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Operational Calculus Approach for Obtaining Periodic and Mean-periodic Solutions of Linear Ordinary Differential Equations with Constant Coefficients

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Abstract. An approach to obtaining periodic and mean-periodic solutions of Linear Ordinary Differential Equation (LODE) with constant coefficients is presented. The use of the Computer Algebra System(CAS) Mathematica for practical application of this approach is considered.
Keywords: Operational calculus, convolution, Duhamel principle, initial value problem, boundary value problem, linear ordinary differential equation with constant coefficients, computer algebra system.

1 The Operational Calculus Approach and Its Application

The main features of the operational calculus approach are presented. Its application for solving problems related to some classes of differential equations is considered.

The main characteristics of the program packages developed with use of the CAS Mathematica and supporting such application are briefly described. They provide the use of:

- the Heaviside algorithm for solving initial value problems for LODE with constant coefficients;
- an extension of the Heaviside algorithm to a class of boundary value problems for LODE with constant coefficients, connected with the problems of obtaining periodic solutions of LODE both in the non-resonance and the resonance cases; the obtaining of mean-periodic solutions of LODE with constant coefficients using such an approach is outlined;

The features of the program tools implemented with use of the CAS Mathematica[20] are briefly described.

1.1 About the Operational Calculus

The essence of the operational calculus consists in transformation of calculus problems to algebraic problems, treating the differentiation operator as an algebraic object.

Some ideas of "symbolic" operational calculus come from the works of Leibnitz, Euler, Cauchy and other mathematicians (see [18], [15], and also [19]). Nevertheless, Oliver Heaviside (1850–1925) is regarded to be the father of the operational calculus. He was the first who successfully applied this method in his research for solving initial value problems related to electromagnetic theory (see [9]). But Heaviside did not established a sound mathematical theory and his calculus was regarded by some scientists as inconsistent. The first justification of his approach was done by means of the Laplace transformation. Quite later – in the middle of the last century – the Polish mathematician Jan Mikusiński (1913–1987) made a return to the original operator viewpoint and developed a direct algebraic approach to the Heaviside operational calculus. He based his calculus on the notion of convolution quotient, without referring it to the Laplace transformation. His calculus is known as Mikusiński's operational calculus. From historical point of view, it is fair to call it as operational calculus of Heaviside – Mikusiński.

Scientists in many countries have published works related to the operational calculus of Mikusiński. Some of them are L. Berg, T.K.Boehme, I.H.Dimovski, V.A. Ditkin, A.P.Prudnikov, K. Yosida, etc. Other names are mentioned in some references, for example in [15]. Some recent results can be found in [21], and others.

Mainly the results of I.H. Dimovski ([2], [3]) on development of operational calculi of Mikusiński's type are considered below.

The operational calculus has been widely used for solving problems in mathematics, physics, mechanics, electrical engineering, etc. The algorithms and the program tools described here are intended to facilitate the use of the operational calculus approach by means of computer.

1.2 Heaviside Algorithm

Since the main idea of the Operational Calculus (OC) of Oliver Heaviside is the conversion of differential equations to algebraic equations by treating of the differentiation operator as an algebraic object, an algorithm for doing that is needed.

The so called Heaviside algorithm based on the operational calculus approach is intended for solving initial value problems for linear ordinary differential equations with constant coefficients. We use it in the frames of Mikusiński's operational calculus.

A description and implementation of the Heaviside algorithm using a CAS are also considered. Special attention is paid to the features making this implementation efficient. Illustrative examples are included The Heaviside algorithm for solving initial value problems for LODE with constant coefficients in the frames of the Mikusinski's operational calculus is described.

The most important role in the Mikusiński's operational calculus plays the classical Duhamel convolution (see [12]):

$$(f * g) = \int_0^t f(t-\tau)g(\tau)d\tau, \tag{1}$$

in the space $\mathcal{C}[0,\infty)$ of the continuous functions on $[0,\infty)$. Mikusiński considers this space as a ring on $\mathbb{R}$ or $\mathbb{C}$. He uses the fact that due to a famous theorem of Titchmarsh the operation (1) has no divisors of zero and hence $(\mathcal{C}[0,\infty), *)$ is an integrity domain. In the same way, as the ring $\mathbb{Z}$ of the integers is extended to the field $\mathcal{Q}$ of the rational numbers, he extends the ring $(\mathcal{C}[0,\infty)*)$ to the smallest field $\mathcal{M}$ containing the initial ring. This field is called Mikusiński's field and it is denoted it by $\mathcal{M}$. The elements of $\mathcal{M}$ are convolution fractions $\dfrac{f}{g} = \dfrac{\{f(t)\}}{\{g(t)\}}$, called "operators".

In Mikusiński's calculus each function $f : [0,\infty) \to \mathbb{R}$ is considered as an algebraic object and the notation $f = \{f(x)\}$ is used.

Basic operators in the Mikusiński approach are the integration operator l: $lf(t) = \int_0^t f(\tau)d\tau$, and the algebraic analogon $s = \dfrac{1}{l}$ of the differentiation operator $\dfrac{d}{dt}$.

The relation between the derivative $f'(t)$ and the product $s\{f(t)\}$ is presented by the basic formula of the Mikusiński operational calculus

$$\{f'(t)\} = s\{f(t)\} - f(0), \tag{2}$$

where $f \in C^1[0,\infty)$ and $f(0)$ is considered as a "numerical operator". If a function $f = \{f(t)\}$ has continuous derivatives to n-th order for $0 \le t < \infty$, a more general formula can be derived:

$$f^{(n)} = s^n f - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0), \; n = 1, 2, 3, \ldots \tag{3}$$

1.3 Solving Initial Value Problems for Linear Ordinary Differential Equation Using the Heaviside Algorithm

Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a non-zero polynomial of n-th degree with real or complex coefficients.

Consider the following initial value problem:

$$P\left(\frac{d}{dt}\right) y = f(t),\, y(0) = \gamma_0,\, y'(0) = \gamma_1,\, \cdots,\, y^{(n-1)}(0) = \gamma_{n-1}. \tag{4}$$

Using the main formula (2)–(3) of the operational calculus of Mikusinski, an "algebraization" of the problem can be made. The problem (4) reduces to the following single algebraic equation of first degree:

$$P(s)\,y = f + Q(s), \tag{5}$$

with $P(s) = \sum\limits_{j=1}^{n} a_j\, s^j,\quad Q(s) = \sum\limits_{j=1}^{n}\left(\sum\limits_{k=j}^{n} a_{n-k}\,\gamma_{k-j}\right)s^{j-1}$, $\deg Q < \deg P$.

The formal solution has the form

$$y = \frac{1}{P(s)}f + \frac{Q(s)}{P(s)}. \tag{6}$$

It can be interpreted as a functional solution if we decompose $1/P(s)$ and $Q(s)/P(s)$ in elementary fractions and interpret these fractions as functions using the formula (see [12]):

$$\frac{1}{(s-\alpha)^n} = \left\{\frac{t^{n-1}}{(n-1)!}\,e^{\alpha\,t}\right\},\ n = 1, 2, \ldots \tag{7}$$

Thus we represent $1/P(s)$ and $Q(s)/P(s)$ as functions:

$$G(t) = 1/P(s),\ \ R(t) = Q(s)/P(s) \tag{8}$$

and the solution takes the form

$$y(t) = G(t) * f(t) + R(t) \tag{9}$$

At last the computation of the convolution product denoted by $*$ in (9) has to be performed.

The solution of an initial value problem for simultaneous ordinary linear differential equations with constant coefficients can be performed in a similar way: algebraization of the problem and reducing it to a system of linear algebraic equations; solving the obtained system using linear algebra methods and functional interpretation of the solution.

1.4 Program Implementation of the Heaviside algorithm

A program implementation of the Heaviside algorithm would allow it to be used by means of computer. Having in mind the kind of the operations of this algorithm and the capabilities of the computer algebra system Mathematica, this system was chosen for development of a program package implementing the Heaviside algorithm.

Information about a full program implementation of the Heaviside algorithm was not found.

Main steps of the algorithm

Formulating once again the steps of the Heaviside algorithm, the features of their program implementation are considered below.

Step 1. *Algebraization of the problem.* The language tools of *Mathematica* allow the transformation of (4) into (5) to be made in a convenient way, using rules for presentation of the formulae (2)–(3) and for the initial conditions of (4).

Step 2. *Solution of the algebraic equation* (5). A polynomial equation (or a system of such equations) has to be solved and *Mathematica* provides such capabilities. The result is obtained in the form (6).

Step 3. *Factorization of the polynomial $P(s)$ and partial fraction decomposition of $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$.* The built-in function named *Factor* of *Mathematica* is used; a presentation of $P(s)$ as a product of factors, each of which is a polynomial of first or second degree, raised to an integer positive number, is obtained. More details are described in [17] and [4]. This process may not finish with success if some of the coefficients of $P(s)$ are parameters and in the same time $\deg P > 4$. In this case the solution of problem (4) is aborted. If the factorization of P is finished successfully, the *Mathematica* function *Apart* represents the rational expressions $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$ as sums of terms with minimal denominators of minimal degrees.

Step 4 *Interpretation of the rational expressions $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$.* Each fraction in these expressions has to be interpreted as a function by means of formulae, such as (7). The main part of the Mikusinski's table is used. The formulae are presented as *Mathematica* rules with appropriate pattern matching. An uniform interpretation of all fractions is obtained. As a result, the presentations (8)–(9) are achieved.

Step 5. *Computation of the Duhamel convolution in the final form of the solution.* The *Mathematica* integratoris used.

Step 6. *Showing the result: solution or a message that the problem can not be solved.* It was mentioned above when the problem will not be solved in case of one equation. In case of solving initial value problem for a system of equations, similar situation may occur, but, in addition, the problem will not be solved if on Step 2 the algebraic system has not solution.

1.5 Program Package for the Heaviside Algorithm

An implementation of the Heaviside algorithm following the steps described above, is developed as a *Mathematica* program package. Its main function *DSolveOC* defines the performance of all steps of the Heaviside algorithm. The call of this function is similar to the call of the *Mathematica* function *DSolve*. The output also has similar form. The solution is presented as a rule or as a list of rules in case of several solutions. The use of options for visualization of the solution and for some additional capabilities is provided.

Illustrative examples

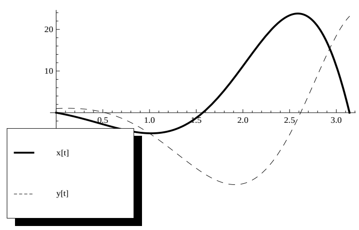
With the following two examples we illustrate the use of the main function *DSolveOC* of the package. The solutions, of two initial value problems - for linear ordinary differential equation and for a system of two linear ordinary differential equations are shown.

Example 1. Initial value problem for one LODE with constant coefficients:

```
task1 = {x'''[t] + x'[t] == e^(2t), x[0] == 0, x'[0] == 0, x''[0] == 0}
{x'[t] + x^(3)[t] == e^(2t), x[0] == 0, x'[0] == 0, x''[0] == 0}
DSolveOC[task1, x[t], t]
DSolveOC[{x'''[t] + x'[t] == e^(2t), x[0] == 0, x'[0] == 0, x''[0] == 0}, x[t], t]
x[t] -> 1/10 (-5 + e^(2t) + 4 Cos[t] - 2 Sin[t])
```

Example 2. Initial value problem for a system of LODE with constant coefficients; an option for visualization of the solution is used.

```
mysyst = {-x[t] + 2 y[t] + x'[t] == -2 e^t, -2 x[t] - y[t] + y'[t] == 0, x[0] == 0, y[0] == 1}
DSolveOC[mysyst, {x[t], y[t]}, t, GraphInterval -> {0, π}]
{x[t] -> -4 e^t Cos[t] Sin[t],
 y[t] -> e^t (-1 + 2 Cos[2 t])}
```



Concluding Remarks

- The Heaviside algorithm gives a closed form solution of an initial value problem for a linear ordinary differential equation or a system of such equations in a direct way, without trying to find partial and general solution.
- An uniform approach is used for homogenous and for non-homogenous equations.
- No special requirements to the right-hand side function are posed (as in the case of Laplace transformation).
- In the Heaviside algorithm the initial value conditions are supposed to be given in the point 0. It is easy to develop an extension of the algorithm allowing the initial value conditions to be given in point $t_0 \neq 0$.
- For solving an initial value problem for a system of ordinary linear differential equations with constant coefficients, all steps of the Heaviside algorithm can be performed in a similar way, as in case of solving initial value problem for one equation.

The presented implementation of the Heaviside algorithm is considered in more details in [4] and [17].

2 Extension of the Heaviside Algorithm to a Class of Boundary Value Problems for LODE with Constant Coefficients. Periodic Solutions of Such Equations

An approach to obtaining periodic and mean-periodic solutions of LODE with constant coefficients is presented.

2.1 An Auxiliary Boundary Value Problem.

An extension of the Heaviside – Mikusiński operational calculus is developed by I. Dimovski and S. Grozdev (see [2], [7], [8]) and in the framework of this operational calculus an extension of the Heaviside algorithm is proposed. It is intended for solving nonlocal initial value problems for LODE with constant coefficients. This approach is used for obtaining periodic solutions of such equations [16].

Let's consider a non-zero polynomial with constant coefficients of degree n:
$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$
and the following ordinary linear differential equation with constant coefficients:

$$P\left(\frac{d}{dt}\right) y = f(t), \ \ -\infty < \ t \ < \infty \tag{10}$$

We are looking for a periodic solution $y(t)$ with period T of this equation, i.e. a solution satisfying the identity:

$$y(t+T) \ = \ y(t), \ \ -\infty \ < \ t \ < \infty \tag{11}$$

An obvious necessary condition for the existance of a periodic solution of (10) with period T is the function $f(t)$ to be periodic with period T, i.e. for each $t \in \mathbb{R}$ the following condition to be satisfied:

$$f(t+T) \ = \ f(t) \tag{12}$$

.

The following **Theorem** could be proven: *A solution of* (10) *with periodic right-hand side* $f(t)$ *with period* T *is* T*-periodic if and only if the following "boundary" conditions are satisfied:*

$$y(T) - y(0) \ = \ 0, \ y'(T) - y'(0) \ = \ 0, \ \ldots \ y^{(n-1)}(T) - y^{(n-1)}(0) \ = \ 0 \tag{13}$$

This theorem allows the problem of obtaining periodic solutions of (10) to be reduced to the problem of finding a solution of this equation in the interval $(-\infty, \infty)$, satisfying the "boundary" conditions (13).

Further we reduce this problem to the following intermediate (auxiliary) boundary-value problem:

$$\begin{aligned} & P\left(\frac{d}{dt}\right) y = f(t), \ \ -\infty < t < \infty \\ & \int_0^T y\,(\tau)\,d\tau \ = \ \alpha_0, \ y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \ \ k = 0, 1, \ldots n-2. \end{aligned} \tag{14}$$

2.2 An Operational Method for Solving the Auxiliary Problem. Convolution of Dimovski.

The Heaviside algorithm is developed for solving initial value problems for LODE with constant coefficients and it can not be used directly for finding periodic solutions of such equations.

Use of Fourier transform and Laplace transform for obtaining periodic solutions can be found in some works of Kaplan [10], Rosenvasser ([13], [14]), Lurie [11] and some others. We use an alternative direct approach, similar to those of Mikusiński, but using another convolution, based on the operational calculus of Dimovski (see [2]) and related to the nonlocal boundary value problem in $C(\mathbb{R})$:

$$y' = f(x), \int_0^T y(\tau)d\tau = 0,$$

where T is a constant.

The solution

$$L\,f(t) = \int_0^t f(\tau)d\tau - \frac{1}{T}\int_0^T \left(\int_0^\tau f(\sigma)d\sigma\right)d\tau$$

is an analogue of the integration operator $lf(t) = \int_0^t f(\tau)d\tau$ of Mikusiński's operational calculus.

The operational calculus of Dimovski for the operator L is an analogue of the operational calculus of Mikusiński, but the following convolution of Dimovski is used:

$$(f \overset{t}{*} g)(t) = \Phi_\tau\{\int_\tau^t f(t+\tau-\sigma)g(\sigma)d\sigma\},$$

with an arbitrary linear functional Φ in $C(\mathbb{R})$. In this case the functional $\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)d\tau$ is used. The convolution

$$(f \overset{t}{*} g)(t) = \frac{1}{T}\int_0^T \left(\int_\tau^t f(t+\tau-\sigma)g(\sigma)d\sigma\right)d\tau$$

has the property $Lf(t) = \{1\}\overset{t}{*} f$.

Dimovski and Grozdev proposed a simpler convolution (without using of repeated integrals):

$$\begin{aligned}(f * g)(t) &= \frac{f(t)}{T}\int_0^T g(\tau)\,d\tau + \frac{g(t)}{T}\int_0^T f(\tau)\,d\tau \\ &-\frac{1}{T}\int_0^t f(t-\tau)\,g(\tau)\,d\tau - \frac{1}{T}\int_t^T f(t+T-\tau)\,g(\tau)\,d\tau,\end{aligned} \tag{15}$$

for which $\{1\}\overset{t}{*} f = f$.

The constant function $\{1\}$ plays the role of a unity in the convolution algebra $(\mathcal{C}(I\!R), *)$. The operator L has the following representation:

$$L\{1\} = t - \frac{T}{2}\,, \text{ i.e. } Lf = \left\{t - \frac{T}{2}\right\} \overset{t}{*} f.$$

Further, convolution fractions of the form f/g are considered (with $f,\, g \in C[0, T]$, g being a nondivisor of 0 of the operation (15)). The ring of the continuous functions on $(-\infty,\, \infty)$ is extended to the smallest ring $\mathcal{M}$, containing the convolution fractions $\dfrac{f}{g}$ with denominators which are nondivisors of 0. The most important convolution fraction

$$S = \frac{1}{L}$$

is considered as an algebraic analogue of d/dt.

The basic formula of the Operational Calculus of Dimovski is:

$$\{f'(t)\} = S\,\{f(t)\} - \frac{1}{T}\int\limits_0^T f(\tau)d\tau. \tag{16}$$

Here $\dfrac{1}{T}\displaystyle\int\limits_0^T f(\tau)d\tau$ is considered as a constant function.

For $f^{(n)}$ the following formula can be derived from (16):

$$f^{(n)} = S^n f - \frac{S^n}{T}\int_0^T f(\tau)d\tau - \sum_{k=1}^{n-1} \frac{S^k}{T}\,\left(f^{(n-1-k)}(T) - f^{(n-1-k)}(0)\right) \tag{17}$$

For the case $T = 1$, the integral operator L is called by Dimovski and Grozdev **Bernoullian integration operator** due to the following relation with the polynomials of Bernoulli:

$$L^n\{1\} = \frac{T^n}{n!}\,B_n\left(\frac{t}{T}\right),\ n = 0,\, 1,\, 2,\, \ldots,$$

where $B_n\,(t)$ is the polynomial of Bernoulli of degree n.

Further the scheme of Mikusiński has to be followed, using the convolution (15) and taking into account the following differences:

1) The operation (15) has a unit element.
2) This operation has divisors of 0.

The eigenfunctions of L are divisors of 0 of (15). These functions have the form $\varphi_n(t) = Ce^{\frac{2\pi int}{T}},\ n \in Z\!\!\!Z \setminus \{0\}$.

For the application of the new operational calculus it is important we to have formulae for convolution fractions of the type $\dfrac{1}{(S-\lambda)^k}$, $k \in I\!N$. They exist iff $S - \lambda$ is a nondivisor of 0 and this is not truth iff $\lambda = \dfrac{2\pi in}{T}$ and $n \in Z\!\!\!Z \setminus \{0\}$.

Thus for each $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z}\setminus\{0\}$ the following formulae hold:

$$\frac{1}{S-\lambda} = -\frac{1}{\lambda} + \frac{T\, e^{t\,\lambda}}{e^{\lambda\,T}-1} \tag{18}$$

$$\frac{S}{S-\lambda} = \frac{T\,\lambda\, e^{t\,\lambda}}{e^{\lambda\,T}-1} \tag{19}$$

Corollary. If $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z}\setminus\{0\}$, more general formulae hold (for each integer $k \geq 1$):

$$\frac{1}{(S-\lambda)^k} = \frac{1}{(k-1)!}\,\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(-\frac{1}{\lambda} + \frac{T\, e^{t\,\lambda}}{e^{\lambda\,T}-1}\right) \tag{20}$$

$$\frac{S}{(S-\lambda)^k} = \frac{1}{(k-1)!}\,\frac{\partial^{k-1}}{\partial\lambda^{k-1}}\left(\frac{T\,\lambda\, e^{t\,\lambda}}{e^{\lambda\,T}-1}\right) \tag{21}$$

The formulae (18)–(21) are intended to be used for interpretation of rational expressions in the extended Heaviside algorithm. For the purposes of the program implementation of this algorithm additional formulae were derived–for the case when the denominator is an integer power of a second degree polynomial.

Non-resonance case. Let's apply the Operational Calculus of Dimovski for solving the auxiliary problem, formulated above:

$$\begin{aligned} &P\left(\frac{d}{dt}\right) y = f(t), \quad -\infty < t < \infty \\ &\int_0^T y\,(\tau)\, d\tau \;=\; \alpha_0,\; y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1},\;\; k = 0, 1, \ldots n-2. \end{aligned} \tag{22}$$

Using the formulae (16)–(17), we can make an "algebraization" of the problem, thus reducing it to one algebraic equation of 1^{st} degree:

$$P(S)y = f + S\,Q(S), \tag{23}$$

where $P(S)$ and $Q(S)$ are polynomials of S and the degree of $Q(S)$ is less than the degree of $P(S)$.

The formal solution of the above equation has the form

$$y = \frac{1}{P(s)} f + S\frac{Q(s)}{P(s)}. \tag{24}$$

The above representation contains division by $P(S)$ and this is possible if $P(S)$ is not a divisor of 0 in $\mathcal{M}$, i.e. iff $P\left(\dfrac{2\pi i m}{T}\right) \neq 0$ for each $m \in \mathbb{Z}\setminus\{0\}$. This is the so–called **non-resonance case**.

Main steps of the extended Heaviside algorithm for solving the intermediate problem in the non-resonance case:

1) Finding the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$.
2) Finding out that none of the roots have the form $\dfrac{2\pi i m}{T}$ with $m \in \mathbb{Z} \setminus \{0\}$.
3) Finding the polynomial $Q(S)$.
4) Expanding $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ into a sum of partial fractions.
5) Interpretation of the fractions $w = \dfrac{1}{P(S)}$ and $v = S\dfrac{Q(S)}{P(S)}$ as functions.
6) Representation of the solution in the form $u = w * f + v$.

A comparison with the classical Heaviside algorithm:

– for algebraization of the problem the formulae (16)-(17) are used now.

– we have here an additional step (step 2);

– new interpretation formulae (such as (18)-(21)) are used on step 5);

– the operation $*$ on step 6) is not the Duhamel convolution; it is the convolution (15).

Resonance case.

If the above condition $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z} \setminus \{0\}$ fails for one or more roots of P, we have the so-called **resonance case** and the corresponding roots are called resonance roots.

Let's denote with $n_1, n_2, \ldots, n_p$ all integer numbers, for which $P\left(\dfrac{2\pi i n_k}{T}\right) = 0$, $k = 1, 2, \ldots, p$, and let $C_{n_1, n_2, \ldots, n_p}$ be the subalgebra of $(C(\mathbb{R}), *)$, such that the convolution (15) plays the role of multiplication in it. It was mentioned above that the eigenfunctions of the operator L have the form $\varphi_n(t) = e^{\frac{2\pi i n t}{T}}$, $n \in \mathbb{Z} \setminus \{0\}$. It is shown in [8] that if $f \in C[0, T]$, then

$$f * \{e^{\frac{2\pi i n t}{T}}\} = \chi_n(f)\, e^{\frac{2\pi i n t}{T}},\ n = \pm 1, \pm 2, \ldots,$$

where

$$\chi_n(f) = \frac{1}{T}\int_0^1 (e^{\frac{2\pi i n t}{T}} - 1) f(t) dt,\ n = \pm 1, \pm 2, \ldots, \tag{25}$$

is a complete system of multiplicative functionals. We call them Fourier coefficients of f with respect to $\left\{e^{\frac{2\pi i n t}{T}}\right\}$, $n \in \mathbb{Z} \setminus \{0\}$.

Due to a theorem proven in the above cited paper, at least one of the Fourier coefficients of the function f has to be equal to zero in order this function to be a divisor of 0 in the algebra $(C(\mathbb{R}), *)$. One can prove that this condition is necessary as well.

Let's denote by $\tilde{L}$ the restriction of the operator L to $C_{n_1,\, n_2,\, \ldots,\, n_p}$. Then instead of $Lf \,=\, r * f$, for $r(t) \,=\, t - \dfrac{T}{2}$ in $[0, T]$, the following presentation in $C_{n_1,\, n_2,\, \ldots,\, n_p}$ will hold: $\tilde{L}f \,=\, \tilde{r} * f$, where

$$\tilde{r}(t) \,=\, r(t) - \sum_{k=1}^{p} \chi_{n_k}(r) e^{\frac{2\pi i n_k t}{T}} \,=\, t - \frac{T}{2} - \sum_{k=1}^{p} \frac{T}{2\pi i n_k t} e^{\frac{2\pi i n_k t}{T}}.$$

We denote by $\mathcal{M}_{n_1,\, n_2,\ldots,n_p}$ the ring of the convolution fractions of $C_{n_1,\, n_2,\, \ldots,\, n_p}$, whose denominators are nondivisors of 0 of the convolution (15).Denote the algebraic inverse element of $\tilde{L}$ by $\tilde{S}$, i.e. $\tilde{S} \,=\, \dfrac{1}{\tilde{L}}$.

Two important theorems, proven by Dimovski and Grozdev, are denoted here by T1 and T2, respectively:

T1. The elements $\tilde{S} - \dfrac{2\pi i n_k}{T}, k \,=\, 1, 2, \ldots, p$ of the ring $\mathcal{M}_{n_1,\, n_2,\ldots,n_p}$ are reversible and

$$\frac{1}{(\tilde{S} - \frac{2\pi i n_k}{T})^m} \,=\, \left\{ \frac{(-1)^{m-1}}{(\frac{2\pi i n_k}{T})^m} + \frac{e^{\frac{2\pi i n_k t}{T}}}{m!} B_m(\frac{t}{T}) \right\} * \tag{26}$$

for $m \,=\, 1, 2,\, \ldots,$ where B_m is the polynomial of Bernoulli of degree m (the sign $*$ means a convolution operator).

T2. If $P(\dfrac{2\pi i n_k}{T}) \,=\, 0$ for $k \,=\, 1, 2, \ldots, p$ and $P(\dfrac{2\pi i n}{T}) \neq 0$ for all other integer numbers $n \neq 0$, an necessary and sufficient condition for solvability of (22) is:

$$\frac{1}{T} \int_0^1 f(t)(e^{\frac{2\pi i n_k t}{T}} - 1)dt \,=\, 0,\; k \,=\, 1,\, 2,\, \ldots,\, p, \tag{27}$$

i.e. the Fourier coefficients of $f(t)$ with numbers $n_1,\, n_2,\, \ldots,\, n_p$ to be equal to 0.

Let's formulate now **the algorithm for solving (22) in the resonance case:**

1) As in the non-resonance case, we can make an algebraization of the problem, i.e. we can reduce it to a single equation but in $C_{n_1,\, n_2,\, \ldots,\, n_p}$:

$$P(\tilde{S})\, \tilde{y} \,=\, f + Q(\tilde{S}). \tag{28}$$

2) We consider the homogenous BVP:

$$P\left(\frac{d}{dt}\right) y = 0,\; \int_0^T y(\tau)d\tau \,=\, 0,\; y^{(j)}(T) - y^{(j)}(0) \,=\, 0, j = 0, 1, \ldots n-2.$$

It is equivalent to the equation $P(\tilde{S})\, y \,=\, 0$ and its solutions have the form:

$$y \,=\, \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \ldots + C_m e^{\frac{2\pi i k_m t}{T}} \right\},$$

where $C_1,\, C_2\, \ldots,\, C_m$ are constants.

3) The solution of (22) has the form:

$$y = \tilde{y} + \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \ldots + C_m e^{\frac{2\pi i k_m t}{T}} \right\}, \tag{29}$$

where $\tilde{y}$ is the solution of (28).

The reducing of the problem for obtaining periodic solutions of LODE with constant coefficients to the auxiliary problem deserves special attention. This consideration is omitted here.

General algorithm for obtaining a periodic solution.
1) Algebraization of the given problem and finding roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$.
2) a) Finding out roots of the form $\dfrac{2\pi i m}{T}$ ($m \in \mathbb{Z} \setminus \{0\}$).
2) b) Verifying whether the roots selected in 2 a) satisfy the conditions (27). If for some of the selected roots these conditions are not satisfied, periodic solutions do not exist.
3) Forming the polynomial $Q(S)$.
4) Partial fraction decomposition of $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ and separation of the resonance and non-resonance parts.
5) Interpretation of the fractions $w = \dfrac{1}{P(S)}$ and $v = \dfrac{Q(S)}{P(S)}$ as functions. As was mentioned above, different groups of formulae are used for interpretation of the fractions from the resonance and the non-resonance parts.
6) Presentation of the solution in the form:

$$\begin{gathered} u_{nr} = w_1 * f + v_1, \;\; u_r = w_2 * f + v_2 \\ u = u_{nr} + u_r, \end{gathered} \tag{30}$$

where w_1 and w_2 are functions, obtained at step 5) after interpretation of the non-resonance and resonance parts respectively of the partial fraction decomposition of w; v_1 and v_2 are functions obtained at step 5) after interpretation of the non-resonance and resonance parts respectively of the partial fraction decomposition of v.

The general solution u is the sum of both parts of the solution–the non-resonance part u_{nr} and the resonance part u_r. It is possible, of course, for each of these parts to be equal to zero.

2.3 Program Implementation of the Algorithm

The program implementation of the general algorithm follows the successive steps formulated above. For obtaining both parts of the solution – the non-resonance and the resonance one, the above described extended algorithm of Heaviside is used. Its implementation is in fact a modified implementation of the classical algorithm of Heaviside. The main differences are as follows:

(i) For algebraization of the problem the formula (17) is used now.

(ii) Other interpretation formulae are used here. The main formulae mentioned above are (18) – (21) and (26). For practical applications more formulae based on them are derived (see [17]).

(iii) The operation denoted by $*$ in (30) is the convolution (15); convolution powers are computed as in case of use of Duhamel convolution.

(iv) The verification of condition (27) here is a part of the algorithm.

The implementation of the general algorithm considered above includes finding of periodic solutions of systems of linear ordinary differential equations with constant coefficients as well.

Main part of the interpretation formulae used by our program implementation:

For the non-resonance case:

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{T\, \partial_{\{a,-1+m\}} \frac{e^{t a}}{-1+e^{T a}}}{(-1+m)!}$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} - \frac{T \operatorname{Cos}[a t - \frac{a T}{2}] \operatorname{Csc}[\frac{a T}{2}]}{2 a}$$

$$\frac{S}{S^2+a^2} = \frac{1}{2} T \operatorname{Csc}\left[\frac{a T}{2}\right] \operatorname{Sin}\left[a t - \frac{a T}{2}\right]$$

$$\frac{c S + d}{S^2+a^2} = c\left(\frac{1}{a^2} - \frac{T \operatorname{Cos}[a t - \frac{a T}{2}] \operatorname{Csc}[\frac{a T}{2}]}{2 a}\right) + d\left(\frac{1}{2} T \operatorname{Csc}\left[\frac{a T}{2}\right] \operatorname{Sin}\left[a t - \frac{a T}{2}\right]\right)$$

$$\frac{1}{S^2+p S+q} = \frac{1}{q} + \frac{\left(\frac{e^{\frac{1}{2}\left(-p+\sqrt{p^2-4 q}\right) t}}{-1+e^{\frac{1}{2}\left(-p+\sqrt{p^2-4 q}\right) T}} - \frac{e^{-\frac{1}{2}\left(p+\sqrt{p^2-4 q}\right) t}}{-1+e^{-\frac{1}{2}\left(p+\sqrt{p^2-4 q}\right) T}}\right) T}{\sqrt{p^2-4 q}}, \quad p^2-4 q \neq 0$$

For $\sqrt{p^2-4 q} = \delta,\ \ p+\delta = \alpha,\ -p+\delta = \beta$:

$$\frac{1}{S^2+p S+q} = \frac{1}{q} + \frac{T}{\delta}\, \frac{\left(-\frac{e^{-\frac{t \alpha}{2}}}{-1+e^{-\frac{T \alpha}{2}}} + \frac{e^{\frac{t \beta}{2}}}{-1+e^{\frac{T \beta}{2}}}\right)}{\sqrt{p^2-4 q}}$$

$$\frac{S}{S^2+p S+q} = \frac{T}{\delta}\left(\frac{e^{-\frac{t \alpha}{2}} \alpha}{2\left(-1+e^{-\frac{T \alpha}{2}}\right)} + \frac{e^{\frac{t \beta}{2}} \beta}{2\left(-1+e^{\frac{T \beta}{2}}\right)}\right)$$

For the resonance case:

$$\frac{1}{S^m} = \frac{T^m}{m!} B\left[m, \frac{t}{T}\right]$$

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{e^{a t}}{m!} T^m B\left[m, \frac{t}{T}\right]$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} + \frac{t \operatorname{Sin}[a t]}{a} - \frac{T \operatorname{Sin}[a t]}{2 a}$$

$$\frac{S}{S^2+a^2} = t \operatorname{Cos}[a t] - \frac{1}{2} T \operatorname{Cos}[a t] + \frac{\operatorname{Sin}[a t]}{a}$$

$$\frac{c S + d}{S^2+a^2} = c\left(\frac{1}{a^2} + \frac{t \operatorname{Sin}[a t]}{a} - \frac{T \operatorname{Sin}[a t]}{2 a}\right) + d\left(t \operatorname{Cos}[a t] - \frac{1}{2} T \operatorname{Cos}[a t] + \frac{\operatorname{Sin}[a t]}{a}\right)$$

$$\frac{1}{S^2+p S+q} = \frac{e^{-\frac{1}{2} t (p+\delta)}}{2 q \delta}\left(e^{t \delta} q\,(2 t - T) + q\,(-2 t + T) + 2 e^{\frac{1}{2} t (p+\delta)} \delta\right), \quad p^2-4 q \neq 0$$

2.4 Program package

The developed program package for Mathematica, provides application of all described operations of the general algorithm for obtaining periodic solutions of LODE with constant coefficients.

The main function of the package is called $DSolveOCP$ and its use is similar to the use of the considered above function $DSolveOC$. An additional argument is the period T. Due to the above considerations, the boundary conditions have the form $\int_0^T y(\tau)\,d\tau = \alpha_0,\ y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1},\ k = 0, 1, \ldots n-2$. The use of an option for visualization of the solution, together with the right–hand side function is provided.

Some illustrative examples follow – for the resonance case and for the "mixed" case when the solution is a sum of two parts – resonance and non-resonance ones.

Example for the non-resonance case:

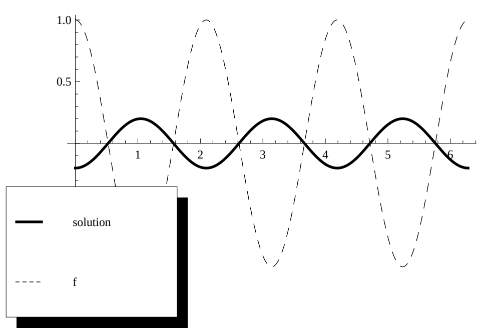
```
<< DSolveOCPpack`

Example1 : {y(t) a^2 + y''(t) == sin(t), α(1) == 0}; T = 2 π

DSolveOCP[{y''[t] + a^2 y[t] == Sin[t], α[1] == 0}, y[t], t, 2 π]
```

$$y[t] \to \frac{1}{2\,a^3\,(-1+a^2)^2\,\pi}\,(2\,a^2\,Cos[t]\,Sin[a\,\pi]\,((-1+a^2)\,\pi\,Cos[a\,\pi] - 2\,a\,Sin[a\,\pi]) + (a\,(1-3\,a^2+2\,a^4)\,\pi + a\,(-1+a^2)\,\pi\,Cos[2\,a\,\pi] + (1-3\,a^2)\,Sin[2\,a\,\pi])\,Sin[t])$$

Example for the resonance case (with option for visualization of the solution):

```
Example 2 : y''[t] + 4 y[t] == Cos[3 t]
DSolveOCP[{y''[t]+4 y[t]==Cos[3 t], α[1]==0}, y[t], t, 2 π, Graph->True]
y[t] -> -1 / 5 Cos[3 t]
```

Out[14]=



Example for a "mixed" case:

Example3 : $\{4\,y(t) + 4\,y'(t) + y''(t) + y^{(3)}(t) == \cos(5\,t),\ \alpha(1) == 0,\ \alpha(2) == 0\};\ T = 2\,\pi;$

```
de = y'''[t] + y''[t] + 4 y'[t] + 4 y[t] == Cos[5 t];

DSolveOCP[{de, α[1] == 0, α[2] == 0}, y[t], t, 2 π]

y[t] → 1/546 (-Cos[5 t] - 5 Sin[5 t])
```

3 Mean-Periodic Solutions of LODE with Constant Coefficients

A more general approach to obtaining periodic solutions of LODE with constant coefficients is considered in the papers [5] and [6].

Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a non-zero polynomial with constant coefficients of degree n and let us consider an ordinary linear differential equation of the form:

$$P\left(\frac{d}{dt}\right) y = f(t), \quad -\infty < t < \infty \tag{31}$$

Let Φ be a linear functional on $\mathcal{C}(I\!R)$. We are looking for solutions of (31) which satisfy the relation

$$\Phi\left\{y(t+\tau)\right\} = 0 \tag{32}$$

for all $t \in I\!R$, i.e. for mean-periodic solutions of (31) with respect to the functional Φ.

Definition 1. *The boundary value problem*

$$P\left(\frac{d}{dt}\right) y = f(t),\ \Phi\{y^{(k)}\} = \alpha_k,\ k = 0,\ 1,\ \ldots\ ,\ n-1,\ f \in C(\Delta) \tag{33}$$

is said to be a non-local Cauchy problem, associated with the functional Φ.

Definition 2. *Let $\Phi \in [C(I\!R)]^*$ be a given linear functional on the space of the continuous functions on the real line. A function $f \in C(I\!R)$ is said to be mean-periodic [1] with respect to the functional Φ if*

$$\Phi_\tau\{f(t+\tau)\} = 0 \text{ for } t \in I\!R.$$

The periodic functions with a period $T > 0$ are mean-periodic with respect to the functional

$$\Phi\{f\} = f(T) - f(0).$$

The antiperiodic functions with an antiperiod $T > 0$, i.e. the functions, satisfying the functional equation $f(T+t) = -f(t)$, are mean-periodic functions with respect to the linear functional

$$\Phi\{f\} = \frac{1}{2}\{f(0) + f(T)\}.$$

Further considerations related to the mean-periodic functions and the use of the Mikusiński type operational calculus of Dimovski (and the Heaviside algorithm with some modifications) for obtaining mean-periodic solutions of LODE with constant coefficients are presented in [5] and [6].

For deriving some formulae and for practical application of the described algorithms the CAS Mathematica is used.

3.1 Advantages of the Presented Approach for Obtaining Periodic Solutions of LODE with Constant Coefficients

The presented approach is more efficient than those in the above mentioned books of Kaplan [10], Rosenvasser [14] and Lurie [11].

The function *DSolve* of Mathematica remains as undetermined the constants appearing in the solution in the resonance case.

In the classical methods for finding periodic solutions, originally the general solution is found and after that the periodicity conditions are used for determi - ning the unknown constants in it. In the above guggested approach these conditions are taken into account at the level of algebraization of the problem.

In case of use the Laplace transformation for finding periodic solutions, the existence of Laplace transform of the right-hand side of the equation is needed.

The presented approach is more efficient (especially in the resonance cases) than the well known (and published) approaches.

A more general approach for the case of mean-periodic solutions is suggested.

All proposed algorithms are convenient for use in the program environment of a CAS (Mathematica in our case).

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