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Quantum Langevin equations for optomechanical systems

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Abstract. We provide a fully quantum description of a mechanical oscillator in the presence of thermal environmental noise by means of a quantum Langevin formulation based on quantum stochastic calculus. The system dynamics is determined by symmetry requirements and equipartition at equilibrium, while the environment is described by quantum Bose fields in a suitable non-Fock representation which allows for the introduction of temperature. A generic spectral density of the environment can be described by introducing its state through a suitable P-representation. Including interaction of the mechanical oscillator with a cavity mode via radiation pressure we obtain a description of a simple optomechanical system in which, besides the Langevin equations for the system, one has the exact input-output relations for the quantum noises. The whole theory is valid at arbitrarily low temperature. This allows the exact calculation of the stationary value of the mean energy of the mechanical oscillator, as well as both homodyne and heterodyne spectra. The present analysis allows in particular to study possible cooling scenarios and to obtain the exact connection between observed spectra and fluctuation spectra of the position of the mechanical oscillator.

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Contents

1. Introduction

Optomechanical systems in the quantum regime are very important for quantum information processing and for testing fundamental issues of quantum mechanics [1–10]. Their theoretical analysis therefore calls for a first principle description. In particular since the focus is on quantum effects, the theoretical models must be fully consistent with quantum mechanics. Actually the correct quantum description of a mesoscopic mechanical oscillator and of the thermal noise affecting it is not a trivial task, and there is not a unique accepted model for them [11–24].

The first aim of this paper is therefore to obtain an accurate quantum mechanical description of a mechanical oscillator taken to be part of an opto-mechanical device. The oscillator cannot be considered as a Brownian particle, but rather as a mesoscopic mechanical system, say a movable mirror mounted on a vibrating structure. Dissipative effects are essentially due to the interaction with phonons. Our strategy will be to introduce reasonable physical requirements leading to a master equation in Lindblad form, valid for any temperature of the thermal bath. We then translate these results into quantum Langevin equations and we show how to obtain a suitable non-Markovian generalization at this level of description. Relying on these results we can consider the description of the simplest optomechanical system, that is a moving mirror interacting with an electromagnetic mode in a cavity via radiation pressure [1, 5–7, 25]. Again a suitable analysis of the composite system and of the monitoring of the emitted light calls for a consistent quantum description. We shall obtain this result by the use of quantum Langevin equations, directly deducing them from a unitary dynamics, and exploiting the theory of measurements in continuous time.

The paper is organized as follows. In Section 2 we determine the reduced dynamics of the mechanical oscillator. Here, the basic assumption is the use of a Markovian master equation with a quadratic generator and having a unique equilibrium state. Its structure is further determined by suitable symmetry requirements and by physical constraints on the behaviour of the mean values of position and momentum. In Section 3 we introduce the quantum Langevin equations for the mechanical oscillator alone; the whole presentation is based on the notions of quantum noise [26, 27] and of input-output fields [28–30], as well as on the use of quantum stochastic calculus [31, 32]. The Bose fields entering in the unitary dynamics play the role of phonon fields. By modifying their state without changing the time evolution operator it is possible to introduce non Markovian effects, namely a non-flat noise spectrum. The differences with respect to usual approaches are of relevance especially at low temperatures, where the zero-point fluctuations play an essential role.

A quantum optomechanical system is studied in Section 4 by using the quantum Langevin approach, within a fully consistent formulation valid at any temperature. Firstly, the typical effect of laser cooling is discussed. Then, the continuous monitoring of the emitted light is introduced in Sections 4.3.1 (homodyne detection) and 4.3.2 (heterodyne detection). The treatment is well based in the theory of measurements in continuous time. Detection of the emitted light is usually assumed to give a direct measurement of the fluctuations of the position of the mechanical component. We show that this is true, but only for not too low temperatures; at very small temperatures, interference terms are important and the direct connection with such fluctuations is lost. This leads to new predictions on the optical spectra at very low temperature. We finally summarize and discuss our results in Section 5.

2. Damped mechanical oscillator: the master equation approach

As a first step towards the construction of models of optomechanical systems valid in the quantum regime at low temperatures, we consider the reduced dynamics of an open mechanical oscillator. A fully consistent quantum description of a massive nanomechanical component, kept at the simplest possible level, will be our basic building block in order to consider more complex dynamics. We therefore formulate in the first instance a Markovian description for the mechanical oscillator, which we build up relying on general physical constraints and symmetry requirements.

A standard approach often considered in the literature is to derive the master equation for the harmonic oscillator from effective environmental models of bosonic oscillators [13, 14, 20–24]. However, previous work [16, 17] has shown that, while using careful approximations a positive Markovian dynamics can be obtained in this framework, the final results are valid only from medium to high temperatures of the thermal bath. To ask for a Markovian dynamics based on symmetry arguments allows to get simpler and more universal models, but again serious problems appear. The requirement that the equilibrium state should be the canonical thermal state determined by the standard Hamiltonian of a harmonic oscillator is known to be incompatible with positivity and translational invariance [11, 33, 34]. This incompatibility induced some authors to renounce to translational invariance [15,34], or to accept non-positive dynamical equations and to give more relevance to obtaining time evolutions very close to the classical ones [13, 14, 35]. A non positive dynamics can be satisfactory when the system is near the classical regime, but this approach becomes questionable when quantum effects are searched for [36, 37]. We shall show that it is possible to maintain positivity and translational invariance by weakening the requests on the equilibrium state. The key point will be a weak formulation of energy equipartition at equilibrium. Our result is therefore to single out from the many proposals appearing in the literature a unique consistent dynamics in the Markov approximation.

2.1. Physical constraints and symmetry requirements

We formulate now our assumptions, starting from the existence of a well defined positive Markovian dynamics, describing damping and translationally invariant apart from the harmonic potential. A weak equipartition condition and the existence of a unique stationary state in Gibbs form, as we shall see, will essentially fix the structure of the reduced dynamics.

Assumption 1 (Positive Markovian dynamics with quadratic generator)**.** *The evolution of the statistical operator of the oscillator is governed by a Markovian master equation preserving the positivity of the states. The generator of the dynamics is at most quadratic in the position and momentum operators of the mechanical oscillator.*

The first assumption is to consider a time-homogeneous and linear time evolution. Such a dynamics can be expressed in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\rho(t) = \mathcal{L}[\rho(t)], \tag{1}$$

with $\mathcal{L}$ a suitable generator or Liouville operator, at most quadratic in the position and momentum operators of the mechanical oscillator q and p, so as to have at most a quadratic potential term and a friction effect proportional to the momentum of the mechanical oscillator. In the case of linear systems it is known that positivity and complete positivity of the dynamics are actually equivalent [11], therefore according to [38, 39] the generator $\mathcal{L}$ must have the standard Lindblad structure. The most general quadratic Liouville operator is obtained in terms of two Lindblad operators [11]

$$R_j = \frac{1}{\sqrt{\hbar}}\left(u_j q + v_j p\right), \qquad u, v \in \mathbb{C}^2, \tag{2}$$

and a generic selfadjoint quadratic Hamiltonian for the mechanical system

$$H_{\mathrm{m}} = \frac{h_q}{2}\,q^2 + \frac{\kappa_0}{4}\,\{q,p\} + \frac{h_p}{2}\,p^2 + f_q q + f_p p,$$

where all the constants are taken to be real, so that $\mathcal{L}$ takes the form

$$\mathcal{L}[\rho] = -\frac{\mathrm{i}}{\hbar}\left[H_{\mathrm{m}},\rho\right] + \sum_{j=1}^{2}\left(R_j\rho R_j^\dagger - \frac{1}{2}\left\{R_j^\dagger R_j,\rho\right\}\right). \tag{3}$$

We ask now to have a damped oscillator, but not an *overdamped* one.

Assumption 2 (Damping)**.** *The "kinetic energy" term is non negative and the mean values of position and momentum decay to zero with an oscillating behaviour.*

Apart from the trivial requirement of a positive *kinetic energy* term, we further look for a dynamics describing the oscillating decay of the mean values of q and p to zero. This condition complies with the Markovian and quadratic approximations, which are expected to be good only for small damping. Denoting by $\langle X\rangle_t$ the mean value of a quantum operator with the state $\rho(t)$ solution of the master equation we have for position and momentum

$$\frac{\mathrm{d}\langle q\rangle_t}{\mathrm{d}t} = h_p\langle p\rangle_t + \left(\frac{\kappa_0}{2} - \mathrm{Im}\,\langle u|v\rangle\right)\langle q\rangle_t + f_p,$$
$$\frac{\mathrm{d}\langle p\rangle_t}{\mathrm{d}t} = -h_q\langle q\rangle_t - \left(\frac{\kappa_0}{2} + \mathrm{Im}\,\langle u|v\rangle\right)\langle p\rangle_t - f_q,$$

with $\langle u|v\rangle$ the inner product in $\mathbb{C}^2$. The eigenvalues of the associated dynamical matrix are $-\operatorname{Im}\langle u|v\rangle \pm \sqrt{\kappa_0^2/4 - h_p h_q}$, so that in order to have an underdamped dynamics we need $\operatorname{Im}\langle u|v\rangle > 0$ and $\kappa_0^2/4 < h_p h_q$. In particular h_p and h_q have the same sign and are non-vanishing. Positivity of the kinetic energy leads to $h_p > 0$ and therefore $h_q > 0$. Then, we can write $h_p = 1/m$ and $h_q = m\Omega_{\rm m}^2$; the above inequality on κ_0 becomes $\kappa_0^2 < 4\Omega_{\rm m}^2$. Finally, the vanishing of the equilibrium means imply $f_q = 0$, $f_p = 0$. Introducing the positive coefficients

$$\gamma_{\rm m} = 2\operatorname{Im}\langle u|v\rangle, \qquad D_{qp} = \operatorname{Re}\langle u|v\rangle, \qquad D_{qq} = \|v\|^2, \qquad D_{pp} = \|u\|^2,$$

the generator can be written in the form

$$\begin{aligned}\hbar\mathcal{L}[\rho] = &-\frac{\rm i}{2m}[p,\{p,\rho\}] - \frac{{\rm i}m\Omega_{\rm m}^2}{2}[q,\{q,\rho\}] - \frac{D_{pp}}{2}[q,[q,\rho]] - \frac{D_{qq}}{2}[p,[p,\rho]] \\ &- D_{qp}[p,[q,\rho]] - \frac{{\rm i}(\kappa_0+\gamma_{\rm m})}{4}[q,\{p,\rho\}] - \frac{{\rm i}(\kappa_0-\gamma_{\rm m})}{4}[p,\{q,\rho\}], \end{aligned} \tag{4}$$

where in particular the constraints

$$D_{qq} \geq 0, \qquad D_{pp} \geq 0, \qquad D_{qq}D_{pp} - D_{qp}^2 - \left(\frac{\gamma_{\rm m}}{2}\right)^2 \geq 0 \tag{5}$$

hold, which provide the necessary and sufficient conditions for the dynamics described by (4) to be in Lindblad form and therefore completely positive [11, 38]. An alternative way to get the same positivity condition is to ask the *generalized* Heisenberg uncertainty relation $\langle q^2\rangle_t\langle p^2\rangle_t - (\langle\{p,q\}\rangle_t/2)^2 \geq \hbar^2/4$ to hold for any time and any initial state [40].

The first two assumptions are implicitly or explicitly taken in all the Markovian approaches. A further natural requirement is that the interaction with the environment does not depend on the position of the oscillator.

Assumption 3 (Translational invariance)**.** *The reduced dynamics is invariant under translations apart from the harmonic potential term. This requirement is equivalent to the validity of the classical equations of motion for the mean values of position and momentum.*

By applying the generic translation $q \mapsto q + x$, $p \mapsto p$ to the generator (4) we see that all the terms are invariant with the exclusion of the term containing the harmonic potential $-\frac{{\rm i}m\Omega_{\rm m}^2}{2}[q,\{q,\rho\}]$ and the last term $-\frac{{\rm i}(\kappa_0-\gamma_{\rm m})}{4}[p,\{q,\rho\}]$. Therefore, the above assumption is satisfied if and only if $\kappa_0 = \gamma_{\rm m}$. The same conclusion is reached by considering the equations of motion for position and momentum and asking them to be equivalent to the classical equations in which the momentum is proportional to the derivative of the position.

The result of the first three assumptions is therefore that the Liouville operator has the structure (4) with $\kappa_0 = \gamma_{\rm m} > 0$ and $\Omega_{\rm m}^2 > \gamma_{\rm m}^2/4$; moreover, the constraints (5) hold. In particular, the Hamiltonian part of $\mathcal{L}$ turns out to be

$$H_{\rm m} = H_0 + \frac{\gamma_{\rm m}}{4}\{q,p\}, \qquad H_0 = \frac{p^2}{2m} + \frac{1}{2}m\Omega_{\rm m}^2 q^2, \tag{6}$$

where, besides a contribution in the form of the free Hamiltonian of a harmonic oscillator with a strictly positive frequency $\Omega_{\rm m}$, one has an additional term in the form of an anticommutator proportional to the damping constant.

The evolution equations for the mean values and second moments of position and momentum then read:

$$\frac{{\rm d}\langle q\rangle_t}{{\rm d}t} = \frac{\langle p\rangle_t}{m}, \qquad \frac{{\rm d}\langle p\rangle_t}{{\rm d}t} = -m\Omega_{\rm m}^2\langle q\rangle_t - \gamma_{\rm m}\langle p\rangle_t, \tag{7}$$

$$\begin{aligned}
\frac{\mathrm{d}\langle q^2\rangle_t}{\mathrm{d}t} &= \frac{\langle\{p,q\}\rangle_t}{m} + \hbar D_{qq}, \\
\frac{\mathrm{d}\langle p^2\rangle_t}{\mathrm{d}t} &= -m\Omega_{\mathrm{m}}^2\langle\{p,q\}\rangle_t - 2\gamma_{\mathrm{m}}\langle p^2\rangle_t + \hbar D_{pp}, \\
\frac{\mathrm{d}\langle\{q,p\}\rangle_t}{\mathrm{d}t} &= \frac{2\langle p^2\rangle_t}{m} - 2m\Omega_{\mathrm{m}}^2\langle q^2\rangle_t - \gamma_{\mathrm{m}}\langle\{q,p\}\rangle_t - 2\hbar D_{qp}.
\end{aligned} \tag{8}$$

The dynamical matrix giving the evolution of the mean values (7) has eigenvalues $-\gamma_{\mathrm{m}}/2$ and $-\gamma_{\mathrm{m}}/2 \pm \mathrm{i}\sqrt{\Omega_{\mathrm{m}}^2 - \gamma_{\mathrm{m}}^2/4}$, which naturally leads to introduce the *damped* frequency ω_{m} of the mechanical oscillator in terms of its *bare* frequency Ω_{m}:

$$\omega_{\mathrm{m}} = \sqrt{\Omega_{\mathrm{m}}^2 - \frac{\gamma_{\mathrm{m}}^2}{4}}. \tag{9}$$

Let us note that according to $\Omega_{\mathrm{m}}^2 > \gamma_{\mathrm{m}}^2/4 > 0$ we have ruled out the case $\Omega_{\mathrm{m}} = 0$, which corresponds to a quantum Brownian particle, that is a massive particle not bounded by a potential in a translation invariant environment [18, 41–45] (see [46] for a recent review).

At this stage we further have to determine the diffusion coefficients D_{qp}, D_{qq} and D_{pp} appearing in (4). We will rely on the study of features of the equilibrium state, but to avoid the known incompatibilities with translation invariance [11] we formulate the equipartition condition in a weaker form, not asking the existence of an equilibrium Gibbs state generated by H_0.

Assumption 4 (Equipartition)**.** *At equilibrium the mean kinetic energy and the mean potential energy have to be equal.*

Since the eigenvalues of the dynamical matrix associated to (8) have a positive real part, *existence of a unique attractive equilibrium state* is granted, and thanks to the linearity of the equations the equilibrium state is actually *Gaussian* and determined by the mean values at equilibrium. Then, our *equipartition* condition is

$$\frac{\langle p^2\rangle_{\mathrm{eq}}}{2m} = \frac{1}{2}\, m\Omega_{\mathrm{m}}^2\langle q^2\rangle_{\mathrm{eq}}, \tag{10}$$

which gives equal weight to the mean kinetic and potential energy at equilibrium. By setting in (8) the time derivatives equal to zero we come to

$$D_{qp} = \frac{m\gamma_{\mathrm{m}}}{2}\, D_{qq}. \tag{11}$$

Moreover, the second moments at equilibrium turn out to be

$$\begin{aligned}
\langle q^2\rangle_{\mathrm{eq}} &= \frac{\hbar}{2\gamma_{\mathrm{m}}}\left(D_{qq} + \frac{D_{pp}}{m^2\Omega_{\mathrm{m}}^2}\right), \qquad \langle\{q,p\}\rangle_{\mathrm{eq}} = -\hbar m D_{qq}, \\
\langle p^2\rangle_{\mathrm{eq}} &= \frac{\hbar}{2\gamma_{\mathrm{m}}}\left(D_{pp} + m^2\Omega_{\mathrm{m}}^2 D_{qq}\right).
\end{aligned} \tag{12}$$

We exploit finally the residual freedom we have in the choice of the diffusion coefficients to get a Gibbs state as equilibrium state. However, as we already noticed, it cannot be state generated by H_0, and we replace it by a generic effective Hamiltonian.

Assumption 5 (Gibbs state and temperature dependence)**.** *The equilibrium state has a Gibbs form which is determined by an effective Hamiltonian independent from the temperature.*

As we shall prove below, this assumption implies that the diffusion coefficients have the expressions

$$D_{qq} = \frac{\gamma_{\mathrm{m}}\,(2N+1)}{2m\omega_{\mathrm{m}}}, \qquad D_{pp} = \frac{\gamma_{\mathrm{m}} m\Omega_{\mathrm{m}}^2}{2\omega_{\mathrm{m}}}\,(2N+1)\,, \qquad D_{qp} = \frac{\gamma_{\mathrm{m}}^2}{4\omega_{\mathrm{m}}}\,(2N+1) \tag{13}$$

and that the equilibrium state has the form

$$\rho_{\rm eq} = \frac{{\rm e}^{-\beta H_{\rm m}}}{{\rm Tr}\left\{{\rm e}^{-\beta H_{\rm m}}\right\}} \,. \tag{14}$$

In these expressions, $H_{\rm m}$ is the mechanical Hamiltonian (6) and $N \geq 0$ represents the *mean number of excitations* in the equilibrium state, namely

$$N = \frac{1}{{\rm e}^{\beta\hbar\omega_{\rm m}} - 1}. \tag{15}$$

Let us prove the above statements. Thanks to the Gaussianity of the equilibrium state, Assumption 5 means $\rho_{\rm eq} \propto \exp\left\{-c\tilde{H}\right\}$ for a suitable quadratic Hamiltonian (without the liner terms, because the means of position and momentum have to be zero), say $\tilde{H} = \frac{p^2}{2\tilde{m}} + \frac{1}{2}\,\tilde{m}\tilde{\Omega}^2 + \frac{\tilde{\gamma}^2}{4}\,\{q,p\}$, with $\tilde{\omega}^2 := \tilde{\Omega}^2 - \frac{\tilde{\gamma}^2}{4} > 0$ in order $\rho_{\rm eq}$ to be a trace-class operator. Then, the eigenvalues of $\tilde{H}$ have the form $\hbar\tilde{\omega}\,(n+1/2)$ and, without changing $\rho_{\rm eq}$, we can redefine c and $\tilde{H}$ in such a way that $\tilde{\omega} = \omega_{\rm m}$ and $c = \beta$, a positive constant which can be interpreted as the inverse temperature of the equilibrium state of the mechanical oscillator. Then, the mean number of excitations has the expression (15). By equating the mean values determined by $\rho_{\rm eq}$ with the expressions (12), after some algebraic manipulations we get

$$\tilde{m}^2\tilde{\Omega}^2 = m^2\Omega_{\rm m}^2, \qquad \frac{m\gamma_{\rm m}}{\tilde{m}\tilde{\gamma}} = \frac{mD_{qq} + \frac{D_{pp}}{m\Omega_{\rm m}^2}}{2mD_{qq}},$$

$$(2N+1)^2 = \frac{\Omega_{\rm m}^2}{\gamma_{\rm m}^2}\left(\frac{D_{pp}}{m\Omega_{\rm m}^2} - mD_{qq}\right)^2 + \frac{4}{\gamma_{\rm m}^2}\left(D_{qq}D_{pp} - \frac{m^2\gamma_{\rm m}^2}{4}\,D_{qq}^2\right).$$

The right hand side of the last equation is greater or equal to 1 by (5) and (11). To have $\tilde{H}$ independent from the temperature implies that the coefficients D_{qq} and D_{pp} are both proportional to $2N+1$ with temperature independent coefficients. The equations above, together with $\tilde{\omega} = \omega_{\rm m}$, give the expressions (13) and $\tilde{H} = H_{\rm m}$.

2.2. Master equation for the mechanical oscillator

From the previous results we see that a central role is played by the mechanical Hamiltonian $H_{\rm m}$, which appears in the commutator part of the Liouville operator and determines the equilibrium state (14). It will be very useful to diagonalize explicitly $H_{\rm m}$ by introducing a suitable mode operator. By defining

$$a_{\rm m} = \frac{1}{\sqrt{2m\hbar\omega_{\rm m}}}\left(m\Omega_{\rm m}\,q + {\rm i}\tau\,p\right), \qquad \tau = \frac{\omega_{\rm m}}{\Omega_{\rm m}} - \frac{\rm i}{2}\frac{\gamma_{\rm m}}{\Omega_{\rm m}}, \tag{16}$$

we get that the mechanical Hamiltonian (6) can be written as

$$H_{\rm m} = \hbar\omega_{\rm m}\left(a_{\rm m}^\dagger a_{\rm m} + \frac{1}{2}\right). \tag{17}$$

The dimensionless quantity τ has modulus equal to one and $[a_{\rm m}, a_{\rm m}^\dagger] = 1$ is satisfied. The inverse formulae giving q, p in terms of $a_{\rm m}$, $a_{\rm m}^\dagger$ are

$$q = \sqrt{\frac{\hbar}{2m\omega_{\rm m}}}\left(\overline{\tau}\,a_{\rm m} + \tau a_{\rm m}^\dagger\right), \qquad p = {\rm i}\sqrt{\frac{m\hbar\Omega_{\rm m}^2}{2\omega_{\rm m}}}\left(a_{\rm m}^\dagger - a_{\rm m}\right). \tag{18}$$

Let us note that the form (14) of the equilibrium state does not come from a direct requirement, but rather it follows from all the considered assumptions. In particular we stress the fact that the operator $H_{\rm m}$ is not the Hamiltonian of the isolated oscillator, but includes a term containing $\gamma_{\rm m}$ which comes from the interaction with the bath. Combining (12) and (13) we have in particular

$$\frac{\langle p^2\rangle_{\rm eq}}{2m} = \frac{1}{2}\, m\Omega_{\rm m}^2\, \langle q^2\rangle_{\rm eq} = \frac{\hbar\Omega_{\rm m}^2}{4\omega_{\rm m}}\,(2N+1)\,, \qquad \frac{\gamma_{\rm m}}{4}\langle\{q,p\}\rangle_{\rm eq} = -\frac{\hbar\gamma_{\rm m}^2}{8\omega_{\rm m}}\,(2N+1)\,, \quad (19)$$

indeed satisfying (5). We see that the term related to damping gives a negative contribution to the equilibrium mean value $\langle H_{\rm m}\rangle_{\rm eq}$ arising from energy exchange with the bath. The Lindblad operators R_j appearing in (2) now read $R_1 = \sqrt{\gamma_{\rm m}\,(N+1)}\, a_{\rm m}$, $R_2 = \sqrt{\gamma_{\rm m} N}\, a_{\rm m}^\dagger$ so that the Liouville operator can be finally written as

$$\mathcal{L}[\rho] = -\frac{\rm i}{\hbar}\,[H_{\rm m},\rho] + \gamma_{\rm m}(N+1)\left(a_{\rm m}\rho a_{\rm m}^\dagger - \frac{1}{2}\left\{a_{\rm m}^\dagger a_{\rm m},\rho\right\}\right) + \gamma_{\rm m} N\left(a_{\rm m}^\dagger \rho a_{\rm m} - \frac{1}{2}\left\{a_{\rm m} a_{\rm m}^\dagger,\rho\right\}\right). \quad (20)$$

Let us stress that, despite the fact that the expression (20) has the form of the generator for an optical oscillator [26], the relations (16), (18) connecting $a_{\rm m}$, $a_{\rm m}^\dagger$ with q, p account for the description of a mechanical oscillator. A master equation for a mechanical oscillator with the Liouville operator (20) and the relation (16) between mode and position/momentum operators was already proposed in [12, Sects. 6, 7], inside a scheme of canonical quantization of dissipative classical systems. The introduction of (16) in order to obtain a quantum description of the mechanical oscillator complying with all the natural physical requirements is a key result of this section, which we will later exploit to consistently treat optomechanical systems.

3. Langevin equations for the mechanical oscillator

So far we have obtained a quantum master equation in Lindblad form for the mechanical oscillator, only relying on general physical constraints and symmetry requirements. However, optomechanical systems are typically dealt with making use of quantum Langevin equations, which provide a suitable and powerful approach for linear systems [37]; in such a framework not only the system of interest appears, but also some quantum noises representing the environment. It is a general result that for any master equation in Lindblad form it is possible to introduce in a rigourous way a unitary dynamics involving the system of interest and suitable quantum Bose fields, which at the level of the reduced dynamics of the system exactly reproduces the master equation. That is, these quantum Bose fields effectively describe the thermal environment affecting the mechanical oscillator and the system/field dynamics is given by a unitary time evolution operator satisfying a quantum stochastic differential equation of the type introduced by Hudson and Parthasarathy [31]. Within this formalism the Heisenberg equations for the system operators provide the quantum Langevin equations, while, as shown in [29], the Heisenberg equations for the Bose fields give the input-output relation of Gardiner and Collet [26, 28]. We thus obtain in a unified framework all relevant physical information [30]. Finally, we shall show in Section 3.2 that this approach allows to treat also non Markovian effects and to introduce noises with non flat spectrum.

Let us start introducing the Hudson-Parthasarathy equation or quantum stochastic Schrödinger equation [31], which gives the evolution equation for the unitary dynamics involving the system of interest and a quantum Bose field. The proper mathematical formulation of this equation relies on the formalism of quantum stochastic calculus [32].

For the Liouville operator (20) the associated Hudson-Parthasarathy equation reads (see e.g. [29,47,48] or [26, Sections 11.2.2, 11.2.7])

$$\mathrm{d}U(t) = \Big\{ -\frac{\mathrm{i}}{\hbar} H_{\mathrm{m}} \mathrm{d}t + \sqrt{\gamma_{\mathrm{m}}} \left(a_{\mathrm{m}} \mathrm{d}B_{\mathrm{th}}^{\dagger}(t) - a_{\mathrm{m}}^{\dagger} \mathrm{d}B_{\mathrm{th}}(t) \right) - \frac{\gamma_{\mathrm{m}}}{2} \left((2N+1)\, a_{\mathrm{m}}^{\dagger} a_{\mathrm{m}} + N \right) \mathrm{d}t \Big\} U(t), \tag{21}$$

with $U(0) = \mathbb{1}$, H_{m} given by (17), and $B_{\mathrm{th}}(t)$ a Bose thermal field satisfying the canonical commutation relations

$$[B_{\mathrm{th}}(t), B_{\mathrm{th}}^{\dagger}(s)] = \min\{t, s\}, \qquad [B_{\mathrm{th}}(t), B_{\mathrm{th}}(s)] = 0, \tag{22}$$

and the *quantum Itô table*

$$\begin{aligned} &\mathrm{d}B_{\mathrm{th}}(t)\, \mathrm{d}B_{\mathrm{th}}^{\dagger}(t) = (N+1)\, \mathrm{d}t, && \mathrm{d}B_{\mathrm{th}}^{\dagger}(t)\, \mathrm{d}B_{\mathrm{th}}(t) = N\, \mathrm{d}t, \\ &\mathrm{d}B_{\mathrm{th}}(t)\, \mathrm{d}B_{\mathrm{th}}(t) = 0, && \mathrm{d}B_{\mathrm{th}}(t)\mathrm{d}t = \mathrm{d}B_{\mathrm{th}}^{\dagger}(t)\mathrm{d}t = 0, \end{aligned} \tag{23}$$

with N the positive quantity introduced in (15). The commutation rules are better understood by introducing the formal field densities: $\mathrm{d}B_{\mathrm{th}}(t) = b_{\mathrm{th}}(t)\, \mathrm{d}t$. Then, these densities satisfy the standard canonical commutation relations

$$[b_{\mathrm{th}}(t), b_{\mathrm{th}}^{\dagger}(s)] = \delta(t-s), \qquad [b_{\mathrm{th}}(t), b_{\mathrm{th}}(s)] = 0. \tag{24}$$

Equation (21) is a quantum stochastic differential equation in Itô sense and the second line of (21) corresponds to the *Itô correction*. The solution $U(t)$ of (21) is a family of unitary operators on the overall Hilbert space which represent the dynamics of the closed system corresponding to “mechanical oscillator plus field”. An heuristic, but more familiar, picture can be obtained by using the field densities introduced above. The formal expression of the unitary evolution is indeed [49]

$$U(t) = \overleftarrow{\mathrm{T}}\, \exp\Big\{ \int_0^t \left[-\frac{\mathrm{i}H_{\mathrm{m}}}{\hbar} + \sqrt{\gamma_{\mathrm{m}}} \left(a_{\mathrm{m}} b_{\mathrm{th}}^{\dagger}(s) - a_{\mathrm{m}}^{\dagger} b_{\mathrm{th}}(s) \right) \right] \mathrm{d}s \Big\}, \tag{25}$$

where $\overleftarrow{\mathrm{T}}$ denotes the time ordered product. From this formal expression one sees that $U(t)$ is the time evolution operator for system and field in the interaction picture with respect to the free field dynamics. The thermal field B_{th} therefore represents the environment, say the phonon field interacting with the mechanical oscillator.

It is possible to show that the physical thermal field $B_{\mathrm{th}}(t)$ does not admit a Fock representation. However, it is useful for computations to have a hand a mathematical representation of $B_{\mathrm{th}}(t)$ in terms of two commuting Bose fields A_1 and A_2 in the Fock representation [29,48]. This means that such fields satisfy the canonical commutations rules $[A_i(t), A_j^{\dagger}(s)] = \delta_{ij} \min\{t, s\}$, $[A_i(t), A_j(s)] = 0$ and that there exists a common Fock vacuum, i.e. a normalized vector $e(0)$ annihilated by all these operators: $A_k(t)e(0) = 0$ for $k = 1, 2$. The field defined by

$$B_{\mathrm{th}}(t) = \sqrt{N+1}\, A_1(t) - \sqrt{N}\, A_2^{\dagger}(t), \tag{26}$$

satisfies the canonical commutation relations (22) and the Itô table (23). It is known in quantum field theory that there exist non unitarily equivalent representations of the canonical commutation relations; indeed, $B_{\mathrm{th}}(t)$ cannot be obtained by unitary transformations of Fock fields.

Let us now consider as state of the field the A-field vacuum $e(0)$. In such a case taking the partial trace over the Fock space of the fields, which corresponds to take the trace over the environmental degrees of freedom in open quantum system theory, the reduced system state is given by $\rho(t) = \mathrm{Tr}_{\mathrm{env}}\left\{U(t)\,\rho(0)\otimes|e(0)\rangle\langle e(0)|U(t)^\dagger\right\}$, with $\rho(0)$ the initial state of the oscillator. Thanks to (21) the reduced dynamics of the mechanical oscillator can be shown to obey exactly the master equation (20) [30]. Further, we have the important relations

$$\begin{aligned}
&\langle e(0)|B_{\mathrm{th}}(t)B_{\mathrm{th}}^\dagger(s)e(0)\rangle = (N+1)\min\{t,s\},\\
&\langle e(0)|B_{\mathrm{th}}^\dagger(t)B_{\mathrm{th}}(s)e(0)\rangle = N\min\{t,s\},\\
&\langle e(0)|B_{\mathrm{th}}(t)B_{\mathrm{th}}(s)e(0)\rangle = 0.
\end{aligned} \tag{27}$$

It is worth noticing that the thermal parameter N does not appear in the commutation rules of the field B_{th}, but rather in the quantum correlations (27). This expresses the fact that N depends on the "state" of the field or, more precisely, N determines a non-Fock representation of the canonical commutation relations. Note furthermore that the vacuum $e(0)$ is not annihilated by the fields $B_{\mathrm{th}}(t)$, but it plays the role of a thermal state [48, Sect. 6]; no vacuum state exists for a non-Fock Bose field.

3.1. Quantum Langevin equations and input-output relations

Relying on the previously introduced formalism we are now in the position to obtain the so-called quantum Langevin equations, providing the stochastic evolution for the system observables in the Heisenberg picture. For a generic system operator X we denote as usual the Heisenberg picture as $X(t) = U(t)^\dagger X U(t)$, with $U(t)$ the unitary operator describing the closed dynamics of system and environment. Differentiating this expression by the rules of quantum stochastic calculus, essentially summarized by the Itô table (23), one obtains the quantum Langevin equations for the relevant system operators, namely for the mode operator

$$\mathrm{d}a_{\mathrm{m}}(t) = -\left(\mathrm{i}\omega_{\mathrm{m}} + \frac{\gamma_{\mathrm{m}}}{2}\right)a_{\mathrm{m}}(t)\mathrm{d}t - \sqrt{\gamma_{\mathrm{m}}}\,\mathrm{d}B_{\mathrm{th}}(t). \tag{28}$$

By (18) we get also the equivalent equations for position and momentum

$$\mathrm{d}q(t) = \frac{p(t)}{m}\,\mathrm{d}t + \mathrm{d}C_q(t), \tag{29}$$

$$\mathrm{d}p(t) = -\left(m\Omega_{\mathrm{m}}^2 q(t) + \gamma_{\mathrm{m}}p(t)\right)\mathrm{d}t + \mathrm{d}C_p(t), \tag{30}$$

in which we have introduced the Hermitian quantum noises

$$\begin{aligned}
C_q(t) &= -\sqrt{\frac{\hbar\gamma_{\mathrm{m}}}{2m\omega_{\mathrm{m}}}}\left(\overline{\tau}\,B_{\mathrm{th}}(t) + \tau B_{\mathrm{th}}^\dagger(t)\right),\\
C_p(t) &= \mathrm{i}\Omega_{\mathrm{m}}\sqrt{\frac{m\hbar\gamma_{\mathrm{m}}}{2\omega_{\mathrm{m}}}}\left(B_{\mathrm{th}}(t) - B_{\mathrm{th}}^\dagger(t)\right),
\end{aligned} \tag{31}$$

where τ is the pase factor defined in (16). By (22) the new noises obey the commutation rules

$$[C_q(t), C_p(s)] = \mathrm{i}\hbar\gamma_{\mathrm{m}}\min\{t,s\}, \qquad [C_q(t), C_q(s)] = [C_p(t), C_p(s)] = 0. \tag{32}$$

A fundamental advantage of the considered formalism is that, thanks to the unitarity of $U(t)$, the transformation $X \mapsto U(t)^\dagger X U(t)$ preserves all the commutation rules among system observables, in particular the Heisenberg relations between position and momentum, as can be checked also directly relying on (32). Warranting preservation of these fundamental commutation relations is indeed a basic step in providing a true quantum description of a dissipative dynamics [26, Chapts. 1, 3].

We now consider the Heisenberg picture for the thermal fields and we define

$$B_{\mathrm{th}}^{\mathrm{out}}(t) = U(t)^\dagger B_{\mathrm{th}}(t)U(t). \tag{33}$$

While $B_{\rm th}(t)$ represents the field before the interaction with the oscillator, the so-called *input field*, $B^{\rm out}_{\rm th}(t)$ is the field after the interaction, the so-called *output field*. We stress in particular that an important consequence of the Hudson-Parthasarathy equation is the identity $B^{\rm out}_{\rm th}(t) = U(T)^\dagger B_{\rm th}(t)U(T)$, $\forall T \geq t$, which warrants the fact that the output fields obey the same commutation relations as the input fields, namely (22); in other words, both the input and the output fields behave as free fields. By differentiating the three contributions in $U(t)^\dagger B_{\rm th}(t)U(t)$ according to the Itô rules, one gets the *input-output relation*

$$\mathrm{d}B^{\rm out}_{\rm th}(t) = \mathrm{d}B_{\rm th}(t) + \sqrt{\gamma_{\rm m}}\, a_{\rm m}(t)\, \mathrm{d}t. \tag{34}$$

The linearity of the Heisenberg equations of motion allows for an explicit solution

$$a_{\rm m}(t) = \mathrm{e}^{-\left(\mathrm{i}\omega_{\rm m}+\frac{\gamma_{\rm m}}{2}\right)t} a_{\rm m} - \sqrt{\gamma_{\rm m}} \int_0^t \mathrm{e}^{-\left(\mathrm{i}\omega_{\rm m}+\frac{\gamma_{\rm m}}{2}\right)(t-s)} \mathrm{d}B_{\rm th}(s), \tag{35}$$

$$B^{\rm out}_{\rm th}(t) = -\frac{\frac{\gamma_{\rm m}}{2} - \mathrm{i}\omega_{\rm m}}{\frac{\gamma_{\rm m}}{2} + \mathrm{i}\omega_{\rm m}}\, B_{\rm th}(t) + \frac{\gamma_{\rm m}}{\frac{\gamma_{\rm m}}{2} + \mathrm{i}\omega_{\rm m}} \int_0^t \mathrm{e}^{-\left(\mathrm{i}\omega_{\rm m}+\frac{\gamma_{\rm m}}{2}\right)(t-s)} \mathrm{d}B_{\rm th}(s) \\ + \frac{\sqrt{\gamma_{\rm m}}}{\frac{\gamma_{\rm m}}{2} + \mathrm{i}\omega_{\rm m}} \left(1 - \mathrm{e}^{-\left(\mathrm{i}\omega_{\rm m}+\frac{\gamma_{\rm m}}{2}\right)t}\right) a_{\rm m}. \tag{36}$$

The explicit expressions for $q(t)$ and $p(t)$ can be easily obtained from (18) and (35).

3.2. Field state and non-Markovian dynamics

In the Markovian approximation considered so far, the temperature enters the theory only through the parameter N defined in (15). This approximation can be described stating that the system actually sees a flat noise spectrum, or more precisely the system is only affected by the value of the bath spectrum at the frequency $\omega_{\rm m}$. A more general and physically more realistic situation is to allow for a structured noise spectrum and this can be achieved without any modification of the unitary dynamics (21) and of the related Langevin equations and input-output relations. To this aim it is enough to change the state of the field by taking mixtures of coherent states [30, 50]. Let us note that considering such a mixture of coherent states for the description of the state of the field is actually analogous to consider a state with a regular P-representation in the case of discrete modes (see e.g. [26]), as explained below. This modification is new in the context of quantum stochastic calculus and will imply that the reduced dynamics of the oscillator is no more Markovian, in the sense that a closed master equation in Lindblad form for the statistical operator cannot be obtained.

3.2.1. The field state. In order to consider a more general field state let us first introduce the Weyl operators [30, 32] for the Fock A-fields, defined as

$$\mathcal{W}_A(g) = \exp\left\{\sum_{k=1}^{2} \int_0^{+\infty} g_k(s)\, \mathrm{d}A_k^\dagger(s) - \text{h.c.}\right\},$$

with g_k square integrable functions. The operator $\mathcal{W}_A(g)$ is unitary and the property $A_k(t)\mathcal{W}_A(g)e(0) = \int_0^t \mathrm{d}s g_k(s)\mathcal{W}_A(g)e(0)$ holds, so that the action of a Weyl operator on the Fock vacuum gives a coherent state. Therefore $\mathcal{W}_A(g)$ is nothing but a displacement operator for the Bose fields [49]. Relying on (26), we can introduce a Weyl operator also for the B-field

$$\mathcal{W}_T(f) = \exp\left\{\int_0^T f(s)\, \mathrm{d}B^\dagger_{\rm th}(s) - \text{h.c.}\right\}, \tag{37}$$

where f is a locally square integrable function and T denotes a suitable large time, which we will let tend to infinity in the final formulae describing the quantities of direct physical interest. Now, $\mathcal{W}_T(f)e(0)$ is not a coherent state for the $B_{\rm th}$-field, but its relevant moments can be computed by using the A-field representation (26):

$$\begin{aligned}
\langle \mathcal{W}_T(f)e(0)|B_{\rm th}(t)\mathcal{W}_T(f)e(0)\rangle &= \int_0^t f(r)\,\mathrm{d}r,\\
\langle \mathcal{W}_T(f)e(0)|B_{\rm th}(t)B_{\rm th}(s)\mathcal{W}_T(f)e(0)\rangle &= \int_0^t f(u)\,\mathrm{d}u\int_0^s f(r)\,\mathrm{d}r,\\
\langle \mathcal{W}_T(f)e(0)|B^\dagger_{\rm th}(s)B_{\rm th}(t)\,\mathcal{W}_T(f)e(0)\rangle &= N\min\{t,s\}+\int_0^t f(u)\,\mathrm{d}u\int_0^s \overline{f(r)}\,\mathrm{d}r,\\
\langle \mathcal{W}_T(f)e(0)|B_{\rm th}(t)B^\dagger_{\rm th}(s)\,\mathcal{W}_T(f)e(0)\rangle &= (N+1)\min\{t,s\}+\int_0^t f(u)\,\mathrm{d}u\int_0^s \overline{f(r)}\,\mathrm{d}r.
\end{aligned} \tag{38}$$

A crucial step is now to consider f to be a random process and to take the state of the field characterizing the environment to be

$$\sigma_{\rm env} = \mathbb{E}\left[\mathcal{W}_T(f)|e(0)\rangle\langle e(0)|\mathcal{W}_T(f)^\dagger\right]. \tag{39}$$

Again, in the final formulae we will take the limit $T\to+\infty$. This is nothing but an analogue of the regular P-representation for the case of discrete modes. Indeed, in the case of a single mode the Glauber-Sudarshan P-representation of a state ρ [26, Section 4.4.3] is defined by $\rho=\int \mathrm{d}^2\alpha\,P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|$. If the pseudo-density P is allowed to become negative and singular, then any state can be represented in this form. When P is a true probability density, one speaks of a regular P-representation and the Glauber-Sudarshan formula describes mixtures of coherent states, including in particular thermal states [26, p. 113]. In a probabilistic language, which is more suitable for generalizations to stochastic processes and fields, the fact that a state ρ has a regular P-representation can be rephrased by saying that it can be written as the expectation value $\rho=\mathbb{E}[|\alpha\rangle\langle\alpha|]$, with α a complex random variable. In order to construct a thermal state it is enough to consider the case in which the distribution of α is Gaussian with vanishing mean $\mathbb{E}[\alpha]=0$ and second moments $\mathbb{E}[\alpha^2]=0$, $\mathbb{E}[|\alpha|^2]=\sigma^2$. Then, $\rho=\mathbb{E}[|\alpha\rangle\langle\alpha|]$ turns out to be a thermal state [26, Section 4.4.5].

By analogy, to construct a thermal field state with a general thermal spectrum we take f to be a Gaussian stationary stochastic process with vanishing mean, $\mathbb{E}[f(t)]=0$, and correlation functions

$$\mathbb{E}[f(t)\,f(s)]=0, \qquad \mathbb{E}[\overline{f(t)}\,f(s)]=:G(t-s). \tag{40}$$

Thanks to stationarity, the function $G(t)$ is positive definite, so that according to Bochner's theorem [51] its Fourier transform

$$\hat{G}(\nu)=\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\nu t}G(t)\,\mathrm{d}t \tag{41}$$

is a positive function, which we assume to be absolutely integrable, thus implying a finite power spectral density for the process. Since the field state $\sigma_{\rm env}$, defined by (39), turns out to be Gaussian, we can characterize it through the means and the correlations of the thermal field $B_{\rm th}$, which are immediately obtained from (38) and the properties of the process f. The only non zero contributions are given by

$$\begin{aligned}
\langle B^\dagger_{\rm th}(s)B_{\rm th}(t)\rangle_{\rm env} &= N\min\{t,s\}+\int_0^t \mathrm{d}u\int_0^s \mathrm{d}r\,G(r-u),\\
\langle B_{\rm th}(t)B^\dagger_{\rm th}(s)\rangle_{\rm env} &= (N+1)\min\{t,s\}+\int_0^t \mathrm{d}u\int_0^s \mathrm{d}r\,G(r-u).
\end{aligned} \tag{42}$$

To better grasp the physical content of the new state and of the formulae (42) let us introduce a set of field modes as in [49]. Using a complete orthonormal set $\{h_n\}$ in $L^2(\mathbb{R})$, we can expand the field in terms of discrete independent temporal modes by defining them as

$$c_{h_n} = \int_{-\infty}^{+\infty} \overline{h_n(t)}\, \mathrm{d}B_{\mathrm{th}}(t).$$

We then obtain $\langle c_{h_n} \rangle_{\text{env}} = 0$, $\langle c_{h_n}^2 \rangle_{\text{env}} = 0$, together with

$$\lim_{T\to+\infty} \langle c_{h_n}^\dagger c_{h_n} \rangle_{\text{env}} = N + \mathbb{E}\left[|\langle h_n | f \rangle_{L^2}|^2 \right] = \int_{-\infty}^{+\infty} \left| \hat{h}_n(\nu) \right|^2 N(\nu)\, \mathrm{d}\nu,$$

where we have defined the positive quantity

$$N(\nu) = N + \hat{G}(\nu) \tag{43}$$

and $\hat{h}_n(\nu)$ is the Fourier transform of $h_n(t)$; by normalization $\int_{-\infty}^{+\infty} |\hat{h}_n(\nu)|^2 \mathrm{d}\nu = 1$. So, the reduced state of the single mode c_{h_n} is exactly a thermal state expressed in the P-representation. If we take h_1 and h_2 having non overlapping Fourier transforms we also get $\lim_{T\to+\infty} \langle c_{h_1}^\dagger c_{h_2} \rangle_{\text{env}} = 0$, which means that these two modes are independent. Then, $N(\nu)$ *is naturally interpreted as the mean number of phonons in a given field mode* c_h *well peaked around the value* ν of the frequency and field modes with different frequencies are independent. A value of $N(\nu)$ different from zero in a neighbourhood of ν implies that the mechanical oscillator can absorb from the bath phonons with energy around $\hbar\nu$. On the contrary, the approximations are such that the oscillator can emit phonons of any frequency, even when $N(\nu) = 0$. The physically relevant quantity is now the combination of the two non negative contributions N and $\hat{G}(\nu)$, rather than the values of the individual quantities. Note that the Markovian reduced dynamics of Section 2 can be obtained either by considering the non-Fock representation for the thermal field, thus assuming a strictly positive $N > 0$ in (26) and taking $\hat{G}(\nu) \equiv 0$, or equivalently by considering a standard Fock representation and formally taking the limit of constant spectrum $\hat{G}(\nu)$ in all the physical quantities.

3.2.2. The equilibrium state of the mechanical oscillator. According to the definition of reduced dynamics, the time evolved state of the mechanical oscillator is still obtained by taking the partial trace with respect to the field degrees of freedom $\rho(t) = \lim_{T\to+\infty} \mathrm{Tr}_{\text{env}} \left\{ U(t) \left(\rho(0) \otimes \sigma_{\text{env}} \right) U(t)^\dagger \right\}$. However, at variance with the case in which the state of the field was taken to be the A-field vacuum, by taking the time derivative of this expression no closed evolution equation is obtained unless $N(\nu)$ is constant. Not to have a closed time-homogeneous equation for the reduced dynamics is indeed a signature of the non-Markov features of such a dynamics.

In spite of the difficulty of not having a closed master equation, the study of the reduced equilibrium state, namely $\rho_{\text{eq}} = \lim_{t\to+\infty} \rho(t)$, can still be afforded and its expression enlightens the physical role of the various parameters. Indeed, thanks to the requirement $\mathbb{E}[f(t)\, f(s)] = 0$, one has that equipartition in the sense of (10) still holds. Starting from the explicit forms of position and momentum in the Heisenberg picture (see (35), (18)) one can check that the equilibrium mean values of position and momentum remain equal to zero, while the variances are still of the form (19) with N replaced by the effective mean number of excitations

$$N_{\text{eff}} = \frac{\gamma_{\mathrm{m}}}{2\pi} \int_{-\infty}^{+\infty} \frac{N(\nu)}{\frac{\gamma_{\mathrm{m}}^2}{4} + (\nu - \omega_{\mathrm{m}})^2}\, \mathrm{d}\nu. \tag{44}$$

Notice that if the quantity $N(\nu)$ introduced in (43) is taken to be the constant N, corresponding to the Markovian case, then $N_{\text{eff}} = N$. This result suggests that the final Markov approximation should be valid when $\hat{G}(\nu)$ is approximately constant in a neighbourhood of $\omega_{\rm m}$. In fact equation (44) represents a smearing of $N(\nu)$ around the frequency of the mechanical oscillator $\omega_{\rm m}$, the more peaked the smaller the damping constant $\gamma_{\rm m}$. Non-Markovian effects can only be relevant if $\hat{G}(\nu)$ appreciably varies in a neighbourhood of width $\gamma_{\rm m}$ around $\omega_{\rm m}$, being suppressed with decreasing $\gamma_{\rm m}$.

Since the equilibrium state is necessarily Gaussian, by comparing (44) with (14) we get that the new equilibrium state is again a Gibbs state with respect to the same Hamiltonian $H_{\rm m}$, but with an effective inverse temperature $\beta_{\rm eff}$ defined by setting $N_{\rm eff} = ({\rm e}^{\beta_{\rm eff}\hbar\omega_{\rm m}} - 1)^{-1}$.

3.3. Properties of the quantum noises and quantum stochastic Newton equation

Let us now come back to the quantum Langevin equations for the position and momentum operators, so as to better understand their physical meaning and the role of the noises. In order to study the properties of the noises we transform the Langevin equations (29), (30) in the form of a stochastic Newton equation.

To this aim we first have to consider the quantum noises (31) appearing in these quantum Langevin equations. The commutation relations (32) for these noises, which are state independent, guarantee the preservation of the canonical Heisenberg commutation relations. Their quantum correlations do instead reflect the physical properties of the field state $\sigma_{\rm env}$ and can be obtained starting from the B-correlations (42). Note that Langevin equations for a mechanical oscillator of the same form and with two noises obeying the same commutations rules (32) were used also in [36]; however, the two point correlations used in [36], taken from [17], arise from approximations in the Caldeira-Legget model which are valid only for medium/high temperatures, while in the present treatment they are deduced from the state of the phonon environment and are valid at any temperature.

We stress the fact that in the present formulation the momentum operator is not related to the time derivative of the position operator according to the classical relation, but rather through (29) where the quantum noise $C_q(t)$ explicitly appears. However, the connection to the classical formulation is not completely lost. In fact from (29) we can derive the relation

$$\frac{q(t_2) - q(t_1)}{t_2 - t_1} - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \frac{p(t)}{m}\,{\rm d}t = \frac{C_q(t_2) - C_q(t_1)}{t_2 - t_1}.$$

By (31) and (42), the mean value of the r.h.s. of the equation above vanishes, while its variance is given by

$$\frac{\langle (C_q(t_2) - C_q(t_1))^2\rangle_{\rm env}}{(t_2 - t_1)^2} = \frac{\hbar\gamma_{\rm m}}{m\omega_{\rm m}\,(t_2 - t_1)}\left(\frac{1}{2} + \int_{-\infty}^{+\infty} \frac{2\left(\sin\frac{\nu(t_2 - t_1)}{2}\right)^2}{\pi\nu^2\,(t_2 - t_1)}\,N(\nu)\,{\rm d}\nu\right),$$

so that in particular also the variance goes to zero for growing $t_2 - t_1$. Then the quantity $v(t) = p(t)/m$ can actually be interpreted as the "coarse grained" velocity of the mechanical oscillator.

If we use the formal field densities $b_{\rm th}(t)$, $b^\dagger_{\rm th}(t)$, with commutation rules (24), take as starting point the quantum Langevin equations (29) and (30) and eliminate the momentum, we can rewrite the quantum Langevin equations in the Newton form:

$$m\ddot{q}(t) + m\gamma_{\rm m}\dot{q}(t) + m\Omega_{\rm m}^2 q(t) = \xi(t), \tag{45}$$

where we have introduced the formally Hermitian quantum noise $\xi(t)$

$$\xi(t) = \dot{C}_p(t) + m\gamma_{\rm m}\dot{C}_q(t) + m\ddot{C}_q(t). \tag{46}$$

Most importantly the commutation relations for this noise take the singular expression

$$[\xi(t),\xi(s)] = 2\mathrm{i}m\hbar\gamma_{\mathrm{m}}\,\frac{\partial}{\partial t}\,\delta(t-s). \tag{47}$$

While the expectation value of this noise with respect to the field state σ_{env} is zero, its symmetrized correlation function can be computed from the relations

$$\begin{aligned}\frac{m}{\hbar\gamma_{\mathrm{m}}}\frac{\partial^2}{\partial t\partial s}\langle C_q(t)C_q(s)\rangle_{\text{env}} &= \frac{1}{m\hbar\gamma_{\mathrm{m}}\Omega_{\mathrm{m}}^2}\frac{\partial^2}{\partial t\partial s}\langle C_p(t)C_p(s)\rangle_{\text{env}}\\ &= \frac{1}{\omega_{\mathrm{m}}}\left\{\left(N+\frac{1}{2}\right)\delta(t-s)+\operatorname{Re}G(t-s)\right\},\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{1}{\hbar\gamma_{\mathrm{m}}^2}\,\frac{\partial^2}{\partial t\partial s}\langle\{C_q(t),C_p(s)\}\rangle_{\text{env}} &= -\frac{1}{\omega_{\mathrm{m}}}\left\{\left(N+\frac{1}{2}\right)\delta(t-s)+\operatorname{Re}G(t-s)\right\}\\ &\quad+\frac{2}{\gamma_{\mathrm{m}}}\,\operatorname{Im}G(t-s)\end{aligned} \tag{49}$$

and has the expression

$$\begin{aligned}\frac{1}{2}\langle\{\xi(t),\xi(s)\}\rangle_{\text{env}} &= \frac{m\hbar\gamma_{\mathrm{m}}}{\omega_{\mathrm{m}}}\left(\Omega_{\mathrm{m}}^2+\frac{\partial^2}{\partial t\partial s}\right)\left[\left(N+\frac{1}{2}\right)\delta(t-s)+\operatorname{Re}G(t-s)\right]\\ &\quad+2m\hbar\gamma_{\mathrm{m}}\,\frac{\partial}{\partial t}\operatorname{Im}G(t-s).\end{aligned} \tag{50}$$

Note that (45) and (47) were already introduced in [26, Sect. 3.1.2] and [37], where the commutation rules (47) were actually enforced by the requirement of preservation of the commutation rules between position and momentum. However, at variance with previous approaches, here we have provided an explicit construction of the quantum noise $\xi(t)$ in terms of a quantum Bose field, based on a rigorous mathematical construction.

We stress the fact that the stochastic Newton equation (45) is mathematically purely formal due to the presence in (46) of $\ddot{C}_q(t)$, which contains the formal derivative $\dot{b}_{\mathrm{th}}(t)$ and its adjoint. Moreover, if one were to take (45), (47) and (50) as starting point for the construction of the quantum Langevin equations for position and momentum, then one should complete (45), which is an equation for $q(t)$ only, with a suitable definition of $p(t)$. The standard choice in this respect, considered for instance in [1, 25, 37], is to take $p(t) = m\dot{q}(t)$. This works out fine as far as the commutation relations of position and momentum are concerned. However, in this case the equation of motion (45) and the structure of the noise $\xi(t)$ obeying (46) imply that $\dot{q}(t)$ contains singular quantum fluctuations, so that it is not a well defined operator. Also p is then not a well defined operator and its variance is actually infinite. Then, one has to regularize the momentum, by subtracting the noise responsible of this divergence; this is what our construction does. The identification of the momentum is given implicitly through the first canonical equation (29), which corresponds to the coarse grained velocity, as discussed above. No divergency appears because the whole construction is based on the well defined unitary evolution (21).

3.3.1. Consistency of the quantum noises. It is important to stress that if a set of quantum Langevin equations is considered as starting point for the description of a stochastic quantum dynamics, commutations rules and symmetrized correlations of the noises cannot be given arbitrarily. In particular, independently of the considered system, if $\{\xi_i(t)\}$ is a set of operator valued noises, the quantum correlation function $\langle \xi_i(t)^\dagger \xi_j(t')\rangle_{\text{env}}$ has to be *positive definite* [51], in the sense that

$$\sum_{ij}\int_0^{+\infty}\mathrm{d}t\int_0^{+\infty}\mathrm{d}t'\,\overline{h_i(t)}\langle \xi_i(t)^\dagger \xi_j(t')\rangle_{\text{env}} h_j(t') \geq 0, \tag{51}$$

for every choice of the "smooth" test functions $\{h_i\}$. Since we can always write

$$\xi_i(t)^\dagger \xi_j(t') = \frac{1}{2}\left\{\xi_i(t)^\dagger,\, \xi_j(t')\right\} + \frac{1}{2}\left[\xi_i(t)^\dagger,\, \xi_j(t')\right], \tag{52}$$

the necessary positivity condition introduced above becomes a consistency condition between commutation rules and symmetrized correlations.

Relying on (48), (49), as well as the commutation relations (32) for the noises C_q and C_p, one can immediately check this fact for the model at hand. Also for the singular noise ξ constrained by (46) one can show that the expression $\langle \xi(t)\xi(s)\rangle_{\text{env}}$ is positive definite. These results are due to the fact that the noise fields have here been explicitly constructed in terms of the quantum Bose fields, so that commutation rules and correlations are not postulated, but rather follow from the mathematical expression of the model.

3.3.2. The noise correlations. For the model at hand we denote the Fourier transform of the correlation of the noise ξ by

$$\hat{R}(\nu) = \frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\nu t}\langle\{\xi(t+s),\xi(s)\}\rangle_{\text{env}}, \tag{53}$$

so that according to (43) and (50) it reads

$$\hat{R}(\nu) = \frac{m\hbar\gamma_{\text{m}}}{2\omega_{\text{m}}}\left(\frac{\gamma_{\text{m}}^2}{4} + (\omega_{\text{m}}+\nu)^2\right)\left(N(\nu)+\frac{1}{2}\right) + (\nu\to-\nu), \tag{54}$$

where $(\nu \to -\nu)$ means to add the same contribution with ν replaced by $-\nu$. Note in particular that $\hat{R}(\nu)$ is an even function of the frequency.

In our treatment, which gives rise to the expression (54), the interaction with the environment is described in terms of exchange of quanta with the bosonic field representing the phonons, see (25). In models in which the system of interest is coupled to other harmonic oscillators, by some approximations it is possible to arrive to a quantum stochastic Newton equation like (45), but with a different noise spectrum. A reference expression often considered in the literature [26, (3.3.9)], [37] for the quantity $\hat{R}$ is given by

$$\hat{R}_{GZ}(\nu) = \pi\hbar J(\nu)\coth\frac{\beta\hbar\nu}{2}, \qquad J(\nu) := \frac{m}{\pi}\,k(\nu)\nu. \tag{55}$$

The quantity $k(\nu)$ contains information on both coupling constant and density of modes of the bath in a neighbourhood of the frequency ν; $J(\nu)$ is often called *spectral density*. Also in the case of this choice of the noise correlations, it is possible to show that the equilibrium mean of $\dot{q}(t)^2$ diverges and therefore the identification of the momentum with $m\dot{q}(t)$ is not possible, but some regularization is needed. A typical choice in this context is $k(\nu) = \gamma_{\text{m}}$ [26, (3.1.1)], [1, 2, 25, 37]; this is equivalent to $J(\nu) \propto \nu$, which is known as *Ohmic spectral density*. The function $k(\nu)$ must be even due to the definition (53) and stationarity. Also the commutations rules (47) and the positivity requirement (51) still have to hold, leading to the requirement $\hat{R}_{GZ}(\nu) - m\hbar\gamma_{\text{m}}\,|\nu| \geq 0$, satisfied at any temperature by taking $k(\nu) \geq \gamma_{\text{m}} > 0$ for all ν. This requirement tells us that in order to have a consistent model satisfying (45), at least at large times, and preserving Heisenberg commutation relations one cannot consider a spectral density with gaps inside the expression (55) of $\hat{R}_{GZ}$.

By a suitable choice of $N(\nu)$, it is possible to get $\hat{R}(\nu)$ very similar to $\hat{R}_{GZ}(\nu)$, apart from the low temperature limit. For instance, by taking

$$N(\nu) = |\nu|\, k(\nu)\, \frac{2\omega_{\rm m}}{(\Omega_{\rm m}^2+\nu^2)\,\gamma_{\rm m}\left({\rm e}^{\beta\hbar|\nu|}-1\right)}, \tag{56}$$

we obtain

$$\hat{R}(\nu) = \hat{R}_{GZ}(\nu) + m\hbar\left[\frac{\gamma_{\rm m}^2}{2\omega_{\rm m}}\left(\Omega_{\rm m}^2+\nu^2\right) - k(\nu)\,|\nu|\right]; \tag{57}$$

note that the difference is independent from the temperature. In (57) the compatibility with the commutation relations is guaranteed by the first term in the square brackets; so, in (56) there is no restriction on the choice of $k(\nu)$, apart from $k(\nu)\geq 0$, and even spectral gaps can be introduced.

Besides the case (56), the freedom in the choice of $N(\nu)$ in (54) allows to model quite different environments, for instance with a sub-Ohmic or super-Ohmic spectral density [22–24], or with a structured occupation spectrum. In optomechanical systems the quantity $\hat{R}(\nu)$ enters experimental quantities, as in the cases of homodyne/heterodyne detection discussed in Section 4.3; so, in principle it is possible to test the form of $\hat{R}(\nu)$. However, essentially a frequency window of width $\gamma_{\rm m}$ around $\omega_{\rm m}$ is experimentally relevant. Some results at room temperature [52] seem to indicate a non-Ohmic spectral density around $\omega_{\rm m}$, a very interesting possibility, but not enough to discriminate between $\hat{R}_{GZ}(\nu)$ and the form (57) for $\hat{R}(\nu)$. At zero temperature, corresponding to $N(\nu)=0$ in our case (54) and to $\beta\to+\infty$ in (55), the difference is most evident, namely

$$\hat{R}(\nu) = m\hbar\gamma_{\rm m}\frac{\Omega_{\rm m}^2+\nu^2}{2\omega_{\rm m}} \tag{58}$$

versus

$$\hat{R}_{GZ}(\nu) = m\hbar k(\nu)\,|\nu|\,, \qquad k(\nu)\geq\gamma_{\rm m}. \tag{59}$$

These expressions are very different, but to discriminate between them one has to consider very low temperatures and a ratio $\gamma_{\rm m}/\omega_{\rm m}$ that is not too small.

4. Cooling and emission spectra of an optomechanical system

As an application of the quantum description of a mechanical oscillator developed so far we consider the simplest optomechanical system [1, 2, 5, 6, 25, 37], namely the mechanical oscillator is a mirror mounted on a cantilever and coupled to the light in an optical cavity by radiation pressure. The cavity is of high quality, without thermal dissipation other than the one due to the coupling between cantilever and phonons and tuned in such a way that only one electromagnetic mode is relevant. Strong laser light is injected and some light is allowed to leave the cavity so that its spectrum can be analysed.

4.1. The optomechanical model

The micro-mechanical oscillator (the mirror) is described by the operators q, p as in (18) and by the Hamiltonian $H_{\rm m}$ (6). The cavity mode is described by the operators $a_{\rm c}$, $a_{\rm c}^\dagger$ and by the free Hamiltonian $\hbar\omega_{\rm c}a_{\rm c}^\dagger a_{\rm c}$. The free electromagnetic field is in a coherent state describing a perfectly monochromatic laser of frequency ω_0; however we use the equivalent description of inserting a source term for the cavity mode in the Hamiltonian and of taking the external field in the vacuum. The final optomechanical Hamiltonian takes the form

$$H_{\rm om}(t) = H_{\rm m} + \hbar\omega_{\rm c}a_{\rm c}^\dagger a_{\rm c} - \hbar g_0 q a_{\rm c}^\dagger a_{\rm c} + {\rm i}\hbar E\left(a_{\rm c}^\dagger {\rm e}^{-{\rm i}\omega_0 t} - a_{\rm c}{\rm e}^{{\rm i}\omega_0 t}\right). \tag{60}$$

Note the trilinear term giving the interaction between the position of the mirror and the number operator of the photons in the cavity, which represents the radiation pressure interaction; the coupling constant is usually expressed as $g_0 = \omega_{\rm c}/L$, where L is the length of the cavity. The laser power is $P = \hbar\omega_0 E^2/\gamma_{\rm c}$, where $\gamma_{\rm c}$ is the cavity decay rate and E the laser amplitude.

In order to include the cavity mode interacting through radiation pressure with the mechanical oscillator, as well as the emission and absorption of the light from the free electromagnetic field, the Hudson-Parthasarathy equation (21) is modified as follows:

$$\mathrm{d}U(t) = \Big\{-\frac{\mathrm{i}}{\hbar}\, H_{\rm om}(t)\mathrm{d}t + \Big(\sqrt{\gamma_{\rm m}}\, a_{\rm m}\, \mathrm{d}B^\dagger_{\rm th}(t) + \sqrt{\gamma_{\rm c}}\, a_{\rm c}\, \mathrm{d}B^\dagger_{\rm em}(t) - \text{h.c.}\Big) \\ - \frac{\gamma_{\rm m}}{2}\left((2N+1)\, a^\dagger_{\rm m} a_{\rm m} + N\right)\mathrm{d}t - \frac{\gamma_{\rm c}}{2}\, a^\dagger_{\rm c} a_{\rm c} \mathrm{d}t\Big\} U(t). \tag{61}$$

Here $B_{\rm th}$ is the thermal field satisfying (22), while $B_{\rm em}$ is an independent Bose field in the Fock representation, describing the electromagnetic field outside the cavity. The relevant Itô rule is $\mathrm{d}B_{\rm em}(t)\mathrm{d}B^\dagger_{\rm em}(t) = \mathrm{d}t$, while all the other possible products vanish. Now $U(t)$ is the unitary operator describing the dynamics of the two interacting oscillators and the fields. The latter are in a factorized state given by the tensor product of the thermal environment state (39) and the electromagnetic vacuum:

$$\tilde{\sigma}_{\rm env} = \sigma_{\rm env} \otimes |e_{\rm em}(0)\rangle\langle e_{\rm em}(0)|. \tag{62}$$

It is convenient to eliminate the laser frequency working in the rotating frame and introducing the unitary operator $V(t) = \mathrm{e}^{\mathrm{i}\omega_0 a^\dagger_{\rm c} a_{\rm c} t} U(t)$, which upon differentiation obeys an equation of the form (61) albeit with $H_{\rm om}(t)$ substituted by

$$H_{\rm m} + \hbar\Delta_0 a^\dagger_{\rm c} a_{\rm c} - \hbar g_0 q a^\dagger_{\rm c} a_{\rm c} + \mathrm{i}\hbar E\left(a^\dagger_{\rm c} - a_{\rm c}\right), \tag{63}$$

with $\Delta_0 = \omega_{\rm c} - \omega_0$ the nominal detuning. For a generic system operator X we define $X(t) = V(t)^\dagger X V(t)$, so that by differentiating according to the rules of quantum stochastic calculus, as done in Section 3.1, we get the following quantum Langevin equations

$$\mathrm{d}a_{\rm c}(t) = \left(-\left(\mathrm{i}\Delta_0 + \frac{\gamma_{\rm c}}{2}\right) a_{\rm c}(t) + \mathrm{i}g_0 q(t) a_{\rm c}(t) - \mathrm{i}E\right)\mathrm{d}t - \sqrt{\gamma_{\rm c}}\, \mathrm{e}^{\mathrm{i}\omega_0 t}\mathrm{d}B_{\rm em}(t), \tag{64}$$

as well as

$$\begin{aligned} \mathrm{d}q(t) &= \frac{p(t)}{m}\,\mathrm{d}t + \mathrm{d}C_q(t), \\ \mathrm{d}p(t) &= \left(-m\Omega^2_{\rm m} q(t) - \gamma_{\rm m} p(t) + \hbar g_0 a^\dagger_{\rm c}(t) a_{\rm c}(t)\right)\mathrm{d}t + \mathrm{d}C_p(t), \end{aligned} \tag{65}$$

where C_q and C_p are given by (31). Defining the output fields as in (33) of Section 3.1 we have besides (34) the input-output relation for the electromagnetic field

$$\mathrm{d}B^{\rm out}_{\rm em}(t) = \mathrm{d}B_{\rm em}(t) + \sqrt{\gamma_{\rm c}}\, \mathrm{e}^{-\mathrm{i}\omega_0 t} a_{\rm c}(t)\mathrm{d}t. \tag{66}$$

In the case of a very intense laser, that is E^2 large, the dynamics can be linearized in a neighbourhood of the equilibrium mean values, determined by self-consistency from the means of the linearized form of the quantum Langevin equations. The equilibrium mean value of the momentum is zero, while setting $\zeta = \langle a_{\rm c}(t)\rangle_{\rm eq}$, we find

$$\zeta = -\frac{\mathrm{i}E}{\frac{\gamma_{\rm c}}{2} + \mathrm{i}\Delta}, \qquad \langle q\rangle_{\rm eq} = \frac{\hbar g_0 \left|\zeta\right|^2}{m\Omega^2_{\rm m}}, \tag{67}$$

where we have introduced the effective detuning Δ,

$$\Delta = \Delta_0 - g_0\langle q\rangle_{\rm eq} = \omega_{\rm c} - g_0\langle q\rangle_{\rm eq} - \omega_0. \tag{68}$$

By inserting the equations (67) into (68) we obtain the self-consistency condition

$$m\Omega_{\rm m}^2\left(\Delta-\Delta_0\right)\left(\frac{\gamma_{\rm c}^2}{4}+\Delta^2\right)+\hbar g_0^2E^2=0; \tag{69}$$

this cubic equation determines Δ as a function of the laser parameters Δ_0 and E.

In writing and solving the linearized quantum Langevin equations it is useful to have dimensionless and selfadjoint system operators. It is therefore convenient to set

$$\hat q(t)=\sqrt{\frac{m\Omega_{\rm m}}{\hbar}}\big(q(t)-\langle q\rangle_{\rm eq}\big),\qquad \hat p(t)=\frac{p(t)}{\sqrt{m\hbar\Omega_{\rm m}}}, \tag{70}$$

$$X(t)=\frac{\zeta a_{\rm c}^\dagger(t)+\overline{\zeta}\,a_{\rm c}(t)}{\sqrt2\,|\zeta|}-\sqrt2\,|\zeta|\,,\qquad Y(t)=\frac{{\rm i}\left(\zeta a_{\rm c}^\dagger(t)-\overline{\zeta}\,a_{\rm c}(t)\right)}{\sqrt2\,|\zeta|}. \tag{71}$$

Then, the linearized quantum Langevin equations turn out to be

$${\rm d}\vec w(t)=A\vec w(t){\rm d}t-{\rm d}\vec Q(t),\qquad \vec w(t)=\left(\hat q(t),\,\hat p(t),\,X(t),\,Y(t)\right)^{\rm T}, \tag{72}$$

where the superscript T means transposition and the dynamical matrix is given by

$$A=\begin{pmatrix}A_{\rm m}&A_{\rm mc}\\ A_{\rm mc}&A_{\rm c}\end{pmatrix},\qquad A_{\rm mc}=\begin{pmatrix}0&0\\ G\sqrt{\omega_{\rm m}/\Omega_{\rm m}}&0\end{pmatrix}, \tag{73}$$

$$A_{\rm m}=\begin{pmatrix}0&\Omega_{\rm m}\\ -\Omega_{\rm m}&-\gamma_{\rm m}\end{pmatrix},\qquad A_{\rm c}=\begin{pmatrix}-\gamma_{\rm c}/2&\Delta\\ -\Delta&-\gamma_{\rm c}/2\end{pmatrix}. \tag{74}$$

The quantity G, having the dimension of a frequency, will play the role of effective coupling constant and is given by

$$G=g_0\,|\zeta|\,\sqrt{\frac{2\hbar}{m\omega_{\rm m}}}\,, \tag{75}$$

so that in particular it depends on the effective detuning Δ through ζ given in (67). The vector of noises is given by the following field combinations:

$$Q_1(t)=\tau\sqrt{\frac{\gamma_{\rm m}\Omega_{\rm m}}{2\omega_{\rm m}}}\,B_{\rm th}^\dagger(t)+\text{h.c.},\qquad Q_2(t)={\rm i}\sqrt{\frac{\gamma_{\rm m}\Omega_{\rm m}}{2\omega_{\rm m}}}\,B_{\rm th}^\dagger(t)+\text{h.c.}, \tag{76}$$

$$\begin{aligned}Q_3(t)&={\rm e}^{{\rm i}\arg\zeta}\sqrt{\frac{\gamma_{\rm c}}{2}}\int_0^t{\rm e}^{-{\rm i}\omega_0 s}{\rm d}B_{\rm em}^\dagger(s)+\text{h.c.},\\ Q_4(t)&={\rm i}{\rm e}^{{\rm i}\arg\zeta}\sqrt{\frac{\gamma_{\rm c}}{2}}\int_0^t{\rm e}^{-{\rm i}\omega_0 s}{\rm d}B_{\rm em}^\dagger(s)+\text{h.c.},\end{aligned} \tag{77}$$

where τ is the phase factor defined in (16) and the quadratures $Q_1(t)$ and $Q_2(t)$, apart from a multiplicative factor due to the change of dimensions, coincide with the noises (31).

Note the different structure of the two dynamical sub-matrices in (74). Indeed the former describes a mechanical oscillator and the latter an optical mode, corresponding to different interactions as discussed in Section 2. The same choice is taken, for instance, in [1, 2, 7, 25, 36, 37], but not in [5, 9, 10].

The linearization around the equilibrium state is meaningful provided one can ensure the existence of such a state. Its stability conditions can be obtained by applying the Routh-Hurwitz criterion to the equations for the mean values, which correspond to the system (72) with the noise term ${\rm d}\vec Q(t)$ suppressed; the result is the couple of conditions

$$G^2\omega_{\rm m}\Delta<\Omega_{\rm m}^2\left(\frac{\gamma_{\rm c}^2}{4}+\Delta^2\right), \tag{78}$$

for $\Delta > 0$, and

$$G^2\omega_{\rm m}\,|\Delta| < \frac{\gamma_{\rm c}\gamma_{\rm m}}{\gamma_{\rm c}+\gamma_{\rm m}}\left[\gamma_{\rm c}\Omega_{\rm m}^2+\gamma_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4}+\Delta^2\right)+\frac{\left(\Omega_{\rm m}^2-\frac{\gamma_{\rm c}^2}{4}-\Delta^2\right)^2}{\gamma_{\rm m}+\gamma_{\rm c}}\right], \tag{79}$$

for $\Delta < 0$; there is no restriction for $\Delta = 0$. The same stability conditions have been found in [25], as their equations for the mean values agree with ours.

4.2. *Energy fluctuations and laser cooling*

To introduce the fluctuation spectra of position and momentum of the mechanical oscillator we use a formulation tailored for (classical or quantum) processes starting at time zero and we define the *gated Fourier transforms* [9]

$$\hat{B}_i^T(\nu) = \frac{1}{\sqrt{T}}\int_0^T {\rm e}^{{\rm i}\nu t}{\rm d}B_i(t), \qquad i = {\rm th},\ {\rm em}, \tag{80}$$

for the Bose fields as well as for the relevant system operators

$$F_i(T;\nu) = \frac{1}{\sqrt{T}}\int_0^T {\rm e}^{{\rm i}\nu t}w_i(t){\rm d}t, \quad i = 1,2,3,4. \tag{81}$$

Here T is a large time going to infinity in the final formulae to recover a stationary situation. Then, the spectra of the fluctuations of position and momentum of the mechanical oscillator are defined, in analogy with the classical case [54], by the quantum expectations

$$\begin{aligned} S_q(\nu) &= \lim_{T\to+\infty}\frac{1}{2}\langle\{F_1(T;\nu),\ F_1(T;-\nu)\}\rangle,\\ S_p(\nu) &= \lim_{T\to+\infty}\frac{1}{2}\langle\{F_2(T;\nu),\ F_2(T;-\nu)\}\rangle, \end{aligned} \tag{82}$$

$$S_{qp}(\nu) = \lim_{T\to+\infty}\frac{1}{4}\langle\{F_1(T;\nu),\ F_2(T;-\nu)\}+\{F_1(T;-\nu),\ F_2(T;\nu)\}\rangle. \tag{83}$$

Let us stress that, while useful, these definitions do not correspond to some continuous monitoring of position and momentum, even though $S_q(\nu)$ is directly related to the observed optical spectra as we shall see in Section 4.3.

The Fourier transformed equations of motion corresponding to (72) can be solved by purely algebraic manipulations and the vector $\vec{F}(T;\nu)$ can be computed; due to the length of the expressions the result is reported in Appendix A. To compute the spectra above we need also the field correlations, which we give in (A.9).

Due to the vanishing of the field cross-correlations, the spectra (82), (83) decompose in a thermal and a radiation pressure contribution according to

$$\begin{aligned} S_q(\nu) &= S_q^{\rm rp}(\nu) + S_q^{\rm th}(\nu), \qquad S_{qp}(\nu) = S_{qp}^{\rm th}(\nu),\\ S_p(\nu) &= \frac{\nu^2}{\Omega_{\rm m}^2}\,S_q^{\rm rp}(\nu) + S_p^{\rm th}(\nu). \end{aligned} \tag{84}$$

By inserting the expressions (A.1), (A.2) into the definitions (82), (83) and by using (A.9) we get, by some computations, the expressions for the spectra of the fluctuations:

$$S_q^{\rm rp}(\nu) = \frac{\Omega_{\rm m}\omega_{\rm m}G^2\gamma_{\rm c}}{2\,|d(\nu)|^2}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}+\nu^2\right), \tag{85}$$

$$S_q^{\rm th}(\nu) = \frac{\Omega_{\rm m}\hat{R}(\nu)}{\hbar m\,|d(\nu)|^2}\left(\frac{\gamma_{\rm c}^2}{4}+(\nu-\Delta)^2\right)\left(\frac{\gamma_{\rm c}^2}{4}+(\nu+\Delta)^2\right), \tag{86}$$

$$S_p^{\rm th}(\nu) = \frac{\gamma_{\rm m}}{2\omega_{\rm m}\Omega_{\rm m}\,|d(\nu)|^2}\left\{\left|\left(\Omega_{\rm m}^2+\nu\left(\omega_{\rm m}-{\rm i}\,\frac{\gamma_{\rm m}}{2}\right)\right)\left(\Delta^2+\left(\frac{\gamma_{\rm c}}{2}-{\rm i}\nu\right)^2\right)\right.\right.$$
$$\left.\left. - G^2\omega_{\rm m}\Delta\right|^2\left(N(\nu)+\frac{1}{2}\right)+(\nu\to-\nu)\right\}, \tag{87}$$

$$S_{qp}^{\rm th}(\nu) = -\frac{\gamma_{\rm m}}{2\Omega_{\rm m}}\,S_q^{\rm th}(\nu)+\frac{\gamma_{\rm m}G^2\Delta}{2\,|d(\nu)|^2}\left\{\left(N(\nu)+\frac{1}{2}\right)\right.$$
$$\left.\times\left[\frac{\gamma_{\rm m}}{2}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}-\nu^2\right)-\nu\gamma_{\rm c}(\omega_{\rm m}+\nu)\right]+(\nu\to-\nu)\right\}, \tag{88}$$

where $(\nu\to-\nu)$ means to add the same contribution with ν replaced by $-\nu$ and the quantity $\hat{R}(\nu)$ is the Fourier transform of the quantum correlations of the noise given in (54). The denominator $d(\nu)$ is the characteristic polynomial of the dynamical matrix A given in (73) and (74):

$$d(\nu) = \det\left(A+{\rm i}\nu\mathbb{1}\right) = \left(\left(\nu+{\rm i}\,\frac{\gamma_{\rm c}}{2}\right)^2-\Delta^{\,2}\right)\left(\left(\nu+{\rm i}\,\frac{\gamma_{\rm m}}{2}\right)^2-\omega_{\rm m}^2\right)-G^2\omega_{\rm m}\Delta. \tag{89}$$

Note that the quantities (85)–(90) are non-negative as they should be in order to give a sensible decomposition of the spectra. Another useful way to write $S_p^{\rm th}(\nu)$ is by putting in evidence its difference from $S_q^{\rm th}(\nu)$; the resulting expression is

$$S_p^{\rm th}(\nu)-S_q^{\rm th}(\nu) = \frac{\omega_{\rm m}\gamma_{\rm m}G^2\Delta}{\Omega_{\rm m}\,|d(\nu)|^2}\left\{\left(N(\nu)+\frac{1}{2}\right)\left[\frac{1}{2}\,G^2\Delta+\nu^2\,\frac{\gamma_{\rm m}\gamma_{\rm c}}{2\omega_{\rm m}}\right.\right.$$
$$\left.\left.+\left(\frac{\Omega_{\rm m}^2}{\omega_{\rm m}}+\nu\right)\left(\nu^2-\Delta^2-\frac{\gamma_{\rm c}^2}{4}\right)\right]+(\nu\to-\nu)\right\}. \tag{90}$$

4.2.1. The peaks in the fluctuation spectra. A relevant role in determining the properties of the system is played by the denominator $d(\nu)$ (89); indeed, the quantity $\Omega_{\rm m}[\Delta^2-(\nu+{\rm i}\gamma_{\rm c}/2)^2]/d(\nu)$ is sometimes interpreted as the effective mechanical susceptibility [25, Eq. (17)]. Most importantly note that the zeros of $d(\nu)$ determine the positions and the widths of the peaks of the fluctuation spectra: even though in principle they can be obtained by solving the fourth order algebraic equation $d(\nu)=0$, it is much more convenient to have simple expressions, even if approximate. An analysis of these zeros is given in Appendix A.1 in the case in which $d(\nu)$ can be written in the form

$$d(\nu) = \left(\left(\nu+{\rm i}\,\frac{\Gamma_{\rm c}}{2}\right)^2-\Delta_{\rm eff}^2\right)\left(\left(\nu+{\rm i}\,\frac{\Gamma_{\rm m}}{2}\right)^2-{\omega_{\rm eff}^{\rm m}}^2\right). \tag{91}$$

The stability conditions (78), (79) guarantee the strict positivity of the *effective damping constants* $\Gamma_{\rm c}$ and $\Gamma_{\rm m}$. The quantity $\omega_{\rm eff}^{\rm m}$ is known as *optical spring rigidity*, while the ratio $(\Gamma_{\rm m}-\gamma_{\rm m})/\gamma_{\rm m}$ is called *co-operativity* [5, 9].

An exact expression for the zeros is found when $\Delta=\omega_{\rm m}$, which allows us to put into evidence a crossing of the frequencies of the hybridized optical and mechanical modes [9]. If also the condition

$$4G^2 < (\gamma_{\rm c}-\gamma_{\rm m})^2 \tag{92}$$

holds, the result is

$$\Gamma_{\rm c} = \frac{\gamma_{\rm c}+\gamma_{\rm m}}{2} + \epsilon\sqrt{2u^2 - 2\omega_{\rm m}^2 + \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{8}}, \tag{93}$$

$$\Gamma_{\rm m} = \frac{\gamma_{\rm c}+\gamma_{\rm m}}{2} - \epsilon\sqrt{2u^2 - 2\omega_{\rm m}^2 + \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{8}}, \tag{94}$$

$$\Delta_{\rm eff}^2 = {\omega_{\rm eff}^{\rm m}}^2 = \frac{\omega_{\rm m}^2 + u^2}{2} - \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{32}, \tag{95}$$

where

$$u^2 = \sqrt{\left(\omega_{\rm m}^2 + \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{16}\right)^2 - G^2\omega_{\rm m}^2}, \qquad \epsilon = \begin{cases} +1 & \text{if } \gamma_{\rm c} > \gamma_{\rm m} \\ -1 & \text{if } \gamma_{\rm c} < \gamma_{\rm m} \end{cases}. \tag{96}$$

Always for $\Delta = \omega_{\rm m}$, under the conditions

$$\frac{\omega_{\rm m}^2(\gamma_{\rm c}-\gamma_{\rm m})^2}{4} < G^2\omega_{\rm m}^2 < \left(\omega_{\rm m}^2 + \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{16}\right)^2, \qquad \omega_{\rm m}^2 > \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{16}, \tag{97}$$

we get instead the result

$$\Gamma_{\rm c} = \Gamma_{\rm m} = \frac{\gamma_{\rm c}+\gamma_{\rm m}}{2}, \qquad \Delta_{\rm eff} = \sqrt{x_\pm}, \qquad \omega_{\rm eff}^{\rm m} = \sqrt{x_\mp}, \tag{98}$$

$$x_\pm = \omega_{\rm m}^2 - \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{16} \pm \omega_{\rm m}\sqrt{G^2 - \frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{4}}. \tag{99}$$

The two alternatives in (98) are completely equivalent; there is no reason to associate the frequency $\sqrt{x_+}$ to the cavity and $\sqrt{x_-}$ to the mechanical oscillator, or viceversa. A striking feature of the case $\Delta = \omega_{\rm m}$ is the change in the behaviour of the zeros at the critical point $\gamma_{\rm c} = \bar{\gamma}_{\rm c}$, solution of $G^2 = (\gamma_{\rm c}-\gamma_{\rm m})^2/4$; recall that G^2 depends on $\gamma_{\rm c}$ due to (67) and (75).

In the general case, an approximate expression can be obtained under the conditions

$$\frac{\gamma_{\rm m}}{\gamma_{\rm c}} \ll 1, \qquad |\chi(\Delta)| \ll 1, \qquad |\chi(\Delta)|\left|1 - \frac{\Delta^2}{\omega_{\rm m}^2} - \frac{\gamma_{\rm c}^2}{4\omega_{\rm m}^2}\right| \ll 1, \tag{100}$$

where

$$\chi(\Delta) = \frac{G^2\omega_{\rm m}\Delta}{\left(\frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{4} + (\Delta-\omega_{\rm m})^2\right)\left(\frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{4} + (\Delta+\omega_{\rm m})^2\right)}. \tag{101}$$

The result for the damping constants is

$$\Gamma_{\rm m} \simeq \gamma_{\rm m} + \chi(\Delta)\,(\gamma_{\rm c}-\gamma_{\rm m}), \qquad \Gamma_{\rm c} \simeq \gamma_{\rm c} - \chi(\Delta)\,(\gamma_{\rm c}-\gamma_{\rm m}). \tag{102}$$

The expressions for ${\omega_{\rm eff}^{\rm m}}^2$ and $\Delta_{\rm eff}^2$ are then obtained by inserting $\Gamma_{\rm m}$ and $\Gamma_{\rm c}$ in the equations (A.12). The compatibility conditions (A.13) have to hold. As one can see, when $\Delta > 0$ (*red detuning*), we have an increasing of the mechanical damping constant, $\Gamma_{\rm m} > \gamma_{\rm m}$, and a decreasing of the spring rigidity, $\omega_{\rm eff}^{\rm m} < \omega_{\rm m}$.

4.2.2. The mean values at equilibrium. By integrating in their frequency dependence the fluctuation spectra one obtains the second moments of position and momentum in the equilibrium state:

$$\langle q^2\rangle_{\rm eq} - \langle q\rangle_{\rm eq}^2 = \frac{\hbar}{2\pi m\Omega_{\rm m}}\int_{\mathbb{R}} S_q(\nu){\rm d}\nu, \qquad \langle p^2\rangle_{\rm eq} = \frac{m\hbar\Omega_{\rm m}}{2\pi}\int_{\mathbb{R}} S_p(\nu){\rm d}\nu, \tag{103}$$

$$\frac{1}{2}\langle\{q,p\}\rangle_{\rm eq} = \frac{\hbar}{2\pi}\int_{\mathbb{R}} S_{qp}(\nu){\rm d}\nu. \tag{104}$$

All these quantities are finite since the integrands behave as ν^{-2} for $|\nu| \to +\infty$. Moreover, the reduced equilibrium state of the mechanical oscillator is a Gaussian state characterized by (103) and (104) together with $\langle q\rangle_{\rm eq} = \hbar g_0 \left|\zeta\right|^2/(m\Omega_{\rm m}^2),\ \langle p\rangle_{\rm eq} = 0$.

On the contrary the integral of $\nu^2 S_q(\nu)$, which would give the fluctuations at equilibrium of $\sqrt{m\Omega_{\rm m}/\hbar}\,\dot q$, does not exist. This fact is related to the features of the noise in the thermal part and, as already noticed right before Section 3.3.1, this noticeably implies that the standard identification of $m\dot q$ with momentum is not possible. The expression of $S_q(\nu)$ coincides with the one given in [1, 25], where however $\hat R_{GZ}(\nu)$ with Ohmic spectral density appears instead of $\hat R(\nu)$. While in the case of [1, 25] $S_q(\nu) \asymp \nu^{-3}$, still $\dot q^2$ does not have a finite mean and also in this case the identification of momentum and velocity is not possible. Notice that the expressions for $S_p(\nu)$ and $S_{qp}(\nu)$ have not been obtained before. In particular $S_{qp}(\nu) \neq 0$ implies that the fluctuations of position and momentum are actually correlated.

The mean energy of the harmonic oscillator at equilibrium takes the form

$$\langle H_{\rm m}\rangle_{\rm eq} = \frac{1}{2}\, m\Omega_{\rm m}^2\langle q\rangle_{\rm eq}^2 + \langle H\rangle_{\rm fl}, \tag{105}$$

where the contribution due to fluctuations is given by

$$\langle H\rangle_{\rm fl} = \frac{\hbar}{4\pi}\int_{\mathbb{R}} {\rm d}\nu \left[\Omega_{\rm m}\big(S_q(\nu)+S_p(\nu)\big) + \gamma_{\rm m}S_{qp}(\nu)\right]. \tag{106}$$

It is convenient and natural to split this contribution into three distinct terms, distinguishing a radiation pressure term from the rest and further dividing the thermal contributions into two, putting into evidence a contribution which is not proportional to position fluctuations and does not have a definite sign. We thus introduce the dimensionless quantities

$$\mathcal{N}_{\rm rp} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\Omega_{\rm m}^2+\nu^2}{2\omega_{\rm m}\Omega_{\rm m}}\, S_q^{\rm rp}(\nu)\,{\rm d}\nu, \qquad \mathcal{N}_{\rm th} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\omega_{\rm m}}{\Omega_{\rm m}}\, S_q^{\rm th}(\nu)\,{\rm d}\nu, \tag{107}$$

as well as

$$\begin{aligned}\mathcal{M}_{\rm th}(\Delta) = \frac{1}{2\pi}\int_{\mathbb{R}} {\rm d}\nu\, \frac{G^2\gamma_{\rm m}\Delta}{2\left|d(\nu)\right|^2}\Bigg\{ \Bigg[\frac{1}{2}\,G^2\Delta - \nu\,\frac{\gamma_{\rm c}\gamma_{\rm m}}{2} + (\omega_{\rm m}+\nu) \\ \times\left(\nu^2 - \Delta^2 - \frac{\gamma_{\rm c}^2}{4}\right)\Bigg]\left(N(\nu)+\frac{1}{2}\right) + (\nu\to-\nu)\Bigg\},\end{aligned} \tag{108}$$

so that the fluctuation contribution can be written as

$$\langle H\rangle_{\rm fl} = \hbar\omega_{\rm m}\big(\mathcal{N}_{\rm rp} + \mathcal{N}_{\rm th} + \mathcal{M}_{\rm th}(\Delta)\big); \tag{109}$$

by construction we have $\mathcal{N}_{\rm rp}+\mathcal{N}_{\rm th}+\mathcal{M}_{\rm th}(\Delta) \geq 1/2$. As it appears, the mean energy density cannot be obtained from the knowledge of S_q alone, but extra terms are present. Moreover, the contribution proportional to $\mathcal{M}_{\rm th}(\Delta)$ can be negative. Depending on the parameter values, the extra terms can be actually quite small. It is important to stress that the given expression for the mean energy of the resonator holds for any temperature of the phonon bath, including the case of zero temperature.

We further stress that there is not strict energy equipartition. This can be expected since the mechanical oscillator is coupled to the cavity through its position and also the counter-rotating terms contribute to the final result. In the thermal part the lack of equipartition is due to the terms proportional to Δ, which are present in $S_p^{\mathrm{th}}(\nu)$ and not in $S_q^{\mathrm{th}}(\nu)$. In the radiation pressure part the term with Ω_{m}^2 comes from the position and the one with ν^2 comes from the momentum and give different contributions to the mean energy.

4.2.3. Vanishing effective detuning. For a vanishing effective detuning $\Delta = 0$ all the computations can be performed analytically. The second thermal contribution $\mathcal{M}_{\mathrm{th}}(\Delta)$ vanishes and the coupling constant takes the value $G^2 = 8\hbar g_0^2 E^2 / \left(m\omega_{\mathrm{m}}\gamma_{\mathrm{c}}^2\right)$. For the spectra of the fluctuations the explicit expressions reduce to

$$
\begin{aligned}
S_q^{\mathrm{rp}}(\nu) &= \frac{\Omega_{\mathrm{m}}\omega_{\mathrm{m}}G^2\gamma_{\mathrm{c}}}{2\left(\nu^2+\frac{\gamma_{\mathrm{c}}^2}{4}\right)\left[\left(\nu-\omega_{\mathrm{m}}\right)^2+\frac{\gamma_{\mathrm{m}}^2}{4}\right]\left[\left(\nu+\omega_{\mathrm{m}}\right)^2+\frac{\gamma_{\mathrm{m}}^2}{4}\right]}, \\
S_q^{\mathrm{th}}(\nu) &= \frac{\Omega_{\mathrm{m}}\gamma_{\mathrm{m}}}{2\omega_{\mathrm{m}}}\left[\frac{N(\nu)+\frac{1}{2}}{\left(\nu-\omega_{\mathrm{m}}\right)^2+\frac{\gamma_{\mathrm{m}}^2}{4}}+(\nu\to-\nu)\right],
\end{aligned}
\tag{110}
$$

leading upon integration to

$$
\begin{aligned}
\frac{1}{2}\,m\Omega_{\mathrm{m}}^2\left(\langle q^2\rangle_{\mathrm{eq}}-\langle q\rangle_{\mathrm{eq}}^2\right) &= \frac{\hbar\Omega_{\mathrm{m}}^2}{4\omega_{\mathrm{m}}}\left(2N_{\mathrm{eff}}+1\right)+\frac{\hbar\omega_{\mathrm{m}}G^2\left(2\gamma_{\mathrm{m}}+\gamma_{\mathrm{c}}\right)}{8\gamma_{\mathrm{m}}\left(\frac{(\gamma_{\mathrm{m}}+\gamma_{\mathrm{c}})^2}{4}+\omega_{\mathrm{m}}^2\right)}, \\
\frac{1}{2m}\langle p^2\rangle_{\mathrm{eq}} &= \frac{\hbar\Omega_{\mathrm{m}}^2}{4\omega_{\mathrm{m}}}\left(2N_{\mathrm{eff}}+1\right)+\frac{\hbar\omega_{\mathrm{m}}G^2\gamma_{\mathrm{c}}}{8\gamma_{\mathrm{m}}\left(\frac{(\gamma_{\mathrm{m}}+\gamma_{\mathrm{c}})^2}{4}+\omega_{\mathrm{m}}^2\right)}, \\
\frac{\gamma_{\mathrm{m}}}{4}\langle\{q,p\}\rangle_{\mathrm{eq}} &= -\frac{\hbar\gamma_{\mathrm{m}}^2}{8\omega_{\mathrm{m}}}\left(2N_{\mathrm{eff}}+1\right),
\end{aligned}
$$

with N_{eff} as in (44). These expressions show that equipartition of the mean energy is not valid just due to the radiation pressure contributions. However equipartition approximately holds for $\gamma_{\mathrm{c}} \gg 2\gamma_{\mathrm{m}}$, which is the case typically considered in many theoretical studies and experiments. We further have for the fluctuation contributions to the mean energy

$$
\mathcal{N}_{\mathrm{rp}} = \frac{G^2\left(\gamma_{\mathrm{m}}+\gamma_{\mathrm{c}}\right)}{4\gamma_{\mathrm{m}}\left(\frac{(\gamma_{\mathrm{m}}+\gamma_{\mathrm{c}})^2}{4}+\omega_{\mathrm{m}}^2\right)}, \qquad \mathcal{N}_{\mathrm{th}} = N_{\mathrm{eff}}+\frac{1}{2}, \qquad \mathcal{M}_{\mathrm{th}}(\Delta) = 0.
$$

The mean equilibrium energy of the mechanical oscillator is thus increased due to the interaction with the cavity as a consequence of the presence of the strong laser in resonance. For the values considered in Figure 1 we have $\mathcal{N}_{\mathrm{rp}} \simeq 1.6\times10^4$ corresponding to a temperature of about 7.9 K.

4.2.4. Laser cooling. As discussed in many papers [1, 3, 5, 6, 53], an important effect which can be described by the quantum models of cavity optomechanics is the laser cooling of the mechanical resonator. Since, as already discussed, we cannot expect equipartition of the mean mechanical energy, we cannot speak of temperature in a strict sense. A natural way to speak about laser cooling is the comparison of the mean energy of the fluctuations of the mechanical oscillator in the presence or the absence of the stimulating laser (corresponding to $\zeta = 0$). So, we have to study the value of the fluctuation contribution (109) and to compare it to its value for $\zeta = 0$, which is given by $\langle H\rangle_{\mathrm{fl}}\big|_{\zeta=0} = \langle H_{\mathrm{m}}\rangle_{\mathrm{eq}}\big|_{\zeta=0} = \hbar\omega_{\mathrm{m}}\left(N_{\mathrm{eff}}+\frac{1}{2}\right)$.

To obtain explicit analytical formulae for the mean energy we consider the case of a constant noise spectrum, that is $N(\nu) = \mathrm{const} = N_{\mathrm{eff}}$. To actually perform the calculations we need the expressions of the zeros of $d(\nu)$; here we consider the generic case given by the Ansatz (91). By lengthy computations the integrals over ν can be exactly performed, leading to involved formulae explicitly given in Appendix A.2. In order to describe cooling effects the relevant contributions can be written in the form

$$\mathcal{N}_{\rm th} = \frac{\gamma_{\rm m}}{\Gamma_{\rm m}}\,\mathcal{Q}\left(N_{\rm eff}+\frac{1}{2}\right), \qquad \mathcal{M}_{\rm th}(\Delta) = \frac{\gamma_{\rm m}}{\Gamma_{\rm m}}\,\mathcal{K}\left(N_{\rm eff}+\frac{1}{2}\right), \quad (111)$$

where the quantities $\mathcal{Q}$ and $\mathcal{K}$ are given in equations (A.16) and (A.17). The expression for $\mathcal{N}_{\rm rp}$ is given in (A.15). Note that, while $\mathcal{Q}$ is always positive, depending on the values of the parameters the quantity $\mathcal{K}$ can be also negative. For a large choice of the parameters $\mathcal{Q}$ turns out to be close to 1.

In the following figures we describe the effective cooling of the mechanical oscillator, by considering as a figure of merit the *cooling factor*

$$\mathcal{C} = \frac{\gamma_{\rm m}}{\Gamma_{\rm m}}\left(\mathcal{Q}+\mathcal{K}\right). \quad (112)$$

We study two cases, corresponding to the parameter regions for which an exact or approximate analytic evaluation of the different contributions to the mean energy has been provided. In both cases mass and bare frequency of the mechanic oscillator are taken to be $m = 2.5\times10^{-10}$ kg and $\Omega_{\rm m} = 2\pi\times10^7$ Hz, while the mechanical damping factor is $\gamma_{\rm m} = 2\pi\times10^2$ Hz. We consider a cavity of length 5×10^{-4} m and resonance frequency $\omega_{\rm c} = 2\pi c/(1064\times10^{-9})$ Hz, driven by a laser with a power of 5×10^{-2} W. For the sake of comparison the values of the fixed parameters are taken from [25].

We start by studying the case $\Delta = \omega_{\rm m}$, studied in Section 4.2.1, in which the location of the poles can be evaluated exactly, provided one distinguishes two regions according to the value of the ratio $(\gamma_{\rm c}-\gamma_{\rm m})^2/4G^2$. No approximation is taken in the expression of the integrals giving the mean energy. If this ratio is above one, verified for a cavity damping $\gamma_{\rm c} > \bar{\gamma}_{\rm c}$, where the critical damping $\bar{\gamma}_{\rm c}$ is introduced in the comments after (99) and corresponds for the considered parameters to $\bar{\gamma}_{\rm c} \simeq 4.1\times10^7$, the effective damping rates $\Gamma_{\rm m}$ (93) and $\Gamma_{\rm c}$ (94) are actually distinct, while $\Delta_{\rm eff} = \omega^{\rm m}_{\rm eff}$ and their expression is given by (95). By numerical computations we see that the cooling factor $\mathcal{C}$ is a monotonic increasing function of the cavity damping rate $\gamma_{\rm c}$, and around the starting point $\bar{\gamma}_{\rm c}$ the cooling factor takes the value 2.9×10^{-5}. In the complementary region, corresponding to $(\gamma_{\rm c}-\gamma_{\rm m})^2/4G^2$ below one, the cooling factor is a decreasing function of the cavity damping rate, so that the optimal cooling is obtained for $\gamma_{\rm c} = \bar{\gamma}_{\rm c}$. In this region, corresponding to $\gamma_{\rm c} < \bar{\gamma}_{\rm c}$, we have $\Gamma_{\rm m} = \Gamma_{\rm c}$ with value given in (98), while the effective frequencies $\Delta_{\rm eff}$ and $\omega^{\rm m}_{\rm eff}$ are given by the expressions (98). To assess the relevance of the various contributions in (112) we report the values for $\gamma_{\rm c} = \bar{\gamma}_{\rm c}$: we have $\gamma_{\rm m}/\Gamma_{\rm m} \simeq 3.05\times10^{-5}$, $\mathcal{Q} \simeq .997$ and $\mathcal{K} \simeq -4.18\times10^{-2}$; then, (112) gives $\mathcal{C} \simeq 2.91\times10^{-5}$, which is a very strong cooling factor.

Instead, in Figure 1 we consider the case $\gamma_{\rm c} \gg \omega_{\rm m}$, that is a cavity damping much bigger than the mechanical oscillator frequency. In the exact formulae for the integrals we use the approximate expressions for $\Gamma_{\rm m}$ and $\Gamma_{\rm c}$ given in (102), relying on the conditions (100). The stationary value of the energy of the mechanical system has a marked dependence on the effective detuning Δ and the optimal cooling region, corresponding to $\mathcal{C}$ of the order of 10^{-3}, is obtained for $\Delta \lesssim \gamma_{\rm c}$. In this parameter region $\mathcal{N}_{\rm rp}$ can be neglected with respect to $\mathcal{C}N_{\rm eff}$, unless the phonon bath is below 1 K, so that indeed the quantity $\mathcal{C}$ given in (112) properly describes the cooling effect. When the detuning Δ goes to zero the cooling factor rapidly increases in agreement with the discussion in Section 4.2.3 showing the presence of heating at $\Delta = 0$; in this parameter region, the cooling effect disappears also when Δ grows.

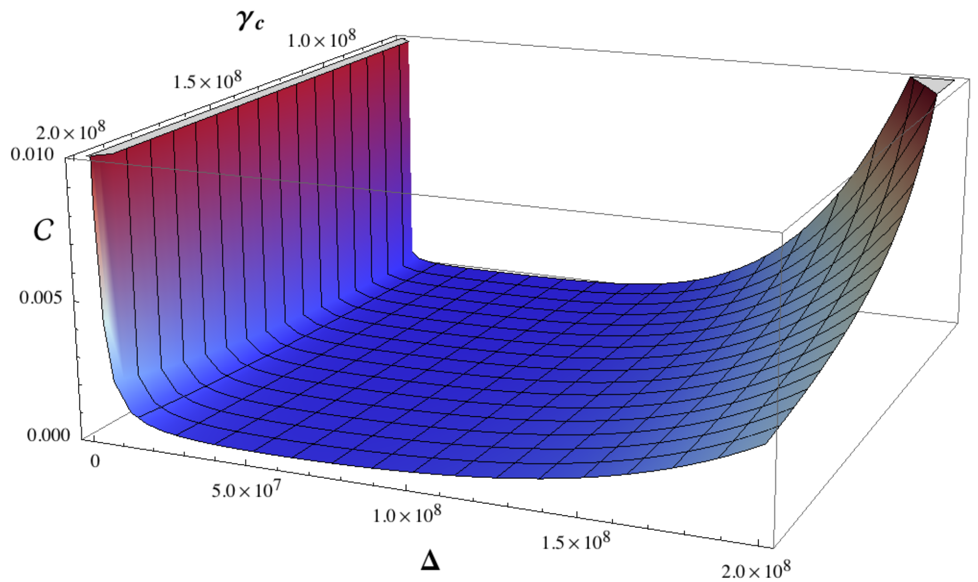

Figure 1. Plot of the cooling factor $\mathcal{C}$ for the case in which the cavity damping is much bigger than the mechanical oscillator frequency. We explore the dependence of the cooling factor on both the effective detuning Δ and the cavity damping rate $\gamma_{\rm c}$, both expressed in Hz. It appears that the best cooling factor is of the order 10^{-3} and corresponds to $\Delta \lesssim \gamma_{\rm c}$.

4.3. Optical spectra

We consider now the monitoring of the emitted light by balanced homodyne and heterodyne detection [56, Sect. 7.2]. The aim is to see which kind of information on the mechanical oscillator can be obtained by detection of the emitted light.

4.3.1. Homodyne spectrum. The case of a perfect coherent monochromatic local oscillator of frequency ω_0 with detection of the whole emitted light [49, 55] corresponds to the continuous measurement of a field quadrature of the type

$$Q(t;\vartheta) = \mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}\mathrm{e}^{\mathrm{i}\arg\zeta}\int_0^t \mathrm{e}^{-\mathrm{i}\omega_0 r}\mathrm{d}B^\dagger_{\rm em}(r) + \text{h.c.}; \tag{113}$$

ϑ is a free parameter which depends on the optical path and determines the observed quadrature. As a consequence of the definition we have that $[Q(t;\vartheta), Q(s;\vartheta)] = 0$. By the properties of the output fields, discussed after Eq. (33), the commutation rules are preserved; this gives that the output current $Q^{\rm out}(t;\vartheta) := U(t)^\dagger Q(t;\vartheta)U(t)$ satisfies $[Q^{\rm out}(t;\vartheta), Q^{\rm out}(s;\vartheta)] = 0$. This is the key property expressing the fact that $Q^{\rm out}(t;\vartheta)$ can be measured with continuity in time. Similarly to (80) we introduce the gated Fourier transforms

$$Q_T(\nu;\vartheta) = \frac{1}{\sqrt{T}}\int_0^T \mathrm{e}^{\mathrm{i}\nu t}\mathrm{d}Q(t;\vartheta), \qquad Q^{\rm out}_T(\nu;\vartheta) = \frac{1}{\sqrt{T}}\int_0^T \mathrm{e}^{\mathrm{i}\nu t}\mathrm{d}Q^{\rm out}(t;\vartheta). \tag{114}$$

From the above relations we obtain the second key relation which guarantees the presence of commuting observables and therefore the consistency of the theory:

$$[Q^{\rm out}_T(\nu;\vartheta), Q^{\rm out}_T(\nu';\vartheta)] = 0. \tag{115}$$

The homodyne spectrum is then given by the expression

$$S(\nu;\vartheta)=\lim_{T\to+\infty}\operatorname{Tr}\left\{Q_T^{\rm out}(-\nu;\vartheta)Q_T^{\rm out}(\nu;\vartheta)\rho_0\otimes\tilde\sigma_{\rm env}\right\}, \tag{116}$$

where the environmental state is given by (62) and ρ_0 is any initial state for the mechanical oscillator and the cavity mode. Note that this expression is nothing but the spectrum of the classical stochastic process representing the output, and not an ad-hoc quantum definition [49, Sect. 4]. The commutation property (115) implies that *the homodyne spectrum $S(\nu;\vartheta)$ is an even function of ν.*

As shown in Appendix B.1, the homodyne spectrum has both an elastic and an inelastic component

$$S(\nu;\vartheta)=S_{\rm el}(\nu;\vartheta)+S_{\rm inel}(\nu;\vartheta), \tag{117}$$

which turn out to have the expressions

$$\begin{aligned}S_{\rm el}(\nu;\vartheta)&=8\pi\gamma_{\rm c}\left|\zeta\right|^2(\sin\vartheta)^2\,\delta(\nu),\\ S_{\rm inel}(\nu;\vartheta)&=S_{\rm th}(\nu;\vartheta)+S_{\rm rp}(\nu;\vartheta),\end{aligned} \tag{118}$$

with

$$S_{\rm th}(\nu;\vartheta)=\frac{2\gamma_{\rm c}\omega_{\rm m}G^2\left[\left(\frac{\gamma_{\rm c}}{2}\cos\vartheta+\Delta\sin\vartheta\right)^2+(\nu\cos\vartheta)^2\right]}{\Omega_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta-\nu)^2\right)\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta+\nu)^2\right)}\,S_q^{\rm th}(\nu), \tag{119}$$

$$\begin{aligned}S_{\rm rp}(\nu;\vartheta)=1&+\frac{2\gamma_{\rm c}\omega_{\rm m}G^2\left[\left(\frac{\gamma_{\rm c}}{2}\cos\vartheta+\Delta\sin\vartheta\right)^2+(\nu\cos\vartheta)^2\right]}{\Omega_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta-\nu)^2\right)\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta+\nu)^2\right)}\,S_q^{\rm rp}(\nu)\\ &+\gamma_{\rm c}\omega_{\rm m}G^2\,\mathrm{Re}\left[\frac{\left(\frac{\gamma_{\rm c}^2}{4}+\nu^2-\Delta^2\right)\sin2\vartheta-\Delta\left(\gamma_{\rm c}\cos2\vartheta-2\mathrm{i}\nu\right)}{d(\nu)\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta-\nu)^2\right)\left(\frac{\gamma_{\rm c}^2}{4}+(\Delta+\nu)^2\right)}\right.\\ &\qquad\left.\times\left(\frac{\gamma_{\rm c}^2}{4}-\nu^2+\Delta^2-\mathrm{i}\gamma_{\rm c}\nu\right)\right].\end{aligned} \tag{120}$$

Note that all the contributions are indeed positive as shown in Appendix B.1. It is important to stress that the connection between $S_q(\nu)$ and $S_{\rm inel}(\nu;\vartheta)$ is far from simple. In particular the last contribution in (120) comes from the interference of the electromagnetic part of the signal with the shot noise, as detailed in Appendix B.1. This is a completely new term, in principle detectable in experiments at very low temperatures.

Let us further stress that different quadratures are incompatible and actually one can prove the general inequality [49, 55]

$$S_{\rm inel}(\nu;\vartheta)S_{\rm inel}(\nu;\vartheta\pm\pi/2)\geq1, \tag{121}$$

which is just a form of the Heisenberg-Robertson uncertainty relations coming from the canonical commutation relations of the involved Bose fields. As a result quite different physical information can be extracted from the different quadratures.

The case $\Delta = 0$. The first striking example of strong dependence on ϑ is in the case $\Delta = 0$. For $\vartheta = \pi/2$ we get $S_{\rm el}(\nu;\pi/2) = 32\pi E^2 \delta(\nu)/\gamma_{\rm c}$ and $S_{\rm inel}(\nu;\pi/2) = 1$: only the shot noise contributes to the inelastic spectrum.

On the contrary, for the quadrature with $\vartheta = 0$ we get $S_{\rm el}(\nu;\pi/2) = 0$ and

$$S_{\rm inel}(\nu;0) = 1 + \frac{2\gamma_{\rm c}\omega_{\rm m}G^2}{\Omega_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4} + \nu^2\right)}\, S_q(\nu), \tag{122}$$

where $S_q(\nu)$ is now explicitly given by (110). An important point is that in this case the interference term vanishes exactly and we have a direct connection of the homodyne spectrum with the fluctuation spectrum of the position of the mirror. This result has been found also in [37], but with the substitution $\hat{R}(\nu) \to \hat{R}_{GZ}(\nu)$ in the expression (86) for $S_q^{\rm th}(\nu)\big|_{\Delta=0}$. As a result, at least in principle, when $\Delta = 0$ the homodyne observation of the quadrature with $\vartheta = \pi/2$ can give direct experimental information on the correct expression for $\hat{R}(\nu)$. At zero temperature one could experimentally discriminate between our result (58) and the standard proposal (59).

In other cases the interference term does not vanish, but it can be negligible at high temperatures. For instance, when the interference term is negligible, at least in the region where $N(\nu) \gg 1$, we recover for $S_{\rm inel}(\nu;0)$ the result given in [1, Sect. 3]. At high temperatures the inelastic homodyne spectrum allows to reconstruct the fluctuation spectrum of position, while no direct information on the fluctuation of the momentum and on the cross-correlation is obtained. Moreover, at high temperatures we have also $S_q(\nu) \simeq S_q^{\rm th}(\nu)$; by using the explicit expressions of $S_q^{\rm th}(\nu)$ (86) and $\hat{R}(\nu)$ (54) we get

$$S_{\rm inel}(\nu;0) \simeq \gamma_{\rm c}\gamma_{\rm m}G^2\, \frac{\left(\frac{\gamma_{\rm c}^2}{4} + \nu^2\right)\left(\frac{\gamma_{\rm m}^2}{4} + (\omega_{\rm m} + \nu)^2\right)}{|d(\nu)|^2} \left(N(\nu) + \frac{1}{2}\right) + (\nu \to -\nu).$$

This expression highlights the dependence of the homodyne spectrum on the thermal spectrum $N(\nu)$ and the characteristic polynomial $d(\nu)$ (89) of the dynamical matrix (73) of the full optomechanical system.

Squeezing. An important information about the non classical nature of the light generated by optomechanical systems can be obtained considering the quadrature with $\vartheta = -\pi/4$. In the simple case of vanishing detuning $\Delta = 0$ and vanishing temperature $N(\nu) \equiv 0$, it is possible to show from (117)–(120) that we have $S_{\rm inel}(0;-\pi/4) < 1$, at least in a certain region of the parameters. This means that in a neighbourhood of $\nu = 0$ we have $S_{\rm inel}(\nu;-\pi/4) < 1$ and the emitted light is squeezed. This result shows that such a kind of optomechanical systems can generate non classical light [3,7]. Note that, if light squeezing is present for certain values of the parameters, then the inequality (121) implies that the complementary quadrature is anti-squeezed. Of course, experimentally it could be difficult to tune the values of the various free parameters in order to have squeezing; moreover, the elastic peak in the spectrum tends to hide the squeezing around $\nu = 0$ in the inelastic spectrum.

4.3.2. Heterodyne spectrum. In the case of heterodyne detection the local oscillator and the stimulating light are produced by different laser sources; now, the stimulating laser frequency ω_0 and the local oscillator frequency, say μ, are in general different. Moreover, the phase difference cannot be maintained stable and this erases some interference terms. It can be shown [50], [30, Sect. 3.5] that the balanced heterodyne detection scheme corresponds to the measurement in continuous time of the observables

$$I(\mu;t)=\int_0^t \sqrt{\varkappa}\,\mathrm{e}^{-\varkappa(t-s)/2}\,\mathrm{e}^{\mathrm{i}\mu s+\mathrm{i}\alpha}\,\mathrm{d}B_{\mathrm{em}}(s)+\text{h.c.},\tag{123}$$

where α is a phase depending on the optical paths and $\sqrt{\varkappa}\,\mathrm{e}^{-\varkappa t/2}$, $\varkappa>0$, represents the detector response function. As we shall see, the heterodyne spectrum does not depend on α. In the Heisenberg description the observables become the "output current"

$$\begin{aligned}I_{\mathrm{out}}(\mu;t)&=U(t)^\dagger I(\mu;t)U(t)\\&=\sqrt{\varkappa}\int_0^t \mathrm{e}^{-\frac{\varkappa}{2}(t-s)+\mathrm{i}\alpha}\left(\mathrm{e}^{\mathrm{i}\mu s}\mathrm{d}B_{\mathrm{em}}(s)+\sqrt{\gamma_{\mathrm{c}}}\,\mathrm{e}^{\mathrm{i}(\mu-\omega_0)s}a_{\mathrm{c}}(s)\mathrm{d}s\right)+\text{h.c.}\end{aligned}$$

By the definition of $I(\mu;t)$ and the properties of $U(t)$ we get $[I_{\mathrm{out}}(\mu;t),I_{\mathrm{out}}(\mu;s)]=0$, which says that the output current at time t and the current at time s are compatible observables.

While in the homodyne scheme the spectrum of the output is analysed, in the heterodyne scheme it is usual to register only the output power as a function of the frequency μ of the local oscillator. The mean output power of the detection apparatus at large times is proportional to

$$P(\mu)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T \mathrm{d}t\ \mathrm{Tr}\left\{I_{\mathrm{out}}(\mu;t)^2\rho_0\otimes\tilde{\sigma}_{\mathrm{env}}\right\};\tag{124}$$

the limit is in the sense of the distributions in μ. As a function of μ, $P(\mu)$ is known as *power spectrum*. Note that to change μ means to change the frequency of the local oscillator, that is to change the measuring apparatus. In general $I_{\mathrm{out}}(\mu;t)$ and $I_{\mathrm{out}}(\mu';s)$ do not commute, even for $t=s$. Then, there is no reason for the power spectrum to have some symmetry in μ. The heterodyne power spectrum can be decomposed in an elastic and an inelastic part

$$P(\mu)=\Sigma_{\mathrm{el}}(\mu)+\Sigma_{\mathrm{inel}}(\mu),\tag{125}$$

$$\begin{aligned}\Sigma_{\mathrm{el}}(\mu)&=\lim_{T\to+\infty}\frac{\varkappa\gamma_{\mathrm{c}}}{T}\int_0^T\mathrm{d}t\left[2\,\mathrm{Re}\left(\zeta\mathrm{e}^{\mathrm{i}\alpha}\int_0^t \mathrm{e}^{-\frac{\varkappa}{2}(t-s)+\mathrm{i}(\mu-\omega_0)s}\mathrm{d}s\right)\right]^2\\&=\frac{\varkappa\gamma_{\mathrm{c}}\left|\zeta\right|^2}{\frac{\varkappa^2}{4}+(\mu-\omega_0)^2}\xrightarrow{\varkappa\downarrow 0}4\pi\gamma_{\mathrm{c}}\left|\zeta\right|^2\,\delta(\mu-\omega_0),\end{aligned}\tag{126}$$

$$\Sigma_{\mathrm{inel}}(\mu)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\mathrm{d}t\ \mathrm{Tr}\left\{I_{\mathrm{inel}}(\mu;t)^2\rho_0\otimes\tilde{\sigma}_{\mathrm{env}}\right\},\tag{127}$$

where

$$\begin{aligned}I_{\mathrm{inel}}(\mu;t)=\sqrt{\varkappa}\int_0^t \mathrm{e}^{-\frac{\varkappa}{2}(t-s)}&\Big(\mathrm{e}^{\mathrm{i}\mu s+\mathrm{i}\alpha}\mathrm{d}B_{\mathrm{em}}(s)\\&+\sqrt{\frac{\gamma_{\mathrm{c}}}{2}}\,\mathrm{e}^{\mathrm{i}(\mu-\omega_0)s+\mathrm{i}\vartheta}\big(Y(s)-\mathrm{i}X(s)\big)\mathrm{d}s\Big)+\text{h.c.}\end{aligned}$$

The inelastic part of the spectrum is computed in Appendix B.2. Again it is possible to identify a radiation pressure contribution and a thermal part

$$\Sigma_{\mathrm{inel}}(\mu)=\Sigma_{\mathrm{rp}}(\mu)+\Sigma_{\mathrm{th}}(\mu).\tag{128}$$

For simplicity we give only the expressions for $\varkappa\downarrow 0$:

$$\Sigma_{\mathrm{rp}}(\mu)=1+\frac{\gamma_{\mathrm{c}}\omega_{\mathrm{m}}G^2S_q^{\mathrm{rp}}(\mu-\omega_0)}{\Omega_{\mathrm{m}}\left(\frac{\gamma_{\mathrm{c}}^2}{4}+(\mu-\omega_0-\Delta)^2\right)}-\mathrm{Im}\,\frac{\gamma_{\mathrm{c}}\omega_{\mathrm{m}}G^2}{d(\mu-\omega_0)}\,\frac{\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0+\Delta\right)}{\frac{\gamma_{\mathrm{c}}}{2}+\mathrm{i}\left(\mu-\omega_0-\Delta\right)},\tag{129}$$

$$\Sigma_{\rm th}(\mu) = \frac{\gamma_{\rm c}\omega_{\rm m} G^2 S_q^{\rm th}(\mu-\omega_0)}{\Omega_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4} + (\mu-\omega_0-\Delta)^2\right)}\,. \tag{130}$$

Both contributions are positive as it follows from the expressions (130) and (B.7). Note the presence of the interference term in (129). By simple computations one can check that

$$\Sigma_{\rm inel}(\nu+\omega_0) + \Sigma_{\rm inel}(\omega_0-\nu) = S_{\rm inel}(\nu;\vartheta) + S_{\rm inel}(\nu;\vartheta+\pi/2); \tag{131}$$

this is a fundamental relation [56, Eq. (9.61)] connecting heterodyne and homodyne spectra. Moreover, by inserting the definitions of the relevant quantities given in (54), (85) and (86), an explicit expression for $\Sigma_{\rm inel}$ can be obtained from which it is apparent that $\Sigma_{\rm inel}(\mu) > 1$: in the heterodyne detection the phase dependencies are lost and it is impossible to detect squeezing in the emitted light.

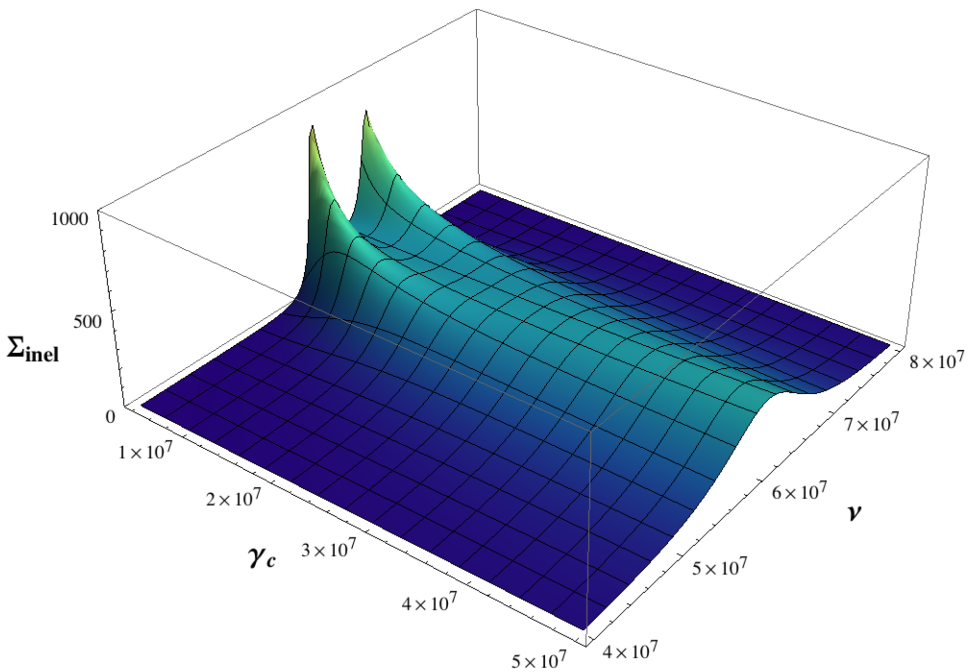

Figure 2. Plot of the inelastic heterodyne spectrum $\Sigma_{\rm inel}$ as a function of ν for a range of values of the cavity damping $\gamma_{\rm c}$ around the critical value $\bar\gamma_{\rm c}$ discussed in Section 4.2.4. It appears how the two distinct peaks of the spectrum coalesce at critical value. The spectrum is plotted for $\Delta = \omega_{\rm m}$, while the other parameters are as in Section 4.2.4.

As in the homodyne case, the interference term in (129) is negligible when $N \gg 1$ and we get

$$\Sigma_{\rm inel}(\mu) \simeq 1 + \frac{\gamma_{\rm c}\omega_{\rm m} G^2}{\Omega_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4} + (\mu-\omega_0-\Delta)^2\right)}\, S_q(\mu-\omega_0). \tag{132}$$

When this approximation holds, the inelastic heterodyne spectrum too allows to reconstruct the asymptotic dynamics of the mirror through the position fluctuations.

To explore the behaviour of the spectrum we take $N(\nu)$ as given by (56) with a Ohmic spectral density. Then, by using the explicit expressions of $S_q^{\rm rp}$ and $S_q^{\rm th}$ and by setting

$\nu = \mu - \omega_0$, we get

$$\Sigma_{\rm inel}(\nu+\omega_0) = 1 + \frac{\gamma_{\rm c} G^2}{2\left|d(\nu)\right|^2}\Bigg\{\gamma_{\rm c}\omega_{\rm m}^2 G^2 + \gamma_{\rm m}\left(\frac{\gamma_{\rm c}^2}{4}+(\nu+\Delta)^2\right)\left[\frac{\gamma_{\rm m}^2}{4}+(\nu-\omega_{\rm m})^2+\frac{4\omega_{\rm m}\left|\nu\right|}{{\rm e}^{\beta\hbar\left|\nu\right|}-1}\right]\Bigg\}. \tag{133}$$

From this expression we see that the main features of the spectrum will be determined by the zeros of the denominator $\left|d(\nu)\right|^2$; for instance, as discussed in Section 4.2.1, for $\Delta = \omega_{\rm m}$ we can have one or two resonance frequencies depending on the value of the cavity decay rate $\gamma_{\rm c}$. In Figure 2 we show this phenomenon: the two distinct peaks coalesce as $\gamma_{\rm c}$ increases. For these values of the parameters one can check that the main contribution to the inelastic heterodyne spectrum comes from the thermal part $\Sigma_{\rm th}$. It can be checked that in this parameter region the behaviour of the inelastic homodyne spectrum $S_{\rm inel}(\nu; 0)$ is very close to the heterodyne one as depicted in Figure 2. Let us notice that the behaviour shown in Figure 2 does not uncover the whole rich structure of the spectrum which appears by exploring other parameter regions.

5. Summary and outlook

In this article we have shown how to give a fully quantum description of a dissipative mechanical oscillator. The combined use of master equations and quantum Langevin equations allows for the construction of a dissipative dynamics respecting symmetries and physical constraints, such as the energy equipartition at equilibrium, and subject to dissipation with an arbitrary noise spectrum. A crucial feature allowing for these results is that for a mechanical oscillator the definition of the creation and annihilation operators $a_{\rm m}$ and $a_{\rm m}^\dagger$ in terms of position and momentum is not the usual one, but rather depends on the damping constant $\gamma_{\rm m}$, as discussed in Section 2.2; the standard result is only recovered for a vanishing damping constant as can be seen from (16). Moreover, the quantum Langevin equations for the system, and the input-output relations for the noises, for both the mechanical oscillator and for the optomechanical system, given in Section 3.1 and Section 4.1 respectively, need not be postulated: they are nothing but the Heisenberg equations of motion determined by the Hudson-Parthasarathy unitary evolutions (21) and (61). In this framework it appears that, in order to preserve the Heisenberg uncertainty relations, the momentum operator can be interpreted as the time derivative of the position operator only in a "coarse grained" picture. An help in comparing our approach to others and in discussing the structure of the noises comes from the quantum Langevin equation in Newton form, see Section 3.3, which at the price of introducing singular noises does not contain the momentum operator. Indeed in the quantum case important constraints on the correlation functions of the operator noises come from the fact that they need to be positive definite and compatible with the commutation rules of such noises. In this formalism, we are further able to introduce a field analog of the P-representation for the state of the environment and this opens the possibility of treating an arbitrary noise spectrum as done in Section 3.2.

Our description of the mechanical oscillator is not very different from other proposals at medium and high temperatures of the phonon bath. Differences become relevant for very small temperatures. Indeed the dynamics we have constructed is fully "quantum" at all temperatures and this opens the possibility of constructing models of optomechanical systems which are reliable also in a deep quantum regime. As an example we have studied a prototypical system: a mechanical resonator interacting via radiation pressure with a single optical mode in a cavity. For this case we have given explicit general formulae for the fluctuation spectra of position and momentum of the mechanical resonator and for the mean mechanical energy at equilibrium. By using detection theory in continuous time, we have obtained the full expressions of the homodyne and heterodyne spectra of the emitted light. For not too low temperatures, usual results are recovered, such as laser cooling and connection between the light spectra and the fluctuations of position of the mechanical component. However, our description is valid also at very low temperatures, when semi-classical reasoning is not valid and the observation of the spectra of the emitted light is not giving a direct measurement of the mechanical fluctuations.

Many generalizations are possible [57–60], which could benefit of a systematic and consistent treatment. The simplest generalization is to include imperfections in the detection scheme and noise in the stimulating laser light [5, 7, 30, 50]. But also direct detection can be included [30] or the entanglement between resonator and optical mode can be studied. Moreover, the whole theory has in some sense "modular" properties and can be applied to more complicated systems, say when more mechanical resonators and more optical modes are involved.

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Appendix A. Computation of the fluctuations

The Fourier transformed equations of motion corresponding to (72) can be solved by purely algebraic manipulations; essentially the problem reduces to compute the inverse of the matrix $A+\mathrm{i}\nu\mathbb{1}$. By using the characteristic polynomial $d(\nu)$ of the dynamical matrix A (89) the final result for large T turns out to be

$$F_1(T;\nu)\simeq\frac{\Delta^2+\left(\mathrm{i}\nu-\frac{\gamma_\mathrm{c}}{2}\right)^2}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{m}\Omega_\mathrm{m}}{2\omega_\mathrm{m}}}\,Z_1^\mathrm{th}(T;\nu)+\frac{G}{d(\nu)}\sqrt{\frac{\omega_\mathrm{m}\Omega_\mathrm{m}\gamma_\mathrm{c}}{2}}\,Z_1^\mathrm{rp}(T;\nu), \tag{A.1}$$

$$F_2(T;\nu)\simeq\frac{\sqrt{\gamma_\mathrm{m}}}{d(\nu)\sqrt{2\omega_\mathrm{m}\Omega_\mathrm{m}}}\,Z_2^\mathrm{th}(T;\nu)-\frac{\mathrm{i}\nu G}{d(\nu)}\sqrt{\frac{\omega_\mathrm{m}\gamma_\mathrm{c}}{2\Omega_\mathrm{m}}}\,Z_1^\mathrm{rp}(T;\nu), \tag{A.2}$$

$$F_3(T;\nu)\simeq\frac{G\Delta}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{m}}{2}}\,Z_1^\mathrm{th}(T;\nu)+\frac{\mathrm{i}\nu\left(\mathrm{i}\nu-\gamma_\mathrm{m}\right)+\Omega_\mathrm{m}^2}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{c}}{2}}\,Z_1^\mathrm{rp}(T;\nu), \tag{A.3}$$

$$\begin{aligned}F_4(T;\nu)\simeq{}&-\frac{\left(\mathrm{i}\nu-\frac{\gamma_\mathrm{c}}{2}\right)G}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{m}}{2}}\,Z_1^\mathrm{th}(T;\nu)+\frac{\mathrm{i}\nu\left(\mathrm{i}\nu-\gamma_\mathrm{m}\right)+\Omega_\mathrm{m}^2}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{c}}{2}}\,Z_2^\mathrm{rp}(T;\nu)\\&-\frac{G^2\omega_\mathrm{m}}{d(\nu)}\sqrt{\frac{\gamma_\mathrm{c}}{2}}\left(\frac{\zeta}{|\zeta|}\,\hat{B}_\mathrm{em}^T(\omega_0-\nu)^\dagger+\frac{\overline{\zeta}}{|\zeta|}\,\hat{B}_\mathrm{em}^T(\nu+\omega_0)\right),\end{aligned} \tag{A.4}$$

$$Z_1^\mathrm{th}(T;\nu)=\left(\mathrm{i}\left(\nu+\omega_\mathrm{m}\right)-\frac{\gamma_\mathrm{m}}{2}\right)\overline{\tau}\,\hat{B}_\mathrm{th}^T(\nu)+\left(\mathrm{i}\left(\nu-\omega_\mathrm{m}\right)-\frac{\gamma_\mathrm{m}}{2}\right)\tau\hat{B}_\mathrm{th}^T(-\nu)^\dagger, \tag{A.5}$$

$$\begin{aligned}Z_2^\mathrm{th}(T;\nu)={}&\left[\left(\Delta^2+\left(\frac{\gamma_\mathrm{c}}{2}-\mathrm{i}\nu\right)^2\right)\left(\Omega_\mathrm{m}^2-\nu\left(\omega_\mathrm{m}+\mathrm{i}\,\frac{\gamma_\mathrm{m}}{2}\right)\right)-G^2\omega_\mathrm{m}\Delta\right]\tau\hat{B}_\mathrm{th}^T(-\nu)^\dagger\\&+\left[\left(\Delta^2+\left(\frac{\gamma_\mathrm{c}}{2}-\mathrm{i}\nu\right)^2\right)\left(\Omega_\mathrm{m}^2+\nu\left(\omega_\mathrm{m}-\mathrm{i}\,\frac{\gamma_\mathrm{m}}{2}\right)\right)-G^2\omega_\mathrm{m}\Delta\right]\overline{\tau}\,\hat{B}_\mathrm{th}^T(\nu),\end{aligned} \tag{A.6}$$

$$Z_1^{\rm rp}(T;\nu) = \left(\mathrm{i}\,(\nu-\Delta) - \frac{\gamma_{\rm c}}{2}\right)\frac{\zeta}{|\zeta|}\,\hat{B}^T_{\rm em}(\omega_0-\nu)^\dagger + \left(\mathrm{i}\,(\nu+\Delta) - \frac{\gamma_{\rm c}}{2}\right)\frac{\overline{\zeta}}{|\zeta|}\,\hat{B}^T_{\rm em}(\nu+\omega_0), \tag{A.7}$$

$$Z_2^{\rm rp}(T;\nu) = \left(\Delta-\nu-\mathrm{i}\,\frac{\gamma_{\rm c}}{2}\right)\frac{\zeta}{|\zeta|}\,\hat{B}^T_{\rm em}(\omega_0-\nu)^\dagger + \left(\Delta+\nu+\mathrm{i}\,\frac{\gamma_{\rm c}}{2}\right)\frac{\overline{\zeta}}{|\zeta|}\,\hat{B}^T_{\rm em}(\nu+\omega_0). \tag{A.8}$$

To compute the spectra (82), (83) we need also the field correlations. From the correlations (42) and the fact that $B_{\rm em}$ is a Fock field in the vacuum state we get

$$\begin{aligned} &\langle \hat{B}^T_{\rm em}(\nu)\hat{B}^T_{\rm em}(\nu)^\dagger\rangle_{\rm env} = 1, \qquad &&\langle \hat{B}^T_{\rm th}(\nu)^\dagger\hat{B}^T_{\rm th}(\nu)\rangle_{\rm env} = N(\nu),\\ &\langle \hat{B}^T_{\rm em}(\nu)^\dagger\hat{B}^T_{\rm em}(\nu)\rangle_{\rm env} = 0, \qquad &&\langle \hat{B}^T_{\rm th}(\nu)\hat{B}^T_{\rm th}(\nu)^\dagger\rangle_{\rm env} = N(\nu)+1, \end{aligned} \tag{A.9}$$

while the cross-correlations involving both $B_{\rm th}$ and $B_{\rm em}$ vanish. Also the fluctuations of the cavity mode operators and the correlations oscillator/mode could be computed by using all the components (A.1)–(A.4), but we do not study them in this work.

Appendix A.1. The zeros of $d(\nu)$

Let us assume that $d(\nu)$ has two zeros of the form $\nu_{\rm m} = \omega^{\rm m}_{\rm eff} - \mathrm{i}\Gamma_{\rm m}/2$ and $\nu_{\rm m} = \Delta_{\rm eff} - \mathrm{i}\Gamma_{\rm c}/2$ with $\omega^{\rm m}_{\rm eff} \neq 0$ and $\Delta_{\rm eff} \neq 0$; by the property $\overline{d(\nu)} = d(-\overline{\nu})$, the other two zeros are $-\overline{\nu_{\rm m}}$ and $-\overline{\nu_{\rm c}}$. Therefore, we can write $d(\nu)$ in the form (91) or

$$d(\nu) = (\nu-\nu_{\rm m})\,(\nu+\overline{\nu_{\rm m}})\,(\nu-\nu_{\rm c})\,(\nu+\overline{\nu_{\rm c}})\,. \tag{A.10}$$

By equating this expression to (89) we get the algebraic system

$$\begin{cases} \Gamma_{\rm m}+\Gamma_{\rm c} = \gamma_{\rm c}+\gamma_{\rm m},\\ \Gamma_{\rm c}\,|\nu_{\rm m}|^2 + \Gamma_{\rm m}\,|\nu_{\rm c}|^2 = \gamma_{\rm c}\Omega_{\rm m}^2 + \gamma_{\rm m}\left(\Delta^2+\dfrac{\gamma_{\rm c}^2}{4}\right),\\ |\nu_{\rm m}|^2 + |\nu_{\rm c}|^2 + \Gamma_{\rm m}\Gamma_{\rm c} = \Omega_{\rm m}^2 + \Delta^2 + \dfrac{\gamma_{\rm c}^2}{4} + \gamma_{\rm c}\gamma_{\rm m},\\ |\nu_{\rm m}|^2\,|\nu_{\rm c}|^2 = \Omega_{\rm m}^2\left(\Delta^2+\dfrac{\gamma_{\rm c}^2}{4}\right) - G^2\omega_{\rm m}\Delta. \end{cases} \tag{A.11}$$

The stability conditions (78), (79) guarantee $\Gamma_{\rm m} > 0$ and $\Gamma_{\rm c} > 0$. By assuming $\Gamma_{\rm c} \neq \Gamma_{\rm m}$, from this system we get in particular

$$\begin{aligned} \Delta_{\rm eff}^2 &= \frac{\Gamma_{\rm c}-\gamma_{\rm m}}{\Gamma_{\rm c}-\Gamma_{\rm m}}\,\Delta^2 - \frac{\gamma_{\rm c}-\Gamma_{\rm c}}{\Gamma_{\rm c}-\Gamma_{\rm m}}\,\omega_{\rm m}^2 - (\gamma_{\rm c}-\Gamma_{\rm c})\,\frac{\Gamma_{\rm c}-\gamma_{\rm m}}{4},\\ \omega^{\rm m\;2}_{\rm eff} &= \frac{\Gamma_{\rm c}-\gamma_{\rm m}}{\Gamma_{\rm c}-\Gamma_{\rm m}}\,\omega_{\rm m}^2 - \frac{\gamma_{\rm c}-\Gamma_{\rm c}}{\Gamma_{\rm c}-\Gamma_{\rm m}}\,\Delta^2 - (\gamma_{\rm c}-\Gamma_{\rm c})\,\frac{\Gamma_{\rm c}-\gamma_{\rm m}}{4}. \end{aligned} \tag{A.12}$$

The case $\Delta = \omega_{\rm m}$. An exact expression for $\Gamma_{\rm m}$ and $\Gamma_{\rm c}$ can be found when $\Delta = \omega_{\rm m}$. We study only the case of $d(\nu)$ of the form (A.10) with four distinct zeros.

In the case $\Gamma_{\rm c} \neq \Gamma_{\rm m}$ we set $x = \Gamma_{\rm c} - \Gamma_{\rm m}$ and insert (A.12) and $\Gamma_{\rm c} + \Gamma_{\rm m} = \gamma_{\rm c} + \gamma_{\rm m}$ into the last equation of the system (A.11); in such a way we get

$$x^4 + \left[16\omega_{\rm m}^2 - (\gamma_{\rm c}-\gamma_{\rm m})^2\right]x^2 + 64G^2\omega_{\rm m}^2 - 16\omega_{\rm m}^2(\gamma_{\rm c}-\gamma_{\rm m})^2 = 0.$$

Then, by using the solution of the equation for x^2 and Eqs. (A.12), we find the result (93)–(96). By imposing $\Gamma_{\rm c}$, $\Gamma_{\rm m}$, $\Delta_{\rm eff}^2$ to be real and strictly positive and $\Gamma_{\rm c} \neq \Gamma_{\rm m}$, we get the necessary and sufficient condition (92).

By the choice $\Gamma_{\rm c} = \Gamma_{\rm m}$, from the system (A.11) we get directly the result (98), (99), together with the conditions (97).

An approximate expression. To compute approximately $\Gamma_{\rm m}$ we adapt a suggestion given in [5, 25] and based on an approximation of the mechanical susceptibility. In the expression of $d(\nu_{\rm m})$ taken from (89) we make the approximation $\left(\nu_{\rm m}+\Delta+{\rm i}\frac{\gamma_{\rm c}}{2}\right)\left(\nu_{\rm m}-\Delta+{\rm i}\frac{\gamma_{\rm c}}{2}\right)\simeq \left(\omega_{\rm m}+\Delta+{\rm i}\,\frac{\gamma_{\rm c}-\gamma_{\rm m}}{2}\right)\left(\omega_{\rm m}-\Delta+{\rm i}\,\frac{\gamma_{\rm c}-\gamma_{\rm m}}{2}\right)$ and we solve $d(\nu_{\rm m})=0$ for $\Gamma_{\rm m}$ under the conditions (100), (101). By using also the first equation of the system (A.11) we get the expression (102) for the damping constants. Once we have $\Gamma_{\rm m}$ and $\Gamma_{\rm c}$, we can compute ${\omega^{\rm m}_{\rm eff}}^2$ and $\Delta_{\rm eff}^2$ from the equations (A.12), which do not contain approximations.

For consistency we need the positivity of $\Delta_{\rm eff}^2$ and ${\omega^{\rm m}_{\rm eff}}^2$, which means the positivity of the right hand sides of equations (A.12). Under the approximations (100)–(102), this gives

$$\begin{aligned}\frac{\Delta^2}{\omega_{\rm m}^2}&\gtrsim \frac{\chi(\Delta)}{1-\chi(\Delta)}+\chi(\Delta)\big(1-2\chi(\Delta)\big)\frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{4\omega_{\rm m}^2},\\ \chi(\Delta)&\left[\frac{\Delta^2}{\omega_{\rm m}^2}+\big(1-\chi(\Delta)\big)\big(1-2\chi(\Delta)\big)\frac{(\gamma_{\rm c}-\gamma_{\rm m})^2}{4\omega_{\rm m}^2}\right]\lesssim 1-\chi(\Delta);\end{aligned}\qquad\text{(A.13)}$$

because $\chi(\Delta)$ has the same sign as Δ, conditions (A.13) give true restrictions only for $\Delta>0$. We see also that conditions (A.13) are violated for Δ positive and small. In this situation the cavity is overdamped and the decomposition of $d(\nu)$ takes the form $d(\nu)=\left(\nu-\omega^{\rm m}_{\rm eff}+{\rm i}\,\frac{\Gamma_{\rm m}}{2}\right)\left(\nu+\omega^{\rm m}_{\rm eff}+{\rm i}\,\frac{\Gamma_{\rm m}}{2}\right)\left(\nu+{\rm i}\,\frac{\Gamma_1}{2}\right)\left(\nu+{\rm i}\,\frac{\Gamma_2}{2}\right)$; we do not study this case.

Appendix A.2. Computation of the mean mechanical energy

The integrals over ν in (107), (108) can be performed by the residue method, under the Ansatz (91) and $N(\nu)\equiv N_{\rm eff}$. First we set

$$\begin{aligned}D^2&=\left(\Delta_{\rm eff}^2+{\omega^{\rm m}_{\rm eff}}^2+\frac{(\gamma_{\rm c}+\gamma_{\rm m})^2}{4}\right)^2-4{\omega^{\rm m}_{\rm eff}}^2\Delta_{\rm eff}^2,\\ L_\pm&=\frac{\gamma_{\rm c}^2\mp\Gamma_{\rm c}^2}{4}-\Delta^2\pm\Delta_{\rm eff}^2,\qquad {\Omega^{\rm m}_{\rm eff}}^2={\omega^{\rm m}_{\rm eff}}^2+\Gamma_{\rm m}^2/4.\end{aligned}\qquad\text{(A.14)}$$

With this notation we have

$$\begin{aligned}\mathcal{N}_{\rm rp}=\frac{G^2\gamma_{\rm c}}{4\Gamma_{\rm m}\Gamma_{\rm c}D^2}\Bigg\{&\frac{G^2\omega_{\rm m}\Delta}{2\left|\nu_{\rm c}\right|^2\left|\nu_{\rm m}\right|^2}\left[\gamma_{\rm m}\Omega_{\rm m}^2+\gamma_{\rm c}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)+\gamma_{\rm m}\gamma_{\rm c}\left(\gamma_{\rm m}+\gamma_{\rm c}\right)\right]\\ &+\left(\Delta^2+\omega_{\rm m}^2+\frac{(\gamma_{\rm c}+\gamma_{\rm m})^2}{4}\right)(\gamma_{\rm c}+\gamma_{\rm m})\Bigg\},\end{aligned}\qquad\text{(A.15)}$$

where $\left|\nu_{\rm c}\right|^2\left|\nu_{\rm m}\right|^2$ is given by the last of (A.11). The thermal contributions $\mathcal{N}_{\rm th}$ and $\mathcal{M}_{\rm th}(\Delta)$ are given in (111) in terms of the expressions

$$\begin{aligned}\mathcal{Q}=\frac{\Omega_{\rm m}^2+{\Omega^{\rm m}_{\rm eff}}^2}{2{\Omega^{\rm m}_{\rm eff}}^2}+\frac{L_+}{2\Gamma_{\rm c}D^2}\Bigg\{&(\gamma_{\rm c}+\gamma_{\rm m})\,\frac{L_-+2\Omega_{\rm m}^2}{16}+2\gamma_{\rm c}\Omega_{\rm m}^2+2\gamma_{\rm m}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)\\ &+\frac{\Omega_{\rm m}^2L_-}{\left|\nu_{\rm c}\right|^2\left|\nu_{\rm m}\right|^2}\left[\gamma_{\rm c}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)+\gamma_{\rm m}\Omega_{\rm m}^2+\gamma_{\rm c}\gamma_{\rm m}\left(\gamma_{\rm c}+\gamma_{\rm m}\right)\right]\Bigg\},\end{aligned}\qquad\text{(A.16)}$$

$$\begin{aligned}\mathcal{K}=\frac{G^2\Delta}{2\Gamma_{\rm c}D^2}\Bigg\{&\frac{\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)\left(\frac{\gamma_{\rm m}^2}{4}-\omega_{\rm m}^2\right)}{2\omega_{\rm m}\left|\nu_{\rm c}\right|^2\left|\nu_{\rm m}\right|^2}\left[\gamma_{\rm m}\Omega_{\rm m}^2+\gamma_{\rm c}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)+\gamma_{\rm c}\gamma_{\rm m}\left(\gamma_{\rm c}+\gamma_{\rm m}\right)\right]\\ &+\omega_{\rm m}\left(\frac{\gamma_{\rm m}}{2}+\gamma_{\rm c}\right)-\frac{\gamma_{\rm c}\left(\Delta^2+\frac{\gamma_{\rm c}^2}{4}\right)+\gamma_{\rm m}\left(\frac{\gamma_{\rm m}}{2}+\gamma_{\rm c}\right)^2}{2\omega_{\rm m}}\Bigg\}.\end{aligned}\qquad\text{(A.17)}$$

Note that, while $\mathcal{Q}$ is always positive, $\mathcal{K}$ can also take on negative values.

Appendix B. Computation of the optical spectra

Appendix B.1. The homodyne spectrum

The homodyne spectrum (116) involves the quantity $Q_T^{\mathrm{out}}(\nu;\vartheta)$ (114); by the rules of quantum stochastic calculus we can compute $\mathrm{d}Q_T^{\mathrm{out}}(\nu;\vartheta)$, which turns out to contain the quantities (A.3), (A.4). Then, by integration we obtain

$$Q_T^{\mathrm{out}}(\nu;\vartheta) \simeq 4\sqrt{\gamma_{\mathrm{c}}}\,|\zeta|\sin\vartheta\;\mathrm{e}^{\mathrm{i}\nu T/2}\frac{\sin\nu T/2}{\nu\sqrt{T}} + Q_T^{\mathrm{th}}(\nu;\vartheta) + Q_T^{\mathrm{em}}(\nu;\vartheta), \tag{B.1}$$

$$Q_T^{\mathrm{th}}(\nu;\vartheta) = \overline{E_{\mathrm{th}}(\nu;\vartheta)\tau}\,\hat{B}_{\mathrm{th}}^T(\nu) + E_{\mathrm{th}}(-\nu;\vartheta)\tau\hat{B}_{\mathrm{th}}^T(-\nu)^\dagger,$$

$$\begin{aligned} Q_T^{\mathrm{em}}(\nu;\vartheta) = {}& -\overline{E_{\mathrm{em}}(\nu;\vartheta)}\,\mathrm{i}\mathrm{e}^{\mathrm{i}(\vartheta-\arg\zeta)}\hat{B}_{\mathrm{em}}^T(\nu+\omega_0) \\ &+ E_{\mathrm{em}}(-\nu;\vartheta)\mathrm{i}\mathrm{e}^{-\mathrm{i}(\vartheta-\arg\zeta)}\hat{B}_{\mathrm{em}}^T(\omega_0-\nu)^\dagger, \end{aligned}$$

$$E_{\mathrm{th}}(\nu;\vartheta) = -G\sqrt{\gamma_{\mathrm{m}}\gamma_{\mathrm{c}}}\left(\frac{\gamma_{\mathrm{m}}}{2} + \mathrm{i}\,(\nu+\omega_{\mathrm{m}})\right)L(\nu;\vartheta), \tag{B.2}$$

$$E_{\mathrm{em}}(\nu;\vartheta) = -\frac{\frac{\gamma_{\mathrm{c}}}{2} - \mathrm{i}\,(\nu-\Delta)}{\frac{\gamma_{\mathrm{c}}}{2} + \mathrm{i}\,(\nu-\Delta)} + \frac{\mathrm{i}\omega_{\mathrm{m}}\gamma_{\mathrm{c}}G^2\mathrm{e}^{\mathrm{i}\vartheta}L(\nu;\vartheta)}{\frac{\gamma_{\mathrm{c}}}{2} + \mathrm{i}\,(\nu-\Delta)}, \tag{B.3}$$

$$L(\nu;\vartheta) = \frac{\Delta\sin\vartheta + \left(\frac{\gamma_{\mathrm{c}}}{2} + \mathrm{i}\nu\right)\cos\vartheta}{d(-\nu)}. \tag{B.4}$$

Note that $L(-\nu;\vartheta) = \overline{L(\nu;\vartheta)}$. The key relation (115) together with $[\hat{B}_i^T(\nu),\,\hat{B}_i^T(\nu)^\dagger] = 1$ implies

$$[Q_T^{\mathrm{th}}(\nu;\vartheta) + Q_T^{\mathrm{em}}(\nu;\vartheta),\, Q_T^{\mathrm{th}}(-\nu;\vartheta) + Q_T^{\mathrm{em}}(-\nu;\vartheta)] = 0,$$

which is equivalent to

$$|E_{\mathrm{th}}(\nu;\vartheta)|^2 - |E_{\mathrm{th}}(-\nu;\vartheta)|^2 + |E_{\mathrm{em}}(\nu;\vartheta)|^2 - |E_{\mathrm{em}}(-\nu;\vartheta)|^2 = 0. \tag{B.5}$$

By long computations this relation can be verified also explicitly by using the expressions of $E_{\mathrm{th}}(\nu;\vartheta)$ and $E_{\mathrm{em}}(\nu;\vartheta)$.

By using (B.1), (B.5) and (A.9), from (116) we get the decomposition of the homodyne spectrum expressed by Eqs. (117), (118) with

$$S_{\mathrm{th}}(\nu;\vartheta) = |E_{\mathrm{th}}(\nu;\vartheta)|^2\left(N(\nu) + \frac{1}{2}\right) + |E_{\mathrm{th}}(-\nu;\vartheta)|^2\left(N(-\nu) + \frac{1}{2}\right),$$

$$S_{\mathrm{rp}}(\nu;\vartheta) = \frac{1}{2}\left(|E_{\mathrm{em}}(\nu;\vartheta)|^2 + |E_{\mathrm{em}}(-\nu;\vartheta)|^2\right). \tag{B.6}$$

Note that $S_{\mathrm{th}}(\nu;\vartheta) \geq 0$ and $S_{\mathrm{rp}}(\nu;\vartheta) \geq 0$. To compute the thermal part we note that $|E_{\mathrm{th}}(\nu;\vartheta)|^2$ can be written by using $\hat{R}(\nu)$ (54); by taking $S_q^{\mathrm{th}}(\nu)$ from (86), we get Eq. (119).

To compute the radiation pressure component of the spectrum, we need the square modulus of E_{em} (B.3), which is the sum of two terms. So, we have the square modulus of the first term (the shot noise), the square modulus of the second term (the signal) and the double product (the interference term):

$$\begin{aligned} |E_{\mathrm{em}}(\nu)|^2 = 1 + {}& \frac{\omega_{\mathrm{m}}^2\gamma_{\mathrm{c}}^2G^2\,|L(\nu;\vartheta)|^2}{\frac{\gamma_{\mathrm{c}}^2}{4} + (\nu-\Delta)^2} \\ &+ \omega_{\mathrm{m}}\gamma_{\mathrm{c}}G^2\,\mathrm{Re}\,\frac{\mathrm{i}\mathrm{e}^{-2\mathrm{i}\vartheta}\left(\frac{\gamma_{\mathrm{c}}}{2} - \mathrm{i}\,(\nu-\Delta)\right) + \mathrm{i}\left(\frac{\gamma_{\mathrm{c}}}{2} - \mathrm{i}\,(\nu+\Delta)\right)}{d(\nu)\left(\frac{\gamma_{\mathrm{c}}}{2} + \mathrm{i}\,(\nu-\Delta)\right)}. \end{aligned}$$

By inserting this expression into (B.6) we get

$$S_{\mathrm{rp}}(\nu;\vartheta) = 1 + \frac{\omega_{\mathrm{m}}^2\gamma_{\mathrm{c}}^2 G^4\left(\frac{\gamma_{\mathrm{c}}^2}{4}+\Delta^2+\nu^2\right)}{\left(\frac{\gamma_{\mathrm{c}}^2}{4}+(\Delta-\nu)^2\right)\left(\frac{\gamma_{\mathrm{c}}^2}{4}+(\Delta+\nu)^2\right)}\left|\frac{\Delta\sin\vartheta+\left(\frac{\gamma_{\mathrm{c}}}{2}+\mathrm{i}\nu\right)\cos\vartheta}{d(\nu)}\right|^2$$

$$+\,\omega_{\mathrm{m}}\gamma_{\mathrm{c}}G^2\left[\mathrm{Re}\,\frac{\mathrm{i}\mathrm{e}^{-2\mathrm{i}\vartheta}\left(\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\nu-\Delta\right)\right)^2+\mathrm{i}\left(\left(\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\nu\right)^2+\Delta^2\right)}{2d(\nu)\left(\frac{\gamma_{\mathrm{c}}^2}{4}+(\Delta-\nu)^2\right)}+(\nu\to-\nu)\right].$$

Finally, by elaborating the argument of the real part and by using the expression (85) for $S_q^{\mathrm{rp}}(\nu)$ we get Eq. (120).

Appendix B.2. The heterodyne spectrum

By a procedure similar to the one used in Appendix A and Appendix B.1, in the limit of $\varkappa\downarrow 0$, $\varkappa t\to+\infty$, we get

$$I_{\mathrm{inel}}(\nu;t)\simeq \mathrm{e}^{\mathrm{i}\alpha}\sqrt{\varkappa}\int_0^t \mathrm{e}^{-\frac{\varkappa}{2}(t-s)+\mathrm{i}\mu s}\Bigg\{\Bigg[-\frac{\frac{\gamma_{\mathrm{c}}}{2}+\mathrm{i}\left(\mu-\omega_0-\Delta\right)}{\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0-\Delta\right)}-\frac{\mathrm{i}\hbar g_0^2\gamma_{\mathrm{c}}\left|\zeta\right|^2}{md(\mu-\omega_0)}$$
$$\times\frac{\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0+\Delta\right)}{\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0-\Delta\right)}+\frac{\mathrm{i}\hbar g_0^2\gamma_{\mathrm{c}}\overline{\zeta}^2}{md(\omega_0-\mu)}\,\mathrm{e}^{-2\mathrm{i}(\mu-\omega_0)s-2\mathrm{i}\alpha}\Bigg]\mathrm{d}B_{\mathrm{em}}(s)$$
$$+\,\mathrm{i}\mathrm{e}^{-\mathrm{i}\omega_0 s}g_0\overline{\tau}\sqrt{\frac{\hbar\gamma_{\mathrm{m}}\gamma_{\mathrm{c}}}{2m\omega_{\mathrm{m}}}}\Bigg[\frac{\overline{\zeta}}{d(\omega_0-\mu)}\left(\frac{\gamma_{\mathrm{c}}}{2}+\mathrm{i}\left(\mu-\omega_0+\Delta\right)\right)$$
$$\times\left(\frac{\gamma_{\mathrm{m}}}{2}+\mathrm{i}\left(\mu-\omega_0-\omega_{\mathrm{m}}\right)\right)\mathrm{e}^{-2\mathrm{i}(\mu-\omega_0)s-2\mathrm{i}\alpha}-\frac{\zeta}{d(\mu-\omega_0)}$$
$$\times\left(\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0+\Delta\right)\right)\left(\frac{\gamma_{\mathrm{m}}}{2}-\mathrm{i}\left(\mu-\omega_0+\omega_{\mathrm{m}}\right)\right)\Bigg]\mathrm{d}B_{\mathrm{th}}(s)\Bigg\}+\text{h.c.}.$$

By using the field correlations (A.9) we can compute the heterodyne spectrum; again, by the vanishing of the field cross-correlations, the thermal contribution and the electromagnetic contributions decouple in the expression of the spectrum. By some long manipulations and by recalling that the limit in (127) is in the sense of distributions, we get Eq. (128) with the thermal part given by (130) and

$$\Sigma_{\mathrm{rp}}(\mu)=\left|1+\frac{\mathrm{i}\omega_{\mathrm{m}}\gamma_{\mathrm{c}}G^2}{2d(\mu-\omega_0)}\frac{\frac{\gamma_{\mathrm{c}}}{2}-\mathrm{i}\left(\mu-\omega_0+\Delta\right)}{\frac{\gamma_{\mathrm{c}}}{2}+\mathrm{i}\left(\mu-\omega_0-\Delta\right)}\right|^2+\frac{\omega_{\mathrm{m}}^2\gamma_{\mathrm{c}}^2G^4}{4\left|d(\mu-\omega_0)\right|^2}\,, \tag{B.7}$$

which becomes (129) by expanding the absolute value and using (110).

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