Note that by Definition 2.4, if (Ω, ρ, dx) is a local homogeneous space, $\Theta \Subset \Omega$ and $r_1(x)$ satisfies a local uniformity condition in Θ with constant $A_* = A_*(\Theta)$ (see (2.6)), then condition weak-D_{q^*} automatically holds with $q^* = d_0$ on Θ, for some constant $C_7 > 0$ and with $\alpha = A_*/2$. See the Appendix for a proof of this result.

The fact that property (3.19) holds with $q^* = d_0$ for suitable families of pseudometric balls $B(x, r)$ with small radii is used repeatedly in the proofs of our results, starting from Proposition 3.3 (see Steps I and III of the proof in the Appendix) and in all the theorems and corollaries that follow it.

It is interesting to note that in the proof of Proposition 3.3, only condition (3.19) with $q^* = d_0$ is used to estimate terms involving the structural coefficients b, c, d, h, while the local Doubling Condition (2.4) is directly used to estimate terms involving some local averages of e, f, g (see Step 6 of the proof in the Appendix).

4. Some Calculus for Degenerate Sobolev Spaces

Lemma 4.1. *Let* $\Theta \subset \Omega$ *be an open set,* $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ *with* $u \in L^\infty(\Theta)$ *and let* $\varepsilon > 0$. *Let*

$$m = \operatorname*{ess\,inf}_{\Theta} \ u, \qquad M = \operatorname*{ess\,sup}_{\Theta} \ u.$$

Then there exists a sequence $\{\varphi_j\}_{j \in \mathbf{N}} \subset Lip_{loc}(\Omega) \cap L^\infty(\Omega)$ *such that* $(\varphi_j, \nabla \varphi_j) \in W^{1,p}_Q(\Omega)$ *and*

i) $(\varphi_j, \nabla\varphi_j) \to (u, \nabla u)$ *in* $W^{1,p}_Q(\Theta)$,

ii) $\varphi_j(x) \in [m-\varepsilon, M+\varepsilon]$ *for every* $x \in \Omega$ *and every* $j \in \mathbf{N}$.

Proof: By definition of $W^{1,p}_Q(\Omega)$, there exists a sequence $\{\hat\varphi_j\}_{j\in\mathbf{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ such that $(\hat\varphi_j, \nabla\hat\varphi_j)$ converges to $(u, \nabla u)$ in $W^{1,p}_Q(\Omega)$. By choosing a subsequence, we may assume that

$$\begin{array}{rclll} \hat\varphi_j & \to & u & & \text{in } L^p(\Omega), \ \text{in } W^{1,p}_Q(\Omega) \text{ and a.e. in } \Omega, \\ \sqrt{Q}\nabla\hat\varphi_j & \to & \sqrt{Q}\nabla u & & \text{in } [L^p(\Omega)]^n, \text{ and a.e. in } \Omega. \end{array} \tag{4.1}$$

Now for every $j \in \mathbf{N}$ and $x \in \Omega$ define

$$\varphi_j(x) = \begin{cases} \hat\varphi_j(x) & \text{if } m-\varepsilon \le \hat\varphi_j(x) \le M+\varepsilon, \\ M+\varepsilon & \text{if } \hat\varphi_j(x) > M+\varepsilon, \\ m-\varepsilon & \text{if } \hat\varphi_j(x) < m-\varepsilon. \end{cases} \tag{4.2}$$

This immediately yields that $\varphi_j \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and that

$$m-\varepsilon \le \varphi_j(x) \le M+\varepsilon$$

for every $j \in \mathbf{N}$ and $x \in \Omega$. Then $\varphi_j \in L^\infty(\Omega)$ for every $j \in \mathbf{N}$. From (4.2) it follows that

$$\nabla\varphi_j(x) = \begin{cases} \nabla\hat\varphi_j(x) & \text{if } m-\varepsilon < \hat\varphi_j(x) < M+\varepsilon, \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

for each $j \in \mathbf{N}$ and almost every $x \in \Omega$. Hence, $|\sqrt{Q}\nabla\varphi_j| \le |\sqrt{Q}\nabla\hat\varphi_j|$ for every $j \in \mathbf{N}$ and a.e. $x \in \Omega$. We conclude that $(\varphi_j, \nabla\varphi_j) \in W^{1,p}_Q(\Omega)$ for every $j \in \mathbf{N}$.

Since $u(x) \in [m, M]$ for a.e. $x \in \Theta$ and $\hat\varphi_j \to u$ for a.e. $x \in \Omega$ by (4.1), we have that $\hat\varphi_j(x) \in (m-\varepsilon, M+\varepsilon)$ for a.e. $x \in \Theta$ when j is large enough. It follows from (4.2) that one also has $\varphi_j(x) = \hat\varphi_j(x)$ pointwise a.e. in Θ when j is large enough. Therefore,

$$\varphi_j \to u \qquad \text{a.e. in } \Theta.$$

Moreover, by (4.3), $\nabla\varphi_j = \nabla\hat\varphi_j$ a.e. in Θ when j is large enough. Hence, by (4.1),

$$\sqrt{Q}\nabla\varphi_j \to \sqrt{Q}\nabla u \qquad \text{a.e. in } \Theta.$$

Since

$$|u(x) - \varphi_j(x)|^p \le |u(x) - \hat\varphi_j(x)|^p \le 2^{p-1}\left[|u(x)|^p + |\hat\varphi_j(x)|^p\right]$$

for a.e. $x \in \Theta$ and $|u|^p + |\hat\varphi_j|^p \to 2|u|^p$ for a.e. $x \in \Omega$ and in $L^1(\Omega)$ by (4.1), Lebesgue's sequentially dominated convergence theorem implies that

$$\varphi_j \to u \qquad \text{in } L^p(\Theta).$$

In a similar way, for a.e. $x \in \Theta$ we have

$$\begin{aligned} |\sqrt{Q}\nabla u(x) - \sqrt{Q}\nabla\varphi_j(x)|^p &\le 2^{p-1}\left[|\sqrt{Q}\nabla u(x)|^p + |\sqrt{Q}\nabla\varphi_j(x)|^p\right] \\ &\le 2^{p-1}\left[|\sqrt{Q}\nabla u(x)|^p + |\sqrt{Q}\nabla\hat\varphi_j(x)|^p\right]. \end{aligned}$$

Further, we have that $|\sqrt{Q}\nabla u|^p + |\sqrt{Q}\nabla\hat\varphi_j|^p \to 2|\sqrt{Q}\nabla u|^p$ a.e. in Ω and in $L^1(\Omega)$ by (4.1). Lebesgue's theorem gives

$$\sqrt{Q}\nabla\varphi_j \to \sqrt{Q}\nabla u \qquad \text{in } [L^p(\Theta)]^n.$$

We conclude that

$$(\varphi_j, \nabla\varphi_j) \to (u, \nabla u) \qquad \text{in } W^{1,p}_Q(\Theta).$$

Proposition 4.2. *Let $\Theta \subset \Omega$ be an open set, $(u, \nabla u) \in W_Q^{1,p}(\Omega)$ with $u \in L^\infty(\Theta)$ and*

$$m = \operatorname*{ess\,inf}_{\Theta} \ u, \qquad M = \operatorname*{ess\,sup}_{\Theta} \ u.$$

Let $F \in C^1\big((m - \varepsilon_0, M + \varepsilon_0)\big)$ for some $\varepsilon_0 > 0$. Then $\big(F(u), \nabla(F(u))\big) \in W_Q^{1,p}(\Theta)$ with

$$\sqrt{Q}\nabla\big(F(u)\big) = F'(u)\sqrt{Q}\nabla u \tag{4.4}$$

almost everywhere in Θ.

Proof: The proof is a straightforward adaptation of the techniques used in the proof of Lemma 4.1 in [MRW]. Fix any $\varepsilon \in (0, \varepsilon_0)$ and consider the sequence $\{\varphi_j\}_{j\geq 1} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ provided by Lemma 4.1. Notice that $\varphi_j(x) \in [m - \varepsilon, M + \varepsilon]$ for every $x \in \Omega$ and every j, that $u(x) \in [m - \varepsilon, M + \varepsilon]$ for a.e. $x \in \Theta$ and that

$$\sup_{t\in[m-\varepsilon,M+\varepsilon]} |F(t)| < \infty, \qquad \sup_{t\in[m-\varepsilon,M+\varepsilon]} |F'(t)| < \infty.$$

Arguing as in Lemma 4.1 in [MRW], it is easy to see that $\{F(\varphi_j)\}_{j\in\mathbf{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ and $\{\big(F(\varphi_j), \nabla(F(\varphi_j))\big)\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $W_Q^{1,p}(\Theta)$. Thus, $\{\big(F(\varphi_j), \nabla(F(\varphi_j))\big)\}_{j\in\mathbf{N}}$ defines an element $\big(F(u), \nabla(F(u))\big)$ of $W_Q^{1,p}(\Theta)$ that satisfies (4.4).

Corollary 4.3. *Let $\Theta \Subset \Omega$ be an open set and fix a quasimetric ball B with $B \Subset \Theta$. Suppose that for some $t \in [1, \infty]$, condition* (2.12) *holds for B and condition* (2.11) *holds for a particular function $\eta \in Lip_0(B)$. Let $\theta \geq 1$, $(u, \nabla u) \in W_Q^{1,p}(\Omega)$ with $u \in L^\infty(\Theta)$,*

$$m = \operatorname*{ess\,inf}_{\Theta} \ u, \qquad M = \operatorname*{ess\,sup}_{\Theta} \ u,$$

and $F \in C^1\big((m - \varepsilon_0, M + \varepsilon_0)\big)$ for some $\varepsilon_0 > 0$. Then $\big(\eta^\theta F(u), \nabla(\eta^\theta F(u))\big) \in (W_Q^{1,p})_0(B)$ and

$$\sqrt{Q}\nabla\big(\eta^\theta F(u)\big) = \theta\eta^{\theta-1}F(u)\sqrt{Q}\nabla\eta + \eta^\theta F'(u)\sqrt{Q}\nabla u \quad \textit{pointwise a.e. in } \Omega.$$

Proof: This is a simple consequence of Proposition 4.2 together with Proposition 2.2 in [MRW].

Remark 4.4. *Let $(u, \nabla u) \in W_Q^{1,p}(\Omega)$ be such that $u \geq m$ a.e. in an open set $\Theta \subset \Omega$, and assume that $F : (m - \varepsilon_0, \infty) \to \mathbf{R}$ is C^1 with $\sup_{(m-\varepsilon_0,\infty)} |F'| < \infty$ for some $\varepsilon_0 > 0$. Then the conclusions of Proposition 4.2 and Corollary 4.3 still hold. We omit the proofs of these facts as they use ideas similar to those used in the previous proofs.*

5. The Inequality of John and Nirenberg

This section develops a local version of the inequality of John and Nirenberg adapted to the class $[cR]BMO(E)$ defined in the next paragraph. The arguments to follow are adaptations of ones in [SW1], where $R(x)$ is a small fixed multiple of $\mathrm{dist}(x, \partial\Omega)$.

Let Ω be an open subset in $\mathbb{R}^n$. Let ρ be a quasimetric in Ω and fix $R : \Omega \to (0, \infty)$. For each $x \in \Omega$ and $0 < c < \infty$, we say that a ρ-ball $B(y, t)$ is a $cR(x)$-ball if $0 < t < cR(x)$, $\overline{B(y, \gamma^* t)} \subset \Omega$, and $B(y, \gamma^* t) \subset B(x, cR(x))$ where γ^* is as in Lemma 2.2. It is useful to note that if $0 < c_1 < c_2$ then a $c_1R(x)$-ball B is also a $c_2R(x)$-ball. Let $E \subset \Omega$, E open. A function $f \in L^1_{loc}(\Omega)$ is said to belong to the class $[cR]BMO(E)$ if

$$||f||_{[cR]BMO(E)} = \sup_{x\in E} \sup_{B} \frac{1}{|B|} \int_B |f(y) - f_B| dy < \infty, \tag{5.1}$$

where the second supremum is taken over all $cR(x)$-balls B.

The main result of this section is as follows.

Proposition 5.1. *Let (Ω, ρ, dx) be a local homogeneous space as in Definition 2.4. Let $R : \Omega \to (0, \infty)$ satisfy $R(x) \le \min\{R_0(x)/(\gamma^*)^2, R_1(x)/\gamma^*\}$ for all x, where R_0 is as above Remark 2.1 and R_1 is as in Definition 2.4. Fix an open set $E \subset \Omega$ and assume that R satisfies a local uniformity condition with respect to ρ in E with constant $A_* = A_*(R, E)$; see* (2.6)*. Then there are positive constants $\delta_0 = \delta_0(R, E), C_8, C_9, c_\rho$ with $\delta_0 < 1$ and $c_\rho > 1$ such that for all $x \in E$, all $\delta_0 R(x)$-balls B, all $f \in [c_\rho R]BMO(E)$ and all $\alpha > 0$,*

$$|\{y \in B \; : \; |f(y) - f_B| > \alpha\}| \le C_8 e^{\frac{-C_9 \alpha}{||f||_{[c_\rho R]BMO(E)}}} |B|. \tag{5.2}$$

Remark 5.2. *The constants C_8, C_9 and c_ρ in Proposition 5.1 depend only on the quasimetric ρ, while the dependence of δ_0 on E occurs only through A_*. As the proof of Proposition 5.1 shows, $c_\rho = 8(\gamma^*)^2\kappa^5$ and $\delta_0 = A_*^2 \min\{A_*^2, (8\kappa^5)^{-1}\}/8(\gamma^*)^3\kappa^5$, where κ is the constant in* (1.9) *and $\gamma^* = \kappa + 2\kappa^2$ as in Lemma 2.2.*

The significance of Proposition 5.1 is its consequence for a special class of A_2 weights. Given $0 < c < \infty$ and a set $E \subset \Omega$, a nonnegative function $w \in L^1_{loc}(\Omega)$ is said to be a $[cR]A_2(E)$ weight if

$$||w||_{[cR]A_2(E)} = \sup_{x \in E} \sup_{B} \left(\frac{1}{|B|}\int_B w dy\right)\left(\frac{1}{|B|}\int_B w^{-1} dy\right) < \infty, \tag{5.3}$$

where the second supremum is taken over all $cR(x)$-balls B. We will use the following corollary of Proposition 5.1 in the proof of Proposition 3.1.

Corollary 5.3. *Under the hypotheses of Proposition 5.1, there are constants $C_8, C_9 > 0$ and $c_\rho > 1$ such that for any open set $E \subset \Omega$, there is a $\delta_0 = \delta_0(R, E) > 0$ for which*

$$||e^f||_{[\delta_0 R]A_2(E)} \le \left(1 + \frac{C_8 ||f||_{[c_\rho R]BMO(E)}}{C_9 - ||f||_{[c_\rho R]BMO(E)}}\right)^2 \tag{5.4}$$

for every $f \in [c_\rho R]BMO(E)$ with $||f||_{[c_\rho R]BMO(E)} < C_9$. The constants δ_0, C_8, C_9, c_ρ are the same as those in Proposition 5.1.

Except for simple changes, the proof of Corollary 5.3 is identical to the proof of [SW1, Corollary 61], and we refer the reader there for its proof.

Proof of Proposition 5.1: The proof is an adaptation to $[c_\rho R]BMO(E)$ of the one in [SW1, Lemma 60]. We begin by recalling the "dyadic grids" defined in [SW3]. Note by (2.2) that the quasimetric space (Ω, ρ) is separable since Ω is separable with respect to Euclidean distance in $\mathbb{R}^n$. Define $\mathbf{N}_\ell = \{1, ..., \ell\}$ for each $\ell \in \mathbf{N}$, and let $\mathbf{N}_\infty = \mathbf{N}$. Set $\lambda = 8\kappa^5$ with κ as in (1.9). Then for each $m \in \mathbb{Z}$ and every $k \ge m$, there are points $\{x_j^k\}_{j=1}^{n_k} \subset \Omega$ ($n_k \in \mathbf{N} \cup \{\infty\}$) and Borel sets $\{Q_j^k\}_{j=1}^{n_k}$ satisfying

$$B(x_j^k, \lambda^k) \subset Q_j^k \subset B(x_j^k, \lambda^{k+1}) \text{ if } j \in \mathbf{N}_{n_k}, \tag{5.5}$$

$$\Omega = \cup_{j=1}^{n_k} Q_j^k, \tag{5.6}$$

$$Q_i^k \cap Q_j^k = \emptyset \text{ if } i, j \in \mathbf{N}_{n_k} \text{ and } i \ne j, \tag{5.7}$$

$$\text{either } Q_j^k \subset Q_i^l \text{ or } Q_j^k \cap Q_i^l = \emptyset \text{ if } k < l, j \in \mathbf{N}_{n_k} \text{ and } i \in \mathbf{N}_{n_l}. \tag{5.8}$$

This dyadic grid of Borel sets depends on the integer m, and there may be different grids for each m. We fix a single grid for each $m \in \mathbb{Z}$ and denote it by $\mathcal{F}_m$:

$$\mathcal{F}_m = \{Q_j^k \; : \; k, j \in \mathbb{Z}, \; k \ge m \text{ and } j \in \mathbf{N}_{n_k}\}. \tag{5.9}$$

For fixed $m, k, j \in \mathbb{Z}$ with $k \ge m$ and $j \in \mathbf{N}_{n_k}$ we will refer to the Borel set $Q_j^k \in \mathcal{F}_m$ as the j^{th} "*cube*" at level k. For $0 < \delta_0 \le 1$, we will call a cube $Q_j^k \in \mathcal{F}_m$ "δ_0-*local*" if $B(x_j^k, \lambda^{k+1})$ satisfies

$\lambda^{k+1} < \delta_0 R(x_j^k)$. For each $m \in \mathbb{Z}$ and $\delta_0 \in (0,1]$ we define

(5.10)
- $\mathcal{F}_{m,\delta_0} = \{Q \in \mathcal{F}_m \ : \ Q \text{ is } \delta_0\text{-local}\}$, and
- $\mathcal{E}_{m,\delta_0} = \{Q = Q_j^k \in \mathcal{F}_{m,\delta_0} \ : \ B(x_j^k, \lambda^{k+1}) \text{ is an } R(x)\text{-ball for some } x \in E\}$.

Set $c_\rho = (\gamma^*)^2\lambda$ and fix $f \in [c_\rho R]BMO(E)$ with $||f||_{[c_\rho R]BMO(E)} = 1$. Let f^m be the discrete expectation of f on the dyadic grid at level m:

$$f^m(z) = \sum_{j \in \mathbf{N}_m} \Big(\frac{1}{|Q_j^m|}\int_{Q_j^m} f dy\Big)\chi_{Q_j^m}(z). \tag{5.11}$$

For the moment, we will assume each of the following.

- There are positive constants C_8', C_9' and δ_1 with $\delta_1 \le A_*\lambda^{-1}$ such that for each $m \in \mathbb{Z}$, $\alpha > 0$ and $Q \in \mathcal{E}_{m,\delta_1}$, we have

$$|\{y \in Q \ : \ |f^m - f_Q| > \alpha\}| \le C_8' e^{-C_9'\alpha}|Q|. \tag{5.12}$$

  Note that C_8', C_9' and δ_1 are independent of m, and C_8', C_9' are also independent of E.
- For almost every $y \in \Omega$,

$$f^m(y) \to f(y) \text{ as } m \to -\infty. \tag{5.13}$$

Taking (5.12) and (5.13) temporarily for granted, let us now prove Proposition 5.1 by using a packing argument and Fatou's lemma. To begin, we will use (5.12) to derive its analogue where the δ_1-local cube Q is replaced by any $\delta_0 R(x)$-ball, for any $x \in E$, provided δ_0 is sufficiently small in terms of δ_1 above. Indeed, fix $x \in E$, set $\delta_0 = A_*\delta_1/[(\gamma^*)^3\lambda]$, let $B = B(z,r)$ be a $\delta_0 R(x)$-ball and let $m \in \mathbb{Z}$ with $\lambda^{m+1} < r$. Choose $k \in \mathbb{Z}$ with $k > m$ such that $\lambda^k < r \le \lambda^{k+1}$. As $\Omega = \cup_j Q_j^k$, there is a nonempty collection $\mathcal{G} \subset \mathbf{N}_{n_k}$ such that

$$\begin{aligned} Q_j^k \cap B \ &\ne \ \emptyset \text{ for all } j \in \mathcal{G}, \text{ and} \\ B \subset \cup_{j\in\mathcal{G}} Q_j^k \subset \cup_{j\in\mathcal{G}} B(x_j^k, \lambda^{k+1}) \ &\subset \ B(z, \gamma^*\lambda r) = B^* \subset B(x, (\gamma^*)^2\lambda\delta_0 R(x)). \end{aligned} \tag{5.14}$$

Here, the third and fourth containments in (5.14) follow from (2.3) since $\lambda^{k+1} < \lambda r$. We now prove that the set $\mathcal{E}_{m,\delta_1}$ is nonempty.

Lemma 5.4. *With $x, B, m, k, \delta_1, \mathcal{G}$ and δ_0 as above, $Q_j^k \in \mathcal{E}_{m,\delta_1}$ for every $j \in \mathcal{G}$.*

Proof of Lemma 5.4: It is enough to show that for each $j \in \mathcal{G}$,

(5.15) $\lambda^{k+1} < R(x)$ and $B(x_j^k, \gamma^*\lambda^{k+1}) \subset B(x, R(x))$ (showing that $B(x_j^k, \lambda^{k+1})$ is an $R(x)$-ball), and

(5.16) $\lambda^{k+1} < \delta_1 R(x_j^k)$ (showing that $Q_j^k \in \mathcal{F}_{m,\delta_1}$).

Fix $j \in \mathcal{G}$. To see that (5.15) holds, we begin by noting that our choice of δ_0 guarantees that $(\gamma^*)^3\lambda\delta_0 < 1$. Due to our choice of k and using that $B(z,r)$ is a $\delta_0 R(x)$-ball we have

$$\lambda^{k+1} < \lambda r < \lambda\delta_0 R(x) < R(x).$$

Also, since $x_j^k \in B^*$, swallowing gives $B(x_j^k, \gamma^*\lambda^{k+1}) \subset B(x, (\gamma^*)^3\lambda\delta_0 R(x)) \subset B(x, R(x))$ establishing (5.15). Next, since $R(x)$ satisfies the uniformity condition (2.6) on E with constant A_* and $x_j^k \in B(x, R(x))$, $A_* R(x) < R(x_j^k)$. Our choice of δ_0 then guarantees that

$$\lambda^{k+1} < \lambda r < \delta_1 A_* R(x) < \delta_1 R(x_j^k),$$

giving (5.16) and proving the lemma.

Next, since $x_j^k \in B^* \cap B(x_j^k, \gamma^* \lambda r)$, we have by the swallowing lemma, the local doubling property (2.4) and the dyadic structure that

$$|B^*| \le |B(x_j^k, (\gamma^*)^2 \lambda r)| \le C_0 (\gamma^* \lambda)^{2d_0} |B(x_j^k, \lambda^k)| \le C' |Q_j^k|$$

for any $j \in \mathcal{G}$ with $C' = C_0(\gamma^*\lambda)^{2d_0}$. Therefore, for each $j \in \mathcal{G}$,

$$\begin{aligned}
|f_B - f_{Q_j^k}| &\le |f_{B^*} - f_B| + |f_{B^*} - f_{Q_j^k}| \\
&= \Big|\frac{1}{|B|}\int_B (f - f_{B^*})dy\Big| + \Big|\frac{1}{|Q_j^k|}\int_{Q_j^k} (f - f_{B^*})dy\Big| \\
&\le \frac{C_0(\gamma^*\lambda)^{d_0} + C'}{|B^*|}\int_{B^*} |f - f_{B^*}|dy \\
&\le C_0(\gamma^*\lambda)^{d_0} + C' = C'' \qquad (5.17)
\end{aligned}$$

since $||f||_{[c_\rho R]BMO(E)} = 1$ and B^* is an $R(x)$-ball (due to our choice of δ_0) with $x \in E$, and hence B^* is also a $c_\rho R(x)$-ball.

Consequently, if $y \in B$ and $\alpha > 2C''$, then (5.17) and the standard triangle inequality imply that $|f^m(y) - f_{Q_j^k}| > \alpha/2$ provided $|f^m(y) - f_B| > \alpha$. Since $Q_j^k \in \mathcal{E}_{m,\delta_1}$ for $j \in \mathcal{G}$, the disjointness in j of the Q_j^k and (5.12) yield

$$\begin{aligned}
|\{y \in B \,:\, |f^m(y) - f_B| > \alpha\}| &\le \sum_{j\in\mathcal{G}} |\{y \in Q_j^k \,:\, |f^m(y) - f_{Q_j^k}| > \alpha/2\}| \\
&\le \sum_{j\in\mathcal{G}} C_8' e^{-C_9'\alpha/2} |Q_j^k| \\
&\le C_8' e^{-C_9'\alpha/2}|B^*|, \text{ as } \cup_{j\in\mathcal{G}} Q_j^k \subset B^*, \\
&\le C_8' e^{-C_9'\alpha/2} C_0(\gamma^*\lambda)^{d_0}|B| \text{ by (2.4).}
\end{aligned}$$

In case $\alpha \le 2C''$, we simply use that $|\{y \in B \,:\, |f^m(y) - f_B| > \alpha\}| \le |B|$ and replace C_8' with $e^{C_9' C''}$ if necessary. Hence, there is a constant $C > 0$ independent of $x \in E$ such that for any $\alpha > 0$ and any $\delta_0 R(x)$-ball B,

$$|\{y \in B \,:\, |f^m(y) - f_B| > \alpha\}| \le C e^{-C_9'\alpha/2}|B|. \qquad (5.18)$$

Next we use the pointwise convergence of f^m to f as $m \to -\infty$ and Fatou's lemma. Set $E_{m,\alpha} = \{y \in B \,:\, |f^m(y) - f_B| > \alpha\}$. Then

$$\begin{aligned}
|\{y \in B \,:\, |f(y) - f_B| > \alpha\}| &= \int \chi_{\{y\in B\,:\,|f(y)-f_B|>\alpha\}}(z)dz \\
&\le \int \liminf_{m\to-\infty} \chi_{E_{m,\alpha}}(z)dz \\
&\le \liminf_{m\to-\infty} |E_{m,\alpha}| \\
&\le C e^{-C_9'\alpha/2}|B|. \qquad (5.19)
\end{aligned}$$

This proves (5.2) with $C_8 = C$ and $C_9 = C_9'/2$ in case $||f||_{[c_\rho R]BMO(E)} = 1$ and $\delta_0 = A_*\delta_1/[(\gamma^*)^3\lambda]$. The general case follows by replacing f and α by $f/||f||_{[c_\rho R]BMO(E)}$ and $\alpha/||f||_{[c_\rho R]BMO(E)}$ as in [SW1].

The proof now rests on the validity of (5.12) and (5.13). We first prove (5.13); the verification of (5.12) is contained in Lemma 5.5 to follow. Given a fixed $x \in \Omega$, the dyadic structure provides a sequence $\{x_m\}_{m=-1}^{-\infty} \subset \Omega$ such that

(i) $\quad x_m = x_{j_m}^m$ for some $j_m \in \mathbf{N}_{n_m}$, and

(ii) $\quad x \in Q_m = Q_{j_m}^m \subset B(x_m, \lambda^{m+1})$ for each m.

By standard homogeneous space theory (see the proof of Lemma 5.5 for further details), almost every point $x \in \Omega$ is a Lebesgue point of f:

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| dy = 0. \tag{5.20}$$

Fix such an x. By Lemma 2.2, there exist m_0 and C such that $B(x_m, \lambda^{m+1}) \subset B(x, C\lambda^{m+1})$ if $m \leq m_0$. Thus $Q_m \subset B(x, C\lambda^{m+1})$. Also, by (5.5), there exists m_1 such that $|Q_m| \approx |B(x_m, \lambda^{m+1})| \approx |B(x, C\lambda^{m+1})|$ uniformly in m if $m \leq m_1$. By choosing $r = C\lambda^{m+1}$, we obtain

$$\lim_{m\to -\infty} \frac{1}{|Q_m|} \int_{Q_m} |f(y) - f(x)| dy = 0.$$

But $f^m(x) = (1/|Q_m|) \int_{Q_m} f(y) dy$, so $f^m(x) \to f(x)$ as $m \to -\infty$. This proves (5.13).

The next lemma verifies (5.12).

Lemma 5.5. *Let (Ω, ρ, dx) be a local homogeneous space as in Definition 2.4, and E be an open set in Ω. Let $R : \Omega \to (0,\infty)$ satisfy $R(x) \leq \min\{R_0(x)/(\gamma^*)^2, R_1(x)/\gamma^*\}$ where R_0 is as above Remark (2.1) and R_1 is as in* (2.4). *Furthermore, assume that $R(x)$ satisfies a local uniformity condition on E with constant A_*. Then there are positive constants $C_8', C_9', \delta_1, c_\rho$ with $\delta_1 \in (0, A_*\lambda^{-1}]$ and $c_\rho > 1$ such that for every $\alpha > 0$, $m \in \mathbb{Z}$, and $Q \in \mathcal{E}_{m,\delta_1}$,*

$$|\{y \in Q \ : \ |f^m(y) - f_Q| > \alpha\}| \leq C_8' e^{-C_9'\alpha} |Q| \tag{5.21}$$

for all $f \in [c_\rho R]BMO(E)$ with $||f||_{[c_\rho R]BMO(E)} = 1$. The constants C_8', C_9' and c_ρ depend only on ρ.

Proof: The proof is broken into five steps.

I: Recall the dyadic structure described in (5.5)–(5.8), and set $c_\rho = (\gamma^*)^2\lambda = 8(\gamma^*)^2\kappa^5$. Let $f \in [c_\rho R]BMO(E)$ with $||f||_{[c_\rho R]BMO(E)} = 1$. Fix $m \in \mathbb{Z}$ and a cube $Q_0 = Q_j^k \in \mathcal{F}_{m,\lambda^{-1}}$; see (5.10). Our first step compares the average of f on Q_0 with its average on the related ρ-ball $B(x_j^k, \lambda^{k+1})$. Indeed,

$$\begin{aligned} \left|f_{B(x_j^k,\lambda^{k+1})} - f_{Q_0}\right| &= \left|\frac{1}{|Q_0|}\int_{Q_0} (f - f_{B(x_j^k,\lambda^{k+1})})dx\right| \\ &\leq \frac{1}{|B(x_j^k,\lambda^k)|}\int_{B(x_j^k,\lambda^{k+1})} |f - f_{B(x_j^k,\lambda^{k+1})}|dx. \end{aligned} \tag{5.22}$$

Thus, as Q_0 is λ^{-1}-local and $R(x_j^k) \leq R_1(x_j^k)$, the local doubling condition (2.4) gives

$$\begin{aligned} \frac{1}{|Q_0|}\int_{Q_0} |f - f_{Q_0}|dx &\leq \frac{1}{|Q_0|}\int_{Q_0} |f - f_{B(x_j^k,\lambda^{k+1})}|dx + |f_{B(x_j^k,\lambda^{k+1})} - f_{Q_0}| \\ &\leq \frac{2}{|B(x_j^k,\lambda^k)|}\int_{B(x_j^k,\lambda^{k+1})} |f - f_{B(x_j^k,\lambda^{k+1})}|dx \text{ , by (5.22)}, \\ &\leq \frac{2C_0\lambda^{d_0}}{|B(x_j^k,\lambda^{k+1})|}\int_{B(x_j^k,\lambda^{k+1})} |f - f_{B(x_j^k,\lambda^{k+1})}|dx. \end{aligned}$$

Therefore, if $B(x_j^k, \lambda^{k+1})$ is also a $c_\rho R(x)$-ball for some $x \in E$, we may write

$$\frac{1}{|Q_0|}\int_{Q_0} |f - f_{Q_0}|dx \ \ \leq \ \ 2C_0\lambda^{d_0} := 2c_0. \tag{5.23}$$

Now, further restrict $Q_0 \in \mathcal{E}_{m,\lambda^{-1}}$. Setting $h = h(x, Q_0) = (f(x) - f_{Q_0})\chi_{Q_0}(x)$, (5.23) gives

$$\frac{1}{|Q|}\int_Q |h|dx \leq \frac{1}{|Q_0|}\int_{Q_0} |f - f_{Q_0}|dx \leq 2c_0 \tag{5.24}$$

for any dyadic cube Q for which $Q_0 \subset Q$.

II: The dyadic maximal function on local cubes nearby E, acting on $g \in L^1_{loc}(\Omega)$, is defined by

$$M_E^{\triangle} g(x) = \sup \frac{1}{|Q'|} \int_{Q'} |g(y)| dy, \tag{5.25}$$

where the supremum is taken over all cubes $Q' \in \mathcal{E}_{m,A_*\lambda^{-1}}$ such that $x \in Q'$. If $x \in \Omega$ is not a member of any cube in $\mathcal{E}_{m,A_*\lambda^{-1}}$ then we set $M_E^{\triangle} g(x) = 0$. Next, we note the weak-type $(1,1)$ inequality for $M_E^{\triangle}$:

$$|\{x \in \Omega : M_E^{\triangle} g(x) > \alpha\}| \leq \frac{1}{\alpha} \int_{\Omega} |g(x)| dx. \tag{5.26}$$

This is a consequence of the analogous inequality in [SW1] for a larger dyadic maximal operator.

For $\alpha > 0$, let $E_\alpha = \{x \in \Omega : M_E^{\triangle} h(x) > \alpha\}$. If $\alpha \geq 2c_0$, then $E_\alpha \subset Q_0$. Indeed, if $M_E^{\triangle} h(x) > \alpha$ there is a cube $Q' \in \mathcal{E}_{m,A_*\lambda^{-1}}$ containing x such that

$$\frac{1}{|Q'|} \int_{Q'} |h| dy > 2c_0.$$

Thus by the definition of h, Q' must intersect Q_0. Therefore, (5.24) and the dyadic structure give that $Q' \subsetneq Q_0$, and so $x \in Q_0$.

For $\alpha > 0$, let C_α be the collection of all cubes $Q \in \mathcal{E}_{m,A_*\lambda^{-1}}$ for which $|h|_Q > \alpha$. By the above, if $\alpha \geq 2c_0$ then $Q \subsetneq Q_0$ for each $Q \in C_\alpha$. Denote the collection of maximal cubes in C_α by $S_\alpha = \{Q_{\alpha,j}\}$. Then

- *(i)* If $\alpha \geq 2c_0$, the cubes of S_α are pairwise disjoint.
- *(ii)* If $\alpha \geq 2c_0$, then $\cup_{Q \in S_\alpha} Q = E_\alpha \subset Q_0$.
- *(iii)* If $2c_0 \leq \alpha < \beta$ and i, j are given then either $Q_{\beta,j} \subset Q_{\alpha,i}$ or $Q_{\beta,j} \cap Q_{\alpha,i} = \emptyset$.

To see (i), note that if two cubes $Q_{\alpha,j}$, $Q_{\alpha,i}$ intersect, the dyadic structure implies that one is contained in the other, violating maximality. For (ii), let $x \in E_\alpha$. Then there is a cube Q' containing x for which $|h|_{Q'} > \alpha$ and so $Q' \subset Q_{\alpha,j}$ for some j. Thus $E_\alpha \subset \cup_j Q_{\alpha,j}$ and (ii) follows. (iii) follows from a similar argument as for (i) using the dyadic structure and maximality of the cubes in S_α.

III: The local doubling condition (2.4) translates to a similar property for the Lebesgue measure of dyadic cubes. Indeed, fix a λ^{-1}-local cube $Q = Q_j^l$ and denote its dyadic predecessor Q_i^{l+1} by Q_1. By (5.5),

$$B(x_j^l, \lambda^l) \subset Q \subset B(x_j^l, \lambda^{l+1}) \text{ and} \tag{5.27}$$

$$B(x_i^{l+1}, \lambda^{l+1}) \subset Q_1 \subset B(x_i^{l+1}, \lambda^{l+2}) \tag{5.28}$$

where $\lambda^{l+1} < \lambda^{-1} R(x_j^l) \leq (\gamma^* \lambda)^{-1} R_1(x_j^l)$. Thus, as $Q \subset Q_1$ and $\lambda^{l+2} < R_1(x_j^l)/\gamma^*$, Lemma 10.2 of the appendix implies that

$$|B(x_i^{l+1}, \lambda^{l+2})| \leq C_0 (\gamma^* \lambda^2)^{d_0} |B(x_j^l, \lambda^l)|. \tag{5.29}$$

Therefore, (5.27) and (5.28) together give

$$|Q_1| \leq C_0 (\gamma^* \lambda^2)^{d_0} |Q| = c_1 |Q|. \tag{5.30}$$

Next, restrict $Q \in \mathcal{E}_{m,A_*\lambda^{-1}}$. Then an inequality similar to (5.23) holds for Q_1, the predecessor of Q. Indeed, for such Q there is an $x \in E$ such that (keeping the same labels as in (5.27) and (5.28))

$\lambda^{l+1} < \min\{R(x), A_*\lambda^{-1}R(x_j^l)\}$ and $B(x_j^l, \gamma^*\lambda^{l+1}) \subset B(x, R(x))$. Since $x_j^l \in B(x_i^{l+1}, \lambda^{l+2})$, we have that

$$B(x_i^{l+1}, \gamma^*\lambda^{l+2}) \subset B(x, (\gamma^*)^2\lambda R(x)).$$

Also, since $R(x)$ satisfies a local uniformity condition on E with constant A_*, it follows that $\lambda^{l+1} < A_*\lambda^{-1}R(x_j^l) < \lambda^{-1}R(x)$, giving $\lambda^{l+2} < R(x)$. This together with the containment above shows that $B(x_i^{l+1}, \lambda^{l+2})$ is a $c_\rho R(x)$-ball. Therefore,

$$\frac{1}{|Q_1|}\int_{Q_1} |f - f_{Q_1}|dx \le 2c_1 \tag{5.31}$$

using a familiar argument.

***IV*:** For each $\alpha \ge 2c_0$, define $\gamma = \gamma(\alpha) = 1 + (4c_1^2/\alpha)$. We claim that for all j,

$$|E_{\gamma\alpha} \cap Q_{\alpha,j}| \le \frac{1}{2}|Q_{\alpha,j}|. \tag{5.32}$$

Indeed, let Q be the dyadic predecessor of $Q_{\alpha,j}$. Then, $Q \subset Q_0$ since $Q_{\alpha,j}$ is a proper subcube of Q_0. The maximality of $Q_{\alpha,j}$ then gives

$$|h|_Q \le \alpha. \tag{5.33}$$

Set $g = (h - h_Q)\chi_Q$ and fix $x \in E_{\gamma\alpha} \cap Q_{\alpha,j}$. Then, since $\gamma \ge 1$, (ii) and (iii) (see step ***II***) give that $x \in Q_{\gamma\alpha,i}$ for some i and $Q_{\gamma\alpha,i} \subset Q_{\alpha,j}$. Using this, we have

$$\gamma\alpha < |h|_{Q_{\gamma\alpha,i}} \quad \le \quad |g|_{Q_{\gamma\alpha,i}} + \alpha. \tag{5.34}$$

Consequently, for every $x \in E_{\gamma\alpha} \cap Q_{\alpha,j}$, $M_E^\Delta g(x) > (\gamma - 1)\alpha$. Further, for each $x \in Q \subset Q_0$ we have

$$\begin{aligned} g(x) = h(x) - h_Q &= f(x) - f_{Q_0} - (f - f_{Q_0})_Q \\ &= f(x) - f_Q. \end{aligned} \tag{5.35}$$

Therefore, by (5.26),

$$\begin{aligned} |E_{\gamma\alpha} \cap Q_{\alpha,j}| &\le |\{x : M_E^\Delta g(x) > (\gamma-1)\alpha\}| \\ &\le \frac{1}{(\gamma-1)\alpha}\int_\Omega |g|dx \\ &= \frac{1}{(\gamma-1)\alpha}\int_Q |f - f_Q|dx \\ &\le \frac{2c_1}{(\gamma-1)\alpha}|Q|, \end{aligned} \tag{5.36}$$

where the last inequality is due to (5.31). Inequality (5.30) combined with our choice of γ gives

$$|E_{\gamma\alpha} \cap Q_{\alpha,j}| \le \frac{2c_1^2}{(\gamma-1)\alpha}|Q_{\alpha,j}| \le \frac{1}{2}|Q_{\alpha,j}|, \tag{5.37}$$

proving (5.32).

***V*:** For $\alpha > 0$, define the distribution function $\omega(\alpha) = |E_\alpha \cap Q_0|$. We add (5.37) over j to obtain a useful inequality for $\alpha \ge 2c_0$ (note that $\omega(\alpha) = |E_\alpha|$ for $\alpha \ge 2c_0$, and that $\gamma\alpha = \alpha + 4c_1^2$):

$$\omega(\alpha + 4c_1^2) = \sum_j |E_{\gamma\alpha} \cap Q_{\alpha,j}| \le \frac{1}{2}\sum_j |Q_{\alpha,j}| = \frac{1}{2}|E_\alpha| = \frac{1}{2}\omega(\alpha). \tag{5.38}$$

We now iterate (5.38). Fix $\alpha \ge 2c_0$. Then there is a $k \in \mathbf{N}$ such that $\alpha \in [2c_0 + 4(k-1)c_1^2, 2c_0 + 4kc_1^2]$. Therefore, there is a $\beta \in [2c_0, 2c_0 + 4c_1^2]$ for which

$$\omega(\alpha) \le \frac{1}{2^{k-1}}\omega(\beta) \le 2e^{-k\log 2}|Q_0| \tag{5.39}$$

as $\omega(s) \le |Q_0|$ for $s > 0$. Since

$$k \ge \frac{\alpha - 2c_0}{4c_1^2},$$

we obtain

$$\omega(\alpha) \le C_8' e^{-C_9' \alpha} |Q_0| \tag{5.40}$$

where C_8' and C_9' depend on c_0, c_1. Finally, if $\alpha \in (0, 2c_0)$ we use that $\omega(\alpha) \le |Q_0|$ to obtain a similar estimate. Hence, for all $\alpha > 0$,

$$|\{x \in Q_0 : M_E^{\Delta}[(f - f_{Q_0})\chi_{Q_0}](x) > \alpha\}| \le C_8' e^{-C_9' \alpha} |Q_0|. \tag{5.41}$$

The proof will then be complete if we show that

$$|f^m - f_{Q_0}|\chi_{Q_0} \le M_E^{\Delta}\Big[(f - f_{Q_0})\chi_{Q_0}\Big]. \tag{5.42}$$

Using the dyadic structure it is easy to see that (5.42) holds provided $Q_i^m \in \mathcal{E}_{m, A_* \lambda^{-1}}$ whenever $Q_i^m \subset Q_0$. This proviso is true by further restricting the size of δ_1. Set $\delta_1 = \min\{A_*^3, A_* \lambda^{-1}\}$ and suppose $Q_0 = Q_j^k \in \mathcal{E}_{m, \delta_1}$. Omitting as we may the case when $k = m$, suppose that $Q_i^m \subset Q_0$ and $m < k$. Recalling that $\gamma^* = \kappa + 2\kappa^2 < 8\kappa^5 = \lambda$, we have

$$\begin{aligned} B(x_i^m, \gamma^* \lambda^{m+1}) &\subset B(x_i^m, \frac{\gamma^* \lambda^{k+1}}{\lambda}) \\ &\subset B(x_i^m, \lambda^{k+1}) \\ &\subset B(x_j^k, \gamma^* \lambda^{k+1}) \subset B(x, R(x)) \end{aligned}$$

for some $x \in E$. Thus $B(x_i^m, \lambda^{m+1})$ is an $R(x)$-ball. Since $Q_j^k \in \mathcal{F}_{m, \delta_1}$ and $x_i^m, x_j^k \in B(x, R(x))$, the uniformity condition gives

$$\lambda^{m+1} < \frac{\delta_1}{\lambda} R(x_j^k) \le \frac{\delta_1}{A_* \lambda} R(x) \le \frac{\delta_1}{A_*^2 \lambda} R(x_i^m) \le \frac{A_*}{\lambda} R(x_i^m), \tag{5.43}$$

and therefore $Q_i^m \in \mathcal{E}_{m, A_* \lambda^{-1}}$. This concludes the proof of both Lemma 5.5 and Proposition 5.1 with $\delta_1 = \min\{A_*^3, A_* \lambda^{-1}\}$.

6. The Proof of Proposition 3.1

Proposition 3.1 will be proved using the results of three lemmas and Corollary 5.3. The lemmas give mean-value estimates for positive and negative powers of weak solutions as well as a logarithmic estimate. In order to simplify their statements, we list now some assumptions to remain in force for the rest of the section. We always assume that (Ω, ρ, dx) is a local homogeneous space as in Definition 2.4, that the Sobolev inequality (2.8) is valid, that (2.10) holds for some $\tau \in (0,1)$ and $s^* > p\sigma'$ with σ as in (2.8), and that (2.11) and (2.12) are valid for some $t \ge 1$. Our first lemma concerns positive powers of weak solutions.

Lemma 6.1. *Noting the assumptions in the paragraph above, let $(u, \nabla u)$ be a weak solution in Ω of (1.1), where* (1.2) *holds with exponents γ, δ, ψ satisfying* (3.5). *Let $s = (s^*/p)' \in [1, \sigma')$. Fix $x_0 \in \Omega$, $k > 0$, $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1]$, a ρ-ball $B(x_0, r)$ with $0 < r < \tau^2 r_1(x_0)$, set $\bar{u} = |u| + k$, and assume that $\bar{Z}(B(x_0, r/\tau), \bar{u}) < \infty$. If $u \ge 0$ in $B(x_0, r)$ then for each $\alpha > 0$ there exists $\alpha_1 \in [\alpha \sigma^{-\frac{1}{2}}, \alpha]$ such that*

$$\operatorname*{ess\,sup}_{B(x_0, \tau r)} \bar{u} \quad \le \quad C_{10} \Big(C_{11} \bar{Z}(B(x_0, r), \bar{u}) \Big)^{\frac{p\Psi_0}{\alpha_1}} ||\bar{u}^{\alpha_1}||_{s, B(x_0, r); \overline{dx}}^{\frac{1}{\alpha_1}}. \tag{6.1}$$

Here $\Psi_0 = \frac{\sigma}{\sigma - s}$, C_{10} depends only on p, σ, s and on $\epsilon_1, \epsilon_2, \epsilon_3$ appearing in the definition (3.4) *of $\bar{Z}$, while C_{11} depends on $p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3, a$ and the constants C_1 from* (2.8) *and N, C_{s^*} in* (2.10).

Remark 6.2. i) α_1 *is defined by*

$$\alpha_1 = \begin{cases} \alpha & \text{if } \log_{\frac{\sigma}{s}} \frac{p-1}{\alpha} \le -\frac{1}{4} \\ \alpha & \text{if } \log_{\frac{\sigma}{s}} \frac{p-1}{\alpha} \in \left[K+\frac{1}{4}, K+\frac{3}{4}\right] \text{ for some } K \in \mathbf{N}\cup\{0\}, \\ \alpha\left(\frac{\sigma}{s}\right)^{-\frac{1}{2}} & \text{if } \log_{\frac{\sigma}{s}} \frac{p-1}{\alpha} \in \left(K-\frac{1}{4}, K+\frac{1}{4}\right) \text{ for some } K \in \mathbf{N}\cup\{0\}. \end{cases}$$

ii) *We explicitly note that the constants C_{10}, C_{11} in* (6.1) *are independent of $(u, \nabla u)$, k, $B(x_0, r)$, b, c, d, e, f, g, h, and α.*

Proof: By [MRW, Theorem 1.2], the weak solution $(u, \nabla u)$ satisfies

$$||\bar{u}||_{L^\infty(B(x_0,r))} \le C\bar{Z}\big(B(x_0, \frac{r}{\tau}), \bar{u}\big)^{\Psi_0} ||\bar{u}||_{sp, B(x_0, \frac{r}{\tau}); \overline{dx}} \tag{6.2}$$

where $\bar{u} = |u| + k$ and $C > 0$ depends only on p, a and ψ. Therefore, as $\bar{Z}\big(B(x_0, r/\tau), \bar{u}\big) < \infty$ by hypothesis, [MRW, Proposition 2.3] gives that $\bar{u}$ is bounded on $B(x_0, r)$. The proof of the lemma will be completed by following the proof of [MRW, Theorem 1.2], but now using a modified test function that exploits boundedness of $\bar{u}$. As in [MRW], we may assume that $\overline{u}$ satisfies the following modified structure conditions in terms of the functions $\bar{b}$, $\bar{d}$, and $\bar{h}$ (see [MRW, (3.1)]):

$$\begin{aligned} \xi \cdot A(x, z, \xi) &\ge a^{-1}|\sqrt{Q(x)}\,\xi|^p - \bar{h}(x)\bar{z}^p, \\ \left|\widetilde{A}(x, z, \xi)\right| &\le a|\sqrt{Q(x)}\,\xi|^{p-1} + \bar{b}(x)\bar{z}^{p-1}, \\ |B(x, z, \xi)| &\le c|\sqrt{Q(x)}\,\xi|^{\psi-1} + \bar{d}(x)\bar{z}^{p-1} \end{aligned} \tag{6.3}$$

for all $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ where $A, \tilde{A}$ and B are as in (1.2) and $\bar{z} = z + k$. For simplicity, we will often not indicate the dependence of $A, \tilde{A}, B$, etc. on their variables.

Choose a nonnegative $\eta \in Lip_0(B(x_0, r))$ and set $v = \eta^p \bar{u}^q$ for $q \in (1-p, 0) \cup (0, \infty)$. By Corollary 4.3, v is a feasible nonnegative test function for any value of q in the indicated range. Corollary 4.3 implies that

$$\begin{aligned} \nabla v \cdot A(x, u, \nabla u) + vB(x, u, \nabla u) &= \sqrt{Q}\nabla v \cdot \tilde{A} + vB \\ &= \left(p\eta^{p-1}\bar{u}^q\sqrt{Q}\nabla\eta + q\eta^p\bar{u}^{q-1}\sqrt{Q}\nabla\bar{u}\right) \cdot \tilde{A} + \eta^p\bar{u}^q B. \end{aligned} \tag{6.4}$$

We now use (6.4) to derive some pointwise estimates. If $q > 0$, we apply (6.3) to (6.4), giving

$$\begin{aligned} \nabla v \cdot A(x, u, \nabla u) + vB(x, u, \nabla u) &\ge q\eta^p\bar{u}^{q-1}\left[a^{-1}|\sqrt{Q}\nabla\bar{u}|^p - \bar{h}\bar{u}^p\right] \\ &\quad -p\eta^{p-1}\bar{u}^q|\sqrt{Q}\nabla\eta||\tilde{A}| - \eta^p\bar{u}^q|B| \\ &\ge a^{-1}q\eta^p\bar{u}^{q-1}|\sqrt{Q}\nabla\bar{u}|^p - q\bar{h}\eta^p\bar{u}^{q+p-1} \\ &\quad -ap\eta^{p-1}\bar{u}^q|\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}|^{p-1} - p\eta^{p-1}\bar{b}\bar{u}^{q+p-1}|\sqrt{Q}\nabla\eta| \\ &\quad -c\eta^p|\sqrt{Q}\nabla\bar{u}|^{\psi-1}\bar{u}^q - \bar{d}\eta^p\bar{u}^{q+p-1}. \end{aligned} \tag{6.5}$$

If $q < 0$ we arrange (6.4) differently. For the second term inside the parentheses on the right side of (6.4), since $q < 0$, the first estimate of (6.3) gives

$$\begin{aligned} q\eta^p\bar{u}^{q-1}\sqrt{Q}\nabla\bar{u} \cdot \tilde{A} &= q\eta^p\bar{u}^{q-1}\nabla\bar{u} \cdot A \\ &\le -a^{-1}|q|\eta^p\bar{u}^{q-1}|\sqrt{Q}\nabla\bar{u}|^p + |q|\eta^p\bar{u}^{q+p-1}\bar{h}. \end{aligned} \tag{6.6}$$

After estimating the other terms of (6.4) as before, we move the first term on the right of (6.6) to the left and obtain

$$\begin{aligned} &\nabla v \cdot A(x, u, \nabla u) + vB(x, u, \nabla u) + a^{-1}|q|\eta^p\bar{u}^{q-1}|\sqrt{Q}\nabla\bar{u}|^p \le \\ &\quad |q|\eta^p\bar{u}^{p+q-1}\bar{h} + ap\eta^{p-1}\bar{u}^q|\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}|^{p-1} + p\bar{b}\eta^{p-1}\bar{u}^{q+p-1}|\sqrt{Q}\nabla\eta| \\ &\quad +c\eta^p\bar{u}^q|\sqrt{Q}\nabla\bar{u}|^{\psi-1} + \bar{d}\eta^p\bar{u}^{q+p-1}. \end{aligned} \tag{6.7}$$

Since u is a weak solution of (1.1) and v is a feasible test function, we have that

$$\int_\Omega \nabla v \cdot A + vB = \int_{B(x_0,r)} \nabla v \cdot A + vB = 0.$$

Integrating either (6.5) or (6.7) over $B = B(x_0, r)$, we obtain that for any $q \in (1-p, 0) \cup (0, \infty)$,

$$\begin{aligned} |q| \fint_B \eta^p \bar{u}^{q-1} |\sqrt{Q}\nabla \bar{u}|^p dx \quad &\le \quad C\Big\{ |q| \fint_B \bar{h}\eta^p \bar{u}^{p+q-1} dx + \fint_B \eta^{p-1}\bar{u}^q |\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}|^{p-1} dx \\ &\quad + \fint_B \bar{b}\eta^{p-1}|\sqrt{Q}\nabla\eta|\bar{u}^{p+q-1}dx + \fint_B c\eta^p \bar{u}^q |\sqrt{Q}\nabla\bar{u}|^{\psi-1}dx \\ &\quad + \fint_B \bar{d}\eta^p\bar{u}^{p+q-1}dx\Big\}, \end{aligned} \tag{6.8}$$

where the constant C in (6.8) depends only on a, p. Now use Young's inequality (10.1) with $\beta = p'$ and $\theta = p'|q|/(4C)$, where C is as in (6.8), on the second term of the right side of (6.8). This gives

$$\begin{aligned} \fint_B \eta^{p-1}\bar{u}^q|\sqrt{Q}\nabla\eta||\sqrt{Q}\nabla\bar{u}|^{p-1}dx \quad &\le \quad \frac{|q|}{4C}\fint_B |\sqrt{Q}\nabla\bar{u}|^p\eta^p\bar{u}^{q-1}dx \\ &\quad + c_2|q|^{1-p}\fint_B |\sqrt{Q}\nabla\eta|^p\bar{u}^{q+p-1}dx. \end{aligned} \tag{6.9}$$

Here c_2 depends only on p, a. Applying Young's inequality (10.1) to the fourth term on the right side of (6.8) with $\beta = \frac{p}{\psi-1}$, $\beta' = \frac{p}{p+1-\psi}$ and $\theta = \frac{|q|p}{4(\psi-1)C}$ yields

$$\begin{aligned} \fint_B c\eta^p\bar{u}^q|\sqrt{Q}\nabla\bar{u}|^{\psi-1}dx \quad &= \quad \fint_B |\sqrt{Q}\nabla\bar{u}|^{\psi-1}\eta^{\psi-1}\bar{u}^{\frac{(\psi-1)(q-1)}{p}} \cdot c\eta^{p+1-\psi}\bar{u}^{q-\frac{(\psi-1)(q-1)}{p}}dx \\ &\le \quad \frac{|q|}{4C}\fint_B |\sqrt{Q}\nabla\bar{u}|^p\eta^p\bar{u}^{q-1}dx + c_3|q|^{\frac{1-\psi}{p+1-\psi}}\fint_B c^{\frac{p}{p+1-\psi}}\eta^p\bar{u}^{q+\frac{\psi-1}{p+1-\psi}}dx. \end{aligned} \tag{6.10}$$

Since under our hypotheses $\psi \in [p, p+1-\sigma^{-1})$, the constant c_3 can be chosen as to depend only on p, a, σ. Inserting (6.9) and (6.10) into (6.8) and absorbing two terms, we obtain

$$\begin{aligned} |q|\fint_B \eta^p\bar{u}^{q-1}|\sqrt{Q}\nabla\bar{u}|^p dx \quad &\le \quad C\Big\{|q|^{1-p}\fint_B |\sqrt{Q}\nabla\eta|^p\bar{u}^{p+q-1}dx \\ &\quad + \fint_B \bar{b}\eta^{p-1}|\sqrt{Q}\nabla\eta|\bar{u}^{p+q-1}dx \\ &\quad + |q|^{\frac{1-\psi}{p+1-\psi}}\fint_B c^{\frac{p}{p+1-\psi}}\eta^p\bar{u}^{q+\frac{\psi-1}{p+1-\psi}}dx \\ &\quad + |q|\fint_B \bar{h}\eta^p\bar{u}^{p+q-1}dx + \fint_B \bar{d}\eta^p\bar{u}^{p+q-1}dx\Big\}, \end{aligned} \tag{6.11}$$

with C depending only on a, p, σ. This inequality is identical to [MRW, (3.8)] with $\mu = 0$. Therefore, we follow the proof of [MRW, Theorem 1.2] through steps 5 and 6 with $Y = p+q-1$, $t = \frac{p}{p+1-\psi}$ and $T = \frac{1-\psi}{p+1-\psi}$. Note that when dealing with term III in step 5 of [MRW], the exponent $T+p$ may be negative for some values of $\psi \in [p, p+1-\sigma^{-1})$. Thus we replace $|q|^{T+p}$ with the larger term $(|q| + |q|^{-1})^{|T|+p}$ to arrive at an analogous inequality to [MRW, (3.22)], recalling the notation given in (3.1):

$$||\eta\bar{u}^{\frac{Y}{p}}||_{p\sigma,B;\overline{dx}} \quad \le \quad C\Big(|q| + \frac{1}{|q|}\Big)^{\tilde{b}_*}\bar{Z}\left\{||\eta\bar{u}^{\frac{Y}{p}}||_{p,B;\overline{dx}} + r||\bar{u}^{\frac{Y}{p}}\sqrt{Q}\nabla\eta||_{p,B;\overline{dx}}\right\} \tag{6.12}$$

where $B = B(x_0, r)$, $\bar{Z} = \bar{Z}\big(B(x_0,r), \bar{u}\big)$ and $\tilde{b}_* \ge b_* > 0$ with b_* as in [MRW, (3.22)]. We explicitly note that C now depends on p, a, σ and on the constant C_1 appearing in (2.8), while $\tilde{b}_*$ depends on p, σ and on $\epsilon_1, \epsilon_2, \epsilon_3$ that appear in the definition (3.4) of $\bar{Z}$.

We now choose $\eta = \eta_j, j \geq 1$, as in (2.10). Let $S_j = supp\, \eta_j$ for $j \geq 1$ and $S_0 = B = B(x_0, r)$. Recall that $\eta_j = 1$ on S_{j+1} and $B(x_0, \tau r) \subset \cap_j S_j$. Since $s^* > p\sigma'$ and $s'p = s^*$, Hölder's inequality and (2.10) give

$$||\bar{u}^{\frac{Y}{p}} \chi_{S_{j+1}}||_{p\sigma,B;\overline{dx}} \;\; \leq \;\; C\Big(|q| + \frac{1}{|q|}\Big)^{\tilde{b}_*} \bar{Z} N^j ||\bar{u}^{\frac{Y}{p}} \chi_{S_j}||_{sp,B;\overline{dx}}, \tag{6.13}$$

which is analogous to inequality [MRW, (3.23)] and where C depends on p, a, σ, on the constants C_1 appearing in (2.8) and N, C_{s^*} in (2.10). Now for $\omega \neq 0$ and $j \in \mathbf{N} \cup \{0\}$ define

$$\Phi(j;\omega) = \Big(\int_{B(x_0,r)} \bar{u}^{\omega} \chi_{S_j} dx \Big)^{1/\omega}. \tag{6.14}$$

By (6.13), noting that $Y = p + q - 1 > 0$ for all $q \in (1-p, 0) \cup (0, \infty)$, we have

$$\Phi(j+1; Y\sigma) \leq C^{\frac{p}{Y}} \Big(|q| + \frac{1}{|q|}\Big)^{\frac{p\tilde{b}_*}{Y}} \bar{Z}^{\frac{p}{Y}} N^{\frac{jp}{Y}} \Phi(j; sY). \tag{6.15}$$

Inequality (6.15) will be iterated to finish the proof. Indeed, set $\mathcal{X} = \sigma/s > 1$ and fix $\alpha_1 > 0$ as in Remark 6.2. Set $q_j = \alpha_1 \mathcal{X}^j + 1 - p$ and $Y_j = \alpha_1 \mathcal{X}^j$ for each $j \in \mathbf{N} \cup \{0\}$.

Claim: We claim that $q_j \in (1-p, 0) \cup (0, \infty)$ for $j \in \mathbf{N} \cup \{0\}$ and that

$$|q_j| + \frac{1}{|q_j|} \;\; \leq \;\; \mathcal{X}^j \left[\alpha_1 \mathcal{X}^{\frac{1}{2}} + p - 1 + \frac{1}{(p-1)(1 - \mathcal{X}^{-\frac{1}{4}})} \right]. \tag{6.16}$$

We start noting that from $(\mathcal{X}^{\frac{1}{8}} - \mathcal{X}^{-\frac{1}{8}})^2 \geq 0$ it follows that

$$\mathcal{X}^{\frac{1}{4}} - 1 \geq 1 - \mathcal{X}^{-\frac{1}{4}}. \tag{6.17}$$

If $\log_{\mathcal{X}} \frac{p-1}{\alpha} \leq -\frac{1}{4}$, then $\alpha_1 = \alpha \geq (p-1)\mathcal{X}^{\frac{1}{4}}$. Thus for every $j \in \mathbf{N} \cup \{0\}$ we have

$$q_j = \alpha \mathcal{X}^j + 1 - p \geq \alpha + 1 - p > \alpha \mathcal{X}^{-\frac{1}{4}} + 1 - p \geq 0$$

and hence, also using (6.17),

$$\begin{aligned} |q_j| + \frac{1}{|q_j|} \;\; &\leq \;\; \alpha \mathcal{X}^j + p - 1 + \frac{1}{\alpha + 1 - p} \\ &\leq \;\; \alpha \mathcal{X}^j + p - 1 + \frac{1}{(p-1)(\mathcal{X}^{\frac{1}{4}} - 1)} \;\; \leq \;\; \alpha \mathcal{X}^j + p - 1 + \frac{1}{(p-1)(1 - X^{-\frac{1}{4}})}. \end{aligned}$$

Since $\mathcal{X} > 1$ and $\alpha = \alpha_1$, (6.16) easily follows.

If $\log_{\mathcal{X}} \frac{p-1}{\alpha} > -\frac{1}{4}$, there exists a unique $K \in \mathbf{N} \cup \{0\}$ such that either $\log_{\mathcal{X}} \frac{p-1}{\alpha} \in (K - \frac{1}{4}, K + \frac{1}{4})$ or $\log_{\mathcal{X}} \frac{p-1}{\alpha} \in [K + \frac{1}{4}, K + \frac{3}{4}]$. In the first case $\alpha_1 = \alpha \mathcal{X}^{-\frac{1}{2}}$ and

$$\alpha \mathcal{X}^{K - \frac{1}{4}} < p - 1 < \alpha \mathcal{X}^{K + \frac{1}{4}}.$$

Thus if $j \leq K$ one has

$$q_j = \alpha \mathcal{X}^{j - \frac{1}{2}} + 1 - p \leq \alpha \mathcal{X}^{K - \frac{1}{2}} + 1 - p = \alpha \mathcal{X}^{K - \frac{1}{4}} \mathcal{X}^{-\frac{1}{4}} - (p-1) < -(p-1)(1 - \mathcal{X}^{-\frac{1}{4}}) < 0.$$

On the other hand, if $j \geq K + 1$, we have

$$q_j = \alpha \mathcal{X}^{j - \frac{1}{2}} + 1 - p \geq \alpha \mathcal{X}^{K + \frac{1}{2}} + 1 - p = \alpha \mathcal{X}^{K + \frac{1}{4}} \mathcal{X}^{\frac{1}{4}} - (p-1) > (p-1)(\mathcal{X}^{\frac{1}{4}} - 1) > 0.$$

Thus $q_j \in (1-p, 0) \cup (0, \infty)$, and moreover for $j \leq K$

$$|q_j| + \frac{1}{|q_j|} \leq \alpha \mathcal{X}^j + p - 1 + \frac{1}{(p-1)(1 - \mathcal{X}^{-\frac{1}{4}})}, \tag{6.18}$$

while for $j \geq K + 1$ we have

$$|q_j| + \frac{1}{|q_j|} \leq \alpha \mathcal{X}^j + p - 1 + \frac{1}{(p-1)(\mathcal{X}^{\frac{1}{4}} - 1)} \leq \alpha \mathcal{X}^j + p - 1 + \frac{1}{(p-1)(1 - \mathcal{X}^{-\frac{1}{4}})}. \tag{6.19}$$

From the previous inequalities, since $\mathcal{X} > 1$ and $\alpha = \alpha_1 \mathcal{X}^{\frac{1}{2}}$, (6.16) follows.

It remains to consider the case when $\log_{\mathcal{X}} \frac{p-1}{\alpha} \in [K + \frac{1}{4}, K + \frac{3}{4}]$ for some $K \in \mathbf{N} \cup \{0\}$. Then we have $\alpha_1 = \alpha$ and

$$\alpha \mathcal{X}^{K+\frac{1}{4}} \leq p - 1 \leq \alpha \mathcal{X}^{K+\frac{3}{4}}.$$

Now if $j \leq K$ we have

$$q_j = \alpha \mathcal{X}^j + 1 - p \leq \alpha \mathcal{X}^K + 1 - p = \alpha \mathcal{X}^{K+\frac{1}{4}} \mathcal{X}^{-\frac{1}{4}} - (p-1) \leq -(p-1)(1 - \mathcal{X}^{-\frac{1}{4}}) < 0,$$

while if $j \geq K + 1$

$$q_j = \alpha \mathcal{X}^j + 1 - p \geq \alpha \mathcal{X}^{K+1} + 1 - p = \alpha \mathcal{X}^{K+\frac{3}{4}} \mathcal{X}^{\frac{1}{4}} - (p-1) \geq (p-1)(\mathcal{X}^{\frac{1}{4}} - 1) > 0.$$

Hence also in this last case $q_j \in (1-p, 0) \cup (0, \infty)$ for every $j \in \mathbf{N} \cup \{0\}$, and for $j \leq K$ we have (6.18), while for $j \geq K + 1$ we have (6.19). The proof of the *claim* is complete.

Let $c_4 = p - 1 + ((p-1)(1 - \mathcal{X}^{-\frac{1}{4}}))^{-1}$. By (6.15) and (6.16), for each $j \in \mathbf{N} \cup \{0\}$,

$$\Phi(j+1; \alpha_1 s \mathcal{X}^{j+1}) \quad \leq \quad \Big[C^{\mathcal{X}^{-j}} (\alpha_1 \mathcal{X}^{\frac{1}{2}} + c_4)^{\tilde{b}_* \mathcal{X}^{-j}} \bar{Z}^{\mathcal{X}^{-j}} \mathcal{X}^{\tilde{b}_* j \mathcal{X}^{-j}} N^{j \mathcal{X}^{-j}} \Big]^{\frac{p}{\alpha_1}} \Phi(j; \alpha_1 s \mathcal{X}^j).$$

Iterating this inequality we see that

$$\Phi(j+1; \alpha_1 s \mathcal{X}^{j+1}) \quad \leq \quad \Big[C^{\Psi_0} (\alpha_1 \mathcal{X}^{\frac{1}{2}} + c_4)^{\tilde{b}_* \Psi_0} \mathcal{X}^{\tilde{b}_* \Psi_1} N^{\Psi_1} \bar{Z}^{\Psi_0} \Big]^{\frac{p}{\alpha_1}} \Phi(0; \alpha_1 s) \tag{6.20}$$

for each $j \in \mathbf{N} \cup \{0\}$ where we have set $\Psi_0 = \sum_{j=0}^{\infty} \mathcal{X}^{-j}$ and $\Psi_1 = \sum_{j=0}^{\infty} j \mathcal{X}^{-j}$. Now, since the function $z^{\frac{1}{z}}$ achieves its maximum for $z \in (0, \infty)$ at $z = e$, we have

$$(\alpha_1 \mathcal{X}^{\frac{1}{2}} + c_4)^{\frac{\tilde{b}_* \Psi_0 p}{\alpha_1}} \quad \leq \quad (2 \max\{\alpha_1 \mathcal{X}^{\frac{1}{2}}, c_4\})^{\frac{\tilde{b}_* \Psi_0 p}{\alpha_1}} \quad = \quad \left[\big(\max\{2\alpha_1 \mathcal{X}^{\frac{1}{2}}, 2c_4\} \big)^{\frac{1}{2\alpha_1 \mathcal{X}^{\frac{1}{2}}}} \right]^{2\tilde{b}_* \mathcal{X}^{\frac{1}{2}} \Psi_0 p}$$

$$\leq \quad \left[\max\left\{ e^{\frac{1}{e}}, (2c_4)^{\frac{1}{2\alpha_1 \mathcal{X}^{\frac{1}{2}}}} \right\} \right]^{2\tilde{b}_* \mathcal{X}^{\frac{1}{2}} \Psi_0 p} \quad \leq \quad c_5 c_6^{\frac{\Psi_0 p}{\alpha_1}}, \tag{6.21}$$

with c_5, c_6 depending on $p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3$. Next, we set $\Phi(\infty; \infty) = \limsup_{j \to \infty} \Phi(j; \alpha_1 s \chi^j)$. Since the right side of (6.20) is independent of j, we may allow $j \to \infty$ in (6.20) and use (6.21) to obtain

$$\Phi(\infty; \infty) \quad \leq \quad C_{10} \Big[C_{11} \bar{Z} \Big]^{\frac{\Psi_0 p}{\alpha_1}} \Phi(0; \alpha_1 s), \tag{6.22}$$

with C_{10} depending on $p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3$, C_{11} depending on $p, \sigma, s, \epsilon_1, \epsilon_2, \epsilon_3, a$ and both also depending on the constants C_1 from (2.8) and N, C_{s^*} in (2.10). Since $B(x_0, \tau r) \subset S_j$ for all $j \geq 1$ we have $\operatorname{ess\,sup}_{B(x_0,\tau r)} \bar{u} \leq \Phi(\infty; \infty)$, see for instance [GT], and therefore we conclude that

$$\operatorname*{ess\,sup}_{B(x_0,\tau r)} \bar{u} \quad \leq \quad C_{10} \Big[C_{11} \bar{Z} \Big]^{\frac{\Psi_0 p}{\alpha_1}} ||\bar{u}^{\alpha_1}||^{\frac{1}{\alpha_1}}_{s, B(x_0,r); \overline{dx}}, \tag{6.23}$$

which completes the proof of (6.1).

Lemma 6.3. *Let the assumptions in the opening paragraph of this section hold, let s be as in Lemma 6.1 and suppose that* $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ *is a weak solution in* Ω *of*

$$\operatorname{div}(A(x, u, \nabla u)) \leq B(x, u, \nabla u) \tag{6.24}$$

where A, B satisfy (1.2) with exponents γ, δ, ψ satisfying (3.5). Fix $x_0 \in \Omega$, $k > 0$, $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1]$ and a quasimetric ρ-ball $B(x_0, r)$ with $0 < r < \tau^2 r_1(x_0)$, set $\bar{u} = |u| + k$, and assume that $\bar{Z}(B(x_0, r/\tau), \bar{u}) < \infty$. If $u \geq 0$ in $B(x_0, r)$, then

$$(6.25) \qquad ||\bar{u}^\alpha||^{\frac{1}{\alpha}}_{s,B(x_0,r);\overline{dx}} \leq C_{10}\Big(C_{11}\bar{Z}(B(x_0,r),\bar{u})\Big)^{\frac{p\Psi_0}{|\alpha|}} \operatorname*{ess\ inf}_{B(x_0,\tau r)} \bar{u},$$

where the constants C_{10}, C_{11} can be chosen as in (6.1).

Proof: Since by our assumptions $\bar{Z}(B(x_0, r/\tau), \bar{u}) < +\infty$, $B(x_0, r) \subset B(x_0, r/\tau)$ and $r/\tau < \tau r_1(x_0) < R_1(x_0)$, we have that $\bar{Z} := \bar{Z}(B(x_0, r), \bar{u})$ is also finite. See the comment following the definition (3.4) of $\bar{Z}$.

Following the same argument as in Lemma 6.1 but now with $q < 1 - p$ and using that $(u, \nabla u)$ is a weak solution of (6.24), applying Remark 4.4 we obtain the following inequality, similar to (6.15):

$$\Big[\Phi(j+1; Y\sigma)\Big]^{\frac{Y}{p}} \leq C\Big(|q| + \frac{1}{|q|}\Big)^{\tilde{b}_*} \bar{Z} N^j \Big[\Phi(j; sY)\Big]^{\frac{Y}{p}},$$

with $\bar{Z} = \bar{Z}(B(x_0, r), \bar{u})$. Since $Y = p + q - 1 < 0$ for any $q \in (-\infty, 1 - p)$, we have

$$(6.26) \qquad \Phi(j+1; Y\sigma) \geq C^{\frac{p}{Y}}\Big(|q| + \frac{1}{|q|}\Big)^{\frac{p\tilde{b}_*}{Y}} \bar{Z}^{\frac{p}{Y}} N^{\frac{jp}{Y}} \Phi(j; sY).$$

Let $\alpha < 0$, set $Y_j = \alpha\mathcal{X}^j$ and $q_j = \alpha\mathcal{X}^j + 1 - p$, where $\mathcal{X} = \sigma/s > 1$ as in Lemma 6.1. Then $Y_j < 0$, $q_j < -(p-1)$ and, with C as in (6.15), we have

$$(6.27) \qquad \Phi(j; s\alpha\mathcal{X}^j) \leq \Big[C^{\mathcal{X}^{-j}}\Big(|q_j| + \frac{1}{|q_j|}\Big)^{\tilde{b}_*\mathcal{X}^{-j}} \bar{Z}^{\mathcal{X}^{-j}} N^{j\mathcal{X}^{-j}}\Big]^{\frac{p}{|\alpha|}} \Phi(j+1; s\alpha\mathcal{X}^{j+1}).$$

Now note that

$$\begin{aligned} |q_j| + \frac{1}{|q_j|} &\leq |\alpha|\mathcal{X}^j + p - 1 + \frac{1}{p-1} < |\alpha|\mathcal{X}^j + p - 1 + \frac{1}{(p-1)(1-\mathcal{X}^{-\frac{1}{4}})} \\ &\leq \mathcal{X}^j\left[|\alpha|\mathcal{X}^{\frac{1}{2}} + p - 1 + \frac{1}{(p-1)(1-\mathcal{X}^{-\frac{1}{4}})}\right] = \mathcal{X}^j\Big[|\alpha|\mathcal{X}^{\frac{1}{2}} + c_4\Big], \end{aligned}$$

with $c_4 = p - 1 + ((p-1)(1-\mathcal{X}^{-\frac{1}{4}}))^{-1}$ as in Lemma 6.1. Then from (6.27) we have

$$\Phi(j; \alpha s\mathcal{X}^j) \leq \Big[C^{\mathcal{X}^{-j}}(|\alpha|\mathcal{X}^{\frac{1}{2}} + c_4)^{\tilde{b}_*\mathcal{X}^{-j}} \bar{Z}^{\mathcal{X}^{-j}} \mathcal{X}^{\tilde{b}_* j\mathcal{X}^{-j}} N^{j\mathcal{X}^{-j}}\Big]^{\frac{p}{|\alpha|}} \Phi(j+1; \alpha s\mathcal{X}^{j+1}).$$

Iterating the previous inequality we obtain

$$\Phi(j; \alpha s\mathcal{X}^j) \leq \Big[C^{\Psi_0}(|\alpha|\mathcal{X}^{\frac{1}{2}} + c_4)^{\tilde{b}_*\Psi_0} \mathcal{X}^{\tilde{b}_*\Psi_1} N^{\Psi_1} \bar{Z}^{\Psi_0}\Big]^{\frac{p}{|\alpha|}} \Phi(\infty; -\infty),$$

where $\Phi(\infty; -\infty) = \limsup_{j\to\infty} \Phi(j; \alpha s\mathcal{X}^j)$, $\Psi_0 = \sum_{j=0}^\infty \mathcal{X}^{-j}$ and $\Psi_1 = \sum_{j=0}^\infty j\mathcal{X}^{-j}$. Also using (6.21) we conclude

$$\Phi(j; \alpha s\mathcal{X}^j) \leq C_{10}\Big[C_{11}\bar{Z}\Big]^{\frac{\Psi_0 p}{|\alpha|}} \Phi(\infty, -\infty),$$

with C_{10}, C_{11} as in (6.22). Since this holds for all $j \in \mathbf{N} \cup \{0\}$, we obtain

$$(6.28) \qquad \Phi(0, \alpha s) \leq C_{10}\Big[C_{11}\bar{Z}\Big]^{\frac{\Psi_0 p}{|\alpha|}} \Phi(\infty, -\infty).$$

Since $B(x_0, \tau r) \subset S_j$ for every $j \geq 1$ we have $\operatorname{ess\,inf}_{B(x_0,\tau r)}\bar{u} \geq \Phi(\infty; -\infty)$, see [GT]; hence

$$(6.29) \qquad ||\bar{u}^\alpha||^{\frac{1}{\alpha}}_{s,B(x_0,r);\overline{dx}} \leq C_{10}\Big[C_{11}\bar{Z}\Big]^{\frac{\Psi_0 p}{|\alpha|}} \operatorname*{ess\ inf}_{B(x_0,\tau r)} \bar{u},$$

which proves (6.25).

Lemma 6.4. *Let the assumptions in the opening paragraph of this section hold, and suppose that* $(u, \nabla u) \in W^{1,p}_Q(\Omega)$ *is a weak solution in* Ω *of*

$$\operatorname{div}(A(x,u,\nabla u)) \leq B(x,u,\nabla u) \tag{6.30}$$

where A, B *satisfy* (1.2) *with exponents* δ, γ, ψ *as in* (3.5)*. Furthermore, suppose that the Poincaré inequality* (2.9) *holds. Fix* $\hat{x} \in \Omega$, $k > 0$, $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1]$ *and let* $\bar{u} = |u| + k$ *and* $w = \log \bar{u}$*. Fix a quasimetric ball* $B(\hat{x}, \mathfrak{b}l/\tau)$ *for* $\mathfrak{b} > 1$ *as in* (2.9) *and* $0 < l < \tau r_1(\hat{x})/\mathfrak{b}$*. If* $u \geq 0$ *in* $B(\hat{x}, \mathfrak{b}l/\tau)$*, then*

$$ {}_{B(\hat{x},l)} |w - w_{B(\hat{x},l)}|\, dx \leq C_{12} \bar{Z}(B(\hat{x}, \mathfrak{b}l/\tau), \bar{u}), \tag{6.31}$$

where $\bar{Z}\big(B(\hat{x}, \mathfrak{b}l/\tau), \bar{u}\big)$ *may be infinite and where* C_{12} *depends on* a, p, σ, *on* $\epsilon_1, \epsilon_2, \epsilon_3$ *in the definition* (3.4) *of* $\bar{Z}$, *on* $\mathfrak{b}, C_2$ *in* (2.9), *on* d_0, C_0 *in* (2.4) *and on* C_{s^*}, τ, N *in* (2.10).

Proof: We can assume that $\bar{Z} := \bar{Z}\big(B(\hat{x}, \mathfrak{b}l/\tau), \bar{u}\big)$ is finite, otherwise (6.31) is trivial. Let $\eta = \eta_1$ be as in (2.10) relative to $B(\hat{x}, \mathfrak{b}l/\tau)$, and set $v = \eta^p \bar{u}^{1-p}$. Applying Remark 4.4 on the quasimetric ball $B(\hat{x}, \mathfrak{b}l/\tau)$, we have that $v \in W^{1,p}_{Q,0}(\Omega)$ with $\operatorname{supp} v \subset B(\hat{x}, \mathfrak{b}l/\tau)$ and using (6.3) we have

$$\begin{aligned} \nabla v \cdot A + vB \;\leq\; & a^{-1}(1-p)\eta^p \bar{u}^{-p} |\sqrt{Q}\nabla \bar{u}|^p + (p-1)\bar{h}\eta^p \\ & + ap\eta^{p-1}|\sqrt{Q}\nabla\eta| \bar{u}^{1-p} |\sqrt{Q}\nabla\bar{u}|^{p-1} + p\eta^{p-1}\bar{b}|\sqrt{Q}\nabla\eta| \\ & + c\eta^p \bar{u}^{1-p}|\sqrt{Q}\nabla\bar{u}|^{\psi-1} + \eta^p \bar{d} \end{aligned} \tag{6.32}$$

a.e. in Ω. We integrate over $B(\hat{x}, \mathfrak{b}l/\tau)$ and use the facts that $(u, \nabla u)$ is a weak solution of (6.30) and that v is a feasible nonnegative test function, obtaining that the left side of the resulting inequality is nonnegative. Also, we move the resulting first term on the right side to the left side and estimate the third and fifth terms on the right in ways like those used to estimate similar terms in (6.8). Then we obtain, as in (6.11) with $q = 1-p$,

$$\begin{aligned} {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \eta^p \bar{u}^{-p} |\sqrt{Q}\nabla\bar{u}|^p dx \;\leq\; & C\Big\{ {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} |\sqrt{Q}\nabla\eta|^p dx + \fint_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \bar{h}\eta^p dx \\ & + {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \eta^{p-1}\bar{b}|\sqrt{Q}\nabla\eta|\, dx + \fint_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \eta^p \bar{d}\, dx \\ & + {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} c^{\frac{p}{p+1-\psi}} \eta^p \bar{u}^{\frac{p(\psi-p)}{p+1-\psi}} dx \Big\}, \end{aligned} \tag{6.33}$$

with C depending only on a, p, σ. Repeating steps 5 and 6 in the proof of [MRW, Theorem 1.2] we obtain

$$\begin{aligned} {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \eta^p \bar{u}^{-p} |\sqrt{Q}\nabla\bar{u}|^p dx \;\leq\; & C\bar{Z}^p \Big\{ {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} |\sqrt{Q}\nabla\eta|^p dx + \frac{1}{l^p} \fint_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} \eta^p dx \Big\} \\ \leq\; & C\bar{Z}^p \Big\{ {}_{B(\hat{x},\frac{\mathfrak{b}l}{\tau})} |\sqrt{Q}\nabla\eta|^p dx + \frac{1}{l^p} \Big\} \end{aligned} \tag{6.34}$$

analogous to [MRW, (3.21)] with $Y = 0$, noting that $0 \leq \eta \leq 1$ on $B(\hat{x}, \frac{\mathfrak{b}l}{\tau})$. We recall that here $\bar{Z} = \bar{Z}(B(\hat{x}, \mathfrak{b}l/\tau), \bar{u})$ and we note that C depends on a, p, σ and on $\epsilon_1, \epsilon_2, \epsilon_3$ appearing in the definition (3.4) of $\bar{Z}$.

Since η is the function η_1 in (2.10) relative to $B(\hat{x}, \mathfrak{b}l/\tau)$, then $\eta \in Lip_0(B(\hat{x}, \mathfrak{b}l/\tau)) \cap Lip(\Omega)$ and $\eta \equiv 1$ on $B(\hat{x}, \mathfrak{b}l)$. Recalling that $\mathfrak{b}l < r_1(y)$, we apply the Poincaré inequality (2.9) to

$w = \log \bar{u}$ (see Remark 4.4) and get

$$
\begin{aligned}
\frac{1}{|B(\hat{x},l)|}\int_{B(\hat{x},l)} |w - w_{B(\hat{x},l)}|dx &\leq Cl\Big(\frac{1}{|B(\hat{x},\mathfrak{b}l)|}\int_{B(\hat{x},\mathfrak{b}l)} |\sqrt{Q}\nabla w|^p dx\Big)^{1/p} \\
&\leq Cl\Big(\frac{1}{|B(\hat{x},\mathfrak{b}l)|}\int_{B(\hat{x},\mathfrak{b}l/\tau)} \eta^p|\sqrt{Q}\nabla w|^p dx\Big)^{1/p} \\
&= Cl\frac{|B(\hat{x},\mathfrak{b}l/\tau)|^{1/p}}{|B(\hat{x},\mathfrak{b}l)|^{1/p}}\Big(\ _{B(\hat{x},\mathfrak{b}l/\tau)} \eta^p\bar{u}^{-p}|\sqrt{Q}\nabla \bar{u}|^p dx\Big)^{1/p} \\
&\leq Cl\Big(\frac{|B(\hat{x},\mathfrak{b}l/\tau)|}{|B(\hat{x},\mathfrak{b}l)|}\Big)^{\frac{1}{p}}\bar{Z}\Big[\Big(\fint_{B(\hat{x},\mathfrak{b}l/\tau)}|\sqrt{Q}\nabla\eta|^p dx\Big)^{\frac{1}{p}} + \frac{1}{l}\Big] \qquad (6.35)
\end{aligned}
$$

where the last line is obtained using (6.34). Also, C in (6.35) depends on $a, p, \sigma, \epsilon_1, \epsilon_2, \epsilon_3$ and on the constants $\mathfrak{b}, C_2$ appearing in (2.9). In (6.35), use Hölder's inequality with exponents $\frac{s^*}{p}, \frac{s^*}{s^*-p}$ together with (2.4) and (2.10) to obtain

$$
\begin{aligned}
\frac{1}{|B(\hat{x},l)|}\int_{B(\hat{x},l)} |w - w_{B(\hat{x},l)}|dx &\leq Cl\bar{Z}\Big(\frac{|B(\hat{x},\mathfrak{b}l/\tau)|}{|B(\hat{x},\mathfrak{b}l)|}\Big)^{1/p}\Big[\Big(\ _{B(\hat{x},\mathfrak{b}l/\tau)}|\sqrt{Q}\nabla\eta|^{s^*}dx\Big)^{\frac{1}{s^*}} + \frac{1}{l}\Big] \\
&\leq C\bar{Z}\tau^{-\frac{d_0}{p}}\Big[\frac{\tau N}{\mathfrak{b}} + 1\Big] \\
&= C_{12}\bar{Z}, \qquad (6.36)
\end{aligned}
$$

where C_{12} depends on $a, p, \sigma, \epsilon_1, \epsilon_2, \epsilon_3, \mathfrak{b}, C_2$, on the constants d_0, C_0 in (2.4) and C_{s^*}, τ, N in (2.10).

Proof of Proposition 3.1: We will use the notation and assumptions of Proposition 3.1 and divide the proof into steps.

Step 1. We have $B(x_0, C_*r) \Subset B(y, \frac{\tau}{2}r_1(y))$. Indeed if $\xi \in B(x_0, C_*r)$, then

$$\rho(\xi, y) \leq \kappa(\rho(\xi, x_0) + \rho(x_0, y)) < \kappa\Big(C_*r + \frac{\tau}{5\kappa}r_1(y)\Big) < \Big(\frac{\tau A_*}{5} + \frac{\tau}{5}\Big)r_1(y) < \frac{\tau}{2}r_1(y).$$

Step 2. By using Lemmas 6.1 and 6.3, let us show that for every $\alpha > 0$ there exists $\alpha_1 \in [\alpha\sigma^{-1/2}, \alpha]$ such that

$$\operatorname*{ess\ sup}_{B(x_0,\tau r)} \bar{u} \leq C_{10}\big[C_{11}\bar{Z}\big(B(x_0,r),\bar{u}\big)\big]^{\frac{p\psi_0}{\alpha_1}}\|\bar{u}^{\alpha_1}\|^{\frac{1}{\alpha_1}}_{s,B(x_0,r);\overline{dx}} \qquad (6.37)$$

and that for every $\alpha_2 < 0$

$$\|\bar{u}^{\alpha_2}\|^{\frac{1}{\alpha_2}}_{s,B(x_0,r);\overline{dx}} \leq C_{10}\big[C_{11}\bar{Z}\big(B(x_0,r),\bar{u}\big)\big]^{\frac{p\psi_0}{|\alpha_2|}}\operatorname*{ess\ inf}_{B(x_0,\tau r)} \bar{u}, \qquad (6.38)$$

with C_{10}, C_{11} and ψ_0 independent of $(u, \nabla u), k, B(x_0, r), y, b, c, d, e, f, g, h, \alpha, \alpha_1, \alpha_2$.

Indeed, by our assumptions, r_1 satisfies a local uniformity condition with respect to ρ on $B = B(y, r_1(y))$ with constant A_*. Since $x_0 \in B(y, r_1(y))$ we have $r_1(x_0) > A_*r_1(y)$, so that $r < \frac{\tau A_*}{5\kappa C_*}r_1(y) < \tau^2 r_1(x_0)$. Moreover $\tau^{-1} < C_*$, so that $r/\tau < C_*r$ and $B(x_0, r/\tau) \subset B(x_0, C_*r)$. Thus by (3.7) we conclude that $\bar{Z}(B(x_0, r/\tau), \bar{u}) \leq M < +\infty$, and hence all the assumptions of Lemmas 6.1 and 6.3 are satisfied.

Step 3. We start implementing the ideas of Section 5. Let

$$R(\xi) = \min\Big\{\frac{16(\gamma^*)^4\kappa^5}{A_*^2\min\{A_*^2, (8\kappa^5)^{-1}\}}r, \frac{\tau^2 A_*}{40\kappa^6(\gamma^*)^4\mathfrak{b}}r_1(\xi)\Big\} \quad \text{for } \xi \in \Omega, \quad \text{and let } B = B(x_0, r).$$

Let us show that if $\xi \in B(y, r_1(y))$ then $R(\xi) = \frac{16(\gamma^*)^4\kappa^5}{A_*^2\min\{A_*^2,(8\kappa^5)^{-1}\}}r$. Indeed for every $\xi \in B(y, r_1(y))$ we have $A_*r_1(y) < r_1(\xi)$ by our assumptions on r_1. Since $r \in (0, \frac{\tau A_*}{5\kappa C_*}r_1(y))$ with

C_* as in (3.6), we obtain

$$\frac{16(\gamma^*)^4\kappa^5}{A_*^2\min\{A_*^2,(8\kappa^5)^{-1}\}}r < \frac{16\tau(\gamma^*)^4\kappa^4}{5C_*A_*\min\{A_*^2,(8\kappa^5)^{-1}\}}r_1(y) = \frac{\tau^2A_*^2}{40\kappa^6(\gamma^*)^4\mathfrak{b}}r_1(y) < \frac{\tau^2A_*}{40\kappa^6(\gamma^*)^4\mathfrak{b}}r_1(\xi).$$

Note that by the second restriction above on $R(\xi)$, we have $R(\xi) < \frac{r_1(\xi)}{(\gamma^*)^4} < \frac{r_1(\xi)}{(\gamma^*)^2}$ for every $\xi \in \Omega$, which meets some of the requirements of Proposition 5.1. Moreover, since $R(\xi)$ is constant on B, it satisfies a local uniformity condition on B with respect to ρ for any constant in $(0,1]$, in particular for A_*. Thus $R(\xi)$ satisfies the requirements in the statement of Proposition 5.1 and consequently can be used in Corollary 5.3. Hence there are constants C_8, C_9, $c_\rho = 8\kappa^5(\gamma^*)^2$, $\delta_0 = \frac{A_*^2\min\{A_*^2,(8\kappa^5)^{-1}\}}{8\kappa^5(\gamma^*)^3}$ (cf. Remark 5.2 for the values of c_ρ and δ_0) such that (5.4) holds for every function $f \in [c_\rho R]BMO(B)$ with $\|f\|_{[c_\rho R]BMO(B)} < C_9$.

Step 4. We claim that if $\xi \in B$ and $B(z,t_0)$ is a $c_\rho R(\xi)$–ball then

$$\text{(6.39)} \qquad \begin{aligned} &i) \quad B(z,t_0) \subset B\Big(z,\frac{\mathfrak{b}}{\tau}t_0\Big) \subset B(x_0,C_*r),\\ &ii) \quad 0 < t_0 < \frac{\mathfrak{b}}{\tau}t_0 < C_*r < \frac{\tau A_*}{5\kappa}r_1(y) < r_1(z)\end{aligned}$$

Since $\xi \in B$ and $B(z,t_0)$ is a $c_\rho R(\xi)$–ball (see the definition above (5.1)), we have

$$\text{(6.40)} \qquad t_0 < c_\rho R(\xi) = \frac{128(\gamma^*)^6\kappa^{10}}{A_*^2\min\{A_*^2,(8\kappa^5)^{-1}\}}r = \frac{2\gamma^*c_\rho}{\delta_0}r$$

and $B(z,\gamma^*t_0) \subset B(\xi,c_\rho R(\xi)) = B\Big(\xi,\frac{2\gamma^*c_\rho}{\delta_0}r\Big)$. Thus, also using the swallowing property of the pseudometric balls as described in Lemma 2.2,

$$\begin{aligned} B(z,t_0) \;&\subset\; B\Big(z,\frac{\mathfrak{b}}{\tau}t_0\Big) \subset B\Big(z,\frac{2\gamma^*c_\rho\mathfrak{b}}{\delta_0\tau}r\Big) \subset B\Big(\xi,\frac{2(\gamma^*)^2c_\rho\mathfrak{b}}{\delta_0\tau}r\Big)\\ &\subset\; B\Big(x_0,\frac{2(\gamma^*)^3c_\rho\mathfrak{b}}{\delta_0\tau}r\Big) = B\Big(x_0,C_*A_*r\Big) \subset B(x_0,C_*r).\end{aligned}$$

Since by Step 1 we have $B(x_0,C_*r) \subset B(y,r_1(y))$ and since r_1 satisfies a local uniformity condition with respect to ρ with constant A_* on that set, we conclude that $A_*r_1(y) < r_1(z)$. Hence, also using (6.40),

$$0 < t_0 < \frac{\mathfrak{b}}{\tau}t_0 < \frac{2\mathfrak{b}\gamma^*c_\rho}{\tau\delta_0}r = \frac{A_*}{(\gamma^*)^2}C_*r < C_*r < \frac{\tau A_*}{5\kappa}r_1(y) < r_1(z).$$

Here the next-to-last inequality is due to the relation between r and $r_1(y)$ that we noted earlier. The proof of our claim is complete.

Step 5. Now we show that $w = \log\bar{u}$ is a function in $[c_\rho R]BMO(B)$, where as above $B = B(x_0,r)$. Let $\xi \in B$ and consider a $c_\rho R(\xi)$–ball $B(z,t_0)$. By Step 4, Lemma 6.4 and condition (3.7) we conclude that

$$\fint_{B(z,t_0)} |w(\zeta) - w_{B(z,t_0)}|\,d\zeta \;\le\; C_{12}\overline{Z}\Big(B\Big(z,\frac{\mathfrak{b}}{\tau}t_0\Big),\bar{u}\Big) \;\le\; C_{12}M,$$

and thus, by the definition given in (5.1), we have $\|w\|_{[c_\rho R]BMO(B)} \le C_{12}M$.

Now we choose $\alpha = \frac{C_9}{2sC_{12}M}$, where C_9 is as in Proposition 5.1 and where s is as in Lemmas 6.1 and 6.3 and also (6.37). Then the corresponding α_1 from inequality (6.37) satisfies $\alpha_1 \in [\alpha\sigma^{-1/2},\alpha]$. Then we have

$$\|s\alpha_1 w\|_{[c_\rho R]BMO(E)} \le s\alpha_1 C_{12}M \le s\alpha C_{12}M \le \frac{C_9}{2},$$

and by Step 3 we can use Corollary 5.3 to conclude that

$$\text{(6.41)} \qquad \|e^{s\alpha_1 w}\|_{[\delta_0 R]A_2(B)} \le (1+C_8)^2.$$

Step 6. We notice here that $x_0 \in B(x_0, r)$ and that $B(x_0, r)$ is a $\delta_0 R(x_0)$–ball, and use this fact in conjunction with (6.41). We start by recalling that since $x_0 \in B(y, r_1(y))$, Step 3 shows that $R(x_0) = \frac{16(\gamma^*)^4\kappa^5}{A_*^2 \min\{A_*^2,(8\kappa^5)^{-1}\}} r$. Now a simple calculation gives $0 < r < \gamma^* r < 2\gamma^* r = \delta_0 R(x_0)$, $B(x_0, \gamma^* r) \subset B(x_0, 2\gamma^* r) = B(x_0, \delta_0 R(x_0))$ and $\overline{B(x_0, \gamma^* r)} \subset \overline{B(x_0, r_1(x_0))} \subset \Omega$.

Since $B(x_0, r)$ is a $\delta_0 R(x_0)$–ball with $x_0 \in B(x_0, r)$, by (6.41) and definition (5.3) we have

$$\left(\quad_{B(x_0,r)} e^{s\alpha_1 w}\, d\zeta \right) \left(\fint_{B(x_0,r)} e^{-s\alpha_1 w}\, d\zeta \right) \leq (1 + C_8)^2,$$

and thus we conclude

$$\|\bar{u}^{\alpha_1}\|_{s,B(x_0,r);\overline{dx}}^{\frac{1}{\alpha_1}} \leq (1 + C_8)^{\frac{2}{s\alpha_1}} \|\bar{u}^{-\alpha_1}\|_{s,B(x_0,r);\overline{dx}}^{-\frac{1}{\alpha_1}} \tag{6.42}$$

Step 7. Now we use (6.37), (6.38) with $\alpha_2 = -\alpha_1 < 0$ and (6.42) to finish the proof:

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_0,\tau r)} \bar{u} &\leq C_{10}\left[C_{11}\bar{Z}\big(B(x_0, r), \bar{u}\big)\right]^{\frac{p\psi_0}{\alpha_1}} \|\bar{u}^{\alpha_1}\|_{s,B(x_0,r);\overline{dx}}^{\frac{1}{\alpha_1}} \\ &\leq C_{10}(1 + C_8)^{\frac{2}{s\alpha_1}} \left[C_{11}\bar{Z}\big(B(x_0, r), \bar{u}\big)\right]^{\frac{p\psi_0}{\alpha_1}} \|\bar{u}^{-\alpha_1}\|_{s,B(x_0,r);\overline{dx}}^{-\frac{1}{\alpha_1}} \\ &\leq C_{10}^2(1 + C_8)^{\frac{2}{s\alpha_1}} \left[C_{11}\bar{Z}\big(B(x_0, r), \bar{u}\big)\right]^{\frac{2p\psi_0}{\alpha_1}} \operatorname*{ess\,inf}_{B(x_0,\tau r)} \bar{u}. \end{aligned}$$

Since $C_8 + 1 \geq 1$, $\bar{Z}\big(B(x_0, r), \bar{u}\big) \geq 1$ and $\alpha_1 \geq \alpha\sigma^{-1/2}$, we see that

$$\operatorname*{ess\,sup}_{B(x_0,\tau r)} \bar{u} \leq C_{10}^2\left[(1 + C_8)^{\frac{1}{sp\Psi_0}} C_{11}\bar{Z}\big(B(x_0, r), \bar{u}\big)\right]^{\frac{2p\psi_0\sqrt{\sigma}}{\alpha}} \operatorname*{ess\,inf}_{B(x_0,\tau r)} \bar{u}$$

which, recalling the definition of α given in Step 5, is inequality (3.8), with $C_4 = C_{10}^2$, $C_5 = (1 + C_8)^{\frac{1}{sp\Psi_0}} C_{11}$ and $C_6 = 4\sqrt{\sigma} p\Psi_0 s C_{12}/C_9$.

7. The Proof of Theorem 3.7

Let $(u, \nabla u)$ be a weak solution of (1.1) in Ω and let $x_0 \in B = B(y, \frac{\tau^2}{5\kappa} r_1(y))$. For $r > 0$ (sufficiently small so that $\overline{B(x_0, r)} \subset \Omega$), define

$$M(r) = \operatorname*{ess\,sup}_{B(x_0,r)} u, \quad m(r) = \operatorname*{ess\,inf}_{B(x_0,r)} u, \text{ and } \omega_{x_0}(r) = M(r) - m(r).$$

We will refer to $\omega_{x_0}(r)$ as the oscillation of u in $B(x_0, r)$. Now, let $r \in (0, \frac{\tau^2 A_*}{5\kappa C_*} r_1(y))$, where C_* is as in (3.6), and set $M_0 = M(C_* r)$, $m_0 = m(C_* r)$, noting that M_0 and m_0 are finite by [MRW, Theorem 1.2] and Proposition 3.3 as $B(x_0, C_* r) \subset B(y, \tau^2 r_1(y))$. Denote

$$(u_1, \nabla u_1) = (M_0 - u, -\nabla u) \quad \text{and} \quad (u_2, \nabla u_2) = (u - m_0, \nabla u).$$

Clearly $(u_1, \nabla u_1), (u_2, \nabla u_2) \in W_Q^{1,p}(\Omega)$ and $u_1, u_2 \geq 0$ in $B(x_0, C_* r)$. For $(x, z, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, let

$$\begin{aligned} A_1(x, z, \xi) &= -A(x, M_0 - z, -\xi), & A_2(x, z, \xi) &= A(x, z + m_0, \xi), \\ \tilde{A}_1(x, z, \xi) &= -\tilde{A}(x, M_0 - z, -\xi), & \tilde{A}_2(x, z, \xi) &= \tilde{A}(x, z + m_0, \xi), \text{ and} \\ B_1(x, z, \xi) &= -B(x, M_0 - z, -\xi), & B_2(x, z, \xi) &= B(x, z + m_0, \xi). \end{aligned}$$

It is not difficult to see that $(u_1, \nabla u_1)$ and $(u_2, \nabla u_2)$ are respectively weak solutions in Ω of

$$\begin{aligned} \operatorname{div}\Big(A_1(x, u, \nabla u)\Big) &= B_1(x, u, \nabla u), \text{ and} \\ \operatorname{div}\Big(A_2(x, u, \nabla u)\Big) &= B_2(x, u, \nabla u). \end{aligned} \tag{7.1}$$

We now check that these equations satisfy (1.2) for coefficients which satisfy $(i)-(iv)$ of Proposition 3.3. The calculations are simple and we only provide an example. Let us show that A_1 satisfies item (ii) of (1.2) for appropriately modified definitions of g,h. Indeed, since A satisfies (1.2), we have

$$\begin{aligned}\xi\cdot A_1(x,z,\xi) &= -\xi\cdot A(x,M_0-z,-\xi)\\ &\geq a^{-1}|\sqrt{Q(x)}\xi|^p - h(x)|M_0-z|^p - g(x)\\ &\geq a^{-1}|\sqrt{Q(x)}\xi|^p - 2^{p-1}h(x)|z|^p - (g(x)+2^{p-1}h(x)|M_0|^p).\end{aligned}$$

Setting $h_1(x)=2^{p-1}h(x)$ and $g_1(x)=g(x)+2^{p-1}|M_0|^p h(x)$ it follows that $A_1(x,z,\xi)$ satisfies (1.2)(ii) with h,g there replaced by h_1,g_1 respectively. Furthermore, $h_1,g_1\in L^{\mathcal{H}}_{loc}(\Omega)$ since $|M_0|<\infty$ and both $h,g\in L^{\mathcal{H}}_{loc}(\Omega)$ by hypothesis. Other verifications are similar using the modified functions

$$\begin{aligned}h_2(x)=2^{p-1}h(x), &\qquad g_2(x)=g(x)+2^{p-1}|m_0|^p h(x),\\ e_1(x)=e(x)+2^{p-1}|M_0|^{p-1}b(x), &\qquad e_2(x)=e(x)+2^{p-1}|m_0|^{p-1}b(x), \text{ and}\\ f_1(x)=f(x)+2^{p-1}|M_0|^{p-1}d(x), &\qquad f_2(x)=f(x)+2^{p-1}|m_0|^{p-1}d(x)\end{aligned}$$

with

$$\begin{aligned}b_1(x) &= b_2(x) = 2^{p-1}b(x),\\ c_1(x) &= c_2(x) = c(x), \text{ and}\\ d_1(x) &= d_2(x) = 2^{p-1}d(x).\end{aligned}$$

Therefore both $(u_1,\nabla u_1)$ and $(u_2,\nabla u_2)$ are weak solutions of equations satisfying the hypotheses of Theorem 3.5. As a consequence, u_1,u_2 satisfy

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)} u_1(z)+k_1(x_0,r) \leq \hat{C}_1\Big[\operatorname*{ess\,inf}_{z\in B(x_0,\tau r)} u_1(z)+k_1(x_0,r)\Big], \text{ and} \tag{7.2}$$

$$\operatorname*{ess\,sup}_{z\in B(x_0,\tau r)} u_2(z)+k_2(x_0,r) \leq \hat{C}_2\Big[\operatorname*{ess\,inf}_{z\in B(x_0,\tau r)} u_2(z)+k_2(x_0,r)\Big]. \tag{7.3}$$

Here $k_1=k_1(x_0,r)$ and $k_2=k_2(x_0,r)$ are defined as k in Proposition 3.3 using the structural coefficient functions $b_1,c_1,d_1,e_1,f_1,g_1,h_1$ and $b_2,c_2,d_2,e_2,f_2,g_2,h_2$ respectively. By Proposition 3.3,

$$k_j(x_0,r)\leq\Lambda_j r^\lambda,\quad j=1,2, \tag{7.4}$$

with λ exactly as in Proposition 3.3 and Λ_1,Λ_2 defined as Λ in Proposition 3.3 using instead the structural coefficients related to A_1,A_2,B_1,B_2. Each of $\hat{C}_1,\hat{C}_2,\Lambda_1,\Lambda_2$ depends on p, ψ, M_0, m_0, $||u||_{p\sigma,\tilde{B};dx}$, $||b||_{\mathcal{B},\tilde{B};dx}$, $||c||_{\mathcal{C},\tilde{B};dx}$, $||d||_{\mathcal{D},\tilde{B};dx}$, $||e||_{\mathcal{B},\tilde{B};dx}$, $||f||_{\mathcal{D},\tilde{B};dx}$, $||g||_{\mathcal{H},\tilde{B};dx}$, $||h||_{\mathcal{H},\tilde{B};dx}$, C_0, d_0, s, a, and N, where $\tilde{B}=B(y,r_1(y))$. It is important to also note that λ is independent of u, and that when $\psi\in(p,p+1-\sigma^{-1})$ the dependence of $\hat{C}_1,\hat{C}_2,\Lambda_1,\Lambda_2$ on $||u||_{p\sigma,B;dx}$ occurs through M_1,M_2 of Proposition 3.3 and through M_0, m_0, see [MRW, Theorem 1.2]. Moreover $\hat{C}_1,\hat{C}_2,M_1,M_2$ are independent of $||u||_{p\sigma,B;dx},M_0,m_0$ when $\psi=p$.

Setting $C=\max\{\hat{C}_1,\hat{C}_2\}$ and rewriting (7.2) and (7.3) in terms of $M(r)$ and $m(r)$ gives

$$\begin{aligned}M_0-m(\tau r)+k_1(x_0,r) &\leq C\Big(M_0-M(\tau r)+k_1(x_0,r)\Big), \text{ and}\\ M(\tau r)-m_0+k_2(x_0,r) &\leq C\Big(m(\tau r)-m_0+k_2(x_0,r)\Big).\end{aligned} \tag{7.5}$$

Adding the inequalities in (7.5), rearranging and inserting the oscillation ω_{x_0}, we obtain

$$\omega_{x_0}(\tau r)(C+1)\leq(C-1)\Big(\omega_{x_0}(C_*r)+(k_1+k_2)\Big)$$

and so

$$\omega_{x_0}(\tau r)\leq\frac{C-1}{C+1}\Big(\omega_{x_0}(C_*r)+\hat{\Lambda}r^\lambda\Big),\quad 0<r<\frac{\tau A_*}{5\kappa C_*}r_1(y), \tag{7.6}$$

where we have used estimate (7.4) and set $\hat{\Lambda} = \Lambda_1 + \Lambda_2$. Define $R = C_* r$, $R_0 = \frac{\tau^2 A_*}{6\kappa} r_1(y)$ and $\hat{\Lambda}_0 = \hat{\Lambda} C_*^{-\lambda}$. Recall that $\tau < 1 < C_*$. Then (7.6) and the monotonicity of ω_{x_0} imply that for every $\nu \leq \tau / C_*$, one has

$$\omega_{x_0}(\nu R) \leq \frac{C-1}{C+1}\Big(\omega_{x_0}(R) + \hat{\Lambda}_0 R^\lambda\Big) \quad \text{for all } R \in (0, R_0]. \tag{7.7}$$

We now iterate (7.7) using powers of ν to obtain essential Hölder continuity of u. Indeed, for any $\nu \leq \tau/C_*$ and $j \geq 1$, we have

$$\omega_{x_0}(\nu^j R_0) \leq \Big(\frac{C-1}{C+1}\Big)^j \Big\{\omega_{x_0}(R_0) + \hat{\Lambda}_0 R_0^\lambda \sum_{i=0}^{j-1} \Big[\frac{C+1}{C-1}\,\nu^\lambda\Big]^i\Big\}. \tag{7.8}$$

Now choose $\nu \leq \frac{\tau}{C_*}$ such that $\frac{C+1}{C-1}\nu^\lambda \leq \frac{1}{2}$ to obtain $\sum_{i=0}^{\infty}\Big[\frac{C+1}{C-1}\,\nu^\lambda\Big]^i \leq 2$. Then (7.8) gives

$$\omega_{x_0}(\nu^j R_0) \leq \Big(\frac{C-1}{C+1}\Big)^j \Big[\omega_{x_0}(R_0) + 2\hat{\Lambda}_0 R_0^\lambda\Big]. \tag{7.9}$$

Now let $0 < R \leq R_0 \nu$ and choose $j \in \mathbb{N}$ such that $\nu^{j+1} R_0 < R \leq \nu^j R_0$. This choice implies that

$$j + 1 > \frac{\ln\Big(\frac{R}{R_0}\Big)}{\ln \nu}. \tag{7.10}$$

Combining (7.10) with (7.9) we obtain

$$\omega_{x_0}(R) \leq \frac{C+1}{C-1}\Big(\frac{R}{R_0}\Big)^\mu \Big(\omega_{x_0}(R_0) + 2\hat{\Lambda}_0 R_0^\lambda\Big), \tag{7.11}$$

where $\mu = \frac{\ln\frac{C-1}{C+1}}{\ln \nu} > 0$. Thus there are positive constants c_7, μ independent of x_0 such that

$$\omega_{x_0}(R) \leq c_7 R^\mu \quad \text{if } 0 < R \leq R_0 \nu. \tag{7.12}$$

As a consequence of (7.12), u is essentially Hölder continuous with respect to ρ in $B = B(y, \frac{\tau^2}{5\kappa} r_1(y))$. To see this, first note that since $u \in L^\infty(B)$, there is a set $E_y \subset B$ with $|E_y| = 0$ such that

$$|u(x)| \leq ||u||_{L^\infty(B)} \tag{7.13}$$

for all $x \in B \setminus E_y$. Choosing $x, w \in B \setminus E_y$, there are two cases to consider.

Case I: $\rho(x, w) < \frac{\nu R_0}{2}$. Applying (7.12) in the ball $B(x, 2\rho(x, w))$ we obtain

$$|u(x) - u(w)| \leq \omega_x(2\rho(x, w)) \leq c_7 2^\mu \rho(x, w)^\mu. \tag{7.14}$$

Case II: $\rho(x, w) \geq \frac{\nu R_0}{2}$. Then

$$|u(x) - u(z)| \leq 2||u||_{L^\infty(B)} \leq \frac{2^{\mu+1}||u||_{L^\infty(B)}}{\nu^\mu R_0^\mu}\rho(x, w)^\mu. \tag{7.15}$$

Setting $c_8 = \max\{c_7 2^\mu, \frac{2^{\mu+1}\|u\|_{L^\infty(B)}}{\nu^\mu R_0^\mu}\}$ and combining estimates, it follows that u is essentially Hölder continuous with respect to ρ in B, which completes the proof.

8. Proof of Corollary 3.9

Fix a compact set $K \subset \Omega$. By hypothesis, there is a positive constants s_0 depending only on K such that $s_0 \leq r_1(y) \leq 1$ for all $y \in K$. As a result, the constants Λ and M of Proposition 3.3 can be chosen larger so that they depend only on K and $S = \bigcup_{y \in K} B(y, r_1(y)) \Subset \Omega$. More precisely, this is achieved by replacing in those definitions all instances of $|B(y, r_1(y))|$ with $\inf_{y \in K} |B(y, r_1(y))| > 0$, expanding all norms calculated on $B(y, r_1(y))$ so that they are calculated over S, and replacing $r_1(y)$ itself by s_0 and 1 as appropriate. Moreover, since $\bar{S}$ is a compact subset of Ω, r_1 satisfies a local uniformity condition on every ball $B(y, r_1(y))$ with $y \in K$, with a uniform constant $A_* = A_*(S)$. In fact, one can choose $A_* = s_0'$ where s_0' satisfies $0 < s_0' \leq r_1(y) \leq 1$ for all $y \in S$. Combining these observations with Theorem 3.5, it follows that any weak solution $(u, \nabla u)$ of (1.1) in Ω satisfies the Harnack inequality (3.11) when $x_0 \in K$ and where the constant C_1 there is chosen to depend only on K and S via the norms of structural coefficients and the $L^{p\sigma}(S)$ norm of u.

Fix now a weak solution $(u, \nabla u)$ of (1.1) in Ω. Working through the proof of Theorem 3.7 with the observations above, one sees that ν (see (7.8)) can now be chosen to depend only on K and S as C there depends only on these quantities, and τ / C_* depends only on K, S through $C_* = C_*(A_*)$. As a result, for every $y \in K$ we have the estimate

$$\sup_{z, w \in B(y, r) \setminus E_y} \frac{|u(z) - u(w)|}{\rho(z, w)^\mu} \leq c_9, \tag{8.1}$$

where $r = \frac{\tau^2}{5\kappa} s_0$ and the constants c_9, μ are independent of y and E_y is as in the proof Theorem 3.7. We now cover K with a finite collection of ρ-balls of the form $B(y_j, \frac{r}{2\kappa})$ and set $E = \bigcup E_{y_j}$. Then $|E| = 0$, and for $x, z \in K \setminus E$ there are two cases to consider:

Case I: $\rho(x, z) < \frac{r}{2\kappa}$. We claim that there exists $y_j \in K$ such that both $x, z \in B(y_j, r)$. Indeed, choose y_j such that $x \in B(y_j, \frac{r}{2\kappa})$. Then

$$\rho(z, y_j) \leq \kappa(\rho(z, x) + \rho(x, y_j)) < r.$$

Since $x, z \in B(y_j, r) \setminus E_{y_j}$, we may apply (8.1) to obtain

$$|u(x) - u(z)| \leq c_9 \rho(x, z)^\mu.$$

Case II: $\rho(x, z) \geq \frac{r}{2\kappa}$. Arguing as at the end of the proof of Theorem 3.7, we have

$$|u(x) - u(z)| \leq 2||u||_{L^\infty(S)} \leq \frac{2^{\mu+1}\kappa^\mu}{r^\mu} ||u||_{L^\infty(S)} \rho(x, z)^\mu.$$

Combining both cases, it follows that u is essentially Hölder continuous in K and, therefore, essentially locally Hölder continuous in Ω.

9. Proofs of results in Subsection 3.3

Proof of Theorems 3.10, 3.11 and of Corollary 3.12. For every $w \in [0, \infty)$ and every $\alpha \in (0, p]$ we have

$$w^\alpha \leq 1 + w^p. \tag{9.1}$$

Now, if $A(x,z,\xi)$ and $B(x,z,\xi)$ satisfy the structural assumptions (1.2) with γ,δ,ψ satisfying (3.14), then by (9.1) they also satisfy the modified structural conditions

$$(9.2)\qquad \begin{cases} (i) \quad A(x,z,\xi) = \sqrt{Q(x)}\tilde{A}(x,z,\xi), \\ (ii) \quad \xi\cdot A(x,z,\xi) \geq a^{-1}\left|\sqrt{Q(x)}\,\xi\right|^{p} - h(x)|z|^{p} - (g(x)+h(x)), \\ (iii) \quad \left|\tilde{A}(x,z,\xi)\right| \leq a\left|\sqrt{Q(x)}\,\xi\right|^{p-1} + b(x)|z|^{p-1} + (b(x)+e(x)), \\ (iv) \quad \left|B(x,z,\xi)\right| \leq c(x)\left|\sqrt{Q(x)}\,\xi\right|^{p-1} + d(x)|z|^{p-1} + (c(x)+d(x)+f(x)) \end{cases}$$

for every $x\in\Omega$, every $z\in\mathbf{R}$ and every $\xi\in\mathbf{R}^n$. Thus, in order to conclude, it is sufficient to apply Theorems 3.5, 3.7 and Corollary 3.9 using the new structural conditions (9.2), i.e. with $\gamma=\delta=\psi=p$ and with e, f, g replaced respectively by $b+e$, $c+d+f$ and $g+h$.

Proof of Theorems 3.13, 3.15 and of Corollary 3.16. As is clear from their proofs, in order to obtain Theorems 3.5, 3.7 and Corollary 3.9, one needs the structural assumptions (1.2) to hold with $\gamma=\delta=p$ and $\psi\in[p,p+1-\sigma^{-1})$ not for every $(x,z,\xi)\in\Omega\times\mathbf{R}\times\mathbf{R}^n$, but only for $(x,z,\xi)=(x,u(x),\nabla u(x))$ for almost every $x\in\Omega$, where u is the weak solution of equation (1.1) under consideration.

If $A(x,z,\xi)$ and $B(x,z,\xi)$ satisfy the structural assumptions (1.2) with $\gamma,\delta,\psi>p$, then we can write

$$(9.3)\qquad \begin{cases} (i) \quad A(x,z,\xi) = \sqrt{Q(x)}\tilde{A}(x,z,\xi), \\ (ii) \quad \xi\cdot A(x,z,\xi) \geq a^{-1}\left|\sqrt{Q(x)}\,\xi\right|^{p} - \big(h(x)|z|^{\gamma-p}\big)|z|^{p} - g(x), \\ (iii) \quad \left|\tilde{A}(x,z,\xi)\right| \leq a\left|\sqrt{Q(x)}\,\xi\right|^{p-1} + \big(b(x)|z|^{\gamma-p}\big)|z|^{p-1} + e(x), \\ (iv) \quad \left|B(x,z,\xi)\right| \leq c(x)\left|\sqrt{Q(x)}\,\xi\right|^{\psi-1} + \big(d(x)|z|^{\delta-p}\big)|z|^{p-1} + f(x). \end{cases}$$

Now, by replacing z with $u(x)$ in (9.3) we can conclude the proof through the application of Theorems 3.5, 3.7 and Corollary 3.9. This is done using the modified structural conditions (9.3) that correspond to (1.2) with $\gamma=\delta=p$ and with h, b, d replaced by $h_1=h|u|^{\gamma-p}$, $b_1=b|u|^{\gamma-p}$ and $d_1=d|u|^{\delta-p}$ respectively. Indeed, note that the map $\mathcal{B}_0\mapsto\frac{p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0}$ is increasing and hence $\frac{p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0}\geq\max\{p'\sigma',\frac{d_0}{p-1}\}$ since $\mathcal{B}_0\geq\max\{\frac{p\sigma}{p\sigma-\sigma-\gamma+1},\frac{d_0p\sigma}{p(p-1)\sigma-d_0(\gamma-p)}\}$. Thus we conclude that

$$\mathcal{B}=\min\left\{\frac{p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0},\mathcal{E}\right\}\geq\max\left\{p'\sigma',\frac{d_0}{p-1}\right\}$$

and, since $\mathcal{B}\leq\mathcal{E}$, we have $e\in L^{\mathcal{B}}_{\rm loc}(\Omega)$. Moreover for any compact subset $\Theta\subset\Omega$ with positive measure

$$\begin{aligned}\|b_1\|_{\mathcal{B},\Theta,\overline{dx}} &\leq \|b_1\|_{\frac{p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0},\Theta,\overline{dx}} = \left(\int_{\Theta} b^{\mathcal{B}}|u|^{(\gamma-p)\mathcal{B}}\,dx\right)^{\frac{1}{\mathcal{B}}}\\ &\leq \left(\int_{\Theta} b^{\frac{p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0}}|u|^{\frac{(\gamma-p)p\sigma\mathcal{B}_0}{p\sigma+(\gamma-p)\mathcal{B}_0}}\,dx\right)^{\frac{p\sigma+(\gamma-p)\mathcal{B}_0}{p\sigma\mathcal{B}_0}}.\end{aligned}$$

By using Hölder's inequality with conjugate exponents $q=\frac{p\sigma+(\gamma-p)\mathcal{B}_0}{p\sigma}$ and $q'=\frac{p\sigma+(\gamma-p)\mathcal{B}_0}{(\gamma-p)\mathcal{B}_0}$ we obtain

$$(9.4)\qquad \|b_1\|_{\mathcal{B},\Theta,\overline{dx}} \leq \left(\int_{\Theta} b^{\mathcal{B}_0}\,dx\right)^{\frac{1}{\mathcal{B}_0}}\left(\int_{\Theta}|u|^{p\sigma}\,dx\right)^{\frac{\gamma-p}{p\sigma}} = \|b\|_{\mathcal{B}_0,\Theta,\overline{dx}}\|u\|^{\gamma-p}_{p\sigma,\Theta,\overline{dx}} < +\infty,$$

and hence $b_1=b|u|^{\gamma-p}\in L^{\mathcal{B}}_{\rm loc}(\Omega)$.

For $\mathcal{H}$ and $\mathcal{D}$ as defined in Theorem 3.13, one can prove in a similar way that $\mathcal{H},\mathcal{D}\geq \frac{d_0}{p}$, that $\mathcal{H},\mathcal{D}>\sigma'$, that $g, h_1 = h|u|^{\gamma-p}\in L^{\mathcal{H}}_{\text{loc}}(\Omega)$, and that $f, d_1 = d|u|^{\delta-p}\in L^{\mathcal{D}}_{\text{loc}}(\Omega)$ with

$$\|h_1\|_{\mathcal{H},\Theta,\overline{dx}} \;\leq\; \|h\|_{\mathcal{H}_0,\Theta,\overline{dx}}\|u\|^{\gamma-p}_{p\sigma,\Theta,\overline{dx}} \qquad\text{and}\qquad \|d_1\|_{\mathcal{D},\Theta,\overline{dx}} \;\leq\; \|d\|_{\mathcal{D}_0,\Theta,\overline{dx}}\|u\|^{\delta-p}_{p\sigma,\Theta,\overline{dx}}. \tag{9.5}$$

Finally, if M is defined as in the statement of Proposition 3.3 (with b,d,h replaced by b_1,d_1,h_1 respectively) and M_2 is as in (3.17), then (9.4) and (9.5) imply that $M\leq M_2$.

Thus we can conclude by applying Theorems 3.5, 3.7 and Corollary 3.9.

10. Appendix

Theorem 10.1. *(Young's Inequality) Let $a_1,a_2,\theta>0$ and $\beta,\beta'\geq 1$ satisfy $\frac{1}{\beta}+\frac{1}{\beta'}=1$. Then*

$$a_1\,a_2 \;\leq\; \theta\,\frac{a_1^{\beta}}{\beta}+\frac{1}{\theta^{\beta'/\beta}}\,\frac{a_2^{\beta'}}{\beta'}. \tag{10.1}$$

Lemma 10.2. *Let (Ω,ρ,dx) be a local homogeneous space and γ^* be as in (2.3). Fix $x,y\in\Omega$, $\lambda\geq 1$, $t>0$, $l\in\mathbf{Z}$, and $k\in\mathbf{N}\cup\{0\}$. Then if $t\leq\lambda^{l+k}<R_1(x)/\gamma^*$ and $B(y,t)\cap B(x,\lambda^{l+k})\neq\emptyset$, we have*

$$|B(y,t)|\leq C_0(\gamma^*\lambda^k)^{d_0}|B(x,\lambda^l)|. \tag{10.2}$$

Proof: The swallowing property (2.3) gives $B(y,t)\subset B(x,\gamma^*\lambda^{l+k})$, and since $\gamma^*\lambda^{l+k}<R_1(x)$, we have that

$$|B(y,t)|\leq|B(x,\gamma^*\lambda^{l+k})|\leq C_0\Big(\frac{\gamma^*\lambda^{l+k}}{\lambda^l}\Big)^{d_0}|B(x,\lambda^l)|=C_0(\gamma^*\lambda^k)^{d_0}|B(x,\lambda^l)|.$$

Proposition 10.3. *Let (Ω,ρ,dx) be a local homogeneous space, see Definition 2.4, let $\Theta\Subset\Omega$ and assume $r_1(x)$ is a function as in (2.13) that satisfies a local uniformity condition in Θ with constant $A_*=A_*(\Theta)$, see (2.6). Then condition weak-D_{q^*}, see Definition 3.17, holds with $q^*=d_0$ on Θ, for some constant $C_7>0$ and with $\alpha=A_*/2$.*

Proof: Since $\Theta\Subset\Omega$ is compact, we can cover it with a finite number of pseudometric balls $B(y_1,r_1(y_1)),\ldots,B(y_P,r_1(y_P))$ with $y_1,\ldots,y_P\in\Theta$. Let $x\in\Theta$, $r\in\big(0,\frac{A_*}{2}r_1(x)\big)$ and choose $y_k\in\{y_1,\ldots,y_P\}$ such that $x\in B(y_k,r_1(y_k))$. Then, conditions (2.6) and (2.13) imply that $0<r<\frac{A_*}{2}r_1(x)<\frac{r_1(y_k)}{2}<R_1(y_k)$. Using (2.4) we conclude that

$$|B(y_k,r_1(y_k))|\leq C_0\left(\frac{r_1(y_k)}{r}\right)^{d_0}|B(x,r)|.$$

It now follows that condition weak-D_{q^*} holds with $q^*=d_0$, $\alpha=A_*/2$ and

$$C_7=\frac{1}{C_0}\min_{k=1,\ldots,N}\left\{\frac{|B(y_k,r_1(y_k))|}{(r_1(y_k))^{d_0}}\right\}.$$

Proof of Proposition 3.3. *Step 1.* We start by recalling that if $x_0\in B\big(y,\frac{\tau}{5\kappa}r_1(y)\big)$, $r\in\big(0,\frac{\tau A_*}{5\kappa C_*}r_1(y)\big)$ and C_* is as in (3.6), then $B(x_0,C_*r)\Subset B(y,r_1(y))$; see Step 1 of the proof of Proposition 3.1. Since by the definition of C_* we have $0<C_*r<r_1(y)\leq R_1(y)$, Definition 2.4 gives

$$|B(y,r_1(y))|\leq C_0\left(\frac{r_1(y)}{C_*r}\right)^{d_0}|B(x_0,C_*r)|.$$

Step 2. We now prove (3.10). By Step 1 and the definition of $k(x_0, r)$,

$$
\begin{aligned}
k(x_0,r) \;=\; & \left[\frac{(C_*r)^{p-1}}{|B(x_0,C_*r)|^{\frac{1}{\mathcal{B}}}}\|e\|_{\mathcal{B},B(x_0,C_*r);dx}\right]^{\frac{1}{p-1}} + \left[\frac{(C_*r)^{p}}{|B(x_0,C_*r)|^{\frac{1}{\mathcal{D}}}}\|f\|_{\mathcal{D},B(x_0,C_*r);dx}\right]^{\frac{1}{p-1}} \\
& + \left[\frac{(C_*r)^{p}}{|B(x_0,C_*r)|^{\frac{1}{\mathcal{H}}}}\|g\|_{\mathcal{H},B(x_0,C_*r);dx}\right]^{\frac{1}{p}} \\
\le\; & \left[\left(\frac{C_0 r_1(y)^{d_0}}{|B(y,r_1(y))|}\right)^{\frac{1}{\mathcal{B}}}(C_*r)^{p-1-\frac{d_0}{\mathcal{B}}}\|e\|_{\mathcal{B},B(y,r_1(y));dx}\right]^{\frac{1}{p-1}} \\
& + \left[\left(\frac{C_0 r_1(y)^{d_0}}{|B(y,r_1(y))|}\right)^{\frac{1}{\mathcal{D}}}(C_*r)^{p-\frac{d_0}{\mathcal{D}}}\|f\|_{\mathcal{D},B(y,r_1(y));dx}\right]^{\frac{1}{p-1}} \\
& + \left[\left(\frac{C_0 r_1(y)^{d_0}}{|B(y,r_1(y))|}\right)^{\frac{1}{\mathcal{H}}}(C_*r)^{p-\frac{d_0}{\mathcal{H}}}\|g\|_{\mathcal{H},B(y,r_1(y));dx}\right]^{\frac{1}{p}}.
\end{aligned}
$$

Thus, by the definitions of λ, Λ and the fact that $C_*r < r_1(y)$,

$$
\begin{aligned}
k(x_0,r) \le\; & \left(\frac{C_0}{|B(y,r_1(y))|}\right)^{\frac{1}{(p-1)\mathcal{B}}} r_1(y)^{1-\lambda}(C_*r)^{\lambda}\|e\|^{\frac{1}{p-1}}_{\mathcal{B},B(y,r_1(y));dx} \\
& + \left(\frac{C_0}{|B(y,r_1(y))|}\right)^{\frac{1}{(p-1)\mathcal{D}}} r_1(y)^{\frac{p}{p-1}-\lambda}(C_*r)^{\lambda}\|f\|^{\frac{1}{p-1}}_{\mathcal{D},B(y,r_1(y));dx} \\
& + \left(\frac{C_0}{|B(y,r_1(y))|}\right)^{\frac{1}{p\mathcal{H}}} r_1(y)^{1-\lambda}(C_*r)^{\lambda}\|g\|^{\frac{1}{p}}_{\mathcal{H},B(y,r_1(y));dx} \;=\; \Lambda r^{\lambda}.
\end{aligned}
$$

Step 3. If $B(x,l) \subseteq B(x_0, C_*r)$ and $0 < l \le C_*r$, then $B(x,l) \Subset B(y, r_1(y))$ and $0 < l < r_1(y) \le R_1(y)$. As in Step 1 we conclude that

$$|B(y,r_1(y))| \le C_0\left(\frac{r_1(y)}{l}\right)^{d_0}|B(x,l)|.$$

Also note that since r_1 satisfies a local uniformity condition on $B(y, r_1(y))$ with respect to ρ with constant A_* and since $x_0 \in B(y, r_1(y))$, then $A_*r_1(y) < r_1(x_0)$. Thus, from

$$0 < l \le C_*r < A_*r_1(y) < r_1(x_0) \le R_1(x_0)$$

and $B(x,l) \subset B(x_0, C_*r)$, we deduce by Definition 2.4 that

$$|B(x_0,C_*r)| \le C_0\left(\frac{C_*r}{l}\right)^{d_0}|B(x,l)|.$$

Step 4. We are now going to prove that $\bar{Z}(B(x,l), \bar{u}) \le M$, with M as in the conclusion of Proposition 3.3. Using the definition of $\bar{Z}$ given in (3.4) and equations (3.3), we have

$$
\begin{aligned}
\text{(10.3)}\qquad \bar{Z}(B(x,l),\bar{u}) \le\; & 1 + l^{p-1}\|b\|_{p'\sigma',B(x,l);\overline{dx}} + \frac{l^{p-1}}{k^{p-1}}\|e\|_{p'\sigma',B(x,l);\overline{dx}} \\
& + \left(l^{p}\|c^{\frac{p}{p+1-\psi}}\bar{u}^{\frac{p(\psi-p)}{p+1-\psi}}\|_{\frac{p\sigma'}{p-\epsilon_1},B(x,l);\overline{dx}}\right)^{\frac{1}{\epsilon_1}} \\
& + \left(l^{p}\|h\|_{\frac{p\sigma'}{p-\epsilon_2},B(x,l);\overline{dx}} + \frac{l^{p}}{k^{p}}\|g\|_{\frac{p\sigma'}{p-\epsilon_2},B(x,l);\overline{dx}}\right)^{\frac{1}{\epsilon_2}} \\
& + \left(l^{p}\|d\|_{\frac{p\sigma'}{p-\epsilon_3},B(x,l);\overline{dx}} + \frac{l^{p}}{k^{p-1}}\|f\|_{\frac{p\sigma'}{p-\epsilon_3},B(x,l);\overline{dx}}\right)^{\frac{1}{\epsilon_3}}.
\end{aligned}
$$

Step 5. Recalling the conditions on $\mathcal{B}$ and by Step 3, we have by Hölder's inequality that

$$
\begin{aligned}
l^{p-1}\|b\|_{p'\sigma',B(x,l);\overline{dx}} &\leq l^{p-1}\|b\|_{\mathcal{B},B(x,l);\overline{dx}} = \frac{l^{p-1}}{|B(x,l)|^{\frac{1}{\mathcal{B}}}}\|b\|_{\mathcal{B},B(x,l);dx} \\
&\leq \frac{C_0^{\frac{1}{\mathcal{B}}} r_1(y)^{\frac{d_0}{\mathcal{B}}} l^{p-1-\frac{d_0}{\mathcal{B}}}}{|B(y,r_1(y))|^{\frac{1}{\mathcal{B}}}}\|b\|_{\mathcal{B},B(y,r_1(y));dx} \\
&\leq C_0^{\frac{1}{\mathcal{B}}} r_1(y)^{p-1}\|b\|_{\mathcal{B},B(y,r_1(y));\overline{dx}}.
\end{aligned}
$$

The terms including norms of h and d are treated in a similar way, also recalling the definitions of ϵ_2, ϵ_3. Thus we obtain

$$
\begin{aligned}
l^p\|h\|_{\frac{p\sigma'}{p-\epsilon_2},B(x,l);\overline{dx}} &\leq C_0^{\frac{1}{\mathcal{H}}} r_1(y)^p\|h\|_{\mathcal{H},B(y,r_1(y));\overline{dx}}, \\
l^p\|d\|_{\frac{p\sigma'}{p-\epsilon_3},B(x,l);\overline{dx}} &\leq C_0^{\frac{1}{\mathcal{D}}} r_1(y)^p\|d\|_{\mathcal{D},B(y,r_1(y));\overline{dx}}.
\end{aligned}
$$

Step 6. Again using the conditions on $\mathcal{B}$, Step 3 and the definition of $k = k(x_0, r)$, we have

$$
\begin{aligned}
\frac{l^{p-1}}{k^{p-1}}\|e\|_{p'\sigma',B(x,l);\overline{dx}} &\leq \frac{l^{p-1}}{(C_*r)^{p-1}\|e\|_{\mathcal{B},B(x_0,C_*r);\overline{dx}}}\|e\|_{\mathcal{B},B(x,l);\overline{dx}} \\
&= \frac{l^{p-1}}{(C_*r)^{p-1}}\frac{|B(x_0,C_*r)|^{\frac{1}{\mathcal{B}}}}{\|e\|_{\mathcal{B},B(x_0,C_*r);dx}}\frac{\|e\|_{\mathcal{B},B(x,l);dx}}{|B(x,l)|^{\frac{1}{\mathcal{B}}}} \\
&\leq \frac{l^{p-1}}{(C_*r)^{p-1}}\frac{|B(x_0,C_*r)|^{\frac{1}{\mathcal{B}}}}{|B(x,l)|^{\frac{1}{\mathcal{B}}}} \leq C_0^{\frac{1}{\mathcal{B}}}\left(\frac{l}{C_*r}\right)^{p-1-\frac{d_0}{\mathcal{B}}} \leq C_0^{\frac{1}{\mathcal{B}}}.
\end{aligned}
$$

The terms involving g and f can be estimated in a similar way, giving

$$
\frac{l^p}{k^p}\|g\|_{\frac{p\sigma'}{p-\epsilon_2},B(x,l);\overline{dx}} \leq C_0^{\frac{1}{\mathcal{H}}} \qquad \text{and} \qquad \frac{l^p}{k^{p-1}}\|f\|_{\frac{p\sigma'}{p-\epsilon_3},B(x,l);\overline{dx}} \leq C_0^{\frac{1}{\mathcal{D}}}.
$$

Step 7. We estimate the remaining term

$$
I := l^p\|c^{\frac{p}{p+1-\psi}}\bar{u}^{\frac{p(\psi-p)}{p+1-\psi}}\|_{\frac{p\sigma'}{p-\epsilon_1},B(x,l);\overline{dx}} = l^p\left(\fint_{B(x,l)} c^{\frac{p^2\sigma'}{(p+1-\psi)(p-\epsilon_1)}}\,\bar{u}^{\frac{p^2\sigma'(\psi-p)}{(p+1-\psi)(p-\epsilon_1)}}\,dx\right)^{\frac{p-\epsilon_1}{p\sigma'}}
$$

starting with an application of Hölder inequality with conjugate exponents

$$
q = \frac{(\sigma-1)(p+1-\psi)(p-\epsilon_1)}{p(\psi-p)} > 1, \qquad q' = \frac{(\sigma-1)(p+1-\psi)(p-\epsilon_1)}{(\sigma-1)(p+1-\psi)(p-\epsilon_1)-p(\psi-p)},
$$

where we will associate q with $\bar{u}$ and q' with c; note also that $q > 1$ due to the definition of ϵ_1 (see Proposition 3.3). Thus, also recalling the conditions on c, we obtain

$$
\begin{aligned}
I &\leq l^p\left(\fint_{B(x,l)} c^{\frac{p^2\sigma}{(\sigma-1)(p+1-\psi)(p-\epsilon_1)-p(\psi-p)}}\,dx\right)^{\frac{(\sigma-1)(p+1-\psi)(p-\epsilon_1)-p(\psi-p)}{p\sigma(p+1-\psi)}}\left(\fint_{B(x,l)}\bar{u}^{\sigma p}dx\right)^{\frac{\psi-p}{(p+1-\psi)\sigma}} \\
&= l^p\|c\|^{\frac{p}{p+1-\psi}}_{\frac{p^2\sigma}{(\sigma-1)(p+1-\psi)(p-\epsilon_1)-p(\psi-p)},B(x,l);\overline{dx}}\|\bar{u}\|^{\frac{p(\psi-p)}{(p+1-\psi)}}_{p\sigma,B(x,l);\overline{dx}} \\
&\leq l^p\|c\|^{\frac{p}{p+1-\psi}}_{\mathcal{C},B(x,l);\overline{dx}}\|\bar{u}\|^{\frac{p(\psi-p)}{(p+1-\psi)}}_{p\sigma,B(x,l);\overline{dx}} = \frac{l^p}{|B(x,l)|^{\frac{p}{(p+1-\psi)\mathcal{C}}+\frac{\psi-p}{\sigma(p+1-\psi)}}}\|c\|^{\frac{p}{p+1-\psi}}_{\mathcal{C},B(x,l);dx}\|\bar{u}\|^{\frac{p(\psi-p)}{(p+1-\psi)}}_{p\sigma,B(x,l);dx},
\end{aligned}
$$

where the last inequality follows from the second part of the minimum in the definition of ϵ_1. Thus, by the first display in Step 3,

$$\begin{aligned} I &\leq \frac{l^{p-\frac{d_0 p}{(p+1-\psi)\mathcal{C}}-\frac{d_0(\psi-p)}{\sigma(p+1-\psi)}}\left(C_0 r_1(y)^{d_0}\right)^{\frac{p}{(p+1-\psi)\mathcal{C}}+\frac{\psi-p}{\sigma(p+1-\psi)}}}{|B(y,r_1(y))|^{\frac{p}{(p+1-\psi)\mathcal{C}}+\frac{\psi-p}{\sigma(p+1-\psi)}}}\|c\|_{\mathcal{C},B(y,r_1(y));dx}^{\frac{p}{p+1-\psi}}\|\bar{u}\|_{p\sigma,B(y,r_1(y));dx}^{\frac{p(\psi-p)}{(p+1-\psi)}} \\ &\leq \frac{r_1(y)^p C_0^{\frac{p\sigma+(\psi-p)\mathcal{C}}{(p+1-\psi)\sigma\mathcal{C}}}}{|B(y,r_1(y))|^{\frac{p\sigma+(\psi-p)\mathcal{C}}{(p+1-\psi)\sigma\mathcal{C}}}}\|c\|_{\mathcal{C},B(y,r_1(y));dx}^{\frac{p}{p+1-\psi}}\left[\|u\|_{p\sigma,B(y,r_1(y));dx}+k(x_0,r)|B(y,r_1(y))|^{\frac{1}{p\sigma}}\right]^{\frac{p(\psi-p)}{(p+1-\psi)}}, \end{aligned}$$

where we have used the facts that $l < r_1(y)$ and $p-\frac{d_0 p}{(p+1-\psi)\mathcal{C}}-\frac{d_0(\psi-p)}{\sigma(p+1-\psi)} \geq 0$, due to the first condition on $\mathcal{C}$ in item (iv) of Proposition 3.3. Finally, since $C_* r < r_1(y)$, Step 2 applies to $k(x_0,r)$ and we have

$$I \leq r_1(y)^p C_0^{\frac{p\sigma+(\psi-p)\mathcal{C}}{(p+1-\psi)\sigma\mathcal{C}}}\|c\|_{\mathcal{C},B(y,r_1(y));\overline{dx}}^{\frac{p}{p+1-\psi}}\left[\|u\|_{p\sigma,B(y,r_1(y));\overline{dx}}+\Lambda r_1(y)^{\lambda}\right]^{\frac{p(\psi-p)}{(p+1-\psi)}}.$$

Step 8. It is now sufficient to insert the estimates from Steps 5,6,7 into inequality (10.3) to conclude the proof.

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