

Maximizing Nash Social Welfare in 2-Value Instances: The Half-Integer Case

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Abstract

We consider the problem of maximizing the Nash social welfare when allocating a set G of indivisible goods to a set N of agents. We study instances, in which all agents have *2-value* additive valuations: The value of a good $g \in G$ for an agent $i \in N$ is either 1 or s , where s is an odd multiple of $1/2$ larger than one. We show that the problem is solvable in polynomial time.

In [ACH⁺22] it was shown that this problem is solvable in polynomial time if s is integral and is NP-hard whenever $s = p/q$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$ are co-prime and $p > q \geq 3$. For the latter situation, an approximation algorithm was also given. It obtains an approximation ratio of at most 1.0345. Moreover, the problem is APX-hard, with a lower bound of 1.000015 achieved at $p/q = 5/4$. *The case $q = 2$ and odd p was left open.*

In the case of integral s , the problem is separable in the sense that the optimal allocation of the heavy goods (= value s for some agent) is independent of the number of light goods (= value 1 for all agents). This leads to an algorithm that first computes an optimal allocation of the heavy goods and then adds the light goods greedily. This separation no longer holds for $s = 3/2$; a simple example is given in the introduction. Thus an algorithm has to consider heavy and light goods together. This complicates matters considerably. Our algorithm is based on a collection of improvement rules that transfers any allocation into an optimal allocation and exploits a connection to matchings with parity constraints.

1 Introduction

The current paper extends [ACH⁺22]. It is therefore appropriate to quote parts of the introduction. “Fair division is an important area at the intersection of economics and computer science. While fair division with divisible goods is relatively well-understood in many contexts, the case of *indivisible* goods is significantly more challenging. Recent work in fair division has started to examine extensions of standard fairness concepts such as envy-freeness to notions such as EF1 (envy-free up to one good) [LMMS04] or EFX (envy-free up to any good) [CKM⁺16], most prominently in the case of non-negative, additive valuations of the agents. In this additive domain, notions of envy-freeness are

closely related to the *Nash social welfare (NSW)*, which is defined by the geometric mean of the valuations. An allocation maximizing the Nash social welfare is Pareto-optimal, satisfies EF1 [CKM⁺16] and in some cases even EFX [ABF⁺20]. An important question is, thus, if we can efficiently compute or approximate an allocation that maximizes NSW. This is the question we study in this paper.

More formally, we consider an allocation problem with a set N of n agents and a set G of m indivisible goods. Each agent $i \in N$ has a valuation function $v_i : 2^G \rightarrow \mathbb{Q}_{\geq 0}$. We assume all functions to be non-negative, non-decreasing, and normalized to $v_i(\emptyset) = 0$. For convenience, we assume every v_i maps into the rational numbers, since for computation these functions are part of the input. The goal is to find an allocation of the goods $A = (A_1, \dots, A_n)$ to maximize the Nash social welfare, given by the geometric mean of the valuations

$$\text{NSW}(A) = \left(\prod_{i=1}^n v_i(A_i) \right)^{1/n}.$$

Clearly, an allocation that maximizes the NSW is Pareto-optimal. By maximizing the NSW, we strike a balance between maximizing the sum-social welfare $\sum_i v_i(A_i)$ and the egalitarian social welfare $\min_i v_i(A_i)$. Notably, optimality and approximation ratio for NSW are invariant to scaling each valuation $v_i(A_i)$ by an agent-specific parameter $c_i > 0$. This is yet another property that makes NSW an attractive objective function for allocation problems. It allows a further normalization – we can assume every $v_i : 2^G \rightarrow \mathbb{N}_0$ maps into the *natural* numbers.

Maybe surprisingly, finding desirable approximation algorithms for maximizing the NSW has recently become an active field of research. For instances with additive valuations, where $v_i(A) = \sum_{g \in A} v_{ig}$ for every $i \in N$, in a series of papers [CDG⁺17, CG18, AGSS17, BKV18a] several algorithms with small constant approximation factors were obtained. The currently best factor is $e^{1/e} \approx 1.445$ [BKV18a]. The algorithm uses prices and techniques inspired by competitive equilibria, along with suitable rounding of valuations to guarantee polynomial running time.

Even for identical additive valuations, the problem is NP-hard, and a greedy algorithm with factor of 1.061 [BKV18b] as well as a PTAS [NR14] were obtained. In terms of inapproximability, the best known lower bound for additive valuations is $\sqrt{8/7} \approx 1.069$ [GHM18]. Notably, this lower bound applies even in the case when the additive valuation is composed of only three values with one of them being 0 (i.e., $v_{ig} \in \{0, p, q\}$ for all $i \in N, g \in G$, where $p, q \in \mathbb{N}$). For the case of two values with one 0 and one positive value (i.e., $v_{ig} \in \{0, q\}$ for all $i \in N, g \in G$), an allocation maximizing the NSW can be computed in polynomial time [BKV18b].” (end of quote).

The paper [ACH⁺22] considers computing allocations with (near-)optimal NSW when every agent has a 2-value valuation with both values non-zero. In such an instance, $v_{ig} \in \{1, s\}$ for every $i \in N$ and $g \in G$, where $s = p/q$, $p, q \in \mathbb{N}$, and $p > q$. In 2-value instances any optimal allocation satisfies EFX, which is not true when agents have 3 or more values [ABF⁺20]. [ACH⁺22] gives a polynomial algorithm for the case of integral s ; for general s , the algorithm guarantees an approximation factor of at most 1.035. This is drastically lower than the constant factors obtained for general additive valuations [CDG⁺17, CG18, BKV18a]. An approximation algorithm for 2-value instances with approximation factor 1.061 has been obtained in [GM21].

Complementing these positive results, the paper also establishes new hardness results for 2-value instances. Maximizing the NSW is NP-hard whenever p and q are co-prime and $q \geq 3$. Maximizing the NSW in 2-value instances can even be APX-hard. A lower bound on the approximation factor of 1.000015 is shown for $s = 5/4$. *The case $q = 2$ and odd p was left open.*

Contribution and Results. We give a polynomial time algorithm for the case $s = p/2$ and p any odd integer greater than 2. Call a good *heavy* if it has value s for some agent and *light* if it has value 1 for all agents. A heavy good is allocated as a heavy good if it is allocated to an agent that considers it heavy. As in [ACH⁺22], we first compute an optimal allocation under the additional assumption that all heavy goods must be allocated as heavy goods. In a second step, we then relax this assumption. It is not always best to allocate heavy goods as heavy as the following example shows: Assume that there are two agents and two goods. One agent values all goods as heavy and the other agent values no good as heavy. The only allocation with non-zero NSW allocates one good to each agent; then a heavy good is allocated as a light good.

Assume now that all heavy goods have to be allocated as heavy goods. In the case of integral s , the problem is separable in the sense that the optimal allocation of the heavy goods is independent of the number of light goods. So one first computes the optimal allocation of the heavy goods and then allocates the light goods greedily. This separation no longer holds. Consider two agents with identical valuations. There are two heavy goods and either two or three light goods. If there are two light goods, each agent should receive a heavy and a light good. Then both have value $5/2$. If there are three light goods, one agent receives two heavy goods and the other agent receives three light goods. Then both have value 3. As the problem is no longer separable, an algorithm has to consider heavy and light goods together. This complicates matters considerably. Our algorithm is based on a collection of improvement rules that transfers any allocation into an optimal allocation and exploits a connection to matchings with parity constraints.

This paper is structured as follows. In Section 2, we give an informal introduction to the improvement rules. The bulk of the paper is Section 3, where we characterize optimal allocations under the assumption that all heavy goods are allocated as heavy goods. We first introduce a set of basic improvement rules (Section 3.1) and show that all not-small-valued bundles are identified after the application of these rules (Section 3.2). More precisely, let A be the allocation maintained by the algorithm and let x be the minimal value of a bundle in A after the application of the basic improvement rules. Then all bundles of value larger than $x + 1$ also exist in an optimal allocation. We remove these bundles and are left with an allocation in which all bundles have values in $\{x, x + 1/2, x + 1\}$. We deal with such allocations in Section 3.3. We introduce an additional improvement rule in Section 3.3.2 and show that whenever A is suboptimal, one of the improvement rules is applicable. In Section 4, we give the algorithm. Here, we make the connection to matchings with parity constraints. Finally, Section 5 removes the assumption that all heavy goods must be allocated as heavy goods. Section H discusses certificates of optimality; it is preliminary.

1.1 Related Work

We again quote [ACH⁺22]. “In addition to additive valuations, the design of approximation algorithms for maximizing NSW with *submodular valuations* has been subject to significant progress very recently. While small constant approximation factors have been obtained for special cases [GHM18, AMGV18] (such as a factor $e^{1/e}$ for capped additive-separable concave [CCG⁺18] valuations), (rather high) constants for the approximation of NSW with Rado valuations [GHV21] and also general non-negative, non-decreasing submodular valuations [LV21] have been obtained.

Interestingly, for *dichotomous* submodular valuations where the marginal valuation of every agent for every good g has only one possible non-negative value (i.e., $v_i(S \cup \{g\}) - v_i(S) \in \{0, p\}$ for $p \in \mathbb{N}$), an allocation maximizing the NSW can be computed in polynomial time [BEF21]. In particular, in this case one can find in polynomial time an allocation that is Lorenz dominating, and simultaneously minimizes the lexicographic vector of valuations, and maximizes both sum social welfare and Nash

social welfare. Moreover, this allocation also has favorable incentive properties in terms of misreporting of agents.

Approximation algorithms for maximizing NSW have also been obtained for subadditive valuations [BBKS20, CGM21] and asymmetric agents [GKK20], albeit thus far not with constant approximation ratios.”

2 Improvement Rules

As already mentioned in the introduction, the goal is to identify improvement rules that transform any suboptimal allocation into an optimal allocation. So if A is a suboptimal allocation, one of the rules applies and improves the NSW of A . In this section, we give an informal introduction to our collection of improvement rules. For this discussion we assume that all heavy goods are allocated as heavy goods, and that $s = 3/2$.

Example 1 Consider a light good. A light good can be given to any agent. So if one has two bundles of values x and y with $x < y$, and an unallocated light good, one should allocate the good to the lighter bundle, since $(x + 1)/x > (y + 1)/y$ and hence allocating the light good to the lighter bundle leads to a greater increase in NSW. This rule is called greedy allocation of light goods. Another way of stating this rule is: If x is the minimum value of any bundle and there is a bundle of value larger than $x + 1$ containing a light good, move the light good to the bundle of value x .

Example 2 We turn to the allocation of heavy goods and the interaction between heavy and light goods. Assume we have two agents i and j owning bundles of value x and $x + 1$, respectively, and i likes a heavy good in j 's bundle and owns a light good. Then moving a heavy good from j to i and a light good from i to j would give both bundles value $x + 1/2$ and improve NSW. The connection between i and j does not have to be direct, but can go through an alternating path as the following example shows:

$$i \xrightarrow{\bar{A}} g \xrightarrow{A} u \xrightarrow{\bar{A}} g' \xrightarrow{A} j.$$

In this example, i owns a light good and likes a heavy good g which u owns, who in turn likes a heavy good g' owned by j . In this diagram, an edge between an agent and a good indicates that the good is heavy for the agent. The superscript A indicates that the good is allocated to the agent, the superscript \bar{A} indicates that the good is not allocated to the agent. We move g' from j to u and g from u to i and move a light good from i to j . The change in the allocation of heavy goods is akin to augmenting the path from i to j to A^H ; here A^H denotes the allocation of the heavy goods in A . Of course, the path from i to j might have more than one intermediate agent.

Augmenting alternating paths is powerful, but not enough. We give four examples to illustrate this point. The first three examples reillustrate that it is advantageous to have bundles of value $x + 1$ containing a light good. In Example 5, we even make a preparatory move to create such a bundle.

Example 3 Assume agents i and j own bundles of value x and agent h owns a bundle of value $x + 1$ containing a light good. i owns a light good and likes a heavy good in j 's bundle. i takes the heavy good from j in return for a light good and j gets another light good from h . There are three agents involved in the update. Before the update, i and j own bundles of value x and h owns a bundle of value $x + 1$. After the update, we have two bundles of value $x + 1/2$ and one bundle of value x . There is another way to interpret this change. We first make a move that does not change NSW: we move a light good from h to j turning h into a bundle of value x and j into a bundle of value $x + 1$. In a second

step, we have the exchange between i and j ; i receives a heavy good from j in return for a light good. We will refer to h as a facilitator, a bundle of value $x+1$ containing a light good.

Example 4 We can also start with two bundles of value $x+1$ and one bundle of value x . One of the bundles of value $x+1$ contains two light goods and its owner likes a heavy good in the other bundle of value $x+1$. He takes the heavy good in return for a light good and gives the other light good to a bundle of value x . As an effect both bundles of value $x+1$ turn into bundles of value $x+1/2$ and the bundle of value x becomes a bundle of value $x+1$.

Example 5 We have four bundles with values x , x , $x+1/2$ and $x+1$, respectively. One of the bundles of value x likes a heavy good in the other bundle of value x and owns a light good. We would be in the situation of the third example, if the bundle of value $x+1$ would contain a light good. Assume it does not, but the bundle of value $x+1/2$ likes a heavy good in the bundle of value $x+1$ and contains two light goods. It pulls a heavy good from the bundle of value $x+1$ in return for a light good. So the two bundles swap values and the utility profile does not change. However, we now have a bundle of value $x+1$ containing a light good. Such bundles facilitate transactions.

Example 6 The final example shows that we need structures that go beyond augmenting paths. Consider the following structure

$$i \xrightarrow{A} g_1 \xrightarrow{\bar{A}} h \xrightarrow{\bar{A}} g_2 \xrightarrow{A} j$$

and assume that h owns three light goods and i and j own bundles of value x and $x+1$ respectively. Note that h is interested in g_1 and g_2 . h gives two light goods to i and one light good to j and we change the heavy part of the allocation to

$$i \xrightarrow{\bar{A}} g_1 \xrightarrow{A} h \xrightarrow{A} g_2 \xrightarrow{\bar{A}} j$$

Note that the value of h 's bundle does not change; h gives away three light goods and obtains two heavy goods; i and j now own bundles of value $x+1/2$.

3 All Heavy Goods are Allocated as Heavy Goods

Throughout this section we assume that all heavy goods are allocated as heavy. We drop this assumption in Section 5. When we refer to the value of a bundle we mean the value to its owner. A is our current allocation which the algorithm tries to change into an optimal allocation by the application of improvement rules, O is either any optimal allocation or an optimal allocation closest to A in a sense to be made precise below. We use x to denote the minimum value of any bundle in A and A_d to denote the set of agents that own a bundle of value $x+d$. We will use this notation only with $d \in \{-1/2, 0, 1/2, 1, 3/2\}$. We will use the short-hand "a bundle in A_d " for a bundle owned by an agent in A_d .

A_i is the bundle owned by agent i . We use h , i , j and sometimes u and v to denote agents and g and g' to denote goods. v_i is the valuation function of agent i and $w_i := v_i(A_i)$ is the value of i 's bundle for i in A . We say that a bundle is *heavy-only* if it does not contain a light good. A bundle of value $x+1$ containing a light good is called a *facilitator*.

Our goal is to transform A into an optimal allocation by the application of improvement rules. Each improvement lexicographically increases the potential function:

$$\Phi = (\text{NSW}(A), \text{number of agents in } A_1 \text{ owning a light good}).$$

Let G^H be the bipartite graph with agents on one side and heavy goods on the other side. There is an edge connecting an agent i and a good g , if g is heavy for i . We use A^H and O^H to denote the heavy part of the allocations A and O . They are subsets of edges of G^H . The allocation O is closest to A if $A^H \oplus O^H$ has minimum cardinality among all optimal allocations.

The improvement rules are based on alternating paths. We will consider two kinds of alternating paths in G^H : A - \bar{A} -alternating paths and A - O -alternating paths. In an A - B -alternating path, the edges in $A^H \setminus B^H$ and $B^H \setminus A^H$ alternate. We will use A - \bar{A} -alternating paths in the algorithm and A - O -alternating paths for showing that some improvement rule applies to any suboptimal A .

Let i and j be agents. An A - B -alternating path from i to j is an A - B -alternating path with endpoints i and j in which i is incident to an edge in A (and hence j is incident to an edge in B). So a B - A -alternating path from i to j uses a B -edge incident to i .

3.1 Basic Improvement Rules

Lemma 1 *Let A be any allocation and let x be the minimum value of any bundle in A . Let i be any agent. For parts b) to g), let j be any other agent, and let p be an A - \bar{A} -alternating path from i to j .*

- a) *If $w_i > x + 1$ and A_i contains a light good, moving the light good to a bundle of value x improves the NSW of A .*
- b) *If $w_i \geq w_j + \lceil s \rceil$, augmenting p to A improves the NSW of A .*
- c) *If $w_i \geq w_j + 1$ and A_j contains more than $s - (w_i - w_j)$ light goods, augmenting p to A and moving $\max(0, \lceil s - (w_i - w_j) + 1/2 \rceil)$ light goods from j to i improves the NSW of A .*
- d) *If $w_i \in \{x + 1, x + 3/2\}$, $w_j = x + 1$, and A_j contains $\lceil s \rceil$ light goods, augmenting p to A and moving $\lceil s \rceil$ light goods from A_j to A_i and another light good from A_j to any bundle of value x improves the NSW of A .*
- e) *If j owns two heavy goods less than i and $w_i \geq x + 3/2$, one of the cases b), c), or d) applies and A can be improved.*
- f) *If $w_i = x$, $w_j = x + 1$, and A_j contains $\lceil s \rceil$ light goods, augmenting p to A and moving $\lceil s \rceil$ light goods from A_j to A_i improves the NSW of A .*
- g) *If $w_i = x + 1$, $w_j = x + 1/2$, A_i is heavy-only, and A_j contains at least $\lceil s \rceil$ light goods, augmenting p to A and moving $\lceil s \rceil$ light goods from A_j to A_i leaves the NSW of A unchanged and increases the number of bundles of value $x + 1$ containing a light good.*

For the proof see Appendix B.

3.2 Range Reduction

We need a finer distinction of the rules in Lemma 1. Let $x = x_A$ be the minimum value of a bundle in A and let k_0 be minimal such that $k_0 s > x + 1$. We call the rules a) to d) when applied with an agent i of value larger than $x + 1$ reduction rules. An allocation A is *reduced* if no reduction rule applies to it. We will show that for a reduced allocation A and an optimal allocation O closest to A , the bundles of value ks in A and O are identical for all $k \geq k_0$. This will allow us to restrict attention to the bundles of value x , $x + 1/2$, and $x + 1$ in A and to the bundles of value x , $x \pm 1/2$, and $x + 1$ in O .

Observation 1 *If O is an optimal allocation closest to A , $A^H \oplus O^H$ is acyclic.*

We decompose $D := A^H \oplus O^H$ into alternating paths. Let $hdeg_i^A$ ($hdeg_i^O$) be the number of A -edges (O -edges) incident to i in A^H (O^H). For i , we form $\min(hdeg_i^A, hdeg_i^O) - |A_i^H \cap O_i^H|$ pairs of A and O -edges incident to i . Then $\max(hdeg_i^A, hdeg_i^O) - \min(hdeg_i^A, hdeg_i^O)$ alternating paths start in i . Depending on which degree is larger, the paths start with an A - or an O -edge. The decomposition of $A^H \oplus O^H$ into alternating paths is not unique.

Lemma 2 *Let A be reduced and let O be an optimal allocation closest to A . Let k_0 be minimal with $k_0s > x + 1$. For $k \geq k_0$ let R_k be the set of agents that own a bundle of value ks in A and let $S_k = \cup_{j \geq k} R_j$ be the agents that own a bundle of value at least ks in A . Agents in R_k own k heavy goods and no light good in A . O agrees with A on S_{k_0} , i.e., $A_i = O_i$ for all $i \in S_{k_0}$. Moreover, in A and O each agent in $[n] \setminus S_{k_0}$ owns at most $k_0 - 1$ heavy goods.*

Proof: We use downward induction on k to show that O and A agree on S_k . Assume that they agree on S_{k+1} ; then agents in S_{k+1} have degree zero in $A^H \oplus O^H$. Since S_k is empty for large enough k , the induction hypothesis holds for large enough k . Throughout this proof we use y as an abbreviation for ks , i.e., $y := ks \geq x + 3/2$ and let

$$R'_k := R_k \cup \{j; j \notin S_{k+1} \text{ and there is an } A\text{-}O\text{-alternating path from } i \in R_k \text{ to } j\}.$$

The proofs of the following claims can be found in Appendix C.

Claim 1 *In A , the bundles in $R'_k \setminus R_k$ contain exactly $k - 1$ heavy goods.*

Claim 2 *In O , each bundle in $[n] \setminus S_{k+1}$ contains at most k heavy goods.*

Claim 3 *All heavy goods assigned to agents in R'_k by A are also assigned to them in O .*

Claim 4 *O assigns k heavy goods to at least $|R_k|$ agents in R'_k .*

Claim 5 $x_O + 1 \leq ks$.

Claim 6 *Agents in R_k own k heavy goods in O .*

Claim 7 *O agrees with A on R_k , i.e., $O_i = A_i$ for all $i \in R_k$. Bundles owned by agents in $[n] \setminus S_k$ do not contain k heavy goods in either A or O .*

We have now shown that A and O agree on S_{k_0} . Also agents in $[n] \setminus S_{k_0}$ own at most $k_0 - 1$ heavy goods in A by definition of S_{k_0} and in O by Claim 7. Finally, $x_O + 1 \leq k_0s$ by Claim 5. ■

At this point, we know that $A_i = O_i$ for all $i \in S_{k_0}$, where k_0 is minimal with $k_0s > x + 1$ and S_{k_0} is the set of agents that own bundles of value at least k_0s in A . We may therefore remove the agents in S_{k_0} and their bundles from further consideration. We call A and O *shrunk* after this reduction. The remaining bundles have value x , $x + 1/2$, and $x + 1$ in A and value x_O , $x_O + 1/2$, \dots in O . Moreover, the remaining bundles contain at most $k_0 - 1$ heavy goods in both A and O and $x_O \leq x + 1/2$. The latter holds since the average value of a bundle in $A_{low} = A_0 \cup A_{1/2} \cup A_1$ is strictly less than $x + 1$ as there is at least one bundle of value x and the average for O is the same.

Theorem 1 *Let A be reduced and let O be an optimal allocation closest to A . Then O consists of A restricted to the agents in S_{k_0} and an optimal allocation for the bundles in $A_{low} = A_0 \cup A_{1/2} \cup A_1$.*

Proof: By the previous lemma, $A_i = O_i$ for all $i \in S_{k_0}$. The remaining bundles in A belong to A_{low} . O allocates the goods in these bundles optimally to the agents in A_{low} . ■

3.3 Only Bundles of Value x , $x + 1/2$, and $x + 1$ in A

We derive further properties of optimal allocations for A_{low} and then introduce an additional reduction rule.

3.3.1 Further Properties of Optimal Allocations

The proofs of all Lemmas in this section can be found in Appendix D.

Lemma 3 *Let A be reduced and shrunken. If there is an optimal allocation having at least as many bundles of value $x + 1$ as A does, A is optimal. In particular, if A has no bundles of value $x + 1$, A is optimal.*

Lemma 4 *Let A be reduced and shrunken and let O be an optimal allocation closest to A . Then $x_O \geq x - 1/2$.*

Lemma 5 *Let A be reduced and shrunken and let O be an optimal allocation closest to A . There is no bundle of value more than $x + 1$ in O .*

In the technical introduction (Section 2) we pointed to the importance of bundles of value $x + 1$ containing a light good. The following Lemma formalizes this observation.

Lemma 6 *Let A be reduced and shrunken and assume further that Lemma 1g) is not applicable. Let O be an optimal allocation closest to A , and consider an agent $i \in A_1$. If A_i is heavy-only, O_i is heavy-only and has value $x + 1$. If all bundles in A_1 are heavy-only, A is optimal.*

In the rest of this section, we briefly summarize what we have obtained so far. Let A be reduced and shrunken and let O be an optimal allocation closest to A . Assume further that Lemma 1g) is not applicable to A . Let x be the minimum value of any bundle in A and let k_0 be minimal such that $k_0s > x + 1$.

- The bundles in A have value x , $x + 1/2$, or $x + 1$, and there is a bundle of value x .
- $x - 1/2 \leq x_O \leq x + 1/2$.
- In A and O bundles contain at most $k_0 - 1$ heavy goods. Any bundle of value more than $(k_0 - 1)s$ must contain a light good.
- If A_i has value $x + 1$ and is heavy-only, O_i has value $x + 1$ and is heavy-only. If all bundles of value $x + 1$ in A are heavy-only, A is optimal. Conversely, if A is suboptimal, there is a bundle of value $x + 1$ in A containing a light good.
- Bundles in O have value at least x_O and at most $x + 1$. Since $x_O \geq x - 1/2$, bundles in O have values in $\{x - 1/2, x, x + 1/2, x + 1\}$.

In the next section, we will introduce improving walks as an additional improvement rule and then show that an allocation to which no improvement is applicable is optimal. For the optimality proof, we consider a suboptimal allocation and an optimal allocation closest to it and then exhibit an applicable improvement rule. In the light of Lemma 6, we may assume that A contains a bundle of value $x + 1$ containing a light good.

3.3.2 Improving Walks

As already mentioned in the introductory section on improvement rules (Section 2), we need more general improving structures than alternating paths. We need improving walks that we introduce in this section. Improving walks are also used in the theory of parity matchings, e.g., generalized matchings in which degrees are constrained to a certain parity; see, for example, the chapter on parity factors in [AK11].

Our goal is to show that, whenever A is suboptimal, an improving walk exists. Improving walks use only edges in $A^H \oplus O^H$. So assume that A is suboptimal. We will first show that there is an agent $i \in (A_0 \cup A_1) \cap O_{1/2}$ and that for such an agent $|A_i^H| \neq |O_i^H|$. The type of an edge is either A or O and we use T for the generic type. If $T = A$, \bar{T} denotes O . If $T = O$, \bar{T} denotes A . Missing proofs are contained in Appendix E.

- Lemma 7**
1. *The parity of the number of heavy goods is the same in bundles of value x and $x + 1$ and in bundles of value $x - 1/2$ and $x + 1/2$ and the former parity is different from the latter.*
 2. *The parity of the number of bundles of value x or $x + 1$ is the same in A and O and equally for the number of bundles of value $x - 1/2$ or $x + 1/2$. More precisely, for $d \in \{-1/2, 0, 1/2, 1\}$ let a_d and o_d be the number of bundles of value $x + d$ in A and O respectively, and let $a_1 = o_1 + z$. Then (the first equation is trivial; it is there for completeness)*

$$\begin{aligned} a_{-1/2} &= 0 = o_{-1/2} - o_{-1/2} & a_0 &= o_0 + 2o_{-1/2} + z \\ a_{1/2} &= o_{1/2} - 2z - o_{-1/2} & a_0 + a_1 &= o_0 + o_1 + 2(o_{-1/2} + z). \end{aligned}$$

3. *Let A be a suboptimal allocation and let O be an optimal allocation. Then $z > o_{-1/2} \geq 0$, $(A_0 \cup A_1) \cap O_{1/2}$ is non-empty. In particular, O contains a bundle of value $x + 1/2$.*

Remark: It is not true that $A_0 \cap O_{\pm 1/2}$ is guaranteed to be non-empty. Same for $A_1 \cap O_{\pm 1/2}$.

We will next prove a number of Lemmas that guarantee ownership of light goods for certain agents. The *heavy parity* of a bundle is the parity of the number of heavy goods in the bundle. A node v is *unbalanced* if $|A_v^H| \neq |O_v^H|$. A node v is *A-heavy* if $|A_v^H| > |O_v^H|$ and *O-heavy* if $|A_v^H| < |O_v^H|$.

Lemma 8 *Let v be unbalanced and let A_v and O_v have the same heavy parity.*

- *If v is A-heavy, O_v contains at least $2s$ light goods (except if $v \in O_0 \cap A_1$ or $v \in A_{1/2} \cap O_{-1/2}$, then $2s - 1$ light goods).*
- *If v is O-heavy, A_v contains at least $2s$ light goods ($2s - 1$ if $v \in O_1 \cap A_0$).*

Lemma 9 *Let $v \in A_0 \cup A_1$ be unbalanced.*

- *If v is A-heavy, O_v contains at least $\lfloor s \rfloor$ light goods if $v \in O_{1/2}$ and at least $2s - 1$ light goods if $v \in O_0 \cup O_1$. If $v \in O_{-1/2}$, O_v contains at least $\lfloor s \rfloor - 1$ light goods.*
- *If v is O-heavy, A_v contains at least $\lfloor s \rfloor$ light goods. If $v \in O_{\pm 1/2}$, v contains at least $\lfloor s \rfloor$ light goods if $v \in A_0$ and at least $\lceil s \rceil$ light goods if $v \in A_1$.*

Lemma 10 *Let $v \in (O_0 \cup O_1) \cap A_{1/2}$ be unbalanced.*

- If v is A -heavy, O_v contains at least $\lfloor s \rfloor$ light goods if $v \in O_0$ and at least $\lceil s \rceil$ light goods if $v \in O_1$.
- If v is O -heavy, A_v contains at least $\lfloor s \rfloor$ light goods. If $v \in O_0$, A_v contains at least $\lceil s \rceil$ light goods.

An A - O -walk from i to j is a sequence of $i = h_0, e_0, h_1, \dots, e_{\ell-1}, h_{\ell-1}, e_\ell, h_\ell = j$ of agents and edges such that:

1. i and j are agents, i is unbalanced and lies in $(A_0 \cup A_1) \cap O_{1/2}$, j is unbalanced and $j \in A_0 \cup A_1 \cup O_0 \cup O_1$ ($j \notin A_{1/2} \cap O_{\pm 1/2}$).
2. All edges belong to $A^H \oplus O^H$.
3. For $1 \leq t < \ell$, if h_t is a good, e_{t-1} and e_t have different types (one in A^H , one in O^H).
4. For $1 \leq t < \ell$, h_t is a through-node if the edges e_t and e_{t+1} have different types and is a T -hinge if both edges have type T . Hinges lie in $A_{1/2} \cap O_{\pm 1/2}$ and are unbalanced.

i and j are the endpoints of the walk and h_1 to $h_{\ell-1}$ are intermediate nodes. The type of i is the type of e_0 and the type of j is the type of e_ℓ . Goods have degree zero or two in $A^H \oplus O^H$. We will augment A - O -walks to either A or O . Augmentation to A will improve A and augmentation to O will either improve O or move O closer to A . We allow $i = j$; we will augment such A - O -walks to O . There is no requirement on ownership of light goods by hinges and endpoints. We will later show that A -hinges own $2s$ light goods in O and O -hinges own $2s$ light goods in A and that endpoints own an appropriate number of light goods.

Lemma 11 *Let W be an A - O -walk. Then $|W \cap A| = |W \cap O|$.*

Lemma 12 *If A is sub-optimal, an A - O -walk exists. If $i = j$ and the walk starts and ends with an edge of the same type, $|A_i^H|$ and $|O_i^H|$ differ by at least two.*

Proof: We construct the walk as follows. The walk uses only edges in $D = A^H \oplus O^H$ and visits each good at most once. We start with a node $i \in (A_0 \cup A_1) \cap O_{1/2}$; by Lemma 7 such a node exists. For such a node the parities of $|A_i^H|$ and $|O_i^H|$ differ. If $|A_i^H| > |O_i^H|$, we start tracing a walk starting at i with an A -edge, otherwise, we start with an O -edge.

Suppose we reach a node h on a T -edge e where $T \in \{A, O\}$. If there is an unused edge, i.e., not part of the walk, of type \bar{T} incident to h , we continue on this edge. This will always be the case for goods. We come back to this claim below.

So assume that there is no unused edge of type \bar{T} incident to h . Then h is an unbalanced T -heavy agent. This can be seen as follows. Any visit to a node uses edges of different types for entering and leaving the node as long as an unused edge of a different type is available for leaving the node. Any later visit either uses the same type for entering and leaving or uses up the last unused edge incident to the node.

If $h = i$, we stop. Note that if the first and the last edge of the walk have the same type, say T , the number of T -edges incident to i is at least two more than the number of \bar{T} -edges.

So assume $h \neq i$. If $h \in A_0 \cup A_1 \cup O_0 \cup O_1$, we stop; $j = h$. Otherwise, $h \in A_{1/2} \cap O_{\pm 1/2}$ and hence the number of heavy edges of both types incident to h has the same parity. Thus there is an unused T -edge incident to h . We pick an unused T -edge incident to h and continue on it.

Since the walk always proceeds on an unused edge and the first visit to a good uses up the A - and the O -edge incident to it, the walk visits each good at most once. ■

A walk is not necessarily a simple path. A walk is *semi-simple* if for different occurrences of the same agent, the incoming edges have different types. In particular, any agent can appear at most twice. Goods appear at most once on a walk.

Lemma 13 *If there is an A - O -walk, there is a semi-simple walk with the same endpoints.*

Hinge nodes lie in $A_{1/2} \cap O_{\pm 1/2}$. If none of the basic improvement rules applies to A , hinge nodes actually lie in $O_{1/2}$ and T -hinges own $2s$ light goods in the allocation \bar{T} for $T \in \{A, O\}$ as we show next.

Lemma 14 *If A is reduced, all hinge nodes belong to $O_{1/2}$, A -hinges own at least $2s$ light goods in O and O -hinges own at least $2s$ light goods in A .*

Proof: By definition, the hinge nodes are unbalanced and lie in $A_{1/2} \cap O_{\pm 1/2}$. Consider two consecutive hinges h and h' and the alternating path p connecting them. Assume that their values in O differ by one, i.e., one has value $x + 1/2$ and the other value $x - 1/2$. The A -endpoint of the path owns at least $2s - 1$ light goods in O according to Lemma 8. When we augment the path to O , the A -endpoint receives an additional heavy good and the \bar{A} -endpoint loses a heavy good. Depending on whether the A -endpoint is the heavier endpoint or not, it moves $\lceil s \rceil$ or $\lfloor s \rfloor$ light goods to the other endpoint. This improves the NSW of O , a contradiction. We have now shown that all hinge nodes have the same value in O .

It remains to show that the first hinge of the walk lies in $O_{1/2}$; call it h . Assume $h \in O_{-1/2}$. We distinguish cases according to whether i is A -heavy or not.

If i is O -heavy, h is A -heavy and hence owns at least $2s - 1$ light goods in O (Lemma 8). We augment p to O and move $\lfloor s \rfloor$ light goods from h to i . After the change i and h_1 belong to O_0 and the NSW of O has improved, a contradiction.

If i is A -heavy, i owns at least $2s - 1$ light goods in O (Lemma 8). We augment p to O and move $\lceil s \rceil$ light goods from i to h . After the change i and h belong to O_0 and the NSW of O has improved, a contradiction.

We now know that all hinges are unbalanced and belong to $A_{1/2} \cap O_{1/2}$. Thus Lemma 8 applies and A -hinges own at least $2s$ light goods in O and O -hinges own at least $2s$ light goods in A . ■

At this point, we have established the existence of an A - O -walk with endpoint $i \in (A_0 \cup A_1) \cap O_{1/2}$. If A is reduced, all hinge nodes belong to $A_{1/2} \cap O_{1/2}$ and T -hinges own $2s$ light goods in the allocation \bar{T} . We will next show that we can use the A - O walk to improve A . We distinguish cases according to whether i is O -heavy or A -heavy.

3.3.3 $i \in (A_0 \cup A_1) \cap O_{1/2}$ and i is O -heavy

The value of A_i is x or $x + 1$ and the value of O_i is $x + 1/2$ and A_i contains fewer heavy goods than O_i . Therefore A_i must contain at least $\lceil s \rceil$ light goods if $i \in A_1$ and at least $\lfloor s \rfloor$ light goods if $i \in A_0$. Let W be an O - A -walk starting in i and let j be the endpoint of the walk. The types of the hinges alternate along the path, the type of the first (last) hinge is opposite to the type of i (j). Each A -hinge holds $2s$ light goods in O and each O -hinge holds $2s$ light goods in A (Lemma 10). If the types of i and j differ, the number of hinges is even, if the types are the same, the number of hinges is odd.

We distinguish three cases: $j \in A_0 \cup A_1$ and $i \neq j$, $j \in A_0 \cup A_1$ and $i = j$, and $j \in (O_0 \cup O_1) \cap A_{1/2}$. In the first case, we show how to improve A and in the other two cases we will derive a contradiction to the assumption that O is closest to A .

Case $j \in A_0 \cup A_1$, and $i \neq j$: We augment the walk to A . The heavy parity of i and j changes; i gains a heavy edge and j gains or loses. The heavy parity of all intermediate nodes does not change. Each O -hinge releases $2s$ light goods and each A -hinge requires $2s$ light goods. If j is an O -endpoint, j gains a heavy edge and the number of A -hinges exceeds the number of O -hinges by one.

Assume first that i and j are O -endpoints. Each of them gives up $\lceil s \rceil$ light goods if in A_1 and $\lfloor s \rfloor$ light goods if in A_0 ; i and j own that many light goods (Lemma 9). So we have between $2s - 1$ and $2s + 1$ light goods; $2s$ of them are needed for the extra A -hinge. If we have $2s + 1$, one goes to an arbitrary bundle in A_0 and if we have $2s - 1$, we take a light from an arbitrary bundle in A_1 . Recall that if A is suboptimal, there is a bundle in A_1 containing a light good.

If j is an A -endpoint, i gains a heavy edge and j loses a heavy edge, i has $\lfloor s \rfloor$ or $\lceil s \rceil$ light goods. We move $\lfloor s \rfloor$ light goods from i to j . If $j \in A_0$, we put an additional light good on j which we either take from i or from a bundle in A_1 containing a light good. If $j \in A_1$ and i had $\lceil s \rceil$ light goods, we put the extra light good on any bundle in A_0 .

In either case, we increased the number of agents in $A_{1/2}$ by two and hence improved A .

Case $j \in A_0 \cup A_1$, and $i = j$: We augment the walk to O . For the intermediate nodes the heavy parity does not change. For i the heavy parity also does not change; it either gains and loses a heavy good or it loses two heavy goods. It remains to show that there are sufficiently many light goods to keep the values of all bundles in O unchanged.

If i loses and gains a heavy good, the number of A - and O -hinges is the same and we use the light goods released by the A -hinges for the O -hinges. If i loses two heavy goods, there is one more A -hinge and we use the light goods from the extra A -hinge for i . The change brings O closer to A , a contradiction to the choice of O . Hence this case cannot arise.

Case $j \in (O_0 \cup O_1) \cap A_{1/2}$: We augment the walk to O . The heavy parity of the intermediate nodes does not change. The heavy parity of i and j changes. i loses a heavy good and j either gains or loses a heavy good. It remains to show that there are sufficiently many light goods to keep the utility profile of O unchanged.

The A - and O -hinges on the walk alternate and their numbers are either the same, if i and j have different types, or there is an extra A -hinge if i and j are both O -endpoints. Each A -hinge releases $2s$ light goods and each O -hinge requires $2s$ light goods.

If j is an O -endpoint, we use the $2s$ light goods provided by the extra A -hinge as follows: If $j \in O_0$, we give $\lceil s \rceil$ light goods to j and $\lfloor s \rfloor$ light goods to i , moving j to $O_{1/2}$ and i to O_0 and if $j \in O_1$, we give $\lfloor s \rfloor$ light goods to j and $\lceil s \rceil$ light goods to i , moving j to $O_{1/2}$ and i to O_1 .

If j is an A -endpoint, it gains a heavy good. By Lemma 10, j owns $\lfloor s \rfloor$ light goods if $j \in O_0$ and owns $\lceil s \rceil$ light goods if $j \in O_1$. We move these goods to i . In either case, j moves to $O_{1/2}$ and i moves to $O_0 \cup O_1$.

In all cases, the utility profile of O does not change and O moves closer to A , a contradiction to our choice of O . So this case cannot arise.

3.3.4 $i \in (A_0 \cup A_1) \cap O_{1/2}$ and i is A -heavy

The value of O_i is $x + 1/2$, the value of A_i is x or $x + 1$, and A_i contains at least one more heavy good than O_i . We observe first that O_i contains at least $\lceil s \rceil$ light goods. If $i \in A_0$, the heavy value of O_i is at most $x - s$ and hence O_i contains at least $\lceil s \rceil$ light goods. If the value of A_i is $x + 1$, A_i cannot be heavy-only since then O_i would also have value $x + 1$ according to Lemma 6 and hence the heavy value of O_i is at most $x + 1 - 1 - s$. So O_i must contain at least $\lceil s \rceil$ light goods. Let W be an A - O -walk starting in i and let j be the other endpoint of the walk. The types of the hinges along the walk alternate, O -hinges hold $2s$ light goods in A , and O -hinges hold $2s$ light goods in O . If the types of i and j differ, there is an equal number of hinges of both types, if i and j are A -endpoints, there is an extra O -hinge on the walk.

Similar to Section 3.3.3, we distinguish three cases: $j \in A_0 \cup A_1$ and $i \neq j$, $j \in A_0 \cup A_1$ and $i = j$, and $j \in (O_0 \cup O_1) \cap A_{1/2}$. In the first case, we show how to improve A and in the other two cases we will derive a contradiction to the assumption that O is closest to A . The arguments are similar to the ones in Section 3.3.3 and hence moved to the Appendix F.

We have now established the existence of improving walks.

Theorem 2 *If A is sub-optimal, an improving A - O -walk exists.*

4 The Algorithm

Let A be reduced. Let x be the minimum value of any bundle and let k_0 be minimal such that $k_0 s > x + 1$. Theorem 1 tells us that an optimal allocation consists of all bundles of value at least $k_0 s$ in A plus an optimal allocation of $A_{low} = A_0 \cup A_{1/2} \cup A_1$. An optimal allocation of A_{low} can be obtained by repeated augmentation of improving walks. The optimal allocation for A_{low} uses only bundles of value x , $x + 1/2$ and $x + 1$.

4.1 Bundles of Value ks with $k \geq k_0$

Lemma 15 *Let ks be the maximum value of any bundle in A and assume $k \geq k_0$. Let T_k be all bundles of value ks and all bundles that can be reached from such a bundle with an A - \bar{A} -alternating path. All bundles in T_k own either k or $k - 1$ heavy goods and all heavy goods owned by the agents in T_k must be owned by them. Let n_k be the number of heavy goods owned by the agents in T_k . Then $n_k - (k - 1)|T_k|$ of the agents in T_k own k heavy goods.*

Proof: If a bundle containing $k - 2$ or less heavy goods can be reached from a bundle containing k heavy goods, the allocation is not optimal (Lemma 1). So all agents in T_k own either k or $k - 1$ heavy goods.

Consider any agent h liking a heavy good g owned by an agent $j \in T_k$. There is an A - \bar{A} -alternating path from an agent i of value ks to j . We can extend this path by $j \xrightarrow{A} g \xrightarrow{\bar{A}} h$. Thus $h \in T_k$ and hence any heavy good owned by an agent in T_k must be owned by an agent in T_k .

Let a be the number of agents in T_k that own k heavy goods. Then $n_k = ak + (|T_k| - a)(k - 1)$. Thus $a = n_k - |T_k|(k - 1)$. ■

We apply this argument repeatedly, i.e., we first identify T_k where ks is the maximum value of any bundle. We remove T_k from the instance and then construct T_{k-1} , and so on. The last step, $k = k_0$, requires more care. The bundles in T_{k_0} containing k_0 heavy goods also contain k_0 heavy goods in an

optimal allocation closest to A . We can thus safely remove these bundles. Altogether, we remove all bundles that contain k_0 or more heavy goods as they are guaranteed to also exist in some optimal allocation and are left with bundles of values x , $x + 1/2$ and $x + 1$. We deal with them in the next section.

4.2 Values x , $x + 1/2$, $x + 1$

We obtain an optimal allocation of A_{low} by repeated augmentation of improving walks. We identify improving walks by exploiting a connection to matchings with parity constraints.

4.2.1 Matchings with Parity Constraints

Consider a generalized bipartite matching problem, where for each node v we have a constraint concerning the degree of v in the matching M . We are interested in parity constraints of the form $\deg_M(v) \in \{p_v, p_v + 2, p_v + 4, \dots, p_v + 2r_v\}$, where p_v and r_v are non-negative integers. Matchings with parity constraints can be reduced to standard matching [Tut52, Tut54, L70, Cor88, Seb93]. For completeness, we review the construction given in [Cor88].

Consider any node v and let t be the degree of v . We replace v by the following gadget. We have t vertices v_1 to v_t . We refer to them as v -vertices. We also create $t - p_v$ vertices z_1 to z_{t-p_v} and connect each v_i with each z_j . Finally, we create the edges $(z_1, z_2), \dots, (z_{2r_v-1}, z_{2r_v})$. This ends the description of the gadget for v . For every edge (v, w) of the original graph, we have the complete bipartite graph between the vertices v_i and w_j of the auxiliary graph.

Lemma 16 ([Cor88]) *The auxiliary graph has a perfect matching if and only if the original graph has a matching satisfying the parity constraints.*

The number of vertices of the auxiliary graph is $O(m)$ and the number of edges of the auxiliary graph is $O(\sum_v \deg_v^2 + \sum_{(v,w) \in E} \deg_v \deg_w) = O(mn)$. Note that the number of edges is maximal if the graph has about m/n vertices of degree n and $n - m/n$ nodes of degree one.

4.2.2 The Reduction to Parity Matching

Let g be the maximum number of heavy goods that a bundle of value $x + 1/2$ may contain. Then g is the maximum integer such that $x + 1/2 - gs$ is non-negative and integral. The following Lemma gives the maximum number of heavy goods in bundles of value x and $x + 1$.

Lemma 17 *Let g be the maximum number of heavy goods that a bundle of value $x + 1/2$ may contain. The following table shows the maximum number of heavy goods in bundles of value x and $x + 1$. We use $s\mathbb{N}$ to denote $\{st; t \in \mathbb{N}\}$.*

x	$x + 1/2$	$x + 1$	Condition
$g - 1$	g	$g + 1$	$x + 1 \in s\mathbb{N}$
$g + 1$	g	$g + 1$	$x + 1 \notin s\mathbb{N}$ and $x + 1 > (g + 1)s$
$g - 1$	g	$g - 1$	$x + 1 \notin s\mathbb{N}$ and $x + 1 < (g + 1)s$

Clearly, $x + 1 \notin s\mathbb{N}$ implies $x + 1 \neq (g + 1)s$.

Proof: For $t \in \{0, 1/2, 1\}$, let m_t be the maximum number of heavy goods in a bundle of value $x + t$. Then $m_0 \leq m_1 \leq m_0 + 2$. The first inequality is obvious (add a light to the lighter bundle) and the second inequality follows because we may remove two heavy goods from the heavy bundle and add $2s - 1$ light goods. Also m_0 and m_1 have the same parity. Finally $m_{1/2} - m_0 = \pm 1$ since the two numbers have different parity and we can switch between the two values by exchanging a heavy good by either $\lfloor s \rfloor$ or $\lceil s \rceil$ light goods.

Let $x + 1/2 = gs + y$ with $y \in \mathbb{N}_0$. Then $x + 1 = gs + y + 1/2$. If $y + 1/2 = s$, $m_1 = g + 1$ and $m_0 = g - 1$. If $y + 1/2 > s$ then $y - 1/2 \geq s$ and hence $m_1 = m_0 = g + 1$. If $y + 1/2 < s$, then $x + 1 = (g - 1)s + (s + y + 1/2)$ and $x - 1 = (g - 1)s + (s + y - 1/2)$ and hence $m_1 = m_0 = g - 1$. ■

We use N_0 , $N_{1/2}$, and N_1 to denote the allowed number of heavy goods in bundles of value A_0 , $A_{1/2}$ and A_1 respectively.

Lemma 18 *Let A be an allocation with all values in $\{x, x + 1/2, x + 1\}$. A is sub-optimal if there is an allocation B^H of the heavy goods in A and a pair of agents i and j in $A_0 \cup A_1$ such that all agents in $A_{1/2} \cup \{i, j\}$ own a number of heavy goods in $N_{1/2}$ and for each of the agents in $A_0 \cup A_1 \setminus \{i, j\}$, the number of owned heavy goods is in the same N -set as in A . In addition, if i and j own bundles of value x in A , there must be a bundle of value $x + 1$ in A containing a light good.*

Proof: If A is sub-optimal there is an improving walk W . Let i and j be the endpoints of the walk. Augmenting the walk and moving the light goods around as described in Section 3.3.2

- adds i and j to $A_{1/2}$,
- reduces the weight of a bundle of value $x + 1$ containing a light good to x if A_i and A_j have value x and increases the weight of a bundle of value x to $x + 1$ if A_i and A_j have value $x + 1$, and
- leaves the value of all other bundles unchanged.

Thus in the new allocation the number of heavy goods owned by i and j lies in $N_{1/2}$. For all other agents the number of owned heavy goods stays in the same N -set. This proves the only-if direction.

We turn to the if-direction. Assume that there is an allocation B^H of the heavy goods in which for two additional agents i and j the number of owned heavy goods lies in $N_{1/2}$ and for all other agents the number of owned heavy goods stays in the same N -set. In addition if i and j own bundles of value $x + 1$ in A , there is an agent k owning a bundle of value $x + 1$ containing a light good. The goal is to allocate the light goods such that B^H becomes an allocation B , in which all bundles have value in $\{x, x + 1/2, x + 1\}$ and $B_{1/2} = A_{1/2} \cup \{i, j\}$. Then the NSW of B is higher than the one of A .

We next define the values of the bundles in B . For i and j , we define $v_i^B = v_j^B = x + 1/2$. If A_i and A_j have both value x , let k be an agent owning a bundle of value $x + 1$ containing a light good and define $v_k^B = x$. If A_i and A_j have both value $x + 1$, let k be an agent owning a bundle of value x and define $v_j^B = x + 1$. Then $v_i^A + v_j^A + v_k^A = v_i^B + v_j^B + v_k^B$ in both cases. If one of A_i and A_j has value x and the other one has value $x + 1$, leave k undefined. Then $v_i^A + v_j^A = v_i^B + v_j^B$. For all ℓ different from i , j , and k , let $v_\ell^A = v_\ell^B$. Then the total value of the bundles in A and B is the same.

For an agent ℓ let h_ℓ and h'_ℓ be the number of heavy goods allocated to ℓ in A and B^H , respectively. Then $\sum_\ell h_\ell = \sum_\ell h'_\ell$. Moreover, $h_\ell \in N_{1/2}$ iff $\ell \in A_{1/2}$ and $h'_\ell \in N_{1/2}$ iff $\ell \in A_{1/2} \cup \{i, j\}$. For all $\ell \in A_0 \cup \{k\} \setminus \{i, j\}$, $h'_\ell \in N_0$ and for all $\ell \in A_1 \setminus \{i, j, k\}$, $h'_\ell \in N_1$. Then $v_\ell^B - h'_\ell s$ is a non-negative integer for all ℓ and

$$\sum_\ell (v_\ell^B - h'_\ell s) = \sum_\ell v_\ell^B + \sum_\ell h'_\ell s = \sum_\ell v_\ell^A + \sum_\ell h_\ell = \sum_\ell (v_\ell^A - h_\ell s).$$

We conclude that the allocation B exists. ■

Example 7 Let $s = 3/2$ and assume we have two agents owning bundles of value 3 and 4 respectively. We have either zero or two or four heavy goods and accordingly seven, four or one light good. Both agents like all heavy goods. If there are two heavy goods the optimal allocation has two bundles of value $7/2$. If we have zero or four heavy goods, the optimal allocation has bundles of value 3 and 4. We have $N_{1/2} = \{1\}$ and there is no way to assign exactly one heavy good to each agent, if the number of heavy goods is zero or four. In the case of four heavy goods, 1 and 3 is possible. Both numbers are odd, but 3 is too large.

Example 8 Let $s = 3/2$ and assume we have two agents owning bundles of value 2 and 3 respectively. The bundle of value 3 consists of two heavy goods and both agents like all heavy goods. We have $N_{1/2} = 1$. Since the two agents have values x and $x + 1$, there is no need for an agent k . In the optimal allocation both bundles contain a heavy and a light good.

In order to check for the existence of the allocation B^H , we set up the following parity matching problem for every pair i and j of agents.

- For goods the degree in the matching must be equal to 1.
- For all agents in $A_{1/2} \cup \{i, j\}$, the degree must be in $N_{1/2}$.
- If A_i and A_j have value x , let A_k be any bundle of value $x + 1$ containing a heavy good. If A_i and A_j have value $x + 1$, let A_k be a bundle of value x . The degree of k must be in N_0 .
- For an $a \in A_0 \setminus \{i, j, k\}$, the degree must be in N_0 , and for an $a \in A_1 \setminus \{i, j, k\}$, the degree must be in N_1 .

If B^H exists for some pair i and j , we improve the allocation. If B^H does not exist for all pairs i and j , A is optimal.

Each improvement increases the size of $A_{1/2}$ by two and hence there can be at most $n/2$ improvements. In order to check for an improvement, we need to solve n^2 perfect matching problems in an auxiliary graph with m vertices and mn edges. Hence the running time is polynomial.

5 The General Case: Heavy Goods can be Allocated as Light

We follow the approach taken in the integral case [ACH⁺22]. We determine allocations A^k for $k = 0, 1, 2, \dots$; A^k is a best (= largest NSW) allocation among all allocations for instances obtained by turning k heavy goods into light goods. We determined A^0 in the previous section.

Let again x be the smallest value of any bundle. If all bundles have value at most $x + 1$, no further improvement is possible by conversions of heavy goods to light goods. We will show this in Section G.2 in the appendix. As long as there is a bundle of value more than $x + 1$, we take a heavy good from a heaviest bundle (the choice of heaviest bundle is arbitrary), convert it to a light good, add it to a bundle of value x (again the choice is arbitrary) and reoptimize. In this way, we will find an optimal allocation. The details are given in Appendix G.

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A Math Preliminaries

The following Lemma is useful for showing that certain reallocations increase the NSW.

Lemma 19 a) Let a , b , and d be non-negative reals with $a \geq b$, and $d \in [0, a - b]$. Then $ab \leq (a - d)(b + d)$ with equality only if $d = 0$ or $d = a - b$.

b) Let a , b , c , d_1 , and d_2 be non-negative reals with $a \geq b \geq c$, $b \geq c + d_2$, and $a \geq c + d_1 + d_2$. Then $abc \leq (a - d_1)(b - d_2)(c + d_1 + d_2)$ with equality only if $d_2 \in \{0, b - c\}$ and $d_1 \in \{0, a - c - d_2\}$.

Proof:

a) We have

$$(a - d)(b + d) - ab = (a - b - d)d \geq 0$$

with equality only if $d = 0$ or $d = a - b$.

b) We apply part a) twice and obtain

$$\begin{aligned} abc &\leq a(b - d_2)(c + d_2) && \text{with equality only if } d_2 = 0 \text{ or } d_2 = (c - b). \\ &\leq (a - d_1)(b - d_2)(c + d_1 + d_2) && \text{with equality only if } d_1 = 0 \text{ or } d_1 = a - c - d_2. \quad \blacksquare \end{aligned}$$

B Proof of Lemma 1

Lemma 1 Let A be any allocation and let x be the minimum value of any bundle in A . Let i be any agent. For parts b) to g), let j be any other agent, and let p be an A - \bar{A} -alternating path from i to j .

a) If $w_i > x + 1$ and A_i contains a light good, moving the light good to a bundle of value x improves the NSW of A .

b) If $w_i \geq w_j + \lceil s \rceil$, augmenting p to A improves the NSW of A .

c) If $w_i \geq w_j + 1$ and A_j contains more than $s - (w_i - w_j)$ light goods, augmenting p to A and moving $\max(0, \lceil s - (w_i - w_j) + 1/2 \rceil)$ light goods from j to i improves the NSW of A .

d) If $w_i \in \{x + 1, x + 3/2\}$, $w_j = x + 1$, and A_j contains $\lceil s \rceil$ light goods, augmenting p to A and moving $\lfloor s \rfloor$ light goods from A_j to A_i and another light good from A_j to any bundle of value x improves the NSW of A .

e) If j owns two heavy goods less than i and $w_i \geq x + 3/2$, one of the cases b), c), or d) applies and A can be improved.

f) If $w_i = x$, $w_j = x + 1$, and A_j contains $\lceil s \rceil$ light goods, augmenting p to A and moving $\lceil s \rceil$ light goods from A_j to A_i improves the NSW of A .

g) If $w_i = x + 1$, $w_j = x + 1/2$, A_i is heavy-only, and A_j contains at least $\lceil s \rceil$ light goods, augmenting p to A and moving $\lfloor s \rfloor$ light goods from A_j to A_i leaves the NSW of A unchanged and increases the number of bundles of value $x + 1$ containing a light good.

Proof:

a) The sum of the values of the two bundles does not change and the new values lie strictly inside the interval defined by the old values. The claim follows from Lemma 19a).

b) The sum of the values of the two bundles does not change and the new values lie strictly in the interval defined by the old values. The claim follows from Lemma 19a).

c) If $w_i \geq w_j + \lceil s \rceil$, the claim follows from part b). So assume $w_i \leq w_j + s$. By assumption, A_j contains at least $r = \lceil s - (w_i - w_j) + 1/2 \rceil$ light goods. After the augmentation and moving the light goods, the weight of A_i is $w_i - s + r > w_i - s + (s - (w_i - w_j)) = w_j$ and similarly the weight of A_j is less than w_i . Thus the NSW increases.

- d) Before the augmentation, we have bundles of value $x + 1 + d$ with $d \in \{0, 1/2\}$, $x + 1$ and x . After the augmentation, we have bundles of value $x + 1 + d - s + \lfloor s \rfloor = x + 1/2 + d$, $x + 1 + s - \lceil s \rceil = x + 1/2$ and $x + 1$. Thus the NSW improves by Lemma 19b).
- e) The heavy weight of j is at most $w_i - 2s$. If j is heavy-only, augmenting p improves A according to part b). If j owns a light good, $w_j \leq x + 1 < w_i$ and j owns at least $w_j - (w_i - 2s) = 2s - (w_i - w_j)$ light goods. If $w_i \geq w_j + 1$, A can be improved according to part c). Otherwise, we must have $w_i = x + 3/2$ and $w_j = x + 1$, and A can be improved according to part d).
- f) Before the augmentation, we have bundles of value x and $x + 1$, after augmentation we have two bundles of value $x + 1/2$. The NSW improves by Lemma 19a).
- g) Before the augmentation we have bundles of value $x + 1$ and $x + 1/2$ and after the augmentation we have bundles of value $x + 1 - s + \lfloor s \rfloor = x + 1/2$, $x + 1/2 + s - \lceil s \rceil = x + 1$. Thus NSW does not change. The bundle of value $x + 1$ now contains at least one light good. ■

C Proofs of Claims 1 to 7 in Lemma 2

Claim 1 *In A , the bundles in $R'_k \setminus R_k$ contain exactly $k - 1$ heavy goods.*

Proof: By definition, bundles in $[n] \setminus S_{k+1}$ contain at most k heavy goods in A . Bundles that contain exactly k heavy goods belong to R_k and hence the bundles in $R'_k \setminus R_k$ contain at most $k - 1$ heavy goods. Each $j \in R'_k \setminus R_k$ is reachable by an A - O -alternating path from an $i \in R_k$ and $w_i = ks > x + 1$ and hence j must contain $k - 1$ heavy goods since otherwise Lemma 1e) would be applicable and A would not be reduced. ■

Claim 2 *In O , each bundle in $[n] \setminus S_{k+1}$ contains at most k heavy goods.*

Proof: Assume that there is an agent i in $[n] \setminus S_{k+1}$ that owns $k + 1$ or more heavy goods in O . Then $w_i^O \geq y + s \geq x + 3/2 \geq x + 3$. There is an O - A -alternating path from i to an agent j containing fewer heavy goods in O than in A . Since we know already that A and O agree on S_{k+1} , j contains at most $k - 1$ heavy goods in O , a contradiction to the optimality of O (Lemma 1e). ■

Claim 3 *All heavy goods assigned to agents in R'_k by A are also assigned to them in O .*

Proof: Let g be any good that A assigns to an agent in R'_k , say j , and let h be the owner of g in O . We need to show $h \in R'_k$. Since A and O agree on S_{k+1} , $h \notin S_{k+1}$. Let p be an A - O -alternating path from $i \in R_k$ to j ; $i = j$ is possible. We extend p by $j \xrightarrow{A} g \xrightarrow{O} h$ and hence h can be reached by an alternating path starting with an A -edge from $i \in R_k$. Thus $h \in R'_k$. ■

Claim 4 *O assigns k heavy goods to at least $|R_k|$ agents in R'_k .*

Proof: In A , the agents in R_k own k heavy goods (by definition) and the agents in $R'_k \setminus R_k$ own $k - 1$ heavy goods (Claim 1). So the number of heavy goods allocated by A to the agents in R'_k is $m_k^A := (k - 1)|R'_k| + |R_k|$. All heavy goods assigned to agents in R'_k by A are also assigned to them by O (Claim 3) and no agent in R'_k is assigned more than k heavy goods in O (Claim 2). Let z be the number of agents in R'_k to which O assigns k heavy goods. Then $zk + (|R'_k| - z)(k - 1) \geq m_k^A$ and hence $z + |R'_k|(k - 1) \geq (k - 1)|R'_k| + |R_k|$. Thus $z \geq |R_k|$. ■

Claim 5 $x_O + 1 \leq ks$.

Proof: Recall that y is a shorthand for ks . Let L be the agents that own bundles of value less than y in A . Then $L = [n] \setminus S_k$ and A assigns all light goods to the agents in L . In A , there is at least one bundle of value x in L . So the average value of a bundle in L is less than $y - 1/2$ in A .

We know already that O and A agree on S_{k+1} , and that O assigns at least k heavy goods to at least $|R_k|$ many heavy agents in R'_k . Choose any $|R_k|$ of them, and let L' be the remaining agents. Then $|L'| = |L|$ and the total value of the O -bundles of the agents in L' is at most the value of the A -bundles of the agents in L since the number of heavy goods assigned to them cannot be larger and all light goods are assigned to agents in L by A . Thus their average value is less than $y - 1/2$ and hence $x_O < y - 1/2$. ■

Claim 6 *Agents in R_k own k heavy goods in O .*

Proof: Let i be an agent in R_k . Then $hdeg_i^O \leq k = hdeg_i^A$; the equality holds by the definition of R_k and the inequality holds by Claim 2. Assume for the sake of a contradiction, that there is an agent $i \in R_k$ with $k = hdeg_i^A > hdeg_i^O$. Consider an A - O -alternating path (in the alternating path decomposition) starting in i . Let j be the other end of the path. Then, $hdeg_j^O > hdeg_j^A$. Also, $j \in R'_k$ and hence $hdeg_j^A \geq k - 1$ (Claim 1). Since $hdeg_j^O \leq k$ (Claim 2), we must have $hdeg_j^O = k$ and $hdeg_j^A = k - 1$. Therefore, the value of j in O is at least y and $j \in R'_k \setminus R_k$.

If $hdeg_i^O = k - 1$, we augment p to O and also exchange the light goods (if any); the utility profile of O does not change and O moves closer to A , a contradiction to the choice of O .

If $hdeg_i^O \leq k - 2$, then the heavy value of i in O is at most $y - 2s$ and hence i owns at least $w_i^O - (y - 2s) = 2s - (w_j^O - w_i^O) > s - (w_j^O - w_i^O)$ light goods. If the value of i is no larger than $y - 1$, O can be improved (Lemma 1c)).

So $w_i^O \geq y - 1/2$ and hence i contains at least $2s$ light goods in O . Thus $w_i^O \leq x_O + 1$ and hence $x_O \geq y - 1/2 - 1 = y - 3/2$. The heavy degree of i in A is at least two more than the heavy degree of i in O . Therefore there must be a second A - O -alternating path starting in i , say q . It ends in a node h . As above for j , we conclude $hdeg_h^O = k$ and $hdeg_h^A = k - 1$. Then, $j \neq h$ as only one alternating path each can end in j as well as in h . It is however possible, that j lies on the path from i to h or that h lies on the path from i to j .

We have $w_j^O \geq ks = y$ and $w_h^O \geq ks = y$. We augment p and q to O and use the $2s$ light goods on i as follows: We give $\lfloor s \rfloor$ each to j and h , and one to a bundle of value x_O . The value of i does not change, the values of j and h go down by $1/2$ each and the value of an x_O bundle goes up by one. We may assume $w_j^O \geq w_h^O$. We apply Lemma 19, part b) with $d_1 := 1/2$ and $d_2 := 1/2$ and conclude that the augmentation improves the NSW of O , a contradiction. Note that $w_j^O \geq ks \geq x_O + 1$ and $w_h^O \geq ks \geq x_O + 1$ by Claim 5. ■

Claim 7 *O agrees with A on R_k , i.e., $O_i = A_i$ for all $i \in R_k$. Bundles owned by agents in $[n] \setminus S_k$ do not contain k heavy goods in either A or O .*

Proof: By the preceding claims, agents in R_k own k heavy goods in A as well as in O . Consider the path decomposition of $A^H \oplus O^H$. If O does not agree with A on R_k , there is an A - O -alternating path passing through an agent $i \in R_k$. Let j and h be the endpoints of the path. Say j is incident to an O -edge and h is incident to an A -edge. Thus $j \in R'_k$. Since $hdeg_j^O > hdeg_j^A \geq k - 1$, we must have $hdeg_j^A = k - 1$ and $hdeg_j^O = k$ and $w_j^O \geq y$.

Since h is incident to an A -edge, $hdeg_h^O < hdeg_h^A$ and hence $hdeg_h^A \leq k - 1$. Recall that $hdeg_h^A = k$ implies $hdeg_h^O = k$ by the preceding claim. Thus $hdeg_h^O \leq k - 2$ and the heavy value of h is at most $y - 2s$. If the value of h is no larger than $y - 1$, we augment p to O and possibly move light goods from h to j ; note that h contains $w_h^O - (y - 2s)$ light goods. This improves O , a contradiction.

So $w_h^O \geq y - 1/2$ and hence h contains at least $2s$ light goods in O . Thus $w_h^O \leq x_O + 1$ and hence $x_O \geq y - 1/2 - 1 = y - 3/2$. If $w_h^O = y - 1/2$ we augment the path to O and move $\lfloor s \rfloor$ light from h to j . This does not change the utility profile of O and moves O closer to A .

We cannot have $w_h^O > y$ as this would imply $x_O + 1 > y$, a contradiction to Claim 5.

This leaves the case $w_h^O = y$, $y = x_O + 1$ (since $w_h^O \leq x_O + 1$ by the above and $y \geq x_O + 1$ by Claim 5) and $x_O = x + 1/2$ (since $y > x + 1$ and $x_O \leq x + 1/2$ (Claim 5)). Also h owns $2s$ light goods. We augment the path to O and move $\lfloor s \rfloor$ light goods from h , $\lfloor s \rfloor$ of them to j and one of them to a bundle of value x_O . Before the

augmentation we have two bundles of value y and one bundle of value x_O , after the change, we have two bundles of value $y - 1/2$ and one bundle of value $x_O + 1$. Since $y = x_O + 1$, the NSW improves, a contradiction.

We have now established $O_i^H = A_i^H$ for all $i \in R_k$; A_i does not contain any light good. O_i does neither since it would have value at least $k_0s + 1 \geq y + 1 > x_O + 1$ otherwise. Thus $O_i = A_i$ for all $i \in R_k$.

Assume next that there is a bundle O_j with $j \notin S_k$ containing k heavy goods. Since $j \notin S_k$, A_k contains fewer than k heavy goods. So there is an O - A -alternating path starting in j . Let h be the other endpoint. The path ends with an A -edge incident to h and hence there are more A -edges than O -edges incident to h . We cannot have $h \in S_k$ because this would imply $hdeg_h^O = hdeg_h^A$. Thus $hdeg_h^A \leq k - 1$ and $hdeg_h^O \leq k - 2$. We now continue as above and derive a contradiction. ■

D Missing Proofs of Section 3.3.1

Lemma 3 *Let A be reduced and shrunken. If there is an optimal allocation having at least as many bundles of value $x + 1$ as A does, A is optimal. In particular, if A has no bundles of value $x + 1$, A is optimal.*

Proof: Assume that there are n_0 bundles of value x and $n_{1/2}$ bundles of value $x + 1/2$ and $n_1 := n - n_0 - n_{1/2}$ bundles of $x + 1$. The total value is $xn + n_{1/2}/2 + n_1$. Split all goods into portions of $1/2$ (i.e. a light good becomes two portions, a heavy good becomes $2s$ portions) and allow them to be allocated freely subject to the constraint that there are at least n_1 bundles of value $x + 1$.

Consider an optimal allocation O under this constraint. Assume we have bundles of value y and y' with $y' \geq y + 1$. We can replace them by bundles of value $y + 1/2$ and $y' - 1/2$ and improve NSW except if $x + 1 \in \{y, y'\}$ and there are exactly n_1 bundles of value $x + 1$. In O , we have $n'_1 \geq n_1$ bundles of value $x + 1$ and at least one bundle of value less than $x + 1$ as the average value of a bundle in A is less than $x + 1$. Thus, if $n'_1 > n_1$, all bundles have value $x + 1/2$ and $x + 1$, and if $n'_1 = n_1$, all bundles have value $z, z + 1/2$ and $x + 1$ for some $z \leq x$. The former case is impossible since $n'_1(x + 1) + (n - n'_1)(x + 1/2) - (nx + n_{1/2}/2 + n_1) = n'_1 + (n - n'_1)/2 - n_{1/2}/2 - n_1 = (n'_1 - n_1 + n - n_1 - n_{1/2})/2 > 0$. In the latter case, we must have $z = x$, and the number of bundles of value x and $x + 1/2$ must be n_0 and $n_{1/2}$, respectively. Thus A is optimal. ■

Lemma 4 *Let A be reduced and shrunken and let O be an optimal allocation closest to A . Then $x_O \geq x - 1/2$.*

Proof: Assume otherwise, i.e., $x_O \leq x - 1$. In O , all light goods are contained in bundles of value at most x , and bundles of value larger than x are heavy-only. Any bundle contains at most $(k_0 - 1)$ heavy goods (Lemma 2) and hence has heavy value at most $(k_0 - 1)s$. Since k_0 is minimal with $k_0s > x + 1$, we have $(k_0 - 1)s \leq x + 1$.

If $(k_0 - 1)s \leq x$, the average value of a bundle in O is strictly less than x (there is a bundle of value x_O and all bundles have value at most x), but the average value of a bundle in A is at least x , a contradiction.

Let $y = (k_0 - 1)s$ and assume $y \in \{x + 1/2, x + 1\}$. In O , bundles of value y are heavy-only. Let S be the owners of the bundles of value y in O . If their bundles in A have value y or more, the average value of a bundle in A is larger than the average value in O , a contradiction. So there must be an agent $i \in S$ whose bundle in A has value less than y . Then $hdeg_i^A \leq (k_0 - 2) < hdeg_i^O$. Consider an O - A -alternating path starting in i . It ends in a node j with $hdeg_j^O < hdeg_j^A$. Then $hdeg_j^O \leq k_0 - 2$ and hence the value of O_j is at most x . We augment the path to O and move $w_j^O - (k_0 - 1)s$ light goods from j to i . This does not change the utility profile of O and moves O closer to A , a contradiction. ■

Lemma 5 *Let A be reduced and shrunken and let O be an optimal allocation closest to A . There is no bundle of value more than $x + 1$ in O .*

Proof: Assume, there is a bundle of value $x + 3/2$ or more. The bundle contains at most $k_0 - 1$ heavy goods and hence its heavy value is at most $x + 1$. So it contains at least one light good and hence $x_O = x + 1/2$ and the bundle under consideration has value $x + 3/2$. The bundles in O have values in $\{x + 1/2, x + 1, x + 3/2\}$. Any

bundle of value $x + 1/2$ can be turned into a bundle of value $x + 3/2$ by moving a light good to it from a bundle of value $x + 3/2$.

We use O_d to denote the set of agents owning bundles of value $x + d$ in O . Assume first $O_{1/2} \cup O_{3/2} \subseteq A_0 \cup A_{1/2}$. Then $A_1 \subseteq O_1$. Since there is at least one bundle of value x in A and at least one bundle of value $x + 3/2$ in O , the average value of a bundle in O must be higher than the average value in A , a contradiction. So there must be an i in $(O_{1/2} \cup O_{3/2}) \cap A_1$. Then the parities of $hdeg_i^A$ and $hdeg_i^O$ differ. By the first paragraph, we may assume $i \in O_{3/2}$.

Assume first that $hdeg_i^O > hdeg_i^A$. Then there exists an O - A -alternating path starting in i . The path ends in j with $hdeg_j^O < hdeg_j^A \leq k_0 - 1$. The heavy value of O_j is at most $(k_0 - 2)s$ which is at most $x + 1 - s$. Since the value of O_j is at least x_O , O_j contains at least $x_O - (x + 1 - s) = s - 1/2 = \lfloor s \rfloor$ light goods. If $w_j^O = x + 1/2$, we augment, move $\lfloor s \rfloor$ light goods from j to i and improve O . If $w_j^O = x + 1$, we augment and move $\lfloor s \rfloor$ light goods from j to i . This does not change the utility profile of O and moves O closer to A , a contradiction. If $w_j^O = x + 3/2$, j contains at least $\lfloor s \rfloor$ light goods. We augment the path, move $\lfloor s \rfloor$ light goods from j to i and one light good from j to a bundle of value x_O . This improves O , a contradiction.

Assume next that $hdeg_i^O < hdeg_i^A \leq k_0 - 1$. Then there exists an A - O -alternating path starting in i . The path ends in j with $hdeg_j^A < hdeg_j^O \leq k_0 - 1$. Since O_i has value $x + 3/2$ and $hdeg_i^O \leq k_0 - 2$, O_i must contain at least $x + 3/2 - (x + 1 - s) = \lfloor s \rfloor$ light goods. We augment the path to O and remove $\lfloor s \rfloor$ light goods from O_i . So the value of O_i becomes $x + 1$. If O_j has value $x + 3/2$, we put $\lfloor s \rfloor$ light goods on j and one light on a bundle of value x_O . If O_j has value $x + 1$ or $x + 1/2$, we put $\lfloor s \rfloor$ light goods on i . This either improves O or does not change the utility profile of O and moves O closer to A , a contradiction. ■

In the technical introduction (Section 2) we pointed to the importance of bundles of value $x + 1$ containing a light good. The following Lemma formalizes this observation.

Lemma 6 *Let A be reduced and shrunken and assume further that Lemma 1g) is not applicable. Let O be an optimal allocation closest to A , and consider an agent $i \in A_1$. If A_i is heavy-only, O_i is heavy-only and has value $x + 1$. If all bundles in A_1 are heavy-only, A is optimal.*

Proof: Consider a heavy-only bundle A_i of value $x + 1$. Then $hdeg_i^A = k_0 - 1$ and $x + 1 = (k_0 - 1)s$. Assume for the sake of a contradiction that O_i has either value less than $x + 1$ or is not heavy-only. In either case, $hdeg_i^A > hdeg_i^O$ and hence O_i contains at most $k_0 - 2$ heavy goods and thus at least $w_i^O - (x + 1 - s) = w_i^O - x - 1 + s$ light goods. If $i \in O_0 \cup O_1$, O_i contains at most $k_0 - 3$ heavy goods as the parity of the number of heavy goods must be the same as for A_i . This holds since the value of A_i and O_i differ by an integer, namely either zero or one.

Consider an A - O -alternating path starting in i and let j be the other end of the path. Then $hdeg_j^A < hdeg_j^O$ and hence A_j contains at most $k_0 - 2$ heavy goods; its heavy value is therefore at most $x + 1 - s$. Since the value of A_j is at least x , A_j must contain at least $\lfloor s \rfloor$ light goods.

If the value of A_j is x , Lemma 1c) applies to A (we augment the path to A and move a light good from j to i), a contradiction. If the value of A_j is $x + 1$, A_j must contain $\lfloor s \rfloor$ light goods, and hence Lemma 1d) applies to A . We augment the path to A , move $\lfloor s \rfloor$ light goods from j to i and one light good from j to a bundle of value x . This improves the NSW of A , a contradiction.

If the value of A_j is $x + 1/2$ and A_j contains at most $k_0 - 3$ heavy goods, A_j must contain at least $\lfloor s \rfloor$ light goods, and hence Lemma 1g) applies. We augment the path to A and move $\lfloor s \rfloor$ light goods from j to i . This does not change the utility profile of A , but increases the number of bundles of value $x + 1$ containing a light good, a contradiction.

So we are left with the case that A_j has value $x + 1/2$ and contains $k_0 - 2$ heavy goods. Then O_j contains $k_0 - 1$ heavy goods and hence is a heavy-only bundle of value $x + 1$. If the value of O_i is $x - 1/2$, O_i contains at least $s - 3/2$ light goods. We augment the path to O and move $s - 3/2$ light goods from i to j . This does not change the utility profile of O and moves O closer to A , a contradiction. If the value of O_i is either x or $x + 1/2$, O_i contains at least $\lfloor s \rfloor$ light goods. We augment the path to O and move $\lfloor s \rfloor$ light goods from i to j . This improves the NSW of O if the value of O_i is x and does not change the utility profile of O and moves O closer to A , otherwise. If the value of O_i is $x + 1$, O_i contains at least $\lfloor s \rfloor$ light goods. We augment the path to O , move

$\lfloor s \rfloor$ light goods from i to j and one light good from i to a bundle of value x . This improves O . In either case, we have obtained a contradiction.

If all bundles in A_1 are heavy-only, $A_1 \subseteq O_1$ and hence A is optimal by Lemma 3. ■

E Missing Proofs of Section 3.3.2

Lemma 7 1. *The parity of the number of heavy goods is the same in bundles of value x and $x+1$ and in bundles of value $x-1/2$ and $x+1/2$ and the former parity is different from the latter.*

2. *The parity of the number of bundles of value x or $x+1$ is the same in A and O and equally for the number of bundles of value $x-1/2$ or $x+1/2$. More precisely, for $d \in \{-1/2, 0, 1/2, 1\}$ let a_d and o_d be the number of bundles of value $x+d$ in A and O respectively, and let $a_1 = o_1 + z$. Then (the first equation is trivial; it is there for completeness)*

$$\begin{aligned} a_{-1/2} &= 0 = o_{-1/2} - o_{-1/2} \\ a_0 &= o_0 + 2o_{-1/2} + z \\ a_{1/2} &= o_{1/2} - 2z - o_{-1/2} \\ a_0 + a_1 &= o_0 + o_1 + 2(o_{-1/2} + z) \end{aligned}$$

3. *Let A be a suboptimal allocation and let O be an optimal allocation. Then $z > o_{-1/2} \geq 0$, $(A_0 \cup A_1) \cap O_{1/2}$ is non-empty. In particular, O contains a bundle of value $x+1/2$.*

Proof: If the values of two bundles differ by an integral amount, the number of heavy goods in both bundles has the same parity. If the values differ by a multiple of $1/2$ which is not an integer, the number of heavy goods has a different parity. This proves the first claim.

For the second claim, observe that $\sum_d a_d = \sum_d o_d$, and $\sum_d a_d d = \sum_d o_d d$. The expressions given for $a_{-1/2}$ to a_1 satisfy these equations and the equations are independent. Further the expressions are the unique solutions and

$$a_0 + a_1 = o_0 + 2o_{-1/2} + z + o_1 + z = o_0 + o_1 + 2(o_{-1/2} + z).$$

So $a_0 + a_1$ and $o_0 + o_1$ have the same parity. This proves the second claim.

We come to the third claim. We have

$$1 < \frac{\text{NSW}(O)}{\text{NSW}(A)} = \frac{(x-1/2)^{o_{-1/2}} (x+1/2)^{2z} (x+1/2)^{o_{-1/2}}}{x^{2o_{-1/2}} x^z (x+1)^z} = \left(\frac{x^2 - 1/4}{x^2} \right)^{o_{-1/2}} \cdot \left(\frac{x^2 + x + 1/4}{x^2 + x} \right)^z,$$

and hence $z > o_{-1/2}$ since $(x^2 - 1/4)(x^2 + x + 1/4)/(x^2(x^2 + x)) < 1$. Thus

$$a_0 + a_1 = o_0 + o_1 + 2(o_{-1/2} + z) > o_0 + o_1 + o_{-1/2}$$

and hence $(A_0 \cup A_1) \cap O_{1/2} \neq \emptyset$ and $o_{1/2} = a_{1/2} + 2z + o_{-1/2} > 0$. ■

Lemma 8 *Let v be unbalanced and let A_v and O_v have the same heavy parity.*

- *If v is A -heavy, O_v contains at least $2s$ light goods (except if $v \in O_0 \cap A_1$ or $v \in A_{1/2} \cap O_{-1/2}$, then $2s-1$ light goods).*
- *If v is O -heavy, A_v contains at least $2s$ light goods ($2s-1$ if $v \in O_1 \cap A_0$).*

Proof: A_v and O_v have the same heavy parity. If v is A -heavy, $|A_v^H| \geq |O_v^H| + 2$. Hence the number of light goods in O_v is at least $2s$ minus the value difference of A_v and O_v . This value difference is non-positive except if $v \in A_1 \cap O_0$ or $v \in O_{-1/2} \cap A_{1/2}$. In these cases, the value difference is 1. If v is O -heavy, a symmetric argument applies. ■

Lemma 9 *Let $v \in A_0 \cup A_1$ be unbalanced.*

- If v is A -heavy, O_v contains at least $\lfloor s \rfloor$ light goods if $v \in O_{1/2}$ and at least $2s - 1$ light goods if $v \in O_0 \cup O_1$. If $v \in O_{-1/2}$, O_v contains at least $\lfloor s \rfloor - 1$ light goods.
- If v is O -heavy, A_v contains at least $\lfloor s \rfloor$ light goods. If $v \in O_{\pm 1/2}$, v contains at least $\lfloor s \rfloor$ light goods if $v \in A_0$ and at least $\lceil s \rceil$ light goods if $v \in A_1$.

Proof: If v is A -heavy, $|O_v^H| < |A_v^H|$. If $v \in O_0 \cup O_1$, $|A_v^H|$ and $|O_v^H|$ have the same parity and hence $|A_v^H| \geq |O_v^H| + 2$. Since the value of O_v is at most one less than the value of A_v , O_v must contain at least $2s - 1$ light goods. If $v \in O_{1/2}$, $|A_v^H| \geq |O_v^H| + 1$ and since the value of O_v is at most $1/2$ less than the value of A_v , O_v must contain at least $\lfloor s \rfloor$ light goods. If $v \in O_{-1/2}$, $|A_v^H| \geq |O_v^H| + 1$ and since the value of O_v is at most $3/2$ less than the value of A_v , O_v must contain at least $\lfloor s \rfloor - 1$ light goods.

If v is O -heavy, $|A_v^H| < |O_v^H|$. If $v \in O_0 \cup O_1$, $|A_v^H|$ and $|O_v^H|$ have the same parity and hence $|A_v^H| \leq |O_v^H| - 2$. Since the value of A_v is at most one less than the value of O_v , A_v must contain at least $2s - 1$ light goods. If $v \in O_{\pm 1/2}$, $|A_v^H| \leq |O_v^H| - 1$, A_v must contain at least $\lfloor s \rfloor$ light goods if $v \in A_0$ and must contain at least $\lceil s \rceil$ light goods if $v \in A_1$. ■

Lemma 10 *Let $v \in (O_0 \cup O_1) \cap A_{1/2}$ be unbalanced.*

- If v is A -heavy, O_v contains at least $\lfloor s \rfloor$ light goods if $v \in O_0$ and at least $\lceil s \rceil$ light goods if $v \in O_1$.
- If v is O -heavy, A_v contains at least $\lfloor s \rfloor$ light goods. If $v \in O_0$, A_v contains at least $\lceil s \rceil$ light goods.

Proof: If v is A -heavy, $|A_v^H| \geq |O_v^H| + 1$. If $v \in O_0$, O_v must contain at least $\lfloor s \rfloor$ light goods, if $v \in O_1$, O_v must contain at least $\lceil s \rceil$ light goods.

If v is O -unbalanced, $|A_v^H| \leq |O_v^H| - 1$. If $v \in O_1$, A_v must contain at least $\lfloor s \rfloor$ light goods, if $v \in O_0$, A_v must contain at least $\lceil s \rceil$ light goods. ■

Lemma 11 *Let W be an A - O -walk. Then $|W \cap A| = |W \cap O|$.*

Proof: The types of the hinges alternate. Moreover, if an endpoint has type T , the type of the first adjacent hinge is \bar{T} . Thus if the types of the endpoints differ, the number of hinges of both types are the same, and if the types of the endpoints agree and is equal to T , there is one more \bar{T} -hinge. We conclude that the number of A -edges on the walk is the same as the number of O -edges. ■

Lemma 13 *If there is an A - O -walk, there is a semi-simple walk with the same endpoints.*

Proof: Consider the walk and assume that an agent v is entered twice on an edge of the same type, say once from g and once from g' ; the second occurrence of v could be the last vertex of the walk.

$$\dots - g \xrightarrow{T} v \xrightarrow{T'} \dots - g' \xrightarrow{T} v \xrightarrow{T''} \dots$$

If the second occurrence of v is the last vertex of the walk, we end the walk at the preceding occurrence of v . Otherwise, the edge of type T'' exists. Either occurrence of v could be a hinge. We cut out the subpath starting with the first occurrence of v and ending with the edge entering the second occurrence of v and obtain

$$\dots - g \xrightarrow{T} v \xrightarrow{T''} \dots$$

If $T'' \neq T$, we still have a walk. If $T'' = T$, the second occurrence of v is a hinge and hence $v \in A_{1/2} \cap O_{\pm 1/2}$. After the removal of the subpath, it is still a hinge. ■

F The Arguments for Section 3.3.4

Case $j \in A_0 \cup A_1$, and $i \neq j$: We augment the walk to A . The heavy parity of i and j changes and the heavy parity of all intermediate nodes does not change.

If i and j are A -endpoints, both lose a heavy edge and there is an extra O -hinge releasing $2s$ light goods. We give $\lfloor s \rfloor$ light goods to any endpoint in A_1 and $\lceil s \rceil$ light goods to any endpoint in A_0 . So we need between $2s - 1$ and $2s + 1$ light goods. If we need only $2s - 1$, we put the extra light good onto any bundle in A_0 , if we need $2s + 1$, we take one light good from any bundle in A_1 with a light good.

If j is an O -endpoint, j gains a heavy good. By Lemma 9, A_j contains $\lfloor s \rfloor$ light goods if $j \in A_0$ and contains $\lceil s \rceil$ light goods if $j \in A_1$. We give $\lfloor s \rfloor$ light goods to i if $i \in A_1$ and $\lceil s \rceil$ light goods if $i \in A_0$. If an extra light is needed, we take it from a bundle in A_1 , if we have one light too many, we put it on a bundle in A_0 .

In either case, we increased the number of agents in $A_{1/2}$ by two and hence improved A .

Case $j \in A_0 \cup A_1$, and $i = j$: We augment the walk to O . The heavy parity of the intermediate nodes does not change and the heavy parity of i changes neither. It either gains and loses a heavy good and then the number of hinges is even or it gains two heavy goods and then the number of hinges is odd and there is an extra O -hinge. We show that there are sufficiently many light goods to keep the values of all bundles in O unchanged.

This is obvious, if i loses and gains a heavy. Then there are an equal number of hinges of both types and we simply move the light goods between them.

If i gains two heavy goods, the first and the last edge of the walk are A -edges. Hence $|A_i^H| \geq |O_i^H| + 2$ (Lemma 12). Since $i \in O_{1/2}$, the parity of the number of heavy goods in A_i and O_i is different. Thus, $|A_i^H| \geq |O_i^H| + 3$. Since the value of O_i is by at most $1/2$ lower than the value of A_i , O_i contains at least $2s$ light goods. We give $2s$ light goods to the extra O -hinge. Note that i gains two heavy goods and hence the value of O_i does not change.

We have now moved O closer to A , a contradiction to our choice of O . Thus this case cannot arise.

Case $j \in (O_0 \cup O_1) \cap A_{1/2}$: We augment the walk to O . For the intermediate nodes, the heavy parity does not change. For i and j the heavy parity changes; i gains a heavy good and j either loses a heavy good and then the number of hinges is even or gains a heavy good and then the number of hinges is odd and there is an extra O -hinge. There are sufficiently many light goods to keep the values of all bundles, except for the bundles of i and j , in O unchanged; i and j change values. Recall that O_i contains at least $\lceil s \rceil$ light goods.

If j is A -heavy, i and j gain a heavy good and there is an extra O -hinge. Since j is A -heavy, the heavy value of j in O is at most $x + 1 - s$. Thus j owns at least $\lceil s \rceil$ light goods if $j \in O_1$ and at least $\lfloor s \rfloor$ light goods if $j \in O_0$. If $j \in O_1$, we move $\lfloor s \rfloor$ light good from i and $\lceil s \rceil$ light goods from j to the extra O -hinge. If $j \in O_0$, we move $\lceil s \rceil$ light goods from i and $\lfloor s \rfloor$ light goods from j to the extra O -hinge. In either case, the values of i and j interchange. Thus the utility profile of O does not change and O moves closer to A , a contradiction.

If j is O -heavy, j loses a heavy good and the number of hinges is even. If $j \in O_1$, we move $\lfloor s \rfloor$ light goods from i to j , if $j \in O_0$, we move $\lceil s \rceil$ light goods from i to j . In either case i and j swap values. Thus the utility profile of O does not change and O moves closer to A , a contradiction.

G The General Case: Heavy Goods can be Allocated as Light

G.1 A Bundle of Value Larger than $x + 1$ Exists

A *facilitator* is a bundle of value $x + 1$ containing a light good.

Lemma 14 *Let A be an optimal allocation in which all bundles have value x , $x + 1/2$ and $x + 1$ and let B be obtained from A by adding a light good to a bundle of value x . If there is an improving walk in B , both endpoints of the walk have value x in A , all bundles of value $x + 1$ in A are heavy-only, and after augmentation of the improving walk, there is no further improving walk. Whether there is an improving walk in B does not depend on the choice of bundle of value x to which we allocate the light good. The maximum gain in NSW after adding a light good is $(x + 1/2)^2/x^2$.*

Proof: Let i be the agent to which we give the light good. i has value x in A and value $x + 1$ in B . Let W be an improving walk in B , and assume, for the sake of a contradiction, that there is a bundle A_f of value $x + 1$ containing a light good in A , i.e., f is a facilitator. Then $i \neq f$.

Assume first that i is not an endpoint of W . If i plays the role of a facilitator for the augmentation of W to B , then both endpoints of W have weight x in B and hence in A and thus are different from f . Thus f could have been the facilitator for the augmentation of W to A , a contradiction to the optimality of A . If i does not play the role of a facilitator, the augmentation was possible in A , a contradiction.

So assume that i is an endpoint of W . It is either an A - or an \bar{A} -endpoint of W . Let j be the other endpoint of the walk. If i is an A -endpoint in B , it gives up a heavy good in return for $\lfloor s \rfloor$ light goods. In A , it needs to receive $\lceil s \rceil$ light goods. If j has weight $x + 1$, no facilitator is needed. If j has weight x , j is different from f and a light good can come from f . In either case the walk can be augmented to A . If i is an \bar{A} -endpoint of value $x + 1$ of W , then i owns at least $\lceil s \rceil$ light goods in B and hence at least $\lfloor s \rfloor$ light goods in A . So i can serve as an \bar{A} -endpoint of value x in A . Let j be the other endpoint of W . If j has value $x + 1$ ($j = f$ is possible), the walk does not need a facilitator when applied to A and hence is applicable to A . If j has value x , j and f are different and hence f can act as a facilitator for the walk in A . In either case, one can augment the walk to A .

We may alternatively phrase the two preceding paragraphs as follows: If at least one endpoint of the walk has weight $x + 1$ in A , the walk can also be augmented to A . If both endpoints have weight x in A , then either i is one of the endpoints of the walk or i acts a facilitator for the augmentation. Since the walk could not be augmented to A , there is no facilitator in A .

We next prove that the existence of a walk is independent of the choice of i . So let i and i' be two distinct bundles of value x and assume that an improving walk exists in B when we allocate the light good to i . Let W with endpoints a and b be an improving walk in B after the allocation of a light good to i . We need to show that we can augment W also when we put the light good on i' .

Assume first that $i \notin \{a, b\}$. If i plays the role of a facilitator and $i' \notin \{a, b\}$, i' can play this role as well. If $i' \in \{a, b\}$, W has an endpoint of weight $x + 1$ in B and does not need a facilitator. So in either case there is an improving walk when the light good is allocated to i' .

Assume next that $i \in \{a, b\}$. If $i' \notin \{a, b\}$, i' can play the role of a facilitator. If $i' \in \{a, b\}$, $\{i, i'\} = \{a, b\}$ and hence W always has an endpoint of weight $x + 1$, no matter where the light good is allocated. So in either case there is an improving walk when the light good is allocated to i' .

Assume now that there is an improving walk in B and let C be obtained from B by augmenting the improving walk. Then all bundles of value $x + 1$ in A are heavy-only and hence all bundles of value $x + 1$ in C are heavy only. Note that the improving walk uses up the newly allocated light good as either i is an endpoint of the walk or acts as a facilitator. If i acts a facilitator, we may equally well put the light item on an endpoint of the walk. So let i and j be the endpoints of the walk. Then $C_{1/2} = A_{1/2} \cup \{i, j\}$, $C_0 = A_0 \setminus \{i, j\}$, and $C_1 = A_1$. Assume that there is an augmenting walk W' with respect to C . We will derive a contradiction to the optimality of A . If W' does not pass through either i or j , it is also an augmenting walk with respect to A . Otherwise, let W'' be a part of W' which starts at an endpoint of weight $x + 1$ and ends at either i or j . Since i and j have weight x in A , W'' is an augmenting walk with respect to A . ■

We next explain how to determine A^{k+1} from A^k . We take a heavy good from any bundle of largest value, turn it into a light good and put it on any bundle of smallest value; as a result x might increase by $1/2$ or 1 . Call the resulting allocation B^{k+1} . In [ACH⁺22] it was shown that in the case of integral s , B^{k+1} can be taken as A^{k+1} . This is no longer true.

However, something weaker is true. Recall that our algorithm consists of two phases: range reduction and optimizing small bundles. Recall that range reduction is the application of the reduction rules, i.e., of rules a) to d) in Lemma 1 with the weight of i greater than $x + 1$. The emphasized suffix played no role in Section 3, but will play a role in this section. *Range reduction has no effect on B^{k+1} .*

Let y be the smallest value of any bundle in B^{k+1} . Then $y \in \{x, x + 1/2, x + 1\}$. Let C^{k+1} be the allocation obtained by running the algorithm of Section 3 with starting allocation B^{k+1} (and of course, the moved good treated as light).

Lemma 15 *Let the weight of the heaviest bundle in A^k be larger than $x + 1$. Range reduction applied to B^{k+1} has no effect. C^{k+1} is optimal among all allocations where exactly $k + 1$ heavy goods are turned into light*

goods.

Proof: Let x be the minimum value of a bundle in A^k . Then the value of any bundle in B^{k+1} is in $\{x, x+1/2, x+1, k_0s, (k_0+1)s, (k_0+2)s, \dots\}$, where k_0s is the smallest multiple of s larger than $x+1$. Let y be the minimum value of a bundle in B^{k+1} . Then $y \in \{x, x+1/2, x+1\}$.

Assume we took the heavy good from the bundle of agent a and put it on the bundle of agent b . We use w_ℓ for the value of ℓ 's bundle in B^{k+1} and w_ℓ^A for its value in A^k . Then $w_a = w_a^A - s$, w_a^A is the maximum value of any bundle in A , $w_a^A > x+1$, $w_b = w_b^A + 1 = x+1$, and $w_\ell = w_\ell^A$ for $\ell \notin \{a, b\}$. For the first part, we need to show that none of the reduction rules of Lemma 1 is applicable to B^{k+1} . Recall that the reduction rules are rules a) to d) of Lemma 1 when applied to an agent i with $w_i \geq y + 3/2$.

Since b is the only bundle that contains an additional light good and $w_b \leq x+1 \leq y+1$, rule a) is not applicable to B^{k+1} .

We turn to rules b) to d). Consider any alternating path from an agent i with $w_i \geq x + 3/2$ to an agent j starting with an A -edge incident to i with respect to B^{k+1} . This path also exists with respect to A^k and none of the rules is applicable with respect to A^k since A^k is optimal. Since $w_i > x+1$, we have $i \neq b$ and $w_i^A \geq w_i \geq x + 3/2$.

Assume first that b) is applicable to B^{k+1} , i.e., $w_i \geq w_j + \lceil s \rceil$. Since b) is not applicable to A^k , $w_i^A < w_j^A + \lceil s \rceil$. Since a is the only agent that loses value, we must have $j = a$. But in B^{k+1} no bundle has value more than $w_a + s$ since a owned a heaviest bundle in A^k and hence $w_i \geq w_a + \lceil s \rceil$ is impossible with respect to B^{k+1} . For the further discussion, we may assume $w_i \leq w_j + s$.

Assume next that c) is applicable with respect to B^{k+1} , i.e., $w_j \leq w_i - 1$, $w_i \geq y + 3/2 \geq x + 3/2$ and j owns more than $s - (w_i - w_j)$ light goods in B^{k+1} . Note that $s - (w_i - w_j) \geq 0$ and hence j owns at least one light good in B^{k+1} . If j owns a light good in A^k , $w_j^A \leq x+1$ and hence $j \neq a$ and $w_j^A \leq w_j$. Thus $w_j^A \leq w_j \leq w_i - 1 \leq w_i^A - 1$. The number of light goods available to j in A^k is more than $s - w_i + w_j - u_{jb}$, where u_{jb} is one if j equals b and is zero otherwise. Since $w_i^A \geq w_i$ and $w_j^A = w_j - u_{jb}$, j owns more than $s - w_i^A + w_j^A$ light goods in A^k and hence c) is applicable. If j does not own a light good in A^k , $j = b$, j owns one light good in B^{k+1} , and $w_j = x+1 = w_j^A + 1$. Thus $w_j^A = w_j - 1 \leq w_i - 2 \leq w_i^A - 2$. Since c) is applicable to B^{k+1} and j owns exactly one light good in B^{k+1} , $s - w_i + w_j < 1$. Thus $w_i > s + w_j - 1 = s + x$ and hence $w_i^A \geq w_i \geq x + \lceil s \rceil = w_j^A + \lceil s \rceil$ and hence b) is applicable to A^k .¹

Assume finally, that d) is applicable to B^{k+1} , i.e., $w_i = y + 3/2$, $w_j = y + 1$, and j owns $\lceil s \rceil$ light goods in B^{k+1} and hence at least $\lceil s \rceil$ light goods in A^k . Thus $w_j^A \leq x+1$ and hence $j \neq a$ and therefore $w_j^A \leq w_j$. If $w_j^A \leq w_i^A - 1$, either b) or c) is applicable to A^k . So $w_i^A \leq w_j^A + 1/2$ and $w_i = w_j + 1/2$. Since $w_i^A \geq w_i$, this is only possible if $w_i = w_i^A$ and $w_j = w_j^A$ and hence $\{i, j\} \cap \{a, b\} = \emptyset$. Thus j owns $\lceil s \rceil$ light goods already in A^k and d) is applicable provided that $x = y$. But $y > x$ would imply that we have bundles of value $y+1$ and $y+1/2$ in A^k for some $y > x$. However, such bundles must be heavy-only, a contradiction.²

We have now shown that range reduction does not change B^{k+1} . Let C^{k+1} be the allocation obtained by optimizing the bundles of weight y , $y+1/2$ and $y+1$. We have shown in Lemma 14 that the optimization of the small valued bundles does not depend on the choice of b , the bundle of value x to which the light good is added. So the change of NSW is the product of two effects: the decrease due to the effect of removing a heavy good from a and the increase due to reoptimizing the small valued bundles after the addition of a light good. The decrease is smallest if a is a bundle of maximum value and the increase is independent of the choice of b . This concludes the proof. ■

Let z be the value of the heaviest bundle in A^k . Removal of a heavy good from a bundle of value z multiplies the NSW by a factor $z-s/z$. We add a light good to a bundle of value x and reoptimize. This might multiply the NSW by a factor $(x+1/2)^2/x^2$. So the combined effect is

$$\frac{z-s}{z} \left(\frac{x+1/2}{x} \right)^2$$

¹It is important here that we require $w_i \geq y + 3/2$ in the reduction rules. Otherwise, we might have $y = x$, $i = b$ and j an agent with value $x+1$ in A^k .

²It is important that we exclude $w_i = y+1$. Otherwise $y = x$, $i = b$, and j any agent owning $\lceil s \rceil$ light goods and reachable from b by an alternating path starting with an A -edge would be possible.

This is larger than one if $(z-s)(x+1/2)^2 > zx^2$ or $z(x+1/4) > s(x^2+x+1/4)$ or

$$z > s \frac{x^2+x+1/4}{x+1/4} > sx.$$

Lemma 16 *Conversion of a heavy good in a bundle of value $z > x+1$ to a light good can improve NSW only if $z > sx$.*

Polynomial Time: We construct iteratively allocations A^1, A^2, \dots . Each time we convert a heavy good to a light good and reoptimize. There are at most m conversions and each reoptimization takes polynomial time. Thus the overall time is polynomial.

G.2 All Bundles Have Value at most $x+1$

The values of all bundles lie between x and $x+1$. We want to show that conversion of a heavy into a light good followed by reoptimization cannot improve the NSW of the allocation.

For the sake of a contradiction, assume otherwise. We consider all g for which conversion of g and reoptimization results in an allocation with larger NSW than A . Consider any such g and let $g \in A_i$. Converting g into a light good results in an allocation $C(g)$; $C_i^H(g) = A_i^H \setminus g$, $C_i(g)$ contains one more light good than $A_i(g)$, $C_j(g) = A_j$ for $j \neq i$. We then reoptimize. Let $B(g)$ be an optimal allocation closest to $C(g)$, i.e., with minimal $|C^H(g) \oplus B^H(g)|$. We choose g such that

- $\text{NSW}(B(g))$ is maximum and
- among the g that maximize $\text{NSW}(B(g))$, $|C^H(g) \oplus B^H(g)|$ is smallest.

For simplicity, let us write B and C instead of $B(g)$ and $C(g)$ for this choice of g .

Lemma 17 $B_i^H = A_i^H \setminus g$ and B_i contains no light good.

Proof: If B_i contains a light good, we convert this light good back to g and obtain an allocation for the original instance whose NSW is better than the NSW of B . Thus the NSW of B cannot be larger than the NSW of A .

Assume next that $B_i^H \setminus C_i^H$ is non-empty. Then there is an B - C -alternating path P starting in i and ending in an agent j with an edge $(g', j) \in C^H \setminus B^H$. We augment P to B , re-allocate g to i , convert g' to a light, and allocate the light goods in the same way as in B . Note augmentation of P reduces the number of heavy goods allocated to i by one which we compensate by the re-allocation of g , and increases the number of heavy goods allocated to j by one which we compensate by converting g' to a light good. In this way, we obtain an allocation D with the same NSW as B which is a candidate for $B(g')$. Also $|D^H \oplus C^H(g')| < |B^H \oplus C^H|$, a contradiction to the choice of g .

Assume finally that B_i^H is a proper subset of $A_i^H \setminus g$. Then $v_i(B_i) \leq v_i(A_i) - 2s \leq x+1 - 2s \leq x-2$. We can obtain B from A as follows. We convert g into a light and allocate the light good to an agent different from i , we move all goods in $(A_i^H \setminus g) \setminus B_i^H$ to their owners in B , and then apply the optimization rules to the bundles different than i . None of the optimization rules decreases the value of the minimum bundle. Thus all bundles in B except for B_i have value at least x and one of them contains a light good. Thus B is not optimum as Lemma 1a) allows to improve it. ■

We can now estimate the NSW of B . $B_i^H = A_i^H \setminus g$ and B_i does not contain a light good. Thus $v_i(B_i) \leq v_i(A_i) - s$. The conversion of the heavy good into a light good can also be viewed as the deletion of the heavy good followed by an addition of a light good to an agent different from i . The maximum increase in NSW obtainable after addition of the light is $(x+1/2)^2/x^2$ according to Lemma 14. Thus

$$\frac{\text{NSW}(B)}{\text{NSW}(A)} \leq \frac{v_i(B_i)}{v_i(A_i)} \cdot \frac{(x+1/2)^2}{x^2} \leq \frac{x-1/2}{x+1} \cdot \frac{(x+1/2)^2}{x^2} \leq \frac{x^2-1/4}{x^2} \cdot \frac{x+1/2}{x+1} < 1,$$

where the second inequality follows from $v_i(A_i) \leq x+1$ and $s \geq \frac{3}{2}$.

H Certificates of Optimality

Our algorithm for finding an optimal allocation consists of two parts: range reduction and optimization of A_{low} . The optimization of A_{low} exploits a connection to parity matchings. It would be nice to have a compact and easy-to-check certificate for the optimality of the output of the algorithm. Such certificates are available for (generalized) matching problems and we conjecture that related certificates are also available for our problem. The book by Akiyama and Kano [AK11] on Factors and Factorizations of Graphs contains a wealth of relevant results.

The basis is Tutte's characterization of the existence of a perfect matching [Tut52].

Theorem 3 (Tutte) *An undirected graph G has a perfect matching if for every subset U of vertices*

$$\text{odd}(G-U) \leq |U|,$$

where $\text{odd}(G-U)$ is the number of components of $G-U$ of odd cardinality. The minimum number of unmatched vertices is $\max_{U \subseteq V} \text{odd}(G-U) - |U|$.

There are implementations of matching algorithms [MN99, MMNS11] that either return a perfect matching or a certificate that none exists, i.e., a set U with $\text{odd}(G-U) > |U|$.

More general theorems deal with factors in graphs with prescribed degrees. Let $f : V \rightarrow \mathbb{N}_0$ be an integer-valued function defined on the vertices of a graph G . An f -factor of G is a subgraph F with $\deg_F(v) = f(v)$ for all v . Tutte [Tut54] generalized the theorem above to f -factors. For disjoint subsets S and T of V , $e_G(S, T)$ is the number of edges of G having one endpoint each in S and T .

Theorem 4 (Tutte) *G has an f -factor if and only if for all disjoint subsets S and T of V*

$$\delta(S, T) = \sum_{v \in S} f(v) + \sum_{v \in T} (\deg_G(v) - f(v)) - e_G(S, T) - q(S, T) \geq 0,$$

where $q(S, T)$ denotes the number of components C of $G - (S \cup T)$ such that

$$\sum_{v \in V(C)} f(v) + e_G(C, T) = 1 \pmod{2}.$$

Lovász [L70] generalized the condition further to take parity constraints into account. Let G be a graph and let $g, f : V(G) \rightarrow \mathbb{N}_0$ be functions such that

$$g(v) \leq f(v) \quad \text{and} \quad f(v) = g(v) \pmod{2}$$

for all $v \in V$. A spanning subgraph F of G is a parity (g, f) -factor if

$$g(v) \leq \deg_F(v) \leq f(v) \quad \text{and} \quad \deg_F(v) = f(v) \pmod{2}$$

for all $v \in V$.

Theorem 5 (Lovász) *G has a parity (g, f) -factor if and only if for all disjoint subsets S and T of G ,*

$$\eta(S, T) = \sum_{v \in S} f(v) + \sum_{v \in T} (\deg_G(v) - g(v)) - e_G(S, T) - q(S, T) \geq 0,$$

where $q(S, T)$ is the number of components C of $G - (S \cup T)$ such that

$$\sum_{v \in C} f(v) + e_G(C, T) = 1 \pmod{2}.$$

Components for which the latter condition holds are called *odd components*.

The proof is a reduction to the f -factor theorem. One adds $(f(v) - g(v))/2$ self-loops to v and then looks for an f -factor.

In our reduction to parity matching, we use $f(v) = g(v) = 1$ for goods. For the agents $g(v) = f(v) \pmod{2}$ and $f(v)$ depends on N -set to which v belongs. There are only three possible values for $f(v)$. Are there simple characterizations for S and T ?