Stronger Nanoscale EM and BEM Solutions by CICT Phased Generators

R. A. Fiorini

Politecnico di Milano University, Italy

Abstract— The addiction to IC (Infinitesimal Calculus), in the mathematical treatment of EM (electromagnetic) and BEM (bioelectromagnetic) modeling problems, is such that, since the digital computer requires an algebraic formulation of physical laws, it is preferred to discretize the differential equations, rather than considering other more convenient tools for problem mathematical description like, for instance, FDC (Finite Differences Calculus) or more sophisticated algebraic methods. Unfortunately, even traditional FDC, FDTD, etc., approaches are unable to conserve overall system information description. As a matter of fact, current Number Theory and modern Numeric Analysis still use mono-directional interpretation for numeric group generator and relations, so information entropy generation cannot be avoided in current computational algorithm and application. Furthermore, traditional digital computational resources are unable to capture and to manage not only the full information content of a single Real Number $\mathbb{R}$, but even Rational Number $\mathbb{Q}$ is managed by information dissipation (e.g. finite precision machine, truncating, rounding, etc.). *CICT* PG approach can offer an effective and convenient "Science 2.0" universal framework, by considering information not only on the statistical manifold of model states but also on the combinatorial manifold of low-level discrete, phased generators and empirical measures of noise sources, related to experimental high-level overall perturbation. We present an effective example; how to unfold the full information content hardwired into Rational OpeRational (OR) representation (nano-microscale discrete representation) and to relate it to a continuum framework (meso-macroscale) with no information dissipation. This paper is a relevant contribute towards arbitrary multi-scale computer science and systems biology modeling, to show how *CICT* PG approach can offer a powerful, effective and convenient "Science 2.0" universal framework to develop innovative, antifragile application and beyond.

1. INTRODUCTION

There is no doubt that the IC (Infinitesimal Calculus) has played in the past and will play in future a major role in the mathematical treatment of EM (electromagnetic) and BEM (bioelectromagnetic) modeling problems. Nevertheless, we must be aware that its addiction specifically on multi-scale system modeling hides some fundamental features of the natural phenomenon being represented, such as both its intrinsic invariant features (ontological, topological, geometrical, etc.) and its extrinsic interaction with observer, to get overall phenomenon formal observation, description and representation by humans. The addiction is such that, since the digital computer requires a finite formulation of physical laws, it is preferred to discretize the differential equations, rather than considering other more convenient tools like FDC (Finite Differences Calculus), Finite Difference Time Domain (FDTD) or more sophisticated algebraic methods. FDC and FDTD deal especially with discrete functions, but they may be applied to continuous function too and to continuum problems, with no loss of generality. They can deal with both discrete and continuum problem categories conveniently. Unfortunately, even traditional FDC, FDTD and more sophisticated and advanced algebraic approaches are unable to conserve overall system information description in a multi-scale modeling environment. As a matter of fact, current Number Theory and modern Numeric Analysis still use LTR (Left-To-Right) mono-directional interpretation for numeric group generator and relations, so information entropy generation cannot be avoided in current computational algorithm and application [1]. Furthermore, traditional digital computational resources are unable to capture and to manage not only the full information content of a single Real Number $\mathbb{R}$, but even Rational Number $\mathbb{Q}$ is managed by information dissipation (e.g. finite precision machine, truncating, rounding, etc.). So, paradoxically if you don't know the code used to communicate a message you can't tell the difference between an information-rich message and a random jumble of letters [3]. This is the information double-bind (IDB) problem in contemporary classic information theory and in current science [3]. Scientific community laid itself in this double-bind situation. Even the most sophisticated instrumentation system is completely unable to reliably discriminate so called "random noise" (RN) from any combinatorially optimized encoded message, which CICT called "deterministic noise", DN for short [3]. It is a problem to solve clearly and reliably, before taking any quantum leap to more competitive and convenient, at first sight, post-human cybernetic approaches in science and technology. As a matter of fact, to grasp a more reliable representation of reality, researchers and scientists need two intelligently articulated hands: both stochastic and combinatorial approach synergically articulated by natural coupling [4]; let's say we need a fresh "Science 2.0" approach. To get stronger solution to advanced problems, like resonant nanoparticle, nanophotonic, optifluidics structure modeling, etc., we have to look for convenient arbitrary multi-scaling BU (bottom-up) point-of-view (POV) (from discrete to continuum view ≡ BU POV) to start from first, and NOT the other way around! We present an effective example; how to unfold the full information content hardwired into Rational OpeRational (OR) representation (nano-microscale discrete representation) and to relate it to a continuum framework (meso-macroscale) with no information dissipation.

2. COMPLEX SYSTEM MODELING

As an example of complex system (hierarchical heterogenous multi-scale system) modeling with important implications, let us consider classical relativistic electrodynamics, applied to biological system modeling (e.g. fullwave electromagnetic modeling of brain waves). Neural activity inside the brain results in low frequency waves known as brain waves. These brain waves can be further classified into delta (0.1 to 3 Hz), theta (4 to 7 Hz), alpha (8 to 12 Hz), beta (12 to 30 Hz) and gamma (30 to 100 Hz) waves based on the rate of neural activity inside the brain. The success of neuroscience in the study of the structural and biochemical properties of neurons, glia cells, and all the biological units and cellular structures in the brain have not yet filled the gap between the behavior understood at cellular level (microscale) and the macroscopic dynamics involved in the traffic between the brain and the world which it is immersed within. There is an essential problem in the study of brain function (mesoscale dynamics) that even today, after so many years since Karl Lashley posed his dilemma [5], still waits for a solution. The quasi-static approach is not sufficiently accurate for the head-modeling problem since the quasi-static potential differs from the full-wave potential by nearly 30 % to 50 % [6], supporting the argument that a full-wave solution should be derived even at low frequencies [7]. In fact, in Quantum Theory (QT), the space-time distribution of matter and energy has a coarse-grained structure which allows its representation as an ensemble of quanta (particle representation). The local phase invariance is shown to hold if a field exists which is connected to the space-time derivatives of the phase. In a previous paper we show how Geometric Algebra (GA) can be used to relate spacetime invariant physical quantities to the variables employed by an inertial observer quite easily [8], if we take into consideration a generic electromagnetic field F, described by Riemann-Silberstein vector [9], and we follow the line of thought reported in spacetime algebra (STA) [10, 11, 12]. In the case of a system made up of electrically charged components (nuclei and electrons of atoms), as a biological system, this is just the electromagnetic (e.m.) potential $\mathbf{A}_\mu$, where μ is the index denoting the usual four space-time coordinates $\gamma_0 = ct, \gamma_1, \gamma_2, \gamma_3$. The electric and magnetic fields are suitable combinations of the space-time derivatives of $\mathbf{A}_\mu$. In order to get the local phase invariance, we should assume that the system Lagrangian is invariant with respect to specific changes of the field $\mathbf{A}_\mu$. Thus a specific principle of invariance, named "gauge invariance," emerges; hence the name "gauge field" denotes $\mathbf{A}_\mu$. It is well known that the Maxwell equations just obey the gauge invariance, which in quantum physics becomes the natural partner of the phase invariance to produce our world. Quantum fluctuations give rise to e.m. potentials which spread the phase fluctuations beyond the system at the phase velocity. This gives an intrinsic nonlocalizability to the system and prevents a direct observation of quantum fluctuations, producing therefore a cooperative behavior. For this reason, the description of a physical system is given in terms of a matter field, which is the space-time distribution of atoms/molecules, coupled to the gauge field with the possible supplement of other fields describing the nonelectromagnetic interactions. According to the principle of complementarity, there is also another representation where the phase assumes a precise value; this representation which focuses on the wave-like features of the system cannot be assumed simultaneously with the particle representation. The relation between these two representations is expressed by connecting the uncertainty of the number of quanta (particle structure of the system) ΔN and the uncertainty of the phase (which describes the rhythm of fluctuation of the system) $\Delta\Phi$:

$$\Delta N \Delta\Phi \geq 1/2 \tag{1}$$

obtaining the above uncertainty relation, quite similar to the better-known Heisenberg relation between position and momentum. In conclusion, a complex system involves two kinds of interaction:

- (1) If $\Delta N = 0$, the number of quanta is well defined, so that we obtain an atomistic description of the system, but lose the information on its capability to fluctuate, since $\Delta\Phi$ becomes infinite. We have an interaction similar to that considered by Classical Physics, where objects interact by exchanging energy. These exchanges are connected with the appearance of forces. Since energy cannot travel faster than light, this interaction obeys the principle of causality;

- (2) If $\Delta\Phi = 0$, the phase is well defined, so that we obtain a description of the movement of the system, but lose the information on its particle-like features which become undefined since ΔN becomes infinite. Such a system having a well-defined phase is termed "coherent" in the physical jargon. We have an interaction where a common phase arises among different objects because of their coupling to the quantum fluctuations and hence to an e.m. potential. In this case there is no propagation of matter and/or energy taking place, and the components of the system "talk" to each other through the modulations of the phase field travelling at the phase velocity, which has no upper limit and can be larger than c, the speed of light.

3. CICT PHASED GENERATORS

$CICT$ is a natural framework for arbitrary multi-scale (AMS) computer science and systems biology modeling in the current landscape of modern QFT [3, 4] created to overcome the computational limitations focused in previous sections. Thanks to this line of generative thinking, it is possible to realize that traditional rational number system can be even regarded as a highly sophisticated open logic, powerful and flexible LTR and RTL (Right-To-Left) formal language of languages, with self-defining consistent words and rules, starting from elementary generators and relations [1]. Further, $CICT$ ODR (Observation, Description, Representation) approach can take advantage immediately from those properties to develop system computational functional closures to achieve information conservation countermeasure at each operative step automatically [3]. Then, all computational information usually lost by classic information approach, based on the traditional noise-affected data stochastic model only, can be captured and fully recovered to arbitrary precision by a corresponding complementary codomain, step-by-step. Theoretically, codomain information can be used to correct any computed result, achieving computational information conservation (virtually noise-free data), according to $CICT$ Infocentric World-view [3]. In this way, overall system resilience and antifragility can be developed quite easily [4]. $CICT$ new awareness of a discrete HG (hyperbolic geometry) subspace (reciprocal space) of coded heterogeneous hyperbolic structures [4], underlying the familiar $\mathbb{Q}$ Euclidean (direct space) surface representation can open the way to AMS information conservation by the CICT phased generator (PG) approach [1, 3]. First, let us introduce a LTR symbolic compression operator SCO $\equiv < M \mid DS >$, where DS is a finite digit string of length W and M is the number of times DS is repeated to get the final unfolded digit string in full (e.g. $< 4 \mid 1 > \equiv 1111$ or $< 2 \mid 123 > \equiv 123123$). Usual symbolic string operations can be applied to SCO. Then, we can write usual rational number OpeRational Representation (OR) corresponding to their Symbolic Representation (SR) as [1]:

$$
\begin{aligned}
Q_1 = \tfrac{1}{D_1} = \tfrac{1}{9} \quad &= \textstyle\sum_{k=0}^{\infty} \frac{1}{10^1}\left(\frac{1}{10^1}\right)^k = 0.1111111111111111111111111\ldots \\
Q_2 = \tfrac{1}{D_2} = \tfrac{1}{99} \quad &= \textstyle\sum_{k=0}^{\infty} \frac{01}{10^2}\left(\frac{1}{10^2}\right)^k = 0.0101010101010101010101010\ldots \\
Q_3 = \tfrac{1}{D_3} = \tfrac{1}{999} \quad &= \textstyle\sum_{k=0}^{\infty} \frac{001}{10^3}\left(\frac{1}{10^3}\right)^k = 0.0010010010010010010010010\ldots \\
\ldots \quad &\ldots,
\end{aligned}
\tag{2}
$$

in a more compact RFD (Representation Fundamental Domain, [1]) Q_W format as:

$$
\begin{aligned}
Q_1 = \tfrac{1}{D_1} = \tfrac{1}{9} \quad &\equiv 0.<\infty|1>= 0.<\infty|(<0 \mid 0><1 \mid 1>)> \\
Q_2 = \tfrac{1}{D_2} = \tfrac{1}{99} \quad &\equiv 0.<\infty|01>= 0.<\infty|(<1 \mid 0><1 \mid 1>)> \\
Q_3 = \tfrac{1}{D_3} = \tfrac{1}{999} \quad &\equiv 0.<\infty|001>= 0.<\infty|(<2 \mid 0><1 \mid 1>)> \\
\ldots \quad &\ldots \\
Q_n = \tfrac{1}{D_n} \equiv \tfrac{1}{<n|9>} \quad &= 0.<\infty|(<n \mid 0><1 \mid 1>)> \\
\ldots \quad &\ldots
\end{aligned}
\tag{3}
$$

In the same way, we can write:

$$\begin{array}{rll}
P_1 = \frac{1}{DD_1} = \frac{1}{1} & = \sum_{k=0}^{\infty} \frac{9}{10^1}\left(\frac{1}{10^1}\right)^k = 0.9999999999999999999999999\ldots \\
P_2 = \frac{1}{DD_2} = \frac{1}{11} & = \sum_{k=0}^{\infty} \frac{09}{10^2}\left(\frac{1}{10^2}\right)^k = 0.0909090909090909090909090\ldots \\
P_3 = \frac{1}{DD_3} = \frac{1}{111} & = \sum_{k=0}^{\infty} \frac{009}{10^3}\left(\frac{1}{10^3}\right)^k = 0.0090090090090090090090090\ldots \\
\ldots & \ldots \\
P_1 = \frac{1}{DD_1} = \frac{1}{1} & \equiv 0. <\infty|9>= 0. <\infty|(<0 \mid 0><1 \mid 9>)> \\
P_2 = \frac{1}{DD_2} = \frac{1}{11} & \equiv 0. <\infty|09>= 0. <\infty|(<1 \mid 0><1 \mid 9>)> \\
P_3 = \frac{1}{DD_3} = \frac{1}{111} & \equiv 0. <\infty|009>= 0. <\infty|(<2 \mid 0><1 \mid 9>)> \\
\ldots & \ldots \\
P_n = \frac{1}{DD_n} \equiv \frac{1}{<n|1>} & = 0. <\infty|(<n \mid 0><1 \mid 9>)> \\
\ldots & \ldots
\end{array} \tag{4}$$

Now, we can realize that P_1 RFD is related by Q_1 RFD, P_2 RFD is related by Q_2 RFD, P_3 RFD is related by Q_1 RFD, and vice-versa by periodic scale relativity (precision length) $W = 1, 2, 3,$, respectively. So, to conserve the full information content of rational correspondence between Q_1 and P_1, Q_2 and P_2, Q_3 and P_3, , we realize that we have to take into account not only the usual Q_1 and P_1, etc., modulus information, but even their related periodic precision length information $W = 1, 2, 3,$, respectively (external or extrinsic phase representation). Furthermore we see that:

$$\begin{array}{rll}
O_1 = \frac{1}{DDD_1} = \frac{1}{9} & \equiv 0. <\infty|1>= 0. <\infty|(<1 \mid 1>)> \\
O_2 = \frac{1}{DDD_2} = \frac{1}{09} & \equiv 0. <\infty|11>= 0. <\infty|(<2 \mid 1>)> \\
O_3 = \frac{1}{DDD_3} = \frac{1}{009} & \equiv 0. <\infty|111>= 0. <\infty|(<3 \mid 1>)> \\
\ldots & \ldots \\
O_n = \frac{1}{DDD_n} \equiv \frac{1}{(<n|0><1|9>)} & = 0. <\infty|(<n \mid 1>)> \\
\ldots & \ldots
\end{array} \tag{5}$$

The coherent representations DDD_1, DDD_2, DDD_3, etc. emerge out of a LTR infinity of symbolic structured infinite length sequences as in (5). So, DDD_n in (5) is the coherent relation representation of traditional scalar modulus D_1 in (2) as denominator at precision W, while scalar modulus D_1 in (2) has to be interpreted as the decoherenced relation representation of DDD_n denominators in (5). In general, for any Natural number $D \in \mathbb{N}$ we can write:

$$\begin{aligned}
&\frac{1}{<n \mid RFD(D)>} \equiv \\
&\equiv <(<k+W(n-1) \mid 0><1 \mid D>)> . <(<k+W(n-1) \mid 0><1 \mid D>)>,
\end{aligned} \tag{6}$$

where RFD is at precision W and k is the number of leading zeros in RFD (if any) at precision W. Now, for $n \to \infty$, we get:

$$\begin{aligned}
&\frac{1}{<n \mid RFD(D)>} \equiv \\
&\equiv <(<k+W(\infty-1) \mid 0><1 \mid D>)> . <(<k+W(\infty-1) \mid 0><1 \mid D>)> = \\
&= \overline{0}D.\overline{0}.
\end{aligned} \tag{7}$$

Then, Natural numbers $\mathbb{N}$ emerge from (7) as a conceptual abstraction, just from the rightmost approximated part of those structured sequences when $n \to \infty$, for $n \in \mathbb{N}$, with truncated decimal part (represented by zero on any finite computational machine). In this way we loose the basic relationship with its fundamental string generators (i.e. information dissipation and system decoherence) completely. Leading zeros in positional notation representation system for $CICT$ $\mathbb{Q}$ Arithmetic do count effectively, and can model the system quantum-classical transition quite effectively [13]. If we do not take into account leading zeros information, we lose the correct rational RFD correspondence information (coherence) which an inner or intrinsic phase for each RTL (right-to-left) string generator can be computed from (i.e. from their optimized exponential cyclic sequences (OECS) of R_W [12]). The $CICT$ fundamental relationship that ties together numeric body information of RTL divergent monotonic power series in any base (in this case decimal, with no loss of generality) with D ending by digit 9 is given by the following equation:

$$\frac{1}{\sum_{k=0}^{\infty}(D+1)^k} = \frac{1}{D}. \quad (8)$$

Further generalizations of (8) related to D ending by digit 1, 3 and 7 are straightforward [13]. Therefore, Rational numeric representations are able to capture two different types of information at the same time, modulus (usual quotient information) and associated outer or extrinsic phase (period) information, which inner generators phase (generator intrinsic period) can be computed from. So, rational information can be better thought to be isomorphic to vector information rather than to usual scalar one, at least. According to our SCO approach, the correct coherent relation representation of traditional scalar modulus D = 7 as denominator of Egyptian fraction, is given by:

$$CQ_1 = \frac{1}{CD_1} \equiv \frac{1}{<\infty|(<\infty|0><1|7>)>} \equiv 0.<\infty|RFD(7)> \equiv 0.<\infty|142857> . \quad (9)$$

To conserve the full information content of rational correspondence at higher level, we realize that we have to take into account not only the usual modulus information, but even the related periodic precision length information $W = 6$ (numeric period or external phase representation) in this case (i.e. $CD_1 = 000007$ as base RFD). We can use Euler's formula to establish the usual fundamental relationship between trigonometric functions and the complex exponential function:

$$e^{ix} = cosx + isinx, \quad (10)$$

where e is the base of the natural logarithm and $i = \sqrt{-1}$. The final result is:

$$CQQ_1 = \frac{1}{CDD_1} = \frac{1}{7}e^{i\frac{\pi(2n+1)}{3}} = \frac{1}{7}(cos(\frac{2\pi(n+1)}{6}) + isin(\frac{2\pi(n+1)}{6})) \quad (11)$$

and

$$CDD_1 = \frac{1}{CQQ_1} = 7(e^{-i\frac{\pi(2n+1)}{3}}) = 7(cos(\frac{\pi(2n-1)}{3}) + isin(\frac{\pi(2n-1)}{3})) = 7(\frac{1}{2} - i\frac{\sqrt{3}}{2}) \quad (12)$$

for $n = 1, 2, 3, ...$in $\mathbb{N}$. $CICT$ shows that any natural number $D \in \mathbb{N}$ has associated a specific, non-arbitrary external phase relationship (OECS Optimized Exponential Cyclic Sequence coherence information, [13]) that we must take into account to full conserve its information content by computation in Euclidean space [4]. The interested reader will have already guessed the relationship of our result to de Moivre number or root of unity (i.e. any complex number that gives 1.0 when raised to some integer power of n). In this way, we can exploit Rational numbers $\mathbb{Q}$ full information content to get stronger solutions to current AMS system modeling problems [3].

4. CONCLUSION

According to QFT, the gauge invariance in quantum physics becomes the natural partner of the phase invariance to produce our world. Quantum fluctuations give rise to e.m. potentials which spread the phase fluctuations beyond the system at the phase velocity. This gives an intrinsic nonlocalizability to the system and prevents a direct observation of quantum fluctuations. Through the e.m. potential, the system gets a chance to communicate with other systems and subsystems. The presence of this field has received experimental corroboration by the discovery of the so-called "Lamb shift," named after the Nobel prize winner Lamb [14]. He discovered as far back as in 1947 that the energy level of the electron orbiting around the proton in the hydrogen atom is slightly shifted (about one part per million) with respect to the value estimated when assuming that no e.m. field is present. Further corroboration for the existence of vacuum fluctuations is provided by the Casimir effect [15]. Therefore a weak e.m. field is always present, just the one arising from the vacuum quantum fluctuations. We see that the correct AMS modeling of complex system must involve two kinds of interaction:

- 1- an interaction similar to that considered by Classical Physics, where objects interact by exchanging energy. These exchanges are connected with the appearance of forces measured by their magnitude (modulus) only, in an assumed continuum manifold that may be approached and studied by traditional stochastic and probabilistic tools offered by the large arena of the Geometric Science of Information (GSI). Since energy cannot travel faster than light, this interaction obeys the principle of causality (Science 1.0 approach). The missing part of this worldview is usually called ”system noise”, ”background radiation”, etc... on cosmic scale by human being;

- 2- an interaction where a common phase arises among different objects because of their coupling to the quantum fluctuations and hence to an e.m. potential. In this case there is no propagation of matter and/or energy taking place, and the components of the system ”talk” to each other through the modulations of the phase field travelling at the phase velocity, which has no upper limit and can be larger than c, the speed of light (Science 2.0 approach). $CICT$ new awareness of a discrete HG (hyperbolic geometry) subspace (reciprocal space) of coded heterogeneous hyperbolic structures [3], underlying the familiar $\mathbb{Q}$ Euclidean (direct space) surface representation, shows that any natural number $n \in \mathbb{N}$ has associated a specific, non-arbitrary phase relationship that we must take into account to full conserve overall system information content by computation in Euclidean space. This awareness opens the way to system AMS information conservation by the $CICT$ PG approach [1, 3].

$CICT$ PG approach can offer an effective and convenient ”Science 2.0” universal framework, by considering information not only on the statistical manifold of model states but also on the combinatorial manifold of low-level discrete, phased generators and empirical measures of noise sources, related to experimental high-level overall perturbation. This paper is a relevant contribute towards arbitrary multi-scale computer science and complex system modeling, towards a more sustainable economy and wellbeing, in a global competition scenario.

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