SCREW DISLOCATIONS IN PERIODIC MEDIA: VARIATIONAL COARSE GRAINING OF THE DISCRETE ELASTIC ENERGY

R. ALICANDRO, M. CICALESE, AND L. DE LUCA

Abstract. We study the asymptotic behavior, as the lattice spacing ε tends to zero, of the discrete elastic energy induced by topological singularities in an inhomogeneous ε periodic medium within a two-dimensional model for screw dislocations in the square lattice. We focus on the $|\log\varepsilon|$ regime which, as $\varepsilon \to 0$, allows the emergence of a finite number of limiting topological singularities. We prove that the Γ-limit of the $|\log\varepsilon|$ scaled functionals as $\varepsilon \to 0$ equals to the total variation of the so-called "limiting vorticity measure" times a factor depending on the homogenized energy density of the unscaled functionals.

Keywords: Topological Singularities; Discrete Systems; Homogenization; Γ-convergence

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Contents

Introduction

This paper deals with the rigorous coarse graining of the discrete elastic energy induced by topological singularities in an inhomogeneous periodic medium. As a paradigmatic example, we consider a two-dimensional periodically inhomogeneous model for screw dislocations (see [11]).

Screw dislocations are straight line defects in the periodic structure of the crystal for which the *Burgers vector*, which is a vector of the underlying lattice measuring defects of the crystalline order, is parallel to the dislocation line [19]. In the framework of anti-plane elasticity, a finite distribution of dislocations is identified by a finite sum of Dirac deltas, representing the positions of the defects, with integer weights, representing the signed moduli of the Burgers vectors. In this context, screw dislocations are topological singularities of the vertical displacement $u : \Omega \cap \varepsilon\mathbb{Z}^2 \to \mathbb{R}$, namely, points around which the elastic part of the discrete gradient of u has non trivial circulation (see Section 1). Here and in what follows $\Omega \cap \varepsilon\mathbb{Z}^2 \subset \mathbb{R}^2$ denotes the reference configuration of the crystal lattice, of lattice spacing $\varepsilon > 0$. According to the theories of Nabarro-Peierls and Frenkel-Kontorova [18], plastic deformations, corresponding to integer jumps of the displacement u, do not store elastic energy. As a consequence of that, one can model the energy stored by the deformation of the crystal introducing periodic potentials, which, in view of the Hooke's law, are quadratic close to the wells. In this respect, the simplest elastic energy can be taken of the form

$$\mathcal{F}_\varepsilon(u) := \sum_{i\in\Omega\cap\varepsilon\mathbb{Z}^2} \Big[\mathrm{dist}^2(u(i+\varepsilon e_1)-u(i),\mathbb{Z}) + \mathrm{dist}^2(u(i+\varepsilon e_2)-u(i),\mathbb{Z})\Big]\,. \tag{0.1}$$

The energy $\mathcal{F}_\varepsilon$ shares the same asymptotic behavior as other functionals describing the emergence of topological singularities, both in discrete and continuum framework, such as the XY model for spin systems [4] and the Ginzburg-Landau (GL) functional for superconductors [13]. For the three models above the picture is nowadays very clear; in the $|\log\varepsilon|$ regime, the discrete vorticity measures (corresponding to the Jacobians of the order parameter in the Ginzburg-Landau context) are pre-compact (in an appropriate sense known as flat convergence) and converge, up to a subsequence, to a finite sum of Dirac deltas with integer weights. Up to a prefactor, the Γ-limit of the family of energies agrees with the total variation of such a limit measure [25, 21, 22, 1, 23, 4, 16]. The asymptotic equivalence of the three models at energy regimes $|\log\varepsilon|^p$ for every $p\geq 1$ has been provided in [5] in terms of Γ-convergence. Moreover, in [6] (see also [7, 8]) it has been shown that the next order term in the expansion of the energy is the so-called renormalized energy [25, 13] governing the interactions among the singularities.

The analysis of a variant of the GL functional in a periodic inhomogeneous medium has been recently provided in [2]. There, it is considered a GL model that depends on two scales, namely, the coherence length ε (playing the same role as the lattice spacing here) and the periodic inhomogeneity scale δ_ε. In such a framework, it is shown that the Γ-limit combines both homogenization and concentration effects depending on the ratio $\frac{|\log\delta_\varepsilon|}{|\log\varepsilon|}$. Our aim is to extend such a continuum analysis to the discrete setting, showing that the techniques are robust enough in order to neglect, at the $|\log\varepsilon|$ regime, further discrete-to-continuum effects. Here we consider a discrete analog of the problem in [2], focussing on the analysis of energies of the type (0.1) which includes periodic inhomogeneities at scale $T\varepsilon$ for some finite $T\in\mathbb{N}$. More precisely, we investigate elastic energies of the form (see Section 1 for details)

$$\mathcal{E}_\varepsilon(u) := \sum_{i\in\Omega\cap\varepsilon\mathbb{Z}^2} \Big[a^1\Big(\frac{i}{\varepsilon}\Big)\mathrm{dist}^2(u(i+\varepsilon e_1)-u(i),\mathbb{Z}) + a^2\Big(\frac{i}{\varepsilon}\Big)\mathrm{dist}^2(u(i+\varepsilon e_2)-u(i),\mathbb{Z})\Big]\,, \tag{0.2}$$

where $a^1,a^2:\mathbb{Z}^2\to[0,+\infty)$ are two functions satisfying the following assumptions: for $k=1,2$

(**P**) Periodicity: There exists $T\in\mathbb{N}$ such that $a^k(\cdot)$ is T-periodic,

(**G**) Growth: There exist two constants γ_1,γ_2 with $0<\gamma_1\leq\gamma_2$ such that
$\gamma_1\leq a^k(y)\leq\gamma_2$, for every $y\in\mathbb{Z}^2$.

We aim at determining the asymptotic behavior of $\mathcal{E}_\varepsilon$ as $\varepsilon\to 0$ in the $|\log\varepsilon|$ energy regime. In view of assumption (**G**) we have that the functionals $\mathcal{E}_\varepsilon$ are bounded from below by (γ_1 times) the functional $\mathcal{F}_\varepsilon$, and hence they share the same compactness property as $\mathcal{F}_\varepsilon$. As for the Γ-limit, one expects that "far" from the singularities the discrete deformation gradient of the displacement u is "small", hence coinciding with its elastic part; therefore, $\mathcal{E}_\varepsilon(u)\sim\mathcal{G}_\varepsilon(u)$, where

$$\mathcal{G}_\varepsilon(u) := \sum_i \Big[a^1\Big(\frac{i}{\varepsilon}\Big)|u(i+\varepsilon e_1)-u(i)|^2 + a^2\Big(\frac{i}{\varepsilon}\Big)|u(i+\varepsilon e_2)-u(i)|^2\Big]\,.$$

The asymptotic behavior of the functionals $\mathcal{G}_\varepsilon$ has been provided in [3], where it has been shown that $\mathcal{G}_\varepsilon$ Γ-converge with respect to the strong L^2-convergence to the functional

$$\mathcal{G}_{\mathrm{hom}}(u) := \int \langle \mathbb{A}_{\mathrm{hom}}\nabla u\,,\nabla u\rangle\,\mathrm{d}x\,,$$

where $\mathbb{A}_{\mathrm{hom}}$ is a two-by-two symmetric matrix defined by a suitable homogenization formula in the discrete setting (see formula (2.2)). Since at the $|\log\varepsilon|$ regime the energy of an isolated singularity concentrates at any scale between ε and 1, by the reasoning above we can deduce that, outside a "small" region enclosing the singularities, the functional $\mathcal{E}_\varepsilon$ behaves as $\mathcal{G}_\varepsilon$ which, in turn, can be approximated by the homogenized energy $\mathcal{G}_{\mathrm{hom}}$. In this framework, we may describe the energy around a singularity of degree z by an asymptotic formula of the type (see Remark 3.1)

$$\begin{aligned}\psi(z) := \lim_{\frac{R}{r}\to+\infty} \min\Big\{ \int_{B_R\setminus\overline{B}_r} \langle \mathbb{A}_{\mathrm{hom}}\nabla u, \nabla u\rangle \,\mathrm{d}x \,:\, u\in SBV^2(B_R\setminus\overline{B}_r)\,,\\ e^{2\pi\imath u}\in H^1(B_R\setminus\overline{B}_r)\,,\ \deg(e^{2\pi\imath u},\partial B_r)=z\Big\}\\ =\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\,,\end{aligned}$$

from which the Γ-limit is obtained by locally optimizing the degree (approximating a singularity of degree z by $|z|$ singularities of degree ±1). This heuristics suggests that

$$\Gamma\text{-}\lim_{\varepsilon\to0}\frac{\mathcal{E}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} = \frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\sum_k |z^k|\,, \tag{0.3}$$

where the z^k's denote the degrees of the limiting singularities. We prove (0.3) in our main result, stated in Theorem 1.2.

Although the heuristic argument described above looks somehow elementary, making it rigorous is not an easy task. The proof of the lower bound follows, up to some vanishing discrete-to-continuum errors, along the lines developed in the continuum setting treated in [2] and recently exploited in a simplified version in the discrete setting in [12]. The strategy consists in applying the ball construction, introduced by Sandier [24] and Jerrard [20] for the GL functional. Such a technique allows us to prove the existence of a finite (i.e., independent of ε) number of balls outside of which the main energy concentrates. For what concerns the upper bound, we need some extra care with respect to the continuum case since in the construction of the recovery sequence we cannot exclude the presence of short dipoles at the discrete level. Nevertheless, the energy bound shows that there can be at most $|\log\varepsilon|$ such dipoles of length ε and that, therefore, they cannot further contribute to the limit energy (see Propositions 3.4 and 3.6).

We finally remark that it would be interesting to extend, at least in the $|\log\varepsilon|$ regime, the variational equivalence between screw dislocations functional, GL model and XY energy from the homogeneous setting studied in [5] to the periodic inhomogeneous setting considered here. This requires, in particular, to extend the main result of this paper to the XY model, that amounts to consider energy as in (0.2) with $\mathrm{dist}^2(\cdot,\mathbb{Z})$ replaced by $f_{XY}(\cdot) := 1-\cos(2\pi\cdot)$. We notice that the potential f_{XY} is still one-periodic and quadratic close to the wells. However, although heuristics suggests that the only behavior to look at is the one close to the wells, generalization of our strategy to such a potential seems to be non-trivial. Finally, we remark that in our analysis the inhomogeneity coefficients a^k are T-periodic for some finite $T\in\mathbb{N}$. The case where $T=T_\varepsilon\to\infty$ as $\varepsilon\to0$ is also interesting and could be treated using the techniques introduced in [2].

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1. The discrete model

In this section we introduce the main objects we deal with and we state our main result.

The discrete setting. Let $Q=[0,1]^2$ be the closed unit square in $\mathbb{R}^2$ with bottom left corner at the origin and let $\Omega\subset\mathbb{R}^2$ be an open bounded set with Lipschitz continuous boundary. For every $\varepsilon>0$ we set

$$\Omega^2_\varepsilon := \{i\in\varepsilon\mathbb{Z}^2 \,:\, i+\varepsilon Q\subset\Omega\}\,,\quad \Omega_\varepsilon := \bigcup_{i\in\Omega^2_\varepsilon}(i+\varepsilon Q)\,,$$

$$\Omega^0_\varepsilon := \Omega_\varepsilon\cap\varepsilon\mathbb{Z}^2\,,\quad \Omega^1_\varepsilon := \{(i,j)\in\Omega^0_\varepsilon\times\Omega^0_\varepsilon \,:\, |i-j|=\varepsilon\}\,.$$

Moreover, we define the *ε-discrete boundary* of Ω as $\partial_\varepsilon\Omega := \Omega^0_\varepsilon\setminus\Omega^2_\varepsilon$.

Furthermore, we set

$$\mathscr{D}_\varepsilon(\Omega) := \{u : \Omega^0_\varepsilon \to \mathbb{R}\}$$

and for every $(i,j) \in \Omega^1_\varepsilon$ we define the *discrete partial derivative* of u in the direction $\frac{j-i}{\varepsilon}$ as $\mathrm{d}u(i,j) := u(j) - u(i)$. Denoting by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$, for every $k \in \{1,2\}$ and for every $u \in \mathscr{D}_\varepsilon(\Omega)$ we set

$$\mathrm{d}^{e_k}_\varepsilon u(i) := \mathrm{d}u(i, i+\varepsilon e_k)\,, \quad \text{for } (i, i+\varepsilon e_k) \in \Omega^1_\varepsilon\,.$$

The energy functional. Let $a^1, a^2 : \mathbb{Z}^2 \to [0,+\infty)$ be two functions satisfying assumptions (**P**) and (**G**) in the introduction. For every $\varepsilon > 0$ we define the energy functional $\mathcal{E}_\varepsilon : \mathscr{D}_\varepsilon(\Omega) \to [0,+\infty)$ as

$$\mathcal{E}_\varepsilon(u) := \sum_{i\in\Omega^2_\varepsilon} \sum_{k=1}^{2} a^k\Big(\frac{i}{\varepsilon}\Big)\mathrm{dist}^2(\mathrm{d}^{e_k}_\varepsilon u(i), \mathbb{Z})\,. \tag{1.1}$$

In the following we will adopt also localized versions of the functional $\mathcal{E}_\varepsilon$; more precisely, for every open bounded set $A \subset \mathbb{R}^2$ with Lipschitz continuous boundary the functional $\mathcal{E}_\varepsilon(\cdot; A) : \mathscr{D}_\varepsilon(A) \to [0,+\infty)$ is defined as in (1.1) with Ω replaced by A. Let $A \subset \mathbb{R}^2$ be open and bounded. For every $\varepsilon > 0$ and for every $u \in \mathscr{D}_\varepsilon(A)$ we denote by $\Pi(u)$ the piecewise affine interpolation of u on the cells $i + \varepsilon Q$ (with $i \in A^2_\varepsilon$). Moreover, we set $w_u := e^{2\pi\imath u}$ and we denote by $\Pi(w_u)$ the piecewise affine interpolation of w_u. The following lemma, whose proof is straightforward, relates the discrete energy $\mathcal{E}_\varepsilon$ defined in (1.1) with the continuum Dirichlet energy of $\Pi(u)$ and $\Pi(w_u)$.

Lemma 1.1. *Let $A \subset \mathbb{R}^2$. There exists a universal constant $C > 0$ depending only on γ_1 and γ_2 such that the following facts hold true for ε small enough:*

(i) *for every $u \in \mathscr{D}_\varepsilon(A)$*

$$\mathcal{E}_\varepsilon(u; A) \le C \int_{A_\varepsilon} |\nabla\, \Pi(w_u)|^2 \,\mathrm{d}x\,;$$

(ii) *for every $u \in \mathscr{D}_\varepsilon(A)$*

$$\mathcal{E}_\varepsilon(u; A) \le \int_{A_\varepsilon} |\nabla\, \Pi(u)|^2 \,\mathrm{d}x\,; \tag{1.2}$$

moreover, if $v \in C^2(\overline{A})$ and $u = v \llcorner A^0_\varepsilon$, then

$$\int_{A_\varepsilon} |\nabla\, \Pi(u)|^2 \,\mathrm{d}x \le C \int_A |\nabla v|^2 \,\mathrm{d}x\,; \tag{1.3}$$

(iii) *for every $u \in \mathscr{D}_\varepsilon(A)$*

$$\mathcal{E}_\varepsilon(u; A) \ge \frac{\gamma_1}{4\pi^2}\mathcal{XY}_\varepsilon(u; A) = \frac{\gamma_1}{4\pi^2}\int_{A_\varepsilon} |\nabla \Pi(w_u)|^2 \,\mathrm{d}x\,, \tag{1.4}$$

where $\mathcal{XY}_\varepsilon(\cdot, A) : \mathscr{D}_\varepsilon(A) \to [0,+\infty)$ is the functional defined by

$$\mathcal{XY}_\varepsilon(u; A) := \sum_{i\in A^2_\varepsilon} \sum_{k=1}^{2} (1 - \cos(2\pi \mathrm{d}^{e_k}_\varepsilon u(i)))\,. \tag{1.5}$$

The discrete topological singularities. In what follows we introduce the notion of discrete vorticity measure associated to a displacement $u \in \mathscr{D}_\varepsilon(\Omega)$. To this purpose, we let $\mathrm{P} : \mathbb{R} \to \mathbb{Z}$ be the function defined by

$$\mathrm{P}(t) := \mathrm{argmin}\{|t - s| \,:\, s \in \mathbb{Z}\}\,,$$

with the convention that, if the argmin is not unique, then we choose the smallest one. Let $u \in \mathscr{D}_\varepsilon(\Omega)$ be fixed. For every $i \in \Omega^2_\varepsilon$ we define the vorticity of u at i as

$$\alpha_u(i) := -\mathrm{P}(\mathrm{d}^{e_1}_\varepsilon u(i)) - \mathrm{P}(\mathrm{d}^{e_2}_\varepsilon u(i+\varepsilon e_1)) + \mathrm{P}(\mathrm{d}^{e_1}_\varepsilon u(i+\varepsilon e_2)) + \mathrm{P}(\mathrm{d}^{e_2}_\varepsilon u(i))\,.$$

Notice that we can alternatively write

$$
\begin{aligned}
\alpha_u(i) =& \big(\mathrm{d}_\varepsilon^{e_1} u(i) - \mathrm{P}(\mathrm{d}_\varepsilon^{e_1} u(i))\big) + \big(\mathrm{d}_\varepsilon^{e_2} u(i+\varepsilon e_1) - \mathrm{P}(\mathrm{d}_\varepsilon^{e_2} u(i+\varepsilon e_1))\big)\\
&- \big(\mathrm{d}_\varepsilon^{e_1} u(i+\varepsilon e_2) - \mathrm{P}(\mathrm{d}_\varepsilon^{e_1} u(i+\varepsilon e_2))\big) - \big(\mathrm{d}_\varepsilon^{e_2} u(i) - \mathrm{P}(\mathrm{d}_\varepsilon^{e_2} u(i))\big)\,.
\end{aligned} \tag{1.6}
$$

One can easily check that the vorticity takes values in the set $\{-1,0,1\}$. Furthermore, we define the discrete vorticity measure $\mu(u)$ as

$$
\mu(u) := \sum_{i\in\Omega_\varepsilon^2} \alpha_u(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)}\,.
$$

For every $u \in \mathscr{D}_\varepsilon(\Omega)$, one can check that if A is an open subset of Ω such that $|\Pi(w_u)| \ge \eta$ on ∂A_ε for some $\eta > 0$, then

$$
\mu(u)(A_\varepsilon) = \deg(\Pi(w_u), \partial A_\varepsilon)\,. \tag{1.7}
$$

Here and below, for every open bounded set E with Lipschitz continuous boundary and for every $h \in H^{\frac{1}{2}}(\partial E;\mathbb{R}^2)$ with $|h| \ge \eta > 0$, $\deg(h,\partial E)$ is the winding number of the map $\frac{h}{|h|} : \partial E \to \mathcal{S}^1$, i.e.,

$$
\deg(h,\partial E) := \frac{1}{2\pi}\int_{\partial E} \frac{h}{|h|}\cdot\frac{\partial}{\partial\tau}\Big(\frac{h_2}{|h|};-\frac{h_1}{|h|}\Big)\,\mathrm{d}\mathcal{H}^1\,,
$$

where τ is the tangent field to ∂E and the product in the above formula is understood in the sense of the duality between $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$ (see [14, 15]).

We set

$$
\begin{aligned}
X(\Omega) &:= \Big\{\mu = \sum_{k=1}^{K} z^k\delta_{x^k} \,:\, K\in\mathbb{N}\,,\, z^k\in\mathbb{Z}\setminus\{0\}\,,\, x^k\in\Omega\Big\}\,,\\
X_\varepsilon(\Omega) &:= \Big\{\mu\in X(\Omega) \,:\, \mu = \sum_{i\in\Omega_\varepsilon^2}\alpha(i)\delta_{i+\frac{\varepsilon}{2}(e_1+e_2)} \,:\, \alpha(i)\in\{-1,0,1\}\Big\}\,, \quad \text{for every } \varepsilon>0\,.
\end{aligned}
$$

The Γ-convergence result. The main result proved in this paper is stated in the next theorem whose statement requires the definition of flat convergence of measures. The latter, together with the usual weak star convergence of measures, are recalled below for the reader's convenience.

Let $C_c(\Omega)$ denote the space of continuous functions compactly supported in Ω endowed with the supremum norm. We say that a family $\{\mu_\varepsilon\}_\varepsilon$ of measures converge weakly star in Ω to a measure μ, and we write $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ if for any $\varphi \in C_c(\Omega)$

$$
\langle\mu_\varepsilon,\varphi\rangle \to \langle\mu,\varphi\rangle \qquad \text{as } \varepsilon\to 0\,.
$$

Let $C^{0,1}(\Omega)$ denote the space of Lipschitz continuous functions on Ω endowed with the norm

$$
\|\psi\|_{C^{0,1}} := \sup_{x\in\Omega}|\psi(x)| + \sup_{\substack{x,y\in\Omega\\ x\ne y}}\frac{|\psi(x)-\psi(y)|}{|x-y|},
$$

and let $C_c^{0,1}(\Omega)$ be the subspace of its functions with compact support in Ω. The norm in the dual of $C_c^{0,1}(\Omega)$ is denoted by $\|\cdot\|_{\mathrm{flat}}$ and referred to as *flat norm*, while $\overset{\mathrm{flat}}{\to}$ denotes the convergence with respect to this norm.

Theorem 1.2. *Let $\mathbb{A}_{\mathrm{hom}}$ be the two-by-two symmetric matrix defined in formula* (2.2) *below. The following Γ-convergence result holds true.*

(i) *(Γ-liminf inequality) For any family $\{u_\varepsilon\}_\varepsilon$ with $u_\varepsilon \in \mathscr{D}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \overset{\mathrm{flat}}{\to} \mu$ with $\mu \in X(\Omega)$, we have*

$$
\liminf_{\varepsilon\to0}\frac{\mathcal{E}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \ge \frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}|\mu|(\Omega).
$$

(ii) *(Γ-limsup inequality) For every $\mu \in X(\Omega)$, there exists a sequence $\{u_\varepsilon\}_\varepsilon$ with $u_\varepsilon \in \mathscr{D}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon) \stackrel{\mathrm{flat}}{\to} \mu$ and*

$$\limsup_{\varepsilon\to 0} \frac{\mathcal{E}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \le \frac{1}{2\pi}\sqrt{\det \mathbb{A}_{\mathrm{hom}}}|\mu|(\Omega)\,. \tag{1.8}$$

The Γ-convergence result above is complemented by the following compactness statement that, in view of (1.4), is a corollary of [6, Theorem 3.1].

Theorem 1.3. *Let $\{u_\varepsilon\}_\varepsilon$ with $u_\varepsilon \in \mathscr{D}_\varepsilon(\Omega)$ be such that $\mathcal{E}_\varepsilon(u_\varepsilon) \le C|\log\varepsilon|$. Then, up to a subsequence, $\mu(u_\varepsilon) \stackrel{\mathrm{flat}}{\to} \mu$ for some $\mu \in X(\Omega)$.*

2. Periodic homogenization of discrete energies

In this section we specialize to our case a homogenization result in the framework of discrete-to-continuum limits that has been proven in [3]. We first introduce some notation. For every open bounded set $A \subset \mathbb{R}^2$ and for every $\delta > 0$ we define the functional $\mathcal{G}_\delta : \mathscr{D}_\delta(A) \to [0,+\infty)$ as

$$\mathcal{G}_\delta(u;A) := \sum_{i\in A^2_\delta}\sum_{k=1}^{2} a^k\Big(\frac{i}{\delta}\Big)(\mathrm{d}^{e_k}_\delta u(i))^2\,, \tag{2.1}$$

where a^1 and a^2 satisfy properties (**P**) and (**G**). With a little abuse of notation, we say that a family of discrete functions $\{u_\delta\}_\delta$ with $u_\delta \in \mathscr{D}_\delta(A)$ for every $\delta > 0$ converge in $L^2(A)$ to a function $u \in L^2(A)$ if the family of its piecewise affine interpolations $\{\Pi(u_\delta)\}_\delta$ extended to 0 outside A_δ converge in $L^2(A)$ to the function u. Furthermore, for every $\xi \in \mathbb{R}^2$ and for every $\delta > 0$ we define the set

$$\mathscr{D}^\xi_\delta(A) := \{u \in \mathscr{D}_\delta(A)\,:\, u(i) = \xi\cdot i \text{ for every } i \in \partial_\delta A\}\,,$$

where $\cdot$ denotes the standard scalar product in $\mathbb{R}^2$. The following result is an immediate consequence of [3, Theorem 4.1 & Remark 5.2].

Theorem 2.1. *The functional $\mathcal{G}_\delta(\cdot;A)$ Γ-converge as $\delta \to 0$ with respect to the strong convergence in $L^2(A)$ to the functional $\mathcal{G}_{\mathrm{hom}}(\cdot;A) : L^2(A) \to [0,+\infty)$ defined as*

$$\mathcal{G}_{\mathrm{hom}}(u;A) := \begin{cases} \displaystyle\int_A \langle \mathbb{A}_{\mathrm{hom}}\nabla u\,,\nabla u\rangle\,\mathrm{d}x & \text{if } u \in H^1(A)\\ +\infty & \text{elsewhere}\,,\end{cases}$$

where $\mathbb{A}_{\mathrm{hom}}$ is the two-by-two symmetric matrix defined by the following homogenization formula

$$\langle \mathbb{A}_{\mathrm{hom}}\xi\,,\xi\rangle := \lim_{t\to+\infty}\frac{1}{t^2}\min\Big\{\sum_{i\in(tQ)^0_1}\sum_{k=1}^{2} a^k(i)(\mathrm{d}^{e_k}_1 u(i))^2\,:\, u\in\mathscr{D}^\xi_1(A)\Big\}\,. \tag{2.2}$$

For every $\varphi \in C^{0,1}(\mathbb{R}^2)$ we define the functionals

$$\mathcal{G}^\varphi_\delta(u;A) := \begin{cases} \mathcal{G}^\varphi_\delta(u;A) & \text{if } u(i) = \varphi(i) \text{ for every } i \in \partial_\delta A\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

and

$$\mathcal{G}^\varphi_{\mathrm{hom}}(u;A) := \begin{cases} \mathcal{G}^\varphi_{\mathrm{hom}}(u;A) & \text{if } u - \varphi \in H^1_0(A)\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

Proposition 2.2. *The functionals $\mathcal{G}^\varphi_\delta(\cdot;A)$ Γ-converge as $\delta \to 0$ with respect to the strong convergence in $L^2(A)$ to the functional $\mathcal{G}^\varphi_{\mathrm{hom}}(\cdot;A)$.*

Remark 2.3. Notice that, in view of properties (**P**) and (**G**), the matrix $\mathbb{A}_{\mathrm{hom}}$ defined in (2.2) satisfies

$$\gamma_1|M|^2 \le \langle \mathbb{A}_{\mathrm{hom}}M, M\rangle \le \gamma_2|M|^2 \qquad \text{for every } M \in \mathbb{R}^2\,.$$

In the following, with a little abuse of notations, we extend the functional $\mathcal{G}_{\mathrm{hom}}$ by setting

$$\mathcal{G}_{\mathrm{hom}}(u;A) := \begin{cases} \displaystyle\int_A \langle \mathbb{A}_{\mathrm{hom}}\nabla u, \nabla u\rangle\,\mathrm{d}x & \text{if } u \in SBV^2(A)\\ +\infty & \text{elsewhere}\,,\end{cases} \tag{2.3}$$

where ∇u denotes the absolutely continuous part of the measure derivative $\mathrm{D}u$ with respect to the Lebesgue measure in $\mathbb{R}^2$.

3. Asymptotic analysis on annuli

In this section we prove some auxiliary results on the asymptotic behavior of the minimal energy $\mathcal{E}_\varepsilon$ on annuli, showing that it converges in a suitable sense to the minimal energy of the functional $\mathcal{G}_{\mathrm{hom}}$ defined in (2.3). We start by introducing some notations. For every $0<r<R$ and for every $x\in\mathbb{R}^2$ we set $A_{r,R}(x):=B_R(x)\setminus\overline{B}_r(x)$ and $A_{r,R}:=A_{r,R}(0)$. Moreover, we set

$$\mathscr{A}_{r,R}(z):=\{u\in SBV^2(A_{r,R})\,:\,w=e^{2\pi\imath u}\in H^1(A_{r,R};\mathcal{S}^1)\,,\,\deg(w,\partial B_r)=z\}\,.$$

Remark 3.1. Let $z\in\mathbb{Z}\setminus\{0\}$ and let $\sqrt{\mathbb{A}_{\mathrm{hom}}}$ be the unique two-by-two positive definite symmetric matrix such that $\sqrt{\mathbb{A}_{\mathrm{hom}}}\sqrt{\mathbb{A}_{\mathrm{hom}}}=\mathbb{A}_{\mathrm{hom}}$, where $\mathbb{A}_{\mathrm{hom}}$ is defined in (2.2). For every $0<r'<R'$, we set

$$\begin{aligned}\mathscr{A}^{\mathbb{A}_{\mathrm{hom}}}_{r',R'}(z):=&\{u\in SBV^2\big(\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r',R'})\big)\,:\,w=e^{2\pi\imath u}\in H^1\big(\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r',R'});\mathcal{S}^1\big)\,,\\&\qquad\qquad\deg\big(w,\partial(\sqrt{\mathbb{A}_{\mathrm{hom}}}(B'_r))\big)=z\}\,.\end{aligned}$$

By a change of variable for every $0<r'<R'$ and for every $u\in\mathscr{A}^{\mathbb{A}_{\mathrm{hom}}}_{r',R'}(z)$ we have

$$\begin{aligned}\mathcal{G}_{\mathrm{hom}}\big(u;\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r',R'})\big)=&\int_{\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r',R'})}\big|\sqrt{\mathbb{A}_{\mathrm{hom}}}\nabla u\big|^2\,\mathrm{d}x\\=&\int_{\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r',R'})}\Big|\nabla\big(u\circ\sqrt{\mathbb{A}_{\mathrm{hom}}}\big)\Big|^2\,\mathrm{d}x\\\geq&\det\sqrt{\mathbb{A}_{\mathrm{hom}}}\min_{v\in\mathscr{A}_{r',R'}(z)}\int_{A_{r',R'}}|\nabla v|^2\,\mathrm{d}y\\=&\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\int_{A_{r',R'}}|\nabla\theta|^2\,\mathrm{d}x=\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\log\frac{R'}{r'}\,,\end{aligned}\tag{3.1}$$

where $\theta\in SBV^2(\mathbb{R}^2\setminus\{0\})$ is the function defined by

$$\theta(x):=\frac{1}{2\pi}\begin{cases}\arctan\frac{x_2}{x_1}+\frac{3}{2}\pi & \text{if } x_1<0\\ \pi & \text{if } x_1=0 \text{ and } x_2>0\\ \arctan\frac{x_2}{x_1}+\frac{\pi}{2} & \text{if } x_1>0\\ 2\pi & \text{if } x_1=0\text{ and } x_2<0\,.\end{cases}\tag{3.2}$$

By (3.1) we have, in particular, that the function

$$z\theta_{\mathbb{A}_{\mathrm{hom}}}:=z\theta\circ\big(\sqrt{\mathbb{A}_{\mathrm{hom}}}\big)^{-1}\tag{3.3}$$

is a minimizer of $\mathcal{G}_{\mathrm{hom}}$ in $\mathscr{A}^{\mathbb{A}_{\mathrm{hom}}}_{r',R'}(z)$. Denoting by λ and Λ the minimal and the maximal eigenvalue of $\sqrt{\mathbb{A}_{\mathrm{hom}}}$, respectively, we have that $0<\lambda<\Lambda$ and $\lambda A_{r,R}\subset\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r,R})\subset\Lambda A_{r,R}$. It follows that, there exists a constant $C_{\mathbb{A}_{\mathrm{hom}},z}>0$ depending only on $\mathbb{A}_{\mathrm{hom}}$ and z (and independent of r and R) and a function $f(r,R,\mathbb{A}_{\mathrm{hom}},z)$ with $|f(r,R,\mathbb{A}_{\mathrm{hom}},z)|\leq C_{\mathbb{A}_{\mathrm{hom}},z}$ such that

$$\begin{aligned}\min_{u\in\mathscr{A}_{r,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{r,R})=&\mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}};\sqrt{\mathbb{A}_{\mathrm{hom}}}(A_{r,R}))+f(r,R,\mathbb{A}_{\mathrm{hom}},z)\\=&\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\log\frac{R}{r}+f(r,R,\mathbb{A}_{\mathrm{hom}},z)\,,\end{aligned}$$

whence we deduce that

$$\lim_{\frac{R}{r}\to+\infty}\frac{1}{\log\frac{R}{r}}\min_{u\in\mathscr{A}_{r,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{r,R})=\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\,.\tag{3.4}$$

Note that the same argument above shows also that

$$\begin{aligned}\min_{u\in\mathscr{A}_{r,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{r,R})\leq&\mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}};A_{r,R})+C'_{\mathbb{A}_{\mathrm{hom}},z}\\\leq&\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\log\frac{R}{r}+C'_{\mathbb{A}_{\mathrm{hom}},z}\,,\end{aligned}\tag{3.5}$$

for some constant $C'_{\mathbb{A}_{\mathrm{hom}},z}$ depending only on $\mathbb{A}_{\mathrm{hom}}$ and z.

3.1. **Lower bound.** We show that the asymptotical minimal energy $\mathcal{E}_\varepsilon$ induced by a singularity at ξ with weight $z\in\mathbb{Z}\setminus\{0\}$ in an annulus $A_{r,R}(\xi)$ is bounded from below by $\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\log\frac{R}{r}$. We prove that this bound holds true also when the radii are powers of ε. To this purpose, for every $z\in\mathbb{Z}\setminus\{0\}$, for every $0<r<R$ and for every $\delta>0$, we define

$$\mathscr{A}_{r,R,\delta}(z):=\{u\in\mathscr{D}_\delta(A_{r,R})\,:\,\mu(u)(B_{r+\sqrt{2}\delta})=z\text{ and }\mu(u)(i+\delta Q)=0\text{ for every }i\in(A_{r,R})^2_\delta\}\,.$$

Furthermore, we say that a family of bonds $\mathsf{J}:=\{(j_{m-1},j_m)\}_{m=1,\dots,M}\subset(A_{r,R})^1_\varepsilon$ is a simple and closed path if $j_m\neq j_n$ for every pair $(m;n)\in\{0,1,\dots,M\}^2\setminus\{(0;M),(M;0)\}$ with $m\neq n$ and $j_0\equiv j_M$; we denote by A_{J} the finite subset of $(A_{r,R})^0_\varepsilon$ enclosed by the union of the segments $[j_{m-1},j_m]$ as $m=1,\dots,M$.

Proposition 3.2. *Let $z\in\mathbb{Z}\setminus\{0\}$ and let $0<r<R$. Let $\mathcal{E}_\delta$ be the functional defined in* (1.1) *for $\varepsilon=\delta$ with a^k satisfying* (**P**) *and* (**G**)*. Then,*

$$\liminf_{\delta\to0}\min_{u\in\mathscr{A}_{r,R,\delta}(z)}\mathcal{E}_\delta(u;A_{r,R})\geq\min_{u\in\mathscr{A}_{r,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{r,R})\,.\tag{3.6}$$

Proof. We set

$$S:=\{(0,x_2)\,:\,x_2<0\}\,.\tag{3.7}$$

For every $\delta>0$ let u_δ be a minimizer of $\mathcal{E}_\delta(\cdot;A_{r,R})$ in $\mathscr{A}_{r,R,\delta}(z)$. We aim at constructing a function $\tilde u_\delta$, which in an open set U compactly supported in $A_{r,R}\setminus S$ satisfies

$$\mathcal{G}_\delta(\tilde u_\delta;U)=\mathcal{E}_\delta(u_\delta;U)\,.$$

Since $u_\delta\in\mathscr{A}_{r,R,\delta}(z)$, using a Poincaré lemma type argument, we can construct a function $\tilde u_\delta\in\mathscr{D}_\delta(A_{r,R})$ such that

$$\mathrm{d}\tilde u_\delta(i,j)=\mathrm{d}u_\delta(i,j)-\mathrm{P}(\mathrm{d}u_\delta(i,j))\qquad\text{for every }(i,j)\in(A_{r,R})^1_\delta\setminus S^1_\delta\,,\tag{3.8}$$

where $S^1_\delta:=\{(i,i+\delta e_1)\,:\,i\in S\cap(A_{r,R})^2_\delta\}$. By (3.8) we have that

$$\mathrm{d}\tilde u_\delta(i,j)-\mathrm{P}(\mathrm{d}\tilde u_\delta(i,j))=\mathrm{d}\tilde u_\delta(i,j)\qquad\text{for every }(i,j)\in(A_{r,R})^1_\delta\setminus S^1_\delta\,;$$

moreover, we claim that

$$\mathrm{d}^{e_1}_\delta\tilde u_\delta(i)+z=\mathrm{d}^{e_1}_\delta u_\delta(i)-\mathrm{P}(\mathrm{d}^{e_1}_\delta u_\delta(i))\quad\text{for every }(i,i+\delta e_1)\in S^1_\delta\,,\tag{3.9}$$

whence we deduce that

$$\mathrm{P}(\mathrm{d}\tilde u_\delta(i,i+\delta e_1))=-z\qquad\text{for every }(i,i+\delta e_1)\in S^1_\delta\,.$$

To prove (3.9) we fix $(i,i+\delta e_1)\in S^1_\delta$ and we consider a generic family of bonds $\{(j_{m-1},j_m)\}_{m=1,\dots,M}\subset(A_{r,R})^1_\delta\setminus S^1_\delta$ such that $j_0=i+\delta e_1$, $j_M=i$, and such that $\{(j_{m-1},j_m)\}_{m=1,\dots,M}\cup(i,i+\delta e_1)$ is a simple and closed counterclockwise oriented path in $A_{r,R}$. Since, by (1.6),

$$\sum_{m=1}^M\big(\mathrm{d}u_\delta(j_{m-1},j_m)-\mathrm{P}(\mathrm{d}u_\delta(j_{m-1},j_m))\big)+\mathrm{d}^{e_1}_\delta u_\delta(i)-\mathrm{P}(\mathrm{d}^{e_1}_\delta u_\delta(i))=z\,,$$

by (3.8), using that $u_\delta\in\mathscr{A}_{r,R,\delta}(z)$, we get

$$\begin{aligned}\mathrm{d}^{e_1}_\delta\tilde u_\delta(i)=&-\sum_{m=1}^M\mathrm{d}\tilde u_\delta(j_{m-1},j_m)=-\sum_{m=1}^M(\mathrm{d}u_\delta(j_{m-1},j_m)-\mathrm{P}(\mathrm{d}u_\delta(j_{m-1},j_m)))\\=&-z+\mathrm{d}^{e_1}_\delta u_\delta(i)-\mathrm{P}(\mathrm{d}^{e_1}_\delta u_\delta(i))\,,\end{aligned}\tag{3.10}$$

i.e., (3.9). We set

$$\bar S_\delta:=\{x\in(A_{r,R})_\delta\,:\,x=y+te_1\,,y\in S\,,\,t\in[0,\delta)\}\,,$$

and we define $\beta_\delta:\bar S_\delta\to\mathbb{R}^2$ as $\beta_\delta(x)=\mathrm{d}^{e_1}_\delta\tilde u_\delta(i)+z$ for every $x\in i+\delta Q$ with $i\in S\cap(A_{r,R})^2_\delta$. Moreover, we define $v_\delta:(A_{r,R})_\delta\to\mathbb{R}$ as

$$v_\delta(x):=\begin{cases}\Pi(\tilde u_\delta)(x)&\text{if }x\in(A_{r,R})_\delta\setminus\bar S_\delta\\ v_\delta(y)+\displaystyle\int_0^t\beta_\delta(y+se_1)\,\mathrm{d}s&\text{if }x=y+te_1\,,y\in S\,,t\in[0,\delta]\,.\end{cases}\tag{3.11}$$

By construction, $v_\delta \in SBV^2((A_{r,R})_\delta)$, $S_{v_\delta} \subset S+\delta e_1$ and $[v_\delta^+ - v_\delta^-] = -z$. Furthermore, by (3.8), (3.9) and (3.11), for every $r<r'<R'<R$ and for δ small enough, we have

$$\int_{A_{r',R'}} |\nabla v_\delta|^2 \,\mathrm{d}x \le \mathcal{E}_\delta(u_\delta; A_{r,R}) \le C\,.$$

Therefore, we can apply Ambrosio compactness result [10] to deduce that, up to a (not relabeled) subsequence $v_\delta \to v$ in $L^2(A_{r',R'})$ for some function $v \in SBV^2(A_{r',R'})$; notice that, by [2, Theorem 1.5], $v \in \mathscr{A}_{r',R'}(z)$. Now, for every $\eta>0$, we get that $\Pi(\tilde u_\delta) = v_\delta \to v$ in $L^2(A_{r',R'} \setminus I^\eta)$, where we have set

$$I^\eta := \{(x_1,x_2)\,:\, -\eta \le x_1 \le \eta\,,\ -R \le x_2 \le -r\}\,;$$

we can thus apply Theorem 2.1 to deduce that for every $r<r'<R'<R$

$$\begin{aligned}\mathcal{G}_{\mathrm{hom}}(v;A_{r',R'}) - \omega(\eta) \le & \mathcal{G}_{\mathrm{hom}}(v;A_{r',R'}\setminus I^\eta) \le \liminf_{\delta\to 0} \mathcal{G}_\delta(\tilde u_\delta; A_{r',R'}\setminus I^\eta)\\ &= \liminf_{\delta\to0}\mathcal{E}_\delta(u_\delta;A_{r',R'}\setminus I^\eta) \le \liminf_{\delta\to 0}\mathcal{E}_\delta(u_\delta;A_{r,R})\,,\end{aligned}$$

for some $\omega(\eta)$ tending to 0 as $\eta\to0$, where the equality above is a consequence of (3.8); therefore, sending $r'\to r$ and $R'\to R$ we get (3.6). □

Proposition 3.3. *Let $\mathcal{E}_\varepsilon$ be the functional defined in* (1.1) *with a^k satisfying* (**P**) *and* (**G**) *and let $\mathbb{A}_{\mathrm{hom}}$ be the matrix defined in* (2.2)*. Then for any $0 \le s_1 < s_2 < 1$ we have*

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\inf_{u\in\mathscr{A}_{\varepsilon^{s_2},\varepsilon^{s_1},\varepsilon}(z)}\mathcal{E}_\varepsilon(u;A_{\varepsilon^{s_2},\varepsilon^{s_1}}) \ge \frac{1}{2\pi}(s_2-s_1)\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\,. \tag{3.12}$$

Proof. The proof follows along the lines of the one of [2, Proposition 3.2]. We fix $R>1$, set $K_{\varepsilon,R} = \lfloor (s_2-s_1)\frac{|\log\varepsilon|}{\log R}\rfloor$, and note that $A_{\varepsilon^{s_2},\varepsilon^{s_1}} \supset \bigcup_{k=1}^{K_{\varepsilon,R}} A_{R^{k-1}\varepsilon^{s_2},R^k\varepsilon^{s_2}}$. Let moreover $u_\varepsilon \in \mathscr{A}_{\varepsilon^{s_2},\varepsilon^{s_1},\varepsilon}(z)$ be such that

$$\mathcal{E}_\varepsilon(u_\varepsilon;A_{\varepsilon^{s_2},\varepsilon^{s_1}}) \le \inf_{u\in\mathscr{A}_{\varepsilon^{s_2},\varepsilon^{s_1},\varepsilon}(z)}\mathcal{E}_\varepsilon(u;A_{\varepsilon^{s_2},\varepsilon^{s_1}}) + C, \tag{3.13}$$

for some constant C (independent of ε) and let $\bar k = \bar k_{\varepsilon,R} \in \{1,\dots,K_{\varepsilon,R}\}$ be such that

$$\mathcal{E}_\varepsilon(u_\varepsilon;A_{R^{\bar k-1}\varepsilon^{s_2},R^{\bar k}\varepsilon^{s_2}}) \le \mathcal{E}_\varepsilon(u_\varepsilon;A_{R^{k-1}\varepsilon^{s_2},R^k\varepsilon^{s_2}})\,,\qquad \text{for all } k=1,\dots,K_{\varepsilon,R}\,.$$

Therefore

$$\mathcal{E}_\varepsilon(u_\varepsilon;A_{\varepsilon^{s_2},\varepsilon^{s_1}}) \ge \sum_{k=1}^{K_{\varepsilon,R}}\mathcal{E}_\varepsilon(u_\varepsilon;A_{R^{k-1}\varepsilon^{s_2},R^k\varepsilon^{s_2}}) \ge K_{\varepsilon,R}\,\mathcal{E}_\varepsilon(u_\varepsilon;A_{R^{\bar k-1}\varepsilon^{s_2},R^{\bar k}\varepsilon^{s_2}})\,. \tag{3.14}$$

By the change of variable $y = R^{1-\bar k}\varepsilon^{-s_2}x$, $u'_{\varepsilon,\bar k}(y) := u_\varepsilon(R^{\bar k-1}\varepsilon^{s_2}y)$, we have

$$\mathcal{E}_\varepsilon(u_\varepsilon;A_{R^{\bar k-1}\varepsilon^{s_2},R^k\varepsilon^{s_2}}) = \mathcal{E}_{R^{1-\bar k}\varepsilon^{1-s_2}}(u'_{\varepsilon,\bar k};A_{1,R})\,. \tag{3.15}$$

Therefore, by using (3.13), (3.14), (3.15), and Theorem 3.2, we deduce that

$$\begin{aligned}\liminf_{\varepsilon\to0}&\frac{1}{|\log\varepsilon|}\inf_{u\in\mathscr{A}_{\varepsilon^{s_2},\varepsilon^{s_1},\varepsilon}(z)}\mathcal{E}_\varepsilon(u;A_{\varepsilon^{s_2},\varepsilon^{s_1}})\\
&\ge \liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(u_\varepsilon;A_{\varepsilon^{s_2},\varepsilon^{s_1}})\\
&\ge \liminf_{\varepsilon\to0}\frac{K_{\varepsilon,R}}{|\log\varepsilon|}\inf\{\mathcal{E}_{R^{1-\bar k}\varepsilon^{1-s_2}}(u;A_{1,R})\,:\, u\in\mathscr{A}_{1,R,R^{1-\bar k}\varepsilon^{1-s_2}}(z)\}\\
&\ge \lim_{\varepsilon\to0}\Big(\frac{s_2-s_1}{\log R}-\frac{1}{|\log\varepsilon|}\Big)\liminf_{\varepsilon\to0}\inf\{\mathcal{E}_{R^{1-\bar k}\varepsilon^{1-s_2}}(u;A_{1,R})\,:\, u\in\mathscr{A}_{1,R,R^{1-\bar k}\varepsilon^{1-s_2}}(z)\}\\
&= \frac{s_2-s_1}{\log R}\min_{u\in\mathscr{A}_{1,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{1,R})\,.\end{aligned}$$

Formula (3.12) follows from the estimate above as $R\to+\infty$ thanks to (3.4). □

3.2. **Upper bound.** The analysis carried out in this subsection will be crucial for the proof of the Γ-limsup inequality in Theorem 1.2(ii).

Proposition 3.4. *Let $\mathcal{E}_\varepsilon$ be the functional defined in* (1.1) *with a^k satisfying* (**P**) *and* (**G**). *Let moreover $\mathbb{A}_{\mathrm{hom}}$ be the matrix defined in* (2.2) *and θ_{hom} be the function in* (3.3). *Then, for every $\rho > 0$ and for every $0 < s < 1$, there exists a family $\{u_{\varepsilon,s}\}_\varepsilon$ with $u_{\varepsilon,s} \in \mathscr{D}_\varepsilon(A_{\varepsilon^s,\rho})$ and $u_{\varepsilon,s}(\cdot) = z\theta_{\mathrm{hom}}(\cdot)$ on $\partial_\varepsilon A_{\varepsilon^s,\rho}$ and a constant C independent of ε such that*

$$\|\mu(u_{\varepsilon,s}) \llcorner A_{\varepsilon^s,\rho}\|_{\mathrm{flat}} \leq C\varepsilon|\log\varepsilon| \tag{3.16}$$

and

$$\limsup_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(u_{\varepsilon,s}; A_{\varepsilon^s,\rho}) \leq \frac{s}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\, z^2 . \tag{3.17}$$

The proof of Proposition 3.4 needs some auxiliary results. The next lemma shows that, if $\{u_\delta\}_\delta$ is a family of functions with uniformly bounded energy $\mathcal{E}_\delta$ which satisfies suitable boundary conditions, then the measures $\mu(u_\delta)$ can have at most a finite number K of dipoles of size δ, with K independent of δ.

Lemma 3.5. *Let $A, A' \subset \mathbb{R}^2$ be open and bounded with $A \subset\subset A'$. For every $\delta > 0$, let $\mathcal{G}_\delta(\cdot; A)$ be the functional defined in* (2.1) *for $\varepsilon = \delta$ with a^k satisfying* (**P**) *and* (**G**). *Let $\{u_\delta\}_\delta$ be such that, for every $\delta > 0$, $u_\delta \in \mathscr{D}_\delta(A)$ and $u_\delta = \bar{u}$ on $\partial_\delta A$ for some $\bar{u} \in C^\infty(A')$. If*

$$\sup_{\delta>0} \mathcal{E}_\delta(u_\delta; A) \leq C , \tag{3.18}$$

for some constant $C > 0$ (independent of δ), then there exist $K = K(C)$ and $L = L(C)$ in $\mathbb{N}$ such that:

(a) $\sharp\,\mathrm{supp}\,\mu(u_\delta) \leq K$,
(b) $\mu(u_\delta)(B_{L\delta}(\xi)) = 0$ *for every $\xi \in \mathrm{supp}\,\mu(u_\delta)$.*

Proof. For every $\delta > 0$ let $\widehat{u}_\delta \in \mathscr{D}_\delta(A')$ be the function defined by

$$\widehat{u}_\delta(i) := \begin{cases} u_\delta(i) & \text{if } i \in A^0_\delta \\ \bar{u}(i) & \text{if } i \in (A')^0_\delta \setminus A^0_\delta . \end{cases}$$

Since $\bar{u} \in C^\infty(A')$, by (3.18) we have

$$\sup_{\delta>0} \mathcal{E}_\delta(\widehat{u}_\delta; A') \leq C . \tag{3.19}$$

For every $\delta > 0$ we set

$$\bar{E}_\delta := \left\{(i,j) \in (A')^1_\delta \,:\, \mathrm{dist}(\mathrm{d}\widehat{u}_\delta(i,j), \mathbb{Z}) \geq \frac{1}{8}\right\},$$

and we notice that, by construction, for δ small enough

$$\bar{E}_\delta = \left\{(i,j) \in A^1_\delta \,:\, \mathrm{dist}(\mathrm{d}u_\delta(i,j), \mathbb{Z}) \geq \frac{1}{8}\right\}.$$

By property (**G**) and by (3.18), we have that

$$\sharp\,\mathrm{supp}\,\mu(u_\delta) \leq \sharp\bar{E}_\delta \leq \frac{64}{\gamma_1}C , \tag{3.20}$$

hence (a).

In order to prove (b), we fix $\hat{\imath}^\delta \in A^2_\delta$ such that $\mu(u_\delta)(\hat{\imath}^\delta + \delta Q) \neq 0$ and we set

$$L := \sup\left\{\bar{l} \in \mathbb{N} \,:\, B_{2^l\delta}(\hat{\imath}^\delta) \subset A' \text{ and } \mu(\widehat{u}_\delta)(B_{2^l\delta}(\hat{\imath}^\delta)) \neq 0 \quad \text{for every } l = 1, \dots, \bar{l}\right\}.$$

In view of (3.20) it is enough to show that L is bounded by a constant depending only on C. To this purpose, we define $\mathcal{L}_{\mathrm{bad}}$ as the set of indices $l = 1, \dots, L$ such that the annulus $A_{2^{l-1}\delta, 2^l\delta}(\hat{\imath}^\delta)$ contains at least one bond in $\bar{E}_\delta$. Notice that, if $l \notin \mathcal{L}_{\mathrm{bad}}$, then $\mu(\widehat{u}_\delta)(B_\rho(\hat{\imath}^\delta)) = \mu(\widehat{u}_\delta)(B_{2^{l-1}\delta}(\hat{\imath}^\delta))$ for every $2^{l-1}\delta < \rho < 2^l\delta$. In view of (3.20), we have that $\sharp\mathcal{L}_{\mathrm{bad}} \leq K$. If $L \leq K + 3$, the claim is proven. Otherwise, noticing that there exists a universal constant $\eta > 0$ such that

$$\min_{A_{(2^{l-1}+\sqrt{2})\delta, (2^l-\sqrt{2})\delta}} |\Pi(w_{\widehat{u}_\delta})| \geq \eta \qquad \text{for every } l \notin \mathcal{L}_{\mathrm{bad}} ,$$

by (1.4) and by (1.7), for δ small enough, we get

$$\begin{aligned}\mathcal{E}_\delta(\widehat{u}_\delta, A') \geq & \mathcal{E}_\delta(\widehat{u}_\delta; A_{\delta,2^L\delta}(\hat{\imath}^\delta)) \\ \geq & \sum_{\substack{l=1\\ l\notin\mathcal{L}_{\mathrm{bad}}}}^{L} \mathcal{E}_\delta(\widehat{u}_\delta; A_{2^{l-1}\delta,2^l\delta}(\hat{\imath}^\delta)) \\ \geq & \frac{\gamma_1}{4\pi^2}\sum_{\substack{l=3\\ l\notin\mathcal{L}_{\mathrm{bad}}}}^{L} \int_{A_{(2^{l-1}+\sqrt{2})\delta,(2^l-\sqrt{2})\delta}(\hat{\imath}^\delta)} |\nabla\Pi(w_{\widehat{u}_\delta})|^2\,\mathrm{d}x \\ = & \frac{\gamma_1}{4\pi^2}\sum_{\substack{l=3\\ l\notin\mathcal{L}_{\mathrm{bad}}}}^{L} \int_{A_{(2^{l-1}+\sqrt{2}),(2^l-\sqrt{2})}(\hat{\imath}^\delta)} |\nabla\Pi(w_{\widehat{u}_\delta})(\delta x)|^2\,\mathrm{d}x \\ \geq & \frac{\gamma_1}{2\pi}\eta^2\log 2(L-3-K)\,,\end{aligned}$$

which, in view of (3.19), implies the claim. □

For every $A\subset\mathbb{R}^2$ open and for every $\delta>0$, let $\widetilde{\mathscr{A}}^{K,L}_{A,\delta}$ be the set of the functions $u\in\mathscr{D}_\delta(A)$ satisfying properties (a) and (b) of Lemma 3.5 for some $K,L\in\mathbb{N}$ and such that

$$u = z\theta_{\mathbb{A}_{\mathrm{hom}}} \qquad \text{on } \partial_\delta A\,, \tag{3.21}$$

where $\theta_{\mathbb{A}_{\mathrm{hom}}}$ is defined in (3.3).

Theorem 3.6. *For every $\delta>0$ let $\mathcal{E}_\delta$ be the functional defined in (1.1) for $\varepsilon=\delta$ with a^k satisfying (**P**) and (**G**). Then, for every $z\in\mathbb{Z}\setminus\{0\}$ and for every $0<r<R$, there exist $K=K_{r,R,z}$ and $L=L_{r,R,z}$ in $\mathbb{N}$ and a family $\{u_\delta\}_\delta\subset\widetilde{\mathscr{A}}^{K,L}_{A_{r,R},\delta}(z)$ such that*

$$\limsup_{\delta\to 0}\mathcal{E}_\delta(u_\delta;A_{r,R}) \leq \frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\log\frac{R}{r} + C_{\mathbb{A}_{\mathrm{hom}},z}\,, \tag{3.22}$$

for some constant $C_{\mathbb{A}_{\mathrm{hom}},z}$ depending only on $\mathbb{A}_{\mathrm{hom}}$ and z.

Proof. By (3.5), we have that

$$\min_{u\in\mathscr{A}_{r,R}(z)}\mathcal{G}_{\mathrm{hom}}(u;A_{r,R}) \leq \mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}};A_{r,R}) + C'_{\mathbb{A}_{\mathrm{hom}},z}\,;$$

therefore, it is enough to construct a family $\{u_\delta\}_\delta$ with $u_\delta\in\widetilde{\mathscr{A}}^{K,L}_{A_{r,R},\delta}(z)$ for every $\delta>0$, such that

$$\limsup_{\delta\to 0}\mathcal{E}_\delta(u_\delta;A_{r,R}) \leq \mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}};A_{r,R}) + C\,,$$

for some constant depending only on $\mathbb{A}_{\mathrm{hom}}$ and z.

By the very definition of θ in (3.2) we have that $\theta\in C^\infty(\mathbb{R}^2\setminus S_\theta)$, $\nabla\theta\in L^\infty(\mathbb{R}^2\setminus 0)$, $S_\theta\equiv S$ with S defined in (3.7) and

$$\theta^+-\theta^- = 1 \quad \text{ on } S\,; \tag{3.23}$$

therefore, all the properties above are inherited by $\theta_{\mathbb{A}_{\mathrm{hom}}}$, up to replacing S with $\sqrt{\mathbb{A}_{\mathrm{hom}}}(S)$.

For every $\eta>0$, we set

$$I^\eta := \{x\in\mathbb{R}^2\,:\,\mathrm{dist}\big(x,\sqrt{\mathbb{A}_{\mathrm{hom}}}(S)\big)\leq\eta\}\,. \tag{3.24}$$

We claim that for every $\eta>0$, there exists a family $\{\widetilde{u}^\eta_\delta\}_\delta\subset\widetilde{\mathscr{A}}^{K,L}_{A_{r,R},\delta}$ for every $\delta>0$ and for some $K,L\in\mathbb{N}$ depending only on r, R and z, such that

$$\limsup_{\delta\to 0}\mathcal{E}_\delta(\widetilde{u}^\eta_\delta;A_{r,R}) \leq \mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}};A_{r,R}) + \omega(\eta)\,. \tag{3.25}$$

with $\omega(\eta)\to 0$ as $\eta\to 0$. Notice that, suitably choosing $\eta=\eta(\delta)\to 0$ as $\delta\to 0$, by the claim above, we have that the sequence $\{u_\delta\}_\delta$ with $u_\delta=\widetilde{u}^{\eta(\delta)}_\delta$, satisfies the statement of the theorem.

It remains to prove the claim. For any $\eta > 0$, by Proposition 2.2, there exists $\{u_\delta^\eta\}_\delta \subset \mathscr{D}_\delta(A_{r,R} \setminus I^\eta)$ such that $u_\delta^\eta = z\theta_{\mathbb{A}_{\mathrm{hom}}}$ on $\partial_\delta I^\eta$ and

$$\begin{aligned}\limsup_{\delta\to 0}\mathcal{E}_\delta(u_\delta^\eta; A_{r,R}\setminus I^\eta) \leq &\limsup_{\delta\to 0}\mathcal{G}_\delta(u_\delta^\eta; A_{r,R}\setminus I^\eta) \leq \mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}}; A_{r,R}\setminus I^\eta)\\ &\leq \mathcal{G}_{\mathrm{hom}}(z\theta_{\mathbb{A}_{\mathrm{hom}}}; A_{r,R}),\end{aligned} \tag{3.26}$$

where the first inequality follows by the very definitions of $\mathcal{E}_\delta$ and $\mathcal{G}_\delta$. We define the map $\widetilde{u}_\delta^\eta \in \mathscr{D}_\delta(A_{r,R})$ as

$$\widetilde{u}_\delta^\eta(i) := \begin{cases} u_\delta^\eta(i) & \text{if } i \in (A_{r,R}\setminus I^\eta)\cap(A_r,R)_\delta^0 \\ z\theta_{\mathbb{A}_{\mathrm{hom}}}(i) & \text{if } i \in (I^\eta \setminus \sqrt{\mathbb{A}_{\mathrm{hom}}}(S))\cap(A_{r,R})_\delta^0 \\ z\theta^+_{\mathbb{A}_{\mathrm{hom}}}(i) & \text{if } i \in (\sqrt{\mathbb{A}_{\mathrm{hom}}}(S))\cap(A_{r,R})_\delta^0 \,. \end{cases} \tag{3.27}$$

It holds that

$$\limsup_{\delta\to 0}\mathcal{E}_\delta(\widetilde{u}_\delta^\eta; (I^\eta + B_{\sqrt{2}\delta})\cap A_{r,R}) = \omega(\eta)\,, \tag{3.28}$$

for some $\omega(\eta)\to 0$ as $\eta\to 0$. By (3.26) and (3.28), (3.25) follows.

Now we show that there exists $K = K_{r,R,z}$ and $L = L_{r,R,z}$ in $\mathbb{N}$ such that $\widetilde{u}_\delta^\eta \in \widetilde{\mathscr{A}}^{K,L}_{r,R,\delta}$ for every $\delta > 0$. By construction $\widetilde{u}_\delta^\eta$ satisfy (3.21) for every $\delta,\eta > 0$. Moreover, by the very definition of $\widetilde{u}_\delta^\eta$ in (3.27) we have that

$$\mu(\widetilde{u}_\delta^\eta)(i+\delta Q) = 0 \qquad \text{for every } i \in \big(A_{r,R}\cap(I^\eta + B_{\sqrt{2}\delta})\big)^2_\delta\,.$$

Notice that the functions obtained extending $\widetilde{u}_\delta^\eta$ to the larger annuli $A_{r',R'} \supset A_{r,R}$ by setting $\widetilde{u}_\delta^\eta := z\theta_{\mathbb{A}_{\mathrm{hom}}}$ still have equi-bounded energy. Therefore, in view of (3.26), we can apply Lemma 3.5 to the sets $A = A_{r,R}\setminus I^\eta$ and $A' = A_{r',R'}\setminus I^{\frac{\eta}{2}}$ to deduce that there exist $K, L \in \mathbb{N}$ depending only on r, R and z such that $\sharp\mathrm{supp}\,\mu(\widetilde{u}_\delta^\eta) \leq K$ and $\mu(\widetilde{u}_\delta^\eta)(B_{L\delta}(\xi)) = 0$ for every $\xi \in \mathrm{supp}\,\mu(\widetilde{u}_\delta^\eta)$; this proves that $\widetilde{u}_\delta^\eta \in \widetilde{\mathscr{A}}^{K,L}_{A_{r,R},\delta}$ and concludes the proof.

□

Proof of Proposition 3.4. Let $R > 1$ and set $M_{\varepsilon,R} := \lceil \frac{\log\rho + s|\log\varepsilon|}{\log R}\rceil$ for every $\varepsilon > 0$. Moreover, for every $m = 1,\dots,M_{\varepsilon,R}$ we set $\delta_{\varepsilon,m} := \varepsilon^{1-s}R^{1-m}$ and we notice that $\delta_{\varepsilon,m} \leq \delta_{\varepsilon,1} = \varepsilon^{1-s} \to 0$ as $\varepsilon\to 0$ for all $m = 1,\dots,M_{\varepsilon,R}$. We define the function $u_{\varepsilon,s} \in \mathscr{D}_\varepsilon(A_{\varepsilon^s,\rho})$ as

$$u_{\varepsilon,s}(i) := \begin{cases} u_{\delta_{\varepsilon,m}}\Big(\frac{i}{R^{m-1}\varepsilon^s}\Big) & \text{if } i \in \big(A_{R^{m-1}\varepsilon^s, R^m\varepsilon^s}\big)^0_\varepsilon \quad \text{for some } m = 1,\dots,M_{\varepsilon,R} \\ z\theta_{\mathbb{A}_{\mathrm{hom}}}(i) & \text{elsewhere in } (A_{\varepsilon^s,1})^0_\varepsilon\,, \end{cases}$$

where $\{u_\delta\}_\delta$ is the family provided by Theorem 3.6. By (3.22) and by change of variable, for every $m = 1,\dots,M_{\varepsilon,R}$ we have that

$$\begin{aligned}\mathcal{E}_\varepsilon(u_{\varepsilon,s}; A_{R^{m-1}\varepsilon^s, R^m\varepsilon^s}) =& \mathcal{E}_{\delta_{\varepsilon,m}}(u_{\delta_{\varepsilon,m}}; A_{1,R})\\ \leq& \frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\, z^2\log R + C_{\mathbb{A}_{\mathrm{hom}},z} + \omega(\delta_{\varepsilon,m})\\ \leq& \frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\, z^2\log R + C_{\mathbb{A}_{\mathrm{hom}},z} + \max_{m=1,\dots,M_{\varepsilon,R}}\omega(\delta_{\varepsilon,m})\,,\end{aligned} \tag{3.29}$$

where $\max_{m=1,\dots,M_{\varepsilon,R}}\omega(\delta_{\varepsilon,m}) \to 0$ as $\varepsilon\to 0$. Since

$$\begin{aligned}&\sum_{m=1}^{M_{\varepsilon,R}}\sum_{i\in\partial_\varepsilon B_{R^{m-1}\varepsilon^s}}\sum_{k=1}^{2} a^k\Big(\frac{i}{\varepsilon}\Big)\mathrm{dist}^2(\mathrm{d}_\varepsilon^{e_k}(z\theta_{\mathbb{A}_{\mathrm{hom}}}(i),\mathbb{Z})\\ \leq& C_{\mathbb{A}_{\mathrm{hom}}} z^2\sum_{m=1}^{M_{\varepsilon,R}}\log\frac{R^{m-1}\varepsilon^s + \sqrt{2}\varepsilon}{R^{m-1}\varepsilon^s - \sqrt{2}\varepsilon} \leq C_{\mathbb{A}_{\mathrm{hom}}} z^2\sum_{m=1}^{+\infty}R^{1-m}\varepsilon^{1-s} = C_{\mathbb{A}_{\mathrm{hom}}} z^2\varepsilon^{1-s}\frac{R}{R-1}\,,\end{aligned}$$

by summing (3.29) for every $m = 1,\dots,M_{\varepsilon,R}$ we get

$$\limsup_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(u_{\varepsilon,s}; A_{\varepsilon^s,\rho}) \leq \frac{s}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\, z^2 + C_{\mathbb{A}_{\mathrm{hom}},z}\frac{s}{\log R}\,,$$

whence (3.17) follows by taking the limit as $R\to+\infty$. Therefore, by Theorem 3.6, we have that, for every $M=1,\dots,M_{\varepsilon,R}$, the function $u_{\delta_{\varepsilon,m}}\in\widetilde{\mathscr{A}}^{K,L}_{A_{1,R},\delta_{\varepsilon,m}}$ for some $K,\,L\in\mathbb{N}$ depending only on R and z, whence we deduce that $\sharp\operatorname{supp}\big(\mu(u_{\varepsilon,s})\llcorner(A_{R^{m-1}\varepsilon^s,R^m\varepsilon^s})\big)\le K$ and

$$\mu(u_{\varepsilon,s})(B_{L\varepsilon}(\xi))=0\qquad\text{for every }\xi\in\operatorname{supp}\mu(u^s_\varepsilon)\,.$$

It follows that

$$\|\mu(u_{\varepsilon,s})\llcorner A_{R^{m-1}\varepsilon^s,R^m\varepsilon^s}\|_{\mathrm{flat}}\le KL\varepsilon\qquad\text{for every }m=1,\dots,M_{\varepsilon,R}\,,$$

which summed over $m=1,\dots,M_{\varepsilon,R}$, yields (3.16), and concludes the proof. □

Remark 3.7. We notice that Propositions 3.3 and 3.4 hold true also if the center of the annulus is a point ξ_ε depending on ε, since all the estimates in the previous proofs do not depend on the center of the annulus. As for the lower bound in Proposition 3.3, the set $\mathscr{A}_{r,R,\varepsilon}$ should be replaced by the set

$$\begin{aligned}\mathscr{A}_{r,R,\varepsilon}(z;\xi_\varepsilon):=&\{u\in\mathscr{D}_\varepsilon(A_{r,R}(\xi_\varepsilon))\,:\,\mu(u)(B_{r+\sqrt{2}\varepsilon}(\xi_\varepsilon))=z\\&\qquad\text{and }\mu(u)(i+\varepsilon Q)=0\text{ for every }i\in(A_{r,R}(\xi_\varepsilon))^2_\varepsilon\}\,,\end{aligned}$$

and hence the statement becomes

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\min_{u\in\mathscr{A}_{\varepsilon^{s_2},\varepsilon^{s_1},\delta}(z;\xi_\varepsilon)}\mathcal{E}_\varepsilon(u;A_{\varepsilon^{s_2},\varepsilon^{s_1}}(\xi_\varepsilon))\ge\frac{1}{2\pi}(s_2-s_1)\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\,.$$

Analogously, the upper bound in Proposition 3.4 reads as follows: For every $\rho>0$, for every $0<s<1$ and for every $\{\xi_\varepsilon\}_\varepsilon\subset\mathbb{R}^2$, there exists a family $\{u^{\xi_\varepsilon}_{\varepsilon,s}\}_\varepsilon\subset\mathscr{D}_\varepsilon(A_{\varepsilon^s,\rho}(\xi_\varepsilon))$ and $u^{\xi_\varepsilon}_{\varepsilon,s}(\cdot)=z\theta_{\mathbb{A}_{\mathrm{hom}}}(\cdot-\xi_\varepsilon)$ on $\partial_\varepsilon A_{\varepsilon^s,\rho}(\xi_\varepsilon)$ such that

$$\|\mu(u^{\xi_\varepsilon}_{\varepsilon,s})\llcorner A_{\varepsilon^s,\rho}(\xi_\varepsilon)\|_{\mathrm{flat}}\le C\varepsilon|\log\varepsilon|\,,\tag{3.30}$$

(with C independent of ε) and

$$\limsup_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(u^{\xi_\varepsilon}_{\varepsilon,s};A_{\varepsilon^s,\rho}(\xi_\varepsilon))\le\frac{s}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}\,z^2\,.\tag{3.31}$$

4. Proof of the Γ-liminf inequality

This section is devoted to the proof of Theorem 1.2(i). The proof will follow along the lines of [2, Proposition 5.2], with some differences due to the peculiarities of the discrete setting we work with in this paper. We start by introducing some notations and preliminary results that have been proven in [6] and [2] and that will be useful in our analysis.

The next result is [6, Proposition 3.3] (the functional $\mathcal{XY}_\varepsilon$ is defined in (1.5)).

Proposition 4.1. *There exists a positive constant β such that for any $\varepsilon>0$, for any function $u\in\mathscr{D}_\varepsilon(\Omega)$ and for any $i\in\Omega^2_\varepsilon$ such that $\min_{i+\varepsilon Q}|\Pi(w_u)|\le\frac12$, it holds*

$$\mathcal{XY}_\varepsilon(u;i+\varepsilon Q+B_{\sqrt{2}\varepsilon})\ge\beta\,.$$

In what follows, we recall the ball construction procedure introduced by Sandier [24] and Jerrard [20] in the context of the Ginzburg-Landau functional for providing lower bounds of the Dirichlet energy in presence of topological singularities. Here we adopt the notation of [2, Section 4] (see also [17, 9]).

Let $\mathcal{B}=\{B_{r_1}(x^1),\dots,B_{r_n}(x^N)\}$ be a finite family of open balls in $\mathbb{R}^2$ and let $\mu=\sum_{n=1}^N z^n\delta_{x^n}$ with $z^n\in\mathbb{Z}\setminus\{0\}$. Notice that in [2] it was assumed that the balls have disjoint closures but this is actually not necessary up to an additional *merging procedure* (see [2, Section 4]). Let moreover $\mathfrak{E}(\mathcal{B},\mu,\cdot)$ be the increasing set-function defined on open subsets of $\mathbb{R}^2$ in the following way: If $A_{r,R}(x)$ is an annulus that does not intersect any $B\in\mathcal{B}$, we set

$$\mathfrak{G}(\mathcal{B},\mu,A_{r,R}(x)):=|\mu(B_r(x))|\log\Big(\frac{R}{r}\Big)\,,\tag{4.1}$$

For every open set $A \subset \mathbb{R}^2$ we set

$$\mathfrak{E}(\mathcal{B}, \mu, A) := \sup \sum_j \mathfrak{G}(\mathcal{B}, \mu, A_j), \tag{4.2}$$

where the supremum is taken over all finite families of disjoint annuli $A_j \subset A$ that do not intersect any $B \in \mathcal{B}$. For every ball $B \subset \mathbb{R}^2$, let $r(B)$ denote the radius of the ball B; moreover, for every family $\mathscr{B}$ of balls in $\mathbb{R}^2$ we set

$$\mathcal{R}ad(\mathscr{B}) := \sum_{B \in \mathscr{B}} r(B).$$

The following result is [2, Proposition 4.2].

Proposition 4.2. *There exists a one-parameter family of open balls $\mathcal{B}(t)$ with $t \geq 0$ such that, setting $U(t) := \bigcup_{B \in \mathcal{B}(t)} B$, the following conditions are fulfilled:*

1. $\mathcal{B}(0) = \mathcal{B}$ *;*
2. $U(t_1) \subset U(t_2)$ *for any* $0 \leq t_1 < t_2$ *;*
3. *the balls in* $\mathcal{B}(t)$ *are pairwise disjoint;*
4. *for any* $0 \leq t_1 < t_2$ *and for any open set* $U \subset \mathbb{R}^2$ *,*

$$\mathfrak{E}(\mathcal{B}, \mu, U \cap (U(t_2) \setminus \overline{U}(t_1))) \geq \sum_{\substack{B \in \mathcal{B}(t_2) \\ B \subset U}} |\mu(B)| \log \frac{1 + t_2}{1 + t_1} \; ; \tag{4.3}$$

5. $\mathcal{R}ad(\mathcal{B}(t)) \leq (1 + t)\mathcal{R}ad(\mathcal{B})$.

We are now in a position to prove the Γ-liminf inequality in Theorem 1.2(i).

Proof of Theorem 1.2(i). We can assume without loss of generality that

$$\mathcal{E}_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon| \,, \tag{4.4}$$

for some constant $C > 0$ independent of ε. Moreover, by a standard localization argument in Γ-convergence, we can assume that $\mu = z^0 \delta_{x^0}$ for some $z^0 \in \mathbb{Z} \setminus \{0\}$ and $x^0 \in \Omega$.

We divide the proof into three steps. In Step 1, using the ball construction procedure in Proposition 4.2, we show that the sequence $\{\mu_\varepsilon\}_\varepsilon$ is flat-equivalent to a sequence $\{\mu_\varepsilon(p)\}_\varepsilon$ having uniformly bounded total variation. In Step 2, we show how to modify the functions u_ε in order to get rid of the balls containing "short" dipoles far from the limiting singularity. In such a way, we can bound from below $\mathcal{E}_\varepsilon(u_\varepsilon)$ with the energy of the modified functions $\hat{u}_\varepsilon$, up to paying a finite error. As a consequence, it is sufficient to estimate the energy of $\hat{u}_\varepsilon$ outside a uniformly bounded family of balls having non-zero measure. This is done in Step 3, where the analysis developed in Subsection 3.1 is used in order to get the desired lower bound.

We first construct the starting family of balls.

Step 1. For every $\varepsilon > 0$ we set $w_\varepsilon := e^{2\pi\imath u_\varepsilon}$ and we denote by $\tilde{w}_\varepsilon := \Pi(w_\varepsilon)$ its piecewise affine interpolation. Furthermore, we set $\mu_\varepsilon := \mu(u_\varepsilon)$. Let $\mathscr{Q}_\varepsilon$ denote the set of ε-squares $i + \varepsilon Q \subset \Omega_\varepsilon$ such that $\min_{i+\varepsilon Q} |\tilde{w}_\varepsilon| \leq \frac{1}{2}$. Since by (1.4) it holds that

$$\mathcal{E}_\varepsilon(u_\varepsilon) \geq \frac{\gamma_1}{4\pi^2} \mathcal{X}\mathcal{Y}(u_\varepsilon; \Omega) \,,$$

Proposition 4.1 and by (4.4) implies

$$\sharp \mathscr{Q}_\varepsilon \leq C|\log \varepsilon| \quad \text{and} \quad |\mu_\varepsilon|(\Omega) \leq C|\log \varepsilon| \,. \tag{4.5}$$

In view of (4.5), there exists a family $\mathcal{B}_\varepsilon$ of open balls covering

$$Q_\varepsilon := \Big\{ x \in \Omega \, : \, \mathrm{dist}\Big(x, \bigcup_{i+\varepsilon Q \in \mathscr{Q}_\varepsilon} (i + \varepsilon Q)\Big) \leq \sqrt{2}\varepsilon \Big\}$$

such that

$$\mathcal{R}ad(\mathcal{B}_\varepsilon) \leq C\varepsilon|\log \varepsilon| \,. \tag{4.6}$$

We set $U_\varepsilon := \bigcup_{B \in \mathcal{B}_\varepsilon} B$.

Let $\Omega'\subset\subset\Omega$ be such that $x^0\in\Omega'$. By (4.4) and by (1.4), for $\varepsilon>0$ small enough we have

$$\begin{aligned} C|\log\varepsilon| &\geq \mathcal{E}_\varepsilon(u_\varepsilon)\geq C\int_{\Omega'\setminus\overline{U}_\varepsilon}|\nabla\tilde{w}_\varepsilon|^2\mathrm{d}x\geq C\int_{\Omega'\setminus\overline{U}_\varepsilon}\Big|\nabla\frac{\tilde{w}_\varepsilon}{|\tilde{w}_\varepsilon|}\Big|^2\mathrm{d}x\\ &\geq C\,\mathfrak{E}(\mathcal{B}_\varepsilon,\mu_\varepsilon,\Omega')\,, \end{aligned}\tag{4.7}$$

where the set-function $\mathfrak{E}$ is defined in (4.1)-(4.2).

For every $\varepsilon>0$, let $\mathcal{B}_\varepsilon(t)$ be a time-parametrized family of balls introduced as in Proposition 4.2, starting from $\mathcal{B}_\varepsilon=:\mathcal{B}_\varepsilon(0)$. For every $t\geq0$, we set $\mathcal{R}_\varepsilon(t):=\mathcal{R}ad(\mathcal{B}_\varepsilon(t))$, $\mathcal{C}_\varepsilon(t):=\{B\in\mathcal{B}_\varepsilon(t)\,:\,B\subset\Omega'\}$ and $U_\varepsilon(t):=\bigcup_{B\in\mathcal{B}_\varepsilon(t)}B$. Moreover, for any $0<p<1$ we set

$$t_\varepsilon(p):=\frac{1}{\mathcal{R}_\varepsilon^{1-p}(0)}-1\qquad\text{and}\qquad\mu_\varepsilon(p):=\sum_{B\in\mathcal{C}_\varepsilon(t_\varepsilon(p))}\mu_\varepsilon(B)\delta_{x_B}\,,$$

where x_B denotes the center of the ball B. By (4.7), applying (4.3) with $U=\Omega'$, $t_1=0$ and $t_2=t_\varepsilon(p)$, and using (4.6), we obtain

$$\begin{aligned} C|\log\varepsilon|\geq\mathfrak{E}(\mathcal{B}_\varepsilon,\mu_\varepsilon,\Omega'\cap(U_\varepsilon(t_\varepsilon(p))\setminus\overline{U}_\varepsilon(0)))&\geq\sum_{B\in\mathcal{C}_\varepsilon(t_\varepsilon(p))}|\mu_\varepsilon(B)|(1-p)|\log\mathcal{R}_\varepsilon(0)|\\ =(1-p)|\mu_\varepsilon(p)|(\Omega')|\log\mathcal{R}_\varepsilon(0)|&\geq C(1-p)|\mu_\varepsilon(p)|(\Omega')|\log\varepsilon| \end{aligned}$$

for sufficiently small ε. Therefore

$$|\mu_\varepsilon(p)|(\Omega')\leq C_p,\tag{4.8}$$

for some constant $C_p>0$ and on p (but independent of ε). By Proposition 4.2(5) and (4.6), we have that

$$\mathcal{R}_\varepsilon(t_\varepsilon(p))\leq\mathcal{R}_\varepsilon^p(0)\leq C\varepsilon^p|\log\varepsilon|^p\,,$$

whence, by applying [2, Lemma 4.3] with $\nu_1=\mu_\varepsilon(p)$ and $\nu_2=\mu_\varepsilon$, we deduce that

$$\|\mu_\varepsilon-\mu_\varepsilon(p)\|_{\mathrm{flat}(\Omega')}\leq C\mathcal{R}_\varepsilon(t_\varepsilon(p))(|\mu_\varepsilon|+|\mu_\varepsilon(p)|)(\Omega')\leq C\varepsilon^p|\log\varepsilon|^{1+p}\to0\quad\text{as }\varepsilon\to0\,.$$

Combining this relation with (4.8) and the fact that $\mu_\varepsilon\overset{\mathrm{flat}}{\to}\mu$ yields

$$\mu_\varepsilon(p)\overset{*}{\rightharpoonup}\mu=z^0\delta_{x^0}\,,\qquad\text{for every }0<p<1\,.$$

Step 2. Let $c>1$ be such that $\log c<\frac{p}{2}\frac{|\log\mathcal{R}_\varepsilon(0)|}{|\mu_\varepsilon|(\Omega')+1}$. Note that, since $|\log\mathcal{R}_\varepsilon(0)|\geq C|\log\varepsilon|$ and $|\mu_\varepsilon|(\Omega')\leq C|\log\varepsilon|$, we are allowed to take the constant c in the previous inequality independent of ε. Notice moreover that by the very construction of $\mathcal{B}_\varepsilon(t)$ and by (1.7) we have that for every $t>0$

$$\mu_\varepsilon(B)=\deg(\tilde{w}_\varepsilon,\partial B)=\deg\Big(\frac{\tilde{w}_\varepsilon}{|\tilde{w}_\varepsilon|},\partial B\Big)\qquad\text{for every }B\in\mathcal{B}_\varepsilon(t)\,.$$

By [2, Lemma 5.3] applied with $\Omega=\Omega'$, $p_1=p$ and $p_2=\frac{p}{2}$, we have that there exist $t_\varepsilon(p)\leq\hat{t}^1_\varepsilon<\hat{t}^2_\varepsilon\leq t_\varepsilon(\frac{p}{2})$ with $(1+\hat{t}^2_\varepsilon)=c(1+\hat{t}^1_\varepsilon)$ such that $\sharp\mathcal{B}_\varepsilon(t)=\sharp\mathcal{B}_\varepsilon(\hat{t}^1_\varepsilon)$ for every $t\in[\hat{t}^1_\varepsilon,\hat{t}^2_\varepsilon)$ and

$$\begin{aligned} \int_{\Omega'\cap(U(\hat{t}^2_\varepsilon)\setminus\overline{U}(\hat{t}^1_\varepsilon))}\Big|\nabla\frac{\tilde{w}_\varepsilon}{|\tilde{w}_\varepsilon|}\Big|^2\mathrm{d}x&\leq\frac{\log c\int_{\Omega'\setminus\overline{U}_\varepsilon}\Big|\nabla\frac{\tilde{w}_\varepsilon}{|\tilde{w}_\varepsilon|}\Big|^2\mathrm{d}x}{\frac{p}{2}|\log\mathcal{R}_\varepsilon(0)|-\log c(|\mu_\varepsilon|(\Omega')+1)}\\ &\leq\frac{C\log c\,\mathcal{E}_\varepsilon(u_\varepsilon;\Omega)}{\frac{p}{2}(|\log\varepsilon|-\log|\log\varepsilon|+C)-\log c\,(C|\log\varepsilon|+1)}\leq C\,, \end{aligned}\tag{4.9}$$

where the second inequality follows from (4.6) and (4.5) whereas the last inequality is a consequence of (4.4). We classify the balls in $\mathcal{C}_\varepsilon(\hat{t}^1_\varepsilon)$ into two subclasses, namely

$$\begin{aligned} \mathcal{C}^{=0}_\varepsilon(\hat{t}^1_\varepsilon)&:=\big\{B\in\mathcal{C}_\varepsilon(\hat{t}^1_\varepsilon)\,:\,\mu_\varepsilon(B)=0\big\}\,,\\ \mathcal{C}^{\neq0}_\varepsilon(\hat{t}^1_\varepsilon)&:=\big\{B\in\mathcal{C}_\varepsilon(\hat{t}^1_\varepsilon)\,:\,\mu_\varepsilon(B)\neq0\big\}\,. \end{aligned}$$

We first consider the balls in $\mathcal{C}_\varepsilon^{=0}(\hat t_\varepsilon^1)$. For every ball $B\in\mathcal{C}_\varepsilon^{=0}(\hat t_\varepsilon^1)$ we denote by $\hat B$ the only ball in $\mathcal{C}_\varepsilon^{=0}(\hat t_\varepsilon^2)$ containing B. Note that the center x_B of B is the same as the center of $\hat B$. By (4.9), we have that

$$\sum_{B\in\mathcal{C}_\varepsilon^{=0}(\hat t_\varepsilon^1)}\int_{\hat B\setminus B}\Big|\nabla\frac{\tilde w_\varepsilon}{|\tilde w_\varepsilon|}\Big|^2\mathrm{d}x\le C\,. \tag{4.10}$$

Now we construct a function $\hat u_\varepsilon\in\mathscr{D}_\varepsilon(\Omega)$ such that $\operatorname{supp}\mu(\hat u_\varepsilon)\subset\bigcup_{B\in\mathcal{C}_\varepsilon^{\neq0}(\hat t_\varepsilon^1)}B$ and

$$\mathcal{E}_\varepsilon(u_\varepsilon)\ge\mathcal{E}_\varepsilon(\hat u_\varepsilon)-C_p\,, \tag{4.11}$$

for some constant C_p independent of ε.

To this purpose, we consider $B=B_R(\xi)$ and $\hat B=B_{cR}(\xi)$ two balls as above. Since $\mu_\varepsilon(B_{cR}(\xi))=\mu_\varepsilon(B_R(\xi))=0$, there exists $v_\varepsilon:(A_{R,cR}(\xi))\cap\varepsilon\mathbb{Z}^2\to\mathbb{R}$ such that $\mathrm{d}v_\varepsilon(i,j)=\mathrm{d}u_\varepsilon(i,j)-\mathrm{P}(\mathrm{d}u_\varepsilon(i,j))$ for every $(i,j)\in(A_{R,cR}(\xi))^1_\varepsilon$. We let $\bar v_\varepsilon$ denote the average of $\Pi(v_\varepsilon)$ on $A_{R,cR}(\xi)$. Let $\sigma:[R,cR]\to\mathbb{R}$ be the cut-off function defined by $\sigma(\rho)=\frac{\rho-R}{R(c-1)}$. We define the function $\hat u_\varepsilon^{\hat B}:B_{cR}(\xi)\cap\varepsilon\mathbb{Z}^2\to\mathbb{R}$ as

$$\hat u_\varepsilon^{\hat B}(i):=\begin{cases}\sigma(|i-\xi|)v_\varepsilon(i)+(1-\sigma(|i-\xi|))\bar v_\varepsilon & \text{if } i\in A_{R,cR}(\xi)\cap\varepsilon\mathbb{Z}^2,\\ \bar v_\varepsilon & \text{if } i\in B_R(\xi)\cap\varepsilon\mathbb{Z}^2\,.\end{cases}$$

By the Poincaré-Wirtinger inequality, using the fundamental theorem of calculus and (4.10), we have that there exists a constant $\hat C$ (independent of ε) such that

$$\begin{aligned}\mathcal{E}_\varepsilon(\hat u_\varepsilon^{\hat B};B_{cR}(\xi))\le&\gamma_2\mathcal{G}_\varepsilon(\sigma(|\cdot-\xi|)(v_\varepsilon-\bar v_\varepsilon);B_{cR}(\xi))\\ \le&\frac{4}{R^2(c-1)^2}\int_{A_{R-\sqrt2\varepsilon,cR}(\xi)}\big|\Pi(v_\varepsilon)(x)-\bar v_\varepsilon\big|^2\,\mathrm{d}x+2\int_{A_{R,cR}(\xi)}|\nabla\Pi(v_\varepsilon)|^2\,\mathrm{d}x\\ &+\frac{4}{R^2(c-1)^2}\sum_{i\in(A_{R-\sqrt2\varepsilon,cR})^2_\varepsilon}\int_{i+\varepsilon Q}|v_\varepsilon(i)-\Pi(v_\varepsilon)(x)|^2\,\mathrm{d}x\\ \le&C\int_{A_{R,cR}(\xi)}|\nabla\Pi(v_\varepsilon)|^2\,\mathrm{d}x\le\hat C\int_{A_{R,cR}(\xi)}\Big|\nabla\frac{\tilde w_\varepsilon}{|\tilde w_\varepsilon|}\Big|^2\mathrm{d}x\le C_p\,.\end{aligned}$$

Therefore, setting

$$\hat u_\varepsilon(i):=\begin{cases}\hat u_\varepsilon^{\hat B}(i) & \text{if } i\in\hat B\cap\varepsilon\mathbb{Z}^2 \text{ for some } \hat B\in\mathcal{C}_\varepsilon^{=0}(\hat t_\varepsilon^2)\\ u_\varepsilon(i) & \text{elsewhere in } \Omega_\varepsilon^0\,,\end{cases}$$

we have that (4.11) holds true.

Step 3. We now focus on the balls in $\mathcal{C}_\varepsilon^{\neq0}(\hat t_\varepsilon^1)$. In view of (4.8), we have that $\sharp\mathcal{C}_\varepsilon^{\neq0}(\hat t_\varepsilon^1)\le C_p$. Therefore, up to extracting a subsequence we may assume that $\sharp\mathcal{C}_\varepsilon^{\neq0}(\hat t_\varepsilon^1)=L$ for every $\varepsilon>0$ and for some $L\in\mathbb{N}$. For every $l=1,\dots,L$, let x_ε^l be the center of the l-th ball B_ε^l in $\mathcal{C}_\varepsilon^{\neq0}(\hat t_\varepsilon^1)$. Up to a further subsequence, we can assume that the points x_ε^l converge to some points in the finite set $\{\xi^0=x^0,\xi^1,\dots,\xi^{L'}\}\subset\bar\Omega$, where $L'\le L$. Let $\rho>0$ be such that $B_{2\rho}(x^0)\subset\subset\Omega'$ and $B_{2\rho}(\xi^j)\cap B_{2\rho}(\xi^k)=\emptyset$ for all $j\neq k$. Then $x_\varepsilon^l\in B_\rho(\xi^j)$ for some $j=1,\dots,L'$ and for ε small enough. We set

$$\tilde\mu_\varepsilon:=\sum_{x_\varepsilon^l\in B_\rho(x^0)}\mu_\varepsilon(B_\varepsilon^l)\delta_{x_\varepsilon^l}\,.$$

By construction, we have that

$$|\tilde\mu_\varepsilon|(\Omega')\le|\mu_\varepsilon(p)|(\Omega')\quad\text{and}\quad\|\tilde\mu_\varepsilon-\mu_\varepsilon(p)\|_{\mathrm{flat}(\Omega')}\to0\,, \tag{4.12}$$

which, in view of the properties of $\mu_\varepsilon(p)$, implies that, up to a subsequence, $\tilde\mu_\varepsilon\overset{*}{\rightharpoonup}\mu=z^0\delta_{x^0}$. Therefore, for sufficiently small ε,

$$\tilde\mu_\varepsilon(B_{2\rho}(x^0))=\sum_{x_\varepsilon^l\in B_\rho(x_0)}\mu_\varepsilon(B_\varepsilon^l)=z^0\,. \tag{4.13}$$

By (4.11) we thus have

$$\mathcal{E}_\varepsilon(u_\varepsilon)\ge\mathcal{E}_\varepsilon(\hat u_\varepsilon;B_{2\rho}(x^0))-C_p\,. \tag{4.14}$$

It remains to prove the lower bound for the right-hand side of (4.14). To this end, we take $0<p'<p$ such that $\mathcal{R}_\varepsilon(\hat t^1_\varepsilon)\le\varepsilon^{p'}$, choose $0<\bar p<p'$ and let $g_\varepsilon:[\bar p,p']\to\{1,\dots,L\}$ denote the function which associates to any $q\in[\bar p,p']$ the number $g_\varepsilon(q)$ of connected components of the set $\bigcup_{l=1}^L B_{\varepsilon^q}(x^l_\varepsilon)$. For every $\varepsilon>0$, the function g_ε is monotonically non decreasing so that it can have at most $\hat L\le L$ discontinuities. Let q^j_ε, for $j=1,\dots,\hat L$, denote the discontinuity points of g_ε and assume that

$$\bar p\le q^1_\varepsilon<\ldots<q^{\hat L}_\varepsilon\le p'\,.$$

There exists a finite set $\triangle=\{q^1,q^2,\dots,q^{\tilde L}\}$ with $q^i<q^{i+1}$ and $\tilde L\le\hat L$ such that, up to a subsequence, $\{q^j_\varepsilon\}_\varepsilon$ converge to some point in $\triangle$, as $\varepsilon\to0$ for every $j=1,\dots,\hat L$. Without loss of generality we may assume that $q^1=\bar p$, and that $q^{\tilde L}=p'$. Let $\lambda>0$ be such that $4\lambda<\min\{q^{i+1}-q^i\,:\,i\in\{1,2,\dots,\tilde L\}\}$ and let ε be so small that for every $j=1,\dots,\hat L$, $|q^j_\varepsilon-q^i|<\lambda$ for some $q^i\in\triangle$. Then the function g_ε is constant in the interval $[q^i+\lambda,q^{i+1}-\lambda]$, its value being denoted by M^i_ε. For every $i=1,\dots,\tilde L-1$ we construct a family of $M^i_\varepsilon\le\tilde L-1$ annuli that we let $C^{i,m}_\varepsilon:=B_{\varepsilon^{q^i+\lambda}}(y^m_\varepsilon)\setminus\overline{B}_{\varepsilon^{q^{i+1}-\lambda}}(y^m_\varepsilon)$ with $y^m_\varepsilon\in B_\rho(x^0)$ and $m=1,\dots,M^i_\varepsilon$. The annuli $C^{i,m}_\varepsilon$ can be taken pairwise disjoint for all i and m and such that

$$\bigcup_{x^l_\varepsilon\in B_\rho(x^0)}B^l_\varepsilon\subset\bigcup_{m=1}^{M^i_\varepsilon}B_{\varepsilon^{q^{i+1}-\lambda}}(y^m_\varepsilon)$$

for all $i=1,\dots,\tilde L-1$. Note that, for ε small enough, $C^{i,m}_\varepsilon\subset B_{2\rho}(x^0)$ for all i and m. By (4.12) we have that $|\mu_\varepsilon(B_{\varepsilon^{q^{i+1}-\lambda}}(y^m_\varepsilon))|\le C$ for every $i=1,\dots,\tilde L-1$ and $m=1,\dots,M^i_\varepsilon$. Therefore, up to passing to a further subsequence, we can assume that $M^i_\varepsilon=M^i$ and that $\mu_\varepsilon(B_{\varepsilon^{q^{i+1}-\lambda}}(y^m_\varepsilon))=z_{i,m}\in\mathbb{Z}\setminus\{0\}$, with M^i and $z_{i,m}$ independent of ε. Finally, in view of (4.13), we have that

$$\sum_{m=1}^{M^i}z_{i,m}=z^0\,. \tag{4.15}$$

We can apply Proposition 3.3 (see also Remark 3.7) with $s_1=q^i+\lambda<q^{i+1}-\lambda=s_2$ to get that for every i and m there exists a modulus of continuity ω such that

$$\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(\hat u_\varepsilon;C^{i,m}_\varepsilon)\ge\frac{1}{2\pi}(q^{i+1}-q^i-2\lambda)\sqrt{\det\mathbb{A}_{\rm hom}}|z_{i,m}|^2-r(\varepsilon)\,,$$

for some function r with $\lim_{\varepsilon\to0}r(\varepsilon)=0$. Summing the previous inequality over m and i and using (4.14) yields

$$\begin{aligned}
&\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(\hat u_\varepsilon;(B_{2\rho}(x^0))\\
\ge&\frac{1}{2\pi}\sum_{i=1}^{\tilde L-1}\sum_{m=1}^{M^i}(q^{i+1}-q^i-2\lambda)\sqrt{\det\mathbb{A}_{\rm hom}}|z_{i,m}|-r(\varepsilon)\\
\ge&\frac{1}{2\pi}\sum_{i=1}^{\tilde L-1}(q^{i+1}-q^i-2\lambda)\sqrt{\det\mathbb{A}_{\rm hom}}|z^0|-r(\varepsilon)\\
=&\frac{1}{2\pi}(p'-\bar p-2(\tilde L-1)\lambda)\sqrt{\det\mathbb{A}_{\rm hom}}|z^0|-r(\varepsilon)\,,
\end{aligned}\tag{4.16}$$

where the second inequality follows from (4.15) and by the fact that $z_{i,m}\in\mathbb{Z}\setminus\{0\}$. Then, the claim follows by (4.14) and (4.16) taking the limits as $\varepsilon\to0$, $\lambda\to0$, $\bar p\to0$, and $p,p'\to1$. □

5. Proof of the Γ-limsup inequality

In this section we prove Theorem 1.2(ii).

Proof of Theorem 1.2(ii). By standard density arguments, it is enough to prove the claim for $\mu=\sum_{k=1}^K z^k\delta_{x^k}\in X(\Omega)$ with $|z^k|=1$ for every $k=1,\dots,K$. We construct a family of functions $\{u_\varepsilon\}_\varepsilon$ with $u_\varepsilon\in\mathscr{D}_\varepsilon(\Omega)$ such that $\mu(u_\varepsilon)\overset{*}{\rightharpoonup}\mu$ as $\varepsilon\to 0$ and (1.8) holds true. To this purpose, we take $\rho>0$ such that $\overline{B}_{2\rho}(x^k)\subset\Omega$ for every $k=1,\dots,K$, and $\overline{B}_{2\rho}(x^m)\cap\overline{B}_{2\rho}(x^n)=\emptyset$ for every $m,n=1,\dots,K$ with $m\neq n$. Let $\{(x^1_\varepsilon,\dots,x^K_\varepsilon)\}_\varepsilon$ be such that $x^k_\varepsilon=i_k+\frac{\varepsilon}{2}(e_1+e_2)$ for some $i_k\in\Omega^2_\varepsilon$ and $|x^k_\varepsilon-x^k|\le\varepsilon$ for every $k=1,\dots,K$. Moreover, let $0<s<1$.

For every $k=1,\dots,K$, let $\{u^k_{\varepsilon,s}\}_\varepsilon$ be the sequence in Remark 3.7 for $\xi_\varepsilon=x^k_\varepsilon$. Let furthermore $\sigma:[\rho,2\rho]\to[0,1]$ be the function defined by $\sigma(r):=\frac{1}{\rho}(r-\rho)$ and set $\Theta_\varepsilon(\cdot):=\sum_{k=1}^K z^k\theta_{\mathbb{A}_{\mathrm{hom}}}(\cdot-x^k_\varepsilon)$. We define the function $u_{\varepsilon,s}\in\mathscr{D}_\varepsilon(\Omega)$ as

$$u_{\varepsilon,s}(i)=\begin{cases} z^k\theta_{\mathbb{A}_{\mathrm{hom}}}(i-x^k_\varepsilon) & \text{if } i\in B_{\varepsilon^s}(x^k_\varepsilon)\cap\varepsilon\mathbb{Z}^2 \text{ for some } k\,,\\ u^k_{\varepsilon,s}(i) & \text{if } i\in(A_{\varepsilon^s,\rho}(x^k_\varepsilon))^0_\varepsilon \text{ for some } k\,,\\ (1-\sigma(|i-x^k_\varepsilon|))z^k\theta_{\mathbb{A}_{\mathrm{hom}}}(i-x^k_\varepsilon)+\sigma(|i-x^k_\varepsilon|)\Theta_\varepsilon(i) & \text{if } i\in A_{\rho,2\rho}(x^k_\varepsilon)\cap\varepsilon\mathbb{Z}^2 \text{ for some } k\,,\\ \Theta_\varepsilon(i) & \text{elsewhere in } \Omega^0_\varepsilon. \end{cases}$$

By construction (see (3.30)), $\sum_{k=1}^K\|\mu(u_{\varepsilon,s})\llcorner(A_{\varepsilon^s,\rho}(x^k_\varepsilon))\|_{\mathrm{flat}}\le C\varepsilon|\log\varepsilon|$ for some universal constant $C>0$, $\mu(u_{\varepsilon,s})\llcorner\Omega\setminus\bigcup_{k=1}^K B_\rho(x^k_\varepsilon)=0$ and $\mu(u_{\varepsilon,s})(B_{\varepsilon^s}(x^k_\varepsilon))=z^k$, and hence $\mu(u_{\varepsilon,s})\overset{\mathrm{flat}}{\to}\mu$ as $\varepsilon\to 0$ for every $0<s<1$. Note that, since for every $x\in\mathbb{R}^2\setminus\{0\}$, $z\in\mathbb{Z}\setminus\{0\}$ and for every $l=1,2$

$$\left|\nabla\left(\left|\frac{\partial}{\partial x_l}z\theta_{\mathbb{A}_{\mathrm{hom}}}\right|^2\right)(x)\right|\le\frac{C_z}{|x|^3}\,,$$

then, using (1.2), for every $k=1,\dots,K$ we get

$$\begin{aligned}\mathcal{E}_\varepsilon(z^k\theta_{\mathbb{A}_{\mathrm{hom}}}(\cdot-x^k_\varepsilon));B_{2\varepsilon^s}(x^k_\varepsilon))\le& C+C\int_{A_{\varepsilon,2\varepsilon^s}(x^k_\varepsilon)}\frac{1}{|x-x^k_\varepsilon|^2}\mathrm{d}x+C\varepsilon\int_{A_{\varepsilon,2\varepsilon^s}(x^k_\varepsilon)}\frac{1}{|x-x^k_\varepsilon|^3}\mathrm{d}x\\ \le& C(1-s)|\log\varepsilon|\,.\end{aligned}\tag{5.1}$$

Moreover, by (1.2) and (1.3), for every $k=1,\dots,K$ there exists a constant $C=C(\gamma_2,\rho,\Omega,\{z^k\}_k)>0$ such that, for ε small enough,

$$\mathcal{E}_\varepsilon(\Theta_\varepsilon;\Omega\setminus\bigcup_{k=1}^K B_\rho(x^k_\varepsilon))\le C\,,\tag{5.2}$$

and

$$\mathcal{E}_\varepsilon\big((1-\sigma(|\cdot-x^k_\varepsilon|))z^k\theta_{\mathbb{A}_{\mathrm{hom}}}(\cdot-x^k_\varepsilon)+\sigma(|\cdot-x^k_\varepsilon|)\Theta_\varepsilon(\cdot);A_{\frac{\rho}{2},2\rho}(x^k_\varepsilon)\big)\le C\,.\tag{5.3}$$

By (3.31), (5.1), (5.2) and (5.3), we thus get

$$\frac{1}{|\log\varepsilon|}\mathcal{E}_\varepsilon(u_{\varepsilon,s})\le sK\frac{1}{2\pi}\sqrt{\det\mathbb{A}_{\mathrm{hom}}}+C(1-s)K+r(\varepsilon)\,.\tag{5.4}$$

Suitably choosing $s_\varepsilon\to 1$ as $\varepsilon\to 0$, we have that the functions $u_\varepsilon=u_{\varepsilon,s_\varepsilon}$ satisfy (1.8).

□

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