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SOME NOTES ON GENERIC RECTANGULATIONS

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ABSTRACT. A *rectangulation* is a subdivision of a rectangle into rectangles. A *generic rectangulation* is a rectangulation that has no crossing segments. We explain several observations and pose some questions about generic rectangulations. In particular, we show how one may "centrally invert" a generic rectangulation about any given rectangle, analogous to reflection across a circle in classical geometry. We also explore 3-dimensional orthogonal polytopes related to "marked" rectangulations and drawings of planar maps. These observations arise from viewing a generic rectangulation as topologically equivalent to a sphere.

1. INTRODUCTION

This article has some observations concerning subdivisions of rectangles into rectangles, known commonly as rectangulations or floorplans, cf. [19]. Rectangulations arise naturally in architecture, VLSI design, and some topics in combinatorics. The main theme in these notes is that a rectangulation bears some similarity to a Schlegel diagram. Recall (from [21] for instance) that a Schlegel diagram of a convex polytope is an image of a skeleton of the polytope via stereographic projection. As we explain below, a rectangulation can represent a drawing of a planar map, which we regard as a slight generalization of a 3-dimensional convex polytope. We demonstrate a construction for generic rectangulations, showing that, given a rectangulation and one of its constituent rectangles, one may form an equivalent rectangulation where the chosen rectangle serves as the bounding rectangle. We call this transformation "central inversion," as it is analogous to reflection across a circle on the one-point compactification of the cartesian plane.

Another aspect of this theme might be called the Lifting Problem: If one has something that looks like it might be a Schlegel diagram, then the fundamental problem is to determine whether or not it is a Schlegel diagram. In 2 dimensions there is a cluster of closely related results concerning this question. For example, one of these results (attributed to James Clerk Maxwell) has an intuitive physical manifestation: Suppose G is a polyhedral graph

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(meaning that it is simple, planar, and 3-connected). Moreover, suppose the vertices of G are represented by movable points, the edges are represented by ideal springs, and one chooses a 2-dimensional face. Then, by demanding that this face fit a specified polygon, one obtains a straight-line drawing of the rest of the graph which happens to be Schlegel diagram of a corresponding 3-dimensional convex polytope. This story is well known and has been told nicely by various authors, [20, 21]. In these notes, we address an analogous issue with rectangulations. In our view, a rectangulation "lifts" to an orthogonal polytope that represents the face lattice of a planar map. The problem is to assign values to the constituent rectangles that specifies the lift to an orthogonal polytope that is topologically "good" (contractible, for instance).

Another theme of this project is graph drawing. In this regard, the works [13, 14, 15] are relevant. The articles [13] and [15] concern a connection between Schnyder woods and orthogonal surfaces. Briefly, a "Schnyder wood" is a certain kind of marking of a planar map that is related to a drawing of the underlying graph on an orthogonal surface; such a construction is used as part of the proof of an auxiliary result of the Brightwell-Trotter theorem [6], which asserts that the dimension of the incidence order of a planar map is four. The relation of this work to Schnyder woods, in particular, is subtle (and perhaps not very important). However, explaining it will require an understanding of zig-zag paths, which are described below and used to prove our main result about central invertibility. The paper [14] is a comprehensive survey on rectangulations and related orthogonal drawings of graphs.

We also mention a more algebraic perspective on rectangulations which has been overlooked by the researchers mentioned thus far. Given a set T with a pair $\pi = (\prec, <)$ of partial orders on T, the authors of [10] address the problem of determining whether or not (T, π) corresponds to a rectangulation (which they call a "tileorder"). In so doing, they relate the notion to double categories. For another algebraic topic relating to rectangulations, we should also mention [18], where so-called "diagonal" rectangulations appear as elements of a Hopf algebra.

2. Terminology

We assume that every rectangulation is embedded in the cartesian plane in such a way that every edge is parallel to either the horizontal or vertical axis. If $e_1 = (1, 0)$ and $e_2 = (0, 1)$ denote the standard basis vectors in $\mathbb{R}^2$, then the *cardinal directions* are the vectors $\{\pm e_1, \pm e_2\}$, to which we may refer using the terrestrial cardinal directions north, south, east, west.

2.1. **Condensed rectangulations.** We regard a *rectangulation* as a set of rectangles that comprise a subdivision of its bounding rectangle. The word "subdivision" implies that such a set is finite and the intersection of any two of them has empty interior. For practical purposes, we may specify a rectangulation as a list of entries of the form $[a, b] \times [c, d]$ where

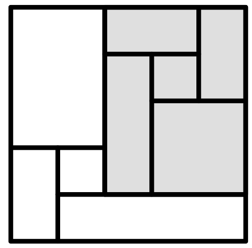

FIGURE 2.1. A rectangulation with a nontrivial subrectangulation (shaded).

the coordinates are real numbers such that $a < b$ and $c < d$ for all i. A *constituent rectangle* of a rectangulation is any rectangle that appears in such a listing. A rectangulation is *generic* if there is no point where two edges intersect in their relative interior.

Let R be a generic rectangulation. A *supporting line* of R is a line that contains an edge of at least one constituent rectangle of R. A *segment* of R is a connected component of a set which is formed as the union of all edges that lie on a given supporting line. Call R *faithful* if every supporting line contains precisely one segment; Figure 2.5 shows a rectangulation that is not faithful. A *rectangle* of R is any rectangle whose edges lie on two vertical and two horizontal segments. A *vertex* of R is any point that appears as a vertex of at least one of its rectangles. A *subrectangulation* of R is a rectangulation of one of its rectangles which is subdivided using constituent rectangles of R. A rectangulation is *trivial* if it has only one rectangle (itself). Thus, with this terminology, a constituent rectangle in R coincides with a trivial subrectangulation. See Figure 2.1 for a rectangulation that has a rectangle that is not a constituent rectangle.

A *sliced box* of R is a triple (r_0, r_1, r) of distinct rectangles of R such that $r = r_0 \cup r_1$. Following similar usage from [2], call R *slicing* if every rectangle appears as r_0, r_1, or r in at least one sliced box (r_0, r_1, r) of R. Call a rectangulation *simple* if it does not have any nontrivial subrectangulations. Figure 2.2 shows a slicing rectangulation and Figure 2.1 shows a rectangulation that is neither slicing nor simple.

Let R be a generic rectangulation. Call two rectangles r_1 and r_2 of R *adjacent* if $(\partial r_1) \cap (\partial r_2)$ is nonempty. Call r_1 and r_2 *aligned* if there is a segment that contains an edge of each of r_1 and r_2. Call two rectangulations *strongly equivalent* (respectively, *weakly equivalent*) if there is a bijection from the subrectangles of one to the subrectangles of the other that preserves adjacency (respectively, alignment).

We also define a more global notion of equivalence. Loosely speaking, two rectangulations are globally equivalent if one may move the supporting lines of one to the other without two parallel supporting lines passing each other. Here is a more precise formulation of this: Let R and R' be rectangulations. Assume that the lines supporting the segments of R are given by $x = x_i$ and

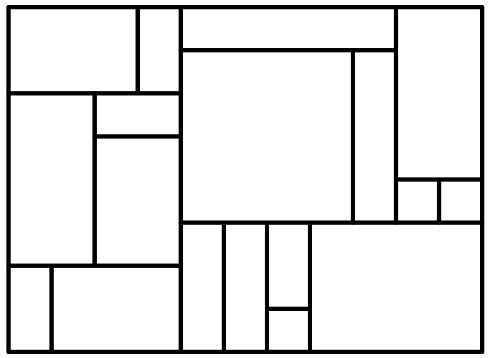

FIGURE 2.2. A condensed slicing rectangulation.

$y = y_j$, where $i \in \{0, 1, 2, \ldots, w\}$ and $j \in \{0, 1, 2, \ldots, h\}$ where $x_i < x_{i+1}$ for all i and $y_j < y_{j+1}$ for all j. Likewise assume that the lines supporting the segments of R' are given by $x = x'_i$ and $y = y'_j$, where $i \in \{0, 1, 2, \ldots, w'\}$ and $j \in \{0, 1, 2, \ldots, h'\}$ where $x'_i < x'_{i+1}$ for all i and $y'_j < y'_{j+1}$ for all j. Call R and R' *globally equivalent* if $w = w'$, $h = h'$, and there are bijections between the segments on $x = x_i$ and $x = x'_i$ and bijections between the segments on $y = y_j$ and $y = y'_j$ that respect adjacency. With this, we accept the following without proof:

Proposition 2.1. *Suppose R and R' are generic rectangulations. (a) If R and R' are globally equivalent, then they are strongly equivalent. (b) If R and R' are strongly equivalent, then they are weakly equivalent.*

Figure 2.4 shows counterexamples to the converses of these statements. (Another way to understand global equivalence is in terms of "contractions" and "elongations" of rectangulations. Thus, suppose R is a rectangulation and L is an axis-parallel line that passes through R but which does not contain a segment of R. Then L passes through a sequence of segments that are perpendicular to L. We obtain an elongation by lengthening these segments while translating the segments parallel to L away from L. Contraction is similar, where one trims sufficiently small congruent intervals out from these segments on pushing the parallel segments towards L. See Figure 2.3. Global equivalence is the coarsest equivalence relation subject to requiring contracted or elongated rectangulations to be equivalent.)

Note that our notion of weak equivalence is also called "R-equivalence" in [3] and [9], and that our notion of strong equivalence is called simply "equivalence" in [19].

Call a rectangulation R *integral* if all of the vertices lie in $\mathbb{Z}^2$. Call R *integrally condensed* or "condensed" for short if it is generic and faithful and all of the coordinates of the vertices are nonnegative integers that are minimal with respect to faithfulness. Thus, in a condensed rectangulation, the vertical line $x = 0$ and the horizontal line $y = 0$ are supporting lines of the bounding rectangle, and $(0, 0)$ is a vertex coinciding with the southwest

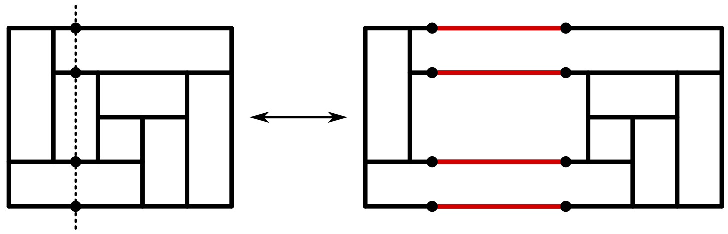

FIGURE 2.3. Elongation and contraction.

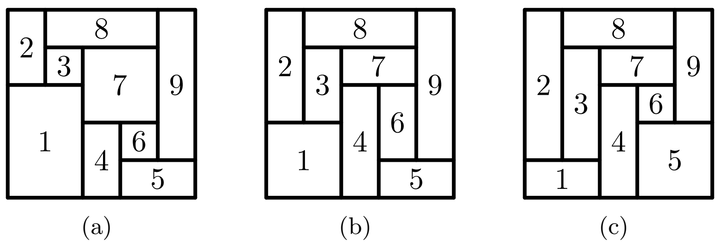

FIGURE 2.4. Rectangulations (a) and (b) are weakly but not strongly equivalent. Rectangulations (b) and (c) are strongly but not globally equivalent.

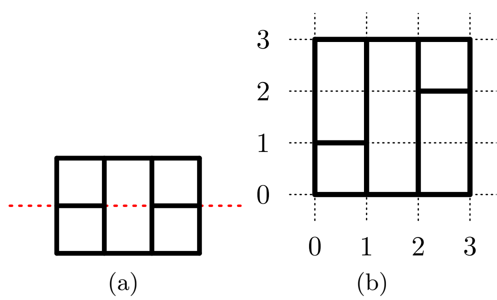

FIGURE 2.5. (a) Nonfaithful rectangulation. (b) Condensed rectangulation.

corner of the bounding rectangle. Moreover, if R is condensed, then there are integers, say w and h, such that each vertical line $x = 0, 1, 2, \ldots, w$ and each horizontal line $y = 0, 1, 2, \ldots, h$ contains exactly one segment of R. The following is elementary (and we accept it without proof), but it emphasizes the significance of global equivalence:

Proposition 2.2. *Suppose R is a faithful, generic rectangulation. There is a unique condensed rectangulation that is globally equivalent to R.*

For the rest of these notes, if it is not otherwise implied, the reader should assume that every given rectangulation is condensed.

We also notice that a rectangulation can be determined by its set of segments. That is, if R is a condensed rectangulation and Σ is a list of its segments, then we may recover R as a list of rectangles from Σ. This will be important later when we discuss transformations of rectangulations via transformations of their segments. One should consult [3] for a focused study of rectangulations with emphasis on their segments.

2.2. **Orthotopes and their faces.** We do not wish to devote too much space to the general theory of d-dimensional orthogonal polytopes, but we should say some things about what they are and especially what we regard as "faces" of orthogonal polytopes. (To this author's best knowledge, such a general theory is severely lacking in the literature at large.) For the purposes of these notes, we say that a *d-dimensional integral orthogonal polytope* (or "orthotope" for short) is any set that can be expressed as

$$P = S + [0,1]^d \text{ (Minkowski sum)},$$

where $S \subset \mathbb{Z}^d$ is a finite set of integer lattice points and $[0,1]^d$ is the d-dimensional unit cube.

Suppose P is a d-dimensional integral orthotope. Then, analogous to the face lattice of a convex polytope, P has a face poset, which we describe as follows. A *supporting plane* of P is a $(d-1)$-dimensional hyperplane Π that is parallel to one of the coordinate hyperplanes and such that the interior of $(\partial P) \cap \Pi$ is nonempty, using the relative topology of $\Pi \approx \mathbb{R}^{d-1}$. A *facet* of P is a subset of P of the form

$$\text{closure}(\text{interior}((\partial P) \cap \Pi)),$$

where Π is a supporting hyperplane of P, using the relative topology of Π. We define a *face* of P recursively as either P itself or any object that is a facet of a higher-dimensional face of P. (We could also define the notion of equivalence of orthotopes, similar to global equivalence of rectangulations above, but this is not of central interest in these notes. Again, a general theory is lacking.)

Call a subset $P \subset \mathbb{R}^d$ *orthogonally convex* if $P \cap L$ is connected for every line L that is parallel to a coordinate axis. Call an orthotope *totally orthogonally convex* if each of its faces is orthogonally convex and *totally spherical* if every face of dimension k is homeomorphic to the disc $[0,1]^k$ for all k. In particular, notice that every face of a totally spherical orthotope is, at the very least, contractible. Figure 2.6 shows a connected 2-dimensional orthotope which is not totally spherical.

2.3. **Planar maps.** The purpose of this section is to explain a relation between planar maps and rectangulations. The significance of this relation emerges when we study central inversion and introduce flag orthotopes below.

For our purposes, a *planar map* is a bridge-free, loop-free, undirected graph (allowing multi-edges) equipped with an embedding in the plane.

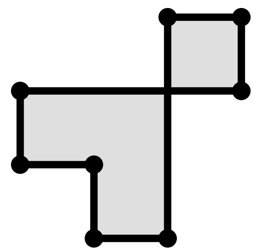

FIGURE 2.6. An orthogonal octagon with a nonspherical 2-face.

Thus, for a planar map, there are well-defined sets of vertices, edges, and faces, with multi-edges distinguished by their placement in the plane. The condition of being loop-free ensures that every edge has two distinct vertices, and the condition of being bridge-free ensures that the dual map has no loops. The *dipole map* is a planar drawing of the graph consisting of 2 vertices and a double edge connecting them. The *theta map* is the planar map consisting of 2 vertices and a triple edge. The *double-triangle* is the dual of the theta graph.

A generic rectangulation offers a way to draw a planar map. Our drawings are somewhat unorthodox, however, as we represent the edges of a map by constituent rectangles (including the bounding rectangle), vertices by vertical lines, and 2-dimensional regions of the map by horizontal segments. This relation respects some familiar constructions in graph theory. Figure 2.7 summarizes how to translate between various terms according to this correspondence. For example, a slicing rectangulation corresponds to a series-parallel graph. Similarly, by Steinitz's theorem, a simple rectangulation corresponds to a 3-dimensional convex polytope. Also notice that dualizing a planar map corresponds to flipping the rectangulation sideways, thereby interchanging horizontal and vertical segments. (It is also entertaining to notice that the rectangulations in the global-equivalence class representing the theta map are all topologically equivalent to the letter θ!)

This relation has been known in one way or another since at least [5]. The key connection seems to come from bipolar orientations of nonseparable maps as they relate to Baxter permutations. With our usage, a "nonseparable map" is a triple $(M, e, *)$, where M is a planar map, e is an edge on the outer face, and $*$ an orientation of e. Thus, in [2] and [19], one sees a direct link between certain markings of rectangulations and Baxter permutations. Also, in [5] one sees a representation of a Baxter permutation as a nonseparable map. Thus, these notes "close the loop" by emphasizing the direct link between rectangulations and planar maps. The reader should consult the survey [14] for more connections between these ideas. Going back a bit further, we should notice the classic article [7], where the authors study tilings of rectangles by squares; although the authors do not explore

Rectangulations	**Planar Maps**
constituent rectangle	edge
vertical segment	vertex
horizontal segment	face/region
vertex	vertex-region incidence
vertical slice	insert series vertex
horizontal slice	insert parallel edge
slicing rectangulation	series-parallel graph
simple rectangulation	polyhedral map
rectangle with a horizontal slice	theta map
rectangle with a vertical slice	double-triangle
trivial rectangulation	dipole map

FIGURE 2.7. Dictionary for rectangulations and planar maps.

this connection in depth, one finds an electrical network corresponding to such a rectangulation, where squares are represented by resistors, etc.

2.4. **Flag orthotopes.** We would like to associate a certain orthotope with a planar map. Ideally, such an object should faithfully display all of the flags in its face lattice. Moreover, we would like to demand that all of the faces should be topologically "nice," meaning that we would hope that the orthotope is totally spherical, and perhaps even totally orthogonally convex.

Suppose M is a planar map. Let V, E, and F be the sets of vertices, edges, and 2-dimensional regions of M, respectively. Then the *face lattice* of M is the set $\mathcal{L}(M) = \{\emptyset\} \cup V \cup E \cup F \cup \{M\}$, equipped with the partial order induced by incidence. A *lattice map* of M is a function

$$\mathcal{L}(M) \xrightarrow{\phi} \mathbb{Z}$$

such that the restrictions of ϕ to V, E, and F are one-to-one. Neglecting the singletons at the top and bottom, let $v \subset e \subset f$ be a complete flag in $\mathcal{L}(M)$ and suppose ϕ is a lattice map. Then $(\phi(v), \phi(e), \phi(f))$ is a point in $\mathbb{R}^3$ which we wish to regard as a vertex of the flag orthotope corresponding to (M, ϕ).

As an example, consider Figure 2.8. This figure shows a rectangulation which is the image of a flag orthotope of the depicted planar map under an orthographic projection. Each of the vertices in the rectangulation thus corresponds to an edge of the corresponding flag orthotope collapsed to a point under projection. The figure has three sets of numbers, colored with red, blue, and green. The blue (respectively green, red) numbers mark the vertices (respectively edges, regions) of the planar map. If we consider these numbers as coordinates in $\mathbb{R}^3$, then this orthotope is good in the sense that every face is both orthogonally convex and homeomorphic to a disk of the appropriate dimension. Figure 4.1 displays another example.

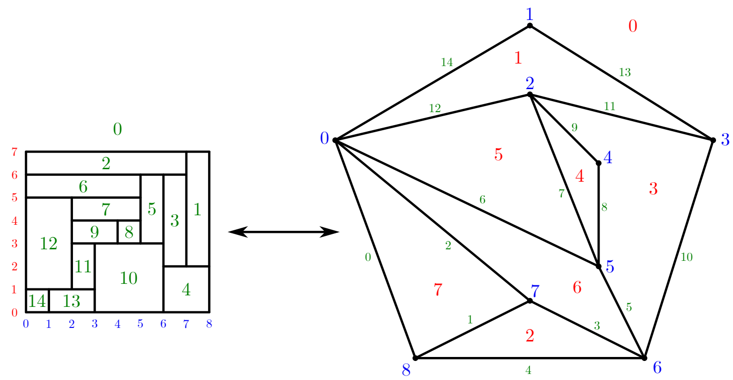

FIGURE 2.8. Flag orthotope of a planar map.

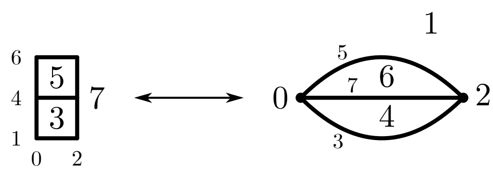

FIGURE 2.9. The theta map.

Below, using certain markings of rectangulations, we will exhibit various functions ϕ for which the orthogonal convex hull of the vertices defined by ϕ is totally spherical and totally orthogonally convex.

3. Central inversion

The purpose here is to study central inversions of rectangulations. In particular, we show how to centrally invert any given rectangulation about any given constituent rectangle. The geometric effect is to interchange the roles of the given rectangle and the bounding rectangle while rearranging the other rectangles in a specific way.

Suppose n is a nonnegative integer and $a, b \in \{0, 1, 2, \ldots, n\}$ satisfy $a < b$. Define a permutation of the set $\{0, 1, 2, \ldots, n\}$ by

$$f_{n,a,b}(x) = \begin{cases} a - x & \text{if} \quad x \leq a, \\ x & \text{if} \quad x > a \text{ and } x < b, \\ b + n - x & \text{if} \quad x \geq b, \end{cases}$$

One checks that (i) $f_{n,a,b}$ is an involutary permutation on $\{0, 1, 2, \ldots, n\}$, (ii) $f_{n,a,b}$ maps the set $\{0, 1, 2, \ldots, a\}$ to itself, (iii) $f_{n,a,b}$ maps $\{b, b+1, \ldots, n\}$ to itself, and (iv) $f_{n,a,b}$ fixes each point strictly between a and b.

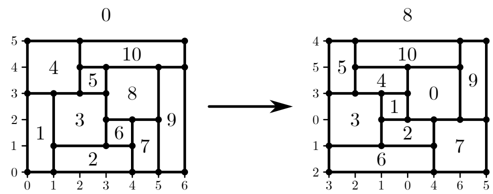

FIGURE 3.1. Central inversion.

We obtain a central inversion by applying involutions of the form $f_{n,a,b}$ to both of the coordinates, thereby effecting a transformation of the constituent rectangles. Thus, suppose R is a rectangulation that is condensed to the set $\{0, 1, 2, \ldots, w\} \times \{0, 1, 2, \ldots, h\}$, and suppose $r = [a, b] \times [c, d]$ is a constituent rectangle. For each vertex (x, y) of R, we define

$$T_r(x, y) = (f_{w,a,b}(x), f_{h,c,d}(y)).$$

By definition, T_r is defined on the vertices of R. We define the effect of T_r on rectangles by saying that if s is a rectangle of R, then the vertices of $T_r(s)$ consist of the points $T_r(v)$, where v is a vertex of s. With this, we can observe that T_r exchanges r with the bounding rectangle $[0, w] \times [0, h]$. Figure 3.1 shows a rectangulation and the rectangulation resulting from a central inversion.

In general, the result of applying T_r to all of the rectangles of R may not yield a bona fide rectangulation. That is, if r_1 and r_2 are constituent rectangles apart from r, then the images $T_r(r_1)$ and $T_r(r_2)$ may intersect with a nonempty interior. If, by contrast, it happens that T_r yields another generic rectangulation with no overlapping constituent rectangles, then we say that T_r *yields a generic rectangulation.*

3.1. **Universally visible rectangles.** Our first result gives necessary and sufficient conditions for when T_r yields a generic rectangulation. In order to state this condition, we introduce a couple more terms. Call constituent rectangles r, s *orthogonally visible* relative to each other if there is a horizontal or vertical line that passes through the interior of both r and s. Figure 3.2 shows an example; the focused rectangle r is bounded by dotted lines, and the rectangles that are not orthogonally visible relative to r are shaded. Call a constituent rectangle r *universally visible* if r is orthogonally visible relative to every other constituent rectangle. We may now state:

Theorem 3.1. *Suppose R is a condensed rectangulation and r is a constituent rectangle. Then central inversion about r yields a generic rectangulation if and only if r is universally visible.*

Proof. This follows from careful scrutiny of the effect of applying T_r to the assembly of constituent rectangles of R. Let δ_1, δ_2 be cardinal directions that are perpendicular to each other. Let S_1 (respectively S_2) be the union of all of the constituent rectangles of R that are visible from r in the direction of

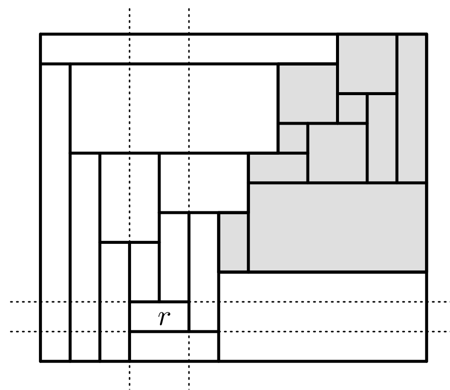

FIGURE 3.2. Orthogonal visibility.

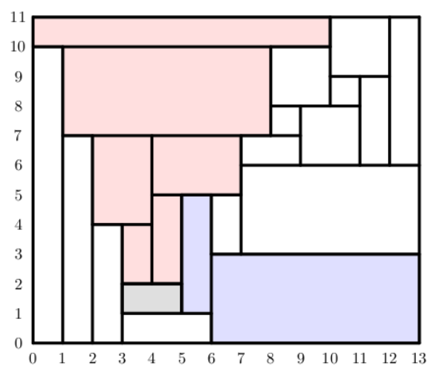

FIGURE 3.3. Visible regions S_1 and S_2.

δ_1 (respectively δ_2). Next, let S be the set of constituent rectangles s of R such that s is not orthogonally visible from r but is visible from a rectangle in S_1 or S_2. (See Figure 3.3.) The main observation is that the region $T_r(S_1) \cap T_r(S_2)$ coincides with the region obtained by rotating the union of the rectangles in S by 180°, as restricted to the quadrant generated by δ_1 and δ_2. Figure 3.4 shows an example.

After this is established, we recall that none of the rectangles of S are visible from r. In more detail, if r is universally visible, then there are no overlapping rectangles in $T_r(S)$. Also, if a constituent rectangle s is not visible from r, then it appears in some set S for some choice of cardinal directions. □

3.2. **Generalized central inversion.** In this section, we show how to centrally invert a weakly equivalent rectangulation about any constituent rectangle. More specifically, we prove:

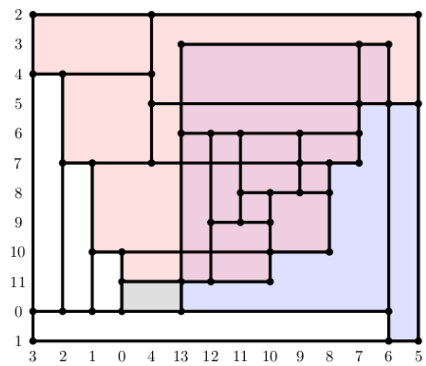

FIGURE 3.4. Overlapping regions caused by central inversion.

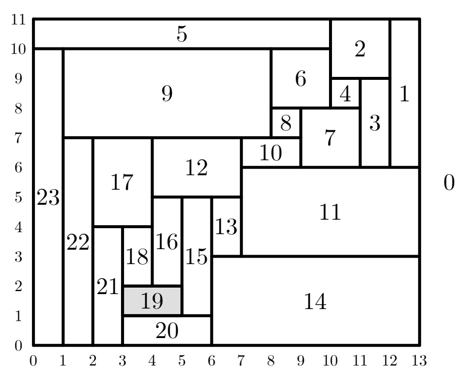

FIGURE 3.5. A condensed rectangulation.

Theorem 3.2. *Suppose R is a condensed rectangulation and r is a constituent rectangle. Then there is a rectangulation R' such that R' is weakly equivalent to R and the constituent rectangle r' of R' which corresponds to r is universally visible.*

We prove this algorithmically. The algorithm inputs a pair (R, r) as above and outputs permutations on the coordinates which cause the focused constituent rectangle to be universally visible in the new rectangulation. We illustrate the algorithm using a running example. In Figure 3.5, one sees a generic rectangulation. The constituent rectangles are marked with integers (whose significance appears later). In particular, we note that the shaded constituent rectangle in Figure 3.5 is not universally visible.

The algorithm depends on the notion of a "zig-zag path," which we define here. Suppose R is rectangulation, r is a constituent rectangle of R, and P_0 is a vertex of r. The zig-zag path determined by the triple (R, r, P_0) is defined recursively as follows. The initial point of the path is P_0. At the vertex P_0, there are two perpendicular cardinal directions, say δ_1 and δ_2,

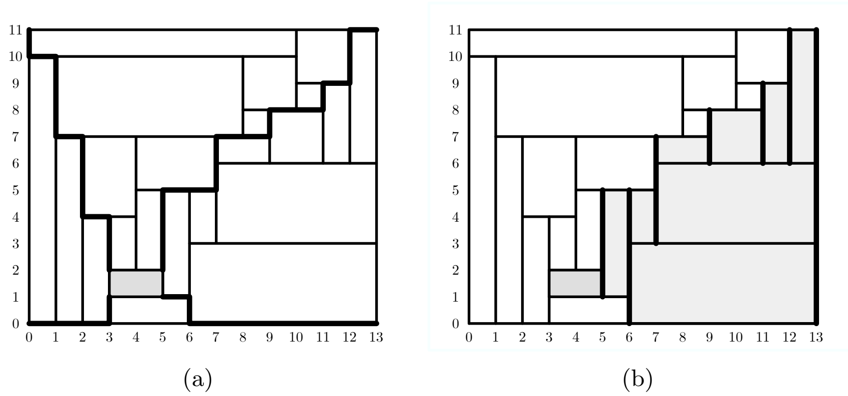

FIGURE 3.6. (a) Zig-zag paths. (b) A sector and its segments.

that point along the edges of r and towards P_0. Next, for each i, assuming P_i is known, let P_{i+1} be the endpoint of the segment that follows one of the cardinal directions, either δ_1 or δ_2, along the segment that contains P_i. Note that (a) the choice between δ_1 and δ_2 is uniquely determined because R is generic, and (b) the directions alternate between the two at each point P_i. This path eventually terminates at a vertex, say P_k of the bounding rectangle. Call this path which starts at P_0, changes cardinal directions at each vertex P_i, and ends at P_k the *zig-zag path* of (R, r, P_0). If r is incident to a bounding corner of R, then the zig-zag path is trivially the point P_0. Similarly, if r is not incident to a bounding corner but incident to a bounding edge, then the zig-zag path is the line segment joining P_0 to P_1. Evidently, there are four zig-zag paths for any given pair (R, r), corresponding to the vertices of r. Figure 3.6 shows the zig-zag paths which start at the shaded rectangle.

For a general choice of constituent rectangle r, the four zig-zag paths starting at r, together with the edges of r, yield a subdivision of the bounding rectangle into 5 regions. Call the four regions abutting r and bounded laterally by two zig-zag paths the *sectors* induced by r. The four sectors thus correspond to the four cardinal directions $\{\pm e_1, \pm e_2\}$. The interior of a sector is empty precisely when the corresponding edge of r lies on an edge of the bounding rectangle. Choose a cardinal direction $\delta \in \{\pm e_1, \pm e_2\}$ and an edge e on the side of r corresponding to δ. If σ is a segment that is perpendicular to δ, then the value of the dot product $\delta \cdot v$ does not depend on the choice of $v \in \sigma$. Thus, the *displacement value of σ relative to δ* is this constant value $\delta \cdot v$ for $v \in \sigma$. If there is no risk of confusion, let $\delta \cdot \sigma$ denote such a displacement value relative to δ. With a choice of (r, δ), there is a sequence $\sigma_0, \sigma_1, \ldots, \sigma_k$ of segments of R with the following properties:

- The segments σ_i are perpendicular to δ, and hence are mutually parallel.

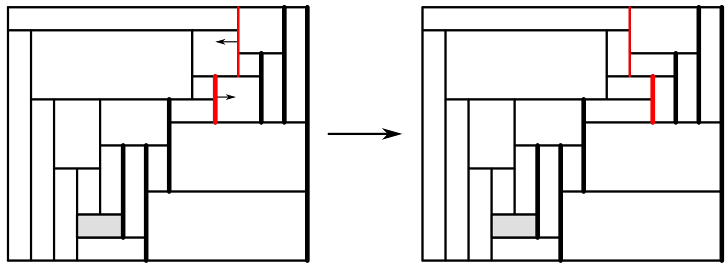

FIGURE 3.7. Segment exchange.

- Each segment σ_i lies in in the sector corresponding to e.
- The displacement values $\delta \cdot \sigma_i$ increase with i.
- The initial segment σ_0 contains the edge e.
- The last segment σ_k coincides with an edge of the bounding rectangle of R.

Figure 3.6 shows a sector and the corresponding sequence of segments.

In order to construct a rectangulation for which the constituent rectangle corresponding to r is universally visible, we apply a permutation to the coordinates that "rearranges" the segments. These permutations depend on the displacement values of the segments. Thus, as above, assume a triple (R, r, δ) is given. Also, let σ_i be the segments determined by (R, r, δ), as described above. Since R is condensed, the displacement values $\delta \cdot \sigma_i$ comprise a sequence of strictly increasing integers. Also, again since R is condensed, every integer in the interval

$$\mathbb{Z} \cap [\delta \cdot \sigma_0, \delta \cdot \sigma_k] = \{\delta \cdot \sigma_0,\ \delta \cdot \sigma_0 + 1,\ \delta \cdot \sigma_0 + 2, \ldots,\ \delta \cdot \sigma_k\}$$

corresponds to precisely one segment that is perpendicular to δ. Define a *gap value* as an integer that lies in this interval but which is not equal to any of the displacement values $\delta \cdot \sigma_i$.

Let R be a condensed rectangulation. Call a segment σ of R *slideable in the cardinal direction* δ if the intersection of the translated segment $\sigma + \delta$ with any other parallel segment has empty relative interior. Since R is condensed, we may say:

Lemma 3.3. *Suppose R is a condensed rectangulation and σ is a segment that is slideable in the cardinal direction δ. Then there is a unique parallel segment σ' of R that is slideable in the direction $-\delta$.*

With that, an *exchangeable pair* is a pair (σ, σ') of segments such that σ and σ' are slideable in opposite cardinal directions. Suppose $\pi = (\sigma, \sigma')$ is an exchangeable pair with respect to δ. Then we define a new rectangulation R_π by exchanging the segments corresponding to the displacement values $\delta \cdot \sigma$ and $\delta \cdot \sigma'$. Figure 3.7 shows the effect of exchanging a pair of vertical segments on our running example.

Exchanging segments is related to the notion of a "wall slide" as found, for instance, in [19] and [9]. A wall slide exchanges two segments, but the segments are abutted by a common perpendicular segment which has one endpoint of each segment. For a general segment exchange, by contrast, the segments are not required to be incident to a common perpendicular segment. Also note that if R and R' are related by a segment exchange, then R and R' are weakly but not strongly equivalent. In particular, two rectangulations that are related by segment exchange are not globally equivalent.

Finally, we describe how to find a sequence of segment that exchanges will result in having a universally visible constituent rectangle. This is based on the observation: If R' is obtained from R by a segment exchange, then R' is weakly equivalent to R. Thus, if $R_0, R_1, \ldots, R_k$ is a sequence of rectangulations where R_i is obtained from R_{i-1} by a segment exchange for all i, then we obtain for each constituent rectangle r_0 of R_0 a sequence $r_0, r_1, \ldots, r_k$ where each r_i is a constituent rectangle of R_i that corresponds to the constituent rectangle r_{i-1} of R_{i-1}. We summarize the key step of the algorithm as follows:

Lemma 3.4. *Suppose R is a condensed rectangulation, r is a constituent rectangle, and δ is a cardinal direction. Let $\sigma_0, \sigma_1, \ldots, \sigma_k$ be the sequence of segments as determined by the triple (R, r, δ). If the set of gap values is nonempty and c is the greatest gap value, then there is a segment σ and $i \in \{0, 1, \ldots, k\}$ such the displacement value of σ relative to δ is c and (σ, σ_i) is an exchangeable pair.*

Proof. Let σ be the segment such that $\delta \cdot \sigma = c$. Existence and uniqueness of σ follows because R is condensed. Also, choose i so that $\delta \cdot \sigma_i$ is the greatest displacement value not exceeding c. We claim that (σ, σ_i) is an exchangeable pair. Notice that the segment σ does not lie in the interior of the sector determined by (R, r, δ). This follows from genericity, as no segment can cross a zig-zag path. Similarly, σ_i lies in this sector. Hence, the relative interior of the set $(\sigma_i + \delta) \cap \sigma$ is empty, and (σ, σ_i) is an exchangeable pair. □

The method of finding a suitable sequence of segment exchanges should now be clear: Whenever there is a gap value, Lemma 3.4 shows how to construct a weakly equivalent rectangulation with a smaller corresponding gap value or with fewer gap values. Since R is finite, one can continue to apply such segment exchanges until no gap values remain. By construction, the focused rectangle will be universally visible in the last rectangulation in this sequence after applying this algorithm to all four sides of r. Figure 3.8 shows the result of applying a sequence of segment exchanges that results in a constituent rectangle having no gap values; equivalently, one may notice that the focused constituent rectangle is universally visible.

Given this theorem, we see that, up to weak equivalence, central inversion is well-defined for every constituent rectangle.

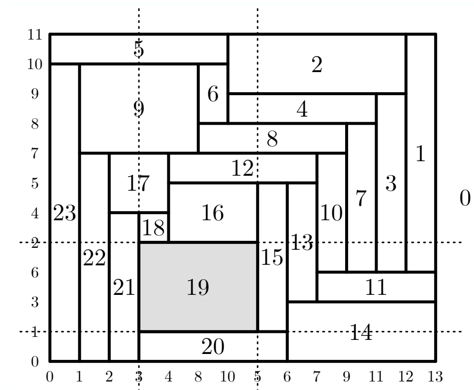

Figure 3.8. No gap values remain.

We note that zig-zag paths and the subdivision they induce have an analogue in the context of Schnyder woods. Suppose G is a planar map. For a general vertex v of a Schnyder wood T of G (see [15] for definitions), there are three distinguished paths joining v to vertices on the outer face of G, analogous to our zig-zag paths. However, whereas the three distinguished paths of a triple (G, T, v) yield a subdivision of the sphere into four regions topologically equivalent to subdividing using a tetrahedron, zig-zag paths yield a subdivision topologically equivalent to that of a cube. One should consult the article [15] and the references therein for details about this construction, and as it relates to orthogonal surfaces and graph drawing.

4. Proper Markings

In this section, we study rectangulations as they relate to flag orthotopes of planar maps.

Let R be a generic rectangulation. A *marking* of R is a linear ordering μ of the set consisting of the constituent rectangles and the bounding rectangle. A *marked rectangulation* is a pair (R, μ), where μ is a marking of R. If R has n constituent rectangles, then we generally use the values $\{0, 1, 2, \ldots, n\}$ for the marks and employ the usual linear ordering of $\mathbb{Z}$.

Suppose (R, μ) is a marked rectangulation. Define the the *lift* of (R, μ) as the intersection of all connected orthotopes which contain the rectangles

$$r \times \{\mu(r)\} \subset \mathbb{R}^3,$$

as r ranges over all constituent rectangles, including the bounding rectangle. Figure 4.1 shows an example of a marked rectangulation and an axonometric drawing of its lift. In general, if the bounding rectangle is marked zero, then the lift is the union of the boxes $r \times [0, \mu(r)]$, as r ranges through the constituent rectangles. Notice that the lift of (R, μ) is a relaxed sort of convex hull of the constituent rectangles of R, where each constituent rectangle lies at the level at which it is marked. We emphasize here that one

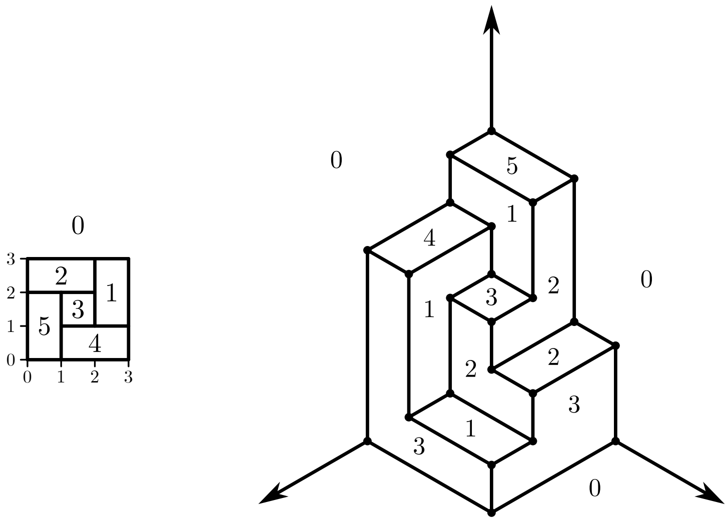

FIGURE 4.1. A marked rectangulation and its lift.

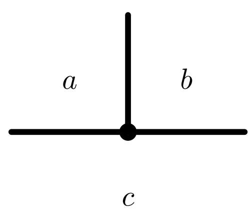

FIGURE 4.2. A local marking.

should not expect the lift of (R, μ), in general, to be orthogonally convex. Indeed, it is the purpose of this section to study markings whose lifts are totally orthogonally convex and/or totally spherical.

Figure 4.1 also shows an example of a "corner orthotope", i.e. an orthogonal polytope that can be expressed as a union of boxes with a common corner at the origin $(0, 0, 0)$ and the opposite corner in the primary octant. 3-dimensional corner orthotopes are essentially the same as corner polyhedra, as studied in [12]. (Another reason why this example is important is that it represents the Coxeter complex of type A_3, corresponding to the symmetric group on 4 letters. This representation is "faithful" to the Coxeter complex in that the rank-r cosets in this complex correspond bijectively with the r-dimensional faces for all $r \in \{0, 1, 2, 3\}$.)

4.1. **The local constraint.** Suppose a rectangulation has a vertex v for which the incident constituent rectangles are marked according to Figure 4.2. The *local constraint* at v is the statement:

$$c < \min(a, b) \text{ or } c > \max(a, b). \tag{4.1}$$

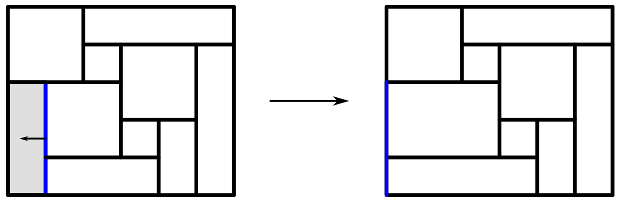

FIGURE 4.3. Deleting a corner rectangle.

Call a marking *proper* if it obeys the local constraint (4.1) at every vertex. Propriety is related to sphericity: If a vertex v does not satisfy the local constraint, then v lifts to a point where the boundary of a 2-dimensional face intersects itself, as, for example, in Figure 2.6. Thus, if the bounding rectangle is marked zero, then every 2-dimensional face of the lift of μ is homeomorphic to the disc $[0,1]^2$ if and only if μ satisfies the local constraint at every vertex. (Curiously, if a rectangulation R is not generic, i.e. if it has a point where two segments cross each other, then no marking of R lifts to a totally spherical orthotope. This is an easy case-by-case analysis that we omit; up to equivalence, there are only 2 cases to consider. In any case, generic rectangulations are the focus of these notes.)

4.2. **Marking via block deletion.** We briefly describe here an algorithm that yields a proper marking of a given rectangulation. One may find more details about this in [2], and it is also implicitly described in [16].

Let R be a rectangulation with at least 2 rectangles. The algorithm proceeds by iteratively removing rectangles from R in a uniquely determined way and marking the order in which they are removed. Here is a description of a single iteration: Choose a constituent rectangle r of R that is adjacent to a corner of the bounding rectangle. Since R is generic, exactly one of the two edges of r that are not incident to this corner must be a segment of R, say σ, while the other edge is a proper subset of another segment of R, say τ. Then we obtain a generic rectangulation R' from R by "sliding" σ across r in the direction parallel to τ until it reaches the boundary of the bounding rectangle. See Figure 4.3 for an illustration of this. Notice that if R has $n \geq 2$ constituent rectangles, then R' has $(n-1)$ constituent rectangles. In successively applying this procedure using the same corner with each iteration, one obtains a marking of R. We refer to this algorithm as the *block-deletion algorithm.* We may state:

Proposition 4.1. *The marking induced by the block-deletion algorithm using in any given corner is proper.*

Proof. Notice that for any segment σ of the rectangulation, all of the markings of constituent rectangles on one side of σ exceed all of the markings on the other side. This marking therefore obeys the local constraint at every vertex, and so it is proper. $\square$

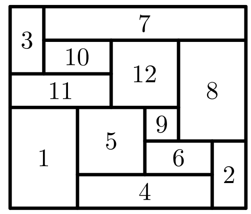

FIGURE 4.4. A proper marking with an interior peak.

See Figures 2.4, 2.8, and 3.5 for examples of rectangulations that were marked using the block-deletion algorithm.

We also observe:

Proposition 4.2. *Suppose R and R' are weakly equivalent rectangulations. Then the block-deletion algorithm yields the same ordering of the corresponding constituent rectangles.*

Another topological observation about the block-deletion algorithm is that it yields a shelling:

Proposition 4.3. *Suppose R is a rectangulation that is marked by the block-deletion algorithm starting in any given corner. Then (a) the union $\bigcup_{i=1}^{k} r_i$ is homeomorphic to the disc $[0,1]^2$ for all k, and (b) the intersection $r_k \cap (\bigcup_{i=1}^{k-1} r_i)$ is homeomorphic to the interval $[0,1]$ for all k.*

4.3. **Peaks.** The main result of this section concerns peaks of a marked rectangulation, which we now introduce.

Suppose (R, μ) is a marked rectangulation and r a constituent rectangle of R. Call r a *peak* of (R, μ) if $\mu(s) < \mu(r)$ for every constituent rectangle s that is adjacent to r. Every proper marking of R has at least one peak. The block-deletion algorithm yields a marking with precisely one peak in the constituent rectangle of R that is opposite the corner used for the block-deletion algorithm. The lift of such a marking is thus a corner orthotope for which every k-dimensional face is orthogonally convex. A proper marking may have a peak that is not incident to a corner of the bounding rectangle, as one may see in Figure 4.4. However, it is surprising that a certain kind of proper marking cannot have more than one peak:

Proposition 4.4. *Every marking of a rectangulation whose lift is orthogonally convex and totally spherical has precisely one peak.*

Proof. We show that no minimal counterexample exists. Assume that R is a rectangulation with a proper marking μ for which every constituent rectangle is marked with the integers $\{1, 2, 3, \ldots, n\}$ and the bounding rectangle is marked zero. Assume that R has two peaks, say r_1 and r_2. Then neither r_1 nor r_2 is marked with 1.

A key observation is that the constituent rectangle marked with 1 must be adjacent to a corner of the rectangle bounding R. This follows from

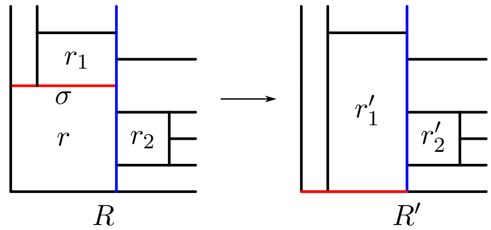

FIGURE 4.5. Detail in Proof.

orthogonal convexity: If r is a rectangle that is not in a corner and for which $\mu(r) = 1$, then there are two rectangles, either above and below or to the left and right that have higher marks. Such a marking would violate orthogonal convexity.

With that, let r be the constituent rectangle such that $\mu(r) = 1$. Let σ be the edge of r that coincides with an entire segment of lying interior to R. The existence and uniqueness of σ is guaranteed by the fact that R is generic and r is adjacent to the bounding rectangle.

Let R' be the generic rectangulation obtained from R by sliding r out from R (necessarily in the direction perpendicular to σ) and let μ' be the marking of R' given by the formula $\mu'(s') = \mu(s) - 1$, where s' is the constituent rectangle of R' corresponding to s. Let r'_1 and r'_2 be the constituent rectangles of R' corresponding to r_1 and r_2. (See Figure 4.5.) Also note that the lift of (R', μ') is also orthogonally convex.

We must consider the possibilities that either r'_1 and r'_2 are both peaks of μ' or they are not. First, suppose r'_1 and r'_2 are both peaks of μ'. This implies that we have a way to reduce a marking with two peaks to a marking with two peaks and fewer rectangles. Thus, since we may apply this reduction repeatedly, we may assume without loss of generality that r'_1 and r'_2 are not both peaks of μ'.

With that, assume r'_1 and r'_2 are not both peaks of μ'. This implies that r'_1 and r'_2 are adjacent in R'. This in turn implies that r_1 and r_2, while not adjacent, must be aligned. Moreover, both r_1 and r_2 must be adjacent to the rectangle r with $\mu(r) = 1$.

Without loss of generality, assume that σ contains an edge of r_1. Thus, r_1 and r are adjacent via σ. Let τ be the segment of R that contains edges of r_1, r_2, and r. Without loss of generality, assume that $\mu(r_1) = n$ and $\mu(r_2) = n - 1$ are the greatest marks, while $\mu(r) = 1$ is the lowest mark. This in turn implies that there is a fourth rectangle, say r_3 that is adjacent to both r and r_1. (See Figure 4.6.) However, any marking $\mu(r_3)$ must then violate the local constraint and we arrive at a contradiction. Hence μ cannot have more than one peak. $\square$

Remarks. We emphasize this theorem as follows. We have three possible statements regarding a marked rectangulation (R, μ) and its lift:

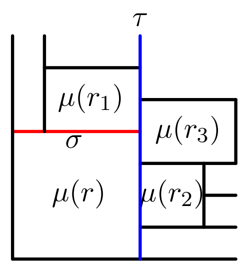

FIGURE 4.6. Detail in Proof.

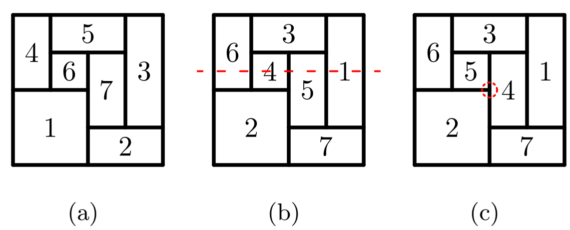

FIGURE 4.7. Marked rectangulations. (a) Orthogonally convex and spherical. (b) Double-peaked and spherical. (c) Double-peaked and orthogonally convex.

(A) The marking has more than 1 peak.
(B) The lift is orthogonally convex.
(C) The lift is totally spherical.

A corollary of the theorem above is that any two of these three statements together imply the negative of the third. Moreover, there are markings that satisfy any two of these statements, but which, according to the theorem, must violate the third. See Figure 4.7.

Also, it is interesting that the imposition of total sphericity and orthogonal convexity leads to the consequence that every orthographic shadow of such an orthotope is also orthogonally convex. That is, it is easy to make a 3-dimensional flag orthotope of a planar map that is orthogonally convex and which has a shadow with two peaks. However, such an orthotope necessarily has at least one 2-dimensional face which is not homeomorphic to a disc. Stated in the contrapositive, by asking a lift to have two peaks, one must sacrifice the local constraint somewhere, which is fundamentally a topological constraint.

5. Some Open Questions

Here are some open questions, most of which are admittedly vague.

5.1. **Generalizations of Baxter Permutations.** Proper markings allow us to generalize Baxter permutations. First recall a construction of Baxter permutations from [2]: Suppose R is a rectangulation with n constituent rectangles, and $\lambda, \mu : R \to \{1, 2, 3, \ldots, n\}$ are the markings of R obtained from the block-deletion algorithm using any two adjacent corners of R. Then, as shown in [2], the permutation $\mu \circ \lambda^{-1}$ is a Baxter permutation on $\{1, 2, 3, \ldots, n\}$. Also, again as demonstrated in [2], every Baxter permutation arises in such a fashion.

With this idea, we generalize Baxter permutations as follows. As above, let R be a generic rectangulation, and let λ be the marking of R obtained by applying the block-deletion algorithm starting in the southwest corner. Let μ be any proper marking of R. Then the function $\mu \circ \lambda^{-1}$ is a permutation of $\{1, 2, 3, \ldots, n\}$ which may or may not be a Baxter permutation. Can we enumerate these or estimate their numbers? Can we find a bijection from these permutations to other well-studied objects? Is there a pattern-avoiding characterization of these permutations, cf. [3, 4, 17, 19]?

For that matter, we should notice that there are several natural variants or strengthenings of propriety, as they all guarantee that an orthotope is topologically nice in one way or another. Here are a few examples, several of which may be combined:

- The lift is orthogonally convex.
- Every facet of the lift is orthogonally convex.
- Every canonical orthographic shadow of the lift is orthogonally convex.
- The marking coincides with a shelling.

Thus, as above, one may ask about counting these or relating them to other objects, etc.

Another closely related type of permutation is obtained through central inversion. Thus, suppose R is a rectangulation with n constituent rectangles, and suppose μ is a marking of R obtained through block deletion starting in, say, the southwest corner. Assume that the bounding rectangle is marked zero. Let r be a constituent rectangle, and let $T_r(R)$ be the generic rectangulation obtained through central inversion about r. Then μ induces a marking of $T_r(R)$ through weak equivalence (although we should not expect it to be proper). Let μ_r be the marking of $T_r(R)$ obtained through block deletion starting in the southwest corner of $T_r(R)$, where the bounding rectangle is marked $\mu(r)$. The map $\mu_r \circ \mu^{-1}$ is a permutation of $\{0, 1, 2, \ldots, n\}$. For example, the permutation obtained from the inversion depicted in Figure 3.1 yields

$$(8, 5, 3, 2, 7, 6, 1, 4, 0, 10, 9).$$

(See Figure 5.1.) As above, there are outstanding analogous combinatorial questions about these permutations as well. It is important to note that, due to the fact that we may centrally invert about any given constituent rectangle, we have such a permutation for every constituent rectangle of a given rectangulation.

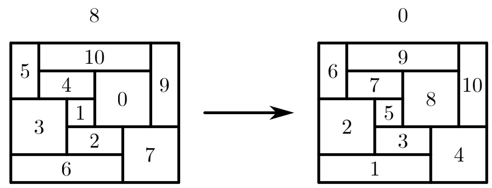

FIGURE 5.1. A permutation resulting from central inversion.

5.2. **The effect of central inversions on proper markings.** Suppose (R, μ) is a properly marked rectangulation, and r is a constituent rectangle of R. What do we have to say about the parameters (R, μ, r) in order to ensure that $T_r(R)$ is properly marked by μ? This author imagines asking this question for the variants of propriety as described above. We can say generally that if μ satisfies the local constraint at a vertex v and v lies interior to a sector determined by (R, r), then μ obeys the local constraint at $T_r(v)$. However, one should not expect μ to obey the local constraint if v lies on a zig-zag path.

5.3. **Single-coordinate inversion.** What happens if we perform central inversion on only one coordinate in a rectangulation? There are many questions that one could ask about these generalized rectangulations, but we will focus on only three.

Figure 5.2 illustrates the application of single-coordinate inversions as applied to the rectangulation appearing in 3.1. In this case, the focused rectangle is universally visible and there is a pleasant "two-layered" property of the resulting rectangles: There are well-defined top and bottom subdivisions of a common orthogonal polygon into rectangles, where the rectangles are determined by the sectors of the focused rectangle. (Compare the markings in the figure.) Is universal visibility a necessary and sufficient condition to ensure that the resulting collection of rectangles has this double-layered property?

Figure 5.2 also illustrates an example of another question: What are the most natural ways to define proper markings of these generalized rectangulations? The lifts of the markings depicted in Figure 5.2 are totally spherical and totally orthogonally convex. What markings of a rectangulation (in the nongeneralized sense) yield single-coordinate inversions corresponding to similarly topologically nice orthotopes?

Finally, what is the relation, if any, to slicings of parallelogram polyominoes, cf. [4]?

5.4. **Promotable rectangles.** Let R be a rectangulation and suppose r is a constituent rectangle. Call r *promotable* if there is a weakly equivalent rectangulation R' and a proper marking μ of R' such that the rectangle r' corresponding to r is peak of μ. Can we characterize these constituent

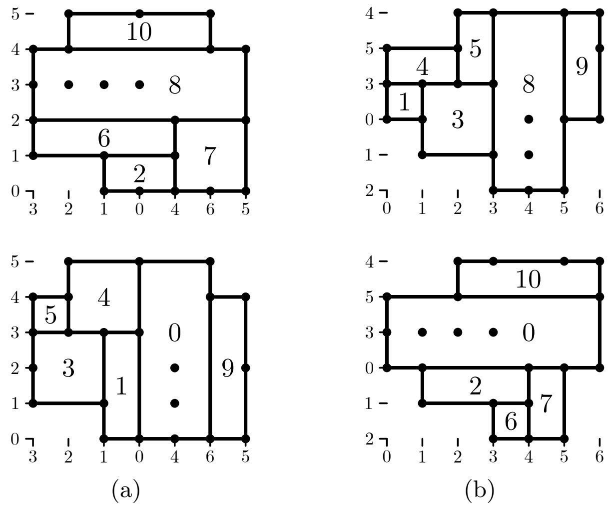

FIGURE 5.2. The results of inverting (a) the x-coordinates (b) the y-coordinates.

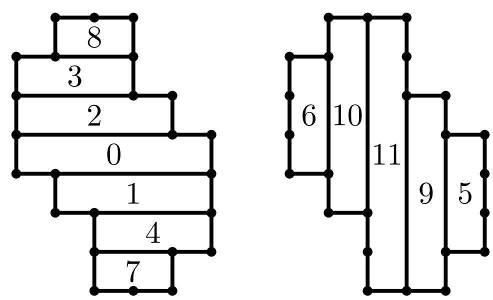

FIGURE 5.3. Semi-hamiltonian orthotope, with a proper marking.

rectangles in some way? This author imagines a criterion similar to characterizing when a constituent rectangle allows a central inversion (i.e. universal visibility).

5.5. **Semi-hamiltonian rectangulations.** Suppose R is a two-layer rectangulation. Call R *semi-hamiltonian* if all of its vertices lie on the bounding orthogonal polygon of R and every integral point that lies on the bounding polygon of R is a vertex of R. In other words, R is semi-hamiltonian when the integral points of the bounding orthogon coincide with the vertices of R. This is somehow "hamiltonian" because the bounding orthogon serves as a "cycle" which passes through every vertex of R. Figure 5.3 shows a semihamiltonian rectangulation that realizes a flag orthotope of the cube $[0,1]^3$. A natural problem is to determine which rectangulations (or planar maps) admit a semihamiltonian rectangulation.

5.6. **Segment-exchange graphs.** Let R be a condensed rectangulation. Define the *segment-exchange* graph of R as $G = (V, E)$, where V is the set of all rectangulations that are weakly equivalent to R and E is the set of pairs $\{R, R'\}$ of rectangulations such that there is an exchangeable pair that transforms R to R'. What structures characterize these graphs? Assuming that the bounding rectangle is $[0, w] \times [0, h]$, can we estimate the cardinalities $|V|$ and $|E|$, functions of w and h? Similarly, what are the extreme values that we should expect of these values?

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