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HIGH ORDER DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PROBLEMS ON SURFACES*

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Abstract. We derive and analyze high order discontinuous Galerkin methods for second order elliptic problems on implicitly defined surfaces in $\mathbb{R}^3$. This is done by carefully adapting the unified discontinuous Galerkin framework of [D. N. Arnold et al., *SIAM J. Numer. Anal.*, 39 (2002), pp. 1749–1779] on a triangulated surface approximating the smooth surface. We prove optimal error estimates in both a (mesh dependent) energy and L^2 norms. Numerical results validating our theoretical estimates are also presented.

Key words. high order discontinuous Galerkin, surface partial differential equations, error analysis

AMS subject classifications. 65N30, 58J05, 65N15

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1. Introduction.

Partial differential equations (PDEs) on manifolds have become an active area of research in recent years due to the fact that, in many applications, mathematical models have to be formulated not on a flat Euclidean domain but on a curved surface. For example, they arise naturally in fluid dynamics (e.g., surface active agents on the interface between two fluids [32, 30]) and material science (e.g., diffusion of species along grain boundaries [16]) but have also emerged in other areas such as image processing (e.g., texture mapping and surface reconstruction [38, 42]) and cell biology (e.g., cell motility involving processes on the cell membrane [39, 1, 29] or phase separation on biomembranes [28]).

Finite element methods (FEMs) for elliptic problems and their error analysis have been successfully applied to problems on surfaces via the intrinsic approach in [24]. This approach has subsequently been extended to parabolic problems [26] as well as evolving surfaces [25]. The literature on the application of FEMs to various surface PDEs is now quite extensive, a review of which can be found in [27]. High order error estimates, which require high order surface approximations, have been derived in [21] for the Laplace–Beltrami operator. However, there are a number of situations where conforming FEMs may not be the appropriate numerical method, for instance, problems which lead to steep gradients or even discontinuities in the solution. Such issues can arise for problems posed on surfaces, as in [43] where the authors analyze a model for bacteria/cell aggregation. Without an appropriate stabilization mechanism artificially added to the surface FEM scheme, the solution can exhibit a spurious oscillatory behavior which, in the context of the above problem, leads to negative densities of on-surface living cells.

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Given the ease with which one can perform hp-adaptivity using high order discontinuous Galerkin (DG) methods and its in-built stabilization mechanisms for dealing with advection dominated problems and solution blowups, it is natural to extend the DG framework for PDEs posed on surfaces. DG methods have first been extended to surfaces in [20], where an interior penalty (IP) method for a linear second-order elliptic problem was introduced and optimal a priori error estimates in the L^2 and energy norms for piecewise linear ansatz functions and surface approximations were derived. A posteriori error estimates have then been derived for this surface IP method in [17], and extensions of the analysis to the advection-diffusion setting have recently been discussed in [18] and [37]. A continuous/discontinuous Galerkin method for a fourth order elliptic PDE on surfaces is considered in [35] and an isogeometric analysis of a DG method for elliptic PDEs on surfaces has been considered in [34, 33, 36, 31] have also derived a priori error bounds for finite volume methods on (evolving) surfaces via the intrinsic approach.

In this paper, we consider a second order elliptic equation on a compact smooth, connected, and oriented surface $\Gamma \subset \mathbb{R}^3$ and, following the unified framework of [4] based on the so-called flux formulation and extending to the nonconforming framework the high order surface approximation approach considered in [21], derive the high order DG formulation on a piecewise polynomial approximation Γ_h^k of Γ, where $k \geq 1$ is the polynomial order of the approximation. Then, by choosing the numerical fluxes appropriately, we derive "surface" counterparts of the various planar DG bilinear forms discussed in [4].

We then perform a unified a priori error analysis of the surface *DG* methods and derive estimates in the L^2 and energy norms by relating Γ_h^k to Γ via the surface lifting operator introduced in [24]. The estimates are a generalization of the a priori error estimates derived in [20] for the surface IP method, which restricted the analysis to the linear case. The geometric error terms arising when approximating the surface involve those present for the surface FEM method given in [21] as well as additional terms arising from the DG methods. The latter are shown to scale with the same order as the former and hence we obtain optimal convergence rates as long as the surface approximation order and the *DG* space order coincide.

The paper is organized in the following way. Section 2 presents the model problem which we investigate, following the approach taken in [24]. In section 3 we present a unified framework for high order DG methods on surfaces and derive the bilinear forms corresponding to each of the classical DG methods outlined in [4]. In section 4 we describe the technical estimates needed to prove the convergence of the surface DG methods, which is then reported in section 5. Section 6 presents some numerical results. Finally, Appendix A contains the proof of a technical result needed in our analysis.

2. Model problem. The notation in this section closely follows that used in [24]. Let Γ be a compact, oriented, C^∞, two dimensional surface without boundary which is embedded in $\mathbb{R}^3$, and let $d(\cdot)$ denote the signed distance function to Γ which we assume to be well-defined in a sufficiently thin open tube U around Γ. The orientation of Γ is set by taking the normal ν of Γ to be in the direction of increasing $d(\cdot)$, i.e.,

$$\nu(\xi) = \nabla d(\xi),\ \ \xi \in \Gamma.$$

We denote by $\pi(\cdot)$ the projection onto Γ, i.e., $\pi : U \to \Gamma$ is given by

$$\pi(x) = x - d(x)\nu(x), \quad \text{where } \nu(x) = \nu(\pi(x)). \tag{2.1}$$

In the following, we assume that there is a one-to-one relation between points $x \in U$ and points $\xi = \pi(x) \in \Gamma$. In particular, (2.1) is invertible in U. We denote by

$$P(\xi) = I - \nu(\xi) \otimes \nu(\xi), \ \xi \in \Gamma,$$

the projection onto the tangent space $T_\xi\Gamma$ on Γ at a point $\xi \in \Gamma$, where $\otimes$ denotes the usual tensor product.

Remark 2.1. It is easy to see that

$$\nabla \pi = P - dH, \tag{2.2}$$

where $H = \nabla^2 d$ [24, Lemma 3].

For any function η defined in an open subset of U containing Γ we define its *tangential gradient* on Γ by

$$\nabla_\Gamma \eta = \nabla \eta - (\nabla \eta \cdot \nu)\, \nu = P \nabla \eta,$$

and the *Laplace–Beltrami* operator by

$$\Delta_\Gamma \eta = \nabla_\Gamma \cdot (\nabla_\Gamma \eta).$$

For an integer $m \geq 0$, we define the surface Sobolev space $H^m(\Gamma) = \{u \in L^2(\Gamma) : D^\alpha u \in L^2(\Gamma)\ \forall |\alpha| \leq m\}$. We endow the Sobolev space with the standard seminorm and norm

$$|u|_{H^m(\Gamma)} = \left(\sum_{|\alpha|=m} \|D^\alpha u\|^2_{L^2(\Gamma)} \right)^{1/2}, \quad \|u\|_{H^m(\Gamma)} = \left(\sum_{k=0}^{m} |u|^2_{H^k(\Gamma)} \right)^{1/2},$$

respectively; cf [44]. Throughout the paper, we write $x \lesssim y$ to signify $x < Cy$, where C is a generic positive constant whose value, possibly different at any occurrence, does not depend on the mesh size. Moreover, we use $x \sim y$ to state the equivalence between x and y, i.e., $C_1 y \leq x \leq C_2 y$, for $C_1,\ C_2$ independent of the mesh size.

Let $f \in L^2(\Gamma)$ be a given function, we consider the following model problem: Find $u \in H^1(\Gamma)$ such that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v + uv \ \mathrm{dA} = \int_\Gamma fv \ \mathrm{dA} \qquad \forall v \in H^1(\Gamma). \tag{2.3}$$

We denote by, respectively, dA and ds the two and one dimensional surface measures over Γ. Throughout the paper, we assume that $u \in H^s(\Gamma)$, $s \geq 2$. Existence, uniqueness, and regularity of such a solution are shown in [5].

3. High order DG approximation. We now follow the high order surface approximation framework introduced in [21]. We begin by approximating the smooth surface Γ by a polyhedral surface $\Gamma_h \subset U$ composed of planar triangles $\widetilde{K}_h$ whose vertices lie on Γ, and denote by $\widetilde{\mathcal{T}}_h$ the associated regular, conforming triangulation of Γ_h, i.e., $\Gamma_h = \bigcup_{\widetilde{K}_h \in \widetilde{\mathcal{T}}_h} \widetilde{K}_h$.

We next describe a family Γ_h^k of polynomial approximations to Γ of degree $k \geq 1$ (with the convention that $\Gamma_h^1 = \Gamma_h$). For a given element $\widetilde{K}_h \in \widetilde{\mathcal{T}}_h$, let $\{\phi_i^k\}_{1 \leq i \leq n_k}$ be the Lagrange basis functions of degree k defined on $\widetilde{K}_h$ corresponding to a set of

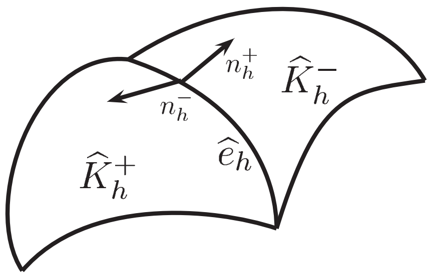

FIG. 1. *Example of two elements in* $\widehat{\mathcal{T}}_h$ *and their respective conormals on the common edge* $\widehat{e}_h$.

nodal points $x_1, \ldots, x_{n_k}$. For $x \in \widetilde{K}_h$, we define the discrete projection $\pi_k : \Gamma_h \to U$ as

$$\pi_k(x) = \sum_{j=1}^{n_k} \pi(x_j)\phi_j^k(x).$$

By constructing π_k elementwise we obtain a continuous piecewise polynomial map on Γ_h. We then define the corresponding discrete surface $\Gamma_h^k = \{\pi_k(x) : x \in \Gamma_h\}$ and the corresponding regular, conforming triangulation $\widehat{\mathcal{T}}_h = \{\pi_k(\widetilde{K}_h)\}_{\widetilde{K}_h \in \widetilde{\mathcal{T}}_h}$. We denote by $\widehat{\mathcal{E}}_h$ the set of all (codimension one) intersections $\widehat{e}_h$ of elements in $\widehat{\mathcal{T}}_h$, i.e., $\widehat{e}_h = \widehat{K}_h^+ \cap \widehat{K}_h^-$, for some elements $\widehat{K}_h^\pm \in \widehat{\mathcal{T}}_h$. Furthermore, we denote by $h_{\widehat{e}_h}$ the length of the edge $\widehat{e}_h \in \widehat{\mathcal{E}}_h$. For any $\widehat{e}_h \in \widehat{\mathcal{E}}_h$, the conormal n_h^+ to a point $x \in \widehat{e}_h$ is the unique unit vector that belongs to $T_x\widehat{K}_h^+$ and that satisfies

$$n_h^+(x) \cdot (x - y) \geq 0 \quad \forall y \in \widehat{K}_h^+ \cap B_\epsilon(x),$$

where $B_\epsilon(x)$ is the ball centered in x with (small enough) radius $\epsilon > 0$. Analogously, one can define the conormal n_h^- on $\widehat{e}_h$ by exchanging $\widehat{K}_h^+$ with $\widehat{K}_h^-$. It is important to notice that, with the above definition,

$$n_h^+ \neq -n_h^-$$

in general and independently of the surface approximation k (see Figure 1). Finally, we denote by ν_h the outward unit normal to Γ_h^k and define for each $\widehat{K}_h \in \widehat{\mathcal{T}}_h$ the discrete projection P_h onto the tangential space of Γ_h^k by

$$P_h(x) = I - \nu_h(x) \otimes \nu_h(x), \ x \in \widehat{K}_h,$$

so that, for v_h defined on Γ_h^k,

$$\nabla_{\Gamma_h^k} v_h = P_h \nabla v_h.$$

Let $K \subset \mathbb{R}^2$ be the (flat) reference element and let $F_{\widehat{K}_h} = \pi_k \circ F_{\widetilde{K}_h} : K \to \widehat{K}_h \subset \mathbb{R}^3$ for $\widehat{K}_h \in \widehat{\mathcal{T}}_h$, where $F_{\widetilde{K}_h} : K \to \widetilde{K}_h$ is the classical affine map from the reference element K to $\widetilde{K}_h$. We define the isoperimetric DG space associated with Γ_h^k by

$$\widehat{S}_{hk} = \left\{ \widehat{\chi} \in L^2(\Gamma_h^k) : \widehat{\chi}|_{\widehat{K}_h} = \chi \circ F_{\widehat{K}_h}^{-1} \text{ for some } \chi \in \mathbb{P}^k(K) \quad \forall \widehat{K}_h \in \widehat{\mathcal{T}}_h \right\}.$$

For $v_h \in \widehat{S}_{hk}$ we adopt the convention that $v_h^{\pm}$ is the trace of v_h on $\widehat{e}_h = \widehat{K}_h^+ \cap \widehat{K}_h^-$ taken within the interior of $\widehat{K}_h^{\pm}$, respectively. In addition, we define the vector-valued function space

$$\widehat{\Sigma}_{hk} = \left\{ \widehat{\tau} \in [L^2(\Gamma_h^k)]^3 : \widehat{\tau}|_{\widehat{K}_h} = \nabla F_{\widehat{K}_h}^{-T} \left(\tau \circ F_{\widehat{K}_h}^{-1} \right) \text{ for some } \tau \in [\mathbb{P}^k(K)]^2 \quad \forall \widehat{K}_h \in \widehat{\mathcal{T}}_h \right\}.$$

Here, $\nabla F_{\widehat{K}_h}^{-1}$ refers to the (left) *pseudoinverse* of $\nabla F_{\widehat{K}_h}$, i.e.,

$$\nabla F_{\widehat{K}_h}^{-1} = \left(\nabla F_{\widehat{K}_h}^T \nabla F_{\widehat{K}_h} \right)^{-1} \nabla F_{\widehat{K}_h}^T.$$

Note that $P_h \nabla F_{\widehat{K}_h}^{-T} = \nabla F_{\widehat{K}_h}^{-T}$, i.e., $\widehat{\tau} \in \widehat{\Sigma}_{hk} \Rightarrow \widehat{\tau} \in T_x \Gamma_h^k$ almost everywhere. This result straightforwardly implies that $\eta \in \widehat{S}_{hk} \Rightarrow \nabla_{\Gamma_h^k} \eta \in \widehat{\Sigma}_{hk}$. Indeed, by the definition of $\widehat{S}_{hk}$ there exists $\chi \in \mathbb{P}^k(K)$ such that $\eta = \chi \circ F_{\widehat{K}_h}^{-1}$ and it holds

$$\nabla_{\Gamma_h^k} \eta = P_h \nabla \left(\chi \circ F_{\widehat{K}_h}^{-1} \right) = P_h \nabla F_{\widehat{K}_h}^{-T} \left(\nabla \chi \circ F_{\widehat{K}_h}^{-1} \right) = \nabla F_{\widehat{K}_h}^{-T} \left(\nabla \chi \circ F_{\widehat{K}_h}^{-1} \right). \tag{3.1}$$

Then, the result follows by taking $\tau = \nabla \chi$ in (3.1).

3.1. Primal formulation. Rewriting (2.3) as a first order system of equations and following the lines of [4], we wish to find $(u_h, \sigma_h) \in \widehat{S}_{hk} \times \widehat{\Sigma}_{hk}$ such that

$$\int_{\widehat{K}_h} \sigma_h \cdot w_h \, \mathrm{dA_{hk}} = -\int_{\widehat{K}_h} u_h \nabla_{\Gamma_h^k} \cdot w_h \, \mathrm{dA_{hk}} + \int_{\partial \widehat{K}_h} \widehat{u} \, w_h \cdot n_{\widehat{K}_h} \, \mathrm{ds_{hk}},$$

$$\int_{\widehat{K}_h} \sigma_h \cdot \nabla_{\Gamma_h^k} v_h + u_h v_h \, \mathrm{dA_{hk}} = \int_{\widehat{K}_h} f_h v_h \, \mathrm{dA_{hk}} + \int_{\partial \widehat{K}_h} \widehat{\sigma} \cdot n_{\widehat{K}_h} \, v_h \, \mathrm{ds_{hk}}$$

for all $w_h \in \widehat{\Sigma}_{hk}$, $v_h \in \widehat{S}_{hk}$, where $\mathrm{dA_{hk}}$ and $\mathrm{ds_{hk}}$ denote the two and one dimensional surface measures over Γ_h^k, respectively, and the discrete right-hand side $f_h \in L^2(\Gamma_h^k)$ will be related to f in section 4.1. Here $\widehat{u} = \widehat{u}(u_h)$ and $\widehat{\sigma} = \widehat{\sigma}(u_h, \sigma_h(u_h))$ are the so-called numerical fluxes which determine the interelement behavior of the solution and will be prescribed later on. In order to deal with these terms, we need to introduce the following discrete surface trace operators.

DEFINITION 3.1. *Suppose there is an element numbering for all $\widehat{K}_h \in \widehat{\mathcal{T}}_h$. For $q \in \Pi_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} L^2(\partial \widehat{K}_h)$, $\{q\}$ and $[q]$ are given by*

$$\{q\} := \frac{1}{2}(q^+ + q^-), \ [q] := q^+ - q^- \ \text{ on } \widehat{e}_h \in \widehat{\mathcal{E}}_h.$$

For $\phi, \tilde{n} \in [\Pi_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} L^2(\partial \widehat{K}_h)]^3$, $\{\phi; \tilde{n}\}$ and $[\phi; \tilde{n}]$ are given by

$$\{\phi; \tilde{n}\} := \frac{1}{2}(\phi^+ \cdot \tilde{n}^+ - \phi^- \cdot \tilde{n}^-), \quad [\phi; \tilde{n}] := \phi^+ \cdot \tilde{n}^+ + \phi^- \cdot \tilde{n}^- \quad \text{on } \widehat{e}_h \in \widehat{\mathcal{E}}_h. \tag{3.2}$$

We now state a useful formula which holds for functions in

$$H^1(\widehat{\mathcal{T}}_h) = \left\{ v|_{\widehat{K}_h} \in H^1(\widehat{K}_h) \ : \ \forall \widehat{K}_h \in \widehat{\mathcal{T}}_h \right\}.$$

Its proof is straighforward and therefore is omitted.

LEMMA 3.2. *Let $\phi \in [H^1(\widehat{\mathcal{T}}_h)]^3$ and $\psi \in H^1(\widehat{\mathcal{T}}_h)$. Then we have that*

$$\sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} \int_{\partial \widehat{K}_h} \psi \phi \cdot n_{\widehat{K}_h} \, ds_{hk} = \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h} \int_{\widehat{e}_h} [\phi; n_h]\{\psi\} + \{\phi; n_h\}[\psi] \, ds_{hk}.$$

We then prooced as in [4] and integrate again by parts the first equation, sum over all elements, and apply Lemma 3.2. We then obtain

$$\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\sigma_h\cdot w_h \ \ \mathrm{dA_{hk}}$$
$$=\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\nabla_{\Gamma_h^k}u_h\cdot w_h \ \ \mathrm{dA_{hk}}$$
$$+\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}[\widehat{u}-u_h]\{w_h;n_h\}+\{\widehat{u}-u_h\}[w_h;n_h] \ \ \mathrm{ds_{hk}}, \tag{3.3}$$
$$\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\sigma_h\cdot\nabla_{\Gamma_h^k}v_h+u_hv_h \ \ \mathrm{dA_{hk}}$$
$$=\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}f_hv_h \ \ \mathrm{dA_{hk}}$$
$$+\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\Big(\{\widehat{\sigma};n_h\}[v_h]+[\widehat{\sigma};n_h]\{v_h\}\Big) \ \ \mathrm{ds_{hk}} \tag{3.4}$$

for every $w_h\in\widehat{\Sigma}_{hk}$ and $v_h\in\widehat{S}_{hk}$.

We now introduce the local DG lifting operators $r_{\widehat{e}_h}:L^2(\widehat{e}_h)\to\widehat{\Sigma}_{hk}$ and $l_{\widehat{e}_h}:L^2(\widehat{e}_h)\to\widehat{\Sigma}_{hk}$ which satisfy

$$\int_{\Gamma_h^k}r_{\widehat{e}_h}(\phi)\cdot\tau_h\ \mathrm{dA_{hk}}=-\int_{\widehat{e}_h}\phi\{\tau_h;n_h\}\ \mathrm{ds_{hk}}\quad\forall\tau_h\in\widehat{\Sigma}_{hk},$$
$$\int_{\Gamma_h^k}l_{\widehat{e}_h}(q)\cdot\tau_h\ \mathrm{dA_{hk}}=-\int_{\widehat{e}_h}q[\tau_h;n_h]\ \mathrm{ds_{hk}}\quad\forall\tau_h\in\widehat{\Sigma}_{hk}.$$

The existence of such operators follows from standard arguments. Moreover, notice that for any edge $\widehat{e}_h$, the support of the operators $r_{\widehat{e}_h}(\cdot)$ and $l_{\widehat{e}_h}(\cdot)$ is confined to the two neighboring elements sharing the edge $\widehat{e}_h$. We then set $r_h:L^2(\widehat{\mathcal{E}}_h)\to\widehat{\Sigma}_{hk}$ and $l_h:L^2(\widehat{\mathcal{E}}_h)\to\widehat{\Sigma}_{hk}$, given by

$$r_h(\phi)=\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}r_{\widehat{e}_h}(\phi),\qquad l_h(\phi)=\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}l_{\widehat{e}_h}(\phi).$$

Using these, we can write σ_h solely in terms of u_h. Indeed, on each element $\widehat{K}_h\in\widehat{\mathcal{T}}_h$ we obtain from (3.3) that

$$\sigma_h=\sigma_h(u_h)=\nabla_{\Gamma_h^k}u_h-r_h([\widehat{u}(u_h)-u_h])-l_h(\{\widehat{u}(u_h)-u_h\}). \tag{3.5}$$

Note that (3.5) does in fact imply that $\sigma_h\in\widehat{\Sigma}_{hk}$ as $\nabla_{\Gamma_h^k}u_h\in\widehat{\Sigma}_{hk}$ and $r_h,l_h\in\widehat{\Sigma}_{hk}$ by construction. Taking $w_h=\nabla_{\Gamma_h^k}v_h$ in (3.3), substituting the resulting expression into (3.4), and using (3.5), we obtain the primal formulation: Find $(u_h,\sigma_h)\in\widehat{S}_{hk}\times\widehat{\Sigma}_{hk}$ such that

$$\mathcal{A}_h^k(u_h,v_h)=\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}f_hv_h\ \mathrm{dA_{hk}}\quad\forall v_h\in\widehat{S}_{hk}, \tag{3.6}$$

where

$$\mathcal{A}_h^k(u_h,v_h) = \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h} \nabla_{\Gamma_h^k} u_h\cdot\nabla_{\Gamma_h^k} v_h + u_h v_h \ \ \mathrm{dA_{hk}} + \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} ([\widehat{u}-u_h]\{\nabla_{\Gamma_h^k} v_h; n_h\} - \{\widehat{\sigma}; n_h\}[v_h]) \ \ \mathrm{ds_{hk}} + \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} (\{\widehat{u}-u_h\}[\nabla_{\Gamma_h^k} v_h; n_h] - [\widehat{\sigma}; n_h]\{v_h\}) \ \ \mathrm{ds_{hk}}. \tag{3.7}$$

3.2. Examples of surface DG methods. For the following methods we introduce the penalization coefficients $\eta_{\widehat{e}_h}$ and $\beta_{\widehat{e}_h}$ defined as

$$\eta_{\widehat{e}_h} = \alpha, \quad \beta_{\widehat{e}_h} = \alpha k^2 h_{\widehat{e}_h}^{-1}, \tag{3.8}$$

where $\alpha > 0$ is a parameter at our disposal.

3.2.1. Surface Bassi–Rebay method. To derive the surface Bassi–Rebay method, based on [7], we choose

$$\widehat{u}^+ = \{u_h\}, \qquad \widehat{u}^- = \{u_h\},$$
$$\widehat{\sigma}^+ = \{\sigma_h; n_h\}n_h^+, \quad \widehat{\sigma}^- = -\{\sigma_h; n_h\}n_h^-.$$

By (3.5) we obtain $\sigma_h = \nabla_{\Gamma_h^k} u_h + r_h([u_h])$. From the definition (3.2) we have

$$\{\widehat{\sigma}; n_h\} = \{\sigma_h; n_h\} = \{\nabla_{\Gamma_h^k} u_h; n_h\} + \{r_h([u_h]); n_h\},$$

which implies

$$\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} \{\widehat{\sigma}; n_h\}[v_h] \ \mathrm{ds_{hk}} = \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} \{\nabla_{\Gamma_h^k} u_h; n_h\}[v_h] \ \mathrm{ds_{hk}} + \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} \{r_h([u_h]); n_h\}[v_h] \ \mathrm{ds_{hk}} = \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} \{\nabla_{\Gamma_h^k} u_h; n_h\}[v_h] \ \mathrm{ds_{hk}} - \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h} r_h([u_h])\cdot r_h([v_h]) \ \mathrm{dA_{hk}}.$$

Therefore, making use of the fact that $\{\widehat{u}-u_h\} = 0$, $[\widehat{u}-u_h] = [u_h]$, and $[\widehat{\sigma}; n_h] = 0$, we have that

$$\mathcal{A}_h^k(u_h,v_h) = \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h} \left(\nabla_{\Gamma_h^k} u_h\cdot\nabla_{\Gamma_h^k} v_h + u_h v_h + r_h([u_h])\cdot r_h([v_h])\right) \mathrm{dA_{hk}} - \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h} \left(\{\nabla_{\Gamma_h^k} u_h; n_h\}[v_h] + \{\nabla_{\Gamma_h^k} v_h; n_h\}[u_h]\right) \mathrm{ds_{hk}}. \tag{3.9}$$

3.2.2. Surface Brezzi et al. method. For the surface Brezzi et al. method, based on [12], we choose

$$\widehat{u}^+ = \{u_h\}, \qquad \widehat{u}^- = \{u_h\},$$
$$\widehat{\sigma}^+ = \{\sigma_h + \eta_{\widehat{e}_h} r_{\widehat{e}_h}([u_h]); n_h\}n_h^+, \quad \widehat{\sigma}^- = -\{\sigma_h + \eta_{\widehat{e}_h} r_{\widehat{e}_h}([u_h]); n_h\}n_h^-.$$

The method is similar to that of Bassi–Rebay, but with an additional term. Indeed,

$$\begin{aligned}
\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{\widehat{\sigma};n_h\}[v_h]\ \mathrm{ds_{hk}}
&=\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{\sigma_h+\eta_{\widehat{e}_h}r_{\widehat{e}_h}([u_h]);n_h\}[v_h]\ \mathrm{ds_{hk}}\\
&=\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{\nabla_{\Gamma_h^k}u_h;n_h\}[v_h]\ \mathrm{ds_{hk}}\\
&\quad+\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{r_h([u_h])+\eta_{\widehat{e}_h}r_{\widehat{e}_h}([u_h]);n_h\}[v_h]\ \mathrm{ds_{hk}}\\
&=\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{\nabla_{\Gamma_h^k}u_h;n_h\}[v_h]\ \mathrm{ds_{hk}}-\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}r_h([u_h])\cdot r_h([v_h])\ \mathrm{dA_{hk}}\\
&\quad-\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\eta_{\widehat{e}_h}r_{\widehat{e}_h}([u_h])\cdot r_{\widehat{e}_h}([v_h])\ \mathrm{dA_{hk}}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{A}_h^k(u_h,v_h)=&\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\nabla_{\Gamma_h^k}u_h\cdot\nabla_{\Gamma_h^k}v_h+u_hv_h\ \mathrm{dA_{hk}}\\
&-\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\{\nabla_{\Gamma_h^k}u_h;n_h\}[v_h]+\{\nabla_{\Gamma_h^k}v_h;n_h\}[u_h]\ \mathrm{ds_{hk}}\\
&+\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}r_h([u_h])\cdot r_h([v_h])+\eta_{\widehat{e}_h}r_{\widehat{e}_h}([u_h])\cdot r_{\widehat{e}_h}([v_h])\ \mathrm{dA_{hk}}.
\end{aligned}\tag{3.10}$$

3.2.3. Surface IP method. To derive the surface IP method, based on [23, 6, 3], we choose the numerical fluxes $\widehat{u}$ and $\widehat{\sigma}$ as follows:

$$\widehat{u}^+=\{u_h\},\qquad \widehat{u}^-=\{u_h\},$$
$$\widehat{\sigma}^+=\Big(\{\nabla_{\Gamma_h^k}u_h;n_h\}-\beta_{\widehat{e}_h}[u_h]\Big)n_h^+,\quad \widehat{\sigma}^-=-\Big(\{\nabla_{\Gamma_h^k}u_h;n_h\}-\beta_{\widehat{e}_h}[u_h]\Big)n_h^-.$$

Substituting them into (3.7), we obtain

$$\begin{aligned}
\mathcal{A}_h^k(u_h,v_h)=&\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h}\nabla_{\Gamma_h^k}u_h\cdot\nabla_{\Gamma_h^k}v_h+u_hv_h\ \ \mathrm{dA_{hk}}+\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\beta_{\widehat{e}_h}[u_h][v_h]\ \ \mathrm{ds_{hk}}\\
&-\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}\big([u_h]\{\nabla_{\Gamma_h^k}v_h;n_h\}+[v_h]\{\nabla_{\Gamma_h^k}u_h;n_h\}\big)\ \mathrm{ds_{hk}}
\end{aligned}\tag{3.11}$$

which is exactly the surface IP method considered in [20].

3.2.4. Surface nonsymmetric IP Galerkin (NIPG) method. For the surface NIPG method, based on [41] (or equivalently the Baumann–Oden method in [10] with $\beta_{\widehat{e}_h}=0$), we choose

$$\widehat{u}^+=\{u_h\}+[u_h],\qquad \widehat{u}^-=\{u_h\}-[u_h],$$
$$\widehat{\sigma}^+=\Big(\{\nabla_{\Gamma_h^k}u_h;n_h\}-\beta_{\widehat{e}_h}[u_h]\Big)n_h^+,\quad \widehat{\sigma}^-=-\Big(\{\nabla_{\Gamma_h^k}u_h;n_h\}-\beta_{\widehat{e}_h}[u_h]\Big)n_h^-.$$

We may derive the surface NIPG bilinear form in a similar way as for the surface IP method.

3.2.5. Surface incomplete IPG (IIPG) method. For the surface IIPG method, based on [15], we choose the numerical fluxes $\widehat{u}$ and $\widehat{\sigma}$ as follows:

$$\widehat{u}^+ = u_h^+, \qquad \widehat{u}^- = u_h^-,$$
$$\widehat{\sigma}^+ = \Big(\{\nabla_{\Gamma_h^k} u_h; n_h\} - \beta_{\widehat{e}_h}[u_h]\Big)n_h^+, \quad \widehat{\sigma}^- = -\Big(\{\nabla_{\Gamma_h^k} u_h; n_h\} - \beta_{\widehat{e}_h}[u_h]\Big)n_h^-.$$

Here again, we may derive the surface IIPG bilinear form in a similar way as for the surface IP method.

3.2.6. Surface Bassi et al. method. For the surface Bassi et al. method, based on [8], we choose

$$\widehat{u}^+ = \{u_h\}, \qquad \widehat{u}^- = \{u_h\},$$
$$\widehat{\sigma}^+ = \Big(\{\nabla_{\Gamma_h^k} u_h + \eta_{\widehat{e}_h} r_{\widehat{e}_h}([u_h]); n_h\}\Big)n_h^+, \quad \widehat{\sigma}^- = -\Big(\{\nabla_{\Gamma_h^k} u_h + \eta_{\widehat{e}_h} r_{\widehat{e}_h}([u_h]); n_h\}\Big)n_h^-.$$

The resulting bilinear surface form can be easily obtained using the contributes of the surface IP and surface Brezzi et al. bilinear forms.

3.2.7. Surface local DG (LDG) method. Finally for the surface LDG method, based on [14], the numerical fluxes are chosen as follows:

$$\widehat{u}^+ = \{u_h\} - \theta \cdot n_h^+[u_h], \quad \widehat{u}^- = \{u_h\} - \theta \cdot n_h^+[u_h],$$
$$\widehat{\sigma}^+ = \Big(\{\sigma_h; n_h\} - \beta_{\widehat{e}_h}[u_h] + \theta \cdot n_h^+[\sigma_h; n_h]\Big)n_h^+,$$
$$\widehat{\sigma}^- = -\Big(\{\sigma_h; n_h\} - \beta_{\widehat{e}_h}[u_h] + \theta \cdot n_h^+[\sigma_h; n_h]\Big)n_h^-,$$

where θ is a (possibly null) uniformly bounded vector of $\mathbb{R}^3$ that does not depend on the discretization parameters. We see that $\{\widehat{u}-u_h\} = -\theta\cdot n_h^+[u_h]$ and $[\widehat{u}-u_h] = -[u_h]$. So, from (3.5), we obtain

$$\widehat{\sigma}^+ = \Big(\{\nabla_{\Gamma_h^k} u_h; n_h\} + \{r_h([u_h]); n_h\} + \{\theta \cdot n_h^+ l_h([u_h]); n_h\} - \beta_{\widehat{e}_h}[u_h]$$
$$+ \theta \cdot n_h^+\Big([\nabla_{\Gamma_h^k} u_h; n_h] + [r_h([u_h]); n_h] + [\theta \cdot n_h^+ l_h([u_h]); n_h]\Big)\Big)n_h^+,$$

and in a similar way $\widehat{\sigma}^-$. Then

$$\sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h^k} \int_{\widehat{e}_h} \{\widehat{\sigma}; n_h\}[v_h]\, \mathrm{ds_{hk}}$$
$$= \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h^k} \int_{\widehat{e}_h} \Big(\{\nabla_{\Gamma_h^k} u_h; n_h\}[v_h] + [\nabla_{\Gamma_h^k} u_h; n_h]\theta \cdot n_h^+[v_h] - \beta_{\widehat{e}_h}[u_h][v_h]\Big)\, \mathrm{ds_{hk}}$$
$$- \sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} \int_{\widehat{K}_h} \Big(r_h([u_h]) + \theta \cdot n_h^+ l_h\big([u_h]\big)\Big) \cdot \Big(r_h([v_h]) + \theta \cdot n_h^+ l_h\big([v_h]\big)\Big)\, \mathrm{dA_{hk}},$$

and the surface LDG form can be written as

$$
\begin{aligned}
&\mathcal{A}_h^k(u_h, v_h) \\
&\quad = \sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} \int_{\widehat{K}_h} \nabla_{\Gamma_h^k} u_h \cdot \nabla_{\Gamma_h^k} v_h + u_h v_h \, \mathrm{dA_{hk}} \\
&\qquad - \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h} \int_{\widehat{e}_h} [u_h]\{\nabla_{\Gamma_h^k} v_h; n_h\} - \{\nabla_{\Gamma_h^k} u_h; n_h\}[v_h] \, \mathrm{ds_{hk}} \\
&\qquad + \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h} \int_{\widehat{e}_h} \Big(- [\nabla_{\Gamma_h^k} u_h; n_h]\theta \cdot n_h^+ [v_h] \\
&\qquad\qquad - \theta \cdot n_h^+ [u_h][\nabla_{\Gamma_h^k} v_h; n_h] + \beta_{\widehat{e}_h}[u_h][v_h] \Big) \, \mathrm{ds_{hk}} \\
&\qquad + \sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} \int_{\widehat{K}_h} \Big(r_h([u_h]) + \theta \cdot n_h^+ l_h\big([u_h]\big) \Big) \cdot \Big(r_h([v_h]) + \theta \cdot n_h^+ l_h\big([v_h]\big) \Big) \, \mathrm{dA_{hk}}.
\end{aligned} \tag{3.12}
$$

Remark 3.3. In the flat case, for which we have $n_h^+ = -n_h^-$, all of the surface DG methods yield the corresponding ones found in [4].

Remark 3.4. Notice that for all of our choices of the numerical fluxes $\widehat{u}$ and $\widehat{\sigma}$, we have that $[\widehat{u}] = 0$ and $[\widehat{\sigma}; n_h] = 0$. In addition, they are consistent with the corresponding fluxes in the flat case given in [4].

4. Technical tools. In this section we introduce the necessary tools and geometric relations needed to work on discrete domains and prove boundedness and stability of the bilinear forms, following the framework introduced in [24].

4.1. Surface lifting. For any function w defined on Γ_h^k we define the surface *lift* onto Γ by

$$w^\ell(\xi) = w(x(\xi)), \ \xi \in \Gamma,$$

where, thanks to the invertibility of (2.1), $x(\xi)$ is defined as the unique solution of

$$x(\xi) = \pi(x) + d(x)\nu(\xi).$$

In particular, for every $\widehat{K}_h \in \widehat{\mathcal{T}}_h$, there is a unique curved triangle $\widehat{K}_h^\ell = \pi(\widehat{K}_h) \subset \Gamma$. We may then define the regular, conforming triangulation $\widehat{\mathcal{T}}_h^\ell$ of Γ given by

$$\Gamma = \bigcup_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \widehat{K}_h^\ell.$$

The triangulation $\widehat{\mathcal{T}}_h^\ell$ of Γ is thus induced by the triangulation $\widehat{\mathcal{T}}_h$ of Γ_h^k via the surface lift operator. Similarly, we denote by $\widehat{e}_h^\ell = \pi(\widehat{e}_h) \in \widehat{\mathcal{E}}_h^\ell$ the unique curved edge associated with $\widehat{e}_h$. The function space for surface lifted functions is chosen to be given by

$$\widehat{S}_{hk}^\ell = \left\{ \chi \in L^2(\Gamma) : \chi = \widehat{\chi}^\ell \text{ for some } \widehat{\chi} \in \widehat{S}_{hk} \right\}.$$

We define the discrete right-hand side f_h such that $f_h^\ell = f$. We also denote by $w^{-\ell} \in \widehat{S}_{hk}$ the *inverse* surface lift of some function $w \in \widehat{S}_{hk}^\ell$ satisfying $(w^{-\ell})^\ell = w$.

As an extension of the results shown in [24, 22], for v_h defined on Γ_h^k, we have that

$$\nabla_{\Gamma_h^k} v_h = P_h(I - dH)P\nabla_\Gamma v_h^\ell. \tag{4.1}$$

Furthermore, let δ_h be the local area deformation when transforming $\widehat{K}_h$ to $\widehat{K}_h^\ell$, i.e.,

$$\delta_h \ \mathrm{dA_{hk}} = \ \mathrm{dA},$$

and let $\delta_{\widehat{e}_h}$ be the local edge deformation when transforming $\widehat{e}_h$ to $\widehat{e}_h^\ell$, i.e.,

$$\delta_{\widehat{e}_h} \ \mathrm{ds_{hk}} = \ \mathrm{ds}.$$

Finally, let

$$R_h = \frac{1}{\delta_h} P(I - dH)P_h(I - dH)P.$$

As a consequence of (4.1) we have that

$$\int_{\Gamma_h^k} \nabla_{\Gamma_h^k} u_h \cdot \nabla_{\Gamma_h^k} v_h + u_h v_h \ \mathrm{dA_{hk}} = \int_\Gamma R_h \nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell + \delta_h^{-1} u_h^\ell v_h^\ell \ \mathrm{dA}. \tag{4.2}$$

4.2. Geometric estimates. We next prove some geometric error estimates relating Γ to Γ_h^k.

LEMMA 4.1. *Let Γ be a compact smooth, connected, and oriented surface in $\mathbb{R}^3$ and let Γ_h^k be its piecewise Lagrange interpolant of degree k. Furthermore, we denote by $n^\pm$ the unit (surface) conormals to, respectively, $\widehat{e}_h^{l+/-}$. Then, for sufficiently small h, we have that*

$$\|d\|_{L^\infty(\Gamma_h^k)} \lesssim h^{k+1}, \tag{4.3a}$$
$$\|1 - \delta_h\|_{L^\infty(\Gamma_h^k)} \lesssim h^{k+1}, \tag{4.3b}$$
$$\|\nu - \nu_h\|_{L^\infty(\Gamma_h^k)} \lesssim h^k, \tag{4.3c}$$
$$\|P - R_h\|_{L^\infty(\Gamma_h^k)} \lesssim h^{k+1}, \tag{4.3d}$$
$$\|1 - \delta_{\widehat{e}_h}\|_{L^\infty(\widehat{\mathcal{E}}_h)} \lesssim h^{k+1}, \tag{4.3e}$$
$$\sup_{\widehat{K} \in \widehat{\mathcal{T}}_h} \|P - R_{\widehat{e}_h}\|_{L^\infty(\partial\widehat{K}_h)} \lesssim h^{k+1}, \tag{4.3f}$$
$$\|n^\pm - Pn_h^\pm\|_{L^\infty(\widehat{\mathcal{E}}_h)} \lesssim h^{k+1}, \tag{4.3g}$$

where $R_{\widehat{e}_h} = \frac{1}{\delta_{\widehat{e}_h}} P(I - dH)P_h(I - dH)$.

For the sake of readability, we postpone the proof of Lemma 4.1 to Appendix A.

4.3. Boundedness and stability. We define the space of piecewise polynomial functions on Γ_h as

$$\widetilde{S}_{hk} = \left\{ \widetilde{\chi} \in L^2(\Gamma_h) : \widetilde{\chi}|_{\widetilde{K}_h} \in \mathbb{P}^k(\widetilde{K}_h) \quad \forall \widetilde{K}_h \in \widetilde{\mathcal{T}}_h \right\}.$$

We recall the following useful result from [21].

LEMMA 4.2. *Let $v \in H^j(\widehat{K}_h)$, $j \geq 2$, and let $\widetilde{v} = v \circ \pi_k$. Then, for h small enough, we have that*

$$\|v^\ell\|_{L^2(\widehat{K}_h^\ell)} \sim \|v\|_{L^2(\widehat{K}_h)} \sim \|\widetilde{v}\|_{L^2(\widetilde{K}_h)}, \tag{4.4a}$$

$$\|\nabla_\Gamma v^\ell\|_{L^2(\widehat{K}_h^\ell)} \sim \|\nabla_{\Gamma_h^k} v\|_{L^2(\widehat{K}_h)} \sim \|\nabla_{\Gamma_h} \widetilde{v}\|_{L^2(\widetilde{K}_h)}, \tag{4.4b}$$

$$\|D^j_{\Gamma_h^k} v\|_{L^2(\widehat{K}_h)} \lesssim \sum_{1\leq m\leq j} \|D^m_\Gamma v^\ell\|_{L^2(\widehat{K}_h^\ell)}, \tag{4.4c}$$

$$\|D^j_{\Gamma_h} \widetilde{v}\|_{L^2(\widetilde{K}_h)} \lesssim \sum_{1\leq m\leq j} \|D^m_{\Gamma_h^k} v\|_{L^2(\widehat{K}_h)}. \tag{4.4d}$$

We will also need the following inverse and trace inequalities. The following result is adapted from [13, Thm 3.2.6].

LEMMA 4.3. *Let l, m be two integers such that $0 \leq l \leq m$. Then,*

$$|v_h|_{H^m(\widetilde{K}_h)} \lesssim h^{l-m}_{\widetilde{K}_h} |v_h|_{H^l(\widetilde{K}_h)} \qquad \forall v_h \in \widetilde{S}_{hk}.$$

LEMMA 4.4. *Let $\widetilde{w} \in H^2(\widetilde{K}_h)$, $\widetilde{K}_h \in \widetilde{\mathcal{T}}_h$. Then, for each $\widetilde{e}_h \in \partial\widetilde{K}_h$, it holds that*

$$\|\widetilde{w}\|^2_{L^2(\widetilde{e}_h)} \lesssim h^{-1}\|\widetilde{w}\|^2_{L^2(\widetilde{K}_h)} + h\|\nabla_{\Gamma_h}\widetilde{w}\|^2_{H^2(\widetilde{K}_h)}.$$

Moreover, combining Lemmas 4.4 and 4.3 we get the following result for polynomial functions.

LEMMA 4.5. *For each $\widetilde{K}_h \in \widetilde{\mathcal{T}}_h$, it holds that*

$$\|\widetilde{v}_h\|^2_{L^2(\widetilde{e}_h)} \lesssim h^{-1}\|\widetilde{v}_h\|^2_{L^2(\widetilde{K}_h)} \qquad \forall\ \widetilde{v}_h \in \widetilde{S}_{hk},$$

with $\widetilde{e}_h \subseteq \partial\widetilde{K}_h$.

Finally, we prove the following trace inequality.

LEMMA 4.6. *For sufficiently small h, we have that*

$$\|\nabla_{\Gamma_h^k}\widehat{w}_h\|^2_{L^2(\partial\widehat{K}_h)} \lesssim h^{-1}\|\nabla_{\Gamma_h^k}\widehat{w}_h\|^2_{L^2(\widehat{K}_h)} \qquad \forall \widehat{w}_h \in \widehat{S}_{hk}.$$

Proof. Defining $\delta_{\widetilde{e}_h} = \mathrm{ds}/\ \mathrm{ds_{h1}}$ and $\delta_{\widetilde{e}_h\to\widehat{e}_h} = \mathrm{ds_{hk}}/\ \mathrm{ds_{h1}}$, using (4.3e) and a Taylor expansion argument, we obtain

$$|1 - \delta_{\widetilde{e}_h\to\widehat{e}_h}| = \left|1 - \frac{\delta_{\widetilde{e}_h}}{\delta_{\widehat{e}_h}}\right| = \left|1 - \frac{1+O(h^2)}{1+O(h^{k+1})}\right| \lesssim h^2.$$

Now let $\widetilde{w}_h \in \widetilde{S}_{hk}$ be such that $\widetilde{w}_h = \widehat{w}_h \circ \pi_k$. From (2.21) and (2.22) in [21] we have that

$$\left|\nabla_{\Gamma_h^k}\widehat{w}_h(\pi_k(\tilde{x}))\right| \lesssim |\nabla_{\Gamma_h}\widetilde{w}_h(\tilde{x})| \tag{4.5}$$

for each $\tilde{x} \in \Gamma_h$, provided h is sufficiently small. Applying Lemma 4.5 we get

$$\int_{\partial\widetilde{K}_h} |\nabla_{\Gamma_h}\widetilde{w}_h|^2\ \mathrm{ds_{h1}} \lesssim \frac{1}{h}\|\nabla_{\Gamma_h}\widetilde{w}_h\|^2_{L^2(\widetilde{K}_h)}.$$

Surface lifting the left-hand side to Γ_h^k, making use of (4.5), and using (4.4b) for the right-hand side we have that

$$\int_{\partial\widehat{K}_h} |\nabla_{\Gamma_h^k}\widehat{w}_h|^2 \delta^{-1}_{\widetilde{e}_h\to\widehat{e}_h}\ \mathrm{ds_{hk}} \lesssim \frac{1}{h}\|\nabla_{\Gamma_h^k}\widehat{w}_h\|^2_{L^2(\widehat{K}_h)}.$$

TABLE 1
Stabilization function of the DG methods considered in our unified analysis.

Method	Stabilization function $S_h(\cdot,\cdot)$
IP [23] NIPG [41] IIPG [15] LDG [14]	(4.6a)
Brezzi et al. [12] Bassi et al. [8]	(4.6b)

We thus obtain, using (4.3e),

$$(1 - Ch^2)\|\nabla_{\Gamma_h^k}\widehat{w}_h\|^2_{L^2(\partial\widehat{K}_h)} \lesssim \frac{1}{h}\|\nabla_{\Gamma_h^k}\widehat{w}_h\|^2_{L^2(\widehat{K}_h)},$$

which yields the desired result for h small enough. ∎

In order to perform a unified analysis of the surface DG methods presented in section 3.2, we introduce the stabilization function

$$S_h(u_h, v_h) = \begin{cases} \displaystyle\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h} \beta_{\widehat{e}_h}\int_{\widehat{e}_h}[u_h][v_h]\,\mathrm{ds_{hk}}, & \text{(4.6a)}\\ \displaystyle\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h} \eta_{\widehat{e}_h}\int_{\Gamma_h^k} r_{\widehat{e}_h}([u_h])\cdot r_{\widehat{e}_h}([v_h])\,\mathrm{dA_{hk}} & \text{(4.6b)}\end{cases}$$

for $u_h,\ v_h \in \widehat{S}_{hk}$; cf. also Table 1.

The next result, together with Lax–Milgram, guarantees that there exists a unique solution $u_h \in \widehat{S}_{hk}$ of (3.7) that satisfies the stability estimate

$$\|u_h\|_{\mathrm{DG}} \lesssim \|f_h\|_{L^2(\Gamma_h^k)}, \tag{4.7}$$

where the DG norm $\|\cdot\|_{\mathrm{DG}}$ is given by

$$\|u_h\|^2_{\mathrm{DG}} = \|u_h\|^2_{1,h} + |u_h|^2_{*,h} \quad \forall u_h \in \widehat{S}_{hk}, \tag{4.8}$$

with

$$\|u_h\|^2_{1,h} = \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\|u_h\|^2_{H^1(\widehat{K}_h)}$$

and

$$|u_h|^2_{*,h} = S_h(u_h, u_h),$$

where $S_h(\cdot,\cdot)$ depends on the method under investigation and is defined as in (4.6a)–(4.6b).

We will now consider boundedness and stability of the bilinear forms $\mathcal{A}_h^k(\cdot,\cdot)$ corresponding to the surface DG methods given in Table 1. We first state some estimates required for the analysis of the surface LDG method.

LEMMA 4.7. *For any $v_h \in \widehat{S}_{hk}$,*

$$\alpha\|r_{\widehat{e}_h}([v_h])\|^2_{L^2(\Gamma_h^k)} \lesssim \beta_{\widehat{e}_h}\|[v_h]\|^2_{L^2(\widehat{e}_h)},$$
$$\alpha\|l_{\widehat{e}_h}([v_h])\|^2_{L^2(\Gamma_h^k)} \lesssim \beta_{\widehat{e}_h}\|[v_h]\|^2_{L^2(\widehat{e}_h)}$$

on each $\widehat{e}_h \in \widehat{\mathcal{E}}_h$.

Proof. The thesis straighforwally follows using the same arguements as in [2, Lemma 2.3] and recalling that here the average and jumps operators appearing in the definition of the local lifting operators are defined in a slightly different way than [2, Lemma 2.3]. ∎

LEMMA 4.8. *The bilinear forms $\mathcal{A}_h^k(\cdot,\cdot)$ corresponding to the surface DG methods given in Table* 1 *are continuous and coercive in the DG norm* (4.8)*, i.e.,*

$$\mathcal{A}_h^k(u_h, v_h) \lesssim \|u_h\|_{DG}\|v_h\|_{DG}, \quad \mathcal{A}_h^k(u_h, u_h) \gtrsim \|u_h\|_{DG}^2$$

for every $u_h, v_h \in \widehat{S}_{hk}$. For the surface IP, Bassi et al., and IIPG methods, coercivity holds provided the penalty parameter α appearing in the definition of $\beta_{\widehat{e}_h}$ or $\eta_{\widehat{e}_h}$ in (3.8) *is chosen sufficiently large.*

Proof. For all the methods stabilized with $S_h(\cdot,\cdot)$ defined as in (4.6a), Lemma 4.6 implies that

$$\begin{aligned}\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}[u_h]\{\nabla_{\Gamma_h^k}v_h; n_h\}\ \mathrm{ds_{hk}} &\leq \sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\left\|\beta_{\widehat{e}_h}^{1/2}[u_h]\right\|_{L^2(\widehat{e}_h)}\left\|\beta_{\widehat{e}_h}^{-1/2}\{\nabla_{\Gamma_h^k}v_h; n_h\}\right\|_{L^2(\widehat{e}_h)}\\ &\lesssim \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\alpha^{-\frac{1}{2}}|u_h|_{*,h}\|\nabla_{\Gamma_h^k}v_h\|_{L^2(\widehat{K}_h)}\\ &\lesssim \alpha^{-\frac{1}{2}}|u_h|_{*,h}\|v_h\|_{1,h},\end{aligned}\tag{4.9}$$

where the hidden constant depends on the degree of the polynomial approximation but not on the penalty parameters $\beta_{\widehat{e}_h}$. Otherwise, if $S_h(\cdot,\cdot)$ is given as in (4.6b), we observe that for $u_h, v_h \in \widehat{S}_{hk}$ we have that

$$\sum_{\widehat{e}_h\in\widehat{\mathcal{E}}_h}\int_{\widehat{e}_h}[u_h]\{\nabla_{\Gamma_h^k}v_h; n_h\}\ \mathrm{ds_{hk}} = \sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\int_{\widehat{K}_h} r_h([u_h])\cdot\nabla_{\Gamma_h^k}v_h\ \mathrm{dA_{hk}}$$

and, making use of the fact that $r_{\widehat{e}_h}$ only has support on $\widehat{K}_h^+\bigcup\widehat{K}_h^-$, where $\partial\widehat{K}_h^+\bigcap\partial\widehat{K}_h^- = \widehat{e}_h$,

$$\|r_h(\phi)\|_{L^2(\widehat{K}_h)}^2 = \left\|\sum_{\widehat{e}_h\subset\partial\widehat{K}_h} r_{\widehat{e}_h}(\phi)\right\|_{L^2(\widehat{K}_h)}^2 \lesssim \sum_{\widehat{e}_h\subset\partial\widehat{K}_h}\|r_{\widehat{e}_h}(\phi)\|_{L^2(\widehat{K}_h)}^2,\tag{4.10}$$

where the last step follows recalling that the support of $r_{\widehat{e}_h}(\cdot)$ is limited to the two neighboring elements sharing the edge $\widehat{e}_h$. Hence, applying the Cauchy–Schwarz inequality, we obtain

$$\sum_{\widehat{K}_h\in\widehat{\mathcal{T}}_h}\|\eta_{\widehat{e}_h}^{1/2}r_h([u_h])\|_{L^2(\widehat{K}_h)}\|\eta_{\widehat{e}_h}^{-1/2}\nabla_{\Gamma_h^k}v_h\|_{L^2(\widehat{K}_h)} \lesssim \alpha^{-\frac{1}{2}}|u_h|_{*,h}\|v_h\|_{1,h},\tag{4.11}$$

where the hidden constant depends on the degree of the polynomial approximation but not on the penalty parameters $\eta_{\widehat{e}_h}$. For the surface LDG method, using Lemmas 4.7 and 4.6 and the $L^\infty(\Gamma_h^k)$ bound on θ, we obtain

$$\begin{aligned}\left|\int_{\widehat{e}_h}[\nabla_{\Gamma_h^k}u_h; n_h]\theta\cdot n_h^+[v_h]\ \mathrm{ds_{hk}}\right| &\lesssim \alpha^{-\frac{1}{2}}\|\beta\|_{L^\infty(\Gamma_h^k)}\|\nabla_{\Gamma_h^k}u_h\|_{L^2(\widehat{K}_h)}|v_h|_{*,h},\\ \left|\int_{\widehat{K}_h} r_h([u_h])\cdot l_h(\theta\cdot n_h^+[u_h])\ \mathrm{ds_{hk}}\right| &\lesssim \alpha^{-1}\|\beta\|_{L^\infty(\Gamma_h^k)}|u_h|_{*,h}|v_h|_{*,h},\end{aligned}$$

and, in a similar way, the remaining quantities. Continuity then follows from the Cauchy–Schwarz inequality and the above estimates. We next show coercivity of the DG bilinear forms. For the surface NIPG method, stability follows straightforwardly from the Cauchy–Schwarz inequality. For the surface LDG method, we have that

$$\begin{aligned}\mathcal{A}_h^k(u_h, u_h) \geq \|u_h\|_{1,h}^2 - 2 \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h^k} \int_{\widehat{e}_h} \left| [u_h]\{\nabla_{\Gamma_h^k} u_h; n_h\} \right| \,\mathrm{ds_{hk}} \\ - 2\|\beta\|_{L^\infty(\Gamma_h^k)} \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h^k} \int_{\widehat{e}_h} \left| [u_h][\nabla_{\Gamma_h^k} u_h; n_h] \right| \,\mathrm{ds_{hk}} + |u_h|_{*,h}^2.\end{aligned}$$

For the other methods involving $S_h(\cdot,\cdot)$ defined as in (4.6a) we obtain

$$\mathcal{A}_h^k(u_h, u_h) \geq \|u_h\|_{1,h}^2 - 2 \sum_{\widehat{e}_h \in \widehat{\mathcal{E}}_h^k} \int_{\widehat{e}_h} \left| [u_h]\{\nabla_{\Gamma_h^k} u_h; n_h\} \right| \,\mathrm{ds_{hk}} + |u_h|_{*,h}^2;$$

otherwise, if $S_h(\cdot,\cdot)$ is given as in (4.6b), we have that

$$\mathcal{A}_h^k(u_h, u_h) \geq \|u_h\|_{1,h}^2 - 2 \sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h^k} \int_{\widehat{K}_h} \left| r_h([u_h]) \cdot \nabla_{\Gamma_h^k} u_h \right| \,\mathrm{dA_{hk}} + |u_h|_{*,h}^2.$$

The result follows by making use of the corresponding boundedness estimates, using the Cauchy–Schwarz inequality and Young's inequalities, and choosing the penalty parameter α sufficiently large. ∎

We now define the DG norm for functions in $\widehat{S}_{hk}^\ell$ as follows:

$$\|u_h^\ell\|_{\mathrm{DG},\ell}^2 = \|u_h^\ell\|_{1,h}^2 + |u_h^\ell|_{*,h}^2 \quad \forall u_h^\ell \in \widehat{S}_{hk}^\ell, \tag{4.12}$$

with

$$\|u_h^\ell\|_{1,h}^2 = \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \|u_h^\ell\|_{H^1(\widehat{K}_h^\ell)}^2$$

and

$$|u_h^\ell|_{*,h}^2 = S_h^\ell(u_h^\ell, u_h^\ell),$$

where $S_h^\ell(\cdot,\cdot)$ is given by

$$S_h^\ell(u_h^\ell, v_h^\ell) = \begin{cases} \displaystyle\sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \beta_{\widehat{e}_h} \int_{\widehat{e}_h^\ell} \delta_{\widehat{e}_h}^{-1} [u_h^\ell][v_h^\ell] \,\mathrm{ds}, & \text{(4.13a)} \\ \displaystyle\sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \eta_{\widehat{e}_h} \int_\Gamma \delta_h^{-1} \big(r_{\widehat{e}_h}([u_h])\big)^\ell \cdot \big(r_{\widehat{e}_h}([v_h])\big)^\ell \,\mathrm{dA} & \text{(4.13b)} \end{cases}$$

for $u_h^\ell,\ v_h^\ell \in \widehat{S}_{hk}^\ell$.

LEMMA 4.9. *Let $u_h \in \widehat{S}_{hk}$ satisfy* (4.7). *Then $u_h^\ell \in \widehat{S}_{hk}^\ell$ satisfies*

$$\|u_h^\ell\|_{DG,\ell} \lesssim \|f\|_{L^2(\Gamma)} \tag{4.14}$$

for h small enough.

Proof. We first show that for any function $v_h \in \widehat{S}_{hk}$, for sufficiently small h,

$$\|v_h^\ell\|_{\mathrm{DG},\ell} \lesssim \|v_h\|_{\mathrm{DG}}. \tag{4.15}$$

The $\|\cdot\|_{1,h}^2$ component of the DG norm is dealt with in exactly the same way as in [21]. For the $|\cdot|_{*,h}^2$ component of the DG norm we have that

$$\int_{\widehat{e}_h} [v_h]^2 \,\mathrm{ds_{hk}} = \int_{\widehat{e}_h^\ell} \delta_{\widehat{e}_h}^{-1} [v_h^\ell]^2 \,\mathrm{ds} \quad \text{and} \quad \int_{\Gamma_h^k} |r_h([v_h])|^2 \,\mathrm{dA_{hk}} = \int_\Gamma \delta_h^{-1} |r_h([v_h])^\ell|^2 \,\mathrm{dA},$$

which straightforwardly yields (4.15). Making use of the discrete stability estimate (4.7) and noting that, by Lemma 4.7, $\|f_h\|_{L^2(\Gamma_h^k)} \lesssim \|f_h^\ell\|_{L^2(\Gamma)} = \|f\|_{L^2(\Gamma)}$, we get the desired result. ∎

For each of the surface DG bilinear forms given in Table 1, we define a corresponding bilinear form on Γ induced by the surface lifted triangulation $\widehat{\mathcal{T}}_h^\ell$ which is well-defined for functions $w, v \in H^2(\Gamma) + \widehat{S}_{hk}^\ell$. For the surface IP bilinear form (3.11), we define

$$\begin{aligned}\mathcal{A}(w,v) = &\sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} \nabla_\Gamma w \cdot \nabla_\Gamma v + wv \,\mathrm{dA} - \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} [w]\{\nabla_\Gamma v; n\} + [v]\{\nabla_\Gamma w; n\} \,\mathrm{ds} \\ &+ \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} \delta_{\widehat{e}_h}^{-1} \beta_{\widehat{e}_h} [w][v] \,\mathrm{ds},\end{aligned} \tag{4.16}$$

where n^+ and n^- are, respectively, the unit surface conormals to $\widehat{K}_h^{\ell+}$ and $\widehat{K}_h^{\ell-}$ on $\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell$. For the Brezzi et al. bilinear form (3.10), we define

$$\begin{aligned}\mathcal{A}(w,v) = &\sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} \nabla_\Gamma w \cdot \nabla_\Gamma v + wv \,\mathrm{dA} \\ &+ \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} \delta_h^{-1} \eta_{\widehat{e}_h} r_{\widehat{e}_h}([w^{-\ell}])^\ell \cdot r_{\widehat{e}_h}([v^{-\ell}])^\ell \\ &+ \delta_h^{-1} \big(r_h([w^{-\ell}])\big)^\ell \cdot \big(r_h([v^{-\ell}])\big)^\ell \,\mathrm{dA} \\ &- \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} [w]\{\nabla_\Gamma v; n\} + [v]\{\nabla_\Gamma w; n\} \; - \delta_{\widehat{e}_h}^{-1} \beta_{\widehat{e}_h} [w][v] \,\mathrm{ds}.\end{aligned} \tag{4.17}$$

For the surface LDG bilinear form (3.12), we define

$$\begin{aligned}\mathcal{A}(w,v) = &\sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} \nabla_\Gamma w \cdot \nabla_\Gamma v + wv \,\mathrm{dA} - \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} [w]\{\nabla_\Gamma v; n\} - \{\nabla_\Gamma w; n\}[v] \,\mathrm{ds} \\ &+ \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} \Big(- \delta_{\widehat{e}_h}^{-1} [\nabla_\Gamma w; n] \theta \cdot n_h^{\ell+} [v] \\ &\qquad - \delta_{\widehat{e}_h}^{-1} \theta \cdot n_h^{\ell+} [w] [\nabla_\Gamma v; n] + \delta_{\widehat{e}_h}^{-1} \beta_{\widehat{e}_h} [w][v] \Big) \,\mathrm{ds} \\ &+ \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} \delta_h^{-1} \Big(r_h([w^{-\ell}]) + \theta \cdot n_h^{\ell+} l_h\big([w^{-\ell}]\big) \Big)^\ell \\ &\qquad \cdot \Big(r_h([v^{-\ell}]) + \theta \cdot n_h^{\ell+} l_h\big([v^{-\ell}]\big) \Big)^\ell \,\mathrm{dA}.\end{aligned} \tag{4.18}$$

The corresponding bilinear forms for the other surface DG methods can be derived in a similar manner. Since we assume that the weak solution u of (2.3) belongs to $H^2(\Gamma)$ they all satisfy

$$\mathcal{A}(u,v) = \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}_h}^\ell} \int_{\widehat{K}_h^\ell} fv \,\mathrm{dA} \qquad \forall v \in H^2(\Gamma) + \widehat{S}_{hk}^\ell. \tag{4.19}$$

Finally, we require the following stability estimate for $\mathcal{A}(\cdot,\cdot)$, which follows by applying similar arguments to those found in the proof of Lemma 4.8.

LEMMA 4.10. *The bilinear forms $\mathcal{A}(\cdot,\cdot)$ induced by the surface DG methods given in Table* 1 *are coercive in the DG norm* (4.12)*, i.e.,*

$$\|w_h^\ell\|_{DG,\ell}^2 \lesssim \mathcal{A}(w_h^\ell, w_h^\ell) \tag{4.20}$$

for all $w_h^\ell \in \widehat{S}_{hk}^\ell$ if, for the surface IP, Bassi et al., and IIPG methods, the penalty parameter α appearing in the definition of $\beta_{\widehat{e}_h}$ or $\eta_{\widehat{e}_h}$ in (3.8) *is chosen sufficiently large.*

5. Convergence.

We now state the main result of this paper.

THEOREM 5.1. *Let $u \in H^{k+1}(\Gamma)$ and $u_h \in \widehat{S}_{hk}$ denote the solutions to* (2.3) *and* (3.6)*, respectively. Let $\eta = 0$ for IIPG, NIPG formulations and let $\eta = 1$ otherwise. Then,*

$$\|u - u_h^\ell\|_{L^2(\Gamma)} + h^\eta \|u - u_h^\ell\|_{DG,\ell} \lesssim h^{k+\eta}(\|f\|_{L^2(\Gamma)} + \|u\|_{H^{k+1}(\Gamma)}),$$

provided the mesh size h is small enough and the penalty parameter α is large enough for the surface IP, Bassi et al., and IIPG methods. Here the hidden constant depends, in particular, on the signed-distance function d and its first/second derivatives.

In order to prove Theorem 5.1 we collect some useful technical results.

For $\widehat{w} \in H^2(\Gamma_h^k)$, we define the interpolant $\widehat{I}_h^k : C^0(\Gamma_h^k) \to \widehat{S}_{hk}$ by

$$\widehat{I}_h^k \widehat{w} = \widetilde{I}_h^k(\widehat{w} \circ \pi_k),$$

where $\widetilde{I}_h^k : C^0(\Gamma_h) \to \widetilde{S}_{hk}$ is the standard Lagrange interpolant of degree k. We also define the interpolant $I_h^k : C^0(\Gamma) \to \widehat{S}_{hk}^\ell$ by

$$I_h^k w = \widehat{I}_h^k(w \circ \pi).$$

LEMMA 5.2. *Let $w \in H^m(\Gamma)$ with $2 \le m \le k+1$. Then for $i = 0, 1$,*

$$|w - I_h^k w|_{H^i(\widehat{K}_h^\ell)} \lesssim h^{m-i} \|w\|_{H^m(\widehat{K}_h^\ell)}.$$

Proof. The proof follows easily by combining standard estimates for the Lagrange interpolant on Γ_h with Lemma 4.2. See [21] for further details. ∎

LEMMA 5.3. *Let $w \in H^m(\Gamma)$ with $2 \le m \le k+1$. Then, for sufficiently small h, we have that*

$$\|w - I_h^k w\|_{L^2(\partial \widehat{K}_h^\ell)}^2 + h^2 \|\nabla_\Gamma (w - I_h^k w)\|_{L^2(\partial \widehat{K}_h^\ell)}^2 \lesssim h^{2m-1} \|w\|_{H^m(\widehat{K}_h^\ell)}^2.$$

Proof. Fix an arbitrary element $\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell$. We then define $\widehat{w} \in H^m(\widehat{K}_h)$ and $\widetilde{w} \in H^m(\widetilde{K}_h)$ such that $w = \widehat{w} \circ \pi$ and $\widetilde{w} = \widehat{w} \circ \pi_k$.

Using Lemma 4.4 on $\widetilde{K}_h \in \widetilde{\mathcal{T}}_h$ we get

$$\int_{\partial \widetilde{K}_h} |\nabla_{\Gamma_h}(\widetilde{w} - \widetilde{I}_h^k \widetilde{w})|^2 \, \mathrm{ds_{h1}} \lesssim \left(\frac{1}{h} \int_{\widetilde{K}_h} |\nabla_{\Gamma_h}(\widetilde{w} - \widetilde{I}_h^k \widetilde{w})|^2 \, \mathrm{dA_{h1}} + h \int_{\widetilde{K}_h} |\nabla^2_{\Gamma_h}(\widetilde{w} - \widetilde{I}_h^k \widetilde{w})|^2 \, \mathrm{dA_{h1}} \right).$$

Applying a classical interpolation result for the right-hand side (see, for example, Theorem 6.4 in [11]), we obtain

$$\int_{\partial \widetilde{K}_h} \left| \nabla_{\Gamma_h} \left(\widetilde{w} - \widetilde{I}_h^k \widetilde{w} \right) \right|^2 \, \mathrm{ds_{h1}} \lesssim h^{2m-3} |\widetilde{w}|^2_{H^m(\widetilde{K}_h)}.$$

Then, lifting the left-hand side on Γ_h^k as in Lemma 4.6 and using (4.4b) with (4.4d) we have

$$(1 - Ch^2) \int_{\partial \widehat{K}_h} \left| \nabla_{\Gamma_h^k} \left(\widehat{w} - \widehat{I}_h^k \widehat{w} \right) \right|^2 \, \mathrm{ds_{hk}} \lesssim h^{2m-3} \|\widehat{w}\|^2_{H^m(\widehat{K}_h)}.$$

In the same way, we lift the left-hand side onto Γ and use (4.4b) with (4.4d):

$$(1 - Ch^{k+1})(1 - Ch^2) \|\nabla_\Gamma (w - I_h^k w)\|^2_{L^2(\partial \widehat{K}_h^\ell)} \lesssim h^{2m-3} \|w\|^2_{H^m(\widehat{K}_h^\ell)}.$$

Similarly, using again Lemma 4.4 for $\widetilde{w} - \widetilde{I}_h^k \widetilde{w}$, we obtain

$$\int_{\partial \widetilde{K}_h} \left| \widetilde{w} - \widetilde{I}_h^k \widetilde{w} \right|^2 \, \mathrm{ds_{h1}} \lesssim \left(\frac{1}{h} \int_{\widetilde{K}_h} |\widetilde{w} - \widetilde{I}_h^k \widetilde{w}|^2 \, \mathrm{dA_{h1}} + h \int_{\widetilde{K}_h} \left| \nabla_{\Gamma_h} \left(\widetilde{w} - \widetilde{I}_h^k \widetilde{w} \right) \right|^2 \, \mathrm{dA_{h1}} \right).$$

Then, using interpolation estimates on $\widetilde{K}_h$ we get

$$\int_{\partial \widetilde{K}_h} \left| \widetilde{w} - \widetilde{I}_h^k \widetilde{w} \right|^2 \, \mathrm{ds_{h1}} \lesssim h^{2m-1} |\widetilde{w}|^2_{H^m(\widetilde{K}_h)}.$$

Finally, doing, as before, the lifting of the left-hand side on Γ_h^k and then on Γ and using (4.4a) with (4.4b), we get the claim for h small enough. ∎

These interpolation estimates allow us to derive the following boundedness estimates for $\mathcal{A}(\cdot, \cdot)$.

LEMMA 5.4. *Let $u \in H^m(\Gamma)$ and $w \in H^n(\Gamma)$ with $2 \leq m, n \leq k+1$. Then, for all $v_h^\ell \in \widehat{S}_{hk}^\ell$, we have that*

$$\mathcal{A}(u - I_h^k u, v_h^\ell) \lesssim h^{m-1} \|u\|_{H^m(\Gamma)} \|v_h^\ell\|_{DG,\ell}, \tag{5.1}$$

$$\mathcal{A}(u - I_h^k u, w - I_h^k w) \lesssim h^{m+n-2} \|u\|_{H^m(\Gamma)} \|w\|_{H^n(\Gamma)}. \tag{5.2}$$

Proof. Since $u \in H^m(\Gamma) \subset C^0(\Gamma)$ for $m \geq 2$ and $I_h^k u \in C^0(\Gamma)$, by the continuity of the *inverse* surface lift operator, we have $[(u - I_h^k u)^{-\ell}] = 0$ on each $\widehat{e}_h \in \widehat{\mathcal{E}}_h$. Then, by the definition of $r_{\widehat{e}_h}$ and $l_{\widehat{e}_h}$, we obtain

$$\|r_{\widehat{e}_h}([(u - I_h^k u)^{-\ell}])\|^2_{L^2(\Gamma_h^k)} = \|l_{\widehat{e}_h}([(u - I_h^k u)^{-\ell}])\|^2_{L^2(\Gamma_h^k)} = 0.$$

Following the proof of Lemma 4.8, it is easy to obtain (5.1) and (5.2) from Lemmas 5.2 and 5.3. ∎

For the first term on the right-hand side of (5.6), we require the following *perturbed* Galerkin orthogonality result.

LEMMA 5.5. *Let $u \in H^s(\Gamma)$, $s \geq 2$, and $u_h \in \widehat{S}_{hk}$ denote the solutions to* (2.3) *and* (3.6), *respectively. We define the functional E_h on $\widehat{S}^\ell_{hk}$ by*

$$E_h(v_h^\ell) = \mathcal{A}(u - u_h^\ell, v_h^\ell).$$

Then, for all considered surface DG methods apart from LDG, E_h can be written as

$$\begin{aligned} E_h(v_h^\ell) = & \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}}_h^\ell} \int_{\widehat{K}_h^\ell} (R_h - P)\nabla_\Gamma u_h^\ell \cdot \nabla_\Gamma v_h^\ell + \left(\delta_h^{-1} - 1\right) u_h^\ell v_h^\ell + \left(1 - \delta_h^{-1}\right) f v_h^\ell \; dA \\ & + \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} [u_h^\ell] \left(\{\nabla_\Gamma v_h^\ell; n\} - \{\delta_{\widehat{e}_h}^{-1} P_h(I - dH)P\nabla_\Gamma v_h^\ell; n_h^\ell\}\right) \; ds \\ & + \zeta \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} [v_h^\ell] \left(\{\nabla_\Gamma u_h^\ell; n\} - \{\delta_{\widehat{e}_h}^{-1} P_h(I - dH)P\nabla_\Gamma u_h^\ell; n_h^\ell\}\right) \; ds, \end{aligned} \tag{5.3}$$

where R_h is given as in Lemma 4.1 *and*

$$\zeta = \begin{cases} -1 & \textit{in NIPG and Baumann–Oden cases,} \\ 0 & \textit{in IIPG case,} \\ 1 & \textit{otherwise.} \end{cases}$$

The functional corresponding to the surface LDG method can be written as

$$\begin{aligned} E_h(v_h^\ell) = (5.3) & + \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} \delta_{\widehat{e}_h}^{-1} \theta \cdot n_h^{\ell+} [v_h^\ell] \left([\nabla_\Gamma u_h^\ell; n] - [P_h(I - dH)P\nabla_\Gamma u_h^\ell; n_h^\ell]\right) \; ds \\ & + \sum_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \int_{\widehat{e}_h^\ell} \delta_{\widehat{e}_h}^{-1} \theta \cdot n_h^{\ell+} [u_h^\ell] \left([\nabla_\Gamma v_h^\ell; n] - [P_h(I - dH)P\nabla_\Gamma v_h^\ell; n_h^\ell]\right) \; ds. \end{aligned} \tag{5.4}$$

Furthermore, for all surface DG methods it holds

$$|E_h(v_h^\ell)| \lesssim h^{k+1} \|f\|_{L^2(\Gamma)} \|v_h^\ell\|_{DG,\ell}. \tag{5.5}$$

Proof. The proof is similar to that of Lemma 4.2 in [20] which considered a piecewise linear approximation of the surface. The expression for the error functional E_h is obtained by first noting that the solution u of (2.3) satisfies (4.19) and then considering the difference between (4.19) and (3.6):

$$\begin{aligned} \mathcal{A}(u, v_h^\ell) - \mathcal{A}_h^k(u_h, v_h) &= \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}_h}^\ell} \int_{\widehat{K}_h^\ell} f v_h^\ell \; \mathrm{dA} - \sum_{\widehat{K}_h \in \widehat{\mathcal{T}}_h} \int_{\widehat{K}_h} f_h v_h \; \mathrm{dA_{hk}} \\ &= \sum_{\widehat{K}_h^\ell \in \widehat{\mathcal{T}_h}^\ell} \int_{\widehat{K}_h^\ell} (1 - \delta_h^{-1}) f v_h^\ell \; \mathrm{dA}. \end{aligned}$$

By lifting $\mathcal{A}_h^k(u_h, v_h)$ onto Γ in a similar fashion to what has been done in (4.2) and using the definition of $\mathcal{A}(\cdot, \cdot)$ we get, after algebraic manipulations, relation (5.3).

The estimate (5.5) is then obtained by making use of the geometric estimates in Lemma 4.1. In particular, following the proof of Lemma 4.2 in [20], we preliminarily get

$$\begin{aligned}|E_h(v_h^\ell)| \lesssim\ & \|R_h - P\|_{L^\infty(\Gamma)} \|u_h^\ell\|_{\mathrm{DG},\ell} \|v_h^\ell\|_{\mathrm{DG},\ell} + \left\|\frac{1}{\delta_h} - 1\right\|_{L^\infty(\Gamma)} \|u_h^\ell\|_{\mathrm{DG},\ell} \|v_h^\ell\|_{\mathrm{DG},\ell} \\ &+ \left\|1 - \frac{1}{\delta_h}\right\|_{L^\infty(\Gamma)} \|f\|_{L^2(\Gamma)} \|v_h^\ell\|_{\mathrm{DG},\ell} \\ &+ \max_{\widehat{e}_h^\ell \in \widehat{\mathcal{E}}_h^\ell} \|n - Pn_h^\ell\|_{L^\infty(\widehat{e}_h^\ell)} \|u_h^\ell\|_{\mathrm{DG},\ell} \|v_h^\ell\|_{\mathrm{DG},\ell} \\ &+ \|d\|_{L^\infty(\Gamma)} \|u_h^\ell\|_{\mathrm{DG},\ell} \|v_h^\ell\|_{\mathrm{DG},\ell}.\end{aligned}$$

Then Lemma 4.1 and the stability estimate (4.14) yield the claimed bound. ☐

Remark 5.6. Note that the error functional E_h in Lemma 5.5 includes all of the terms present in the high order surface FEM setting (see [21]) as well as additional terms arising from the surface DG methods.

Proof of Theorem 5.1. The proof will follow an argument similar to the one outlined in [4]. Using the stability result (4.20), we have that

$$\|\phi_h^\ell - u_h^\ell\|_{\mathrm{DG},\ell}^2 \lesssim \mathcal{A}(\phi_h^\ell - u_h^\ell, \phi_h^\ell - u_h^\ell) = \mathcal{A}(u - u_h^\ell, \phi_h^\ell - u_h^\ell) + \mathcal{A}(\phi_h^\ell - u, \phi_h^\ell - u_h^\ell), \tag{5.6}$$

where $\phi_h^\ell \in \widehat{S}_{hk}^\ell$. Choosing the continuous interpolant $\phi_h^\ell = I_h^k u$, using the error functional estimate (5.5) and the boundedness estimate (5.1), the right-hand side of (5.6) can be bounded by

$$\begin{aligned}\|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell}^2 &\lesssim E_h(I_h^k u - u_h^\ell) + \mathcal{A}(I_h^k u - u, I_h^k u - u_h^\ell) \\ &\lesssim h^{k+1} \|f\|_{L^2(\Gamma)} \|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell} + h^k \|u\|_{H^{k+1}(\Gamma)} \|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell},\end{aligned}$$

which implies

$$\|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell} \lesssim h^k (\|f\|_{L^2(\Gamma)} + \|u\|_{H^{k+1}(\Gamma)}).$$

Recalling that $u - I_h^k u \in C^0(\Gamma)$, using Lemma 5.2 we obtain

$$\|u - u_h^\ell\|_{\mathrm{DG},\ell} \leq \|u - I_h^k u\|_{\mathrm{DG},\ell} + \|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell} \lesssim h^k (\|f\|_{L^2(\Gamma)} + \|u\|_{H^{k+1}(\Gamma)}).$$

This concludes the first part of the proof. In the case of $\eta = 1$, to derive the L^2 estimate, we first observe that the solution $z \in H^2(\Gamma)$ to the dual problem

$$-\Delta_\Gamma z + z = u - u_h^\ell \tag{5.7}$$

satisfies

$$\|z\|_{H^2(\Gamma)} \lesssim \|u - u_h^\ell\|_{L^2(\Gamma)}. \tag{5.8}$$

Then, using the symmetry of the bilinear form $\mathcal{A}(\cdot,\cdot)$, we have that

$$\begin{aligned}\|u - u_h^\ell\|_{L^2(\Gamma)}^2 &= (u - u_h^\ell, u - u_h^\ell)_\Gamma = \mathcal{A}(z, u - u_h^\ell) \\ &= \mathcal{A}(u - u_h^\ell, z) = \mathcal{A}(u - u_h^\ell, z - I_h^k z) + E_h(I_h^k z).\end{aligned} \tag{5.9}$$

Using (5.5), a triangle inequality, and the interpolation estimate in Lemma 5.2, we obtain

$$|E_h(I_h^k z)| \lesssim h^{k+1}\|f\|_{L^2(\Gamma)}\|I_h^k z\|_{H^1(\Gamma)} \lesssim h^{k+1}\|f\|_{L^2(\Gamma)}\|z\|_{H^2(\Gamma)}.$$

Hence, using (5.8),

$$|E_h(I_h^k z)| \lesssim h^{k+1}\|f\|_{L^2(\Gamma)}\|u - u_h^\ell\|_{L^2(\Gamma)}.$$

Making use of the continuity of $I_h^k z - z$ and $I_h^k u - u$, the symmetry of the bilinear form $\mathcal{A}(\cdot,\cdot)$, Lemma 5.4, and the stability estimate (5.8) we get

$$\begin{aligned}
\mathcal{A}(u - u_h^\ell, z - I_h^k z) &= \mathcal{A}(z - I_h^k z, u - u_h^\ell) \\
&\lesssim \mathcal{A}(z - I_h^k z, I_h^k u - u_h^\ell) + \mathcal{A}(z - I_h^k z, u - I_h^k u) \\
&\lesssim h\|z\|_{H^2(\Gamma)}\|I_h^k u - u_h^\ell\|_{\mathrm{DG},\ell} + h^{k+1}\|z\|_{H^2(\Gamma)}\|u\|_{H^{k+1}(\Gamma)} \\
&\lesssim h\|z\|_{H^2(\Gamma)}(\|I_h^k u - u\|_{\mathrm{DG},\ell} + \|u - u_h^\ell\|_{\mathrm{DG},\ell} \\
&\quad + h^{k+1}\|z\|_{H^2(\Gamma)}\|u\|_{H^{k+1}(\Gamma)} \\
&\lesssim (h^{k+1}\|u\|_{H^{k+1}(\Gamma)} + h\|u - u_h^\ell\|_{\mathrm{DG},\ell})\|u - u_h^\ell\|_{L^2(\Gamma)}.
\end{aligned}$$

Combining these estimates with (5.9) yields

$$\|u - u_h^\ell\|_{L^2(\Gamma)}^2 \lesssim \left(h\|u - u_h^\ell\|_{\mathrm{DG},\ell} + h^{k+1}(\|f\|_{L^2(\Gamma)} + \|u\|_{H^{k+1}(\Gamma)})\right)\|u - u_h^\ell\|_{L^2(\Gamma)},$$

which gives us the desired L^2 estimate and concludes the proof. In the case of $\eta = 0$, we can trivially obtain the (suboptimal) bound for the error in the L^2 norm from bounding it by the error in the DG norm. ∎

6. Numerical experiments. We show results for the IP method (cf. section 3.2.3), implemented using DUNE-FEM, a discretization module based on the Distributed and Unified Numerics Environment (DUNE) [9, 19]. For the initial mesh generation we use the Computational Geometry Algorithms Library (CGAL) (see [40]). We consider the following test problem

$$-\Delta_\Gamma u + u = f \tag{6.1}$$

on the unit sphere $\Gamma = \{x \in \mathbb{R}^3 \ : \ \|x\|_{l^2} = 1\}$, choosing f so that the exact solution is $u(x_1, x_2, x_3) = \cos(2\pi x_1)\cos(2\pi x_2)\cos(2\pi x_3)$. Figure 2 shows the DG approximate solutions obtained with $k = 1$ and $k = 4$ DG approximation orders. In Table 2 we report the computed errors measured in the DG norm (4.12) as well as the computed convergence factors for *linear* ($k = 1$), *quadratic* ($k = 2$), and *quartic* ($k = 4$) DG/surface approximation orders. The same results obtained measuring the error in the L^2 norm are shown in Table 3. The numerical results validate the theoretical estimates of Theorem 5.1.

Appendix A. This section is devoted to proving Lemma 4.1.

Proof of Lemma 4.1. Proofs of (4.3a)–(4.3d) can be found in [21, Prop. 2.3 and Prop. 4.1]. The proof of (4.3f) will follow exactly the same lines as (4.3d) once we have proven (4.3e). See Figure 3 for mappings used in this proof. Let e, K be the reference segment [0,1] and the (flat) reference element, respectively, and let $\widetilde{K}_h$, $\widehat{K}_h$, and $\widehat{K}_h^\ell$ be elements in Γ_h, Γ_h^k, and Γ, respectively, such that $\pi_k(\widetilde{K}_h) = \widehat{K}_h$ and $\pi(\widehat{K}_h) = \widehat{K}_h^\ell$. Let

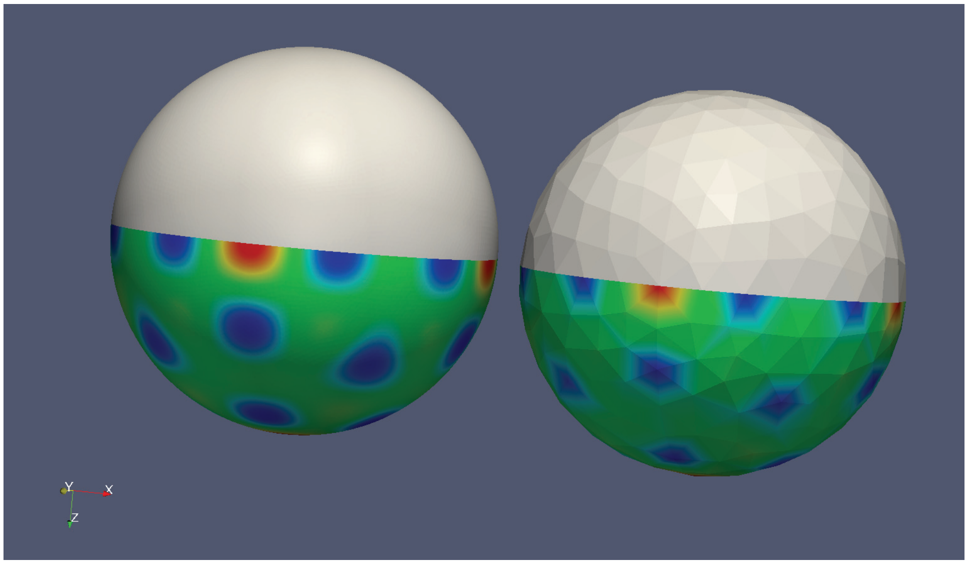

FIG. 2. *Paraview plots of the linear (right) and quartic (left) DG approximations of* (6.1) *on a mesh consisting of* 623 *elements.*

TABLE 2
Computed errors measured in the DG norm (4.12) *and computed convergence factors for the DG approximation of* (6.1) *with linear* ($k = 1$)*, quadratic* ($k = 2$)*, and quartic* ($k = 4$) *approximation orders.*

N. elements	h	$k = 1$		$k = 2$		$k = 4$	
632	0.220	5.0766		1.4205		4.7121e-2	
2528	0.110	2.6427	0.94	3.8696e-1	1.88	3.2585e-3	3.85
10112	0.056	1.3151	1.01	9.8648e-2	1.97	2.0765e-4	3.97
40448	0.028	6.5361e-1	1.01	2.4795e-2	1.99	1.3051e-5	4.00
161792	0.014	3.2596e-1	1.00	6.2087e-3	2.00	-	-
647168	0.007	1.6282e-1	1.00	-	-	-	-

TABLE 3
Computed errors measured in the L^2 *norm and computed convergence factors for the DG approximation of* (6.1) *with linear* ($k = 1$)*, quadratic* ($k = 2$)*, and quartic* ($k = 4$) *approximation orders.*

N. elements	h	$k = 1$		$k = 2$		$k = 4$	
632	0.220	1.7146e-1		3.6978e-2		7.7900e-4	
2528	0.110	5.2882e-2	1.70	4.9040e-3	2.91	2.6808e-5	4.86
10112	0.056	1.4605e-2	1.86	6.1000e-4	3.00	8.4834e-7	4.98
40448	0.028	3.7830e-3	1.95	7.5856e-5	3.01	2.6582e-8	5.00
161792	0.014	9.5800e-4	1.98	9.4598e-6	3.00	-	-
647168	0.007	2.4100e-4	1.99	-	-	-	-

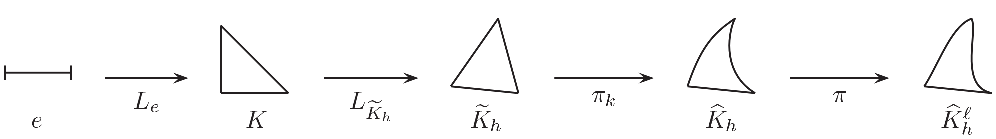

FIG. 3. *Mappings used in the proof of Lemma* 4.1.

also L_e be the inclusion operator that maps e into an edge of K and let $L_{\widetilde{K}_h}(K) = \widetilde{K}_h$. A tangent on an edge $\widehat{e}_h \subset \widehat{K}_h$ in Γ_h^k is given by $\tau_h = \nabla(\pi_k \circ L_{\widetilde{K}_h} \circ L_e)$. Analogously, a tangent on the surface lifted edge $\widehat{e}_h^\ell \subset \widehat{K}_h^\ell$ in Γ is given by $\tau = \nabla\pi\tau_h$. We denote by $\overline{\tau}_h$ and $\overline{\tau}$, respectively, the unit tangents of $\widehat{e}_h$ and $\widehat{e}_h^\ell$, and let $\lambda = \|\tau_h\|_{l^2}$. We will now prove estimate (4.3e). Let dx be the Lebesque measure on the reference interval e. We then have

$$\begin{aligned} \mathrm{ds_{hk}} &= \lambda\ \mathrm{dx}, \\ \mathrm{ds} &= \sqrt{\|(\nabla\pi\tau_h)^T\cdot\nabla\pi\tau_h\|_{l^2}}\ \mathrm{dx} = \lambda\sqrt{\|(\nabla\pi\overline{\tau}_h)^T\cdot\nabla\pi\overline{\tau}_h\|_{l^2}}\ \mathrm{dx} = \underbrace{\|\nabla\pi\overline{\tau}_h\|_{l^2}}_{\delta_{\widehat{e}_h}}\ \mathrm{ds_{hk}}. \end{aligned}$$

Having characterized $\delta_{\widehat{e}_h}$, we wish to show that

$$1 - Ch^{k+1} \leq \|\nabla\pi\overline{\tau}_h\|_{l^2} \leq 1 + Ch^{k+1}.$$

Making use of (2.2) and (4.3a), we have that

$$\|\nabla\pi\overline{\tau}_h\|_{l^2} \leq \|\nabla\pi\|_{l^2}\|\overline{\tau}_h\|_{l^2} \leq \|P - dH\|_{l^2} \leq 1 + Ch^{k+1}. \tag{A.1}$$

Next, to provide a lower bound for $\|\nabla\pi\overline{\tau}_h\|_{l^2}$, we consider

$$\tau - \tau_h = (\nabla\pi - P_h)\tau_h = \lambda(\nabla\pi - P_h)\overline{\tau}_h.$$

Recalling the definition of the projection matrices P and P_h, we have that

$$\|\tau - \tau_h\|_{l^2} \leq \lambda\|(P - P_h) - dH\|_{l^2}\|\overline{\tau}_h\|_{l^2} \leq \lambda Ch^k.$$

Using the reverse triangle inequality, we obtain

$$\lambda\|\nabla\pi\overline{\tau}_h\|_{l^2} = \|\tau\|_{l^2} \geq \|\tau_h\|_{l^2} - \|\tau - \tau_h\|_{l^2} \geq \lambda(1 - Ch^k) \tag{A.2}$$

and, dividing by λ and using (A.1), we obtain the suboptimal estimate

$$1 - Ch^k \leq \|\nabla\pi\overline{\tau}_h\|_{l^2} \leq 1 + Ch^{k+1}. \tag{A.3}$$

The lower bound (A.3) can be improved in an iterative way as follows. We consider

$$\lambda\|\nabla\pi\overline{\tau}_h\|_{l^2} = \|\tau\|_{l^2} \geq \|P\tau_h\|_{l^2} - \|P\tau_h - \tau\|_{l^2}. \tag{A.4}$$

Then, using again the reverse triangular inequality, we have that

$$\|P\tau_h\|_{l^2} = \lambda\|P\overline{\tau}_h\|_{l^2} \geq \lambda(\|\overline{\tau}\|_{l^2} - \|\overline{\tau} - P\overline{\tau}_h\|_{l^2}) = \lambda(1 - \|\overline{\tau} - P\overline{\tau}_h\|_{l^2}). \tag{A.5}$$

Since $\overline{\tau}, n, \nu$ form an orthonormal basis of $\mathbb{R}^3$ and recalling that P maps vectors into the tangential space of Γ (hence have a null normal component), we get

$$\begin{aligned} \lambda(1 - \|\overline{\tau} - P\overline{\tau}_h\|_{l^2}) &= \lambda(1 - \|1 - (\overline{\tau}, P\overline{\tau}_h)\overline{\tau} - (n, P\overline{\tau}_h)n\|_{l^2}) \\ &\geq \lambda(1 - \|(1 - (\overline{\tau}, \overline{\tau}_h))\|_{l^2} - \|(n, \overline{\tau}_h)\|_{l^2}) \\ &\geq \lambda(1 - \|\overline{\tau} - \overline{\tau}_h\|_{l^2}^2 - \|(n, \overline{\tau}_h)\|_{l^2}). \end{aligned} \tag{A.6}$$

Now

$$\overline{\tau}_h - \overline{\tau} = \left(P_h - \frac{\nabla\pi}{\|\nabla\pi\overline{\tau}_h\|_{l^2}}\right)\overline{\tau}_h,$$

so using (A.3) and a Taylor expansion argument, it is easy to see that

$$\|\overline{\tau}_{\widehat{e}_h} - \overline{\tau}_{\widehat{e}_h^\ell}\|_{l^2} \lesssim h^k. \tag{A.7}$$

To deal with the last term of (A.6) we note that

$$(n, \overline{\tau}_h) = (\overline{\tau} \times \nu, \overline{\tau}_h) = (\nu, \overline{\tau}_h \times \overline{\tau}) = \left(\nu, \overline{\tau}_h \times \frac{\nabla\pi\overline{\tau}_h}{\|\nabla\pi\overline{\tau}_h\|_{l^2}}\right).$$

Then, using the suboptimal lower bound (A.3) and a Taylor expansion argument, we get

$$\left(\nu, \overline{\tau}_h \times \frac{\nabla\pi\overline{\tau}_h}{\|\nabla\pi\overline{\tau}_h\|_{l^2}}\right) = \frac{1}{\|\nabla\pi\overline{\tau}_h\|_{l^2}}(\nu, \overline{\tau}_h \times \nabla\pi\overline{\tau}_h) \lesssim |(\nu, \overline{\tau}_h \times \nabla\pi\overline{\tau}_h)|.$$

Using the definition of P and (2.2), we have that

$$\nabla\pi\overline{\tau}_h = (P - dH)\overline{\tau}_h = \overline{\tau}_h - (\nu \cdot \overline{\tau}_h)\nu - dH\overline{\tau}_h. \tag{A.8}$$

Now, using (A.8), we can write

$$(\nu, \overline{\tau}_h \times \nabla\pi\overline{\tau}_h) = \Big(\nu, \overline{\tau}_h \times (\overline{\tau}_h - (\overline{\tau}_h \cdot \nu)\nu - dH\overline{\tau}_h)\Big) = -(\nu, \overline{\tau}_h \times dH\overline{\tau}_h).$$

Hence,

$$\|(n, \overline{\tau}_h)\|_{l^2} \lesssim \|d\|_{L^\infty}\|(\nu, \overline{\tau}_h \times H\overline{\tau}_h)\|_{l^2} \lesssim h^{k+1}. \tag{A.9}$$

Combining (A.9) and (A.7) with (A.6) we obtain that

$$\|P\tau_h\|_{l^2} \geq \lambda(1 - \|(1 - (\overline{\tau}, P\overline{\tau}_h))\overline{\tau} - (n, P\overline{\tau}_h)n\|_{l^2}) \geq \lambda(1 - Ch^{k+1}). \tag{A.10}$$

For the second term in the right-hand side of (A.4), notice that

$$\|\tau - P\tau_h\|_{l^2} = \|\nabla\pi\tau_h - P\tau_h\|_{l^2} = \|dH\tau_h\|_{l^2} \leq \lambda Ch^{k+1}. \tag{A.11}$$

We are now ready to improve the lower bound in (A.3). By making use of (A.11) and (A.10) in (A.4), we get

$$\|\nabla\pi\overline{\tau}_h\|_{l^2} \geq 1 - Ch^{k+1} \tag{A.12}$$

which proves (4.3e).

To prove (4.3g), we need to first prove the following auxiliary estimates:

$$|(\overline{\tau}, n_h)| \lesssim h^{k+1}, \tag{A.13}$$

$$|1 - (n, n_h)| \lesssim h^{2k}. \tag{A.14}$$

We start showing (A.13). Using the property of the cross product, we get

$$(\overline{\tau}, n_h) = (\overline{\tau}, \nu_h \times \overline{\tau}_h) = (\nu_h, \overline{\tau}_h \times \overline{\tau}) = (\nu_h, \overline{\tau}_h \times \nabla\pi\overline{\tau}_h). \tag{A.15}$$

Replacing (A.8) in (A.15), we obtain

$$(\overline{\tau}, n_h) = [\nu \cdot (\overline{\tau}_h - \overline{\tau})](\overline{\tau}_h, \nu \times \nu_h) - (\nu_h, \overline{\tau}_h \times dH\overline{\tau}_h).$$

Taking the absolute value and using (4.3a), (4.3c), and (A.7), we find

$$|(\overline{\tau}, n_h)| \lesssim h^{2k+1} + Ch^{k+1} \lesssim h^{k+1}.$$

In order to prove (A.14), we start showing that the following holds,

$$|(\nu, n_h)| \lesssim h^k. \tag{A.16}$$

Indeed, using again the properties of the cross and scalar products, we obtain

$$|(\nu, n_h)| = |(\nu, \nu_h \times \overline{\tau}_h)| = |(\nu_h, \overline{\tau}_h \times \nu)| = |(\nu_h, \overline{\tau}_h \times (\nu - \nu_h))| \lesssim h^k.$$

Since the vector n_h is of unit length, there exist $a(x), b(x), c(x) \in \mathbb{R}$ satisfying $a^2 + b^2 + c^2 = 1$ such that

$$n_h = a\overline{\tau} + bn + c\nu,$$

where $a = (\overline{\tau}, n_h)$, $b = (n, n_h)$, and $c = (\nu, n_h)$. Hence, using (A.13), (A.16), and a Taylor expansion argument, we get

$$b = \pm\sqrt{1 - a^2 - c^2} = \pm\sqrt{1 + O(h^{2k})} = \pm 1 + O(h^{2k}).$$

The inequality (A.14) follows by assuming that the mesh size h of $\widehat{\mathcal{T}}_h$ is chosen small enough so that $b = 1 + O(h^{2k})$. Finally, writing $Pn_h = (\overline{\tau}, Pn_h)\overline{\tau} + (n, Pn_h)n$, we obtain (4.3g), i.e.,

$$\begin{aligned} \|n - Pn_h\|_{L^\infty(\widehat{e}_h)} &= \|n - (\overline{\tau}, Pn_h)\overline{\tau} + (n, Pn_h)n\|_{L^\infty(\widehat{e}_h)} \\ &\leq |1 - (n, Pn_h)| + |(\overline{\tau}, Pn_h)| \\ &= |1 - (n, n_h)| + |(\overline{\tau}, n_h)| = O(h^{k+1}). \qquad \square \end{aligned}$$

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