Compact thermal models for stochastic thermal analysis

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1. Introduction

Thermal effects in electronics are of increasing concern at both transistor, package and system levels. In order to estimate such effects, in the last decades many methods have been proposed for performing heat diffusion analysis, both analytical and numerical. A particularly effective approach, able to alleviate the computational burden of linear heat diffusion analysis, has turned out to be the construction of compact thermal models [1–8].

All these results rely on the deterministic knowledge of material parameters and boundary conditions. However, for a wide range of examples in electronics, uncertainties should be introduced in the modeling process. In particular the analysis of the effects of parameter uncertainty on thermal performance is crucial in nanoelectronics. The standard approach for stochastic analysis is the Monte Carlo method. Monte Carlo gives accurate results and its implementation is straightforward, but it requires a huge number of simulations. Since simulations are often computationally expensive due to the complexity of the electronic systems, Monte Carlo has a very high computational cost and can be even unfeasible. Recently, new approaches, based on Polynomial Chaos Expansion (PCE) [11,12], have emerged to perform stochastic analysis as efficient alternatives to Monte Carlo.

In this paper, the projection-based multi-point moment matching approach for constructing compact thermal models for deterministic heat diffusion problems is extended to stochastic heat diffusion problems. This approach is limited to linear heat diffusion problems in which thermal parameters are assumed not to depend on temperature. The obtained compact thermal models allow to estimate the PCEs of the space–time temperature rise distribution of the heat diffusion problem, in both the time and frequency domain. These expansions are then used to determine any statis-tical properties of the thermal variables of the problem, such as statistical moments and probability density functions.

The method is robust, since the projection preserves the structure of the equations that are modeling the stochastic heat diffusion problem. The method is efficient, since the projection space is determined by computing the solution to few stochastic heat diffusion problems in the frequency domain, by means of a method which allows to reduce the computational burden. The method also leads to accurate approximations of the statistical properties of the whole space–time temperature rise distribution by means of compact models with state-space dimensions as small as those of the compact thermal models for deterministic heat diffusion problems. The stochastic compact thermal models thus can be numerically solved at negligible computational cost.

As an application example, the stochastic thermal analysis of a Ball Grid Array (BGA) package is performed. In particular, a stochastic compact thermal model is efficiently derived, from which the probability density functions of the temperature rise distribution within the package are estimated when thermal conductivities, heat capacities and heat exchange coefficients are assumed to have known statistical properties.

The rest of this paper is organized as follows. In Section 2 the PCE formulation of stochastic heat diffusion problems for components and packages is introduced. In Section 3, the projection approach for constructing compact thermal models is extended from the deterministic to the stochastic case. In Section 4 the multi-point moment matching technique is extended from the

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deterministic to the stochastic case, by means of an algorithm of reduced computational complexity. The BGA application example is investigated in Section 5.

2. Stochastic thermal analysis by polynomial chaos expansion

The heat diffusion equation in an electronic component or package, discretized by any proper numerical method, such as the Finite Difference (FD) or the Finite Element Method (FEM) assumes the form

$$\mathbf{C}\frac{d\mathbf{u}}{dt}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{g}(t) \tag{1}$$

in which $\mathbf{u}(t)$ is a column vector with N unknowns, usually representing the degrees of freedom for the temperature rise distribution, due to a power density distribution represented by the $N\times 1$ vector $\mathbf{g}(t)$. Matrices $\mathbf{C}$ and $\mathbf{K}$ are N-order square matrices, symmetric and positive definite, which model respectively the effects of volumetric heat capacity and of thermal conductivity with heat exchange coefficients.

When a deterministic thermal model is introduced, for the sake of simplicity here assumed to have only one port, the power density is written as proportional to the junction power $P(t)$

$$\mathbf{g}(t)=\mathbf{G}P(t), \tag{2}$$

in which $\mathbf{G}$ is an $N\times 1$ vector, and the junction temperature rise is introduced as the spatial mean of temperature rise space–time distribution weighted by the power density shape [7]

$$T(t)=\mathbf{G}^T\mathbf{u}(t). \tag{3}$$

The junction power $P(t)$ and the junction temperature rise $T(t)$ are the variables measured at the port of the deterministic thermal model.

If a material parameter p, among the volumetric heat capacities, the thermal conductivities and the heat exchange coefficients, is uncertain, it can be represented by its probability distribution. As a result hereafter it is assumed that the thermal parameters p_k, with $k=1,\dots,q$, of the materials composing the electronics component or package depend on a small number r of random variables $\xi_1,\dots,\xi_r$, which can always be assumed to be statistically independent. Such variables form a vector $\boldsymbol{\xi}$ so that it can be written as $p_k=p_k(\boldsymbol{\xi})$, with $k=1,\dots,q$, and (1) takes the form

$$\mathbf{C}(\boldsymbol{\xi})\frac{d\mathbf{u}}{dt}(t,\boldsymbol{\xi})+\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(t,\boldsymbol{\xi})=\mathbf{g}(t), \tag{4}$$

in which the temperature rise $\mathbf{u}(t,\boldsymbol{\xi})$ and the matrices $\mathbf{C}(\boldsymbol{\xi})$, $\mathbf{K}(\boldsymbol{\xi})$ now depend on the random variables $\boldsymbol{\xi}$. More precisely it can be written as

$$\mathbf{C}(\boldsymbol{\xi})=\sum_{k=1}^{q}p_k(\boldsymbol{\xi})\mathbf{C}_k, \tag{5}$$

$$\mathbf{K}(\boldsymbol{\xi})=\sum_{k=1}^{q}p_k(\boldsymbol{\xi})\mathbf{K}_k, \tag{6}$$

in which the N-order matrices $\mathbf{C}_k$, $\mathbf{K}_k$ do not depend on random variables.

Eq. (4) is a stochastic equation, whose solution allows to compute all stochastic properties of $\mathbf{u}(t,\boldsymbol{\xi})$, such as any probability density function. A solution to this equation can be approximated by PCE [11]. In this approach the unknowns $\mathbf{u}(t,\boldsymbol{\xi})$ are approximated by the expansion

$$\mathbf{u}(t,\boldsymbol{\xi})=\sum_{|\boldsymbol{\alpha}|\le p}\mathbf{u}_{\boldsymbol{\alpha}}(t)\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \tag{7}$$

in which $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)$ are multi-indices of r elements, $|\boldsymbol{\alpha}|$ indicates the sum $\alpha_1+\dots+\alpha_r$ and

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})=\psi^1_{\alpha_1}(\xi_1)\psi^2_{\alpha_1}(\xi_2)\cdots\psi^r_{\alpha_r}(\xi_r), \tag{8}$$

in which $\psi^k_j(\xi_k)$ are polynomials of degrees j, forming an orthonormal basis in the probability space of random variables ξ_k, with $k=1,\dots,r$. Thus the functions $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ form a basis of dimensions

$$n=\binom{p+r}{p}. \tag{9}$$

Projecting (4) onto the space spanned by the basis functions $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, it results in

$$\mathbf{E}\left[\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\left(\mathbf{C}(\boldsymbol{\xi})\frac{d\mathbf{u}}{dt}(t,\boldsymbol{\xi})+\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(t,\boldsymbol{\xi})-\mathbf{g}(t)\right)\right]=0$$

in which $\mathbf{E}[\cdot]$ is the expected value operator. Thus, using (7), it results in

$$\sum_{|\boldsymbol{\beta}|\le p}\mathbf{E}[\mathbf{C}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]\frac{d\mathbf{u}_{\boldsymbol{\beta}}}{dt}(t)+\mathbf{E}[\mathbf{K}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]\mathbf{u}_{\boldsymbol{\beta}}(t)=\mathbf{g}(t)\delta_{\boldsymbol{\alpha}\mathbf{0}},$$

in which δ is Kronecker's delta symbol. Using (5) and (6), this expression can be written in simplified form

$$\sum_{k=1}^{q}\sum_{|\boldsymbol{\beta}|\le p}p^k_{\boldsymbol{\alpha\beta}}\left(\mathbf{C}_k\frac{d\mathbf{u}_{\boldsymbol{\beta}}}{dt}(t)+\mathbf{K}_k\mathbf{u}_{\boldsymbol{\beta}}(t)\right)=\mathbf{g}(t)\delta_{\boldsymbol{\alpha}\mathbf{0}} \tag{10}$$

in which

$$p^k_{\boldsymbol{\alpha\beta}}=\mathbf{E}[p_k(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})] \tag{11}$$

are scalars. The PCE of the junction temperature rise can also be computed

$$T(t,\boldsymbol{\xi})=\sum_{|\boldsymbol{\alpha}|\le p}T_{\boldsymbol{\alpha}}(t)\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \tag{12}$$

in which

$$T_{\boldsymbol{\alpha}}(t)=\mathbf{G}^T\mathbf{u}_{\boldsymbol{\alpha}}(t) \tag{13}$$

Eqs. (10) and (13) define a stochastic thermal model of heat diffusion within the electronics component or package. It has n thermal ports, one for each term in the PCE of the junction temperature rise, as shown in Fig. 1. At the first thermal port the power $P(t)$ is injected and the expected value of the junction

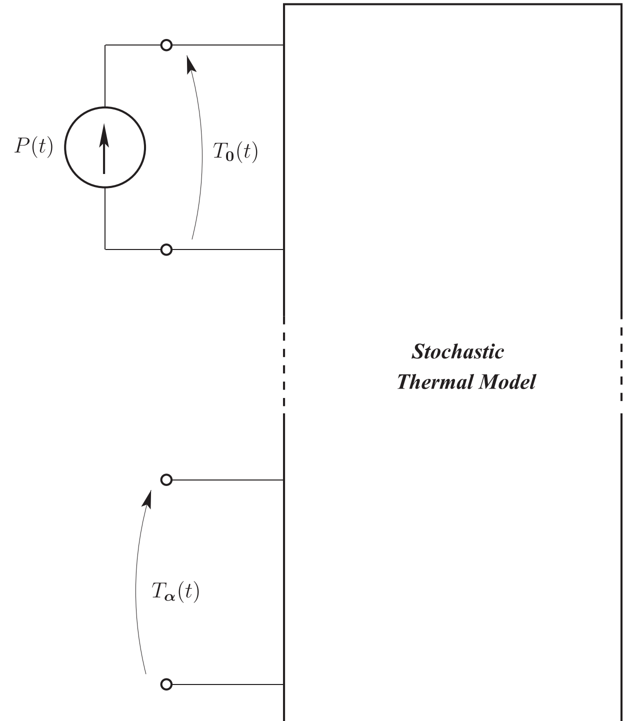

Fig. 1. Stochastic thermal model.

temperature rise $T_{\mathbf{0}}(t)$ is measured. At the other thermal ports no power is injected while the higher terms $T_{\boldsymbol{\alpha}}(t)$ in the PCE of the junction temperature rise are measured. All the terms of the PCE of the junction temperature rise provided by the stochastic thermal model can be used to determine the junction temperature $T(t)$, function of random variables $\boldsymbol{\xi}$, by means of (12). This expression can be used, as a post-processing, to determine any statistic property, such as the mean value or the standard deviation or even the probability density function of $T(t)$.

It is noted that the power $P(t)$ is injected only at the first thermal port since it has been assumed not to be affected by uncertainty. If power is affected by uncertainty, the proposed model could be straightforwardly modified, by introducing the terms of the PCE of the injected power at the thermal ports.

Eqs. (10) and (13) form a linear dynamical system which can be rewritten in the matrix form as

$$\mathsf{C}\frac{d\mathbf{u}}{dt}(t)+\mathsf{K}\mathbf{u}(t)=\mathsf{G}P(t), \tag{14}$$

$$\mathsf{T}(t)=\mathsf{F}^T\mathbf{u}(t). \tag{15}$$

Here

$\mathbf{u}(t)=[\mathbf{u}_{\boldsymbol{\alpha}}(t)]$

are the nN unknowns,

$$\mathsf{C}=[\mathbf{E}[\mathbf{C}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]]=\left[\sum_{k=1}^{q}p_{\boldsymbol{\alpha\beta}}^{k}\mathbf{C}_k\right],$$

$$\mathsf{K}=[\mathbf{E}[\mathbf{K}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]]=\left[\sum_{k=1}^{q}p_{\boldsymbol{\alpha\beta}}^{k}\mathbf{K}_k\right],$$

are nN-order symmetric, positive definite matrices,

$\mathsf{G}=[\mathbf{G}\delta_{\boldsymbol{\alpha}\mathbf{0}}]$

in an $nN\times 1$ vector and

$\mathsf{F}=[\mathbf{G}\delta_{\boldsymbol{\alpha\beta}}]$,

is an $nN\times n$ matrix. The output variables are the PCE terms of the junction temperature $T(t,\boldsymbol{\xi})$, gathered into vector

$\mathsf{T}(t)=[\mathbf{T}_{\boldsymbol{\alpha}}(t)]$.

3. Stochastic compact thermal modeling

From (14), it is noted that the stochastic heat diffusion problem formulated by PCE has the same matrix form of a deterministic heat diffusion problem. As a result the projection-based approach used for constructing compact thermal models for deterministic heat diffusion problems [7] can be extended also to stochastic heat diffusion problems. Thus $\mathbf{u}(t)$ is approximated by a linear combination of basis functions, so that

$$\mathbf{u}(t)=\mathsf{U}\hat{\mathbf{u}}(t) \tag{16}$$

in which $\hat{\mathbf{u}}(t)$ is a vector with a small number $\hat{m}$ of elements, and U has dimensions $nN\times\hat{m}$. Equivalent to (16), it is

$\mathbf{u}_{\boldsymbol{\alpha}}(t)=\mathbf{U}_{\boldsymbol{\alpha}}\hat{\mathbf{u}}(t)$,

$\mathbf{U}_{\boldsymbol{\alpha}}$ being a set of $N\times\hat{m}$ matrices such that

$\mathsf{U}=[\mathbf{U}_{\boldsymbol{\alpha}}]$.

With this assumption, by projecting (10) onto the space spanned by the columns of the U matrix, it results in

$$\hat{\mathbf{C}}\frac{d\hat{\mathbf{u}}}{dt}(t)+\hat{\mathbf{K}}\hat{\mathbf{u}}(t)=\hat{\mathbf{g}}(t) \tag{17}$$

in which

$$\hat{\mathbf{C}}=\mathsf{U}^T\mathsf{C}\mathsf{U}=\sum_{|\boldsymbol{\alpha}|\le p}\sum_{|\boldsymbol{\beta}|\le p}\mathbf{U}_{\boldsymbol{\alpha}}^T\mathbf{E}[\mathbf{C}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]\mathbf{U}_{\boldsymbol{\beta}}$$
$$=\sum_{k=1}^{q}\sum_{|\boldsymbol{\alpha}|\le p}\sum_{|\boldsymbol{\beta}|\le p}p_{\boldsymbol{\alpha\beta}}^{k}\mathbf{U}_{\boldsymbol{\alpha}}^T\mathbf{C}_k\mathbf{U}_{\boldsymbol{\beta}}, \tag{18}$$

$$\hat{\mathbf{K}}=\mathsf{U}^T\mathsf{K}\mathsf{U}=\sum_{|\boldsymbol{\alpha}|\le p}\sum_{|\boldsymbol{\beta}|\le p}\mathbf{U}_{\boldsymbol{\alpha}}^T\mathbf{E}[\mathbf{K}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]\mathbf{U}_{\boldsymbol{\beta}}$$
$$=\sum_{k=1}^{q}\sum_{|\boldsymbol{\alpha}|\le p}\sum_{|\boldsymbol{\beta}|\le p}p_{\boldsymbol{\alpha\beta}}^{k}\mathbf{U}_{\boldsymbol{\alpha}}^T\mathbf{K}_k\mathbf{U}_{\boldsymbol{\beta}}. \tag{19}$$

Also recalling (2) and (3), it results in

$\hat{\mathbf{g}}(t)=\hat{\mathbf{G}}_{\mathbf{0}}P(t)$,

where

$$\hat{\mathbf{G}}_{\mathbf{0}}=\mathbf{U}_{\mathbf{0}}^T\mathbf{G}. \tag{20}$$

Eqs. (17) and (20) are the equations of a stochastic compact thermal model approximating the stochastic thermal model defined in Section 2, by introducing port variables. From the solution $\hat{\mathbf{u}}(t)$ of this model, the PCE of $\mathbf{u}(t,\boldsymbol{\xi})$ can be reconstructed in the form

$$\hat{\mathbf{u}}(t,\boldsymbol{\xi})=\sum_{|\boldsymbol{\alpha}|\le p}\mathbf{U}_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\Bigg)\hat{\mathbf{u}}(t) \tag{21}$$

and, in particular, the PCE of the junction temperature rise $T(t,\boldsymbol{\xi})$ is reconstructed in the form

$$\hat{T}(t,\boldsymbol{\xi})=\sum_{|\boldsymbol{\alpha}|\le p}\mathbf{G}^T\mathbf{U}_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\Bigg)\hat{\mathbf{u}}(t)$$
$$=\sum_{|\boldsymbol{\alpha}|\le p}\hat{\mathbf{G}}_{\boldsymbol{\alpha}}^T\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\Bigg)\hat{\mathbf{u}}(t), \tag{22}$$

where

$\hat{\mathbf{G}}_{\boldsymbol{\alpha}}=\mathbf{U}_{\boldsymbol{\alpha}}^T\mathbf{G}$.

From the PCEs (21) and (22), the statistical properties of both the temperature rise distribution and junction temperature rise can be straightforwardly determined [11]. It is noted that both $\hat{\mathbf{C}}$ and $\hat{\mathbf{K}}$ are symmetric, positive definite matrices. The stochastic compact thermal model equations thus preserve the same structure of the stochastic heat diffusion equation.

4. Multipoint moment matching

The basis vectors, columns of $\mathbf{U}_{\boldsymbol{\alpha}}$, can be computed by generalizing the multi-point moment matching approach [7] from deterministic to stochastic heat diffusion problems. Precisely by writing (4) in the complex frequency domain s, and assuming $P(t)$ equal to a Dirac's delta function, it results in

$$(s\mathbf{C}(\boldsymbol{\xi})+\mathbf{K}(\boldsymbol{\xi}))\mathbf{v}(s,\boldsymbol{\xi})=\mathbf{G} \tag{23}$$

$\mathbf{v}(s,\boldsymbol{\xi})$ being the Laplace transform of $\mathbf{u}(s,\boldsymbol{\xi})$. Approximated the solution to this equation by PCE, it results in

$$\mathbf{v}(s,\boldsymbol{\xi})=\sum_{|\boldsymbol{\alpha}|\le p}\mathbf{v}_{\boldsymbol{\alpha}}(s)\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}). \tag{24}$$

Projecting (23) onto the space spanned by the basis functions $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$, it results in

$\mathbf{E}[\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})((s\mathbf{C}(\boldsymbol{\xi})+\mathbf{K}(\boldsymbol{\xi}))\mathbf{v}(s,\boldsymbol{\xi})-\mathbf{G})]=0.$

Thus using (24) it results in

$$\sum_{|\boldsymbol{\beta}|\le p}(s\mathbf{E}[\mathbf{C}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})]+\mathbf{E}[\mathbf{C}(\boldsymbol{\xi})\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})\psi_{\boldsymbol{\beta}}(\boldsymbol{\xi})])\mathbf{v}_{\boldsymbol{\beta}}(s)=\mathbf{G}\delta_{\boldsymbol{\alpha}\mathbf{0}}$$

or equivalently, after recalling (5) and (6),

$$\sum_{k=1}^{q} \sum_{|\boldsymbol{\beta}|\le p} p^{k}_{\boldsymbol{\alpha\beta}}(s\mathbf{C}_k+\mathbf{K}_k)\mathbf{v}_{\boldsymbol{\beta}}(s)=\mathbf{G}\delta_{\boldsymbol{\alpha}\mathbf{0}}. \tag{25}$$

Eq. (25) forms a linear system, written in the matrix form as

$$(s\mathbf{C}+\mathbf{K})\mathbf{v}(s)=\mathbf{G}, \tag{26}$$

in the nN unknowns

$$\mathbf{v}(s)=[\mathrm{v}_{\boldsymbol{\alpha}}(s)].$$

Eq. (26) has the same matrix form of a deterministic heat diffusion problem in the frequency domain. Proceeding as in the multipoint moment matching method for deterministic problems, (26) is solved for different values s_j of the complex frequency s, with $j=1,\dots\hat{m}$. Following [7], the s_j values are chosen as

$$s_j=\lambda_M \mathrm{dn}\left[\frac{2j-1}{2m}K,k\right],\quad j=1,\dots,\hat{m}$$

in which K is the complete elliptic integral of the first kind of modulus k, dn is the elliptic function of modulus k and

$$k=\sqrt{1-k'^2},$$
$$k'=\frac{\lambda_m}{\lambda_M}. \tag{27}$$

In these expressions λ_m and λ_M are the minimum and maximum eigenvalues of the eigenvalue problem

$$\lambda\mathbf{C}\mathbf{v}+\mathbf{K}\mathbf{v}=0.$$

It is noted that in order to determine the set of values s_j with $j=1$, …, $\hat{m}$, it is sufficient to have a rough estimate of λ_m a λ_M [7]. Such a rough estimate can be achieved by approximating the minimum and maximum eigenvalues of the deterministic heat diffusion problem derived from the stochastic heat diffusion problem by taking the statistical means of the material thermal properties.

The solutions $\mathbf{v}(s_j)$, with $j=1,\dots\hat{m}$, are used for constructing the projection matrix $\mathbf{U}$ directly as

$$\mathbf{U}=[\mathbf{v}(s_1)\cdots\mathbf{v}(s_m)]. \tag{28}$$

Equivalently it is assumed

$$\mathbf{U}_{\boldsymbol{\alpha}}=[\mathbf{v}_{\boldsymbol{\alpha}}(s_1)\cdots\mathbf{v}_{\boldsymbol{\alpha}}(s_m)]. \tag{29}$$

These expressions are used in (18)–(20) for determining the stochastic compact thermal model. The dimension $\hat{m}$ of the stochastic compact thermal model is equal to the number of the chosen values s_j. As shown in [7], for a chosen level of accuracy this number depends on the ratio k' given by (27). Since the eigenvalues λ_m, λ_M can be roughly estimated by the values of the minimum and maximum eigenvalues of the deterministic heat diffusion problem derived from the stochastic heat diffusion problem by taking the statistical means of the material thermal properties, the state-space dimension of the compact thermal model of a deterministic problem, commonly in the range of 1–15, is expected to be also the state-space dimension of the compact thermal model of a stochastic heat diffusion problem.

As a result, the extension of the multipoint moment matching technique from deterministic to stochastic heat diffusion problems requires the solution of the $\hat{m}$linear systems (25) of dimensions nN. These linear systems, for complex electronics applications, are huge. In these cases it can become unfeasible even to store into memory the coefficient matrices. Here an approach is proposed for amply alleviating the computational burden of this problem. The linear systems of Eq. (25) have symmetric, positive definite coefficient matrices

$$\mathbf{A}(s_j)=s_j\mathbf{C}+\mathbf{K},\quad j=1,\dots,\hat{m}.$$

Thus these linear systems can be efficiently solved by any conjugate gradient based algorithm. Using any of such algorithms, it is not necessary to manipulate directly the coefficient matrices $\mathbf{A}(s_j)$, but it is sufficient just to be able to compute the vectors $\mathbf{y}=[\mathbf{y}_{\boldsymbol{\alpha}}]$ obtained by multiplying the coefficient matrix $\mathbf{A}(s_j)$ by any vector $\mathbf{x}=[\mathbf{x}_{\boldsymbol{\alpha}}]$. Recalling (5) and (6), these quantities can be determined by formulae

$$\mathbf{y}_{\boldsymbol{\alpha}}=\sum_{k=1}^{q}(s\mathbf{C}_k+\mathbf{K}_k)\left.\sum_{|\boldsymbol{\beta}|\le p} p^{k}_{\boldsymbol{\alpha\beta}}\mathbf{x}_{\boldsymbol{\beta}}\right) \tag{30}$$

The computation of these expressions requires only to store the N-order square matrices $\mathbf{C}_k$ and $\mathbf{K}_k$ and the scalars $p^{k}_{\boldsymbol{\alpha\beta}}$, for $k=1$, …, q. This implies about the same storage memory for one deterministic heat diffusion problem. As a result in solving (25), the $\mathbf{A}(s_j)$ coefficient matrices are not explicitly constructed but conjugate gradient based methods are used, exploiting (30). In this way the solutions of (25) become affordable, as proven by numerical applications.

5. Numerical application

A BGA package, containing a single die, is considered. Thermal grease is used to connect the upper face of the die to a cap, while the lower face is connected to a substrate via a controlled collapse chip connection (C4). The substrate is then connected to a printed circuit board via a column grid array. The die is modeled by a parallelepiped of 13.65 mm × 11.9 mm × 0.83 mm having thermal conductivity 108 W K^{-1} m^{-1} and volumetric heat capacity 1.794 × 10^6 J K^{-1} m^{-3}. Power $P(t)$ is assumed to be uniformly dissipated inside the die. The aluminum cap is modeled by a parallelepiped of 40.5 mm × 40.5 mm × 3 mm having thermal conductivity 240 W K^{-1} m^{-1} and volumetric heat capacity 2.530× 10^6 J K^{-1} m^{-3}. Similarly a cap protrusion towards the die is modeled using a parallelepiped of 13.7 mm × 13.7 mm × 0.83 mm having thermal conductivity 240 W K^{-1} m^{-1} and volumetric heat capacity 2.530 × 10^6 J K^{-1} m^{-3}. The thermal grease is modeled by a parallelepiped of 34.1 mm × 34.1 mm × 1.7 mm having thermal conductivity 1 W K^{-1} m^{-1} and volumetric heat capacity 1.568× 10^6 J K^{-1} m^{-3}. The ceramic substrate is modeled by a parallelepiped of 42.5 mm × 42.5 mm × 5.85 mm having thermal conductivity 18 W K^{-1} m^{-1} and volumetric heat capacity 3.613 × 10^6 J K^{-1} m^{-3}. The tungsten vias inside the substrate are modeled by multiple parallelepipeds. Large vias are modeled by a parallelepiped of 42.5 mm × 42.5 mm × 2.93 mm and thermal conductivity and volumetric heat capacity 3.598 W K^{-1} m^{-1}. Small vias are modeled by a parallelepiped of 13.65 mm × 11.9 mm × 2.93 mm having thermal conductivity 30.57 W K^{-1} m^{-1} and volumetric heat capacity 3.523 W K^{-1} m^{-1}. The column grid array is modeled by a parallelepiped of 42.5 mm × 42.5 mm × 2.25 mm having equivalent thermal conductivity 12 W K^{-1} m^{-1} and volumetric heat capacity 0.6122 W K^{-1} m^{-1}. The top and side of the BGA package are assumed adiabatic. At the bottom of the BGA package a heat exchange coefficient of 2000 W K^{-1} m^{-2} is assumed.

Three material thermal parameters, shown in Fig. 2, are assumed to be given by independent random variables, with $\pm$30% uniformly distributed variations around their mean values: the thermal conductivity of the small vias, the volumetric thermal capacity of the column grid array, the heat exchange coefficient at the bottom of the package. Without sacrificing accuracy, thermal parameters have been assumed to be independent on temperature. This is a common assumption [9,10] when temperature rises do not exceed 50–100 K.

An accurate FEM discretization by means of $N=107\,911$ degrees of freedom is performed. An accurate PCE is introduced by expanding each of the three independent random variable by 7 basis functions. Such number has been chosen in such a way that

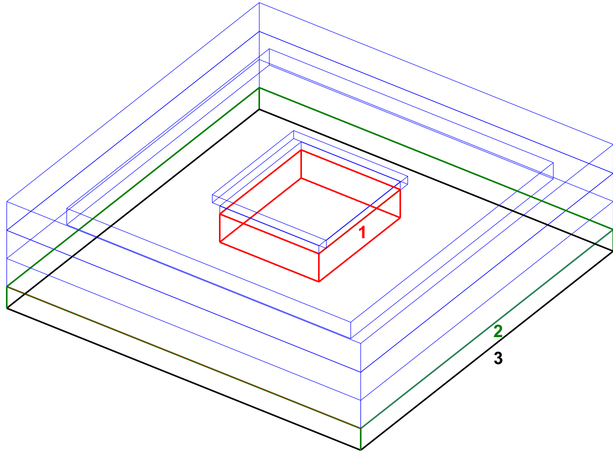

Fig. 2. Regions in which the materials are assumed to have uncertain thermal properties: (1) small vias, (2) column grid array and (3) bottom of the BGA.

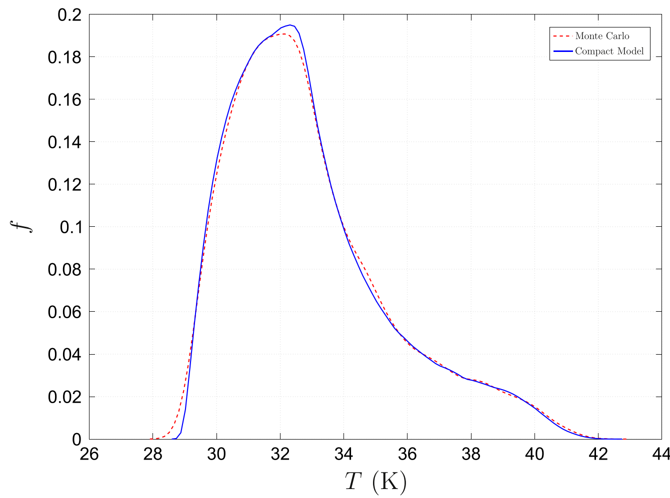

Fig. 3. Probability density function f of junction temperature rise T at steady state, estimated both by the Compact Thermal Model and by Monte Carlo Analysis.

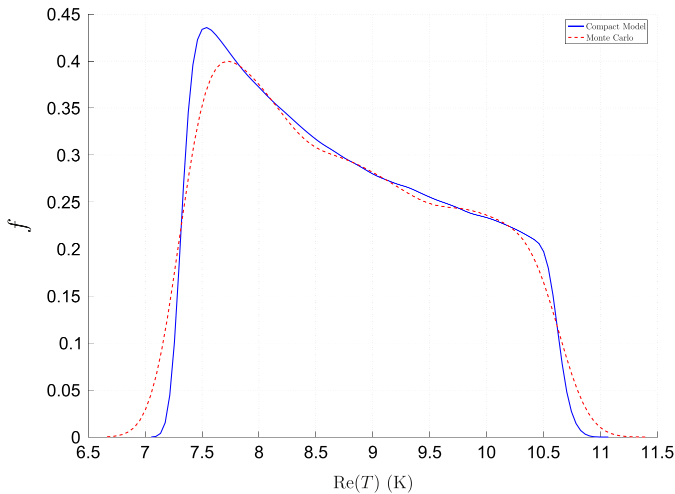

Fig. 4. Probability density function f of the real part of the junction temperature rise T at frequency 0.1 Hz, estimated both by the Compact Thermal Model and by Monte Carlo Analysis.

its further increase affects the thermal simulation results by less than 0.1%. In this way the number of total PCE terms in (7) is 120.

The stochastic multipoint moment matching method is now applied to this problem. A ratio $\lambda_m/\lambda_M \approx 10^{-7}$ is estimated so that

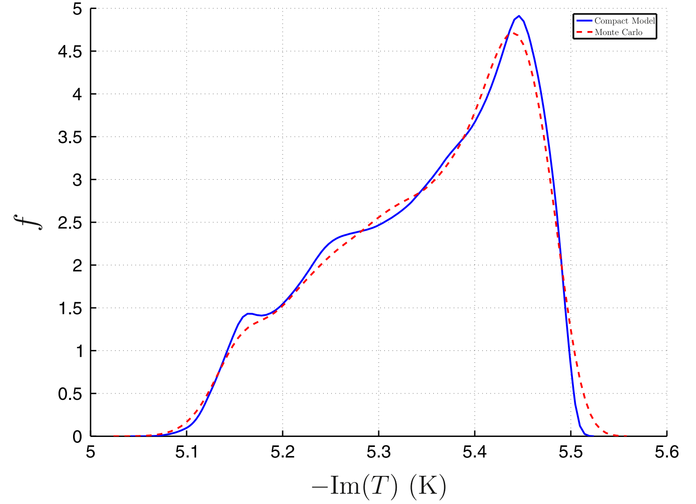

Fig. 5. Probability density function f of the imaginary part of the junction temperature rise T at frequency 0.1 Hz, estimated both by the Compact Thermal Model and by Monte Carlo Analysis.

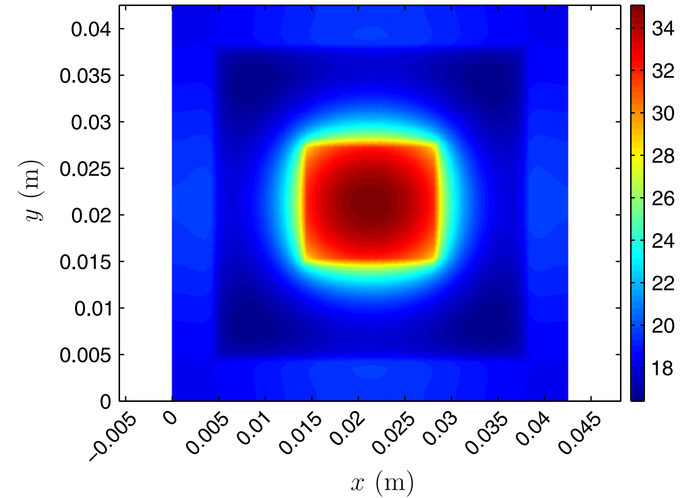

Fig. 6. Mean temperature rise distribution (in K) at steady state, in a horizontal section of the package, cutting the die.

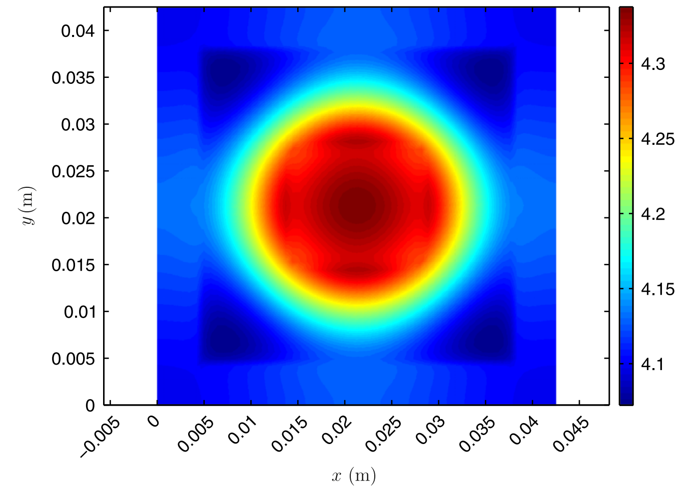

Fig. 7. Standard deviation of the temperature rise distribution (in K) at steady state, in a horizontal section of the package, cutting the die.

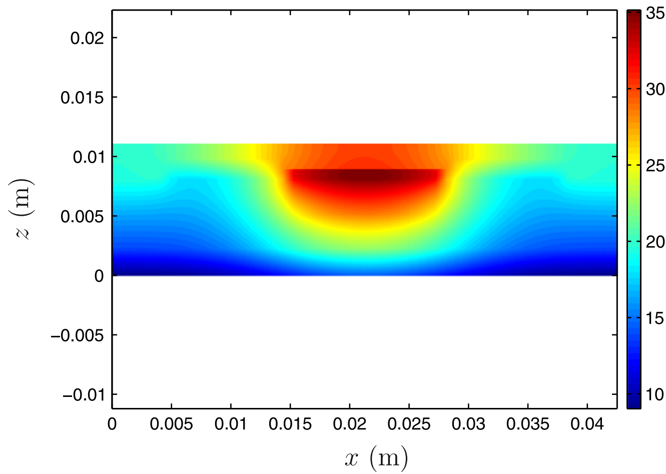

Fig. 8. Mean temperature rise distribution (in K) at steady state, in a vertical section of the package, cutting the die.

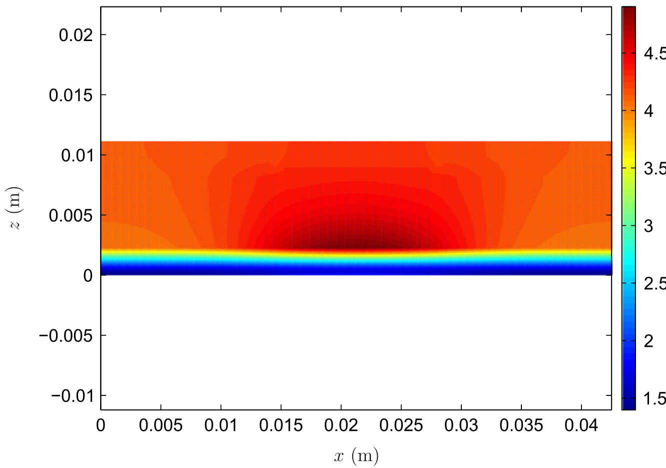

Fig. 9. Standard deviation of the temperature rise distribution (in K) at steady state, in a vertical section of the package, cutting the die.

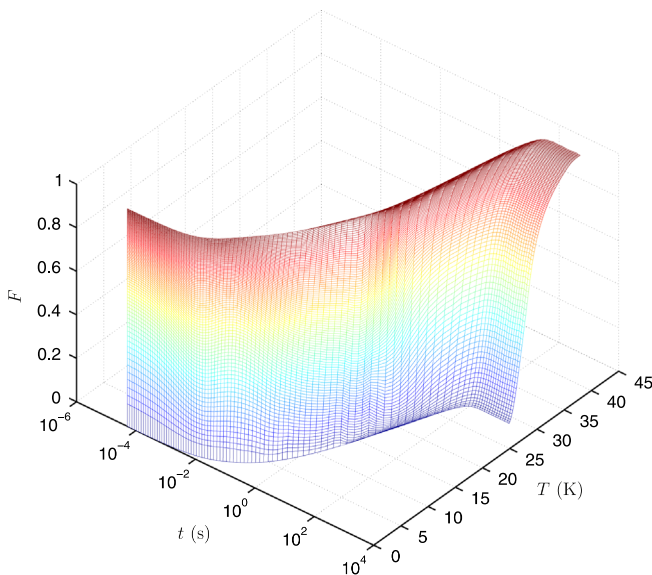

Fig. 10. Cumulative probability distribution function F of the junction temperature rise T, as a function of the time instant t.

a number $\hat{m}$= 10 of expansion points is used for getting an accurate stochastic compact thermal model with relative error less than 0.1% [7] with respect to the FEM model. The corresponding $\hat{m}$= 10 linear systems are solved using the conjugate gradient method without constructing the coefficient matrices, as detailed in the text. The storage requirement is of 1 020 834 double precision numbers, to be compared to 849 010 double precision numbers of the corresponding deterministic problem. The iterative solution of each of these linear systems, with a 10^{-7} relative error requires about 9 min and 1003 iteration steps on a 2.3 GHz Intel Core i7, instead of 6 s and 997 iteration steps for the corresponding deterministic problem. Since the number of the iteration steps of the conjugate gradient algorithm is practically unchanged passing from the deterministic to the stochastic problem, the number of numerical operations of the stochastic problem is about 120 times larger than the deterministic problem. The achieved stochastic compact thermal model has then been used to perform both time and frequency domain simulations. Each of these simulations, shown in Figs. 6–13, has required less than 1 s of simulation time. As a post-processing, stochastic

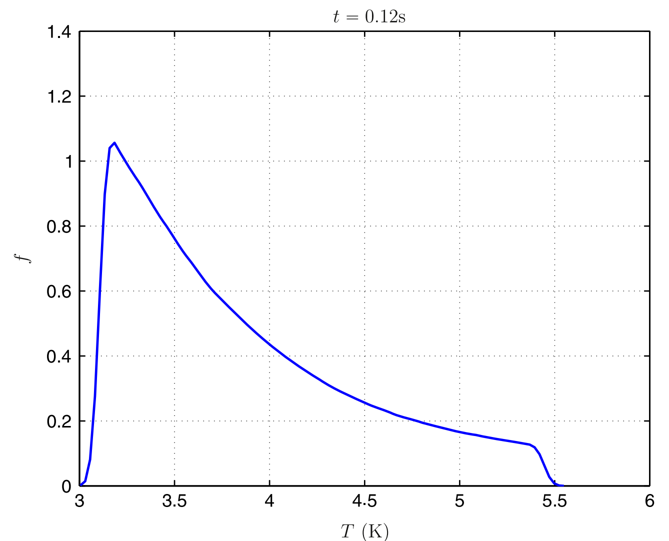

Fig. 11. Probability density function f of the junction temperature rise T at time $t = 0.12$ s.

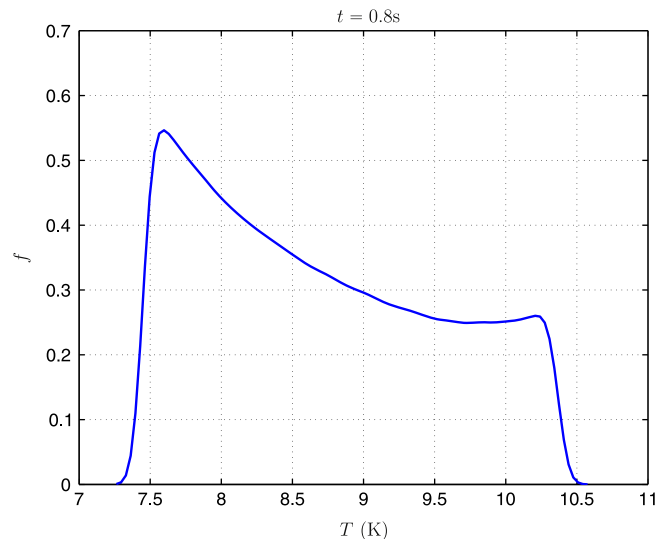

Fig. 12. Probability density function f of the junction temperature rise T at time $t = 0.8$ s.

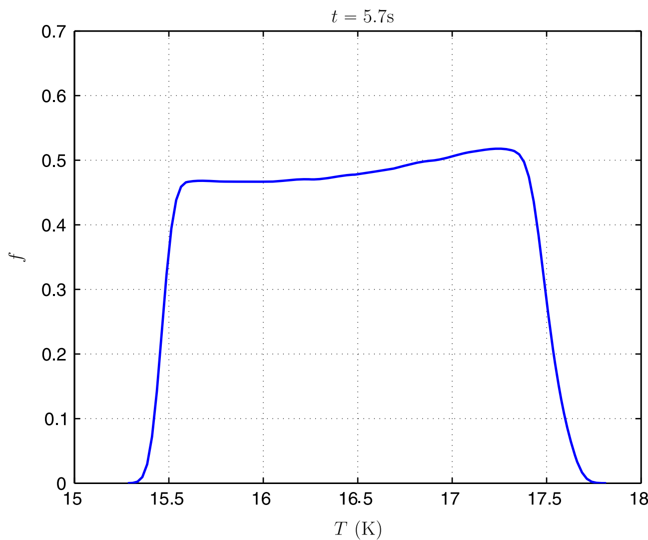

Fig. 13. Probability density function f of the junction temperature rise T at time $t = 5.7$ s.

information not only of junction temperature but also on the whole space–time distribution of temperature within the BGA package is obtained, as shown in Figs. 6–9.

As an alternative to the proposed method, Monte Carlo Analysis of the FEM model could be applied. Using such method, the accuracy practically achievable is low and the computational complexity can easily become prohibitive. For instance the results of the frequency domain analysis are compared in Figs. 3–5 to the much lesser accurate results obtained by Monte Carlo after 10 000 tries with a computational time of 14 h. The result of the time domain analysis shown in the other figures is not feasible by the Monte Carlo method with 10 000 tries, since it would require about six months of computational work.

6. Conclusions

In this paper an approach has been proposed for constructing compact thermal models for the stochastic thermal analysis of electronics components and packages. The approach exhibits high levels of accuracy for state-space dimensions of the model as small as those of the compact models for deterministic heat diffusion problems. It is also efficient thanks to an approach for reducing the computational burden for the solution of the linear systems that are modeling the stochastic heat diffusion problem. Such compact models can be used for accurately determining not only the stochastic properties of junction temperature rises but also of the space–time distributions of temperature rise within the electronics component or package.

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