Subexponential Parameterized Directed Steiner Network Problems on Planar Graphs: a Complete Classification

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Abstract

In the Directed Steiner Network problem, the input is a directed graph G, a set $T \subseteq V(G)$ of k terminals, and a demand graph D on T. The task is to find a subgraph $H \subseteq G$ with the minimum number of edges such that for every $(s,t) \in E(D)$, the solution H contains a directed $s \to t$ path. The goal of this paper is to investigate how the complexity of the problem depends on the demand pattern in *planar graphs*. Formally, if $\mathcal{D}$ is a class of directed graphs closed under identification of vertices, then the $\mathcal{D}$-Steiner Network ($\mathcal{D}$-SN) problem is the special case where the demand graph D is restricted to be from $\mathcal{D}$. For general graphs, Feldmann and Marx [14] characterized those families of demand graphs where the problem is fixed-parameter tractable (FPT) parameterized by the number k of terminals. They showed that if $\mathcal{D}$ is a superset of one of five hard families, then $\mathcal{D}$-SN is W[1]-hard parameterized by k, otherwise it can be solved in time $f(k) \cdot n^{O(1)}$.

For planar graphs, besides the existence of an FPT algorithm, it is also an interesting question whether the W[1]-hard cases can be solved by subexponential parameterized algorithms. For example, Chitnis et al. [8] showed that, assuming the Exponential-Time Hypothesis (ETH), there is no $f(k) \cdot n^{o(k)}$ time algorithm for the general $\mathcal{D}$-Steiner Network problem on planar graphs, but the special case called Strongly Connected Steiner Subgraph (where the demand graph D is a bidirected clique) can be solved in time $f(k) \cdot n^{O(\sqrt{k})}$ on planar graphs. We present a far-reaching generalization and unification of these two results: we give a complete characterization of the behavior of every $\mathcal{D}$-SN problem on planar graphs. We classify every class $\mathcal{D}$ closed under identification of vertices into three cases: assuming ETH, either the problem is

1. solvable in time $2^{O(k)} \cdot n^{O(1)}$, i.e., FPT parameterized by the number k of terminals, but not solvable in time $2^{o(k)} \cdot n^{O(1)}$,
2. solvable in time $f(k) \cdot n^{O(\sqrt{k})}$, but cannot be solved in time $f(k) \cdot n^{o(\sqrt{k})}$, or
3. solvable in time $f(k) \cdot n^{O(k)}$, but cannot be solved in time $f(k) \cdot n^{o(k)}$.

We show that the FPT cases (Case 1) are the same as in the case of general graphs: $\mathcal{D}$ needs to exclude the same five families of hard graphs. We further identify a finite number of hard families that $\mathcal{D}$ needs to exclude if we want to solve $\mathcal{D}$-SN on planar graphs in time $f(k) \cdot n^{O(\sqrt{k})}$ (Case 2). As an important step of our lower bound proof, we discover that, assuming ETH, $\mathcal{D}$-SN on planar graphs has no $f(k) \cdot n^{o(k)}$ time algorithm where $\mathcal{D}$ is the class of all directed bicliques. This corresponds to the following simple problem: given two sets of terminals S and T with $|S| + |T| = k$, find a subgraph with minimum number of edges such that every vertex of T is reachable from every vertex of S. Our result gives a rare example of a genuinely planar problem that cannot be solved in time $f(k) \cdot n^{o(k)}$.

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1 Introduction

Finding Steiner trees and related network design problems were intensively studied in undirected graphs, directed graphs, and planar graphs, from the viewpoint of approximation and parameterized algorithms [1–3, 5–8, 11–14, 16–18, 21, 22, 24, 27, 28, 30, 32, 33]. The simplest problem of this type is Steiner Tree, where given a graph G and set $T \subseteq V(G)$ of terminals, the task is to find a tree with smallest number of edges that contains every terminal. This problem models a network-design scenario where the terminals need to be connected to each other with a network of minimum cost. Steiner Forest is the generalization where we do not require connection between every pair of terminals, but have to satisfy a given set of demands. Formally, the input of Steiner Forest is a graph G with pairs of vertices (s_1, t_1), ..., (s_d, t_d), the task is to find a subgraph with the minimum number of edges that satisfies every request, that is, s_i and t_i are in the same component of the solution for every $i \in [d]$.

On directed graphs, Directed Steiner Tree (DST) is defined by specifying one of the terminals in T to be the root and the task is to find a subgraph with the smallest number of edges such that there is path from the root to every terminal in the solution. This problem models a scenario where we need to construct a network where the root can broadcast to every other terminal. An equally natural network design problem on directed graphs is the Strongly Connected Steiner Subgraph (SCSS) problem, where given a directed graph G and a set $T \subseteq V(G)$ of terminals, the task is to find a subgraph with the smallest number of edges where T is in a single strongly connected component, or in other words, the solution contains a path from every terminal to every other terminal. The directed variant of Steiner Forest generalizes both of these problems: in Directed Steiner Network (DSN), the input is a digraph G with pairs of vertices (s_1, t_1), ..., (s_d, t_d), and the task is to find a subgraph with the minimum number of edges that has an $s_i \to t_i$ path for every $i \in [d]$.

Planar graphs. A well-known phenomenon on planar graphs is that the running time of parameterized algorithms for typical NP-hard problems have exponential dependence on $O(\sqrt{k})$, where k is the parameter, and this dependence is best possible assuming the Exponential-Time Hypothesis (ETH) [8–10, 15, 19, 20, 23, 25–27, 29]. All three of Directed Steiner Tree, Strongly Connected Steiner Subgraph, and Directed Steiner Network remain NP-hard on planar graphs. However, they behave very differently from the viewpoint of parameterized complexity: the dependence of the running time on the number k of terminals is very different.

Our starting point

1. Planar DST can be solved in time $2^k \cdot n^{O(1)}$ [4], but cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$ [27], assuming the ETH.
2. Planar SCSS can be solved in time $2^{O(k \log k)} \cdot n^{O(\sqrt{k})}$ [8], but has no algorithm with running time $f(k) \cdot n^{o(\sqrt{k})}$ for any function f, assuming the ETH [8].
3. Planar DSN can be solved in time $f(k) \cdot n^{O(k)}$ [12], but has no algorithm with running time $f(k) \cdot n^{o(k)}$ for any function f, assuming the ETH [8].

The goal of this paper is to put these results into the context of a wider landscape of directed network design problems. We systematically explore other special cases of Directed Steiner Network and determine their behavior on planar graphs. Our main result is showing that every special case behaves similarly to one of these three problems: assuming ETH, the best possible running time is of the form $2^{O(k)} \cdot n^{O(1)}$, $f(k) \cdot n^{O(\sqrt{k})}$, or $f(k) \cdot n^{O(k)}$. Furthermore, we provide an exact combinatorial characterization of the problems belonging to the three classes.

Dichotomy for general graphs. We explore the different special cases of Directed Steiner Network on planar graphs in a framework similar to how Feldmann and Marx [14] treated the problem on general graphs. We can define various special cases of Directed Steiner Network by looking at what kind of graph the connection demands define on the terminals: it is an out-star for Directed Steiner Tree, a bidirected clique for Strongly Connected Steiner Subgraph, and a matching for Directed Steiner Network. More generally, for every class $\mathcal{D}$ of directed graphs, we investigate the problem where the pattern of demands has to belong to the class $\mathcal{D}$. Our goal is to understand how the graph-theoretic properties of the members of $\mathcal{D}$ influence the resulting special case of Directed Steiner Tree.

Formally, for every class $\mathcal{D}$, Feldmann and Marx [14] defined the restriction of the problem in the following way.

$\mathcal{D}$-Steiner Network
Input: Digraph G, a set of k terminals $T \subseteq V(G)$, and a demand digraph $D \in \mathcal{D}$ with vertex set T.
Question: What is the minimum number of edges in a subgraph H of G where for each $(u, v) \in E(D)$ there is a $u \to v$ path in H?

Note that only the transitive closure of D matters for the problem: if D_1 and D_2 have the same transitive closure, then having D_1 or D_2 in the input results in exactly the same problem. Therefore, it makes sense to consider only classes $\mathcal{D}$ that are *closed under transitive equivalence,* that is, if D_1 and D_2 have the same transitive closure and $D_1 \in \mathcal{D}$, then $D_2 \in \mathcal{D}$ as well. Another natural assumption is that $\mathcal{D}$ is *closed under identifying vertices.* That is, if $G \in \mathcal{D}$ and G' is obtained by merging two vertices $x, y \in V(G)$ to a single vertex whose in- and out-neighbors are the union of the in- and out-neighbors of x and y, then G' is also in $\mathcal{D}$. This closure property models the extension of the problem where we can put multiple terminals at the same vertex and we parameterize by the number of vertices that have terminals.

Feldmann and Marx [14] characterized those classes $\mathcal{D}$ closed under transitive equivalence and identifying vertices where $\mathcal{D}$-Steiner Network is fixed-parameter tractable (FPT) parameterized by the number of terminals, that is, can be solved in time $f(k) \cdot n^{O(1)}$. They identified five classes of graphs that prevent the problem from being FPT. A *pure out-diamond* is a complete bipartite graph $K_{2,t}$ directed from the 2-element side to the t-element side. A *flawed out-diamond* has in addition a vertex v and edges going from v to the 2-element side. The pure in-diamond and flawed in-diamond are defined similarly by reversing the orientation of the edges. Let us denote by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_5$ the class of all pure out-diamonds, flawed out-diamonds, pure in-diamonds, flawed in-diamonds, and directed cycles, respectively.

Theorem 1.1 (Feldmann and Marx [14])**.** *Let $\mathcal{D}$ be a class of graphs closed under transitive equivalence and identifying vertices.*

1. ***FPT:*** *If $\mathcal{A}_i \not\subseteq \mathcal{D}$ for any $i \in [5]$, then $\mathcal{D}$-*Steiner Network *can be solved in time $2^{O(k)} n^{O(1)}$, where k is the number of terminals.*
2. ***Hard:*** *If $\mathcal{A}_i \subseteq \mathcal{D}$ for some $i \in [5]$, then $\mathcal{D}$-*Steiner Network *is W[1]-hard parameterized by the number k of terminals.*

The first part of Theorem 1.1 was proved by a combination of an algorithm that solves the problem in time $2^{O(kw \log w)} \cdot n^{O(w)}$ if there is an optimum solution with treewidth w and a combinatorial result showing that if $\mathcal{D}$ is not the superset of $\mathcal{A}_i$ for any $i \in [5]$, then there is a constant bound on the treewidth of optimum solutions. The second part follows from a W[1]-hardness result for each of the five classes $\mathcal{A}_i$.

Our result: trichotomy for planar graphs. Our main result classifies PLANAR $\mathcal{D}$-STEINER NETWORK into three levels of complexity: $2^{O(k)} \cdot n^{O(1)}$, $f(k) \cdot n^{O(\sqrt{k})}$, or $f(k) \cdot n^{O(k)}$ time. In light of Theorem 1.1 and the earlier results on planar graphs, there are three natural questions that arise:

1. Are there cases that are FPT on planar graphs, but W[1]-hard on general graphs?
2. Are there subexponential FPT cases on planar graphs, that is, where the running time is $2^{o(k)} \cdot n^{O(1)}$?
3. Where is the boundary line between the $f(k) \cdot n^{O(\sqrt{k})}$ and $f(k) \cdot n^{O(k)}$ cases?

We answer the first question negatively: the hard cases remain hard on planar graphs. The answer to the second question is also negative: we show that every (nontrivial) case of PLANAR $\mathcal{D}$-SN is at least as hard as DIRECTED STEINER TREE, hence a known lower bound [27] shows that there is no subexponential FPT algorithm, assuming ETH. To answer the third question, we define a finite number $\kappa \leq 10000$ of classes $\mathcal{C}_i$, $i \in [\kappa]$, and show that these are precisely the classes of patterns that prevent subexponential $f(k) \cdot n^{O(\sqrt{k})}$ time algorithms.

Our main result

Theorem 1.2. *Let $\mathcal{D}$ be a class of directed graphs closed under transitive equivalence and identifying vertices where the number of edges is not bounded.*

1. ***FPT:*** *If $\mathcal{A}_i \not\subseteq \mathcal{D}$ for any $i \in [5]$, then* PLANAR $\mathcal{D}$-STEINER NETWORK
   (i) can be solved in time $2^{O(k)} \cdot n^{O(1)}$,
   (ii) but has no $2^{o(k)} \cdot n^{O(1)}$ time algorithm assuming the ETH.
2. ***Subexponential:*** *If $\mathcal{A}_i \subseteq \mathcal{D}$ for some $i \in [5]$, but $\mathcal{C}_i \not\subseteq \mathcal{D}$ for any $i \in [\kappa]$, then* PLANAR $\mathcal{D}$-STEINER NETWORK
   (iii) can be solved in time $f(k) \cdot n^{O(\sqrt{k})}$,
   (iv) but has no $f(k) \cdot n^{o(\sqrt{k})}$ time algorithm assuming the ETH.
3. ***Hard:*** *If $\mathcal{C}_i \subseteq \mathcal{D}$ for some $i \in [\kappa]$, then* PLANAR $\mathcal{D}$-STEINER NETWORK
   (v) can be solved in time $f(k) \cdot n^{O(k)}$,
   (vi) but has no $f(k) \cdot n^{o(k)}$ time algorithm assuming the ETH.

Hard classes. Let us define now the graph classes $\mathcal{C}_i$ representing the hard-patterns. Given a digraph G and a set $X \subseteq V(G)$, an X*-source* is a vertex $s \in V(G) \setminus X$ such that $N^+(s) = X$. Similarly, an X*-sink* is a vertex $t \in V(G) \setminus X$ such that $N^-(t) = X$. The first 4 classes $\mathcal{C}_1, \ldots, \mathcal{C}_4$ are defined by extending a biclique.

Definition 1.3 (t-hard-biclique-pattern)**.** A t*-hard-biclique-pattern* is an (acyclic) digraph D constructed in the following way. We start with two disjoint sets A and B with $|A| = |B| = t$ and introduce every edge from A to B. Furthermore, we introduce into D any combination of the following items (see Figure 1):

1. an A-source;
2. a B-sink.

In particular, there are $2 \cdot 2$ types of t-hard-biclique patterns: we let $\mathcal{C}_1, \ldots, \mathcal{C}_4$ be the 4 classes that each contain all the t-hard-biclique-patterns of a specific type for every t.

The following definition specifies the remaining classes.

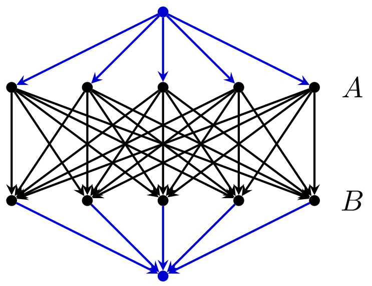

Figure 1: The 5-hard biclique patterns: each blue vertex may or may not be present.

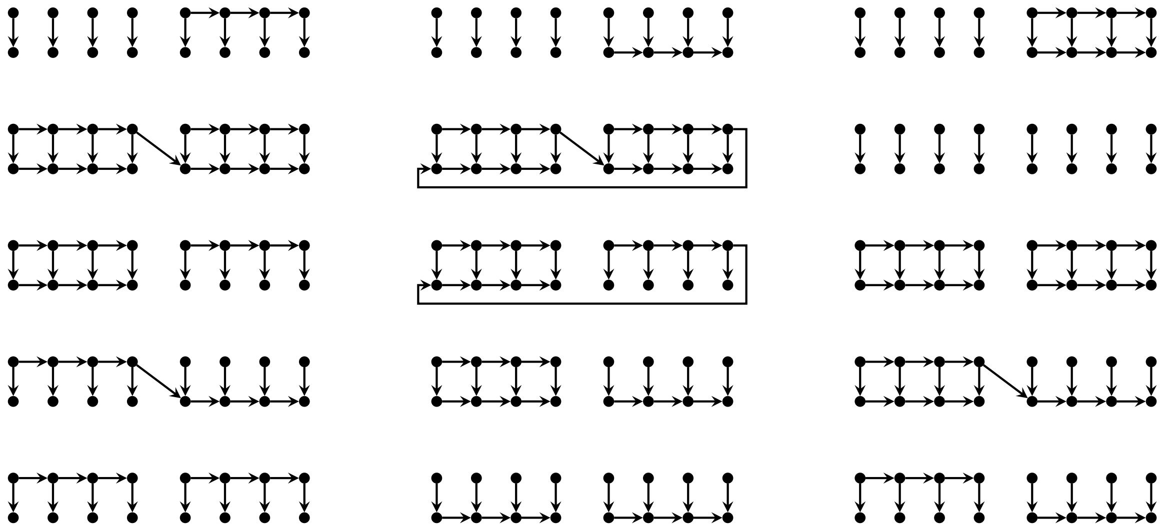

Figure 2: The 4-hard matching patterns (without source, sink, r_{WZ}, or r_{YX}).

Definition 1.4 (t-hard-matching-pattern)**.** A *t-hard-matching-pattern* is an (acyclic) digraph D constructed the following way. We start with disjoint vertex sets $W = \{w_1, \ldots, w_t\}$, $X = \{x_1, \ldots, x_t\}$, $Y = \{y_1, \ldots, y_t\}$ and $Z = \{z_1, \ldots, z_t\}$ and introduce the edges $w_i x_i$ and $y_i z_i$ for every $i \in [t]$. Furthermore, we introduce into D any combination of the following items:

1. either the directed path $w_1 \to w_2 \to \ldots \to w_t \to z_1 \to z_2 \to \ldots \to z_t$, or any of the directed paths $w_1 \to w_2 \to \ldots \to w_t$ and $z_1 \to z_2 \to \ldots \to z_t$;
2. either the directed path $y_1 \to y_2 \to \ldots \to y_t \to x_1 \to x_2 \to \ldots \to x_t$, or any of the directed paths $x_1 \to x_2 \to \ldots \to x_t$ and $y_1 \to y_2 \to \ldots \to y_t$;
3. an S-source for exactly one $S \in \{W, X, Y, Z, W \cup Y, X \cup Z, X \cup Y, W \cup Z\}$;
4. an S-sink for exactly one $S \in \{W, X, Y, Z, W \cup Y, X \cup Z, X \cup Y, W \cup Z\}$;
5. a vertex r_{WZ} such that $N^-(r_{WZ}) = W$ and $N^+(r_{WZ}) = Z$;
6. a vertex r_{YX} such that $N^-(r_{YX}) = Y$ and $N^+(r_{YX}) = X$.

In particular, there are $5 \cdot 5 \cdot 9 \cdot 9 \cdot 2 \cdot 2$ types of t-hard matching patterns: we let $\mathcal{C}_5, \ldots, \mathcal{C}_{8104}$ be the 8100 classes that each contain all the t-hard-matching-patterns of a specific type for every t.

Note that some of these classes are isomorphic. For example, adding the path $x_1 \to x_t$ or the path $z_1 \to z_t$ lead to isomorphic graphs. If we just consider the graph classes where we choose not to add a source, sink, vertex r_{WZ}, or vertex r_{YX}, then we have 15 nonisomorphic classes, as shown in Figure 2. One could think of t-hard-matching-patterns as (the transitive closure of) one of these graphs, potentially extended by appropriate sources and sinks.

Finally, we define a t-hard pattern as any of the patterns defined above.

Definition 1.5 (t-hard-pattern)**.** A *t-hard-pattern* is either a t-hard-biclique-pattern or t-hard-matching-pattern.

1.1 Overview of our main result

Observe that Theorem 1.2 consists of six statements. Let us briefly discuss how these six statements are proved. Note that some of these statements follow from known results, while for others we need to do a substantial amount of new technical work. The proofs of statments *(iii)* and *(vi)* form the main technical part of the paper (see Figure 4).

Statement *(i)*

The FPT result *(i)* follows directly from Theorem 1.1 (here the surprising aspect is that, by statement *(iv)*, there are no further FPT cases).

Statement *(ii)*

The lower bound *(ii)* follows by observing that every relevant class contains either all in-stars or all out-stars, hence the lower bound for Directed Steiner Tree [27] applies. To avoid triviality, we need to assume that the class contains graphs with arbitrarily large number of edges.

Lemma 1.6. *Let $\mathcal{D}$ be a class of graphs closed under identifying vertices and transitive closures where the number of edges of the graphs is not bounded. Then one of the following holds:*

- *$\mathcal{D}$ contains every directed cycle,*
- *$\mathcal{D}$ contains every out-star, or*
- *$\mathcal{D}$ contains every in-star.*

In statement *(ii)* of Theorem 1.2, we assume that $\mathcal{A}_i \not\subseteq \mathcal{D}$, and $\mathcal{A}_5$ is the class of all directed cycles. Thus $\mathcal{D}$ contains either every out-star or every in-star.

Statement *(iii)*

Our main technical result is proving statement *(iii)*: the existence of a $f(k) \cdot n^{O(\sqrt{k})}$ time algorithm if $\mathcal{C}_i \not\subseteq \mathcal{D}$ for any $i \in [\kappa]$ (in the following subsection, we give a more detailed description of the proof). This algorithm is obtained by showing that the treewidth of the optimal solution is always $O(\sqrt{k})$ under these conditions. Then we can use the following result of Feldmann and Marx [14].

Theorem 1.7 (Theorem 1.5 of [14])**.** *If an instance (G, T, D) of* Directed Steiner Network *has an optimum solution H of treewidth w, then it can be solved in $2^{O(kw \log w)} \cdot n^{O(w)}$ time.*

Note that this is a slightly weaker form of the statement, with a simplified bound on the running time. With Theorem 1.7 at hand, our main goal is to prove that every optimum solution of Planar $\mathcal{D}$-SN has treewidth $O(\sqrt{k})$ if $\mathcal{C}_i \not\subseteq \mathcal{D}$ for any $i \in [\kappa]$.

Towards proving this bound, we first translate the question to a problem on acyclic graphs: it is sufficient to show that if the solution is acyclic, then the total degree of the branch vertices (i.e., of degree > 2) is $O(k)$. More formally, for a vertex v of a digraph, let $d^*(v)$ denote the *branch degree* of v, defined as

$$d^*(v) = \max(d^+(v) + d^-(v) - 2, 0),$$

where $d^+(v)$ and $d^-(v)$ denotes the out- and in-degree of v, respectively. The total branch degree of a graph G is the sum of all branch degrees of the vertices of G.

We say that a feasible solution H of (G, T, D) is *edge-minimal* if for all edges $e \in E(H)$ the graph $H - e$ is not feasible. An edge e is *essential* for some demand edge $tt' \in E(D)$ if there is no $t \to t'$ path in $H - e$. Note that all edges of an edge-minimal graph H are essential for some demand edge of D. We say that a pattern class $\mathcal{D}$ is *c-bounded* for some $c = O(1)$ if for any instance (G, T, D) Planar $\mathcal{D}$-Steiner Network where G, D are acyclic, and any edge-minimal solution H, the total branch degree of H is at most ck.

The next theorem moves the problem to the domain of acyclic digraphs: what we need now is a linear bound on the total branch degree of acyclic solutions.

Theorem 1.8. *If the pattern class $\mathcal{D}$ is c-bounded for some $c = O(1)$, then for any instance of* Planar $\mathcal{D}$-Steiner Network *with $|T| = k$ the solution graph H has treewidth $O(\sqrt{k})$.*

Applying Theorem 1.7 implies that c-bounded classes have the desired subexponential algorithm, but we still need to establish a link between non-c-bounded classes and t-hard-patterns. First, we argue that if the total branch degree is too large, then a grid-like structure can be found in the solution. The grid-like structure appears in the solution to satisfy a set of edges in the demand graph D, and this set of demands form a certain hard structure in the demand pattern that we call the t-tough-pair which we define informally here (see Definition 2.3 for a formal definition). We say that two edges e_1 and e_2 are *weakly independent* if there is no directed path from the head of one to the tail of the other. Edges e_1 and e_2 are *strongly independent* if, in addition to being weakly independent, there is no directed path containing the heads of both edges and there is no directed path containing the tails of both edges. An edge e is *minimal* in a digraph D if there is no path from the head of e to the tail of e avoiding e. Let $E_1 \cup E_2$ be a vertex-disjoint set of minimal edges with $|E_1| = |E_2| = t$. We say that (E_1, E_2) is a *t-tough-pair* if

- any two edges in $e, e' \in E_1$ are weakly independent,
- any two edges in $e, e' \in E_2$ are weakly independent, and
- any two edges $e_1 \in E_1$ and $e_2 \in E_2$ are strongly independent.

Observe that in particular the two matchings in a t-hard-matching-pattern form (vertical edges in Figure 2) a t-tough-pair. Similarly, taking two vertex-disjoint matchings of size t each in a t-hard-biclique-pattern is also a t-tough-pair.

Our main structure theorem connects the total branch degree to the existence of these kind of hard structures.

Theorem 1.9 (Structure Theorem)**.** *Let $\mathcal{D}$ be a class of graphs closed under identifying vertices and transitive equivalence. Then either $\mathcal{D}$ has a pattern with a t-tough-pair for each positive integer t, or it is c-bounded for some constant c.*

Theorems 1.8 and Theorem 1.9 show that the existence of arbitrarily large t-tough-pairs is the canonical reason why treewidth is not $O(\sqrt{k})$. The lower bounds ruling out $f(k) \cdot n^{o(k)}$ time algorithms essentially rely on the existence of t-tough-pairs. However, the existence of a t-tough-pair in a demand pattern $D \in \mathcal{D}$ is not sufficient for the lower bound: the t-tough-pair could be only a small part of the pattern D, and hence the lower bounds may not apply. We show, with heavy use of Ramsey's Theorem and other combinatorial arguments, that whenever a large t-tough-pair appears in a graph, then the graph can be "cleaned": we can identify vertices to obtain one of the t-hard-patterns. Therefore, if arbitrary large t-tough-pairs appear in the members of a class $\mathcal{D}$ closed under identifying vertices, then the class is a superset of one of the hard classes $\mathcal{C}_i$.

Theorem 1.10. *Let $\mathcal{D}$ be a class of graphs closed under transitive equivalence and identifying vertices. The following two are equivalent:*

1. *For every t, there is a $D \in \mathcal{D}$ that has a t-tough pair.*

2. $\mathcal{C}_i \subseteq \mathcal{D}$ for some $i \in [\kappa]$.

We can conclude that if $\mathcal{D}$ is not the superset of $\mathcal{C}_i$ for any $i \in [\kappa]$, then the treewidth of the optimum solution is $O(\sqrt{k})$, implying that Planar DSN can be solved in time $f(k) \cdot n^{O(\sqrt{k})}$.

Statement *(iv)*

The statement *(iv)* ruling out $f(k) \cdot n^{o(\sqrt{k})}$ time algorithms follows from the known lower bound for Strongly Connected Steiner Subgraph (i.e., $\mathcal{D} = \mathcal{A}_5$) [8] and from reproving the W[1]-hardness of diamonds (i.e., $\mathcal{D} \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$) for planar graphs [14]. Compared to the W[1]-hardness on general graphs, the proof for planar graphs is more involved. As it is very usual for planar problems, we establish these lower bounds by reducing from $k \times k$-Grid Tiling, which cannot be solved in time $f(k) \cdot n^{o(k)}$, assuming ETH [9]. For statement *(iv)*, we need to reduce from $\sqrt{k} \times \sqrt{k}$ Grid Tiling to a Planar DSN with $O(k)$ terminals forming a pure/flawed in/out-diamond pattern, ruling out $f(k) \cdot n^{o(\sqrt{k})}$ algorithms for such patterns.

In all these reductions, we are reusing and extending the gadget constructions from earlier work [8]. However, the high-level structure of the reduction is substantially different and depends on the pattern class we are considering. In light of Theorem 1.7, we should first verify, as a sanity check, that the treewidth of the solution can be sufficiently large, that is, it can be $\Omega(\sqrt{k})$ in case of diamonds. Typically, one can expect that examples with sufficiently large treewidth shed some light on how the high-level structure of the hardness proof could like. Figure 3 shows that treewidth can be indeed sufficiently large: a $\sqrt{k} \times \sqrt{k}$ grid can be obtained from two "interlocking combs."

Statement *(v)*

The upper bound $f(k) \cdot n^{O(k)}$ (statement *(v)*) follows from the work of Eiben et al. [12], who showed that Planar DSN with k terminals can be always solved within this running time.

Statement *(vi)*

To prove statement *(vi)* ruling out $f(k) \cdot n^{o(k)}$ algorithms, we provide such a lower bound for each class $\mathcal{C}_i$ for $i \in [\kappa]$. Analogously to statement *(iv)*, the proof is by reduction from $k \times k$ Grid Tiling to a Planar DSN instance with a k-hard-matching-pattern or a k-hard-biclique-pattern, ruling out $f(k) \cdot n^{o(k)}$ algorithms. Again, let us verify that the treewidth can be sufficiently large: Figure 3 shows how a $k \times k$ grid can appear in the solution to an instance with k terminals.

For t-hard-matching-patterns, the simplest case is when we have two induced matchings of size t. Then a $t \times t$ grid can arise very easily in the solution if the terminals are on the boundary of a grid. The crucial point here is that the t-hard-matching-pattern was defined in a way that all the additional paths, sources etc. do not interfere with the grid, see the figure for an example. For the t-hard-biclique-pattern, there is a non-obvious and highly delicate way of constructing an instance with $2t$ terminals where a $t \times t$ grid appears. Combining these constructions gives the lower bound.

Theorem 1.11. *Let $\mathcal{D}$ be a class of graphs closed under identifying vertices and transitive equivalence. If $\mathcal{C}_i \subseteq \mathcal{D}$ for some $i \in [\kappa]$, then* Planar $\mathcal{D}$-Steiner Network *has no $f(k) \cdot n^{o(k)}$ time algorithm assuming the ETH.*

Let us observe that if $\mathcal{D}$ consists of bicliques directed from one side to the other, then Planar $\mathcal{D}$-SN corresponds to the following problem: given a planar digraph G with two sets $S, T \subseteq V(G)$ of terminals with $|S| + |T| = k$, find a subgraph with minimum number of edges

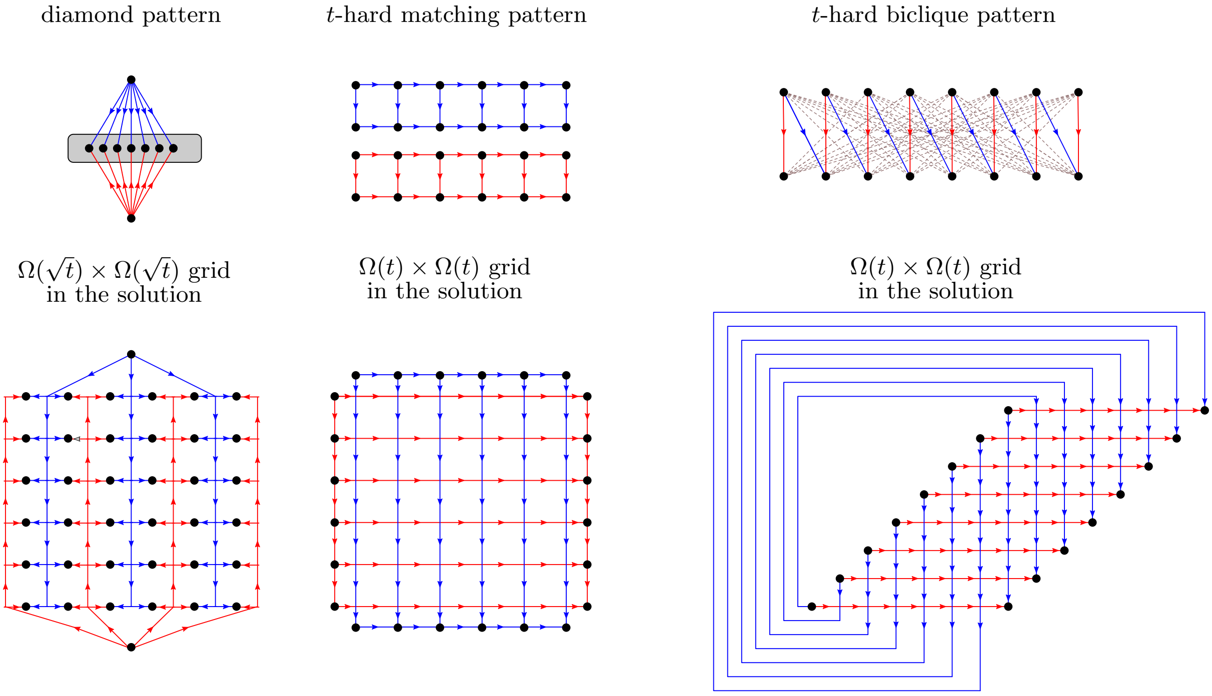

Figure 3: Pattern graphs (top row) and example minimal solution graphs with large grid patterns and large treewidth (bottom row). The red/blue edges show how (some of the) demands are connected in the solution.

such that there is a path from every vertex of S to every vertex of T. Our result shows that, assuming ETH, this problem has no $f(k) \cdot n^{o(k)}$ time algorithm. This result is surprising, as the problem can be considered to be *genuinely planar* in the sense that the input is a planar graph with k terminals and a single bit of annotation at each terminal. To our knowledge, this is the first example of a relatively natural planar problem where $f(k) \cdot n^{O(k)}$ is best possible and cannot be improved to $f(k) \cdot n^{O(\sqrt{k})}$.

1.2 Details of Statement *(iii)*: the $f(k) \cdot n^{O(\sqrt{k})}$ algorithm

In this section, we give a more detailed overview of the technical steps of the proof of *(iii)* sketched above.

From treewidth to total branch degree. Theorem 1.8 translates the question about the treewidth of the solution in general graphs to a question about the total branch degree of the solution in acyclic graphs. Suppose that we have an edge-minimal solution H in a (not necessarily acyclic) graph G with k terminals. Let us contract the strongly connected components of H in both G and H to obtain G' and H', respectively. We can observe that H' is an acyclic graph that is the optimum solution to an instance in G' with at most k terminals. Our goal is to show that if H' has total branch degree d, then H has treewidth $O(\sqrt{d+k})$. Therefore, in the later steps of the proof, we bound the total branch degree of H' by $O(k)$, giving an $O(\sqrt{k})$ bound on the treewidth of H.

We say that a vertex of a strongly connected component of H is a *portal* if it is incident to an edge connecting it to some other component. For simplicity of discussion, let us assume here that every strongly connected component of H has at least 3 edges incident to the portals, that is, every vertex of H' has at least 3 incident edges. (If a component has less than 3 such edges and has no terminal, then it consists only of a single vertex and does not affect treewidth

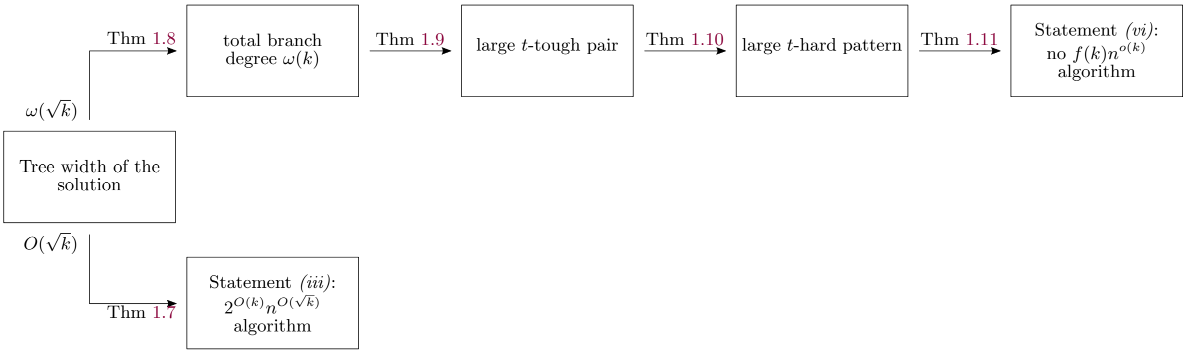

Figure 4: The structure of the proofs of statements *(iii)* and *(vi)*.

anyway; if it has terminals, then it can be taken into account with additional calculations.) By this assumption, the set P of portals have size at most $6d$, where d is the total branch degree of H'.

We want to bound the treewidth of H by showing that there is a set W of $O(d+k)$ vertices such that $H-W$ has treewidth at most 2. It is known that if removing a set W of vertices from a *planar* graph reduces treewidth to a constant, then the planar graph has treewidth $O(\sqrt{|W|})$. Thus the treewidth bound $O(\sqrt{d+w})$ follows from the existence of such a set W.

Let $H[V_i]$ be a strongly connected component of H that has p_i portals and contains k_i terminals. The key observation is that the only role of $H[V_i]$ in the solution is to fully connect the terminals and portals in $H[V_i]$. That is, we can assume that $H[V_i]$ is an optimum solution of a Strongly Connected Steiner Subgraph instance with $p_i + k_i$ terminals. Chitnis et al. [8] showed that we can remove a set W_i of $O(p_i+k_i)$ vertices from such an optimum solution to reduce its treewidth to 2. Therefore, taking the union of P and every W_i, we get a set W of size $O(d) + O(\sum(p_i+k_i)) = O(d+k)$ whose removal reduces treewidth to 2 (as removing P breaks the graph in a way that each component is a subset of some V_i, and the removal of W_i breaks $H[V_i]$ into components of treewidth at most 2).

Building a skeleton. Towards the proof of Theorem 1.9, our goal is to bound the total branch degree by $O(k)$ in an edge-minimal acyclic solution H. At some step of the proof, it will be important to assume that H is a triangulated planar graph (every face has exactly three vertices and edges), which is of course not true in general. Therefore, we introduce artificial undirected edges in the graph H to make it triangulated. As these edges do not play any role in the directed problem, it does not change the nature of the solution. Another simplification step is that we assume that there is no vertex $v \notin T$ with $d^-(v) = d^+(v) = 1$. Such a vertex has branch degree 0 and hence suppressing it (i.e., removing it and adding an edge from its in-neighbor to its out-neighbor) has no effect on the total branch degree and on the connectivity of the terminals.

We start by building a *skeleton* of the solution: a connected subgraph that contains every terminal. The skeleton is composed from *segments* of two types. A *long segment* is a directed path of H of length at least some constant L. A *short segment* is any path in the undirected sense of length at most L, possibly containing both undirected or directed edges of any orientation. Furthermore, we require that any two long segments in the skeleton are *distant,* that is, have distance at least L in the undirected sense.

A skeleton tree consisting of $O(k)$ terminals and containing all the segments can be built the following way. Initially, we start with an edgeless subgraph R containing only the k terminals. For simplicity of discussion, let us assume that the demand pattern is connected (in the undirected sense). Then there has to be a demand t_it_j such that t_i and t_j are in two different components C_i and C_j of R, respectively. This means that H has a directed path P connecting two different components of R. If P has length at most L, then we can introduce it as short segment to reduce the number of components of R. Otherwise, we can shorten P to P' such that every vertex of P' is at distance at least L from R and the two endpoints are at distance exactly L from two different components C and C' of R. Then we can reduce the number of components of R by introducing P' as a long segment and two short segments connecting the endpoints of P' to C and C'. By repeating these steps, we can reduce the number of components to 1 by introducing $O(k)$ segments in total.

Refining the faces. Our next goal is to further refine the skeleton such that every face of the skeleton has at most 35 segments on its boundary, and it is still true that the skeleton consists of $O(k)$ segments. We achieve this goal by iteratively dividing a face into two by introducing to the skeleton a new path consisting of at most 5 segments. We argue below that if the division is not very skewed in a certain sense, then the bound $O(k)$ on the number of segments can be achieved even after iterative applications of this step.

Suppose that we have a face F where $x \geq 36$ segments appear on the boundary. Let P be a path between two segments of the boundary and assume that P consists of at most 5 segments. Introducing the path P into the skeleton creates two new faces F_1 and F_2 that see some number x_1 and x_2 segments on the boundary of F, plus the 5 new segments of P. We have $x_1 + x_2 \leq x + 2$: if the endpoints of P are internal vertices of segments, then we may have up to 2 segments that are now on the boundary of both F_1 and F_2.

For a face seeing $x \geq 13$ segments of the skeleton, let us define $x - 13 \geq 0$ to be the potential of the face. If we chose the path P such that $x_1, x_2 \geq 13$, then the potential of the two new faces F_1 and F_2 are defined. Moreover, the total potential of the two faces is at most

$$(x_1 + 5 - 13) + (x_2 + 5 - 13) \leq x - 14,$$

strictly less than the potential of F. This means that if we start with a face F that sees x segments of the skeleton, then repeated applications of this step can introduce only $O(x)$ new segments.

Finding a division that is not skewed. Next we show that if face F sees $x \geq 36$ segments of the skeleton, then we can find a division with $x_1, x_2 \geq 13$. Then as we have seen above, repeated applications of this step introduces $O(x)$ segments and divide F into faces that see at most 35 segments each.

Let us divide the boundary of F into three parts, red, green, and blue, each containing at least 12 segments (see Figure 5). As every vertex v inside the face F is essential for the solution, there is a directed path P_v from v to some vertex of the boundary; let us fix such a P_v for each v. This defines a color of v according to which of the three parts of the boundary contains the head of P_v. Then by Sperner's Lemma and fact that the graph is triangulated, there is a triangle u_r, u_g, u_b inside F where the three vertices have three different colors. From the assumptions that u_r, u_g, u_b are on three different parts, and each part has length at least 13, it follows that there are two vertices, say u_r and u_b, such that both subpaths of the boundary between the heads of P_{u_r} and P_{u_b} have at least 12 segments. Then putting together P_{u_r} and P_{u_b} creates a path P that divides the face F in the required way. This argument needs to be refined a bit further: as we said earlier, we want a skeleton where the long segments are distant, i.e., are at distance at least L from each other. But this can be easily achieved by appropriately shortening the long segments P_{u_r}, P_{u_b}, and then extending them by three short segments.

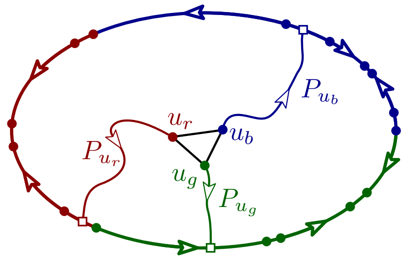

Figure 5: Finding a division that is not skewed.

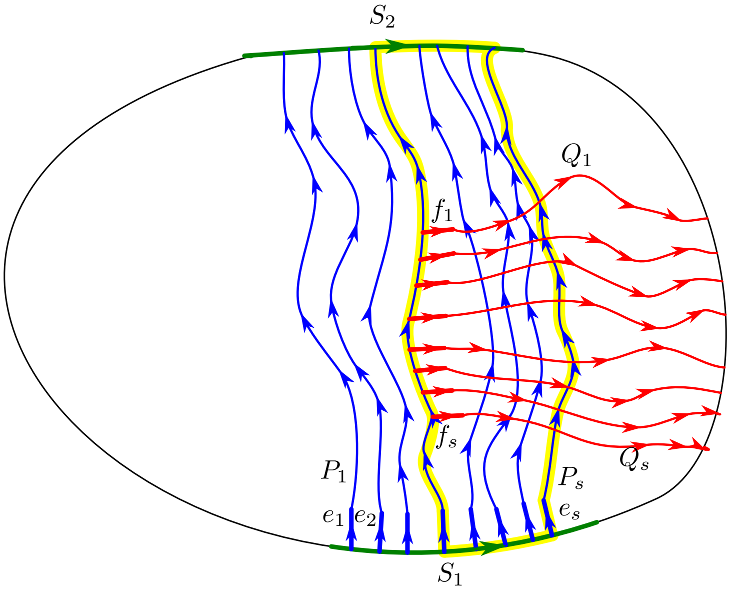

Figure 6: Finding a grid.

Many edges incident to a long path. We assume now that the skeleton has $O(k)$ faces, each seeing at most 35 segments. If we can show that the total branch degree (of the orginal solution H without the artificial edges) is a constant in each face, then we can bound by $O(k)$ the total branch degree of the solution. We can observe, using the acyclicity of the edges inside the face, that we need to bound only the number of edges incident on the boundary.

Let e be an edge inside the face incident to vertex v of the boundary. We say that e is *essential* for demand t_it_j if removing e breaks every path from t_i to t_j. Then we can define a path P_e the following way: let us take any path P from t_i to t_j, and let P_e be the subpath of P starting from e (which has to appear on P) to the first vertex on the boundary of F. Let us consider two edges e_1, e_2 starting from the same vertex v of the boundary. Let us observe that P_{e_1} and P_{e_2} cannot intersect: then we could bypass e.g. e_1 by starting on P_{e_2} and following it until intersection. By a similar argument, P_{e_1} and P_{e_2} cannot go to the same long segment: then one of P_{e_1} and P_{e_2} could be avoided by using the other path and part of the long segment. From these observations, it follows that the only way the boundary can have many edges incident to it is that if there are edges $e_1, \ldots, e_s$ incident to distinct vertices of a long segment S_1, with paths $P_1, \ldots, P_s$ going to distinct vertices of some other long segment S_2 (see Figure 6).

Finding a grid and a t-tough pair. Now comes the point where we use the assumption that long segments are distant. In particular, this means that the "middle path" $P_{e_{s/2}}$ is long. The internal vertices of this path have no terminals (as all the terminals are on the skeleton), hence it is not possible that $d^+(v) = d^-(v) = 1$ for any such internal vertex. Thus either there are many vertices on this path that have an edge leaving the path, or many vertices that have an edge entering the path. Assume without loss of generality the former, let $f_1, \ldots, f_s$ be these edges. Again, each edge is essential for some demand, hence the path satisfiying the demand has a subpath Q_i starting with f_i and going to the boundary. We can observe again that these paths have to be disjoint. Therefore, we can obtain a grid-like structure in the region surrounded by S_1, $P_{s/2}$, S_2, and P_s, see the region highlighted by yellow in Figure 6. (There are some other cases to consider, which we ignore here. For example, the paths Q_i may go to S_1 or S_2.) This region has $s/2 - 1$ "vertical" paths $P_{s/2}, \ldots, P_{s-1}$, intersected by the s "horizontal paths" $Q_1, \ldots, Q_s$.

We observe that if this grid has t horizontal and vertical paths, then we can use it to discover a t-tough pair. Each edge e_i is essential for some minimal demand; let E_1 be the set of these t demands. Similarly, we define E_2 based on chosing a minimal demand for which f_i is essential. Then we can carefully verify that (E_1, E_2) is a t-tough-pair: if there is an edge in the demand graph that is not allowed, then a careful analysis shows that there is a way of bypassing some e_i or f_i in the grid, contradicting the fact that it is essential. This concludes the proof that if we have an upper bound on the size of the largest t-tough pair appearing in the graphs of class $\mathcal{D}$, then we can bound the treewidth of the solution by $O(\sqrt{k})$.

Cleaning. To prove Theorem 1.10, we need to show that if arbitrary large t-tough-pairs appear in the graphs of $\mathcal{D}$, then $\mathcal{C}_i \subseteq \mathcal{D}$ for some $i \in [\kappa]$. The proof is a long combinatorial argument to show that we can find t-tough-pairs that are canonical in some sense, and then we use the assumption that $\mathcal{D}$ is closed under identifying vertices to contract the vertices outside the t-tough-pair into a small constant number of well-behaved vertices.

Suppose that there is a t-tough-pair(E_1, E_2) in a digraph D. The minimality of the edges in E_1 and the fact that they do not appear in directed cycles (as they are weakly independent to themselves) imply that for any two edges $x_iy_i, x_jy_j \in E_1$, at least one of the following holds:

1. exactly the edges x_iy_j, x_jy_i appear between $\{x_i, y_i\}$ to $\{x_j, y_j\}$,
2. there is no edge from $\{x_i, y_i\}$ to $\{x_j, y_j\}$, or
3. there is no edge from $\{x_j, y_j\}$ to $\{x_i, y_i\}$.

Let us consider a complete graph on t vertices $w_1, \ldots, w_t$, and for every $i < j$, color the edge w_iw_j according to which of the three statements hold for the edges x_iy_i and x_jy_j (if more than one statement is true, we can choose arbitrarily). By Ramsey's Theorem, there is a large subset $E_1' \subseteq E_1$ where the same statement holds for any pair of edges. We can find a similar subset $E_2' \subseteq E_2$. We consider two main cases. The first case is when Statement 1 holds either in E_1' or E_2'. Then what we have is a matching x_iy_i of minimal edges that is part of a complete bipartite graph, that is, every x_i is adjacent to every y_j (but note that x_iy_j does not have to be a minimal edge). The second case is where we have Statement 2 or 3 in both E_1' and E_2'. Then we can reorder E_1 and E_2 to have a further ordering property: there is no edge from $\{x_i, y_i\}$ to $\{x_j, y_j\}$ for $j < i$. We handle the two cases separately. With further Ramsey arguments and case distinctions, we show that identifications can be used to find a t'-hard biclique pattern or a t'-hard matching pattern appearing in a graph in $\mathcal{D}$, where t' is some unbounded function of t. It follows that if arbitrarily large t-tough pairs appear in $\mathcal{D}$, then $\mathcal{D}$ is a superclass of some $\mathcal{C}_i$.

2 Formal definition of a t-tough-pair

In this section we give the formal definition of a t-tough-pair. Further definitions, that are specific to the sections. are defined in the beginning of the respective sections.

Given a digraph D and an edge $e = (u, v) \in E(D)$, we say that e is a *minimal* edge of D if D has no (u, v)-path of length strictly greater than 1 in D, where the length of the path is the number of edges in it. We say that a digraph D is *reachability-minimal* if each edge of D is minimal. For an edge $e = (u, v)$ in a directed graph D, v is called the *head* of e and u is called the *tail* of e. For any $E' \subseteq E(D)$, $\text{head}(E')$ (resp. $\text{tail}(E')$) denotes the set of heads (resp. tails) of the edges in E'. Next we define weak independence and strong independence that are crucially to define the t-tough-pair formally.

Definition 2.1 (Weakly independent edges)**.** Given a digraph D and $e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E(D)$, we say that the pair of edges (e_1, e_2) is weakly independent in D, if $u_1 \neq v_1 \neq u_2 \neq v_2$, and D has neither a (v_1, u_2)-path nor a (v_2, u_1)-path. A set of edges $E' \subseteq E(D)$ are weakly independently if every pair of distinct edges in E' are pairwise weakly independent and for each edge $(u_i, v_i) \in E'$, there is no (v_i, u_i)-path in D.

Informally, a pair of edges is weakly independent, if the head of one cannot reach the tail of the other. Therefore, if a pair of edges are weakly independent, then they cannot lie on a directed path.

Definition 2.2 (Strongly independent edges)**.** Given a digraph D and $e_1 = (u_1, v_1), e_2 = (u_2, v_2) \in E(D)$, we say that the pair of edges (e_1, e_2) is strongly independent in D, if they are weakly independent in D, and additionally D has no (u_1, u_2)-path, no (u_2, u_1)-path, no (v_1, v_2)-path and no (v_2, v_1)-path.

Informally, a pair of edges is strongly independent, if they are weakly independent, and the head of one cannot reach the head of the other, and the tail of one cannot reach the tail of the other. That is, the vertices of the heads (resp. vertices of tails) do not lie on any directed path.

Definition 2.3 (t-tough-pair)**.** Given a digraph D, $E_1, E_2 \subseteq E(D)$, we say that (E_1, E_2) is a tough-pair in D if:

1. $|E_1| = |E_2|$,
2. each edge of $E_1 \cup E_2$ is a minimal edge in D,
3. all edges in E_i are pairwise weakly independent in D, for both $i \in \{1, 2\}$, and
4. for each $e_1 \in E_1$ and $e_2 \in E_2$, (e_1, e_2) are strongly independent in D.

Further, for a positive integer t, we say that (E_1, E_2) is a t-tough-pair if $|E_1| = |E_2| = t$.

3 The structure theorem

The goal of this section is to prove Theorem 1.9 (see Section 1 for the statement). We start by giving the algorithm for the c-bounded case, that is the proof of Theorem 1.8, and refocusing our efforts on acyclic instances.

3.1 A subexponential algorithm

In order to prove Theorem 1.8, we will utilize the following result.

Theorem 3.1 (Lemma 2.1 of [8])**.** *Suppose that H is an edge-minimal solution for the* Strongly Connected Steiner Network *problem on the instance (G, T), where $|T| = k$. Then there is a set $W \subseteq V(H)$ of $9k$ vertices such that deleting W from H results in a graph of treewidth at most 2.*

We start by generalizing the above theorem to our setting.

Lemma 3.2. *Let H be an edge-minimal solution to* Planar $\mathcal{D}$-Steiner Network *where $\mathcal{D}$ is a c-bounded class. Then there is a set of $(30c + 29)k$ vertices in $V(H)$ whose deletion from H results in a graph H' of treewidth at most 2.*

Proof. Let $V_1, \dots, V_t$ be the vertex sets of the strongly connected components of H that are not singletons. The degree of V_i, denoted by $d(V_i)$, is the number of edges in H with one endpoint in V_i and one endpoint outside V_i.

We claim that if V_i has no terminals, then $d(V_i) \geq 3$. Suppose the contrary: $V_i \cap T = \emptyset$ and $d(V_i) \leq 2$. Notice that $d(V_i) \neq 1$ as in such a case we could remove all edges of H induced by V_i as well as the edge that enters/exits V_i. Suppose now that $d(V_i) = 2$. Notice that the two edges that have one endpoint in V_i cannot both be entering or both be exiting V_i as again that would be redundant in H. Thus V_i has one edge entering at some vertex $u \in V_i$ and one edge exiting from some vertex $v \in V_i$. Let π be a path in $H[V_i]$ from u to v. Notice that since $|V_i| \geq 2$ and $H[V_i]$ is strongly connected, there must be some edge e in $H[V_i]$ that is not on the path π. However all terminal-to-terminal paths passing through V_i can be realized using π, since if such a path enters V_i then it enters at u and exits at v. Consequently, the edge e is redundant, contradicting the minimality of H.

We say that a vertex v is a *portal* of V_i if it is an endpoint of some edge $e \in E(H)$ that enters or exits V_i. Let P denote the set of all portals, and let P_i be the set of portals in V_i. Consider now a strongly connected subgraph $H[V_i]$ that contains at least one terminal, and let $T_i = V_i \cap T$. Since $H[V_i]$ is strongly connected, it is an edge-minimal solution of Planar Strongly Connected Steiner Network for the terminal set $T_i \cup P_i$: indeed, if some edge e can be removed from $H[V_i]$ to maintain strong connectivity on $T_i \cup P_i$, then $H - e$ would also be feasible for the original problem, which contradicts the edge-minimality of H. We can therefore apply Theorem 3.1 for the graph $H[V_i]$ and terminal set $T_i \cup P_i$: there is a set X_i of at most $9|T_i \cup P_i|$ vertices such that $H[V_i \setminus X_i]$ has treewidth at most 2.

We now delete the following vertices from H:

- the portal set P
- all the sets X_i
- the set Y of vertices outside $\bigcup_i V_i$ that have degree at least 3.

Let H^* be the resulting graph. Each connected component of H^* is either a subgraph of some strongly connected component $H[V_i]$, or it lies outside $\bigcup_i V_i$, thus its vertices have maximum degree at most 2: in both cases, it has treewidth at most 2. Consequently, $\text{tw}(H^*) \leq 2$.

In order to bound the number of deleted vertices, let us first introduce an acyclic instance of Planar $\mathcal{D}$-Steiner Network based on (G, T, D) and H: this is required so that we can use the c-boundedness of $\mathcal{D}$. Let $G/(\bigcup V_i)$ be the graph obtained by contracting each vertex set V_i into a single vertex (deleting all loops and parallel edges). Notice that the picture H' of H under this contraction is acyclic. In the demand graph D, we identify each vertex set T_i individually, resulting in a demand graph D' and terminal set T'. Notice that since $\mathcal{D}$ is closed under identification, we have $D' \in \mathcal{D}$.

We claim that H' is an edge-minimal solution to the Planar $\mathcal{D}$-Steiner Network instance $(G' = H', T', D')$ (where H' and D' are acyclic). First we show that H' is feasible. If D' has an edge $u'v'$, then the edge has an ancestor uv in D, so H contains some path P_{uv} connecting the terminals u and v (where u and v cannot be located in the same set V_i). When we apply the contraction on P_{uv}, we get a path P' connecting u' and v' in H', concluding the proof of feasibility for H'.

To prove the edge-minimality of H', suppose the contrary: that there is some edge $a'b'$ in H' whose deletion does not break feasibility. Let ab be an edge of H that contracts to $a'b'$. Notice that ab is not induced by any component V_i. We claim that $H - ab$ is feasible for (G, T, D). Consider a demand edge $uv \in E(D)$. If the terminals u and v are in the same V_i, then they are connected within $H[V_i]$ and thus they are also connected in $H - ab$. Otherwise these terminals are identified with some distinct terminals u' and v', respectively, and D' has a demand $u'v'$. Thus $H' - a'b'$ connects u' and v' via some path P'. Let u and v be arbitrary vertices of G that contract to (or are equal to) u' and v', respectively.

Using P', we will now build a path P connecting u and v in $H - ab$. First, we pick for each edge $x'y' \in P'$ an arbitrary edge whose image after contraction is $x'y'$. Consider now an edge pair $x'y', y'z'$ on P': if y' comes from the contraction of V_i, then we may have to select xy_1 as the ancestor of $x'y'$ and y_2z as the ancestor of $y'z'$, where $y_1 \neq y_2$ are different portals. But since $(H - ab)[V_i]$ is strongly connected, there is a path P_i connecting the portals y_1 to y_2 in $(H - ab)[V_i]$. We concatenate this path with the edges xy_1 and y_2z. Using the same technique, we can build a path from some u_2 to some v_2 where either $u_2 = u$ or u_2 and u are in the same component V_j (and the analogous statement holds for v and v_2). We can again use the strong connectivity of the components V_i to get a path P from u to v. Consequently, $H - ab$ is feasible for the original problem, which contradicts the edge-minimality of H.

Finally, we note that for any pair V_i, V_j with $i \neq j$ there can be at most one edge going between V_i and V_j. Since these are strongly connected components, there cannot be two edges in different directions, as that would make them into a single strongly connected component. Suppose that uv and $u'v'$ both go from V_i to V_j. Then either of these edges can be removed without affecting feasibility, which contradicts the edge-minimality of H. This property implies that in the proposed contraction the degree of the vertex that we get from contracting V_i is equal to $d(V_i)$.

We can now bound the size of the deleted set. Since $\mathcal{D}$ is c-bounded, we have that $\sum_{v \in V(H')} d^*(v) \leq ck$. Since each non-0 term in the sum comes from some contraction of V_i or a deleted vertex from Y, we obtain the following.

$$\sum_{v \in V(H')} d^*(v) = \sum_i d^*(V_i) + \sum_{v \in Y} d^*(v) \leq ck, \tag{1}$$

where $d^*(V_i) = \max(0, d(V_i) - 2)$. Recall that if V_i has degree at most 2, then it must have at least one terminal. It follows that it can have at most 2 portals, and thus $|P_i| \leq 2|T_i|$. If it has degree at least three, then it has at most $d(V_i) = d^*(V_i) + 2 \leq 3d^*(V_i)$ portals.

Thus the total number of deleted vertices can be bounded as:

$$\begin{aligned}
|Y| + |P| + \left|\bigcup_i X_i \setminus P\right| &\leq |Y| + |P| + \sum_i 9|T_i \cup P_i| \\
&\leq |Y| + 9|T| + 10|P| \\
&= |Y| + 9k + 10 \cdot \left(\sum_{i\,:\,d(V_i) \geq 3} |P_i| + \sum_{i\,:\,d(V_i) \leq 2} |P_i| \right) \\
&\leq |Y| + 9k + 10 \cdot \left(\sum_i 3d^*(V_i) + \sum_{i\,:\,d(V_i) \leq 2} 2|T_i| \right) \\
&\leq (30c + 29)k,
\end{aligned}$$

where the second inequality uses that the sets V_i are disjoint, and the last inequality uses the bound (1). Consequently, we have removed at most $(30c + 29)k$ vertices from $\bigcup_i V_i$, which concludes the proof. □

The next lemma can essentially be found within the proof of Lemma 2.2 in [8]. We reproduce the proof for completeness.

Lemma 3.3. *If G is a planar graph where deleting $k \geq 1$ vertices results in a graph of treewidth $w \geq 1$, then* $\mathrm{tw}(G) \leq 3w\sqrt{k}$.

Proof. By the planar grid theorem [31], there is a constant $\bar{c}$ such that any planar graph of treewidth $\bar{c}\omega$ has a grid minor of size $\omega \times \omega$. If G has treewidth at least $\bar{c} \cdot \lceil 3w\sqrt{k} \rceil$, then it has a grid minor of size at least $\lceil 3w\sqrt{k} \rceil$. This minor can be decomposed into $\left\lfloor \frac{\lceil 3w\sqrt{k} \rceil}{w+1} \right\rfloor \cdot \left\lfloor \frac{\lceil 3w\sqrt{k} \rceil}{w+1} \right\rfloor$ vertex disjoint grid minors, each of size at least $(w+1) \times (w+1)$. Notice that $\left\lfloor \frac{\lceil 3w\sqrt{k} \rceil}{w+1} \right\rfloor \cdot \left\lfloor \frac{\lceil 3w\sqrt{k} \rceil}{w+1} \right\rfloor \geq k+1$ for $k, w \geq 1$ integers, thus there is at least one $(w+1) \times (w+1)$ grid minor where no vertex has been deleted. This intact grid minor has treewidth at least $w+1$, which contradicts our assumptions. □

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. Consider an input (G, T, D). By Lemma 3.2, we can find a set of $(30c+29)k$ vertices in H whose deletion results in a graph of treewidth at most 2. Lemma 3.3 then bounds the treewidth of H as $\operatorname{tw}(H) \leq 3 \cdot 2 \cdot \sqrt{(30c+29)k} < 47\sqrt{ck} = O(\sqrt{k})$. This concludes the proof. □

3.2 A tree of segments

We begin the proof of Theorem 1.9 by supposing that $\mathcal{D}$ is a class that is closed under identification, but it is not c-bounded. The goal is now to show that $\mathcal{D}$ has a pattern with a t-tough pair for all positive integers t. Consider an instance where G and D are acyclic.

First, we show that it sufficient to consider a weakly connected acyclic solution H where all vertices have undirected degree at least 3. To show the minimum degree bound, we contract all edges uv of H (and G) where u or v has undirected degree at most 2. If at least one of u and v is a non-terminal, then such contractions do not influence the feasibility and edge minimality of H, and it also does not change the total branch degree of H. If both u and v are terminals, then let H' denote the new graph after the contraction, and let us identify u and v in D, resulting in the graph D'. Since $|D'| = k-1$ and H and H' have the same total branch-degree, we have that if H has total branch degree at least ck, then also H' has total branch degree at least $c \cdot (k-1)$. It is therefore sufficient to consider graphs H where all vertices of the undirected graph $\bar{H}$ have degree at least 3. (Throughout this section, the notation $\bar{X}$ refers to the undirected graph given by the edges of the graph X, where X may be a directed graph or mixed graph.)

Suppose now that H is disconnected, and let H_i be the connected components of H ($i = 1, 2, \dots$). For each H_i let D_i be the subgraph of D induced by $T \cap V(H_i)$. Notice that the graphs D_i are a partition of the edges of D, where each connected component of D is contained in a single graph D_i. Observe that each H_i is an edge-minimal solution for the instance $(G, T \cap V(H_i), D_i)$. Moreover, if H has branch-degree at least ck, than at least one among the H_i has branch degree at least $c \cdot |T \cap V(H_i)|$. Therefore we can restrict our attention to weakly connected graphs H.

Defining a tree of segments. The skeleton S is a 2-connected mixed graph that we will build based on an optimum solution H of the instance (G, T, D) of Planar $\mathcal{D}$-SN where G and D are acyclic. Suppose now that H is a solution to (G, T, D) that is a connected acyclic graph of minimum degree at least 3. We fix a plane embedding of H, and add undirected edges to $\bar{H}$ in a greedy manner to create a triangulation of the plane with vertex set $V(H)$; let H_Δ denote the resulting mixed graph, where the edges of H are directed, and the newly added triangulation edges are undirected.

A *long segment* is a *directed* path of length at least L in H_Δ. A *short segment* is a path of $\bar{H}_\Delta$ (i.e., of arbitrary orientation edges in H_Δ) that consists of at most L edges. A pair of segments A, B are *distant* if for any pair of vertices $a \in A$ and $b \in B$ the distance of a and b in $\bar{H}$ is at least L. Our goal is to create a skeleton where the boundary of the so-called *relevant* face consists of $O(k)$ (long and short) segments where the long segments are pairwise distant.

Next, we construct a tree R in H_Δ consisting of $O(k)$ segments that contains all terminals.

Lemma 3.4. *There is a tree R in H_Δ that contains all terminals and consists of $O(k)$ segments, such that any pair of long segments of R are distant.*

Proof. Initially, we set R to be the edgeless forest with vertex set T. We add segments to R using the following insertions, until it becomes connected.

Insertion 1. Let C and C' be the closest components of the current graph R, that is, the pair of components where $\delta(C, C') := \min_{u \in V(C), v \in V(C')} \mathrm{dist}_{\bar{H}_\Delta}(u, v)$ is minimized. If $\delta(C, C') \leq 2L$, then let P be a shortest path connecting C and C' in H_Δ. Since $|P| \leq 2L$, we can decompose P into at most two segments, which we add to R, connecting C and C'.

Insertion 2. Suppose that R is not connected, but Insertion 1 can no longer be applied. A vertex v is a *collaborator* of a component C of R if there is an edge e incident to v that is essential for some demand tt' where $t \in V(C)$ or $t' \in V(C)$. Let v be a vertex that collaborates with at least two distinct components. Let P_v and P'_v be directed paths connecting v to these two components. We shortcut the loops of $P_v \cup P'_v$, to find a path P whose internal vertices are outside N_R that connects two distinct neighborhoods N_C and $N_{C'}$. Note that P is the concatenation of at most two directed paths, thus it can be decomposed into at most two short or long segments. If we have two long segments, we also need to separate these with a short segment: let u be the first vertex on the first long segment that is within $\bar{H}$-distance L to the other long segment. We connect u to the last vertex v of the other long segment to which it has distance L using a short segment, and remove the parts of the long segments that fall between u and v. Let P be the final path, which now consists of at most 3 segments, and if it has two long segments, then those are distant. Connecting the endpoints of P to C and C' with short segments, we are able to add at most five segments to R that connect C and C'.

We claim that applying the above insertions exhaustively leads to a tree that contains all terminals. Observe that after each successful insertion the number of connected components of R decreases by at least one (and no cycles can be created), thus after at most $k-1$ insertions we get a connected tree R. In order to show this, we need to show that as long as R is disconnected, there is an insertion that we can apply.

Claim 3.5. *If R is disconnected, then at least one of the insertions can be applied.*

Proof. Suppose for the sake of contradiction that Insertion 1 and 2 cannot be applied, but R is still disconnected. It follows that each vertex collaborates with at most one component. If some edge e is essential for demand $t_i t_j$ and the terminals t_i, t_j lie in different connected components of R, then Insertion 2 can be applied, so suppose that no such edge exists, i.e., for each edge the corresponding demand terminals fall into the same component. Since all edges e of H are essential and H has no isolated vertices, we have that all vertices of H collaborate with at least one component of R.

Let P_0 be a path in $\bar{H}$ connecting the distinct components C and C' of R whose internal vertices are disjoint from $V(R)$: such a path exists because R is a disconnected subgraph of the connected graph H. The starting vertex of P_0 collaborates with C, and its ending vertex collaborates with C', so there must be an edge e along this path whose endpoints collaborate with distinct components. But this contradicts the definition of collaboration, which implies that adjacent vertices collaborate with all components for which the connecting edge is essential. ⌟

Notice that each insertion adds at most 5 new segments to R, and we can do at most $k-1$ insertions to reach the final tree R, thus the final tree R consists of at most $5k-5$ segments.

Moreover, long segments can only be added with Insertion 2 and 3, and each of them adds a long segment outside the current N_R, thus the newly added long segments are distant from all earlier long segments. Thus insertions preserve the property that long segments are pairwise distant. This concludes the proof. □

3.3 Region slicing

We will now slice the plane into smaller regions using segments. We say that a closed region (some connected union of faces of H_Δ) in the plane is *relevant* if it is interior-disjoint from the plane tree R. We work towards the following lemma.

Lemma 3.6 (Skeleton Lemma). *There is a subgraph* Skel $\subset H_\Delta$ *consisting of $O(k)$ segments where the faces of* Skel *are relevant and they partition the plane, each face of* Skel *has at most* 35 *segments on its boundary, and within the subgraph of $\bar{H}$ induced by each face of* Skel *any pair of long segments on the face boundary are either subpaths of the same directed path of H, or they are distant.*

The proof of the lemma requires a good separation, which we will prove next. The separator that we prove relies on Sperner's lemma, which can be phrased as follows.

Theorem 3.7 (Sperner's lemma). *Let G be an undirected simple plane graph where the vertices are assigned one of three colors (red, green, or blue). Suppose that other than the outer face, the rest of the graph is triangulated. Moreover, suppose that the boundary of the outer face has three marked vertices, v_1, v_2, v_3 colored with $1, 2, 3$, and that each vertex on the boundary is assigned one of the colors of the two marked vertices that enclose it on the boundary. Then G has a triangle whose vertices have three different colors.*

If $\mathcal{F}$ is a relevant region bounded by some cycle of $\bar{H}_\Delta$, then let $H^{\mathcal{F}}$ and $H^{\mathcal{F}}_\Delta$ denote the subgraph of H and H_Δ consisting of the edges in $\mathcal{F}$ (including edges on the boundary of $\mathcal{F}$), respectively.

Lemma 3.8 (Separator). *Let $\mathcal{F}$ be relevant region of H that is bounded by a cycle F of $\bar{H}_\Delta$ which consists of $s \geq 36$ segments, where long segments are distant in $\bar{H}^{\mathcal{F}}$. Then there is a path P in $\mathcal{F}$ that splits $\mathcal{F}$ into two regions, $\mathcal{F}_1$ and $\mathcal{F}_2$ with boundary cycles F_1 and F_2 consisting of s_1 and s_2 segments, so that*

- *long segments in $F \cup P = F_1 \cup F_2$ are distant,*
- *$s_1 + s_2 \leq s + 12$,*
- *and $13 \leq s_1, s_2 \leq s - 4$.*

Proof. We will assign colors to the vertices of H_Δ that fall in $\mathcal{F}$. First, we split the boundary cycle $\bar{F}$ into three paths P_1, P_2, P_3 of almost equal number of segments, i.e., so that the number of segments on each of these paths have a difference of at most 1. We color the endpoints of the paths with $1, 2$ and 3 so that P_i goes from the point of color i to the point of color $i+1$, where indices are defined modulo 3. Next, all internal vertices of P_i will be colored by i for each $i = 1, 2, 3$, see Figure 7.

To color the vertices in the interior of $\mathcal{F}$, let v be an arbitrary vertex there. Since v is not in R, it cannot be a terminal, so it has an essential edge incident to it, which is on some terminal-to terminal path. Let P_v be this directed path connecting two terminals that contains v. Since there are no terminals in the interior of $\mathcal{F}$, going forward on the path we will eventually exit $\mathcal{F}$. Before this point, there will be a first vertex w of P after v that is of distance at most L to some point $b \in V(F)$, that is, either $\mathrm{dist}\bar{H}^{\mathcal{F}}_\Delta(v, b) \leq L$ and $v = w$, or $\mathrm{dist}_{\bar{H}^{\mathcal{F}}}(w, b) = L$ and $\max_{x \in P[v,w)} \mathrm{dist}_{\bar{H}^{\mathcal{F}}_\Delta}(x, V(F)) > L$. (If there are multiple vertices $b \in V(F)$ at minimum distance to w in $\bar{H}^{\mathcal{F}}_\Delta$, then we choose an arbitrary such vertex b). We assign to v the color that we have

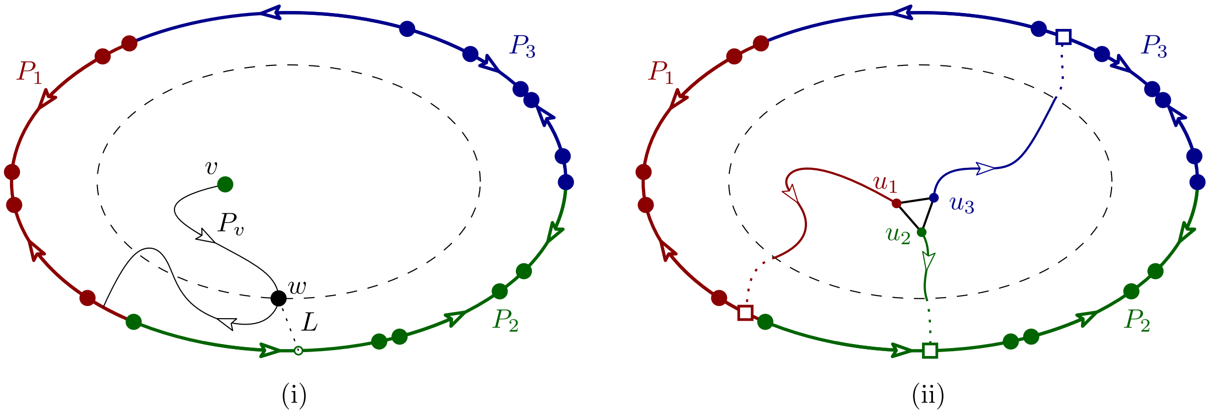

Figure 7: (i) Coloring segments on the boundary of the region $\mathcal{F}$. Long segments are denoted with an arrow. The color of vertex v is defined using a path P_v that arrives in the L-neighborhood of $\partial\mathcal{F}$. (ii) Applying Sperner's lemma in the coloring. Two of the landing vertices in $\partial\mathcal{F}$ (denoted by empty squares) will be from distant segments along $\partial\mathcal{F}$.

assigned to b. Notice that we have a path Q consisting of a directed path (a subpath of P_v) and a short segment that connects v to b, and Q is distant from all long segments of $V(F)$.

By Sperner's lemma there is a triangle $u_1u_2u_3$ in this coloring where u_i has color i. Let Q_i be the path from u_i to the boundary point $b_i \in V(P_i)$ that we used to assign color i to u_i. Since the paths P_i have at least $\lfloor s/3 \rfloor$ segments, we have that there is a pair b', b'' among b_1, b_2, b_3 such that along F there are at least $\lfloor s/3 \rfloor$ complete segments between them (and up to two partial segments). Let $u', u'' \in \{u_1, u_2, u_3\}$ and $Q', Q'' \in \{Q_1, Q_2, Q_3\}$ be the starting points and paths corresponding to b' and b''. Consider the path that is the concatenation of Q', the edge $b'b''$, and Q''. If both Q' and Q'' contain long segments attached to u' and u'', then the path can be shortcut in the middle with a short segment to ensure that they remain distant from each other: let u be the first vertex on the first long segment that is within $\bar{H}$-distance L to the other long segment. We connect u to the last vertex v of the other long segment to which it has distance L using a short segment, and remove the parts of the long segments that fall between u and v.

Let P be the resulting path connecting b' and b'', and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the regions that we get by splitting $\mathcal{F}$ with P. By construction, the resulting path consists of at most 5 segments, and long segments in $F \cup P$ are pairwise distant. Note that the vertices b' and b'' may be internal vertices of some segment of F, and in such a case the corresponding segment will be counted both in F_1 and F_2. We are also counting the at most 5 segments of P both times. Thus we have that $s_1 + s_2 \leq s + 2 + 2 \cdot 5 = s + 12$. Since P has at least one segment, we have that $\min(s_1, s_2) \geq \lfloor s/3 \rfloor + 1 \geq 13$ since $s \geq 36$. Taking into account the partial segments, the upper bound can be written as $\max(s_1, s_2) \leq s - \lfloor s/3 \rfloor + 2 + 5 \leq 2s/3 + 8 \leq s - 4$, since $s \geq 36$. ☐

We can use this separator to prove Lemma 3.6, but first let us consider a walk along the boundary of the unbounded face of the plane tree R. Notice that the walk uses each edge of R twice, once in each direction. Additionally, whenever we encounter a branching vertex (a vertex of degree at least 3) of R that is an interior vertex of the segment we have been walking along, then we slice the current segment into two smaller segments at v, resulting in two (short or long) segments. As a result, we end up with a walk that consists of the original $O(k)$ segments, plus at most the number of branching vertices of R, which is at most $O(k)$, as there are at most $O(k)$ leaves in R. Consequently, the walk has at most $O(k)$ segments.

Proof of Lemma 3.6. If the walk around R consists of at most 35 segments, then we set $\text{Skel} = R$; this has all the desired properties. Otherwise, we recursively slice the region outside R into smaller regions using Lemma 3.8, until all regions have at most 35 segments on their boundary, using the following procedure.

For a partition $\mathcal{P} = \{\mathcal{F}_1, \dots, \mathcal{F}_t\}$ of the plane where each $\mathcal{F}_i$ is a non-empty union of faces of H_Δ and where F_i has s_i boundary segments, we define the potential function

$$f(\mathcal{P}) = \sum_{i=1}^{t} s_i - 13t.$$

Notice that the contribution of region $\mathcal{F}_i$ to $f(\mathcal{P})$ is $s_i - 13$. By Lemma 3.8, for any region $\mathcal{F}_i$ appearing throughout the slicing we have $s_i \geq 13$, therefore each region contributes a non-negative integer to f. It follows that $f(\mathcal{P}) \geq 0$ for all partitions $\mathcal{P}$ that appear in the slicing.

If F is the region boundary of an element of $\mathcal{P}$ with $s \geq 36$ segments on its boundary, then by Lemma 3.8, after the slicing it will be replaced by two regions, and the total number of segments on region boundaries increase by at most 12. Therefore in the potential function f the number t increases by one, while the segment sum increases by at most 12. Thus if $\mathcal{P}'$ is the new partition we get by applying Lemma 3.8 on F, then we have

$$f(\mathcal{P}') \leq f(\mathcal{P}) + 12 - 13 = f(\mathcal{P}) - 1.$$

Thus the potential is decreasing in each step. Initially we have a singleton partition with value $O(k)$, and we know that the potential remains non-negative, thus there can be at most $O(k)$ steps. It follows that Skel consists of $O(k)$ regions, each of which has $O(1)$ segments on their boundary. The long segments of any of the faces $\mathcal{F}$ in Skel are either on the same directed path of the original solution H, or they are distant inside $\bar{H}_\Delta^{\mathcal{F}}$. Indeed, this property is true for the initial singleton partition, the distances are preserved by the slicing, and the distance is not shortened when restricting to a subgraph defined by some region $\mathcal{F}$. □

3.4 Finding a t-tough pair

Let $\mathcal{F}$ be a relevant face of the skeleton Skel given by Lemma 3.6. If F is the unbounded face of Skel, then we change the embedding of H_Δ so that F is not the outer face. Let $H^{\mathcal{F}}$ denote the subgraph of H induced by the vertices in $\mathcal{F}$, and let B denote the vertices that are on the boundary of $\mathcal{F}$. We denote by $\partial\mathcal{F}$ the edges of $H^{\mathcal{F}}$ that are on the boundary of $\mathcal{F}$, and let int$\mathcal{F}$ denote the edge set $E(H^{\mathcal{F}}) \setminus \partial\mathcal{F}$. (Note that $\mathcal{F}$ may contain undirected triangulation edges that are not contained in $\partial\mathcal{F}$ or int$\mathcal{F}$.) By Lemma 3.6 we know that $\mathcal{F}$ consists of $c_{\text{Skel}} = O(1)$ segments. Observe that if we decompose the short segments on the boundary $\partial\mathcal{F}$ into length-0 paths, then we can think of it as a collection of at most $L \cdot c_{\text{Skel}}$ directed paths (where the long segments appear as themselves, and vertices only incident to short segments as singletons).

Given an (essential) edge e in int$\mathcal{F}$ (i.e., not on the boundary $\mathcal{F}$), there exists a directed path in $H^{\mathcal{F}}$ connecting two distinct vertices of B using e to satisfy the demand for which e is essential. Note that the edges of this path are all in int$\mathcal{F}$. Such a path is called an *essential path* through e.

Lemma 3.9. *For each $v \in B$ we have that $\deg_{H^{\mathcal{F}}}(v) \leq 2L \cdot c_{\text{Skel}} + 2$.*

Proof. Let vu be an edge where $v \in B$ and $u \notin B$. Since vu is essential for some demand, there is some essential path P through vu; let v' be the other endpoint of P. Suppose that $w' \in B$ is reachable from v' along $\partial\mathcal{F}$. Then there can be no essential path from v to w' that avoids vu, as the connection is already established by P and the path along $\partial\mathcal{F}$ from v' to w'. In particular, there can be at most one edge leaving v where the corresponding essential path ends on a given directed path of $\partial\mathcal{F}$. Since $\partial\mathcal{F}$ consists of at most $L \cdot c_{\text{Skel}}$ directed paths, we have that there are at most $L \cdot c_{\text{Skel}}$ inner edges leaving v. An analogous argument for the incoming edges plus the at most two boundary edges proves the desired upper bound on the degree of v. □

We will now consider the number of demands for which the edges of $H^{\mathcal{F}}$ are essential. In what follows, let κ be the number of such demands, i.e., suppose that $H \setminus \text{int}\mathcal{F}$ fails to satisfy κ of the demands.

Lemma 3.10. *If $H \setminus \text{int}\mathcal{F}$ fails κ demands, then the total branching degree of $H^{\mathcal{F}}$ is at most $\kappa(\kappa-1)$.*

Proof. First we show that the internal vertices of $H^{\mathcal{F}}$ have bounded total branching degree. To do so, we first bound the number of intersections between paths satisfying different demands.

We claim that there can be at most one intersection point where there are incoming edges of $H^{\mathcal{F}}$ essential for a given pair of demands. Suppose the contrary: that $v, w \in V(H^{\mathcal{F}})$ are distinct vertices such that both of them have an incoming edge essential for demand edge d (denoted by $e(d,v)$ and $e(d,w)$ and an incoming edge essential for demand edge d', denoted by $e(d',v)$ and $e(d',w)$. Note that either v is reachable form w or vice versa, as otherwise we could remove one of $e(d,v)$ and maintain the connection of d via w. Assume without loss of generality that there is a path P from v to w. Note that P cannot use both $e(d,w)$ and $e(d',w)$; suppose that it does not use $e(d,w)$. Since $e(d,w)$ is essential for d, there is a directed path $Q \subset E(H^{\mathcal{F}})$ through it[1] satisfying the demand d. Note that Q must also pass through the edge $e(d,v)$, and in particular, contains vertex v.

We can therefore use an initial part of Q to get to v: note that this part of Q is disjoint from P as otherwise there would be a closed walk and thus a cycle in H. We use P to go from v to w, and continue on Q after w; as Q goes through $e(d,w)$, it can be continued from w. The resulting path Q' essentially replaces $Q[v,w]$ with P, and has the same endpoints as Q, making $e(d,w)$ non-essential for d, a contradiction.

The analogous argument for outgoing edges gives the same bound. Imagine labeling each unordered pair of edges that have the same head with an unordered pair of demands, such that one edge is essential for one demand and the other edge is essential for the other. The above argument implies that these labels must be distinct for all pairs of edges sharing the same head. The analogous can be done for edge pairs that share the same tail. Consequently, the vertices of $H^{\mathcal{F}}$ satisfy

$$\sum_{v \in V(H^{\mathcal{F}})} \binom{\rho(v)}{2} \leq \binom{\kappa}{2}$$
$$\sum_{v \in V(H^{\mathcal{F}})} \binom{\delta(v)}{2} \leq \binom{\kappa}{2},$$

where $\rho(v)$ and $\delta(v)$ denote the in- and outdegree of v in $H^{\mathcal{F}}$, respectively. On the other hand, observe that $\binom{\rho(v)}{2} + \binom{\delta(v)}{2} \geq d^*(v)$, thus we have that the total branching degree is at most $2\binom{\kappa}{2} = \kappa(\kappa-1)$. □

Lemma 3.11. *If for each relevant face $\mathcal{F}$ of* Skel *we have that $H \setminus \mathcal{F}$ fails at most κ demands, then, then the total branch degree of H is at most $O(\kappa^2) \cdot k$.*

Proof. By Lemma 3.10, we have that each relevant face $\mathcal{F}$ has total branch degree at most $\kappa(\kappa-1)$. Since each vertex of H has degree at least 3, the same holds for the vertices of $H^{\mathcal{F}}$ that are in int$\mathcal{F}$. Thus all of the inner vertices contribute at least 1 to the branch degree of $H^{\mathcal{F}}$; it follows that there are at most $\kappa(\kappa-1)$ inner vertices. It follows that the total degree of the inner vertices is at most $\kappa(\kappa-1) + 2\kappa(\kappa-1)$, since the degree and branch degree differs by at most 2. Let $B_{\mathcal{F}}$ denote the vertices of $\partial\mathcal{F}$, and let $\hat{B}_{\mathcal{F}}$ denote those vertices that are adjacent

[1]Here Q may use boundary edges of $H^{\mathcal{F}}$, i.e., it is not necessarily an essential path.

to some inner vertex of $\mathcal{F}$ in $H^{\mathcal{F}}$. Since the total degree of inner vertices is at most $3\kappa(\kappa-1)$, this also bounds the number of vertices in $\hat{B}_{\mathcal{F}}$: we have $|\hat{B}_{\mathcal{F}}| \leq 3\kappa(\kappa-1)$.

By Lemma 3.6 we know that Skel has $O(k)$ faces, thus the sum of the branch degrees of all relevant faces is $m = O(\kappa^2 k)$. We claim that the total branch degree of H is at most $O(k)$ larger than $3m$. Notice that the difference between the two amounts is due to the vertices of H that lie on the shared boundary of some relevant faces of Skel. Suppose now that v has degree 2 in Skel with neighboring relevant faces $\mathcal{F}$ and $\mathcal{F}'$. Then $d^*_H(v) \leq d^*_{H^{\mathcal{F}}}(v) + d^*_{H^{\mathcal{F}'}}(v) + 2$. Therefore, the total contribution from vertices in $\hat{B}_{\mathcal{F}}$ is at most $2\kappa(\kappa-1)$, and all other vertices have all adjacent edges in the other face, so they satisfy $d^*_{\mathcal{F}'}(v) = d^*_H(v)$.

Suppose now that v has degree $d_v \geq 3$ in Skel, with neighboring faces $\mathcal{F}_1, \ldots \mathcal{F}_{d_v}$ and corresponding branch degrees $d^*_1(v), \ldots, d^*_{d_v}(v)$. Then we have

$$d^*_H(v) \leq \sum_{i=1}^{d_v} d^*_i(v) + 2(d_v - 1).$$

Consequently, the branch degree of $\mathcal{H}$ differs by at most $2\sum_v (d_v - 1)$, where the sum goes over vertices of Skel of degree at least 3. Note that Skel is a plane graph with $O(k)$ faces, thus the sum is at most $O(k)$, concluding the proof. □

We say that a directed edge uv enters (resp. exits) a path P if $u \notin V(P)$ and $v \in P$ (resp. $u \in V(P)$ and $v \notin V(P)$).

Lemma 3.12. *Suppose that P and P' are essential paths for the edges e, e' respectively, and that the directed paths Q_s, Q, Q_t do not contain these edges. Suppose that P and P' exit the directed path Q_s before e and e' respectively, and the edges e, e' enter the same directed path Q. Alternatively, suppose that e, e' exit the same directed path Q, and after them P and P' enter the same directed path Q_t. Then P and P' are vertex-disjoint, and the paths Q_s, Q (respectively, Q, Q_t) have the same "direction", that is, their intersections with P and P' appear in the same order.*

Proof. Let $e = uv$ and $f = u'v'$, and let P and P' be essential paths through e and f respectively, and suppose that uv and $u'v'$ both enter a directed path. Let Q denote the directed path from v to v'. Let s, t and s', t' denote the start- and endpoints of P and P', respectively. We observe that the directed path Q_s must be oriented from s to s': indeed, if it is oriented the other way, then P' cannot be an essential path for $u'v'$, as we can use the path $s' \xrightarrow{Q_s} s \xrightarrow{P} v \xrightarrow{Q} v'$ instead of $P'[s', v']$, and we claim that this path avoids $u'v'$. In case of Q_s and Q do not contain $u'v'$, so one only needs to check P. But if $u'v' \in P[s, v]$, then we could create closed walk together with Q and contradict acyclicity.

Now suppose for the sake of contradiction that P and P' intersect at some vertex p. We distinguish three cases:

Case 1: p occurs after the edge uv on P.

This contradicts the essentiality of uv, as one can use $s \xrightarrow{Q_s} s' \xrightarrow{P'} p$ instead of $P[s, p]$.

Case 2: p occurs before the edge uv on P, and before the edge $u'v'$ on P'.

This contradicts the essentiality of $u'v'$, as one can use $p \xrightarrow{P} v \xrightarrow{Q} v'$ instead of $P'[p, v']$.

Case 3: p occurs before the edge uv on P, but after the edge $u'v'$ on P'.

This contradicts the acyclicity of H, as $p \xrightarrow{P} v \xrightarrow{Q} v' \xrightarrow{P'} p$ is a closed walk.

Suppose now that uv and $u'v'$ both exit a directed path. Let Q denote the directed path from u to u'. We observe that the directed path Q_t must be oriented from t to t': indeed, if it is oriented the other way, then P cannot be an essential path for uv, as we can use the path $u \xrightarrow{Q} u' \xrightarrow{P'} t' \xrightarrow{Q_t} t$ instead of $P[u,t]$. Again this path avoids uv as if $uv \in P'[u',t']$ then together with Q we would get a closed walk.

Now suppose for the sake of contradiction that P and P' intersect at some vertex p. We distinguish three cases:

Case 1: p occurs before the edge $u'v'$ on P'.
This contradicts the essentiality of $u'v'$, as one can use $p \xrightarrow{P} t \xrightarrow{Q_t} t'$ instead of $P'[p,t']$.

Case 2: p occurs after the edge $u'v'$ on P', and after the edge uv on P.
This contradicts the essentiality of uv, as one can use $u \xrightarrow{Q} u' \xrightarrow{P'} p$ instead of $P[u,p]$.

Case 3: p occurs after the edge $u'v'$ on P', but before the edge uv on P.
This contradicts the acyclicity of H, as $p \xrightarrow{P} u \xrightarrow{Q} u' \xrightarrow{P'} p$ is a closed walk.

This concludes the proof. □

A *bundle* is a collection of pairwise vertex-disjoint essential paths of $\mathcal{F}$ that exit a directed path P_s of $\partial\mathcal{F}$ and enter a directed path P_t of $\partial\mathcal{F}$. For a cycle C in $\bar{H}^{\mathcal{F}}_{\Delta}$ let H^C denote the edges of H that are inside[2] C or on C.

Definition 3.13 (Grid structure)**.** A grid structure of size λ is a cycle C in $\bar{H}^{\mathcal{F}}_{\Delta}$ and a pair of directed path sets $\mathcal{B}_P = \{P_1, \dots, P_\lambda\}$ and $\mathcal{B}_Q = \{Q_1, \dots, Q_\lambda\}$, where the paths are in H^C, and the following properties hold. See Figure 8 for an illustration.

(i) Path P_i starts at $s_i^P \in V(C)$ and ends at $t_i^P \in V(C)$. Similarly, path Q_j starts at some $s_j^Q \in V(C)$ and ends at $t_j^Q \in V(C)$. Apart from their start- and endpoints, each P_i and Q_j is vertex-disjoint from C. The paths of $\mathcal{B}_P$ are pairwise vertex-disjoint, and the paths of $\mathcal{B}_Q$ are pairwise vertex disjoint.

(ii) The cycle C contains the vertices

$$s_1^P, s_2^P, \dots, s_\lambda^P, \quad t_1^Q, t_2^Q, \dots, t_\lambda^Q, \quad t_\lambda^P, t_{\lambda-1}^P, \dots, t_1^P, \quad s_\lambda^Q, s_{\lambda-1}^Q, \dots, s_1^Q$$

in this cyclic order, or reversed.

(iii) There is a directed path $Q_s \subset C$ through $s_1^P, s_2^P, \dots, s_\lambda^P$. Similarly, the directed path $Q_t \subset C$ goes through $t_1^P, t_2^P, \dots, t_\lambda^P$, the path $P_s \subset C$ goes through $s_1^Q, s_2^Q, \dots s_\lambda^Q$, and the path $P_t \subset C$ goes through $t_1^Q, t_2^Q, \dots, t_\lambda^Q$.

(iv) Each P_i intersects each Q_j in some non-empty connected subpath.

(v) Each P_i is a subpath of an essential path for some edge $e_i \in P_i$, and each Q_j is a subpath of an essential path for some edge $e'_j \in Q_j$. Moreover, $e_i \notin Q_j$ and $e'_j \notin Q_i$ for any $i, j \in [\lambda]$.

(vi) For each $1 \le i_1 < i_2 \le \lambda$ the head of e_{i_2} is reachable from the tail of e_{i_1} within H^C using a path that avoids e_{i_1} or e_{i_2}. Similarly, the head of e'_{i_2} is reachable from the tail of e'_{i_1} within H^C using a path that avoids e'_{i_1} or e'_{i_2}.

Let us fix a vertex $x_{i,j} \in P_i \cap Q_j$ for each $i, j \in [\lambda]$. The embedding ensures that $x_{i,1}, \dots, x_{i,\lambda}$ appear in this order on P_i, and $x_{1,j}, \dots, x_{\lambda,j}$ appear in this order on Q_j. Observe that contracting

[2] That is, in the fixed embedding, these edges lie entirely in the bounded region defined by C. Recall that all relevant faces of Skel are bounded.

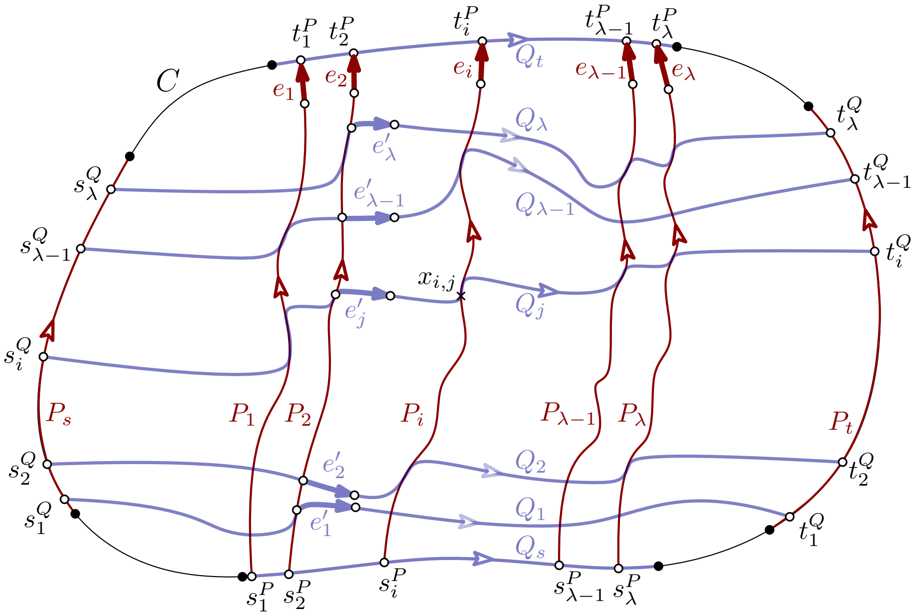

Figure 8: A grid structure with cycle C and path sets $\mathcal{B}_P, \mathcal{B}_Q$. Thick edges are the defining essential edges.

the edges of $P_i \cap Q_j$ as well as all edges that have an incident degree-2 edge results in a plane grid with grid lines P_i and Q_j.

Our task now is to find a large grid structure. Consider a bundle $\mathcal{B}$ consisting of paths $P_1, \dots, P_t$ from long segment A to long segment B where the indices are according to the order of starting points on A. Suppose that $P_{t/4}$ starts at $a_1 \in A$ and ends in $b_1 \in B$, and that $P_{3t/4}$ starts at $a_2 \in A$ and ends in $b_2 \in B$. The *spread* of $\mathcal{B}$ is defined as $\text{dist}_{\bar{A}}(a_1, a_2) + \text{dist}_{\bar{B}}(b_1, b_2)$.

Lemma 3.14. *Suppose that $\kappa > L^4$ and that L is large enough. If $H \setminus \text{int}\mathcal{F}$ fails κ demands, then $H^{\mathcal{F}}$ has a grid structure of size $\Omega(L)$.*

Proof. By Lemma 3.9 we have that each vertex on $\partial\mathcal{F}$ has degree $O(Lc_{\text{Skel}})$. Since there are at most Lc_{Skel} vertices on $\partial\mathcal{F}$ that are on some short segment, we have that at least $\kappa - O(L^2c_{\text{Skel}}^2)$ demands whose essential paths go through long segments. Consequently, there exists a long segment S that has $\frac{\kappa - O(L^2c_{\text{Skel}}^2)}{c_{\text{Skel}}}$ demands going through it. At least half of these demands are exiting S or entering S; suppose the former, i.e., there are at least $\frac{\kappa - O(L^2c_{\text{Skel}}^2)}{2c_{\text{Skel}}}$ demands exiting S (the entering case can be handled analogously). Since vertices of segment S have degree at most $O(Lc_{\text{Skel}})$, we have that among these demands exiting S, there must be at least $\frac{\kappa - O(L^2c_{\text{Skel}}^2)}{2c_{\text{Skel}} \cdot O(Lc_{\text{Skel}})} = \Omega(\kappa/L)$ demands exiting S that have pairwise distinct starting vertices.

Each of these demands have some edge exiting S, and for each of these edges let us fix a corresponding essential path. Among these essential paths, at most $O(L^2c_{\text{Skel}}^2)$ of them end on short segments, and among the rest, at least $\frac{1}{c_{\text{Skel}}}$ proportion of them end on the same long segment of $\partial\mathcal{F}$. Thus there are at least $\frac{\Omega(\kappa/L) - O(L^2c_{\text{Skel}}^2)}{c_{\text{Skel}}} = \Omega(\kappa/L)$ essential paths among them that also end on the same long segment of $\partial\mathcal{F}$. Among these paths, at least $\frac{1}{O(Lc_{\text{Skel}})}$ proportion of them have pairwise distinct ending points. Thus, there exists a bundle of size at least $\Omega(\kappa/L^2)$ where either all defining essential edges exit a long segment of $\partial\mathcal{F}$, or all defining essential edges enter a long segment of $\partial\mathcal{F}$. We fix a constant c^* such that there exists a bundle of size at least $c^*\kappa/L^2$. Let $\mathcal{B}$ be the bundle of minimum spread that has exactly $\lambda := \lfloor L/(4c_{\text{Skel}}^2) \rfloor$ and where the defining edges are all entering or all exiting a long segment. Since L is large enough, we may assume $L > 1/c^*$, so $c^*\kappa/L^2 > \kappa/L^3 > L > \lambda$, so such a bundle $\mathcal{B}$ exists.

By Lemma 3.12, we have that the paths of $\mathcal{B}$ are pairwise vertex-disjoint. Moreover, by the same lemma, we can index the paths of $\mathcal{B}$ by the order in which their starting points occur in $\partial\mathcal{F}$ as $P_1, \ldots, P_\lambda$. By the essentiality of the defining edges of $\mathcal{B}$, we know that the target of a path cannot be reachable from its source, thus the starting and ending long segments of $\mathcal{B}$ are distinct and they cannot be subpaths of the same directed path of H. Thus Lemma 3.6 implies that the starting and ending long segment of $\mathcal{B}$ are distant, therefore there are at least L vertices on the middle path P_m of $\mathcal{B}$, where we set $m = \lfloor \lambda/2 \rfloor$. Note that each vertex of P_m has degree at least 3, thus each of these vertices has an incident edge that is not on P_m. At least half of these edges are on the same side of P_m in the embedding, and at least half of them are all entering or all exiting P_m. Consider now the essential paths of the $L/4$ edges selected this way. At least $1/c_{\mathrm{Skel}}^2$ proportion of them have the same starting and ending segment, thus there at least $\lambda = \lfloor L/(4c_{\mathrm{Skel}}^2) \rfloor$ paths all entering or all exiting P_m in the same direction. Let $\mathcal{B}'$ be this size-λ collection of essential paths.

We can index the paths of $\mathcal{B}'$ by the order in which their essential edge occur on P_m as $P'_1, \ldots, P'_\lambda$. Let e'_j the essential edge of P'_j.

Let $S_\mathcal{B}, T_\mathcal{B}, S'_\mathcal{B}, T'_\mathcal{B}$ be starting and ending segments of $\mathcal{B}$ and $\mathcal{B}'$, respectively. We denote by e_i and e'_i the essential edge of P_i and P'_i. Suppose now that P_i and P'_j intersect more than once, and let x and x' be the first and last intersection along P_i. By acyclicity of H, we know that x and x' are also the first and last intersection along P'_j. Observe that the e_i and e'_j cannot occur between x and x', as that would allow us to circumnavigate an essential edge using a portion of the other path. Consequently, we can change P'_j by exchanging $P'_j[x, x']$ with $P_i[x, x']$; the result is still an essential path for the edge e'_j. By making such changes exhaustively, we can ensure that for each $i, j \in [\lambda]$ if P_i intersects P_j, then their intersection is a connected (possibly one-vertex) subpath.

We now distinguish several cases based on what segment $\mathcal{B}'$ starts and ends on. Note that $S_\mathcal{B} \neq T_\mathcal{B}$ and $S'_\mathcal{B} \neq T'_\mathcal{B}$ by essentiality of e_1 and e'_1, and acyclicity. Assume without loss of generality that $\mathcal{B}$ goes from bottom to top, with both $S_\mathcal{B}$ and $T_\mathcal{B}$ oriented left to right. See Figure 9 for an illustration.

Case 1. $S'_\mathcal{B} \notin \{S_\mathcal{B}, T_\mathcal{B}\}$ (or symmetrically, $T'_\mathcal{B} \notin \{S_\mathcal{B}, T_\mathcal{B}\}$). First we show that the paths of $\mathcal{B}'$ are pairwise disjoint. Suppose that the edges e'_j are entering P_m from the left. Then we can apply Lemma 3.12 with P_1 playing the role of Q_s and P_m playing the role of Q. If they are exiting P_m on the right, and $T'_\mathcal{B} \notin \{S_\mathcal{B}, T_\mathcal{B}\}$, then Lemma 3.12 is applied with $Q = P_m$ and $Q_t = P_\lambda$. If $T'_\mathcal{B} \in \{S_\mathcal{B}, T_\mathcal{B}\}$, then Lemma 3.12 is applied with $Q = P_m$ and $Q_t = T'_\mathcal{B}$. All remaining cases can be handled with analogous invocations of Lemma 3.12.

Suppose that the edges e'_i enter P_m from the left or exit it on the right. Because of the embedding it follows that all paths of $\mathcal{B}'$ intersect $P_1, P_2, \ldots, P_m$. We claim that if the edges e'_i exit P_m, then the paths P'_i also intersect the path $Z = S_\mathcal{B}[m, m+1] \cup P_{m+1}$, where $S_\mathcal{B}[m, m+1]$ denotes the portion of $S_\mathcal{B}$ between the starting point of P_m and P_{m+1}. Note that P'_i enters the inside of the closed curve $P_m \cup S_\mathcal{B}[m, m+1] \cup P_{m+1} \cup T_\mathcal{B}[m, m+1]$, so P'_i must intersect Z or $T_\mathcal{B}[m, m+1]$ after passing e'_i (because of the earlier simplifaction it cannot intersect P_m again). If P'_i enters $T_\mathcal{B}[m, m+1]$, then its portion containing e'_i can be curcumnavigated on a directed subpath of $P_m \cup T_\mathcal{B}[m, m+1]$, contradicting the essentaility of e'_i.

If e'_i enters P_m, then we set $Z = P_m$.

Consider the cycle C of $\bar{H}$ formed by P_1, P'_1, P'_m, Z (Since P_1, Z and P'_1, P'_m are vertex-disjoint, and the other pairs have a connected intersection, there is a unique cycle in the union $P_1 \cup P'_1 \cup P'_m \cup Z$.) One can verify that C, $\{P_2|_C, \ldots, P_{m-1}|_C\}$, and $\{P'_2|_C, \ldots, P'_{m-1}|_C\}$ form a grid structure, where $P|_C$ denotes the portion of the path P that falls in the interior of the bounded region of C.

The case when the edges $\mathcal{B}'$ enter P_m from the right or exit it to the left can be handled

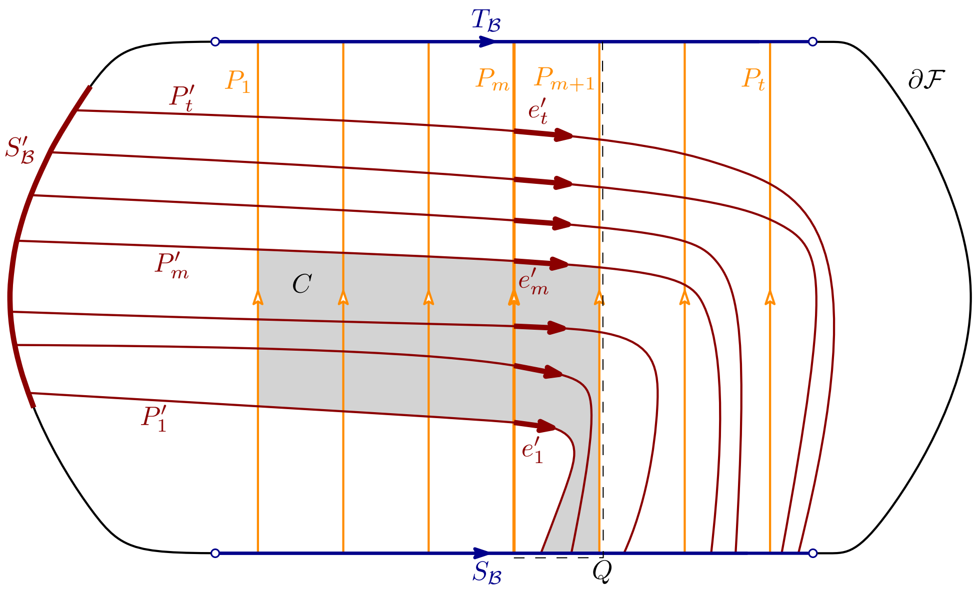

Figure 9: Finding a grid structure based on a bundle $\mathcal{B}$ (orange). The edges e'_j exit the middle path P_m on the right. The bundle $\mathcal{B}'$ (red) has a distinct starting segment $S'_\mathcal{B}$, as in Case 1. The directed path Z is depicted with dashed line, and the cycle C around the found grid is the boundary of the gray shaded region.

analogously, using the paths $P_m, \ldots, P_\lambda$ instead of $P_1, \ldots, P_m$, and setting $Z = P_{m-1} \cup T_\mathcal{B}[m-1, m]$ or $Z = P_m$ for entering/exiting edges e'_i. The cycle C is defined by $Z, P_\lambda, P'_m, P'_\lambda$, and the grid is given by C, $\{P_{m+1}|_C, \ldots, P_{\lambda-1}|_C\}$, and $\{P'_{m+1}|_C, \ldots, P'_{\lambda-1}|_C\}$. In all cases the grid has size at least $\lfloor \lambda/2 \rfloor - 2$.

Case 2. $S_\mathcal{B} = S'_\mathcal{B}$ and $T_\mathcal{B} = T'_\mathcal{B}$. We claim that this case cannot occur. If e'_i exits P_m on the right, then we can exchange the portion of P'_i starting at e'_i with a part of P_m and $T_\mathcal{B}$, contradicting the essentiality of e'_i. If e'_i enters P_m from the left, then we can exchange the portion of P'_i ending at e'_i with a part of $S_\mathcal{B}$ and P_m, contradicting the essentiality of e'_i.

If e'_i exits P_m on the left, or enters P_m from the right, then we can exchange the portion of P_m starting at e_m with a part of $S_\mathcal{B}$ and P'_m, contradicting the essentiality of e_m.

Case 3. $S_\mathcal{B} = T'_\mathcal{B}$ and $T_\mathcal{B} = S'_\mathcal{B}$. First we note that e'_i cannot enter P_m from the right or exit it on the left, as both would create a cycle. We can invoke Lemma 3.12 on any pair of paths of $\mathcal{B}'$ with $Q = P_m$ and either $Q_s = S'_\mathcal{B}$ or $Q_t = T'_\mathcal{B}$ to prove that the paths of $\mathcal{B}'$ are pairwise vertex-disjoint.

Since $\mathcal{B}$ has minimum spread, we have that $\mathcal{B}'$ has a spread at least as big. Recall that if $\mathcal{B}$ goes from $A = S_\mathcal{B}$ to $B = T_\mathcal{B}$, then $P_{\lambda/4}$ starts at $a_1 \in A$ and ends in $b_1 \in B$, while $P_{3\lambda/4}$ starts at $a_2 \in A$ and ends in $b_2 \in B$. Now $\mathcal{B}'$ goes from B to A, so we set the start and endpoint of $P'_{\lambda/4}$ as $b'_1 \in B$ and $a'_1 \in A$, and similarly, the start and end of $P'_{3\lambda/4}$ as $b'_2 \in B$ and $a'_2 \in A$. Now $\mathrm{dist}_{\bar{A}}(a_1, a_2) + \mathrm{dist}_{\bar{B}}(b_1, b_2) \leq \mathrm{dist}_{\bar{A}}(a'_1, a'_2) + \mathrm{dist}_{\bar{B}}(b'_1, b'_2)$, thus at least one of the inequalities $\mathrm{dist}_{\bar{A}}(a_1, a_2) \leq \mathrm{dist}_{\bar{A}}(b'_1, b'_2)$ and $\mathrm{dist}_{\bar{B}}(b_1, b_2) \leq \mathrm{dist}_{\bar{B}}(a'_1, a'_2)$ holds.

Suppose that the latter inequality holds. Then because of the embedding we have that $P'_1, \ldots, P'_{\lfloor \lambda/4 \rfloor}$ all intersect $P_{\lceil \lambda/4 \rceil}, \ldots, P_{\lfloor \lambda/2 \rfloor}$. We can then define a grid for these two smaller bundles of size at least $\lambda/4 - 2$ as seen in Case 1 by imagining that the path $T_\mathcal{B}$ is split into two shorter paths, the first part containing the endpoints of $P'_1, \ldots, P'_{\lfloor \lambda/4 \rfloor}$, and the second containing the starting points of $P_{\lceil \lambda/4 \rceil}, \ldots, P_{\lfloor \lambda/2 \rfloor}$. Similarly, if the former inequality

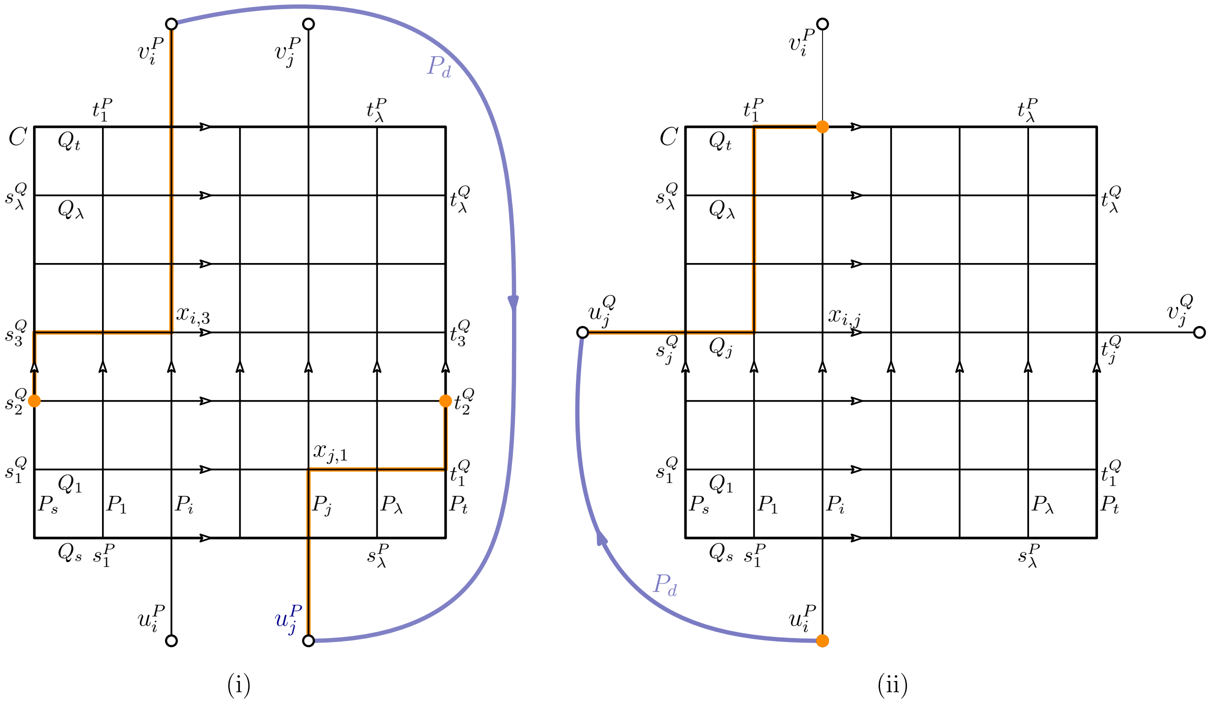

Figure 10: (i) Proving weak independence: a demand $v_i^P u_j^P$ leads to the path Q_2 being avoidable, contradicting essentiality of e'_2. (ii) Proving strong independence. A demand $u_i^P u_j^Q$ leads to the edge e_i being avoidable on P_i^{ex}, contradicting its essentiality for $u_i^P v_i^P$. In both sides the path P^* is represented as the concatenation of the path(s) with orange background and the blue path P_d.

($\text{dist}_{\bar{B}}(b_1, b_2) \le \text{dist}_{\bar{B}}(a'_1, a'_2)$) holds, then $P'_{\lceil 3\lambda/4 \rceil}, \ldots, P'_\lambda$ all intersect $P_{\lceil \lambda/2 \rceil}, \ldots, P_{\lfloor 3\lambda/4 \rfloor}$, and the grid can again be defined analogously to Case 1.

In all possible cases we have shown that a grid of size $\Omega(\lambda) = \Omega(L)$ exists which completes the proof. □

The usefulness of the grid structure is demonstrated by the following lemma.

Lemma 3.15. *If $C, \mathcal{B}$, and $\mathcal{B}'$ form a grid structure of size λ, then the demands corresponding to a subset of their essential paths form a $(\lambda - 2)$-tough pair.*

Proof. First we show that the demands corresponding to the essential paths $\mathcal{B}_P = \{P_1, \ldots, P_\lambda\}$ are pairwise weakly independent. For a path P_i let $d_i^P = u_i^P v_i^P$ be the corresponding minimal demand, which is served by some path P_i^{ex} that is an extension of P_i. (Similarly, the minimal demand of Q_j is $d_j^Q = u_j^Q v_j^Q$, and it is served by the path Q_j^{ex}.) See Figure 10 for an illustration.

Suppose now that d_i^P and d_j^P are not weakly independent, where $i < j$. There cannot be a demand $v_j^P u_i^P$, as a path satisfying this demand together with $P_i^{ex}[u_i^P, x_{i,1}] \cup Q_1[x_{i,1}, x_{j,1}] \cup P_j^{ex}[x_{j,1}, v_j^P]$ forms a closed walk, i.e., contradicts acyclicity. Thus weak independence must be violated by a demand $v_i^P u_j^P$; let P_d be the path corresponding to this demand. We claim that this contradicts the essentiality of e'_2, as $Q^* = Q_2[s_2^Q, t_2^Q]$ can be exchanged with the path

$$P^* = s_2^Q \xrightarrow{P_s} s_3^Q \xrightarrow{Q_3} x_{i,3} \xrightarrow{P_i^{ex}} v_i^P \xrightarrow{P_d} u_j^P \xrightarrow{P_j^{ex}} x_{j,1} \xrightarrow{Q_1} t_1^Q \xrightarrow{P_t} t_2^Q.$$

Unless P^* also contains e'_2, this exchange contradicts the essentiality of e'_2. Suppose now that P^* does contain e'_2; by the properties of the grid we know that e'_2 cannot lie on any of

$P_s, Q_3, P_i^{ex}, P_j^{ex}, Q_1, P_t$, thus it could only be contained in P_d. Notice that if e_2' is not essential for the demand of P_d, then we can change P_d to exclude e_2' and get a contradiction as above. Thus in what follows, we assume that e_2' is an essential edge for P_d.

Notice that the same argument can be repeated for e_3', thus P_d must contain e_3' as an essential edge. By Property (vi) of grid structures, the head of e_3' is reachable from the tail of e_2'. It follows that e_2', e_3' must appear in this order on P_d, as otherwise we could create a closed walk $(\text{tail}(e_2') \to \text{head}(e_3') \xrightarrow{P_d} \text{tail}(e_2'))$, contradicting acyclicity. Consequently, P_d has a subpath

$$\text{tail}(e_2') \xrightarrow{e_2'} \text{head}(e_2') \xrightarrow{P_d} \text{tail}(e_3') \xrightarrow{e_2'} \text{head}(e_3'),$$

which we could replace with the guaranteed path $\text{tail}(e_2') \to \text{head}(e_3')$ from the grid structure that avoids e_2' or e_3', contradicting the essentiality of either e_2' or e_3' for P_d. The weak independence of the paths Q_j can be proven symmetrically (by switching the role of P and Q).

Next we show strong independence of d_i^P and d_j^Q for all $i, j \in \{2, 3, \ldots, \lambda - 1\}$. Notice that this is sufficient, as it shows that the demands $d_2^P, \ldots, d_{\lambda-1}^P$ and $d_2^Q, \ldots, d_{\lambda-1}^Q$ form a $(\lambda - 2)$-tough pair. Observe that having a demand $v_j^Q u_i^P$ or $v_i^P u_j^Q$ for any i, j would create a closed walk:

$$x_{i,j} \xrightarrow{Q_j^{ex}} v_j^Q \to u_i^P \xrightarrow{P_i^{ex}} x_{i,j} \quad \text{and} \quad x_{i,j} \xrightarrow{P_i^{ex}} v_i^P \to u_j^Q \xrightarrow{Q_j} x_{i,j},$$

respectively, contradicting acyclicity. Suppose now that there is a demand $u_i^P u_j^Q$ served by a path P_d, where $i, j \in \{2, 3, \ldots, \lambda - 1\}$. Then the path $P_i^{ex}[u_i^P, t_i^P]$ can be replaced by

$$P^* = u_i^P \xrightarrow{P_d} u_j^Q \xrightarrow{Q_j^{ex}} x_{1,j} \xrightarrow{P_1} t_1^P \xrightarrow{Q_t} t_i^P.$$

Similarly to earlier, the existence of such a path contradicts the essentiality of e_i, unless P^* passes through e_i. The grid properties imply that e_i cannot lie on any of Q_j^{ex}, P_1, Q_t, thus it must lie on P_d, and moreover, that it must be essential for P_d. We now distinguish two cases based on the location of e_i on P_i.

Case 1. Edge e_i comes before $x_{i,j}$ on P_i. Then e_i is inside the bounded region of the non-directed cycle

$$C' = s_{i-1}^P \xrightarrow{Q_s} s_{i+1}^P \xrightarrow{P_{i+1}} x_{i+1,j} \xleftarrow{Q_j} x_{i-1,j} \xleftarrow{P_{i-1}} s_{i-1}^P,$$

see Figure 11 for an illustration. We claim that after P_d passes through e_i, it is "trapped" inside C', i.e., it cannot pass through any vertex of C'.

Entering some vertex x of $Q_j[x_{i-1,j}, x_{i+1,j}]$ after e_i is not possible since it creates a closed walk $x \xrightarrow{P_d} u_j^Q \xrightarrow{Q_j^{ex}} x$. Entering $Q_s[s_{i-1}^P, s_i^P]$ at vertex x would also create a closed walk: $x \xrightarrow{Q_s} s_i^P \xrightarrow{P_i} \text{head}(e_i) \xrightarrow{P_d} x$. Entering some vertex x of $P^\# := s_i^P \xrightarrow{Q_s} s_{i+1}^P \xrightarrow{P_{i+1}} x_{i+1,j}$ contradicts the essentiality of e_i for P_d, as we can exchange $P_d[u_i^P, x]$ with $u_i^P \xrightarrow{P_i^{ex}} s_i^P \xrightarrow{P^\#} x$. Thus P_d has to enter some vertex x of $P_{i-1}[s_{i-1}^P, x_{i-1,j}]$. If x appears before e_{i-1} on P_{i-1}, then we get a closed walk $\text{head}e_i \xrightarrow{P_d} x \xrightarrow{P_{i-1}} \text{tail}(e_{i-1}) \to \text{head}(e_i)$, where the last portion of the walk is supplied by Property (vi) of grids. This contradicts acyclicity. If x appears after e_{i-1}, then e_{i-1} is non-essential for P_{i-1}, as $P_{i-1}[s_{i-1}^P, x]$ can be circumnavigated on $s_{i-1}^P \xrightarrow{Q_s} s_i^P \xrightarrow{P_i} \text{head}(e_i) \xrightarrow{P_d} x$. Thus no vertex of C' can be entered by P_d after passing through e_i.

Case 2. Edge e_i comes after $x_{i,j}$ on P_i. Then e_i is inside the bounded region of the non-directed cycle

$$C' = x_{i-1,j} \xrightarrow{Q_j} x_{i+1,j} \xleftarrow{P_{i+1}} t_{i+1}^P \xleftarrow{Q_t} s_{i-1}^P \xleftarrow{P_{i-1}} x_{i-1,j}.$$

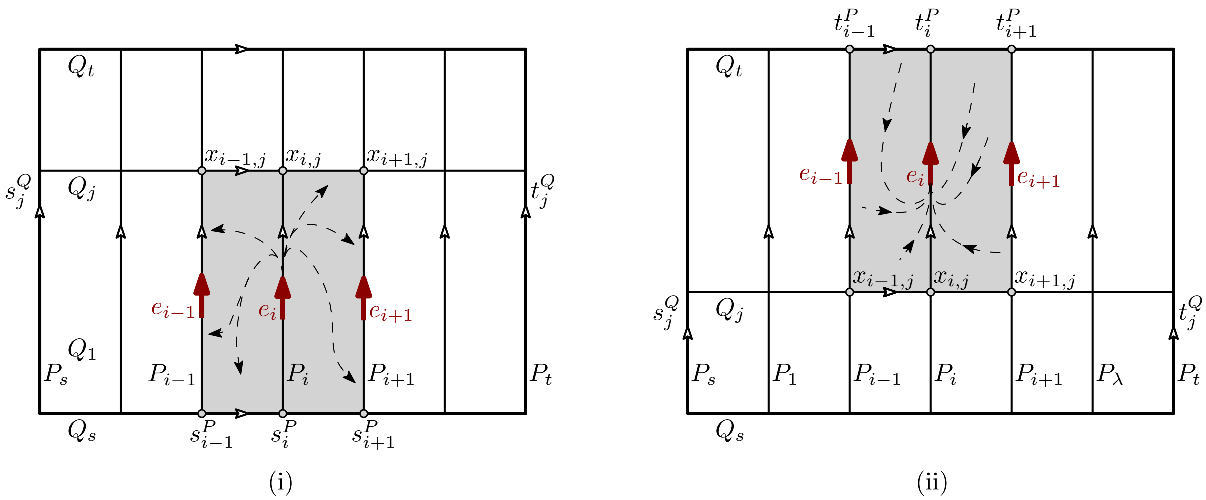

Figure 11: (i) The demand path P_d is trapped in the shaded region after passing through e_i, as entering its boundary is not possible. (ii) The demand path P_d cannot exit the boundary of the shaded region before it passes through e_i.

We claim that after P_d exits C', it cannot pass through e_i. The case can be handled analogously to Case 1: If P_d exits from $Q_j[x_{i-1,j}, x_{i+1,j}]$, $Q^t[t_i^P, t_{i+1}^P]$, or P_{i+1} after e_{i+1}, then it creates a closed walk, contradicting acyclicity. Exiting at vertex x of P_{i+1} before e_{i+1} contradicts the essentiality of e_{i+1} for P_{i+1}, as $P_{i+1}[x, t_{i+1}^P]$ can be exchanged with $x \xrightarrow{P_d} \text{head}(e_i) \xrightarrow{P_i} t_i^P \xrightarrow{Q_t} t_{i+1}^P$. Exiting from vertex x of $P^\# := x_{i-1,j} \xrightarrow{P_{i-1}} t_{i-1}^P \xrightarrow{Q_t} t_i^P$ contradicts essentiality of e_i for P_i^{ex}, as $P_i^{ex}[u_i^P, t_i^P]$ can be exchanged with $u_i^P \xrightarrow{P_d} x \xrightarrow{P^\#} t_i^P$.

The non-existence of a demand $u_j^Q u_i^P$ can be proven as above by exchanging the role of P_i and Q_j. If a demand $v_i^P v_j^Q$ could exist, then in the reversed orientation graph it would be a valid demand $u_i^P u_j^Q$, contradicting the above arguments. The non-existence of demands of the form $v_j^Q v_i^P$ then follows by exchanging the role of P_i and Q_j again. This concludes the proof. □

We are now ready to prove the Structure Theorem (Theorem 1.9).

Proof of Theorem 1.9. Suppose that $\mathcal{D}$ is not c-bounded, that is, for any positive real number γ, there exists an instance $(G_\gamma, T_\gamma, D_\gamma)$ of Planar $\mathcal{D}$-SN and an optimum solution $H := H_\gamma$ such that the total branch degree of H is more than γk, where $k = |T_\gamma|$.

We say that a quantity μ is γ-tied if it can be lower bounded by $\mu > f(\gamma)$ where $\lim_{\gamma \to \infty} f(\gamma) = \infty$. We need to show that H contains a tough pair whose size is γ-tied. This implies that we can find a sequence of patterns that have tough pairs whose size goes to infinity. We can use vertex identifications in these patterns on the tough pairs to get patterns with t-tough pairs for all positive integers t.

As discussed in the beginning of Section 3.2, we may assume without loss of generality that H is acyclic, weakly connected, and the vertices of H have undirected degree at least 3.

Note that if for each face $\mathcal{F}$ of Skel we have that $H \setminus \text{int}\mathcal{F}$ fails at most κ demands for some absolute constant κ, then each $H^{\mathcal{F}}$ has total branch degree at most $O(\kappa^2)k$ by Lemma 3.11, thus we have a face $\mathcal{F}$ where $H \setminus \text{int}\mathcal{F}$ fails at least $\kappa = \Omega(\gamma^{1/2})$ demands. We set $L = \lfloor \kappa^{1/3} \rfloor = \Omega(\gamma^{1/6})$ (in particular, both κ and L are γ-tied). Now we invoke Lemma 3.14, which gives us a grid structure of size $\lambda = \Omega(L)$, i.e., a grid structure whose size is γ-tied. Finally, we use Lemma 3.15 on this grid: we get a $(\lambda - 2)$-tough pair. Since λ is γ-tied, this concludes the proof. □

4 Cleaning: Identifying to a t-hard-pattern

The goal of this section is to prove Theorem 1.10.

Theorem 1.10. *Let $\mathcal{D}$ be a class of graphs closed under transitive equivalence and identifying vertices. The following two are equivalent:*

1. *For every t, there is a $D \in \mathcal{D}$ that has a t-tough pair.*
2. *$\mathcal{C}_i \subseteq \mathcal{D}$ for some $i \in [\kappa]$.*

Towards the proof of Theorem 1.10, we first prove Lemma 4.1.

Lemma 4.1 (Identifying to a t-hard-pattern)**.** *Let $\mathcal{D}$ be a class of graphs that is closed under transitive equivalence and identifying vertices. Let $D \in \mathcal{D}$ and let t be a positive integer. Then there exists t' that depends only on t, such that if D contains a t'-tough-pair then there exists $D' \in \mathcal{D}$ that is a t-hard-pattern.*

From Lemma 4.1, the proof of Theorem 1.10 follows easily. We give this proof before proving Lemma 4.1.

Proof of Theorem 1.10. We first prove the forward direction. From Lemma 4.1, for every positive integer t, $\mathcal{D}$ contains a digraph from some t-hard-pattern, that is, $\mathcal{D}$ contains a digraph from some $\mathcal{C}_i$. Since κ is finite, there exists $i \in [\kappa]$ such that infinitely many digraphs of $\mathcal{C}_i$ belong to $\mathcal{D}$. Further, since $\mathcal{D}$ is closed under vertex identifications and from any member of $\mathcal{C}_i$ every smaller member can be obtained by vertex identifications, we conclude that $\mathcal{C}_i \subseteq \mathcal{D}$.

For the backward direction observe from the definitions of $\{\mathcal{C}_1, \ldots, \mathcal{C}_\kappa\}$, that each t-hard digraph of each $\mathcal{C}_i$ contains in fact a t-tough-pair. □

The remainder of this section is dedicated to the proof of Lemma 4.1 which is divided into five separate steps presented in Sections 4.1, 4.2, 4.3.1, 4.3.2 and 4.4, respectively. Below we give the main results of each of these sections and show how they together imply Lemma 4.1. Before stating the results, we give some important definitions that are used throughout the section.

Definitions. For any digraph D, we denote by $D^\star$ the *transitive closure* of D, that is $D^\star$ is obtained from D by repeatedly adding edges (u, v) whenever (u, v) is not already an edge but there is a (u, v)-path in D.

For a digraph D and two ordered sets $A = (a_1, \ldots, a_t), B = (b_1, \ldots, b_t) \subseteq V(D)$ such that $|A| = |B|$, we say that D has an *(A, B)-matching* if for each $i \in [t]$, $(a_i, b_i) \in E(D)$. We say that D has an *(A, B)-induced-matching* if D has an (A, B)-matching, for each $i \neq j$, $(a_i, b_j) \notin E(D)$, $(b_j, a_i) \notin E(D)$, $(b_i, a_i) \notin E(D)$ and A, B are independent sets in D.

We say that D has an *(A, B)-biclique* if for each $i, j \in [t]$, $(a_i, b_j) \in E(D)$ and $(b_j, a_i) \notin E(D)$. We say that D has an *(A, B)-induced-biclique* if D has an (A, B)-biclique and, A, B are independent sets in D. We say that D has a *t-biclique* if there exists an (A, B)-biclique in D for some $A, B \subseteq V(D)$ and $|A| = |B| = t$. Similarly, we say that D has a *t-induced-biclique* if there exists an (A, B)-induced-biclique in D for some $A, B \subseteq V(D)$ and $|A| = |B| = t$. In all these cases, we call the set of edges $\{(a_i, b_i) : i \in [t]\}$ as the *matching edges* of (A, B).

For positive integers i, j, let $\mathcal{R}(i, j)$ denote the minimum number of vertices such that any $\mathcal{R}(i, j)$-vertex complete graph whose edges are colored with j colors, contains a monochromatic clique of size i. Let $\mathcal{R}'(i)$ denote the minimum integer such that any bipartite graph that has a matching of size $\mathcal{R}'(i)$, either has an induced matching of size t or a t-induced-biclique. From Ramsey Theorem, such numbers always exists and they depend only on i, j, or i, respectively. Recall the definitions of weakly independence, strongly independence and t-tough-pair from Section 2.

Definition 4.2 (Ordered t-tough-pair)**.** Given a digraph D, $E_1, E_2 \subseteq E(D^\star)$, we say that (E_1, E_2) is an ordered t-tough-pair in D if

1. $|E_1| = |E_2| = t$,
2. all edges in E_i are pairwise weakly independent in D, for each $i \in \{1, 2\}$, and
3. for each $e_1 \in E_1$ and $e_2 \in E_2$, (e_1, e_2) are strongly independent in D, and

there exists an ordering of the sets $\text{head}(E_1) = (w_1, \ldots, w_t)$, $\text{tail}(E_1) = (x_1, \ldots, x_t)$, $\text{head}(E_2) = (y_1, \ldots, y_t)$ and $\text{tail}(E_2) = (z_1, \ldots, z_t)$ such that

1. for any $1 \leq i < j \leq t$, there is no (w_j, x_i)-path, no (w_j, w_i)-path and no (x_j, x_i)-path in D,
2. for any $1 \leq i < j \leq t$, there is no (y_j, z_i)-path, no (y_j, y_i)-path and no (z_j, z_i)-path in D.

If (E_1, E_2) is an ordered t-tough-pair then we treat E_1 and E_2 as ordered sets such that their head sets and tail sets satisfies the above properties. We say ordered tough-pair to mean an ordered t-tough-pair for some t.

Observe that, unlike the t-tough-pair, if (E_1, E_2) is an ordered t-tough-pair in D, then the edges of E_1 and E_2 may not be minimal in D.

Simplifying the t-tough-pair. In Section 4.1, using Ramsey arguments we show that it D contains a t-tough-pair for a large enough t, then it contains one of the three structures described in Lemma 4.3.

Lemma 4.3 (Hard sub-structures)**.** *Let D be a reachability-minimal digraph and t be a positive integer. If D contains an $\mathcal{R}(2t, 9)$-tough-pair then one of the following holds.*

1. *D contains an ordered t-tough-pair or*
2. *D contains a t-biclique, or*
3. *there exist ordered sets $A, B \subseteq V(D)$ such that $|A| = |B| = t$, D contains an (A, B)-matching and $D^\star$ contains an (A, B)-induced-biclique.*

Note that if D is reachability-minimal and contains a t-biclique then it contains a t-induced-biclique. From Lemma 4.3 one concludes that, in order to prove Lemma 4.1, it is enough to identify one of the three structures defined in Lemma 4.3 to some t-hard-pattern. The first outcome of Lemma 4.3 is handled in Section 4.3, the second outcome is handled in Section 4.4 and the third outcome is handled in Section 4.2.

Cleaning ordered t-tough-pair. In Section 4.3 we show that if the outcome of Lemma 4.3 is an ordered t-tough-pair, then one can identify D to either a t'-hard-pattern or $D^\star$ contains a t'-induced-biclique whose edges are minimal in D. This is achieved by two rounds of cleaning: in the first round the digraph is cleaned to an intermediate structure, called a *semi-cleaned ordered t-tough-pair* (defined below), and in the second round this semi-cleaned ordered t-tough-pair is either identified to obtain a t'-hard-pattern, or it can be shown that it contains a t'-induced-biclique whose edges are minimal. This is formalized in Lemmas 4.5 and 4.6 which are proved in Section 4.3.1 and 4.3.1, respectively.

Definition 4.4 (Semi-cleaned ordered t-tough-pair)**.** We say that a digraph D is a semi-cleaned ordered t-tough-pair, if it contains an ordered t-tough-pair (A, B) such that the vertex set of D contains at most two vertices, called a source $\mathtt{s}$ and a sink $\mathtt{t}$, outside of the set $V(A \cup B)$, and $N^-_D(\mathtt{s}), N^+_D(\mathtt{t}) = \emptyset$. (Note that from this definition $(\mathtt{t}, \mathtt{s}) \notin E(D)$).

Lemma 4.5 (Semi-cleaning the ordered tough-pair)**.** *Let D be a digraph such that D contains an ordered t^2-tough-pair, then one can obtain $\widehat{D}$ from D by identification, such that $\widehat{D}$ is a semi-cleaned t-ordered tough-pair, for some function g that depends only on t.*

For any positive integer t, let $h(t) = 2\mathcal{R}(2\mathcal{R}(2\mathcal{R}(4t+2,4),4),5)$.

Lemma 4.6 (Cleaning the semi-cleaned ordered tough-pair)**.** *If D is a semi-cleaned ordered $h(t)$-tough-pair then either,*

- *$D^\star$ contains a t-induced-biclique whose edges are minimal in D, or*
- *D can be identified to a digraph $\hat{D}$ such that $\hat{D}$ is transitively equivalent to a t-hard-matching-pattern.*

For any positive integer t, let $\mathtt{cor}(t) = (h(t))^2$. Corollary 4.7 follows from Lemmas 4.5 and 4.6.

Corollary 4.7. *If a digraph D contains an ordered $\mathtt{cor}(t)$-tough-pair, then either D can be identified to a t-hard-matching-pattern, or $D^\star$ contains a t-induced-biclique whose edges are minimal in D.*

If the outcome of Corollary 4.7 a t-hard-matching-pattern, then we are done. Otherwise, we need to clean the t-induced-biclique whose edges are minimal, which is what is done next.

Cleaning a minimal induced-biclique. In Section 4.4 we show that if D contains a $9t$-induced-biclique whose edges are minimal, then D can be identified to a t-hard-biclique-pattern. In this case, Lemma 4.1 is proved.

Lemma 4.8 (Cleaning minimal biclique)**.** *For a positive integer t and a digraph D, if $D^\star$ contains a $9t$-induced-biclique whose edges are minimal in D, then D can be identified to digraph that is transitively equivalent to a t-hard-biclique-pattern.*

Simplifying the third outcome of Lemma 4.3. In Section 4.2, we show that if Lemma 4.3 outputs its third outcome, then one can contract some edges of the input digraph (and hence obtain a digraph in the same pattern class as the input, since the pattern class is closed under identification and contraction is a type of identification) such that the resulting digraph either contains an ordered t'-tough-pair (which we know how to deal using Corollary 4.7), or it contains a t'-biclique whose edges are minimal (which we also know to deal with because of Lemma 4.8).

For positive integers t, p, define $f(t,p) = 2^{\mathcal{R}'(2t)} \cdot f(t,p-1) + \mathcal{R}'(2t)$ when $p > 1$ and $f(t,1) = t$.

Lemma 4.9 (Simplifying a biclique)**.** *Let D be a directed graph and t be a positive integer. Suppose there exists ordered sets $A, B \subseteq V(D)$ such that $|A| = |B| \geq f(t, 2t+2)$, $D^\star$ contains an (A,B)-induced-biclique and the matching edges of (A,B) are minimal in D,. Then, one can obtain $\widehat{D}$ from D by contraction, such that either*

1. *$\widehat{D}$ contains an ordered t-tough-pair, or*
2. *$\widehat{D}^*$ contains a t-induced-biclique whose edges are minimal in $\widehat{D}$.*

Proof of Lemma 4.1. We now give a proof of Lemma 4.1 using the lemmas stated earlier. The proofs of the those lemmas appear in their respective sections.

Set $t_1 = \max\{\mathtt{cor}(9t), 9t\}$, $t_2 = \max\{f(t_1, 2t_1+2), \mathtt{cor}(9t_1), 9t_1\}$ where f is defined as in Lemma 4.9 and $\mathtt{cor}$ is defined in Corollary 4.7. Then, define $t' = \mathcal{R}(2t_2, 9)$. We will now show that if D contains a t'-tough-pair, then there exists $D' \in \mathcal{D}$ that is a t-hard-pattern.

Given D, let (X,Y) be a t'-tough-pair in D. Let D_1 be a spanning subgraph of D such that D_1 is reachability-minimal, D_1 is transitively equivalent to D and (X,Y) is a t'-tough-pair in

D_1. Observe that such a graph D_1 exists and can be obtained by starting with the edge set $E_0 = X \cup Y$ and adding an edge of D to E_0 as long as $D[E_0]$ remains reachability-minimal.

Using Lemma 4.3 on D_1, we conclude that either D_1 contains an ordered t_2-tough-pair, or a t_2-biclique, or there exists $A, B \subseteq V(D_1)$ such that $|A| = |B| = t_2$ and D_1 contains an (A, B)-induced-matching and $D_1^\star$ contains an (A, B)-induced-biclique.

In the first case, by applying Corollary 4.7 on $D_1^\star$, we either get $D_2 \in \mathcal{D}$ such that D_2 is some t-hard-matching-pattern, or conclude that $D_1^\star$ contains a $9t$-induced-biclique whose edges are minimal in D_1. Then applying Lemma 4.8 to D_1 gives some t-hard-biclique-pattern in $\mathcal{D}$. In the second case, again by applying Lemma 4.8 to D_1 gives some t-hard-biclique-pattern in $\mathcal{D}$.

In the third case, by applying Lemma 4.9 we conclude that there exists $D_2 \in \mathcal{D}$ such that either D_2 contains an ordered t_1-tough-pair, in which case a further application of Corollary 4.7 on $D_2^\star$ either yields a $D_3 \in \mathcal{D}$ which is a t-hard-matching-pattern, or we conclude that $D_2^\star$ contains a $9t$-induced-biclique whose edges are minimal in D_2. In the remaining cases, applying Lemma 4.8 yields a $D_4 \in \mathcal{D}$ which is some t-hard-biclique-pattern. ☐

Basic terminology for the remaining section. Throughout the remaining section, we use the following basic notation. For two integers i, j, $[i]$ denotes the set $\{1, \ldots, i\}$ and $[i, j]$ denotes the set $\{i, i+1, \ldots, j\}$. For any (di)graph D and sets $S, T \subseteq V(D)$, an (S, T)-path in D is a path in D from some vertex of S to some vertex of T. If S or T is singleton, say $S = \{v\}$, then we use the notation (v, T)-path. Let $e \in E(D)$, let D/e represents the (di)graph obtained after contracting e, that is, by deleting the endpoints of e and adding a new vertex x_e such that the set of in-neighbours (resp. out-neighbours) of x_e is the union of the set of in-neighbours (resp. out-neighbours) of the end points of e in D. For $E' \subseteq E(D)$, $V(E') \subseteq V(G)$ is the set of vertices that are endpoints of some edge in E'. If D is a digraph, the for any $v \in V(D)$, $N_D^-(v)$ denotes the set of in-neighbours if v in D, $N_D^+(v)$ denotes the set of out-neighbours if v in D and $N_D(v)$ denotes the set of in-neighbours and out-neighbours, called the neighbours, of v in D. We say that a digraph D is connected if its underlying undirected graph is connected.

4.1 Simplifying the t-tough-pair

In this section we prove Lemma 4.3 restated below.

Lemma 4.3 (Hard sub-structures)**.** *Let D be a reachability-minimal digraph and t be a positive integer. If D contains an $\mathcal{R}(2t, 9)$-tough-pair then one of the following holds.*

1. *D contains an ordered t-tough-pair or*
2. *D contains a t-biclique, or*
3. *there exist ordered sets $A, B \subseteq V(D)$ such that $|A| = |B| = t$, D contains an (A, B)-matching and $D^\star$ contains an (A, B)-induced-biclique.*

Proof. Let (E_1, E_2) be an $\mathcal{R}(2t, 9s)$-tough-pair in D. Let $t' = \mathcal{R}(2t, 9)$. Fix an arbitrary ordering of the edges of E_1 and E_2. Let $E_1 = (e_1^1, \ldots, e_{t'}^1)$ and let $E_2 = (e_1^2, \ldots, e_{t'}^2)$. Fix $i \in [2]$. We will use Ramsey arguments to prove the lemma. Towards this, construct an auxiliary undirected, complete, edge-colored graph $\texttt{Aux}_i$ as follows. The vertex set of $\texttt{Aux}_i$ corresponds to the edges of E_i, that is, for each $e \in E_i$ there is a vertex corresponding to e. For the sake of simplicity we denote the vertex of $\texttt{Aux}_i$ that correspond to the edge e of E_i, by e itself. The coloring function $\texttt{col}_i$ on the edges of $\texttt{Aux}_i$ (equivalently on the pair of distinct edges of E_i) is defined based on the following. Fix any two distinct edges $e = (u, v), e' = (u', v') \in E_i$. The coloring function $\texttt{col}_i(e, e')$ is defined based on the existence of the edges (u, v') and (u', v) in D and $D^\star$. Below we describe $\texttt{col}_i(e, e')$.

1. If $(u, v') \in E(D)$, and

   (a) $(u', v) \in E(D)$, then $\texttt{col}_i(e, e') = (1, 1)$,
   (b) $(u', v) \in E(D^\star) \setminus E(D)$, then $\texttt{col}_i(e, e') = (1, 2)$,
   (c) $(u', v) \notin E(D)$, $(u', v) \notin E(D^\star)$, then $\texttt{col}_i(e, e') = (1, 3)$.

2. If $(u, v') \in E(D^\star) \setminus E(D)$, and
   (a) $(u', v) \in E(D)$, then $\texttt{col}_i(e, e') = (2, 1)$,
   (b) $(u', v) \in E(D^\star) \setminus E(D)$, then $\texttt{col}_i(e, e') = (2, 2)$,
   (c) $(u', v) \notin E(D)$, $(u', v) \notin E(D^\star)$, then $\texttt{col}_i(e, e') = (2, 3)$.
3. If $(u, v') \notin E(D)$, $(u, v') \notin E(D^\star)$, and
   (a) $(u', v) \in E(D)$, then $\texttt{col}_i(e, e') = (3, 1)$,
   (b) $(u', v) \in E(D^\star) \setminus E(D)$, then $\texttt{col}_i(e, e') = (3, 2)$,
   (c) $(u', v) \notin E(D)$, $(u', v) \notin E(D^\star)$, then $\texttt{col}_i(e, e') = (3, 3)$.

We now use a Ramsey argument on the t'-vertex graph $\texttt{Aux}_i$ with edge-coloring function $\texttt{col}_i$ (that uses at most 9 different colors). Using Ramsey arguments, we conclude that there exists a monochromatic clique in $\texttt{Aux}_i$ of size $2t$. Below we show how one can get one of the three outcomes in the lemma statement based of the color of the monochromatic clique. Say edges of E_i that correspond to this monochromatic clique in $\texttt{Aux}_i$ are $E'_i = \{e^i_{j_1}, e^i_{j_2}, \dots, e^i_{j_t}\}$ such that $j_1 < \dots < j_t$ (recall we fixed an ordering of the edges of E_i). For the ease of notation later, we assume that the sets E'_i are ordered with the ordering as described in the previous line.

Claim 4.10. *If for each $i \in [2]$, the color of the monochromatic clique in $\texttt{Aux}_i$ is $(j, 3)$ or $(3, j)$ for any $j \in [3]$, then (E'_1, E'_2) form an ordered $2t$-tough-pair in D.*

Proof. Since (E_1, E_2) is a tough-pair and $E'_i \subseteq E_i$, (E'_1, E'_2) is also a tough-pair (of size t). If $\texttt{col}_i$ colors the clique corresponding to the edges of E'_i with the color $(j, 3)$, then consider the ordering $(e^i_{j_1}, e^i_{j_2}, \dots, e^i_{j_t})$ of E'_i, otherwise (when $\texttt{col}_i$ colors the monochromatic clique with color $(3, j)$), then consider the ordering $(e^i_{j_t}, \dots, e^i_{j_1})$ of E'_i. Then from the description of the definition of $\texttt{col}_i$ corresponding to the case when $\texttt{col}_i$ takes value $(3, j)$ or $(j, 3)$, one concludes that (E'_1, E'_2) is indeed an ordered t-tough-pair with the ordering described above. ⌟

Note that if the graph has an ordered $2t$-tough-pair, then it also has an ordered t-tough-pair.

Observation 4.11. *It is easy to observe that if the color of the monochromatic clique in $\texttt{Aux}_i$ is $(1, 1)$, then $(tail(E'_i), head(E'_i))$ form a $2t$-biclique in D. If the color is $(2, 2)$, then D contains the $(tail(E'_i), head(E'_i))$-induced matching and $D^\star$ contains the $(tail(E'_i), head(E'_i))$-biclique (thus, the third outcome of the lemma holds).*

Note that if the graph has $2t$-biclique, then it also has an ordered t-biclique.

Claim 4.12. *If there exists $i \in [2]$ such that the color of the monochromatic clique in $\texttt{Aux}_i$ is $(1, 2)$ or $(2, 1)$, then H contains a t-biclique.*

Proof. Suppose that the color of the monochromatic clique in $\texttt{Aux}_i$ is $(1, 2)$. Recall that $E'_i = ((e^i_{j_1}, e^i_{j_2}, \dots, e^i_{j_t}))$ is the ordered set of edges that correspond to the monochromatic clique. The $(\text{tail}(\{e^i_{j_1}, \dots, e^i_{j_t}\}), \text{head}(\{e^i_{j_{t+1}}, \dots, e^i_{j_{2t}}\}))$ form a t-biclique in D.

Similarly, if the color of the monochromatic clique in $\texttt{Aux}_i$ is $(2, 1)$, then $(\text{tail}(\{e^i_{j_{t+1}}, \dots, e^i_{j_{2t}}\}), \text{head}(\{e^i_{j_1}, \dots, e^i_{j_t}\}))$ form a t-biclique in H. ⌟

Thus, from Claims 4.12 and 4.10, and Observation 4.11, the lemma follows. □

4.2 Simplifying a biclique

In this section, we prove the following lemma. Recall that, from Section 4, for positive integer t, p, $\mathcal{R}'(t)$ is the smallest positive integer such that any bipartite graph with a matching of size $\mathcal{R}'(t)$ either has an induced matching of size t or a t-induced-biclique. Furethermore $f(t,p) = 2^{\mathcal{R}'(2t)} f(t,p-1) + \mathcal{R}'(2t)$ when $p > 1$ and $f(t,1) = t$.

Lemma 4.9 (Simplifying a biclique)**.** *Let D be a directed graph and t be a positive integer. Suppose there exists ordered sets $A, B \subseteq V(D)$ such that $|A| = |B| \geq f(t, 2t+2)$, $D^\star$ contains an (A,B)-induced-biclique and the matching edges of (A,B) are minimal in D,. Then, one can obtain $\widehat{D}$ from D by contraction, such that either*

1. *$\widehat{D}$ contains an ordered t-tough-pair, or*
2. *$\widehat{D}^*$ contains a t-induced-biclique whose edges are minimal in $\widehat{D}$.*

The remainder of this section is dedicated to the proof of Lemma 4.9. Recall that A, B are ordered sets. Let $g(t) = f(t, \mathcal{R}(2t,2))$. Further let $A = (a_1, \ldots, a_{g(t)})$ and $B = (b_1, \ldots, b_{g(t)})$.

We begin by showing that, one can assume without loss of generality that D is acyclic. Suppose that D is not acyclic. First observe that for any $a_i, a_j \in A$, such that $i \neq j$, a_i and a_j belong to different strongly connected components of D. Similarly, each vertex of B belongs to a distinct strongly connected component of D. Further, for any a_i, b_j, a_i and b_j do not belong to the same strongly connected component of D. Indeed, as otherwise there is a (b_j, a_i)-path in D contracting the definition of an (A,B)-biclique. Thus, each vertex in $\{a_1, \ldots, a_{g(t)}, b_1, \ldots, b_{g(t)}\}$ belong to distinct strongly connected components of D. Let D' be obtained from D by contracting each strongly connected component of D into a single vertex. That is, D' has a vertex for each strongly connected component of D and for two vertices u, v in D', there is an edge from u to v in D' if there is vertex in the strongly connected component of D which was contracted onto u, that has an edge in D to a vertex of the strongly connected component of D which was contracted to v. Let us call a vertex of D' a_i if it is obtained by contracting a strongly connected component containing a_i. Similarly, let us call a vertex of D' b_i if it is obtained by contracting a strongly connected component containing b_i. It is easy to observe that D' also contains an (A,B)-matching and $D'^\star$ contains an (A,B)-induced-biclique. Indeed, since an (a_i, b_j)-path in D implies an (a_i, b_j)-path in D' and no (a_i, a_j)-path (resp. no (b_i, b_j)-path) in D implies no (a_i, a_j)-path (resp. no (b_i, b_j)-path) in D'. Further if D' contains an (a_i, b_i)-path of length strictly greater than 1 then so does D. Since D' is obtained by contraction operation from D, we conclude that without loss of generality, we can assume for the rest of the section that the input graph D in Lemma 4.9 is acyclic.

We say that an edge $e \in E(D) \setminus E(A,B)$ is *contraction-redundant* with respect to (A,B) in D if the following holds.

1. If D/e is acyclic,
2. all the edges in $\{(a_i, b_i) : i \in [g(t)]\}$ (that is the matching edges of (A,B)) are minimal edges in D/e, and
3. $(D/e)^\star$ has an (A,B)-induced-biclique..

Let $D_{\texttt{contr}}$ be the graph obtained from D by repeatedly contracting the contraction-redundant edges with respect to (A,B) until $D_{\texttt{contr}}$ has no such edge. For the ease of notation by aa-path we mean a path between two distinct vertices of A, by bb-path we mean a path between two distinct vertices of B and by a_*b_*-path we mean a (a_i, b_i)-path for some $i \in [g(t)]$. We divide the proof of Lemma 4.9 into two independent parts based on the length of a longest path in $D_{\texttt{contr}}$. In Section 4.2.1, we consider the case when the length of a longest path in D_{contr} is at least $2t+2$. In this case, we show that $D_{\texttt{contr}}$ contains an ordered t-tough-pair. In Section 4.2.2, we consider the case when the length of any longest path in $D_{\texttt{contr}}$ is at most $2t+2$. In this case, we get one of the desired outputs.

4.2.1 When $D_{\texttt{contr}}$ has a long path

In this section, we prove Lemma 4.9 when the length of a longest path in $D_{\texttt{contr}}$ is at least $2t+2$. Let $P = (v_1, \ldots, v_p)$ be a directed longest path in $D_{\texttt{contr}}$ (then $p \geq 2t+2$). Recall that none of the edges of $D_{\texttt{contr}}$, and in particular, none of the edges of P, are *contraction-redundant* with respect to (A, B).

Lemma 4.13. *For each $e \in E(P)$, $D_{\texttt{contr}}/e$ is acyclic.*

Proof. Suppose, for the sake of contradiction, that there is a cycle in $D_{\texttt{contr}}/e$. Let $e = (v_i, v_{i+1})$. Then, either there is a (v_i, v_{i+1})-path in $D_{\texttt{contr}} - \{(v_i, v_{i+1})\}$ or a (v_{i+1}, v_i)-path in $D_{\texttt{contr}}$. If there exists a (v_{i+1}, v_i)-path in $D_{\texttt{contr}}$ then together with the edge $e = (v_i, v_{i+1})$, it creates a directed closed walk in $D_{\texttt{contr}}$. This contradicts that $D_{\texttt{contr}}$ is acyclic.

In the other case, suppose there exists a (v_i, v_{i+1})-path, say P' in $D_{\texttt{contr}} - \{(v_i, v_{i+1})\}$. We first claim that the internal vertices of P' are disjoint from P. Suppose not, let $v_j \in V(P)$ be the first internal vertex of P' that belongs to P. Then there is a (v_i, v_j)-path in $D_{\texttt{contr}}$ and (v_j, v_{i+1})-path in $D_{\texttt{contr}}$. If $j < i$, then the (v_i, v_j)-path together with the (v_j, v_i)-subpath of P, gives a closed walk in $D_{\texttt{contr}}$, contradicting its acyclicity. Otherwise, $j > i+1$. In this case, the (v_j, v_{i+1})-path together with the (v_{i+1}, v_j)-subpath of P gives a closed walk in $D_{\texttt{contr}}$, again contradicting its acyclicity. □

Since each edge of $D_{\texttt{contr}}$, in particular, each edge of P, is not contraction-redundant, from Lemma 4.13 and the definition of contraction-redundant, for each $e \in E(P)$ either

1. there exists an *aa*-path, or a *bb*-path in $D_{\texttt{contr}}/e$ (that is, $(D_{\texttt{contr}}/e)^\star$ does not contain the (A, B) -induced-biclique), or
2. there exists $i \in [g(t)]$ an $a_i b_i$-path in $(D_{\texttt{contr}}/e) - \{(a_i, b_i)\}$ ((a_i, b_i) is not a minimal edge in $D_{\texttt{contr}}/e$).

Lemma 4.14. *Let $e = (v_i, v_{i+1}) \in E(P)$ such that H_{contr}/e has an (a_j, b_j)-path. Then there exists an (a_j, v_{i+1})-path and a (v_i, b_j)-path in $D_{\texttt{contr}}$.*

If $D_{\texttt{contr}}/e$ has a (a_j, a_ℓ)-path, for some $a_j, a_\ell \in A$, then there exists an (a_j, v_{i+1})-path and a (v_i, a_ℓ)-path in $D_{\texttt{contr}}$.

If $D_{\texttt{contr}}/e$ has a (b_j, b_ℓ)-path, for some $b_j, b_\ell \in B$, then there exists an (b_j, v_{i+1})-path and (v_i, b_ℓ)-path in $D_{\texttt{contr}}$.

Proof. Let $e = (v_i, v_{i+1}) \in E(P)$ such that D_{contr}/e has an (a_j, b_j)-path. Then either there exists an (a_j, v_i)-path and a (v_{i+1}, b_j)-path in $D_{\texttt{contr}}$, or there exists an (a_j, v_{i+1})-path and (v_i, b_j)-path in $D_{\texttt{contr}}$. In the later case, we are done. The former case implies an (a_j, b_j)-path in $D_{\texttt{contr}}$ of length at least three (with at least three edges), which contradicts that (a_j, b_j) is a minimal edge in $D_{\texttt{contr}}$.

If D_{contr}/e has an (a_j, a_ℓ)-path, then either there exists an (a_j, v_i)-path and a (v_{i+1}, a_ℓ)-path in $D_{\texttt{contr}}$, or there exists an (a_j, v_{i+1})-path and (v_i, a_ℓ)-path in $D_{\texttt{contr}}$. In the later case, we are done. The former case implies an (a_j, a_ℓ)-path in $D_{\texttt{contr}}$ which contradicts that (A, B) is an induced biclique in $D^\star_{\texttt{contr}}$ (in particular, that A is an independent set in $D^\star_{\texttt{contr}}$). One can similarly show that if D_{contr}/e has a (b_j, b_ℓ)-path, then there exists an (b_j, v_{i+1})-path and (v_i, b_ℓ)-path in $D_{\texttt{contr}}$. □

For each $e = (v_i, v_{i+1}) \in E(P)$, we say that:

1. e is *j-irredundant* in $D_{\texttt{contr}}$, if there is an (a_j, v_{i+1})-path and a (v_i, b_j)-path in $D_{\texttt{contr}}$.
2. e is *aa-irredundant* in $D_{\texttt{contr}}$, if there exists $a_j, a_\ell \in A$ such that there is an (a_j, v_{i+1})-path and a (v_i, a_ℓ)-path in $D_{\texttt{contr}}$.

3. e is *bb-irredundant* in $D_{\texttt{contr}}$, if there exists $b_j, b_\ell \in B$ such that there is an (b_j, v_{i+1})-path and (v_i, b_ℓ)-path in $D_{\texttt{contr}}$.

From Lemmas 4.13 and 4.18, for each $e \in E(P)$, either e is j-irredundant, for some $j \in [g(t)]$, or is aa-irredundant or is bb-irredundant.

Lemma 4.15. *Let $e = (v_i, v_{i+1}) \in E(P)$ such that e is aa-irredundant in $D_{\texttt{contr}}$. Then $i = 1$.*

Proof. For the sake of contradiction, suppose that $i > 1$. Let $e' = (v_{i-1}, v_i)$. If e' is aa-irredundant or e' is j-irredundant for some $j \in [g(t)]$, then there exists a path from some vertex of A to v_i in $D_{\texttt{contr}}$. Also since e is aa-irredundant, there exists a path from v_i to some vertex of A in $D_{\texttt{contr}}$. This implies either a directed closed walk in $D_{\texttt{contr}}$, contradicting its acyclicity, or a path between two distinct vertices of A in $D_{\texttt{contr}}$, contradicting that A is an independent set in $D^\star_{\texttt{contr}}$.

If e' is bb-irredundant, then there exists a path from some vertex of B to v_i. Also since e is aa-irredundant, there exists a path from v_i to some vertex of A. This implies a path from some vertex of B to some vertex of A. Since (A, B) is an induced biclique in $D^\star_{\texttt{contr}}$, that is there is a path from every vertex of A to every vertex of B in $D_{\texttt{contr}}$, this implies a directed closed walk in $D_{\texttt{contr}}$, contradicting its acyclicity. □

Lemma 4.16. *Let $e = (v_i, v_{i+1}) \in E(P)$ such that e is bb-irredundant in $D_{\texttt{contr}}$. Then $i+1 = p$.*

Proof. For the sake of contradiction, suppose that $i + 1 < p$. Let $e' = (v_{i+1}, v_{i+2})$. If e' is bb-irredundant or e' is j-irredundant for some $j \in [g(t)]$, then there exists a path from v_{i+1} to some vertex of B. Also since e is bb-irredundant, there exists a path from some vertex of B to v_{i+1}. This either implies a directed closed walk in $D_{\texttt{contr}}$, contradicting its acyclicity, or a path between two distinct vertices of B in $D_{\texttt{contr}}$, contradicting that B is an independent set in $D^\star_{\texttt{contr}}$.

If e' is aa-irredundant, then there exists a path from v_{i+1} to some vertex of A in $D_{\texttt{contr}}$. Also since e is bb-irredundant, there exists a path from some vertex of B to v_{i+1} in $D_{\texttt{contr}}$. This implies a path from some vertex of B to some vertex of A in $D_{\texttt{contr}}$. Since (A, B) is an induced-biclique in $D^\star_{\texttt{contr}}$, this implies a directed closed walk in $D_{\texttt{contr}}$, contradicting its acyclicity. □

Let P' be a subpath obtained from P by without the first and last edge of P, that is $P' = (v_2, \ldots, v_{p-1})$. From Lemmas 4.15 and 4.16, each edge $e \in E(P')$ is j-irredundant for some $j \in [g(t)]$.

Lemma 4.17. *Let e, e' be distinct edges of P' such that e is q-irredundant in $D_{\texttt{contr}}$ and e' is r-irredundant in $D_{\texttt{contr}}$, for some $q, r \in [g(t)]$. Then $q \neq r$.*

Proof. For the sake of contradiction, suppose that $q = r$. Without loss of generality, let $e = (v_i, v_{i+1})$ and $e' = (v_j, v_{j+1})$ such that $i < j$. Then there exists an (a_q, v_{i+1})-path and (v_j, b_q)-path in $D_{\texttt{contr}}$. Since $i < j$, there exists a path from (v_i, v_j) (which is the the (v_{i+1}, v_j)-subpath of P') in $D_{\texttt{contr}}$. This implies an (a_q, b_q)-path of length strictly greater than one in $D_{\texttt{contr}}$, contradicting that (a_q, b_q) is a minimal edge in $D_{\texttt{contr}}$. □

For each $e = (v_i, v_{i+1}) \in E(P')$, fix a $\rho(i) \in [g(t)]$, such e is $\rho(i)$-irredundant in $D_{\texttt{contr}}$. The following observation follows from Lemma 4.17.

Observation 4.18. *For each $v_i \in V(P')$, $i \in [3, p-2]$, there exists a $(a_{\rho(i-1)}, v_i)$-path and a $(v_i, b_{\rho(i)})$-path in $D_{\texttt{contr}}$. Also, for $i, j \in [3, p-2]$, $i \neq j$, $\rho(i) \neq \rho(j)$.*

For each $i \in [3, p-2]$, let P_i^{a} denote an (arbitrarily) fixed $(a_{\rho(i-1)}, v_i)$-path in $D_{\mathtt{contr}}$ and let P_i^{b} denote an (arbitrarily) fixed $(v_i, b_{\rho(i)})$-path in $D_{\mathtt{contr}}$. Let $E_1 = (v_3, b_{\rho(3)}), (v_4, b_{\rho(4)}), \ldots, (v_{p/2-2}, b_{\rho(p/2-2)})$ and let $E_2 = ((a_{\rho(p/2-2)}, v_{p/2-1}), (a_{\rho(p/2-1)}, v_{p/2}), \ldots, (a_{\rho(p-1)}, v_{p-2})$. From Observation 4.18, each edge of $E_1 \cup E_2$ is an edge in $D^\star_{\mathtt{contr}}$ and the endpoints of the edges in $E_1 \cup E_2$ are distinct. We will now show that (E_1, E_2) is an ordered $(p/2-1)$-tough-pair in $D^\star_{\mathtt{contr}}$. Since $p \geq 2t+2$, this proves Lemma 4.9 in the case when the length of a longest path in $D_{\mathtt{contr}}$ is at least $2t+2$.

Lemma 4.19. *(E_1, E_2) is an ordered $(p/2-1)$-tough-pair in $D^\star_{\mathtt{contr}}$.*

Proof. To prove the lemma, we prove the following claims.

Claim 4.20 (Weak independence of E_1 (resp. E_2))**.** *The edges in E_1 are pairwise weakly independent. Similarly, the edges in E_2 are pairwise weakly independent in $D^\star_{\mathtt{contr}}$.*

Proof. Let $(a_{\rho(i-1)}, v_i), (a_{\rho(j-1)}, v_j) \in E_1$. For the sake of contradiction, say $(v_j, a_{\rho(i-1)}) \in D^\star_{\mathtt{contr}}$. Then this implies a $(a_{\rho(j-1}, a_{\rho(i-1)})$-path in $D^\star_{\mathtt{contr}}$ contradicting that A is an independent set in $D^\star_{\mathtt{contr}}$. Using symmetric arguments one can show that all edges in E_2 are pairwise weakly independent. ⌟

Claim 4.21 (Orderedness on E_1 and E_2)**.** *For any $i, j \in [3, p-2]$, $i < j$, $(a_{\rho(j-1)}, v_i)$, $(a_{\rho(j-1)}, a_{\rho(i-1)}), (v_j, b_{\rho(i)}), (v_j, v_i) \notin E(D^\star_{\mathtt{contr}})$.*

Proof. Since A is an independent set in $D^\star_{\mathtt{contr}}$, $(a_{\rho(j-1)}, a_{\rho(i-1)}) \notin E(D^\star_{\mathtt{contr}})$. Since $D^\star_{\mathtt{contr}}$ is acyclic and there is a (v_i, v_j)-path in $D_{\mathtt{contr}}$, we conclude that $(v_j, v_i) \notin E(D^\star_{\mathtt{contr}})$. If there is an $(a_{\rho(j-1)}, v_i)$-path in $D_{\mathtt{contr}}$, then this together with the (v_i, v_{j-1})-subpath of P' and the $(v_{i-1}, b_{\rho(i-1)})$ path P^{b}_{i-1}, implies a $(a_{\rho(j-1), b_{\rho(j-1)}})$-path of length strictly greater than 1 in $D_{\mathtt{contr}}$, which is a contradiction. Using symmetric arguments, one can show that $(v_j, b_{\rho(i)}), (v_j, v_i) \notin E(D^\star_{\mathtt{contr}})$. ⌟

Claim 4.22 (Weak independence between E_1 and E_2)**.** *Let $(v_i, b_{\rho(i)}) \in E_1$ and $(a_{\rho(j-1)}, v_j) \in E_2$. There is $(v_j, b_{\rho(i)}), (b_{\rho(i)}, a_{\rho(i-1)}) \notin E(D^\star_{\mathtt{contr}})$.*

Proof. First observe from the construction of E_1, E_2 that $i < j$. If $(v_j, b_{\rho(i)}) \in E(D^\star_{\mathtt{contr}})$, then the $(a_{\rho(i)}, v_{i+1})$-path P^{a}_{i+1}, together with the (v_{i+1}, v_j)-subpath of P', together with the $(v_j, b_{\rho(i)})$-path, implies a $(a_{\rho(i)}, b_{\rho(i)})$-path of length strictly greater than 1 in $D_{\mathtt{contr}}$, which is a contradiction.

If $(b_{\rho(i)}, a_{\rho(i-1)}) \in E(D^\star_{\mathtt{contr}})$, then the $(a_{rho(i-1)}, b_{\rho(i)})$-path obtained by concatenating P_i^{a} and P_i^{b}, together with the $(b_{\rho(i)}, a_{\rho(i-1)})$-path, implies a $(a_{\rho(i-1)}, a_{\rho(j-1)})$-path in $D_{\mathtt{contr}}$, which contradicts that A is an independent set in $D_{\mathtt{contr}}$. ⌟

Claim 4.23 (Strong independence between E_1 and E_2)**.** *Let $(v_i, b_{\rho(i)}) \in E_1$ and $(a_{\rho(j-1)}, v_j) \in E_2$. Then $(v_i, a_{\rho(j-1)}), (a_{\rho(j-1)}, v_i), (b_{\rho(i)}, v_j), (v_j, b_{\rho(i)}) \notin E(D^\star_{\mathtt{contr}})$.*

Proof. Recall that $i > j$. If $(v_i, a_{\rho(j-1)}) \in E(D^\star_{\mathtt{contr}})$, then together with P_i^{a}, it contradicts that A is independent in $D^\star_{\mathtt{contr}}$. Similarly, if $(b_{\rho(i)}, v_j) \in E(D^\star_{\mathtt{contr}})$, then it contradicts the independence of B in $D^\star_{\mathtt{contr}}$.

If $(a_{\rho(j-1)}, v_i) \in E(D^\star_{\mathtt{contr}})$, then this together with the (v_i, v_{j-1})-subpath of P', together with the path P^{b}_{j-1}, implies a $(a_{\rho(j-1)}, b_{\rho(j-1)})$-path of length strictly greater than 1 in $D_{\mathtt{contr}}$, which is a contradiction. Similarly if, $(v_j, b_{\rho(i)}) \in E(D^\star_{\mathtt{contr}})$, then this implies a $(a_{\rho(i)}, b_{\rho(i)})$-path of length strictly greater than 1 in $D_{\mathtt{contr}}$, which is a contradiction. ⌟

This concludes the proof. □

4.2.2 No long path in $D_{\texttt{contr}}$

In this section, we prove Lemma 4.9 in the case when the length of a longest path in $D_{\texttt{contr}}$ is $p \leq 2t+2$. Recall that $D_{\texttt{contr}}$ contains (A,B)-matching and $D^{\star}_{\texttt{contr}}$ contains (A,B)-induced-biclique. Also, the edges of $A \cup B$ are minimal in $D_{\texttt{contr}}$. We prove Lemma 4.9 by induction on the length of a longest (A,B)-path in $D_{\texttt{contr}}$. For this purpose we essentially restate Lemma 4.9 in a form that is "induction-friendly". Observe, as a base case, that when the length of a longest (A,B)-path is 1, then since $D^{\star}_{\texttt{contr}}$ contains (A,B)-induced-biclique, we conclude that $D_{\texttt{contr}}$ contains (A,B)-induced-biclique and the edges of the (A,B)-induced-biclique are minimal. In this case, we conclude that we get the second output of Lemma 4.9.

Lemma 4.24. *Let $A,B \subseteq V(D_{\texttt{contr}})$ be ordered sets such that the length of a longest (A,B)-path in $D_{\texttt{contr}}$ is p, $|A| = |B| \geq f(t,p)$, $D^{\star}_{\texttt{contr}}$ contains an (A,B)-induced-biclique and the matching edges of (A,B) are minimal in $D_{\texttt{contr}}$, then either*

1. *there exist ordered sets $A',B' \subseteq V(D_{\texttt{contr}})$ such that the length of a longest (A',B')-path in $D_{\texttt{contr}}$ is at most p, $|A'| = |B'| \geq f(t,p-1)$, $D^{\star}_{\texttt{contr}}$ contains an (A',B')-induced-biclique and the matching edges of (A',B') are minimal in $D_{\texttt{contr}}$, or*
2. *$D_{\texttt{contr}}$ contains an ordered t-tough-pair, or*
3. *$D^{\star}_{\texttt{contr}}$ contains a t-induced-biclique whose edges are minimal in $D_{\texttt{contr}}$.*

Proof. Let L be the set of vertices of $D_{\texttt{contr}}$ that contains the first internal vertex on every longest (A,B)-path in $D_{\texttt{contr}}$.

Claim 4.25. *There is no (L,L)-path in $D_{\texttt{contr}}$.*

Proof. For the sake of contradiction, suppose there exist $x,y \in L$ such that there this an (x,y)-path in $D_{\texttt{contr}}$. Note that $x \neq y$, as otherwise, $D_{\texttt{contr}}$ would contain a cycle. From the definition of L, there exists $a_i, a_j \in L$ (i not necessarily distinct from j) such that x is the first internal vertex on some (a_i,B)-path, say P_x, of length p and y is the first internal vertex on some (a_j,B)-path, say P_y, of length p. Let P_{xy} be some (x,y)-path in $D_{\texttt{contr}}$. We first claim that the set of internal vertices of $P_{x,y}$ is disjoint from P_y. Suppose not. Let z be some internal vertex of P_{xy} that is also a vertex of P_y. Since $z \in V(P_y)$, there exists a (y,z)-path in $D_{\texttt{contr}}$. Also since $z \in V(P_{x,y})$, there exists a (z,y)-path in $D_{\texttt{contr}}$. This implies a cycle in $D_{\texttt{contr}}$, which is a contradiction. Thus, we conclude that the internal vertices of P_{xy} are disjoint from that of P_y. Then consider the (x,B)-path obtained by appending P_{xy} and P_y. Note that the length of P_y is $p-1$ and the length of P_{xy} is at least 1. This (x,B)-path, together with the edge (a_i,x), gives an (a_i,B)-path of length strictly greater than p in $D_{\texttt{contr}}$, which contradicts that the length of a longest (A,B)-path in $D_{\texttt{contr}}$ is p. ⌟

Claim 4.26. *For each $(a,x) \in E(D_{\texttt{contr}})$ such that $a \in A$ and $x \in L$, (a,x) is a minimal edge of $D_{\texttt{contr}}$.*

Proof. For the sake of contradiction, say there exists an (a,x)-path in $D_{\texttt{contr}}$ of length at least 2. Let this path be P_{ax}. Since $x \in L$, there exists an (A,B)-path of length p whose first internal vertex is x. Let this path be P_{AB}. Then the internal vertices of P_{ax} are disjoint from the internal vertices of P_{AB}, as otherwise there would be a cycle in $D_{\texttt{contr}}$. By appending the path P_{sx} (which is of length at least 2) with the subpath of P_{ABs} starting from x (which is of length $p-1$), we get an (A,B)-path in $D_{\texttt{contr}}$ of length at least $p+1$, which is a contradiction. ⌟

Claim 4.27. *Let $a \in A$ and $x \in L$, such that $(a,x) \notin E(D_{\texttt{contr}})$. Then $(a,x) \notin E(D^{\star}_{\texttt{contr}})$.*

Proof. For the sake of contradiction, suppose there exists an (a, x)-path, say $P_{a,x}$ in $D_{\mathtt{contr}}$ of length at least two. Since $x \in L$, there exists $a' \in A$ (a' not necessarily distinct from a) such that x is the first internal vertex of a (a', T)-path, say $P_{a'B}$, of length p. First observe that the internal vertices of $P_{a,x}$ are disjoint from that of $P_{a'B}$, as otherwise there would be a cycle in $D_{\mathtt{contr}}$. The path $P_{a,x}$ appended with the (x, B)-subpath of $P_{a'B}$ gives an (A, B)-path in $D_{\mathtt{contr}}$ of length strictly greater than p, which is a contradiction. ⌟

Consider the bipartite graph $D_{\mathtt{bip}} = D_{\mathtt{contr}}[A \cup L]$. Note, from Claim 4.26, that each edge of $D_{\mathtt{bip}}$ is a minimal edge in $D_{\mathtt{contr}}$. We now distinguish into two cases based on the size of the matching in $D_{\mathtt{bip}}$.

Case 1: The size of a maximum matching in $D_{\mathtt{bip}}$ is at least $\mathcal{R}'(2t)$. In this case, from Ramsey's Theorem, either there exists an induced matching of size $2t$ in $D_{\mathtt{bip}}$ or a $2t$-induced-biclique. From Claim 4.26, each edge of $D_{\mathtt{bip}}$ is a minimal edge of D_{contr}.

In the first case, we get sets $A^* \subseteq A$ and $B^* \subseteq L$ such that there is an (A^*, B^*)-induced-matching in $D_{\mathtt{bip}}$. Since A is an independent set in $D^\star_{\mathtt{contr}}$, so is A^*. Since $B^* \subseteq L$, from Claim 4.26, B^* is an independent set in $D^\star_{\mathtt{contr}}$. Thus, (A^*, B^*) is an induced-matching in $D_{\mathtt{contr}}$. Further, from Claim 4.27, (A^*, B^*) is in fact an induced matching in $D^\star_{\mathtt{contr}}$. Observe that if a digraph has an induced-matching of size $2t$ in its transitive closure, then it has a ordered t-tough-pair. Thus, in this case we get the second outcome of the lemma.

In the second case, when $H_{\mathtt{bip}}$ contains a $2t$-induced-biclique, then this is also a $2t$-induced-biclique in $D_{\mathtt{contr}}$ where all edges of the biclique are minimal edges of $D_{\mathtt{contr}}$. Thus, in this case we get the third outcome of the lemma.

Case 2: The size of a maximum matching in $D_{\mathtt{bip}}$ is at most $\mathcal{R}'(2t)$. In this case, we will find ordered sets $A', B' \subseteq V(D_{\mathtt{contr}})$ satisfies the properties stated in the first outcome of the lemma.

Let Z be a minimum vertex cover of $D_{\mathtt{bip}}$ of size at most $\mathcal{R}'(2t)$. Let $Z_A = Z \cap A$ and $Z_L = Z \cap L$. Let $A^* = A \setminus Z_S$. Since $Z_A \cup Z_L$ is a vertex cover of $D_{\mathtt{bip}}$, for any $a \in A^*$, $N_L(a) \subseteq Z_L$. For each subset $L' \subseteq Z_L$, let $A_{L'} \subseteq A^*$ such that for each vertex $a \in A_{L'}$, $N_{Z_L}(a) = L'$ (and hence $N_L(a) = L'$).

Fix $L' \subseteq Z_L$ such that $|S_{L'}|$ is maximized. Recall that the sets $A = (a_1, \ldots, a_{f(t,p)})$ and $T = (t_1, \ldots, t_{f(t,p)})$ are ordered sets. Set $A' = A_{L'}$ and B' be the corresponding vertices of B, that is $b_i \in B'$ if and only if $a_i \in A'$. From the choice of L', $|A'| = |B'| \geq (|A| - \mathcal{R}'(2t))/2^{\mathcal{R}'(2t)} = f(t, p-1)$. We will now show that the length of a longest (A', B')-path in $D_{\mathtt{contr}}$ is at most $p - 1$.

Claim 4.28. *Let $a_i \in A'$ and $b_j \in B'$ such that $i \neq j$. Let $P = (a_i, x_1, \ldots, x_q, b_j)$ be a longest (A', B')-path in $D_{\mathtt{contr}}$. Then $x_1 \notin L$.*

Proof. Since $b_j \in B'$, $a_j \in A'$ (from construction of B'). First observe that since $b_j \in B'$, then $a_j \in A'$. For the sake of contradiction, suppose $x_1 \in L$. Since $(a_i, x) \in E(D_{\mathtt{contr}})$ and all the vertices of A' have the same neighbourhood in Z_L and $a_j \in A'$, we conclude that $(a_j, x) \in E(D_{\mathtt{contr}})$. Thus, there is an (a_j, b_j)-path in $D_{\mathtt{contr}}$ of length strictly greater than one, which contradicts that (a_j, b_j) is a minimal edge of $D_{\mathtt{contr}}$. ⌟

From Claim 4.28, the first internal vertex of every longest (A', B')-path is not contained in L. Now suppose that there exists an (A', B')-path of length p. Then its first internal vertex should be in L by the definition of L. This is a contradiction. □

Lemma 4.24 proves Lemma 4.9 when the length of the longest path $p \leq 2t + 2$.

4.3 Cleaning the ordered tough-pair

The cleaning of ordered tough-pairs is done in two steps: we first show that an ordered tough-pair can be identified to a so-called *semi-cleaned ordered tough-pair* (see Section 4.3.1) and show thereafter how to further identify a semi-cleaned ordered tough-pair to a hard matching pattern (see Section 4.3.2).

4.3.1 Semi-cleaning

Recall the definition of a semi-cleaned ordered t-tough-pair from Section 4. The goal of this section is to prove Lemma 4.5 restated below.

Lemma 4.5 (Semi-cleaning the ordered tough-pair)**.** *Let D be a digraph such that D contains an ordered t^2-tough-pair, then one can obtain $\widehat{D}$ from D by identification, such that $\widehat{D}$ is a semi-cleaned t-ordered tough-pair, for some function g that depends only on t.*

To prove Lemma 4.5, we first define a type of a vertex in D with respect to the ordered t-tough-pair (A, B). This is based on the neighbourhoods of a vertex in the head sets and tail sets of A and B. We then define a notion of a bad type. The vertices having a bad type hinder the semi-cleaning procedure. To overcome this, we show that if a digraph D contains an ordered t^2-tough-pair then it also contains an ordered t-tough-pair (A', B') such that there are no bad vertices with respect to (A', B'). We then show that given such an ordered tough-pair (A', B') we can identify the remaining vertices to one of the endpoints of $A' \cup B'$.

Bad vertices with respect to the ordered tough-pair (A, B). Given an ordered tough-pair (A, B) in a digraph D, we say that a vertex $v \in V(D) \setminus V(A \cup B)$ is *bad* with respect to (A, B) if $v \in N^+_{D^\star}(\text{tail}(A)) \cap N^-_{D^\star}(\text{head}(A)) \cap N^+_{D^\star}(\text{tail}(B)) \cap N^-_{D^\star}(\text{head}(B))$.

Lemma 4.29 (Eliminating bad vertices)**.** *Let D be a digraph such that D contains an ordered t^2-tough-pair. Then D contains an ordered t-tough-pair (A', B') such that there is no bad vertex in $V(D) \setminus V(A' \cup B')$ with respect to (A', B').*

Proof. Let (A, B) be an ordered t^2-tough-pair in D and let $t' = t^2$. Let $A = ((w_1, x_1), \dots, (w_{t'}, x_{t'}))$ and $B = ((y_1, z_1), \dots, (y_{t'}, z_{t'}))$. For each vertex $v \in V(H) \setminus V(A \cup B)$ which is bad with respect to (A, B), let $W(v)$ be the largest index in $[t']$ such that $(a_{W(v)}, v) \in E(D)$, $X(v)$ be the smallest index in $[t']$ such that $(v, b_{X(v)}) \in E(D)$, $Y(v)$ be the largest index in $[t']$ such that $(c_{Y(v)}, v) \in E(D)$, and $Z(v)$ be the smallest index in $[t']$ such that $(v, d_{Z(v)}) \in E(D)$. For every $p, q \in [\sqrt{t'}]$, let $V_{p,q}$ be the set of bad vertices v with respect to (A, B) such that $W(v), X(v) \in \{(p-1)\sqrt{t'} + 1, \dots, p\sqrt{t'}\}$, and $Y(v), Z(v) \in \{(q-1)\sqrt{t'}, \dots, q\sqrt{t'}\}$.

If there exists $p, q \in [\sqrt{t'}]$ such that $V_{pq} = \emptyset$, then let $A' = ((w_{(p-1)\sqrt{t'}+1}, x_{(p-1)\sqrt{t'}} + 1), \dots, (x_{p\sqrt{t'}}, x_{p\sqrt{t'}}))$ and $B' = ((c_{(q-1)\sqrt{t'}+1}, z_{(q-1)\sqrt{t'}+1}), \dots, (c_{q\sqrt{t'}}, z_{q\sqrt{t'}}))$. Then from the definition of $V_{p,q}$, (A', B') is an ordered $\sqrt{t'}$-tough-pair such that there are no bad vertices with respect to it.

Without loss of generality, assume that for each $p, q \in [\sqrt{t'}]$, $V_{pq} \neq \emptyset$. Let $v_{p,q}$ denote an arbitrarily fixed vertex in V_{pq}. Let $A' = ((w_{W(v_{2,1})}, v_{2,1}), \dots, (w_{W(v_{\sqrt{t'},1})}, v_{\sqrt{t'},1}))$ and $B' = ((y_{Y(v_{1,2})}, v_{1,2}), \dots, (y_{Y(v_{\sqrt{t'},1})}, v_{1,\sqrt{t'}}))$. We will now show that (A', B') is an ordered tough-pair in D. Moreover, we will show that there is no $(\text{tail}(B'), \text{head}(A'))$-path in D, thereby concluding that there is no bad vertex with respect to (A', B'). Note that the edges of A', B' are minimal edges because they belong to D which is a reachability minimal digraph.

Weak independence of A' (resp. B'). We show that there is no $(\mathrm{head}(A'), \mathrm{tail}(A'))$ path (resp. no $(\mathrm{head}(B'), \mathrm{tail}(B'))$ path) in D. Recall that every vertex in $\mathrm{head}(A')$ is a bad vertex with respect to (A, B). Thus, each vertex of $\mathrm{tail}(A')$ is an out-neighbour of some vertex of $\mathrm{tail}(B')$. Thus, a $(\mathrm{head}(A'), \mathrm{tail}(A'))$-path in D implies a $(\mathrm{tail}(B), \mathrm{tail}(A'))$-path in D. Since $\mathrm{tail}(A') \subseteq \mathrm{tail}(A)$, this contradicts that every edge of A is strongly independent with every edge of B. The other case is symmetric.

Weak independence between A' and B'. We show that a pair of edges containing an edge of A' and an edge of B' is weakly independent. To see that there is no $(\mathrm{head}(B'), \mathrm{tail}(A'))$-path, recall that every vertex of $\mathrm{head}(B')$ is a bad vertex with respect to (A, B). Thus, each vertex of $\mathrm{head}(B')$ is an out-neighbour of some vertex of $\mathrm{tail}(B)$. Thus, a $(\mathrm{head}(B'), \mathrm{tail}(A'))$-path in D implies a $(\mathrm{tail}(B), \mathrm{head}(A'))$-path in D. Since $\mathrm{tail}(A') \subseteq \mathrm{tail}(A)$, this is a contradiction. One can symmetrically prove that there is no $(\mathrm{head}(A'), \mathrm{tail}(B'))$-path.

Strong independence of A' and B'. We prove the following three statements.

There is no $(tail(A'), tail(B'))$-path and no $(tail(B'), tail(A'))$-path. Since $\mathrm{tail}(A') \subseteq \mathrm{tail}(A)$ and $\mathrm{tail}(B') \subseteq \mathrm{tail}(B)$, and (A, B) is a tough-pair, we conclude that there is no $(\mathrm{tail}(A'), \mathrm{tail}(B'))$-path and no $(\mathrm{tail}(B'), \mathrm{tail}(A'))$-path.

There is no $(head(A'), head(B'))$-path. For the sake of contradiction, suppose there exists $v_{i,1} \in \mathrm{head}(B')$ and $v_{1,j} \in \mathrm{head}(B')$, $i, j \neq 1$, such that there exists a $(v_{i,1}, v_{1,j})$-path in D. By definition $w(v_{i,1}) \in \{2\sqrt{t'}+1, \ldots, t'\}$ and $X(v_{1,j}) \in \{1, \ldots, \sqrt{t'}\}$. Also, $(w_{W(v_{i,1})}, v_{i,1}) \in E(D)$ and $(v_{1,j}, x_{X(v_{1,j})}) \in E(D)$. Thus, a $(v_{i,1}, v_{1,j})$-path in D implies a $(w_{W(v_{i,1})}, y_{Y(v_{1,j})})$-path in D. Since $X(v_{1,j}) < W(v_{i,1})$, this is a contradiction.

There is no $(head(B'), head(A'))$-path. For the sake of contradiction, suppose there exists $v_{i,1} \in \mathrm{head}(A')$ and $v_{1,j} \in \mathrm{head}(B')$, $i, j \neq 1$, such that there exists a $(v_{1,j}, v_{i,1})$-path in D. By definition $Y(v_{1,j}) \in \{2\sqrt{t'}+1, \ldots, t'\}$ and $Z(v_{i,1}) \in \{1, \ldots, \sqrt{t'}\}$. Also, $(y_{Y(v_{1,j})}, v_{1,j}) \in E(D)$ and $(v_{i,1}, z_{Z(v_{i,1})}) \in E(D)$. Thus, a $(v_{1,j}, v_{i,1})$-path in D implies a $(y_{Y(v_{1,j})}, z_{Z(v_{i,1})})$-path in D. Since $Z(v_{i,1}) < Y(v_{1,j})$, this is a contradiction.

Ordered condition on A' (resp. B'). Consider vertices $v_{i,1}$ and $v_{j,1}$ such that $i \leq j$. Suppose for the sake of contradiction that there exists a $(v_{j,1}, v_{i,1})$-path in D. Recall $(w_{W(v_{j,1})}, v_{j,1}) \in E(D)$ and $(v_{i,1}, x_{X(v_{i,1})}) \in E(D)$. Thus, there exists a $(w_{W(v_{j,1})}, y_{Y(v_{i,1})})$-path in D. Since $i \leq j$, $W(v_{j,1}) \leq X(v_{i,1})$. This is a contradiction. The other case can be proved symmetrically.

There is no $(\mathbf{tail}(B'), \mathbf{head}(A'))$-path. For the sake of contradiction, let $v_{i,1} \in \mathrm{head}(A')$, $i \neq 1$, and $y_j \in \mathrm{tail}(B')$ such that there exists a $(y_j, v_{i,1})$-path in D. Since $y_j \in \mathrm{tail}(B')$, $j \in \{2\sqrt{t'}+1, \ldots, t'\}$. Also $Z(v_{i,1}) \in \{1, \ldots, \sqrt{t}\}$. Since $(v_{i,1}, z_{Z(v_{i,1})}) \in E(D)$, we conclude that there is a $(y_j, z_{Z(v_{i,1})})$-path. This is a contradiction which concludes the proof. $\square$

Lemma 4.30. *Let D be a directed graph and let (A, B) be an ordered t-tough-pair in D such that there are no bad vertices with respect to (A, B) in D. Then one can obtain a semi-cleaned ordered t-tough-pair from D by identification.*

The remainder of this section is devoted to the proof of Lemma 4.30. Note that Lemma 4.30, together with Lemma 4.29, proves Lemma 4.5.

Recall that A, B are ordered sets. Let $\mathrm{tail}(A) = W = (w_1, \ldots, w_t)$, $\mathrm{head}(A) = X = (x_1, \ldots, x_t)$, $\mathrm{tail}(B) = Y = (y_1, \ldots, y_t)$ and $\mathrm{head}(B) = Z = (z_1, \ldots, z_t)$. We will now describe a procedure of identifying vertices in $V(D) \setminus V(A \cup B)$ onto the vertices of $V(A \cup B)$, as long as there is a vertex that has both an in-neighbour and an out-neighbour in $V(A \cup B) = W \cup X \cup Y \cup Z$ in the graph D^*, while maintaining the invariants that after the identification (A, B) remains an ordered tough-pair in the new graph, no vertex in the new graph is bad with respect to $(A \cup B)$ and the new graph is acyclic. We will apply these identification rules exhaustively, in order.

Invariants. *If $\widehat{D}$ is the graph obtained after identification, then (A, B) is an ordered tough-pair in $\widehat{D}$, and there are no bad vertices with respect to (A, B) in $\widehat{D}$.*

Identification Rule 1. *If there exists $v \in V(D) \setminus (W \cup X \cup Y \cup Z)$ such that $N^{+}_{D^\star}(v) \cap W \neq \emptyset$ and $N^{-}_{D^\star}(v) \cap W \neq \emptyset$, then let $i \in [t]$ be the largest index such that $(w_i, v) \in E(D^\star)$. Identify v onto w_i.*

Lemma 4.31. *Let $\widehat{D}$ be the graph obtained after the application of Identification Rule 1. Then, the invariants are satisfied.*

Proof. Since $N^{+}_{D^\star}(v) \cap W \neq \emptyset$ and $N^{-}_{D^\star}(v) \cap W \neq \emptyset$, there is a (v, W)-path, say $P_{\mathtt{out}}$, and a (W, v)-path say $P_{\mathtt{in}}$ in D.

Weak independence of A in $\widehat{D}$. For the sake of contradiction, say there is a (X, w_i)-path in $\widehat{D}$. Then there exists a (X, v)-path in D. This, together with the (v, W)-path $P_{\mathtt{out}}$, implies an (X, W)-path in D, which is a contradiction to the weak independence of the edges of A in D.

Weak independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (Z, w_i)-path in $\widehat{D}$. Then there exists a (Z, v)-path in D. This, together with (v, W)-path $P_{\mathtt{out}}$, implies an (Z, W)-path in D, which is a contradiction to the weak independence between the edges of A, B in D.

Strong independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (w_i, Y)-path in $\widehat{D}$. Then there exists a (v, Y)-path in D. This, together with the $(W, v$-path $P_{\mathtt{in}}$ in D, implies an (W, Y)-path in D, which is a contradiction to the strong independence between the edges of A, B in D.

For the sake of contradiction, say there is a (Y, w_i)-path in $\widehat{D}$. Then there exists a (Y, v)-path in D. This, together with the (v, W)-path $P_{\mathtt{out}}$, implies an (Y, W)-path in D, which is a contradiction to the strong independence between the edges of A, B in D.

Orderedness. Fix $j < i$. Suppose, for the sake of contradiction, that there is a (w_i, w_j)-path in $\widehat{D}$. Then, there is a (v, w_j)-path in D. This, together with the (w_i, v)-path in D, implies a (w_i, w_j)-path in D, contradicting the ordered-ness condition on the edges of A.

For the sake of contradiction, for $j < i$, say there is a (w_i, x_j)-path in $\widehat{D}$. Then there is a (v, x_j)-path in D, which together with the (w_i, v)-path in D contradicts the ordered-ness condition on the edges of A.

No bad vertices. For the sake of contradiction, say there exists a bad vertex u with respect to (A, B) in $\widehat{D}$, that is, $N^{-}_{\widehat{D}^\star}(u) \cap W, N^{+}_{\widehat{D}^\star}(u) \cap X, N^{-}_{\widehat{D}^\star}(u) \cap Y, N^{+}_{\widehat{D}^\star}(u) \cap Z \neq \emptyset$. Then, considering that $\widehat{D}$ is obtained from D by only identifying v onto w_i, we get that $N^{+}_{D^\star}(u) \cap X, N^{-}_{D^\star}(u) \cap Y, N^{+}_{D^\star}(u) \cap Z \neq \emptyset$ and $(v, u) \in E(D^\star)$. The (v, u)-path, together with the (W, v)-path $P_{\in}$, implies a (W, u)-path in D, that is $N^{-}_{D^\star}(u) \cap W \neq \emptyset$. That is, u is a bad vertex with respect to (A, B) in D, which is a contradiction. □

Identification Rule 2 is the analogue of Identification Rule 1.

Identification Rule 2. *If there exists $v \in V(D) \setminus (W \cup X \cup Y \cup Z)$ such that $N^{+}_{D^\star}(v) \cap Y \neq \emptyset$ and $N^{-}_{D^\star}(v) \cap Y \neq \emptyset$, then let $i \in [t]$ be the largest index such that $(y_i, v) \in E(D^\star)$. Identify v onto y_i.*

Symmetrically to Lemma 4.31, the following holds.

Lemma 4.32. *Let $\widehat{D}$ be the graph obtained after the application of Identification Rule 2. Then, the invariants are satisfied.*

Identification Rule 3. *If there exists $v \in V(D) \setminus (W \cup X \cup Y \cup Z)$ such that $N^-_{D^\star}(v) \cap X \neq \emptyset$ and $N^+_{D^\star}(v) \cap X \neq \emptyset$, then let $i \in [t]$ be the largest index such that either $(w_i, v) \in E(D^\star)$ or $(x_i, v) \in E(D^\star)$. Identify v onto x_i.*

Lemma 4.33. *Let $\widehat{D}$ be the graph obtained after the application of Identification Rule 3. Then, the invariants are satisfied.*

Proof. Since $N^-_{D^\star}(v) \cap X \neq \emptyset$ and $N^+_{D^\star}(v) \cap X \neq \emptyset$, here is a (X, v)-path, say $P_{\texttt{in}}$, and a (v, X)-path say $P_{\texttt{out}}$ in D.

Weak independence of A in $\widehat{D}$. For the sake of contradiction, say there is a (x_i, w_j)-path in $\widehat{D}$. Then there exists a (v, w_j)-path, say P, in D. From the choice of i and since $N^-_{D^\star}(v) \cap X \neq \emptyset$, there exists $i' \leq i$, such that there is a $(x_{i'}, v)$-path in D. This implies a $(x_{i'}, w_j)$-path in D. Since $i' < i$ and A is weakly independent in D, $j < i$.

From the choice of i, either $(w_i, v) \in E(D^*)$, in which case the existence of P implies a (w_i, w_j)-path, $j < i$, in D, contradicting the weak independence of A in D. Or, $(x_i, v) \in E(D^\star)$, in which case P implies a (x_i, w_j)-path, $i < i$ in D, again contradicting the weak independence of A in D.

Weak independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (x_i, Y)-path in $\widehat{D}$. Then there exists a (v, Y)-path in D. This, together with (X, v)-path $P_{\texttt{in}}$, implies an (X, Y)-path in D, which is a contradiction to the weak independence between the edges of A, B in D.

Strong independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (Z, x_i)-path in $\widehat{D}$. Then there exists a (Z, v)-path in D. This, together with the (v, X)-path $P_{\texttt{out}}$ in D, implies an (Z, X)-path in D, which is a contradiction to the strong independence between the edges of A, B in D.

For the sake of contradiction, say there is a (x_i, Z)-path in $\widehat{D}$. Then there exists a (v, Z)-path in D. This, together with the (X, v)-path $P_{\texttt{in}}$, implies an (X, Z)-path in D, which is a contradiction to the strong independence between the edges of A, B in D.

Orderedness. Suppose, for the sake of contradiction, that there is a (w_j, x_i)-path in $\widehat{D}$ for $j > i$. Then, there is a (w_j, v)-path in D. Since $j > i$, this contradicts the choice of i.

For the sake of contradiction, say there is a (x_j, x_i)-path in $\widehat{D}$ for $j > i$. Then, there is a (x_j, v)-path in D. Since $j > i$, this contradicts the choice of i.

For the sake of contradiction, say there is a (x_i, x_j)-path in $\widehat{D}$ for $j < i$. Then, there is a (v, x_j)-path in D. This together with either the (x_i, v)-path or the (w_i, v)-path, implies either a (x_i, x_j)-path or a (w_i, x_j)-path in D, contradicting the ordered-ness condition on A.

No bad vertices. For the sake of contradiction, say there exists a bad vertex u with respect to (A, B) in $\widehat{D}$, that is, $N^-_{\widehat{D}^\star}(u) \cap W, N^+_{\widehat{D}^\star}(u) \cap X, N^-_{\widehat{D}^\star}(u) \cap Y, N^+_{\widehat{D}^\star}(u) \cap Z \neq \emptyset$. Then, considering that $\widehat{D}$ is obtained from D by only identifying v onto x_i, we get that $N^-_{D^\star}(u) \cap W, N^-_{D^\star}(u) \cap Y, N^+_{D^\star}(u) \cap Z \neq \emptyset$ and $(u, v) \in E(D^\star)$. The (u, v)-path, together with the (v, X)-path $P_{\texttt{out}}$, implies a (u, X)-path in D, that is $N^+_{D^\star}(u) \cap X \neq \emptyset$. That is, u is a bad vertex with respect to (A, B) in D, which is a contradiction. $\square$

Identification Rule 4 is the analogue of Identification Rule 3.

Identification Rule 4. *If there exists $v \in V(D) \setminus (W \cup X \cup Y \cup Z)$ such that $N^-_{D^\star}(v) \cap Z \neq \emptyset$ and $N^+_{D^\star}(v) \cap Z \neq \emptyset$, then let $i \in [t]$ be the largest index such that either $(y_i, v) \in E(D^\star)$ or $(z_i, v) \in E(D^\star)$. Identify v onto z_i.*

Symmetrically to Lemma 4.34, the following holds.

Lemma 4.34. *Let $\widehat{D}$ be the graph obtained after the application of Identification Rule 3. Then, the invariants are satisfied.*

The remaining identification rules are applied on a vertex v when the (in and out) neighbourhood of v is empty in one of the sets W, X, Y or Z.

Identification Rule 5. *If $N_{D^\star}(v) \cap W = \emptyset$ and $\emptyset \neq N^-_{D^\star}(v) \cap (X \cup W \cup Y \cup Z) \subseteq Y$, then let $i \in [t]$ be the largest integer such that $(y_i, v) \in E(D^*)$. Identify v onto y_i.*

Lemma 4.35. *Let $\widehat{D}$ be the graph obtained after the application of Identification 5. Then, $\widehat{D}$ satisfies the invariants.*

Proof. **Weak independence of B in $\widehat{D}$.** For the sake of contradiction, say there is a (Z, y_i)-path in $\widehat{D}$. Then there exists a (Z, v)-path, in D, which is a contradiction (to the assumptions of Identification Rule 5).

Weak independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (X, y_i)-path in $\widehat{D}$. Then there exists a (X, v)-path in D, which is a contradiction (to the assumptions of Identification Rule 5).

Strong independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (W, y_i)-path in $\widehat{D}$. Then there exists a (W, v)-path in D, which is a contradiction (to the assumptions of Identification Rule 5). Similarly, if there is a (y_i, W)-path in $\widehat{D}$, then there exists a (v, W)-path in D, which is again a contradiction.

Orderedness. Fix $j < i$. For the sake of contradiction,say there is a (y_i, y_j)-path in $\widehat{D}$. Then, there is a (v, y_j)-path in D. This together with the (y_i, v)-path implies a (y_i, y_j)-path in D, which contradicts the ordered-ness condition on B.

Similarly, if there is a (y_i, z_j)-path in $\widehat{D}$, then there is a (v, z_j)-path in D. This, together with the (y_i, v)-path, implies a (y_i, z_j)-path in D, again contradicting the ordered-ness condition on B.

No bad vertices. For the sake of contradiction, say there exists a bad vertex u with respect to (A, B) in $\widehat{D}$, that is, $N^-_{\widehat{D}^\star}(u) \cap W, N^+_{\widehat{D}^\star}(u) \cap X, N^-_{\widehat{D}^\star}(u) \cap Y, N^+_{\widehat{D}^\star}(u) \cap Z \neq \emptyset$. Then, considering that $\widehat{D}$ is obtained from D by only identifying v onto y_i, we get that $N^-_{D^\star}(u) \cap W, N^+_{D^\star}(u) \cap X, N^+_{D^\star}(u) \cap Z \neq \emptyset$ and $(v, u) \in E(D^\star)$. The (v, u)-path, together with the (v, y_i)-path, implies a (u, Y)-path in D, that is $N^-_{D^\star}(u) \cap Y \neq \emptyset$. That is, u is a bad vertex with respect to (A, B) in D, which is a contradiction. □

The following identification rule is symmetric to Identification Rule 5.

Identification Rule 6. *If $N_{D^\star}(v) \cap Y = \emptyset$ and $\emptyset \neq N^-_{D^\star}(v) \cap (X \cup W \cup Y \cup Z) \subseteq W$, then let $i \in [t]$ be the largest integer such that $(x_i, v) \in E(D^*)$. Identify v onto w_i.*

Similarly to Lemma 4.35, we can prove the following.

Lemma 4.36. *Let $\widehat{D}$ be the graph obtained after the application of Identification 6. Then, $\widehat{D}$ satisfies the invariants.*

Identification Rule 7. *If $N_{D^\star}(v) \cap X = \emptyset$ and $\emptyset \neq N^+_{D^\star}(v) \cap (X \cup W \cup Y \cup Z) \subseteq Z$, then let $i \in [t]$ be the smallest integer such that $(v, z_i) \in E(D^*)$. Identify v onto z_i.*

Lemma 4.37. *Let $\widehat{D}$ be the graph obtained after the application of Identification 7. Then, $\widehat{D}$ satisfies the invariants.*

Proof. **Weak independence of B in $\widehat{D}$.** For the sake of contradiction, say there is a (z_i, Y)-path in $\widehat{D}$. Then there exists a (v, Y)-path, in D, which is a contradiction (to the assumptions of Identification Rule 7).

Weak independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (z_i, W)-path in $\widehat{D}$. Then there exists a (v, W)-path in D, which is a contradiction (to the assumptions of Identification Rule 7).

Strong independence between A, B in $\widehat{D}$. For the sake of contradiction, say there is a (z_i, X)-path in $\widehat{D}$. Then there exists a (v, X)-path in D, which is a contradiction (to the assumptions of Identification Rule 7). Similarly, if there is a (X, z_i)-path in $\widehat{D}$, then there exists a (X, v)-path in D, which is again a contradiction.

Orderedness. For the sake of contradiction,say there is a (z_i, z_j)-path in $\widehat{D}$ for $j < i$.. Then, there is a (v, z_j)-path in D. This together with the (z_i, v)-path implies a (z_i, z_j)-path in D, which contradicts the ordered-ness condition on B.

If there is a (z_i, y_j)-path in $\widehat{D}$, then there is a (v, y_j)-path in D. This, together with the (y_i, v)-path, implies a (y_i, y_j)-path in D, again contradicting the ordered-ness condition on B.

No bad vertices. For the sake of contradiction, say there exists a bad vertex u with respect to (A, B) in $\widehat{D}$, that is, $N^-_{\widehat{D}^\star}(u) \cap W, N^+_{\widehat{D}^\star}(u) \cap X, N^-_{\widehat{D}^\star}(u) \cap Y, N^+_{\widehat{D}^\star}(u) \cap Z \neq \emptyset$. Then, considering that $\widehat{D}$ is obtained from D by only identifying v onto z_i, we get that $N^-_{D^\star}(u) \cap W, N^+_{D^\star}(u) \cap X, N^-_{D^\star}(u) \cap Y \neq \emptyset$ and $(u, v) \in E(D^\star)$. The (u, v)-path, together with the (v, z_i)-path, implies a (u, Z)-path in D, that is $N^+_{D^\star}(u) \cap Z \neq \emptyset$. That is, u is a bad vertex with respect to (A, B) in D, which is a contradiction. □

The following identification rule is symmetric to Identification Rule 7.

Identification Rule 8. *If $N_{D^\star}(v) \cap Z = \emptyset$ and $\emptyset \neq N^+_{D^\star}(v) \cap (X \cup W \cup Y \cup Z) \subseteq X$, then let $i \in [t]$ be the smallest integer such that $(v, x_i) \in E(D^*)$. Identify v onto x_i.*

Similarly to Lemma 4.37, we can prove the following.

Lemma 4.38. *Let $\widehat{D}$ be the graph obtained after the application of Identification 8. Then, $\widehat{D}$ satisfies the invariants.*

Lemma 4.39. *When none of the Identification Rules 1-8 are applicable, for each vertex of $v \in V(D) \setminus V(A \cup B)$ either $N^-_D(v) = \emptyset$ or $N^+_D(v) = \emptyset$.*

Proof. To prove the lemma, we prove the following claims.

Claim 4.40. *If $N^+_{D^\star}(v) \cap W \neq \emptyset$, then $N^-_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \subseteq W$.*

Proof. If $x_j \in N^-_{D^\star}(v)$ for some $j \in [t]$, then this implies a (X, W)-path in D, contradicting the weak independence of A. If $y_j \in N^-_{D^\star}(v)$ for some $j \in [t]$, then this implies a (Y, W)-path in D, contradicting the strong independence between A, B. If $z_j \in N^-_{D^\star}(v)$ for some $j \in [t]$, then this implies a (Z, W)-path in D, contradicting the weak independence between A, B. ⌟

Symmetrically to Claim 4.40, the following holds.

Claim 4.41. *If $N^+_{D^\star}(v) \cap Y \neq \emptyset$, then $N^-_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \subseteq Y$.*

Claim 4.42. *If $N^-_{D^\star}(v) \cap X \neq \emptyset$, then $N^+_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \subseteq X$.*

Proof. If $w_j \in N^+_{D^\star}(v)$ for some $j \in [t]$, then this implies a (X, W)-path in D, contradicting the weak independence of A. If $y_j \in N^+_{D^\star}(v)$ for some $j \in [t]$, then this implies a (X, Y)-path in D, contradicting the weak independence between A, B. If $z_j \in N^+_{D^\star}(v)$ for some $j \in [t]$, then this implies a (X, Z)-path in D, contradicting the strong independence between A, B. ⌟

Symmetrically to Claim 4.42, the following holds.

Claim 4.43. *If $N^-_{D^\star}(v) \cap Z \neq \emptyset$, then $N^+_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \subseteq Z$.*

From Claims 4.40 and 4.43, if Identification Rules 1-4 are not applicable and there exists $v \in W \cup X \cup Y \cup Z$ such that $N^+_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \neq \emptyset$ and $N^-_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \neq \emptyset$, then since there are no bad vertices with respect to (A, B), we conclude that either $N_{D^\star}(v) \cap W = \emptyset$ or $N_{D^\star}(v) \cap X = \emptyset$ or $N_{D^\star}(v) \cap Y = \emptyset$ or $N_{D^\star}(v) \cap Z = \emptyset$. Thus, Claims 4.44-4.47 prove the lemma.

Claim 4.44. *If Identification Rules 1-4 are no longer applicable, $N_{D^\star}(v) \cap W = \emptyset$ and $N^+_{D^\star}(v) \cap (X \cup Y \cup Z) \neq \emptyset$, then $N^-_{D^\star}(v) \cap (X \cup Y \cup Z) \subseteq Y$.*

Proof. We first show that that $N^-_{D^\star}(v) \cap X = \emptyset$. For the sake of contradiction, say $N^-_{D^\star}(v) \cap X \neq \emptyset$. Then, since Identification Rule 2 is not applicable, $N^+_{D^\star}(v) \cap X = \emptyset$. Also, $N^+_{D^\star}(v) \cap Y = \emptyset$, as otherwise there is a (X, Y)-path in D, contradicting the weak independence between A, B. Further, $N^+_{D^\star}(v) \cap Z = \emptyset$, as otherwise there is a (X, Z)-path in D, contradicting the strong independence between A, B.

Similarly, if $N^-_{D^\star}(v) \cap Z \neq \emptyset$. Then, since Identification Rule 2 is not applicable, $N^+_{D^\star}(v) \cap Z = \emptyset$. Also, $N^+_{D^\star}(v) \cap X = \emptyset$, as otherwise there is a (Z, X)-path in D, contradicting the strong independence between A, B. Further, $N^+_{D^\star}(v) \cap Y = \emptyset$, as otherwise there is a (Z, Y)-path in D, contradicting the weak independence of B. ⌟

Similarly to Claim 4.44, we can prove the following.

Claim 4.45. *If Identification Rules 1-4 are no longer applicable, $N_{D^\star}(v) \cap Y = \emptyset$ and $N^+_{D^\star}(v) \cap (X \cup W \cup Z) \neq \emptyset$, then $N^-_{D^\star}(v) \cap (W \cup X \cup Z) \subseteq W$.*

Claim 4.46. *If Identification Rules 1-4 are no longer applicable, $N_{D^\star}(v) \cap X = \emptyset$ and $N^-_{D^\star}(v) \cap (X \cup Y \cup Z) \neq \emptyset$, then $N^+_{D^\star}(v) \cap (W \cup Y \cup Z) \subseteq Z$.*

Proof. We first show that that $N^+_{D^\star}(v) \cap W = \emptyset$. For the sake of contradiction, say $N^+_{D^\star}(v) \cap W \neq \emptyset$. Then, since Identification Rule 2 is not applicable, $N^-_{D^\star}(v) \cap W = \emptyset$. Also, $N^-_{D^\star}(v) \cap Y = \emptyset$, as otherwise there is a (Y, W)-path in D, contradicting the strong independence between A, B. Further, $N^-_{D^\star}(v) \cap Z = \emptyset$, as otherwise there is a (Z, W)-path in D, contradicting the weak independence between A, B.

Similarly, if $N^+_{D^\star}(v) \cap Y \neq \emptyset$, then since Identification Rule 2 is not applicable, $N^-_{D^\star}(v) \cap Y = \emptyset$. Also, $N^-_{D^\star}(v) \cap W = \emptyset$, as otherwise there is a (W, Z)-path in D, contradicting the strong independence between A, B. Further, $N^-_{D^\star}(v) \cap Z = \emptyset$, as otherwise there is a (Z, Y)-path in D, contradicting the weak independence of B. ⌟

Similarly to Claim 4.46, we can prove the following.

Claim 4.47. *If Identification Rules 1-4 are no longer applicable, $N_{D^\star}(v) \cap Z = \emptyset$ and $N^-_{D^\star}(v) \cap (X \cup Y \cup Z) \neq \emptyset$, then $N^+_{D^\star}(v) \cap (W \cup X \cup Y) \subseteq X$.*

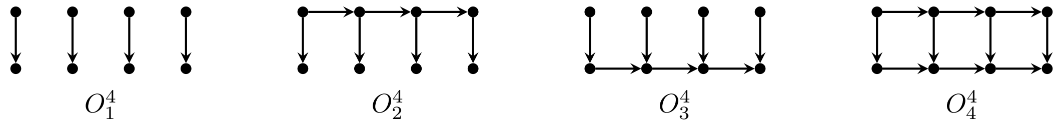

Figure 12: The graphs of Definition 4.49 for $t = 4$.

This concludes the proof of Lemma 4.39. □

Now suppose that Identification Rules 1-8 are no longer applicable. From Lemma 4.39, each vertex of $V(D) \setminus (W \cup X \cup Y \cup Z)$ either has only in-neighbours or only out-neighbours in $D^\star$. Let $V_{\mathtt{s}} = \{v : v \in V(D) \setminus (W \cup X \cup Y \cup Z)$ and $N^+_{D^\star}(v) \cap (W \cup X \cup Y \cup Z) \neq \emptyset\}$ and $V_{\mathtt{t}} = V(D) \setminus (W \cup X \cup Y \cup Z \cup V_{\mathtt{s}})$. Then identify all the vertices of $V_{\mathtt{s}}$ onto a new vertex $\mathtt{s}$ and all the vertices of $V_{\mathtt{t}}$ onto a new vertex $\mathtt{t}$. The resulting graph is a semi-cleaned ordered t-tough-pair.

4.3.2 From semi-cleaned ordered tough-pair to hard-pattern

The aim of this section is to prove Lemma 4.6. We first introduce the *almost t-hard matching patterns* which differ from the t-hard matching patterns in that the source and sink vertices of items 3 and 4, respectively, in Definition 1.4 may not have "full" neighborhoods (recall that an S-source is a source vertex s such that $N^+(s) = S$ and that an S-sink is a sink vertex such that $N^-(s) = S$).

Definition 4.48 (almost t-hard matching pattern)**.** An *almost t-hard matching pattern* is an (acyclic) digraph D constructed the following way. We start with disjoint vertex sets $W = \{w_1, \dots, w_t\}$, $X = \{x_1, \dots, x_t\}$, $Y = \{y_1, \dots, y_t\}$ and $Z = \{z_1, \dots, z_t\}$ and introduce the edges (w_i, x_i) and (y_i, z_i) for every $i \in [t]$. Furthermore, we introduce into D any combination of the following items:

1. either the directed path $w_1 \to w_2 \to \dots \to w_t \to z_1 \to z_2 \to \dots \to z_t$, or any of the directed paths $w_1 \to w_2 \to \dots \to w_t$ and $z_1 \to z_2 \to \dots \to z_t$;
2. either the directed path $y_1 \to y_2 \to \dots \to y_t \to x_1 \to x_2 \to \dots \to x_t$, or any of the directed paths $x_1 \to x_2 \to \dots \to x_t$ and $y_1 \to y_2 \to \dots \to y_t$;
3. a vertex s such that $N^-(s) = \emptyset$ and $N^+(s) \subseteq W \cup X \cup Y \cup Z$;
4. a vertex t such that $N^+(t) = \emptyset$ and $N^-(t) \subseteq W \cup X \cup Y \cup Z$;
5. a vertex r_{WZ} such that $N^-(r_{WZ}) = W$ and $N^+(r_{WZ}) = Z$;
6. a vertex r_{YX} such that $N^-(r_{YX}) = Y$ and $N^+(r_{YX}) = X$.

We further introduce the four canonical graphs below corresponding to "half" a hard matching pattern (see Figure 12 for an illustration).

Definition 4.49. Let $U = \{u_1, \dots, u_t\}$ and $V = \{v_1, \dots, v_t\}$. We define the following four reachability-minimal acyclic digraphs on vertex set $U \cup V$.

- The digraph O_1^t is a (U, V)-induced-matching.
- The digraph O_2^t is a (U, V)-matching which further contains the directed path on U from u_1 to u_t.
- The digraph O_3^t is a (U, V)-matching which further contains the directed path on V from v_1 to v_t.
- The digraph O_4^t is a (U, V)-matching which further contains the directed path on U from u_1 to u_t and the directed path on V from v_1 to v_t.

Lemma 4.6 (Cleaning the semi-cleaned ordered tough-pair)**.** *If D is a semi-cleaned ordered $h(t)$-tough-pair then either,*

- $D^\star$ *contains a t-induced-biclique whose edges are minimal in D, or*
- *D can be identified to a digraph $\hat{D}$ such that $\hat{D}$ is transitively equivalent to a t-hard-matching-pattern.*

Proof. For every $p \geq 1$ and $j \geq 1$, let $\ell_j(p)$ be an integer such that $p \geq \mathcal{R}(\ell_j(p), j)$. For notational convenience, we show, instead of the above statement, that if D is a semi-cleaned ordered t-tough-pair then either D^* contains a $g(t)$-induced-biclique whose edges are minimal in D, or D can be identified to a digraph $\hat{D}$ such that $\hat{D}$ is transitively equivalent to a $g(t)$-hard matching pattern, where $g(t) = (\ell_4(\ell_4(\ell_5(t))/2) - 4)/8$ (equivalently, $h(t) = \mathcal{R}(2\mathcal{R}(2\mathcal{R}(8t + 4, 4), 4), 5))$.

Let D be a semi-cleaned ordered t-tough-pair and let (A, B) be the ordered t-tough-pair contained in D, where $A = ((w_1, x_1), \ldots, (w_t, x_t))$ and $B = ((y_1, z_1), \ldots, (y_t, z_t))$. Denote by $W = \{w_i \mid i \in [t]\}$, $X = \{x_i \mid i \in [t]\}$, $Y = \{y_i \mid i \in [t]\}$ and $Z = \{z_i \mid i \in [t]\}$. Given a digraph H, we denote by $\overline{H}$ the underlying undirected graph and by H^* the transitive closure of H. Before turning to the proof of the lemma, we first show the following claims.

Claim 4.50. *One of the following holds.*

(1) There exist subsets $X' \subseteq X$ and $W' \subseteq W$ such that $|X'| = |W'| = \lfloor \ell_5(t)/2 \rfloor$ and D contains a (W', X')-induced-biclique.

(2) There exist a set $I \subseteq [t]$ of size $\ell_5(t)$ such that $D[\{x_i, w_i \mid i \in I\}]$ is transitively equivalent to one of $O_1^{\ell_5(t)}, O_2^{\ell_5(t)}, O_3^{\ell_5(t)}$ and $O_4^{\ell_5(t)}$.

Proof. Consider the edge-coloring of the complete undirected graph G on vertex set $\{1, \ldots, t\}$ defined as follows: for every $i, j \in [t]$ where $i < j$, the edge $ij \in E(G)$ receives

- color 1 if $(w_i, w_j), (x_i, x_j), (w_i, x_j) \notin E(D^*)$;
- color 2 if $(w_i, w_j) \in E(D)$ and $(x_i, x_j) \notin E(D^*)$;
- color 3 if $(w_i, w_j) \notin E(D)$ and $(x_i, x_j) \in E(D^*)$;
- color 4 if $(w_i, w_j), (x_i, x_j) \in E(D^*)$; and
- color 5 if $(w_i, w_j), (x_i, x_j) \notin E(D^*)$ and $(w_i, x_j) \in E(D^*)$.

Note that for any $1 \leq i < j \leq t$, if D^* contains one of (w_i, w_j) and (x_i, x_j) then D^* also contains (w_i, x_j) and thus, each edge of $E(G)$ receives a color in the above coloring. Now by Ramsey's Theorem, G contains a monochromatic clique of size $\ell_5(t)$: let $1 \leq i_1 < \ldots < i_{\ell_5(t)} \leq t$ be a set of $\ell_5(t)$ vertices inducing a monochromatic clique in G. Since (A, B) is an ordered t-tough-pair, for every $1 \leq p < q \leq \ell_5(t)$, there is no (w_{i_q}, w_{i_p})-path, no (w_{i_q}, x_{i_p})-path and no (x_{i_q}, x_{i_p})-path. It follows that if the clique in G on vertex set $\{i_1, \ldots, i_{\ell_5(t)}\}$ has

- color 1 then $D[\{w_{i_j}, x_{i_j} \mid j \in [\ell_5(t)]\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$;
- color 2 then $D[\{w_{i_j}, x_{i_j} \mid j \in [\ell_5(t)]\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$;
- color 3 then $D[\{w_{i_j}, x_{i_j} \mid j \in [\ell_5(t)]\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$;
- color 4 then $D[\{w_{i_j}, x_{i_j} \mid j \in [\ell_5(t)]\}]$ is transitively equivalent to $O_4^{\ell_5(t)}$;
- color 5 then D contains a $(\{w_{i_j} \mid j \in [\lfloor \ell_5(t)/2 \rfloor]\}, \{x_{i_{\lfloor \ell_5(t)/2 \rfloor + j}} \mid j \in [\lfloor \ell_5(t)/2 \rfloor]\})$-induced-biclique.

Thus, if the clique in G on vertex set $\{i_1, \ldots, i_{\ell_5(t)}\}$ does not have color 5 then we may take $I = \{i_1, \ldots, i_{\ell_5(t)}\}$ to prove our claim. ⌟

Symmetrically to Claim 4.50, we have the following.

Claim 4.51. *One of the following holds.*

(1) There exist subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $|Y'| = |Z'| = \lfloor \ell_5(t)/2 \rfloor$ and D contains a (Y', Z')-induced-biclique.

(2) There exist a set $I \subseteq [t]$ of size $\ell_5(t)$ such that $D[\{y_i, z_i \mid i \in I\}]$ is transitively equivalent to one of $O_1^{\ell_5(t)}, O_2^{\ell_5(t)}, O_3^{\ell_5(t)}$ and $O_4^{\ell_5(t)}$.

Claim 4.52. *For any subsets $W' \subseteq W$ and $Z' \subseteq Z$ where $|W'| = |Z'| = p$, there exist subsets $W'' \subseteq W'$ and $Z'' \subseteq Z'$ such that $|W''| = |Z''| = \lfloor \ell_4(p)/2 \rfloor$ and*
- *either there is no edge in D from a vertex of W'' to a vertex of Z'',*
- *or D^* contains a (W'', Z'')-biclique (not necessarily induced).*

Proof. Let $W' \subseteq W$ and $Z' \subseteq Z$ be two subsets of size p. We distinguish two cases depending on whether the bipartite graph $\overline{D}[W', Z']$ has a matching of size at least $p/2$ or not.

Case 1. The bipartite graph $\overline{D}[W', Z']$ has a matching of size at least $p/2$. Let M be a matching in $\overline{D}[W', Z']$ of size $\lfloor p/2 \rfloor$ and let $e_1, \dots, e_{\lfloor p/2 \rfloor}$ be an arbitrary ordering of the edges in M. Consider the edge-coloring of the complete undirected graph G on vertex set $\{1, \dots, \lfloor p/2 \rfloor\}$ defined as follows: for every $1 \le i < j \le \lfloor p/2 \rfloor$, the edge $ij \in E(G)$ receives
- color 1 if $(\text{tail}(e_i), \text{head}(e_j)), (\text{tail}(e_j), \text{head}(e_i)) \notin E(D^*)$;
- color 2 if $(\text{tail}(e_i), \text{head}(e_j)) \in E(D^*)$ and $(\text{tail}(e_j), \text{head}(e_i)) \notin E(D^*)$;
- color 3 if $(\text{tail}(e_i), \text{head}(e_j)) \notin E(D^*)$ and $(\text{tail}(e_j), \text{head}(e_i)) \in E(D^*)$; and
- color 4 if $(\text{tail}(e_i), \text{head}(e_j)), (\text{tail}(e_j), \text{head}(e_i)) \in E(D^*)$;

Then by Ramsey's Theorem, G contains a monochromatic clique of size $\ell_4(p)$: let $1 \le i_1 < \dots < i_{\ell_4(p)} \le p/2$ be a set of $\ell_4(p)$ vertices inducing a monochromatic clique in G. Now if this clique has
- color 1 then by taking $W'' = \{\text{tail}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$ and $Z'' = \{\text{head}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$, there is no edge from W'' to Y'' in D^* (and a fortiori in D);
- color 2 then by taking $W'' = \{\text{tail}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$ and $Z'' = \{\text{head}(e_{i_{\lfloor \ell_4(p)/2 \rfloor + j}}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$, D^* contains a (W'', Z'')-biclique;
- color 3 then by taking $W'' = \{\text{tail}(e_{i_{\lfloor \ell_4(p)/2 \rfloor + j}}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$ and $Z'' = \{\text{head}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$, D^* contains a (W'', Z'')-biclique;
- color 4 then by taking $W'' = \{\text{tail}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$ and $Z'' = \{\text{head}(e_{i_j}) \mid j \in [\lfloor \ell_4(p)/2 \rfloor]\}$, D^* contains a (W'', Z'')-biclique.

Case 2. The bipartite graph $\overline{D}[W', Z']$ has no matching of size at least $p/2$. Then by König's Theorem, $\overline{D}[W', Z']$ has, in this case, a vertex cover at size at most $p/2$: let $V \subseteq W' \cup Z'$ be a minimum vertex cover of $\overline{D}[W', Z']$. Since then, $\min\{W' \setminus V, Z' \setminus V\} \ge \lfloor \ell_4(p)/2 \rfloor \ge \lfloor \ell_4(p)/2 \rfloor$, we may take any subsets $W'' \subseteq W' \setminus V$ and $Z'' \subseteq Z' \setminus V$ of size $\lfloor \ell_4(p)/2 \rfloor$ to prove the claim, as there are surely no edges from W'' to Z'' in D. ⌟

Symmetrically to Claim 4.52, we have the following.

Claim 4.53. *For any subsets $Y' \subseteq Y$ and $X' \subseteq X$ where $|Y'| = |X'| = p$, there exist subsets $Y'' \subseteq Y'$ and $X'' \subseteq X'$ such that $|Y''| = |X''| = \lfloor \ell_4(p)/2 \rfloor$ and*
- *either there is no edge in D from a vertex of Y'' to a vertex of X'',*
- *or D^* contains a (Y'', X'')-biclique (not necessarily induced).*

We next define partitions with respect to a set and prove thereafter useful properties of these partitions.

Definition 4.54. For any $V \in \{W, X, Y, Z\}$ and any $U \subseteq V$, the *partition of V w.r.t. U* is the partition of $V \setminus U$ into four sets $V^\star, V^\circ, V^+$ and V^- such that the following hold.
- For every $v \in V^\star$, $N^+_{D^*}(v) \cap U \neq \emptyset$ and $N^-_{D^*}(v) \cap U \neq \emptyset$.
- For every $v \in V^\circ$, $(N^+_{D^*}(v) \cup N^-_{D^*}(v)) \cap U = \emptyset$.
- For every $v \in V^+$, $N^+_{D^*}(v) \cap U \neq \emptyset$ and $N^-_{D^*}(v) \cap U = \emptyset$.
- For every $v \in V^-$, $N^+_{D^*}(v) \cap U = \emptyset$ and $N^-_{D^*}(v) \cap U \neq \emptyset$.

Note that since (A, B) is an ordered tough-pair, for any $V \in \{W, X, Y, Z\}$ and any $v \in V$,
- if $u \in N^-_{D^*}(v)$ then u has a smaller index than that of v, and

- if $u \in N^+_{D^*}(v)$ then u has a larger index than that of v.

Claim 4.55. *For any $V \in \{W, X, Y, Z\}$ and any $U \subseteq V$, the following hold. Let $V^\star, V^\circ, V^+$ and V^- be the partition of V w.r.t U. Then there is no edge in D^**
(i) from a vertex of $V^\star$ to a vertex of $V^\circ \cup V^+$,
(ii) from a vertex of V^- to a vertex of $V^\star \cup V^\circ \cup V^+$, and
(iii) from a vertex of V° to a vertex of $V^+ \cup V^\star$.
Furthermore, if $D^*[U]$ has no edge then $V^\star = \emptyset$.

Proof. Consider $u \in V^\star$. If there exists $v \in V^\circ \cup V^+$ such that $(u, v) \in E(D^*)$ then in particular $\emptyset \neq N^-_{D^*}(u) \cap U \subseteq N^-_{D^*}(v)$, a contradiction to the fact that $v \in V^+ \cup V^\circ$. Similarly, there is no edge in D^* from a vertex of V^- to a vertex of $V^+ \cup V^\star$: indeed, if there exist $u \in V^-$ and $v \in V^+ \cup V^\star$ such that $(u, v) \in E(D^*)$ then in particular $\emptyset \neq N^-_{D^*}(u) \cap U \subseteq N^-_{D^*}(v)$, a contradiction to the fact that $v \in V^+$. Furthermore, there is no edge from a vertex of V^- to a vertex of V° for if there exist $u \in V^-$ and $v \in V^\circ$ such that $(u, v) \in E(D^*)$, then in particular $\emptyset \neq N^-_{D^*}(u) \cap U \subseteq N^-_{D^*}(v)$, a contradiction to the fact that $v \in V^\circ$. Finally, if there exist $u \in V^\circ$ and $v \in V^+ \cup V^\star$ such that $(u, v) \in E(D^*)$ then in particular $\emptyset \neq N^+_{D^*}(v) \cap U \subseteq N^+_{D^*}(u)$, a contradiction to the fact that $u \in V^\circ$. Now assume that $D^*[U]$ has no edge and suppose for a contradiction that there exists $u \in V \setminus U$ such that both $N^-_{D^*}(u) \cap U \neq \emptyset$ and $N^+_{D^*}(u) \cap U \neq \emptyset$. Then for any $x \in N^-_{D^*}(u) \cap U$ and $y \in N^+_{D^*}(u) \cap U$, $(x, y) \in E(D^*)$, a contradiction to the fact that U is an independent set of D^*. ⌟

Claim 4.56. *For any $V \in \{W, Y\}$ and any $U \subseteq V$, the following hold:*
- *if there exist $u \in W \cup X \cup Y \cup Z$ and $v \in V^+$ such that $(u, v) \in E(D^*)$ then $u \in V^+$;*
- *if there exist $u \in W \cup X \cup Y \cup Z$ and $v \in V^\circ$ such that $(u, v) \in E(D^*)$ then $u \in V^\circ \cup V^+$,*

where $V^\star, V^\circ, V^+, V^-$ is the partition of V w.r.t U.

Proof. Suppose that there exist $u \in W \cup X \cup Y \cup Z$ and $v \in V^+$ such that $(u, v) \in E(D^*)$. Observe first that since $V \in \{W, Y\}$ and (A, B) is an ordered tough-pair, necessarily $u \in V$. Now since $v \in V^+$, $u \notin U$ by definition and since by Claim 4.55, $u \notin V^\star \cup V^\circ \cup V^-$, we conclude that $u \in V^+$. Suppose next that there exist $u \in W \cup X \cup Y \cup Z$ and $v \in V^\circ$ such that $(u, v) \in E(D^*)$. Then as previously $u \in V$. Now since $v \in V^\circ$, $u \notin U$ by definition and since by Claim 4.55, $u \notin V^\star \cup V^-$, we conclude that $u \in V^\circ \cup V^+$. ⌟

Claim 4.57. *For any $V \in \{X, Z\}$ and any $U \subseteq V$, the following hold:*
- *if there exist $u \in V^-$ and $v \in W \cup X \cup Y \cup Z$ such that $(u, v) \in E(D^*)$ then $v \in V^-$;*
- *if there exist $v \in V^\circ$ and $v \in W \cup X \cup Y \cup Z$ such that $(u, v) \in E(D^*)$ then $u \in V^\circ \cup V^-$,*

where $V^\star, V^\circ, V^+, V^-$ is the partition of V w.r.t U.

Proof. Suppose that there exist $u \in V^-$ and $v \in W \cup X \cup Y \cup Z$ such that $(u, v) \in E(D^*)$. Observe first that since $V \in \{X, Z\}$ and (A, B) is an ordered tough-pair, necessarily $v \in V$. Now since $u \in V^-$, $v \notin U$ by definition and since by Claim 4.55, $v \notin V^\star \cup V^\circ \cup V^+$, we conclude that $v \in V^-$. Suppose next that there exist $v \in V^\circ$ and $v \in W \cup X \cup Y \cup Z$ such that $(u, v) \in E(D^*)$ then $u \in V^\circ \cup V^-$. Then as previously $v \in V$. Now since $u \in V^\circ$, $v \notin U$ by definition and since by Claim 4.55, $v \notin V^+ \cup V^\star$, we conclude that $v \in V^\circ \cup V^-$. ⌟

We now turn to the proof of the lemma. We assume henceforth that D contains no $g(t)$-induced-biclique (we are done otherwise). We first prove that D can be identified to a digraph which is transitively equivalent to an almost $4g(t)$-hard matching pattern and then show how to obtain a $g(t)$-hard matching pattern from an almost $4g(t)$-hard matching pattern.

By Claim 4.50, there exists a set $I_A \subseteq [t]$ of size $\ell_5(t)$ such that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to one of $O_1^{\ell_5(t)}, O_2^{\ell_5(t)}, O_3^{\ell_5(t)}$ and $O_4^{\ell_5(t)}$; and similarly, by Claim 4.51, there exists a set $I_B \subseteq [t]$ of size $\ell_5(t)$ such that $D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to one of $O_1^{\ell_5(t)}, O_2^{\ell_5(t)}, O_3^{\ell_5(t)}$ and $O_4^{\ell_5(t)}$. Now by Claim 4.52 there exist subsets $I_A^1 \subseteq I_A$ and $I_B^1 \subseteq I_B$ such that $|I_A^1| = |I_B^1| = \ell_4(\ell_5(t))/2$ and there is either no edge or every edge in D from $\{w_i \mid i \in I_A^1\}$ to $\{z_i \mid i \in I_B^1\}$. By Claim 4.53, there further exist subsets $I_A^2 \subseteq I_A^1$ and $I_B^2 \subseteq I_B^1$ such that $|I_A^2| = |I_B^2| = \ell_4(\ell_4(\ell_5(t))/2))/2 = 4g(t)+2$ and there is either no edge or every edge in D from $\{y_i \mid i \in I_A^2\}$ to $\{x_i \mid i \in I_B^2\}$.

Now let $W^\star, W^\circ, W^+, W^-$ be the partition of W w.r.t. $\{w_i \mid i \in I_A^2\}$, $X^\star, X^\circ, X^+, X^-$ be the partition of X w.r.t $\{x_i \mid i \in I_A^2\}$, $Y^\star, Y^\circ, Y^+, Y^-$ be the partition of Y w.r.t $\{y_i \mid i \in I_B^2\}$ and $Z^\star, Z^\circ, Z^+, Z^-$ be the partition of Z w.r.t $\{z_i \mid i \in I_B^2\}$. We devise the following identification rules.

Identification Rule 9. *Proceed as follows.*

1. *Identify every vertex in $W^\circ \cup W^+ \cup Y^\circ \cup Y^+$ (together with the source vertex $\mathtt{s}$ of D when it exists) to a single vertex s.*
2. *If $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$, further identity every vertex in X^+ to s.*
3. *If $D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$, further identity every vertex in Z^+ to s.*

Identification Rule 10. *Proceed as follows.*

1. *Identify every vertex in $X^\circ \cup X^- \cup Z^\circ \cup Z^-$ (together with the sink vertex $\mathtt{t}$ of D when it exists) to a single vertex t.*
2. *If $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$, further identity every vertex in W^- to t.*
3. *If $D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$, further identity every vertex in Y^- to t.*

Identification Rule 11. *For every consecutive $i < i' \in I_A^2 \cup \{1,t\}$ (that is, $(I_A^2 \cup \{1,t\}) \cap [i,i'] = \{i,i'\}$) and every $j \in [i,i']$, proceed as follows.*

1. *$D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$: if $w_j \in W^\star \cup W^-$ then identify w_j to w_i and if $x_j \in X^+$ then identify x_j to $w_{i'}$.*
2. *$D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$: if $w_j \in W^-$ then identify w_j to x_i and if $x_j \in X^\star \cup X^+$ then identify x_j to $x_{i'}$.*
3. *$D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_4^{\ell_5(t)}$: if $w_j \in W^\star \cup W^-$ then identify w_j to w_i and if $x_j \in X^\star \cup X^+$ then identify x_j to $x_{i'}$.*

Identification Rule 12. *For every consecutive $i < i' \in I_B^2 \cup \{1,t\}$ and every $j \in [i,i']$, proceed as follows.*

1. *$D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$: if $y_j \in Y^\star \cup Y^-$ then identify y_j to y_i and if $z_j \in Z^+$ then identify z_j to $y_{i'}$.*
2. *$D[\{y_i, x_i \mid i \in I_B\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$: if $y_j \in Y^-$ then identify y_j to z_i and if $z_j \in Z^\star \cup Z^+$ then identify z_j to $z_{i'}$.*
3. *$D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to $O_4^{\ell_5(t)}$: if $y_j \in Y^\star \cup Y^-$ then identify y_j to y_i and if $z_j \in Z^\star \cup Z^+$ then identify z_j to $z_{i'}$.*

Let D_I be the digraph resulting from an exhaustive application of Identification Rules 9-12. We aim to show that D_I can be identified to a digraph which is transitively equivalent to an almost $4g(t)$-hard matching pattern. To this end, we first prove the following claims and then distinguish cases depending on the graphs to which $D[\{w_i, x_i \mid i \in I_A\}]$ and $D[\{y_i, z_i \mid i \in I_B\}]$ are transitively equivalent, and whether there is no edge or every edge in D from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ (from $\{y_i \mid i \in I_A^2\}$ to $\{x_i \mid i \in I_B^2\}$, respectively).

Claim 4.58. *The vertex set of D_I consists of $\{w_i, x_i \mid i \in I_A^2\} \cup \{y_i, z_i \mid i \in I_B^2\}$ together with possibly s or t.*

Proof. We first prove that $V(D_I) \cap W = \{w_i \mid i \in I_A^2\}$. Clearly, after Identification Rule 9.1 has been applied, there are no more vertices from $W^\circ \cup W^+$ left. Now if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ then $W^\star = \emptyset$ by Claim 4.55 and once Identification Rule 10.2 has been applied, there are no more vertices from W^- left. Similarly, if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$ then $W^\star = \emptyset$ by Claim 4.55 and once Identification Rule 11.2 has been exhaustively applied, there are no more vertices from W^- left. Otherwise, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to either $O_2^{\ell_5(t)}$ or $O_4^{\ell_5(t)}$ in which case, an exhaustive application of Identification Rule 11.1 and Identification Rule 11.3, respectively, leaves no vertex from $W^\star \cup W^-$. We conclude symmetrically that $V(D_I) \cap Y = \{y_i \mid i \in I_B^2\}$.

Second, let us show that $V(D_I) \cap X = \{x_i \mid i \in I_A^2\}$. Clearly, after Identification Rule 10.1 has been applied, there are no more vertices from $X^\circ \cup X^-$ left. Now if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ then $X^\star = \emptyset$ by Claim 4.55 and once Identification Rule 9.2 has been applied, there are no more vertices from X^+ left. Similarly, if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$ then $X^\star = \emptyset$ by Claim 4.55 and once Identification Rule 11.1 has been exhaustively applied, there are no more vertices from X^+ left. Otherwise, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to either $O_3^{\ell_5(t)}$ or $O_4^{\ell_5(t)}$ in which case, an exhaustive application of Identification Rule 11.2 and Identification Rule 11.3, respectively, leaves no vertex from $X^\star \cup X^+$. We conclude symmetrically that $V(D_I) \cap Z = \{z_i \mid i \in I_B^2\}$. ⌟

Claim 4.59. *If s exists then $d^-_{D_I}(s) = 0$ and if t exists then $d^+_{D_I}(t) = 0$.*

Proof. Assume that s exists and suppose to the contrary that there exists $u \in V(D_I)$ such that $u \in N^-_{D_I}(s)$. Then there exist $v_s, v_u \in V(D)$ such that v_s has been identified to s, v_u has been identified to u and $(v_u, v_s) \in E(D)$. Since the source vertex $\mathtt{s}$ of D (if any) has in-degree zero in D, necessarily $v_s \neq \mathtt{s}$. Furthermore, v_s cannot belong to $W^\circ \cup W^+ \cup Y^\circ \cup Y^+$: indeed, if, say, $v_s \in W^\circ \cup W^+$ (the case where $v_s \in Y^\circ \cup Y^+$ is symmetric) then by Claim 4.56, $v_u \in W^\circ \cup W^+$, a contradiction as v_u is not identified to s. Thus, it must be that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ and $v_s \in X^+$, or $D[\{y_i, x_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ and $v_s \in Z^+$. Assume without loss of generality that the latter holds (the other case is symmetric). Then since (A, B) is an ordered tough-pair, necessarily $v_u \in Y \cup Z$. However, since v_u is not identified to s, $v_u \notin Y^\circ \cup Y^+$ and since $D * [\{y_i | \ i \in I_B\}]$ has no edge, $Y^\star = \emptyset$ by Claim 4.55. Furthermore, v_u cannot belong to $\{y_i \mid i \in I_A^2\} \cup Y^-$ for otherwise, since $v_s \in Z^+$ and (A, B) is an ordered tough-pair, there exist $i, j \in I_A^2$ such that $i < j$ and $(y_i, z_j) \in E(D^*)$, a contradiction to the fact that $D[\{y_i, x_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$. Thus, it must be that $v_u \in Z$. However, since $v_s \in Z^+$, $v_u \notin Z^\star \cup Z^- \cup Z^\circ$ by Claim 4.55 and $v_u \notin \{z_i \mid i \in I_B^2\}$ by definition of Z^+. Thus, we conclude that $v_u \in Z^+$, a contradiction since v_u is not identified to s.

Assume now that t exists and suppose to the contrary that there exists $u \in V(D_I)$ such that $u \in N^+_{D_I}(t)$. Then there exist $v_t, v_u \in V(D)$ such that v_t is identified to t, v_u is identified to u and $(v_t, v_u) \in E(D)$. Since the sink vertex $\mathtt{t}$ of D (if any) has out0degree zero in D, necessarily $v_t \neq \mathtt{t}$. Furthermore, v_t cannot belong to $X^\circ \cup X^- \cup Z^\circ \cup Z^-$: indeed, if, say, $v_t \in X^\circ \cup X^-$ (the case where $v_t \in Z^\circ \cup Z^-$ is symmetric) then by Claim 4.57, $v_u \in X^\circ \cup X^-$, a contradiction since v_u is not identified to t. Thus, it must be that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ and $v_t \in W^-$, or $D[\{y_i, z_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ and $v_t \in Y^-$. Assume without loss of generality that the latter holds (the other case is symmetric). Then since (A, B) is an ordered tough-pair, necessarily $v_u \in Y \cup Z$. However, since v_u is not identified to t, $v_u \notin Z^\circ \cup Z^-$ and since $D * [\{z_i | \ i \in I_B\}]$ has no edge, $Z^\star = \emptyset$ by Claim 4.55. Furthermore,

v_u cannot belong to $\{z_i \mid i \in I_B^2\} \cup Z^+$ for otherwise, since $v_t \in Y^-$ and (A, B) is an ordered tough-pair, there exist $i, j \in I_B^2$ such that $i < j$ and $(y_i, z_j) \in E(D^*)$, a contradiction to the fact that $D[\{y_i, x_i \mid i \in I_B\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$. Thus, it must be that $v_u \in Y$. However, since $v_t \in Y^-$, $v_u \notin Y^\star \cup Y^\circ \cup Y^+$ by Claim 4.55 and $v_u \notin \{y_i \mid i \in I_B^2\}$ by definition of Y^-. Thus, we conclude that $v_u \in Y^-$, a contradiction since v_u is not identified to t. ⌟

Claim 4.60. *For any $i \in [4]$, if $D[\{w_j, x_j \mid j \in I_A\}]$ is transitively equivalent to $O_i^{\ell_5(t)}$ then $D_I[\{w_j, x_j \mid j \in I_A^2\}]$ is transitively equivalent to $O_i^{4g(t)+2}$.*

Proof. Let us first show that if $D[\{w_i \mid i \in I_A\}]$ has no edge then $D_I[\{w_i \mid i \in I_A^2\}]$ has no edge as well. Assume that $D[\{w_i \mid i \in I_A\}]$ has no edge, that is, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ or $O_3^{\ell_5(t)}$. Then by Claim 4.55, $W^\star = \emptyset$. Suppose first that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$. Then every vertex in $W^\circ \cup W^+$ is identified to s and every vertex in W^- is identified to t where $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59. Since, in this case, every vertex in X is identified either to s, t or itself and no other identification may create an edge between two vertices of $\{w_i \mid i \in I_A^2\}$, we conclude that $D_I[\{w_i \mid i \in I_A^2\}]$ has no edge. Suppose second that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$. Then every vertex in $W^\circ \cup W^+$ is identified to s and every vertex in $X^\circ \cup X^-$ is identified to t where $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59. Since every vertex in $W^- \cup X^\star \cup X^+$ is identified to a vertex in $\{x_i \mid i \in I_A^2\}$ and no other identification may create an edge between two vertices of $\{w_i \mid i \in I_A^2\}$, we conclude that $D_I[\{w_i \mid i \in I_A^2\}]$ has no edge.

Second, let us show that if $D[\{w_i \mid i \in I_A\}]$ is transitively equivalent to the directed path $w_{\min I_A} \to w_{\min I_A + 1} \to \ldots \to w_{\max I_A}$ then $D_I[\{w_i \mid i \in I_A^2\}]$ is transitively equivalent to the directed path $w_{\min I_A^2} \to w_{\min I_A^2 + 1} \to \ldots \to w_{\max I_A^2}$. Assume that $D[\{w_i \mid i \in I_A\}]$ is transitively equivalent to this directed path, that is, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$ or $O_4^{\ell_5(t)}$. Note that since $I_A^2 \subseteq I_A$, it is enough to show that no edge from a vertex in $\{w_i \mid i \in I_A^2\}$ to a vertex in $\{w_i \mid i \in I_A^2\}$ of smaller index is created (we refer to such an edge as a *bad edge* in the following). First, since every vertex $w_j \in W^\star \cup W^-$ is identified to the vertex $w_{j'} \in \{w_i \mid i \in I_A^2\}$ such that $j' = \max\{i \in I_A^2 \ : \ i \leq j\}$, no bad edge is created here. Now if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$ (note that, in this case, $X^\star = \emptyset$ by Claim 4.55), then every vertex $x_j \in X^+$ is identified to the vertex $w_{j'} \in \{w_i \mid i \in I_A^2\}$ such that $j' = \min\{i \in I_A^2 \ : \ j \leq i\}$ and so, no bad edge is created here; and if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_4^{\ell_5(t)}$, then every vertex in $X^\star \cup X^+$ is identified to a vertex in $\{x_i \mid i \in I_A^2\}$ and thus, no bad edge is created here as well. Since every vertex in $W^\circ \cup W+$ is identified to s and every vertex in $X^\circ \cup X^-$ is identified to t where $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59, no bad edge is created with these identifications; and since no other identification may create a bad edge, we conclude that $D_I[\{w_i \mid i \in I_A^2\}]$ is indeed transitively equivalent to the directed path $w_{\min I_A^2} \to \ldots \to w_{\max I_A^2}$.

Consider next X. As in the case of W, let us first show that if $D[\{x_i \mid i \in I_A\}]$ has no edge then $D_I[\{x_i \mid i \in I_A^2\}]$ has no edge as well. Assume that $D[\{x_i \mid i \in I_A\}]$ has no edge, that is, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$ or $O_2^{\ell_5(t)}$. Then by Claim 4.55, $X^\star = \emptyset$. Now if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_1^{\ell_5(t)}$, we conclude, symmetrically to the case of W, that $D_I[\{x_i \mid i \in I_A^2\}]$ has no edge. Suppose therefore that $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_2^{\ell_5(t)}$. Since then, every vertex in $X^\circ \cup X^-$ is identified to t, every vertex in $W^\circ \cup W^+$ is identified to s where $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59, and every vertex in $X^+ \cup W^\star \cup W^-$ is identified to a vertex in $\{w_i \mid i \in I_A^2\}$, no edge between two vertices in $\{x_i \mid i \in I_A^2\}$ is created with these identifications. Since no other identification may create an edge between two vertices in $\{x_i \mid i \in I_A^2\}$, we conclude that $D_I[\{x_i \mid i \in I_A^2\}]$ has no edge in this case as well.

Let us next show that if $D[\{x_i \mid i \in I_A\}]$ is transitively equivalent to the directed path $x_{\min I_A} \to x_{\min I_A+1} \to \ldots \to x_{\max I_A}$ then $D_I[\{x_i \mid i \in I_A^2\}]$ is transitively equivalent to the directed path $x_{\min I_A^2} \to x_{\min I_A^2+1} \to \ldots \to x_{\max I_A^2}$. Assume that $D[\{x_i \mid i \in I_A\}]$ is transitively equivalent to this directed path, that is, $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$ or $O_4^{\ell_5(t)}$. Note that since $I_A^2 \subseteq I_A$, it is enough to show that no edge from a vertex in $\{x_i \mid i \in I_A^2\}$ to a vertex in $\{x_i \mid i \in I_A^2\}$ of smaller index is created (we refer to such an edge as a *bad edge* in the following). First, since every vertex $x_j \in X^\star \cup X^+$ is identified to the vertex $x_{j'} \in \{w_i \mid i \in I_A^2\}$ such that $j' = \min\{i \in I_A^2 \ : \ j \leq i\}$, no bad edge is created here. Now if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_3^{\ell_5(t)}$ (note that, in this case, $W^\star = \emptyset$ by Claim 4.55), then every vertex $w_j \in W^-$ is identified to the vertex $x_{j'} \in \{x_i \mid i \in I_A^2\}$ such that $j' = \max\{i \in I_A^2 \ : \ i \leq j\}$ and so, no bad edge is created here; and if $D[\{w_i, x_i \mid i \in I_A\}]$ is transitively equivalent to $O_4^{\ell_5(t)}$, then every vertex in $W^\star \cup W^-$ is identified to a vertex in $\{w_i \mid i \in I_A^2\}$ and thus, no bad edge is created here as well. Since every vertex in $X^\circ \cup X^-$ is identified to t and every vertex in $W^\circ \cup W+$ is identified to s where $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59, no bad edge is created with these identifications; and since no other identification may create a bad edge, we conclude that $D_I[\{x_i \mid i \in I_A^2\}]$ is indeed transitively equivalent to the directed path $x_{\min I_A^2} \to \ldots \to x_{\max I_A^2}$.

By combining the above four properties, we conclude that if $D[\{w_j, x_j \mid j \in I_A\}]$ is transitively equivalent to $O_i^{\ell_5(t)}$ for some $i \in [4]$, then $D_I[\{w_j, x_j \mid j \in I_A^2\}]$ is transitively equivalent to $O_i^{|I_A^2|}$; and since $|I_A^2| = 4g(t) + 2$, the claims follows. ⌟

Symmetrically to Claim 4.60, we have the following.

Claim 4.61. *For any $i \in [4]$, if $D[\{y_j, z_j \mid j \in I_B\}]$ is transitively equivalent to $O_i^{\ell_5(t)}$ then $D_I[\{y_j, z_j \mid j \in I_B^2\}]$ is transitively equivalent to $O_i^{4g(t)+2}$.*

Now note that since every vertex in A (B, respectively) is identified either to s, t or a vertex in A (B, respectively) and $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$ by Claim 4.59, no edge between A and B is created by these identifications that didn't already exist. This implies in particular that if there is no edge in D from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ and from $\{y_i \mid i \in I_B^2\}$ to $\{x_i \mid i \in I_A^2\}$ then by Claims 4.60 and 4.61, $D_I - \{s,t\}$ is transitively equivalent to the disjoint union of $O_i^{4g(t)+2}$ and $O_j^{4g(t)+2}$, where $O_i^{4g(t)+2}$ ($O_j^{4g(t)+2}$, respectively) is the digraph to which $D[\{w_a, x_a \mid a \in I_A\}]$ ($D[\{y_a, z_a \mid a \in I_B^2\}]$, respectively) is transitively equivalent. Thus, since we may safely identify, e.g., the two first edges of $D[\{w_i, x_i \mid i \in I_A^2\}]$ (that is, $(w_{\min I_A^2}, x_{\min I_A^2})$ and $(w_{\min I_A^2+1}, x_{\min I_A^2+1})$) to the third edge of $D[\{w_i, x_i \mid i \in I_A^2\}]$ and proceed similarly with $D[\{y_i, z_i \mid i \in I_B^2\}]$, we conclude that, in this case, D_I can be identified to an almost $4g(t)$-hard matching pattern. Note that, more generally, if there is no edge in D from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ then $D_I[\{w_i \mid i \in I_A^2\} \cup \{z_i \mid i \in I_B^2\}]$ is transitively equivalent to the disjoint union of $D_I[\{w_i \mid i \in I_A^2\}]$ and $D_I[\{z_i \mid i \in I_B^2\}]$ (in particular, no vertex r_{WZ} may be created by identifications); and the same holds for $\{y_i \mid i \in I_B^2\}$ and $\{x_i \mid i \in I_A^2\}$.

Assume henceforth that there is every edge either from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$, from $\{y_i \mid i \in I_B^2\}$ to $\{x_i \mid i \in I_A^2\}$, or both. Note that since we assume D not to contain a $g(t)$-induced-biclique, if there is every edge from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ then at least one of $D_I[\{w_i \mid i \in I_A^2\}]$ and $D_I[\{z_i \mid i \in I_B^2\}]$ has edges (in fact, is transitively equivalent to a directed path by Claims 4.60 and 4.61); and the same holds for $\{y_i \mid i \in I_B^2\}$ and $\{x_i \mid i \in I_A^2\}$. Now if there is every edge in D from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ and both $D_I[\{w_i \mid i \in I_A^2\}]$ and $D_I[\{z_i \mid i \in I_B^2\}]$ are transitively equivalent to a directed path, then $D_I\{w_i \mid i \in I_A^2\} \cup \{z_i \mid i \in I_B^2\}]$ is transitively equivalent to the directed path $w_{\min I_A^2} \to \dots \to w_{\max I_A^2} \to z_{\min I_B^2} \to \dots \to z_{\max I_B^2}$ in which case we need not introduce the vertex r_{WZ}; and we conclude symmetrically with $\{y_i \mid i \in I_B^2\}$ and $\{x_i \mid i \in I_A^2\}$.

By the above discussion, there remains to consider the case where every edge in D from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ (or from $\{y_i \mid i \in I_B^2\}$ to $\{x_i \mid i \in I_A^2\}$) exist and exactly one of $D_I[\{w_i \mid i \in I_A^2\}]$ and $D_I[\{z_i \mid i \in I_B^2\}]$ ($D_I[\{y_i \mid i \in I_B^2\}]$ and $D_I[\{x_i \mid i \in I_A^2\}]$, respectively) has no edge. In this case, we devise the following identification rules (for simplicity, we still call D_I the graph resulting from these identifications).

Identification Rule 13. *If there is every edge from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ in D_I and exactly one of $D_I^*[\{w_i \mid i \in I_A^2\}]$ and $D_I^*[\{z_i \mid i \in I_B^2\}]$ has no edge, then proceed as follows.*

1. *If $D_I^*[\{w_i \mid i \in I_A^2\}]$ has no edge then identify $y_{\min I_B^2}$ to s and rename $z_{\min I_B^2}$ as r_{WZ}.*
2. *If $D_I^*[\{z_i \mid i \in I_B^2\}]$ has no edge then identify $x_{\max I_A^2}$ to t and rename $w_{\max I_A^2}$ as r_{WZ}.*

Claim 4.62. *If Identification Rule 13 has been applied then it still holds that $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$. Furthermore, $N^-_{D_I}(r_{WZ}) \setminus \{s\} = \{w_i \mid i \in I_A^2\}$ and $N^+_{D_I}(r_{WZ}) \setminus \{t\} = \{z_i \mid i \in I_B^2 \setminus \{\min I_B^2\}\}$.*

Proof. Assume that Identification Rule 13 has been applied and that $D_I^*[\{w_i \mid i \in I_A^2\}]$ has no edge (the case where $D_I^*[\{z_i \mid i \in I_B^2\}]$ has no edge is symmetric). Then, by assumption, $D_I^*[\{z_i \mid i \in I_B^2\}]$ must have edges; and in fact, by Claim 4.61 and definition of I_B^2, $D_I^*[\{z_i \mid i \in I_B^2\}]$ must be transitively equivalent to the directed path $z_{\min I_B^2} \to z_{\min I_B^2+1} \to \dots \to z_{\max I_B^2}$. It follows that $N^+_{D_I}(r_{WZ}) \setminus \{t\} = N^+_{D_I}(z_{\min I_B^2}) \setminus \{t\} = \{z_i \mid i \in I_B^2 \setminus \{\min I_B^2\}\}$ and since, by assumption, there is every from $\{w_i \mid i \in I_A^2\}$ to $\{z_i \mid i \in I_B^2\}$ in D_I, we conclude that $N^-_{D_I}(r_{WZ}) \setminus \{s\} = \{w_i \mid i \in I_A^2\}$. Finally to see that $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$, note that by construction, the only possible in-neighbor for $y_{\min I_B^2}$ is s and the only possible out-neighbor for $x_{\max I_A^2}$ is t. ⌟

Identification Rule 14. *If there is every edge from $\{y_i \mid i \in I_B^2\}$ to $\{x_i \mid i \in I_A^2\}$ in D_I and exactly one of $D_I^*[\{y_i \mid i \in I_B^2\}]$ and $D_I^*[\{x_i \mid i \in I_A^2\}]$ has no edge, then proceed as follows.*

1. *If $D_I^*[\{y_i \mid i \in I_B^2\}]$ has no edge then identify $w_{\min I_A^2}$ to s and rename $x_{\min I_A^2}$ as r_{YX}.*
2. *If $D_I^*[\{x_i \mid i \in I_A^2\}]$ has no edge then identify $z_{\max I_B^2}$ to t and rename $y_{\max I_B^2}$ as r_{YX}.*

Symmetrically to Claim 4.62, we have the following.

Claim 4.63. *If Identification Rule 14 has been applied then it still holds that $d^-_{D_I}(s) = d^+_{D_I}(t) = 0$. Furthermore, $N^-_{D_I}(r_{YX}) \setminus \{s\} = \{y_i \mid i \in I_B^2 \setminus \{\max I_B^2\}\}$ and $N^+_{D_I}(r_{YX}) \setminus \{t\} = \{x_i \mid i \in I_A^2\}$.*

Note finally that Identification Rule 13 reduces the size of $E(D_I - \{s, t\})$ by one as does Identification Rule 14 (it may be that the size of $\{(w_i, x_i) \mid i \in I_A^2\}$ reduces by two while the size of $\{(y_i, z_i) \mid i \in I_B^2\}$ remains the same if Identification Rule 13.2 and Identification Rule 14.1 are applied, and vice-versa). Since we may always safely identify consecutive edges in $\{(w_i, x_i) \mid i \in I_A^2\}$ or $\{(y_i, z_i) \mid i \in I_B^2\}$ to reduce their size to $4g(t)$ (as done above), we conclude that D_I can be identified to a digraph transitively equivalent to an almost $4g(t)$-hard matching pattern.

There remains to show how to obtain a $g(t)$-hard matching pattern from an almost $4g(t)$-hard matching pattern. To this end, let H be an almost $4g(t)$-hard matching pattern. We first show how to handle the source vertex from item 3 in Definition 4.48 and then how to handle the sink vertex from item 4 in Definition 4.48.

Let s be the vertex of H such that $N^-(s) = \emptyset$ and $N^+(s) \subseteq W \cup X \cup Y \cup Z$. Suppose first that $N^+(s) \cap W \neq \emptyset$ and let $I_1 \subseteq [4g(t)]$ be the set of indices such that $N^+(s) \cap W = \{w_i \mid i \in I_1\}$.

By the pigeonhole principle, either (1) $|I_1| \geq 4g(t)/2$ or (2) $|[4g(t)] \setminus I_1| \geq 4g(t)/2$. If (1) holds then we proceed as follows.

- For every $j < \min I_1$, we identify w_j and x_j to $w_{\min I_1}$ and $x_{\min I_1}$, respectively.
- For every consecutive $i < i' \in I_1$ and for every $j \in]i, i'[$, we identify w_j and x_j with w_i and x_i, respectively.
- For every $j > \max I_1$, we identify w_j and x_j to $w_{\max I_1}$ and $x_{\max I_1}$, respectively.

Suppose next that (2) holds. If H contains the path $w_1 \to w_2 \to \ldots \to w_{4g(t)}$ then we simply identify s to w_1. Let us therefore assume that H does not contain this path. If H contains the path $x_1 \to x_2 \to \ldots \to x_{4g(t)}$ then for every $i \in I_1$, we identify w_i to x_i and further identify s to x_1. Finally, if H does not contain this path then we identify every vertex in $\{w_i, x_i \mid i \in I_1\} \cup \{s\}$ to w_1.

Now suppose that $N^+(s) \cap W = \emptyset$ and $N^+(s) \cap X \neq \emptyset$. Let $I_1 \subseteq [4g(t)]$ be the set of indices such that $N^+(s) \cap W = \{w_i \mid i \in I_1\}$. By the pigeonhole principle, either (1) $|I_1| \geq 4g(t)/2$ or (2) $|[4g(t)] \setminus I_1| \geq 4g(t)/2$. If (1) holds then we proceed as follows.

- For every $j < \min I_1$, we identify w_j and x_j to $w_{\min I_1}$ and $x_{\min I_1}$, respectively.
- For every consecutive $i < i' \in I_1$ (that is, $I_1 \cap [i,' i] = \{i, i'\}$) and for every $j \in]i, i'[$, we identify w_j and x_j with w_i and x_i, respectively.
- For every $j > \max I_1$, we identify w_j and x_j to $w_{\max I_1}$ and $x_{\max I_1}$, respectively.

Suppose next that (2) holds. If H contains the path $x_1 \to x_2 \to \ldots \to x_{4g(t)}$ then we simply identify s to x_1. Let us therefore assume that H does not contain this path. If H contains the path $w_1 \to w_2 \to \ldots \to w_{4g(t)}$ then for every $i \in I_1$, we identify x_i to w_i and further identify s to w_1. Finally, if H does not contain this path then we identify every vertex in $\{w_i, x_i \mid i \in I_1\} \cup \{s\}$ to w_1.

By proceeding symmetrically with Y and Z, we may identify H to an almost $2g(t)$-hard matching pattern H_1 in which s satisfies item 3 of Definition 1.5. Now let t be the vertex of H (and of H_1) such that $N^+(t) = \emptyset$ and $N^-(t) \subseteq W \cup X \cup Y \cup Z$. Then using similar arguments, we may further identify H_1 to an almost $g(t)$-hard matching pattern H_2 in which s still satisfies item 3 of Definition 1.5 and t satisfies item 4 of Definition 1.5, that is, H_2 is a $g(t)$-hard matching pattern which concludes the proof. □

4.4 Cleaning the induced-biclique with minimal edges

In this section, we prove Lemma 4.8 restated below.

Lemma 4.8 (Cleaning minimal biclique)**.** *For a positive integer t and a digraph D, if $D^\star$ contains a $9t$-induced-biclique whose edges are minimal in D, then D can be identified to digraph that is transitively equivalent to a t-hard-biclique-pattern.*

Proof. Let (A, B) be a $9t$-biclique in D such the edges of $A \cup B$ are minimal edges of D, and such that (A, B) is a $9t$-induced-biclique in $D^\star$. Then A, B are independent sets in $D^\star$.

We first claim that for any $v \in V(D) \setminus (A \cup B)$, it is not the case that $N^-_{D^\star}(v) \cap (A \cup B) \neq \emptyset$ and $N^+_{D^\star}(v) \cap (A \cup B) \neq \emptyset$. For the sake of contradiction, say it is the case. Let $x \in N^-_{D^\star}(v) \cap (A \cup B)$ and let $y \in N^+_{D^\star}(v) \cap (A \cup B)$. Then there exists an (x, y)-path in D of length strictly greater than 1. If $x, y \in A$ (resp. $x, y \in B$), then this contradicts that A (resp. B) is an independent set in D^*. If $x \in A$ and $y \in B$, then this contradicts the minimality of the edges of the (A, B)-biclique. If $x \in B$ and $y \in A$, then this contradicts the definition of an (A, B)-biclique.

Let $V_{\mathtt{s}} \subseteq V(D) \setminus (A \cup B)$ such that for each $v \in V_{\mathtt{s}}$, $N^+_{D^\star}(v) \neq \emptyset$ and $N^-_{D^\star}(v) \cap (A \cup B) = \emptyset$. Let $V_{\mathtt{t}} = V(D) \setminus (A \cup B \cup V_{\mathtt{s}})$. Then from the claim in the above paragraph, for each $v \in V_{\mathtt{t}}$, $N^+_{D^\star}(v) \cap (A \cup B) = \emptyset$. Observe that $V_{\mathtt{s}} \uplus V_{\mathtt{t}}$ partition the vertex set $V(D) \setminus (A \cup B)$. If $V_{\mathtt{s}} \neq \emptyset$, then identify the vertices of $V_{\mathtt{s}}$ into $\mathtt{s}$, if $V_{\mathtt{t}} \neq \emptyset$, then identify the vertices of $V_{\mathtt{t}}$ into $\mathtt{t}$. Let $\widehat{D}$ be the resulting digraph. We now show that $E(V_{\mathtt{t}}, V_{\mathtt{s}}) = \emptyset$. For the sake of contradiction, say $u \in V_{\mathtt{t}}$ and $v \in V_{\mathtt{s}}$ such that $(u,v) \in E(D^\star)$. Then $N^+_{D^\star}(u) \cap (A \cup B) \neq \emptyset$. Since $u \notin V_{\mathtt{s}}$, $N^-_{D^\star}(u) \neq \emptyset$. But this contradicts the claim in the second paragraph. Thus, we conclude that $N^-_{\widehat{D}}(\mathtt{s}), N^+_{\widehat{D}}(\mathtt{t}) = \emptyset$.

Now let $t' = 9t$. Recall that A, B are ordered sets, say $A = (a_1, \ldots, a_{t'})$ and $B = (b_1, \ldots, b_{t'})$. Let $I_1 \subseteq [t']$ such that for $i \in I_1$, $(\mathtt{s}, a_i) \in E(\widehat{D})$. Let $I_2 \subseteq [t']$ such that for each $i \in I_2$, $(\mathtt{s}, a_i) \notin E(\widehat{D})$ and $(\mathtt{s}, b_i) \in E(\widehat{D})$. Let $I_3 \subseteq [t']$ such that for each $i \in I_3$, $(\mathtt{s}, a_i), (\mathtt{s}, b_i) \notin E(\widehat{D})$. Observe that $[t'] = I_1 \uplus I_2 \uplus I_3$. Let $J_1 \subseteq [t']$ such that for each $j \in J_1$, $(b_j, \mathtt{t}) \in E(\widehat{D})$. Let $J_2 \subseteq [t']$ such that for each $j \in J_2$, $(b_j, \mathtt{t}) \notin E(\widehat{D})$ and $(a_j, \mathtt{t}) \in E(\widehat{D})$. Let $J_3 \subseteq [t']$ such that for each $j \in J_3$, $(b_j, \mathtt{t}), (a_j, \mathtt{t}) \notin E(\widehat{D})$. Observe that $[t'] = J_1 \uplus J_2 \uplus J_3$.

By the pigeon-hole principle, either $|I_1| \geq t'/3$ or $|I_2| \geq t'/3$ or $|I_3| \geq t'/3$. We consider these three cases separately.

Case 1. $|I_1| \geq t'/3$**.** Suppose first that $|J_1 \cap I_1| \geq |I_1|/3$. Let $A' = \{a_i : i \in J_1 \cap I_1\}$ and $B' = \{b_i : i \in J_1 \cap I_1\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $A' \subseteq N^+_{\widehat{D'}}(\mathtt{s}), B' \subseteq N^-_{\widehat{D'}}(\mathtt{t})$ and $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose next that $|J_2 \cap I_1| \geq |I_1|/3$. Let $A' = \{a_i : i \in J_2 \cap I_1\}$ and $B' = \{b_i : i \in J_2 \cap I_1\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $A' \subseteq N^+_{\widehat{D'}}(\mathtt{s}), A' = N^-_{\widehat{D'}}(\mathtt{t})$ and $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose finally that $|J_3 \cap I_1| \geq |I_1|/3$. Let $A' = \{a_i : i \in J_3 \cap I_1\}$ and $B' = \{b_i : i \in J_3 \cap I_1\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B') \cup \mathtt{t}$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $A' \subseteq N^+_{\widehat{D'}}(\mathtt{s})$.

Case 2. $|I_2| \geq t'/3$**.** Suppose first that $|J_1 \cap I_2| \geq |I_2|/3$. Let $A' = \{a_i : i \in J_1 \cap I_2\}$ and $B' = \{b_i : i \in J_1 \cap I_2\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $B' = N^+_{\widehat{D'}}(\mathtt{s}), B' \subseteq N^-_{\widehat{D'}}(\mathtt{t})$ and $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose next that $|J_2 \cap I_2| \geq |I_2|/3$. Let $A' = \{a_i : i \in J_2 \cap I_2\}$ and $B' = \{b_i : i \in J_2 \cap I_2\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $B' = N^+_{\widehat{D'}}(\mathtt{s}), A' = N^-_{\widehat{D'}}(\mathtt{t})$ and $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose finally that $|J_3 \cap I_2| \geq |I_2|/3$. Let $A' = \{a_i : i \in J_3 \cap I_2\}$ and $B' = \{b_i : i \in J_3 \cap I_2\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B') \cup \mathtt{t}$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that $B' = N^+_{\widehat{D'}}(\mathtt{s})$.

Case 3. $|I_3| \geq t'/3$**.** Suppose first that $|J_1 \cap I_3| \geq |I_3|/3$. Let $A' = \{a_i : i \in J_1 \cap I_3\}$ and $B' = \{b_i : i \in J_1 \cap I_3\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$, if any, with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Observe that if $(A \setminus A') \cup (B \setminus B') = \emptyset$, then then $\mathtt{s} \notin V(\widehat{D'})$ and $B \subseteq N^-_{\widehat{D'}}(\mathtt{t})$. Otherwise, additionally, $\mathtt{s} \in V(D)$ and since (A, B) is an induced biclique, $B' = N^+_{\widehat{D'}}(\mathtt{s})$. Moreover, $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose next that $|J_2 \cap I_3| \geq |I_3|/3$. Let $A' = \{a_i : i \in J_2 \cap I_3\}$ and $B' = \{b_i : i \in J_2 \cap I_3\}$. Observe that $|A'| = |B'| \geq t'/3$. Identify the vertices of $(A \setminus A') \cup (B \setminus B')$, if any, with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. If $(A \setminus A') \cup (B \setminus B') = \emptyset$, then $\mathtt{s} \notin V(\widehat{D'})$ and $A = N^-_{\widehat{D'}}(\mathtt{t})$. Otherwise, additionally $\mathtt{s} \in V(\widehat{D'})$ and since (A, B) is an induced biclique, $B = N^+_{\widehat{D'}}(\mathtt{s})$. Moreover, $(\mathtt{t}, \mathtt{s}) \notin E(\widehat{D'})$.

Suppose finally that $|J_3 \cap I_3| \geq |I_3|/3$. Let $A' = \{a_i : i \in J_3 \cap I_3\}$ and $B' = \{b_i : i \in J_3 \cap I_3\}$. Observe that $|A'| = |B'| \geq t'/3$ and $(A \setminus A') \cup (B \setminus B') \neq \emptyset$ since D is connected (without loss of generality). Identify the vertices of $(A \setminus A') \cup (B \setminus B') \cup \mathtt{t}$ with $\mathtt{s}$. Let the resulting digraph be $\widehat{D'}$. Since (A, B) is an induced biclique and $(A \setminus A') \cup (B \setminus B') \neq \emptyset$, $B' = N^+_{\widehat{D'}}(\mathtt{s})$. $\square$

5 Hardness proofs and lower bounds

This section is devoted to presenting the lower bounds on the complexity of Planar $\mathcal{D}$-SN for various classes $\mathcal{D}$. First we prove the combinatorial result Lemma 1.6 that implies that there are no nontrivial class with subexponential FPT algorithms on planar graphs (Section 5.1). Then we present reductions from Grid Tiling to Planar $\mathcal{D}$-SN where $\mathcal{D}$ is the class of diamonds (Section 5.2), hard matching patterns (Section 5.3.1), or hard biclique patterns (Section 5.3.2).

5.1 Finding a star

We present here a short proof that every nontrivial class $\mathcal{D}$ in our setting contains either all cycles, all in-stars, or all out-stars.

Proof (of Lemma 1.6). Suppose that there is an integer α such that $\mathcal{D}$ does not contain a cycle, in-star, or out-star on α vertices. This means in particular that a graph $\mathcal{D}$ cannot have a strongly connected component with α vertices. Indeed, by selecting α vertices of the component and identifying every other vertex of the graph with one of these vertices, we would obtain a strongly connected graph on α vertices, which is transitively equivalent to a cycle on α vertices. It also follows that a graph $D \in \mathcal{D}$ has less than α strongly connected components of size larger than 1: otherwise, identifying one vertex from each such component to a single vertex would create a strongly connected component of size larger than α.

This bound on the size of the strongly connected components implies that identifying the vertex set of each strongly connected component of some $D \in \mathcal{D}$ to a single vertex results in an acyclic graph whose size is at most a factor of α smaller. Therefore, the fact that $\mathcal{D}$ contains arbitrarily large graphs implies that $\mathcal{D}$ contains arbitrary large acyclic graphs. Furthermore, $\mathcal{D}$ contains arbitrarily large graphs without isolated vertices: if we have a graph with c edges and identify every isolated vertex with some vertex v, then we get a graph without isolated vertices and exactly c edges. If we identify the strongly connected components of such a graph to single vertices, then less than α isolated vertices can be created (as there are less than α components of size larger than 1). It follows that $\mathcal{D}$ contains arbitrarily large acyclic graphs with less than α isolated vertices.

Observe that $D \in \mathcal{D}$ cannot have a path longer than $\alpha + 1$: otherwise, identifying the first and last vertices of the path would create a member of $D \in \mathcal{D}$ that has a strongly connected component larger than α. This means that every acyclic $D \in \mathcal{D}$ can be partitioned into $\alpha + 2$ levels, that is, D has a topological ordering with blocks $B_1, \ldots, B_\ell$ with $\ell \leq \alpha + 2$ such that each block B_i is an independent set. If an acyclic graph D_c has at least $(\alpha + 2)c$ vertices, then one such block B_i has size at least c. Let D_c^* be obtained by identifying the blocks $B_1, \ldots, B_{i-1}$ to a single vertex x, and the blocks $B_{i+1}, \ldots, B_\ell$ to a single vertex y. If there are α vertices in B_i adjacent to both x and y, then identifying x and y creates a strongly connected component, a contradiction. If there is a set $I \subseteq B_i$ of α vertices that is adjacent only to x, then identifying $(B_i \setminus I) \cup \{x, y\}$ to a single vertex results in an out-star on $\alpha + 1$ vertices. Similarly, if there is a set $I \subseteq B_i$ of α vertices that is adjacent only to y, then identifying $(B_i \setminus I) \cup \{x, y\}$ to a single vertex results in an in-star on $\alpha + 1$ vertices. Finally, I cannot have α vertices that are adjacent to neither x or y, because they would be isolated vertices. Therefore, if $c \geq 4\alpha$, then we arrive to a contradiction in one of the three cases. $\square$

5.2 Diamonds

The aim of this section is to prove the following.

Theorem 5.1. *For any $\ell \in [4]$,* Planar $\mathcal{A}_\ell$-Steiner Network *is* W[1]*-hard parameterized by the number k of terminals and does not admit an $f(k) \cdot n^{o(\sqrt{k})}$ algorithm for any computable function f, unless* ETH *fails.*

We only formally prove the statement for pure out-diamonds as it will become clear from the proof that to handle

1. flawed out-diamonds, it suffices to add a vertex s and edges (s, r_1) and (s, r_2) in the construction below;
2. pure in-diamonds, it suffices to reverse the direction of every edge in the construction below; and
3. flawed in-diamonds, it suffices to add a vertex t and edges (r_1, t) and (r_2, t), and additionally reverse the direction of every edge in the construction below.

In the following, we assume that the class of all pure out-diamonds is $\mathcal{A}_1$. To prove the statement, we first introduce the $k \times k$-Grid Tiling problem, formally defined below.

$k \times k$-Grid Tiling
Input: Integers k, n and k^2 nonempty set $S_{i,j} \subseteq [n] \times [n]$ where $i, j \in [k]$.
Question: For each $i, j \in [k]$, does there exist an entry $(x_{i,j}, y_{i,j}) \in S_{i,j}$ such that
- for every $i \in [k]$, $x_{i,1} = x_{i,2} = \ldots = x_{i,k}$ and
- for every $j \in [k]$, $y_{1,j} = y_{2,j} = \ldots = y_{k,j}$?

Throughout the paper, we assume that whenever given a set $S \subseteq [n] \times [n]$, $1 < x, y < n$ holds for every $(x, y) \in S$: it suffices to increase n by two and replace (x, y) by $(x+1, y+1)$ otherwise.

It was shown [9, Theorem 14.28] that, under ETH, $k \times k$-Grid Tiling does not admit an algorithm running in time $f(k) \cdot n^{o(k)}$ for any computable function f. To prove Theorem 5.1 for pure out-diamonds, we give a reduction which transforms an instance of $k \times k$-Grid Tiling into an instance of (edge-weighted)[3] Planar $\mathcal{A}_1$-Steiner Network with $O(k^2)$ terminals. To this end, we design three types of gadgets: the *connector gadget*, the *down main gadget* and the *up main gadget*. The reduction represents each set of the $k \times k$-Grid Tiling instance with a copy of the up or down main gadget, and uses the connector gadgets to further connect these gadgets (see Figure 16). The remainder of this section is organized as follows. We first introduce the connector gadget and prove in Lemma 5.2 that it satisfies several desired properties. We then introduce the down main gadget and prove in Lemma 5.7 that it also satisfies several desired properties. The up main gadget is introduced thereafter together with its symmetrical Lemma 5.24. We end the section with the precise description of the reduction and a proof of its correctness.

Connector gadget. Given an integer $n > 0$, the connector gadget CG_n is an edge-weighted planar digraph consisting of $n \times n$ grid, where the vertex lying at the intersection of column i and row j is denoted by $x_{i,j}$, and $2n + 2$ additional vertices $p_1, \ldots, p_n, q_1, \ldots, q_n, p, q$. The adjacencies and edge weights are defined as follows (we fix $N = 3n$).

- *Left source edges:* for every $j \in [n]$, there is an edge $(p_j, x_{1,j})$. Together these edges are called left source edges. The weight of each such edge is set to N^2.
- *Right source edges:* for every $j \in [n]$, there is an edge $(q_j, x_{n,j})$. Together these edges are called right source edges. The weight of each such edge is set to N^2.

[3] We argue at the end of the section that with polynomially bounded integer weights, the edge-weighted version of the problem reduces to its unweighted version.

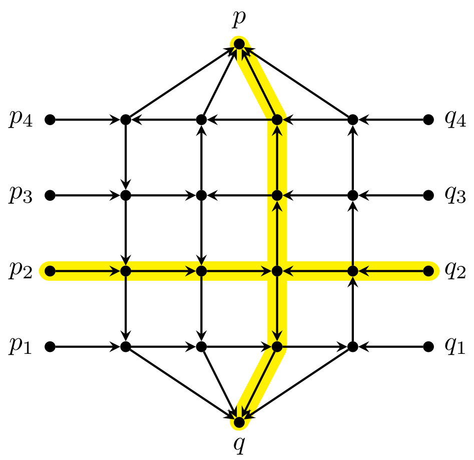

Figure 13: The connector gadget for $n = 4$. A set of edges representing 2 is highlighted.

- *Top sink edges:* for every $i \in [n]$, there is an edge $(x_{i,n}, p)$. Together these edges are called top sink edges. The weight of each such edge is set to N^2.
- *Bottom sink edges:* for every $i \in [n]$, there is an edge $(x_{i,1}, q)$. Together these edges are called bottom sink edges. The weight of each such edge is set to N^2.
- The $n \times n$ grid is divided in two according to the diagonal $j = n + 1 - i$: the vertices on the top-right part (that is, the vertices $x_{i,j}$ such that $j + i \geq n + 1$) are connected with $\leftarrow$ and $\uparrow$ edges, and the vertices of the bottom-left part (that is, the vertices $x_{i,j}$ such that $j + i \leq n + 1$) are connected with $\rightarrow$ and $\downarrow$ edges. More specifically, we define the following edges.
  - *Up edges*: for every $i \in [n]$ and every $j \in [n - 1]$ such that $j + i \geq n + 1$, there is an edge $(x_{i,j}, x_{i,j+1})$. Together these edges are called up edges.
  - *Down edges:* for every $i \in [n]$ and every $j \in [n] \setminus \{1\}$ such that $j + i \leq n + 1$, there is an edge $(x_{i,j}, x_{i,j-1})$. Together these edges are called down edges.

  We set the weight of every up/down edge to N.
  - *Left edges:* for every $i \in [n] \setminus \{1\}$ and every $j \in [n]$ such that $j + i > n + 1$, there is an edge $(x_{i,j}, x_{i-1,j})$. Together these edges are called left edges.
  - *Right edges:* for every $i \in [n - 1]$ and every $j \in [n]$ such that $j + i < n + 1$, there is an edge $(x_{i,j}, x_{i+1,j})$. Together these edges are called right edges.

  We set the weight of every left/right edge to 1.

This concludes the construction of the connector gadget CG_n (see Figure 13 for an illustration of the connector gadget for $n = 4$). In the following, we call the vertices $p_1, \ldots, p_n$ the *left vertices*, the vertices $q_1, \ldots, q_n$ the *right vertices* and the vertices p, q the *terminal vertices*. We further set

$$C_n^* = 4N^2 + (n - 1)N + n - 1.$$

A set $E \subseteq E(CG_n)$ satisfies the *connectedness* property if the following hold in E:
- p can be reached from some right vertex and from some left vertex;
- q can be reached from some right vertex and from some left vertex.

A set $E \subseteq E(CG_n)$ satisfying the connectedness property *represents* an integer $j \in [n]$ if the only left source edge in E is the one incident to p_j and the only right source edge in E is the one incident to q_j (see Figure 13 for a set of edges representing 2).

Lemma 5.2. *For any integer $n > 0$, the connector gadget CG_n satisfies the following properties.*
(1) For every $j \in [n]$, there exists a set $E_j \subseteq E(CG_n)$ of weight C_n^* representing j.

(2) *If there exists a set* $E \subseteq E(CG_n)$ *of weight at most* C_n^* *satisfying the connectedness property, then* E *has weight exactly* C_n^* *and represents some integer* $j \in [n]$*.*

Proof. To prove (1), it suffices to take E_j to be the union of the following sets of edges:

- $\{(p_j, x_{1,j}), (q_j, x_{n,j}), (x_{n+1-j,n}, p), (x_{n+1-j,1}, q)\}$;
- $\{(x_{i,j}, x_{i+1,j}) \mid 1 \leq i \leq n-j\}$;
- $\{(x_{i,j}, x_{i-1,j}) \mid n+2-j \leq i \leq n\}$;
- $\{(x_{n+1-j,\ell}, x_{n+1-j,\ell+1}) \mid j \leq \ell \leq n-1\}$; and
- $\{(x_{n+1-j,\ell}, x_{n+1-j,\ell-1}) \mid 2 \leq \ell \leq j\}$.

It is not difficult to see that E_j represents j and has weight C_n^*.

Next, suppose that $E \subseteq E(CG_n)$ is a set of weight at most C_n^* satisfying the connectedness property. Let us show that E has weight exactly C_n^* and represents some integer $j \in [n]$. To this end, we prove the following claims.

Claim 5.3. *E contains exactly one left source edge, one right source edge, one top sink edge and one bottom sink edge.*

Proof. Since E satisfies the connectedness property, it contains at least one left source edge, one right source edge, one top sink edge and one bottom sink edge. Now if E contains at least two left source edges (the other cases are symmetric), then the weight of E is at least $5N^2$; however, by definition,

$$C_n^* = 4N^2 + (n-1)N + n - 1 < 4N^2 + nN + nN < 5N^2$$

as $N = 3n > 2n$, a contradiction. ⌟

In the following, we let p_{j_1} be the left vertex incident to the left source edge in E and q_{j_2} be the right vertex incident to the right source edge in E.

Claim 5.4. *For every $j \in [n-1]$, E contains exactly one (up or down) edge with one endvertex on row j and one endvertex one row $j+1$. In particular, E contains exactly $n-1$ up/down edges.*

Proof. First note that if there exists $j \in [n-1]$ such that E contains no (up or down) edge with one endvertex on row j and the other endvertex on row $j+1$, then either $j_1 \leq j$ in which case p_{j_1} cannot reach p in E, or $j_1 > j$ in which case p_{j_1} cannot reach q in E, a contradiction in both cases to the connectedness of E. Thus, for every $j \in [n-1]$, E contains at least one (up or down) edge with one endvertex on row j and the other endvertex on row $j+1$; in particular, E contains at least $n-1$ up/down edges. Now if E contains at least n up/down edges then by Claim 5.3, the weight of E is at least $4N^2 + nN$; however, by definition,

$$C_n^* = 4N^2 + (n-1)N + n - 1 < 4N^2 + (n-1)N + n < 4N^2 + nN$$

as $N = 3n > n$, a contradiction. ⌟

Claim 5.5. *E contains exactly $n-1$ left/right edges.*

Proof. Observe first that for every $i \in [n-1]$, E contains at least one (left or right) edge with one endvertex on column i and the other endvertex on column $i+1$: indeed, if this were not the case for some $i \in [n-1]$, then every path in $CG_n[E]$ from p_{j_1} to p would be edge-disjoint from every path in $CG_n[E]$ from q_{j_2} to p and thus, E would contain at least two top sink edges, a contradiction to Claim 5.3. Hence, for every $i \in [n-1]$, E contains at least one (left or right) edge with one endvertex on column i and the other endvertex on column $i+1$; in particular, E contains at least $n-1$ left/right edges. Now if E contains at least n left/right edges then by Claims 5.3 and 5.4, the weight of E is at least $4N^2 + (n-1)N + n > C_n^*$, a contradiction. ⌟

Now by Claims 5.3, 5.4 and 5.5, E has weight exactly C_n^*; we next show that $j_1 = j_2$ which would imply that E represents j_1. Suppose to the contrary that $j_2 < j_1$ (the case where $j_2 > j_1$ is symmetric). By Claim 5.4, E contains exactly one edge e with one endvertex on row j_2 and one endvertex on row $j_2 + 1$; but then, either e is an up edge in which case p_{j_1} cannot reach q in E, or e is a down edge and q_{j_2} cannot reach p in E, a contradiction in both cases to the connectedness of E which concludes the proof. □

Down Main Gadget. Given an integer $n > 0$, the down main gadget dMG_S represents a set $S \subseteq [n] \times [n]$[4] and is constructed as follows. It is an edge-weighted planar digraph consisting of a $2n \times n^2$ grid, where the vertex lying at the intersection of column i and row j is denoted by $x_{i,j}$, and $6n$ additional vertices $\ell_1, \dots, \ell_n, \ell'_1, \dots, \ell'_n, r_1, \dots, r_n, r'_1, \dots, r'_n, t_1, \dots, t_n,\ b_1, \dots, b_n$. The adjacencies and edge weights are defined as follows (we fix $M = 13n^2$).

- *Source edges:* for every $i \in [n]$, there is an edge (t_i, x_{i,n^2}). Together these edges are called source edges. The weight of each such edge is set to M^5.
- *Bottom sink edges:* for every $i \in [n]$, there is an edge $(x_{n+i,1}, b_i)$. Together these edges are called bottom sink edges. The weight of each such edge is set to M^5.
- *Left sink edges:* for every $j \in [n]$, there is an edge (ℓ'_j, ℓ_j) whose weight is set to Mj. Together these edges are called left sink edges.
- *Right sink edges:* for every $j \in [n]$, there is an edge (r'_j, r_j) whose weight is set to $M^2 - Mj$. Together these edges are called right sink edges.
- *Left internal sink edges:* for every $j \in [n]$ and every $(j-1)n+1 \le p \le jn$, there is an edge $(x_{1,p}, \ell'_j)$ whose weight is set to $p - (j-1)n$. For every fixed $j \in [n]$, these edges together are called the j^{th} left internal sink edges.
- *Right internal sink edges:* for every $j \in [n]$ and every $(j-1)n+1 \le p \le jn$, there is an edge $(x_{2n,p}, r'_j)$ whose weight is set to $n + 1 - (p - (j-1)n)$. For every fixed $j \in [n]$, these edges together are called the j^{th} right internal sink edges.
- *Right bridge edges:* for every $j \in [n]$ and every $(j-1)n+1 \le p \le jn$, there is an edge $(x_{n,p}, x_{n+1,p})$. For every fixed $j \in [n]$, these edges together are called the j^{th} right bridge edges. The weight of each j^{th} right bridge edge is set to M^4.
- *Downward bridge edges:* for every $j \in \{pn+1 \mid p \in [n-1]\}$ and every $i \in [2n]$, there is an edge $(x_{i,j}, x_{i,j-1})$. For every fixed $j \in \{pn+1 \mid p \in [n-1]\}$, these edges together are called the j^{th} downward bridge edges. The weight of each j^{th} downward bridge edge is set to M^3.
- *Down edges:* for every $i \in [2n]$, every $j \in [n]$ and every $(j-1)n+2 \le p \le jn$, there is an edge $(x_{i,p}, x_{i,p-1})$. Together these edges are called down edges. The weight of each such edge is set to 4.
- *Right edges:* for every $j \in [n]$, every $(j-1)n+1 \le p \le jn$ and every $jn+1-p \le q \le 2n-1$ such that $q \ne n$, there is an edge $(x_{q,p}, x_{q+1,p})$. Together these edges are called right edges. The weight of each such edge is set to 4.
- *Left edges:* for every $j \in [n]$, every $(j-1)n+1 \le p \le jn-1$ and every $2 \le q \le jn+1-p$, there is an edge $(x_{q,p}, x_{q-1,p})$. Together these edges are called left edges. The weight of each such edge is set to 4.
- *Shortcut edges:* for every $s = (i,j) \in S$, we introduce two shortcut edges e_s^ℓ, e_s^r as follows. Set $p = jn + 1$, then
  - subdivide the edge $(x_{i,p+1-i}, x_{i,p-i})$ by adding a vertex $y_{i,j}$ and the edge $(x_{i,p+1-i}, y_{i,j})$ (of weight 3) with the edge $(y_{i,j}, x_{i,p-i})$ (of weight 1);
  - subdivide the edge $(x_{n+i,p-i}, x_{n+i,p-1-i})$ by adding a vertex $z_{i,j}$ and the edge $(x_{n+i,p-i}, z_{i,j})$ (of weight 3) with the edge $(z_{i,j}, x_{n+i,p-1-i})$ (of weight 1).

  The introduced edges are called *down subdivided edges.* Then

[4] Recall that, by assumption, $1 < x, y < n$ holds for every $(x, y) \in S$.

– $e_s^\ell = (y_{i,j}, x_{i-1,p-i})$ and its weight is set to 2;
– $e_s^r = (z_{i,j}, x_{n+i+1,p-i})$ and its weight is set to 2.

The edges e_s^ℓ are called *left* shortcut edges and the edges e_s^r are called *right* shortcut edges. This concludes the construction of the down main gadget dMG_S (see Figure 14 for an illustration of the down main gadget with $n = 4$ representing $S = \{(2,2),(2,3),(3,2)\}$). In the following, we call the vertices $\ell_1, \ldots, \ell_n$ the *left vertices*, the vertices $r_1, \ldots, r_n$ the *right vertices*, the vertices $t_1, \ldots, t_n$ the *top vertices* and the vertices $b_1, \ldots, b_n$ the *bottom vertices*. Furthermore, we set

$$M_n^* = 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12.$$

We further let $V_\ell = \{x_{i,j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n^2\} \cup \{y_{i,j} \mid (i,j) \in S\}$, $V_r = \{x_{i,j} \mid n+1 \leq i \leq 2n \text{ and } 1 \leq j \leq n^2\} \cup \{z_{i,j} \mid (i,j) \in S\}$ and for every $j \in [n]$, $V_j = \{x_{i,p} \mid 1 \leq i \leq 2n \text{ and } (j-1)n + 1 \leq p \leq jn\}$. Note that, by construction, the following holds.

Observation 5.6. *For every $j \in [n]$ and every $x_{i,p} \in V_j \cap V_\ell$, the following holds.*
- *If $n+1-i > p-(j-1)n$ then $x_{i,p}$ cannot reach a vertex $x_{i',p'} \in V_j$ such that $n+1-i' \leq p'-(j-1)n$.*
- *If $n+1-i < p-(j-1)n$ then $x_{i,p}$ cannot reach a vertex $x_{i',p'} \in V_j$ such that $i' < i$ and $p'-(j-1)n > n+1-i$.*

A set $E \subseteq E(dMG_S)$ satisfies the *connectedness* property if the following hold in E:
- a top vertex can reach a bottom vertex;
- a top vertex can reach a right vertex;
- a top vertex can reach a left vertex.

A set $E \subseteq E(dMG_S)$ satisfying the connectedness property *represents* a pair $(i,j) \in [n] \times [n]$ if the only source edge in E is the one incident to t_i, the only bottom sink edge in E is the one incident to b_i, the only left sink edge in E is the one incident to ℓ_j and the only right sink edge in E is the one incident to r_j (see Figure 14 for a set of edges representing $(2,2)$).

Lemma 5.7. *For any $n > 0$ and any $S \subseteq [n] \times [n]$, the down main gadget dMG_S satisfies the following properties.*
(1) For every $(i,j) \in S$, there exists a set $E_{i,j} \subseteq E(dMG_S)$ of weight M_n^* representing (i,j).
(2) If there exists a set $E \subseteq E(dMG_S)$ of weight at most M_n^* satisfying the connectedness property, then E has weight exactly M_n^* and represents a pair $(i,j) \in S$.

Proof. To prove (1), it suffices to take $E_{i,j}$ to be the union of the following set of edges:
- $\{(t_i, x_{i,n^2}), (x_{n+i,1}, b_i), (x_{0,jn+1-i}, \ell'_j), (\ell'_j, \ell_j), (x_{2n+1,jn+1-i}, r'_j), (r'_j, r_j)\}$;
- $\{(x_{i,p}, x_{i,p-1}) \mid jn+2-i \leq p \leq n^2\} \cup \{(x_{n+i,p}, x_{n+i,p-1}) \mid 2 \leq p \leq jn+1-i\}$;
- $\{(x_{p,jn+1-i}, x_{p-1,jn+1-i}) \mid 1 \leq p \leq i-1\} \cup \{e^\ell_{(i,j)}\}$;
- $\{(x_{p,jn+1-i}, x_{p+1,jn+1-i}) \mid i \leq p \leq 2n \text{ and } p \neq n+i\} \cup \{e^r_{(i,j)}\}$.

It is not difficult to see that $E_{i,j}$ represents (i,j) and has weight M_n^*.

Next, suppose that $E \subseteq E(dMG_S)$ is a set of weight at most M_n^* satisfying the connectedness property. Let us show that E has weight exactly M_n^* and represents some $(i,j) \in S$. To this end, we first prove the following claims.

Claim 5.8. *E contains exactly one source edge and one bottom sink edge.*

Proof. Since E satisfies the connectedness property, it contains at least one source edge and one bottom sink edge. Now if E contains at least two source edges, then the weight of E is at least $3M^5$; however, by definition,

$$\begin{aligned} M_n^* &= 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12 \\ &< 2M^5 + n^2M^4 + n^2M^4 + n^2M^4 + 9n^2M^4 < 3M^5 \end{aligned}$$

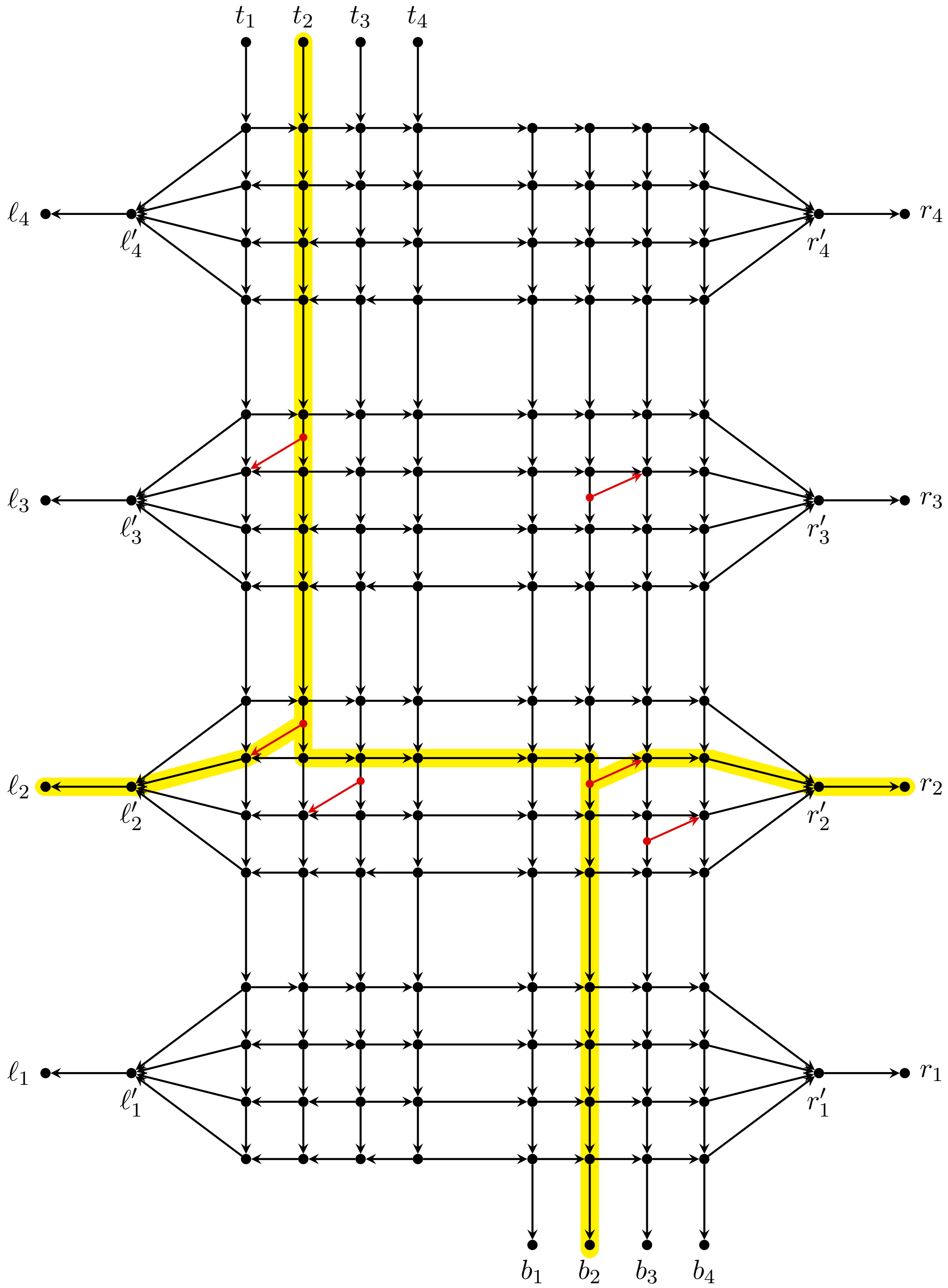

Figure 14: The down main gadget dMG_S with $n = 4$ representing $S = \{(2, 2), (2, 3), (3, 2)\}$ (the red edges are the shortcut edges). A set of edges representing $(2, 2)$ is highlighted.

as $M = 13n^2 > 12n^2$, a contradiction. We conclude similarly if E contains at least two bottom sink edges. ⌟

In the following, we let t_{i_1} be the top vertex incident to the source edge in E and b_{i_2} be the bottom vertex incident to the bottom sink edge in E.

Claim 5.9. *E contains exactly one right bridge edge.*

Proof. Since t_{i_1} can reach b_{i_2} in E, E contains at least one right bridge edge. Now if E contains at least two right bridge edges then by Claim 5.8, the weight of E is at least $2M^5 + 2M^4$; however, by definition,

$$\begin{aligned} M_n^* &= 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12 \\ &< 2M^5 + M^4 + n^2M^4 + n^2M^4 + 9n^2M^4 \\ &< 2M^5 + 2M^4 \end{aligned}$$

as $M = 13n^2 > 11n^2$, a contradiction. ⌟

Claim 5.10. *For every $j \in [n-1]$, E contains exactly one j^{th} downward bridge edge. In particular, E contains exactly $n-1$ downward bridge edges.*

Proof. Since t_{i_1} can reach b_{i_2} in E, E contains at least one j^{th} downward bridge edge for every $j \in [n-1]$. Now if E contains at least n downward bridge edges then by Claims 5.8 and 5.9, the weight of E is at least $2M^5 + M^4 + nM^3$; however, by definition,

$$\begin{aligned} M_n^* &= 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12 \\ &< 2M^5 + M^4 + (n-1)M^3 + n^2M^2 + 9n^2M^2 \\ &< 2M^5 + M^4 + nM^3 \end{aligned}$$

as $M = 13n^2 > 10n^2$, a contradiction. ⌟

Since E satisfies the connectedness property, it contains at least one left sink edge $(\ell'_{j_1}, \ell_{j_1})$ and at least one right sink edge (r'_{j_2}, r_{j_2}); we next show that E contains in fact no other left or right sink edge.

Claim 5.11. *It holds that $j_1 = j_2$. Furthermore, E contains a j_1^{th} right bridge edge.*

Proof. Towards a contradiction, suppose that $j_1 \neq j_2$. Assume first that $j_1 < j_2$ and let P be a path from t_{i_1} to r_{j_2} in $dMG_S[E]$. Then P must contain a j^{th} right bridge edge e for some $j \geq j_2$, and since E contains exactly one right bridge edge by Claim 5.9, it follows that every path in $dMG_S[E]$ from t_{i_1} to b_{i_2} contains e. But t_{i_1} can reach ℓ_{j_1} in E as well and so, E must contain at least $n-j$ downward bridge edges for t_{i_1} to reach the tail of e, plus $j-1$ downward bridge edges for the head of e to then reach b_{i_2}, plus $j - j_1$ downward bridge edges for t_{i_1} to also reach ℓ_{j_1}, a contradiction to Claim 5.10. Suppose next that $j_2 < j_1$. Then since E contains the edge $(\ell'_{j_1}, \ell_{j_1})$ and the edge (r'_{j_2}, r_{j_2}), it follows from Claims 5.8, 5.9 and 5.10 that the weight of E is at least $2M^5 + M^4 + (n-1)M^3 + Mj_1 + M^2 - Mj_2 \geq 2M^5 + M^4 + (n-1)M^3 + M^2 + M$. However, by definition,

$$\begin{aligned} M_n^* &= 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12 \\ &< 2M^5 + M^4 + (n-1)M^3 + M^2 + 9n^2 \\ &< 2M^5 + M^4 + (n-1)M^3 + M^2 + M \end{aligned}$$

as $M = 13n^2 > 9n^2$, a contradiction. Thus, $j_1 = j_2$ and by arguing as above, we can show that E must then contain a j_1^{th} right bridge edge. ⌟

In the following, we say that a path P in dMG_S *crosses a row* j if $V(P)\cap\{x_{i,j} \mid i\in[2n]\}\neq\emptyset$ and that P *crosses a column* i if $V(P)\cap\{x_{i,j} \mid j\in[n^2]\}\neq\emptyset$. Furthermore, we define

- a $2d$*-move* to be the union of two down subdivided edges e, e' such that the head of e coincides with the tail of e';
- a $d\ell$*-move* to be the union of a down subdivided edge e and a left shortcut edge e' such that the head of e coincides with the tail of e';
- a dr*-move* to be the union of a down subdivided edge e and a right shortcut edge e' such that the head of e coincides with the tail of e'.

For convenience, we also call

- a right edge an r*-move*,
- a left edge an ℓ*-move*,
- a down edge a d*-move* and
- a downward bridge edge a Db*-move*.

Note that no dr-move is possible in $dMG_S[V_\ell]$ and that no x-move with $x\in\{d\ell,\ell\}$ is possible in $dMG_S[V_r]$. Furthermore, for any $x\in\{2d,r,\ell,d\}$, the weight of an x-move is 4 and for any $x\in\{dr,d\ell\}$, the weight of an x-move is 5. Given a path P of dMG_S and $x\in\{2d,d\ell,dr,r,\ell,d,Db\}$, we let $m(P,x)$ be the number of x-moves in P.

Let (x_{1,j_0},ℓ'_{j_1}) be a j_1^{th} left internal sink edge contained in E (recall that $\ell_{j_1}\in V(E)$). Further let x_{i_0,j_1n} be the head of the j_1^{th} downward bridge in E if $j_1\le n-1$ (recall that by Claim 5.10, E contains exactly one such edge) and the head of the source edge in E otherwise (note that in this case $i_0=i_1$). Since x_{i_0,j_1n} can reach x_{1,j_0}, the following holds.

Observation 5.12. $j_0-(j_1-1)n\le n+1-i_0$.

Now by connectedness of E, there exists a path P^t in $dMG_S[E]$ from the head of the source edge in E to x_{i_0,j_1n}; we next lower bound the weight of P^t.

Claim 5.13. *The weight of P^t is at least $(n-j_1)M^3+4(n-1)(n-j_1)+|i_1-i_0|$.*

Proof. Observe first that since by Claims 5.9 and 5.11, E contains one j_1^{th} right bridge edge and no other right bridge edge, necessarily $i_0\le n$. Now P^t must cross every row $j_1n\le j\le n^2$ and thus,

$$m(P^t,Db)+m(P^t,d)+m(P^t,2d)+m(P^t,d\ell)=n^2-j_1n.$$

By Claim 5.10, P^t contains exactly $n-j_1$ downward bridge edges, that is, $m(P^t,Db)=n-j_1$. We next distinguish cases depending on whether $i_1\le i_0$ or $i_1>i_0$. Suppose first that $i_1\le i_0$. Then since P^t must cross every column $i_1\le i\le i_0$, $m(P^t,r)\ge i_0-i_1$. It follows that the weight of P^t is at least

$$\begin{aligned}
&M^3m(P^t,Db)+4m(P^t,d)+4m(P^t,2d)+5m(P^t,d\ell)+4m(P^t,r)\\
&=(n-j_1)M^3+4m(P^t,d)+4m(P^t,2d)+5m(P^t,d\ell)+4m(P^t,r)\\
&\ge(n-j_1)M^3+4(m(P^t,d)+m(P^t,2d)+m(P^t,d\ell))+4(i_0-i_1)\\
&\ge(n-j_1)M^3+4(n^2-j_1n-(n-j_1))+4(i_0-i_1)\\
&\ge(n-j_1)M^3+4(n-1)(n-j_1)+(i_0-i_1)
\end{aligned}$$

Second, suppose that $i_1>i_0$. Then since P^t must cross every column $i_0\le i\le i_1$,

$$m(P^t,\ell)+m(P^t,d\ell)=m(P^t,r)+i_1-i_0$$

and so, the weight of P^t is at least

$$\begin{aligned}
&M^3 m(P^t, Db) + 4m(P^t, d) + 4m(P^t, 2d) + 5m(P^t, d\ell) + 4m(P^t, \ell) + 4m(P^t, r) \\
&= (n - j_1)M^3 + 4m(P^t, d) + 4m(P^t, 2d) + 5m(P^t, d\ell) + 4m(P^t, \ell) + 4m(P^t, r) \\
&\geq (n - j_1)M^3 + 4(n^2 - j_1 n - (n - j_1)) + m(P^t, d\ell) + 4m(P^t, \ell) + 4(m(P^t, \ell) \\
&+ m(P^t, d\ell) - (i_1 - i_0)) \\
&\geq (n - j_1)M^3 + 4(n - 1)(n - j_1) + 5m(P^t, d\ell) + 8m(P^t, \ell) - 4(i_1 - i_0) \\
&\geq (n - j_1)M^3 + 4(n - 1)(n - j_1) + 5(m(P^t, d\ell) + m(P^t, \ell)) - 4(i_1 - i_0) \\
&\geq (n - j_1)M^3 + 4(n - 1)(n - j_1) + (i_1 - i_0)
\end{aligned}$$

as $m(P^t, d\ell) + m(P^t, \ell) \geq i_1 - i_0$ which proves our claim. ⌟

Let $(x_{2n,j_0'}, r_{j_1}')$ be a j_1^{th} right internal sink edge contained in E (recall that $r_{j_2} \in V(E)$ and $j_2 = j_1$ by Claim 5.11). Further let $x_{i_0',(j_1-1)n+1}$ be the tail of the $(j_1 - 1)^{th}$ downward bridge in E if $j_1 > 1$ (recall that by Claim 5.10, E contains exactly one such edge) and the tail of the bottom sink edge contained in E otherwise (note that in this case $i_0' = n + i_2$). Then by connectedness of E, there exists a path P^b in $dMG_S[E]$ from $x_{i_0',(j_1-1)n+1}$ to the tail of the bottom sink edge in E; we next lower bound the weight of P^b.

Claim 5.14. *The weight of P^b is at least $(j_1 - 1)M^3 + 4(n - 1)(j_1 - 1) + 4(n + i_2 - i_0')$.*

Proof. Observe first that since by Claims 5.9 and 5.11, E contains a j_1^{th} right bridge edge and no other right bridge edge, necessarily $i_0' \geq n + 1$. Furthermore, since $dMG_S[V_r]$ contains no edge r-move, the following holds.

Observation 5.15. $i_0' \leq n + i_2$.

Now P^b must cross every column $i_0' \leq i \leq n + i_2$ and every row $1 \leq j \leq (j_1 - 1)n + 1$; thus

$$m(P^b, Db) + m(P^b, d) + m(P^b, 2d) = (j_1 - 1)n \text{ and } m(P^b, r) + m(P^b, dr) = n + i_2 - i_0'.$$

Since by Claim 5.10, P^b contains exactly $j_1 - 1$ downward bridge edges, it follows that the weight of P^b is at least

$$\begin{aligned}
&M^3 m(P^b, Db) + 4m(P^b, d) + 4m(P^b, 2d) + 4m(P^b, r) + 5m(P^b, dr) \\
&= (j_1 - 1)M^3 + 4(m(P^b, d) + m(P^b, 2d)) + 4m(P^b, r) + 5m(P^b, dr) \\
&\geq (j_1 - 1)M^3 + 4((j_1 - 1)n - (j_1 - 1)) + 4(m(P^b, r) + m(P^b, dr)) \\
&\geq (j_1 - 1)M^3 + 4(j_1 - 1)(n - 1) + 4(n + i_2 - i_0')
\end{aligned}$$

which proves our claim. ⌟

Let $x_{n+1,t}$ be the head of the j_1^{th} right bridge edge in E (recall that by Claims 5.9 and 5.11, E contains exactly one j_1^{th} right bridge edge and no other right bridge edge). Then by connectedness of E, there exist in $dMG_S[E]$ a path P and a path P' from $x_{i_0,j_1 n}$ to x_{1,j_0} and from $x_{i_0,j_1 n}$ to $x_{n,t}$ respectively; we next lower bound the weight of $P \cup P'$. To this end, denote by $x_{i',t'}$ the last vertex in $V(P)$ belonging to $V(P') \cap V_{j_1}$.

Observation 5.16. $\max\{t, j_0\} \leq t'$ *and* $n + 1 - i' \leq t' - (j_1 - 1)n$.

Indeed, since $x_{i',t'}$ can reach both x_{1,j_0} and $x_{n,t}$, $t' \geq j_0$ and $t' \geq t$. Now if $n + 1 - i' > t' - (j_1 - 1)n$ then by Observation 5.6, $x_{i',t'}$ cannot reach $x_{n,t}$ as $t - (j_1 - 1)n \geq 1 = n + 1 - n$, a contradiction.

Observation 5.17. $i_0 \le i'$.

Indeed, if $i_0 > i'$ then, in particular, $i_0 > 1$ and so, $n+1-i_0 < n = j_1 n - (j_1-1)n$; but then, by Observation 5.6, $x_{i_0,j_1 n}$ cannot reach $x_{i',t'}$ as $n+1-i_0 < n+1-i' \le t' - (j_1-1)n$ by Observation 5.16, a contradiction.

Claim 5.18. *The weight of $P \cup P'$ is at least*

$$4(n-1) + 4(t'-t) + 4(j_1 n - j_0) + 4(i'-i_0) - 3(p_1 + m(P, d\ell))$$

where

$$p_1 = \begin{cases} 1 & \text{if } (i', j_1) \in S,\ e^{\ell}_{i',j_1} \in E(P) \text{ and } x_{i',t'-1} \in V(P') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Denote by e the down or down subdivided edge with tail $x_{i',t'}$ (note that e is a down subdivided edge if and only if $(i', j_1) \in S$). Then by definition of $x_{i',t'}$, the weight of $P \cup P'$ is at least the sum of the weights of P' and $P[x_{i',t'}, x_{1,j_0}]$ minus the weight of e whenever $(i', j_1) \in S$ and $e \in E(P') \cap E(P[x_{i',t'}, x_{1,j_0}])$. Let us therefore lower bound the weights of P' and $P[x_{i',t'}, x_{1,j_0}]$.

First note that since for any left shortcut edge f with head $x_{p,q} \in V_j$, it holds that $n+1-p > q-(j-1)n$, $x_{p,q}$ cannot reach any vertex $x_{p',q'} \in V_j$ such that $n+1-p' \le q'-(j-1)n$ by Observation 5.6. Further note that since the tail of f can only be reached through $x_{p-1,q-1}$ where $n+1-(p-1) < (q-1)-(j-1)n$, no vertex $x_{p',q'} \in V_j$ such that $n+1-p' > q'-(j-1)n$ can reach the tail of f by Observation 5.6. It follows that P' and $P[x_{i_0,j_1 n}, x_{i',t'}]$ contain no left shortcut edge (recall that $n+1-i' \le t'-(j_1-1)n$ by Observation 5.16) and that $P[x_{i',t'}, x_{1,j_0}]$ contains at most one left shortcut edge, that is, the following holds.

Observation 5.19. $m(P', d\ell) = m(P[x_{i_0,j_1 n}, x_{i',t'}], d\ell) = 0$ *and* $m(P, d\ell) \in \{0, 1\}$.

Now P' must cross every row $t \le j \le j_1 n$ and $P[x_{i',t'}, x_{1,j_0}]$ must cross every row $j_0 \le j \le t'$ (recall that $\max\{t, j_0\} \le t'$ by Observation 5.16); thus

$$\begin{aligned} &m(P', d) + m(P', 2d) = j_1 n - t \text{ and} \\ &m(P[x_{i',t'}.x_{1,j_0}], d) + m(P[x_{i',t'}.x_{1,j_0}], 2d) + m(P[x_{i',t'}.x_{1,j_0}], d\ell) = t' - j_0. \end{aligned}$$

Similarly, P' must cross every column $i_0 \le i \le n$ and $P[x_{i',t'}, x_{1,j_0}]$ must cross every column $1 \le i \le i'$ (recall that $i_0 \le i'$ by Observation 5.17); thus

$$\begin{aligned} &m(P', \ell) + m(P', r) \ge n - i_0 \text{ and} \\ &m(P[x_{i',t'}.x_{1,j_0}], \ell) + m(P[x_{i',t'}.x_{1,j_0}], d\ell) \ge i' - 1. \end{aligned}$$

Now note that $p_1 = 1$ if and only if $(i', j_1) \in S$ and e belongs to both P and $P[x_{i',t'}, x_{1,j_0}]$: indeed, if $p_1 = 1$ then clearly $(i', j_1) \in S$ and $e \in E(P) \cap E(P[x_{i',t'}, x_{1,j_0}])$. Conversely, if $(i', j_1) \in S$ and e belongs to both P and $P[x_{i',t'}, x_{1,j_0}]$ then since by definition of $x_{i',t'}$, $x_{i',t'-1}$ cannot belong to both P' and $P[x_{i',t'}, x_{1,j_0}]$, and P' contains no left shortcut edge by Observation 5.19, it must be that P' contains $x_{i',t'-1}$ while $P[x_{i',t'}, x_{1,j_0}]$ contains the left shortcut edge $e^{\ell}_{(i',j_1)}$, that is, $p_1 = 1$. It follows that the weight of $P \cup P'$ is at least

$$\begin{aligned} &4(m(P', d) + m(P', 2d)) + 4(m(P', \ell) + m(P', r)) + 4(m(P[x_{i',t'}.x_{1,j_0}], d) + m(P[x_{i',t'}.x_{1,j_0}], 2d)) \\ &+ 5(m(P[x_{i',t'}.x_{1,j_0}], d\ell) - p_1) + 2p_1 + 4m(P[x_{i',t'}.x_{1,j_0}], \ell) \\ &\ge 4(j_1 n - t) + 4(n - i_0) + 4(t' - j_0) + m(P[x_{i',t'}.x_{1,j_0}], d\ell) - 3p_1 \\ &+ 4(i' - 1 - m(P[x_{i',t'}.x_{1,j_0}], d\ell)) \\ &\ge 4(n-1) + 4(t'-t) + 4(j_1 n - j_0) + 4(i' - i_0) - 3(p_1 + m(P[x_{i',t'}.x_{1,j_0}], d\ell)) \end{aligned}$$

which proves our claim as $m(P[x_{i',t'}.x_{1,j_0}], d\ell) = m(P, d\ell)$ by Observation 5.19. ⌟

Note that if $p_1 = 1$ then, since $x_{i'-1,t'-1} \in V(P)$ can reach x_{1,j_0} and $x_{i',t'-1} \in V(P')$ can reach $x_{n,t}$, the following holds.

Observation 5.20. *If $p_1 = 1$ then $j_0 \leq t' - 1$ and $t \leq t' - 1$.*

Now by connectedness of E, there exist in $dMG_S[E]$ a path Q and a path Q' from $x_{n+1,t}$ to $x_{i'_0,(j_1-1)n+1}$ and from $x_{n+1,t}$ to x_{2n,j'_0}, respectively; we next lower bound the weight of $Q \cup Q'$. To this end, denote by $x_{i'',t''}$ the last vertex in $V(Q')$ belonging to $V(Q) \cap V_{j_1}$.

Claim 5.21. *The weight of $Q \cup Q'$ is at least*

$$4(n-2) + 4(i'_0 - i'') + 4(t'' - j'_0) + 4(t - (j_1 - 1)n) - 2p_2$$

where

$$p_2 = \begin{cases} 1 & \text{if } (i''-n, j_1) \in S \text{ and } e^r_{i''-n,j_1} \in E(Q'), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that since $x_{n+1,t}$ can reach $x_{i'',t''}$, and $x_{i'',t''}$ can reach both x_{2n,j'_0} and $x_{i'_0,(j_1-1)n+1}$, the following holds.

Observation 5.22. *$j'_0 \leq t'' \leq t$ and $i'' \leq i'_0$.*

Now denote by e the down or down subdivided edge with tail $x_{i'',t''}$ (note that e is a down subdivided edge if and only if $(i''-n, j_1) \in S$). Then by definition of $x_{i'',t''}$, the weight of $Q \cup Q'$ is at least the sum of the weights of Q and $Q'[x_{i'',t''}, x_{2n,j'_0}]$ minus the weight of e whenever $(i''-n, j_1) \in S$ and $e \in E(Q) \cap E(Q'[x_{i'',t''}, x_{2n,j'_0}])$. Let us therefore lower bound the weights of Q and $Q'[x_{i'',t''}, x_{2n,j'_0}]$.

Since Q must cross every row $(j_1 - 1)n + 1 \leq j \leq t$ and $Q'[x_{i'',t''}, x_{2n,j'_0}]$ must cross every row $j'_0 \leq j \leq t''$ (recall that $j'_0 \leq t'' \leq t$ by Observation 5.22), it follows that

$$m(Q, d) + m(Q, 2d) = t - ((j_1 - 1)n + 1) \text{ and}$$
$$m(Q'[x_{i'',t''}, x_{2n,j'_0}], d) + m(Q'[x_{i'',t''}, x_{2n,j'_0}], 2d) = t'' - j'_0.$$

Similarly, Q must cross every column $n + 1 \leq i \leq i'_0$ and $Q'[x_{i'',t''}, x_{2n,j'_0}]$ must cross every column $i'' \leq i \leq 2n$ (recall that $i'' \leq i'_0$ by Observation 5.22); thus

$$m(Q, r) + m(Q, dr) = i'_0 - (n + 1) \text{ and}$$
$$m(Q'[x_{i'',t''}, x_{2n,j'_0}], r) + m(Q'[x_{i'',t''}, x_{2n,j'_0}], dr) = 2n - i''.$$

Now note that $p_2 = 1$ if and only if $(i''-n, j_1) \in S$ and e belongs to both Q and Q': indeed, if $p_2 = 1$ then by definition $(i''-n, j_1) \in S$, $e \in E(Q')$ and $x_{i''+1,t''} \in V(Q')$ which implies, by definition of $x_{i'',t''}$, that $x_{i''+1,t''} \notin V(Q)$ and so, $e \in E(Q)$. Conversely, if $(i''-n, j_1) \in S$ and e belongs to both Q and Q' then, by definition of $x_{i'',t''}$, it must be that $x_{i'',t''-1}$ belongs to one of Q and Q' while $x_{i''+1,t''}$ belongs to the other. However, if $x_{i''+1,t''}$ belongs to Q (and so, $x_{i'',t''-1} \in V(Q')$) then, since Q crosses row $(j_1 - 1)n + 1$ and Q' crosses column $2n$, necessarily $Q[x_{i''+1,t''}, x_{i'_0,(j_1-1)n+1}] \cap Q'[x_{i'',t''-1}, x_{2n,j'_0}] \cap V_{j_1} \neq \emptyset$, a contradiction to the definition of $x_{i'',t''}$. Thus, $x_{i''+1,t''} \in V(Q')$ and so, $e^r_{(i''-n,t'')} \in E(Q')$, that is, $p_2 = 1$. It follows that the weight of $Q \cup Q'$ is at least

$$\begin{aligned}
&4(m(Q,d) + m(Q,2d)) + 4m(Q,r) + 5m(Q,dr) + 4(m(Q'[x_{i'',t''}, x_{2n,j'_0}], d) \\
&+ m(Q'[x_{i'',t''}, x_{2n,j'_0}], 2d)) + 4m(Q'[x_{i'',t''}, x_{2n,j'_0}], r) + 5(m(Q'[x_{i'',t''}, x_{2n,j'_0}], dr) - p_2) + 2p_2 \\
&\geq 4(t - ((j_1 - 1)n + 1)) + 4(m(Q,r) + m(Q,dr)) + 4(t'' - j'_0) + 4(2n - i'') \\
&+ m(Q'[x_{i'',t''}, x_{2n,j'_0}], dr) - 3p_2 \\
&\geq 4(t - ((j_1 - 1)n + 1)) + 4(i'_0 - (n + 1)) + 4(t'' - j'_0) + 4(2n - i'') \\
&+ m(Q'[x_{i'',t''}, x_{2n,j'_0}], dr) - 3p_2 \\
&\geq 4(n-2) + 4(i'_0 - i'') + 4(t'' - j'_0) + 4(t - (j_1 - 1)n) - 2p_2
\end{aligned}$$

which proves our claim. ⌟

In the following, we write $j_0 = (j_1 - 1)n + p_0$ and $j_0' = (j_1 - 1)n + p_0'$ and further let

$$W = |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j_0') - (8 + 3(p_1 + m(P, d\ell)) + 2p_2).$$

Observe that by Claims 5.13, 5.14, 5.18 and 5.21, the weight of the union of P^t, P^b, P, P', Q and Q' is at least

$$(n-1)M^3 + 4n(n+1) + W$$

and so, by Claims 5.8, 5.9, 5.10 and 5.11, the weight of E is at least

$$2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) + p_0 - p_0' + W. \tag{2}$$

Claim 5.23. *The following hold.*
(1) $i_1 = i_2$ *and* $(i_1, j_1) \in S$*.*
(2) $W = -12$ *and* $p_0 = p_0'$*.*

Proof. Since the weight of E is at most M_n^*, it follows from Equation (2) that

$$\begin{aligned} M_n^* &= 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) - 12 \\ &\geq 2M^5 + M^4 + (n-1)M^3 + M^2 + (4n+1)(n+1) + p_0 - p_0' + W. \end{aligned}$$

Thus, by definition of W,

$$\begin{aligned} &p_0 - p_0' + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j_0') \\ &\leq -4 + 3(p_1 + m(P, d\ell)) + 2p_2 \\ &\leq 4 \end{aligned} \tag{3}$$

as $p_1, p_2 \in \{0, 1\}$ by definition and $m(D, d\ell) \in \{0, 1\}$ by Observation 5.19. Now suppose to the contrary that $p_0' > p_0$ or, equivalently, that $j_0' > j_0$. Then since $t' \geq t \geq t'' \geq j_0'$ by Observations 5.16 and 5.22, $p_0' - p_0 = j_0' - j_0 \leq t' - j_0$ and so,

$$\begin{aligned} &p_0 - p_0' + 4(t' - j_0) + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t'' - j_0') \\ &\geq 3(t' - j_0) + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t'' - j_0'). \end{aligned}$$

But $n + i_2 \geq i''$, $i' \geq i_0$ and $t' \geq t'' \geq j_0' > j_0$ by Observations 5.15, 5.16, 5.17 and 5.22, hence

$$3(t' - j_0) + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t'' - j_0') \geq 3.$$

It then follows from Equation (3) and the above that $p_1 = m(P, d\ell) = p_2 = 1$ and $t' = j_0 + 1$; however, by Observation 5.20, if $p_1 = 1$ then $t' \geq t + 1$ and so, by Observation 5.16, $j_0' \leq t \leq t' - 1 = j_0$, a contradiction. Thus, $p_0 \geq p_0'$ and so,

$$p_0 - p_0' + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j_0') \geq 0 \tag{4}$$

by Observations 5.15, 5.16, 5.17 and 5.22. Now suppose for a contradiction that $p_1 = 0$. Then by Equation (3),

$$p_0 - p_0' + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j_0') \leq 1$$

and so, by Observations 5.15, 5.16, 5.17 and 5.22, we must have $n + i_2 = i''$, $t' = j_0$, $t'' = j_0'$ and $i_0 = i'$. But then, by Observations 5.12 and 5.16, $p_0 = n + 1 - i_0$ which implies that P contains no left shortcut edge as $t' = j_0$; thus, by Equation (3),

$$p_0 - p_0' + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j_0') \leq -2,$$

as $m(P, d\ell) = 0$, a contradiction to Equation (4). Hence, $p_1 = 1$ and so, $m(D, d\ell) = 1$ as well. Now by Observation 5.20, $t' \geq j_0 + 1$ and so, by Equation (3),

$$4 \leq 4(t' - j_0) \leq p_0 - p'_0 + |i_1 - i_0| + 4(n + i_2 - i'') + 4(i' - i_0) + 4(t' - j_0) + 4(t'' - j'_0) \leq 2 + 2p_2.$$

Therefore, $p_2 = 1$ and $t' - 1 = j_0 = j'_0 = t''$, $i_1 = i_0 = i'$ and $n + i_2 = i''$ by Observations 5.15, 5.16, 5.17 and 5.22; in particular, $W = -12$. Now since $p_1 = 1$, $(i', j_1) \in S$ by definition and so, $t' = j_1 n + 2 - i'$ by construction; and since $p_2 = 1$, $(i'' - n, j_1) \in S$ by definition and so, $t'' = j_1 n + 1 - (i'' - n)$ by construction. As $t' - 1 = t''$, $i' = i_1$ and $i_2 = i'' - n$, we conclude that $i_1 = i_2$ and $(i_1, j_1) \in S$. ⌟

It now follows from Claims 5.11 and 5.23(1) that E represents the pair $(i_1, j_1) \in S$; and by combining Equation (2) with Claim 5.23(2), we conclude that the weight of E is at least M_n^* and thus exactly M_n^* which completes the proof. □

Up Main Gadget. Given an integer $n > 0$, the up main gadget uMG_S represents a set $S \subseteq [n] \times [n]$[5] and is constructed as follows. It is an edge-weighted planar digraph consisting of a $2n \times n^2$ grid, where the vertex lying at the intersection of column i and row j is denoted by $x_{i,j}$, and $6n$ additional vertices $\ell_1, \ldots, \ell_n, \ell'_1, \ldots, \ell'_n, r_1, \ldots, r_n$, $r'_1, \ldots, r'_n, t_1, \ldots, t_n$, $b_1, \ldots, b_n$. The adjacencies and edge weights are defined as follows.

- *Source edges:* for every $i \in [n]$, there is an edge $(b_i, x_{n+i,1})$. Together these edges are called source edges. The weight of each such edge is set to M^5.
- *Top sink edges:* for every $i \in [n]$, there is an edge (x_{i,n^2}, t_i). Together these edges are called top sink edges. The weight of each such edge is set to M^5.
- *Left sink edges:* for every $j \in [n]$, there is an edge (ℓ'_j, ℓ_j) whose weight is set to Mj. Together these edges are called left sink edges.
- *Right sink edges:* for every $j \in [n]$, there is an edge (r'_j, r_j) whose weight is set to $M^2 - Mj$. Together these edges are called right sink edges.
- *Left internal sink edges:* for every $j \in [n]$ and every $(j-1)n + 1 \leq p \leq jn$, there is an edge $(x_{1,p}, \ell''_j)$ whose weight is set to $p - (j-1)n$. For every fixed $j \in [n]$, these edges together are called the j^{th} left internal sink edges.
- *Right internal sink edges:* for every $j \in [n]$ and every $(j-1)n + 1 \leq p \leq jn$, there is an edge $(x_{2n,p}, r''_j)$ whose weight is set to $n + 1 - (p - (j-1)n)$. For every fixed $j \in [n]$, these edges together are called the j^{th} right internal sink edges.
- *Left bridge edges:* for every $j \in [n]$, and every $(j-1)n + 1 \leq p \leq jn$, there is an edge $(x_{n+1,p}, x_{n,p})$. For every fixed $j \in [n]$, these edges together are called the j^{th} right bridge edges. The weight of each j^{th} left bridge edge is set to M^4.
- *Upward bridge edges:* for every $j \in \{pn \mid p \in [n-1]\}$ and every $i \in [2n]$, there is an edge $(x_{i,j}, x_{i,j+1})$. For every fixed $j \in \{pn \mid p \in [n-1]\}$, these edges together are called the j^{th} upward bridge edges. The weight of each j^{th} downward bridge edge is set to M^3.
- *Up edges:* for every $i \in [2n]$, every $j \in [n]$ and every $(j-1)n + 1 \leq p \leq jn - 1$, there is an edge $(x_{i,p}, x_{i,p+1})$. Together these edges are called up edges. The weight of each such edge is set to 4.
- *Left edges:* for every $j \in [n]$, every $(j-1)n+1 \leq p \leq jn-1$ and every $2 \leq q \leq jn+1-p$, there is an edge $(x_{q,p}, x_{q-1,p})$; and for every $j \in [n]$, every $(j-1)n + 1 \leq p \leq jn - 1$ and every $2 \leq i \leq n$, there is an edge $(x_{i,p}, x_{i-1,p}$. Together these edges are called left edges. The weight of each such edge is set to 4.
- *Right edges:* for every $j \in [n]$, every $(j-1)n+2 \leq p \leq jn$ and every $jn+1-p \leq q \leq n-1$, there is an edge $(x_{n+q,p}, x_{n+q+1,p})$. Together these edges are called right edges. The weight of each such edge is set to 4.

[5]Recall that, by assumption, $1 < x, y < n$ holds for every $(x, y) \in S$.

- *Shortcut edges:* for every $s = (i, j) \in S$, we introduce two shortcut edges e^ℓ_s, e^r_s as follows. Set $p = jn + 1$ then
  - subdivide the edge $(x_{i,p-i}, x_{i,p+1-i})$ by adding a vertex $y_{i,j}$ and the edge $(x_{i,p-i}, y_{i,j})$ (of weight 3) with the edge $(y_{i,j}, x_{i,p+1-i})$ (of weight 1);
  - subdivide the edge $(x_{n+i,p-1-i}, x_{n+i,p-i})$ by adding a vertex $z_{i,j}$ and the edge $(x_{n+i,p-1-i}, z_{i,j})$ (of weight 3) with the edge $(z_{i,j}, x_{n+i,p-i})$ (of weight 1).

  The introduced edges are called *down subdivided edges.* Then
  - $e^\ell_s = (y_{i,j}, x_{i-1,p-i})$ and its weight set to 2;
  - $e^r_s = (z_{i,j}, x_{n+i+1,p-i})$ and its weight set to 2.

The edges e^ℓ_s are called *left* shortcut edges and the edges e^r_s are called *right* shortcut edges. This concludes the construction of the up main gadget uMG_S (see Figure 15 for an illustration of the up main gadget uMG_S with $n = 4$ representing $S = \{(2, 2), (2, 3), (3, 2)\}$). We call the vertices $\ell_1, \ldots, \ell_n$ the *left vertices*, the vertices $r_1, \ldots, r_n$ the *right vertices*, the vertices $t_1, \ldots, t_n$ the *top vertices* and the vertices $b_1, \ldots, b_n$ the *bottom vertices.*

A set $E \subseteq E(uMG_S)$ satisfies the *connectedness* property if the following hold in E:
- a bottom vertex can reach a top vertex;
- a bottom vertex can reach a left vertex;
- a bottom vertex can reach a right vertex.

A set $E \subseteq E(uMG_S)$ satisfying the connectedness property *represents* a pair $(i, j) \in [n] \times [n]$ if the only source edge in E is the one incident to b_i, the only top sink edge in E is the one incident to t_i, the only left sink edge in E is the one incident to ℓ_j and the only right sink edge in E is the one incident to r_j (see Figure 15 for a set of edges representing $(2, 2)$). Symmetrical to Lemma 5.7, we have the following.

Lemma 5.24. *For any $n > 0$ and any $S \subseteq [n] \times [n]$, the up main gadget uMG_S satisfies the following properties.*
(1) For every $(i, j) \in S$, there exists a set $E_{i,j} \subseteq E(uMG_S)$ of weight M^*_n representing (i, j).
(2) If there exists a set $E \subseteq E(uMG_S)$ of weight at most M^*_n satisfying the connectedness property then E has weight exactly M^*_n and represents a pair $(i, j) \in S$.

Reduction. Given an instance $(k, n, \{S_{i,j} \mid i, j \in [k]\})$ of Grid Tiling, we construct an equivalent instance (G, T, D) of (edge-weighted) Planar $\mathcal{A}_1$-Steiner Network where G is defined as follows (see Figure 16).
- We introduce a total of k^2 (down/up) main gadgets and $k(k + 1)$ connector gadgets.
- For every set $S_{i,j}$ of the Grid Tiling instance such that i is odd, we introduce an up main gadget $uMG_{i,j}$ representing $S_{i,j}$. The up main gadget $uMG_{i,j}$ is surrounded by two connector gadgets: $CG_{i,j}$ lying to its left and $CG_{i+1,j}$ lying to its right. We identify each right vertex of $CG_{i,j}$ with the left vertex of $uMG_{i,j}$ of the same index, and each left vertex of $CG_{i+1,j}$ with the right vertex of $uMG_{i,j}$ of the same index. Furthermore, for every $j \in [k - 1]$, $uMG_{i,j}$ lies above $uMG_{i,j+1}$ and we identify each bottom vertex of $uMG_{i,j}$ with the top vertex of $uMG_{i,j+1}$ of the same index.
- For every set $S_{i,j}$ of the Grid Tiling instance such that i is even, we introduce a down main gadget $dMG_{i,j}$ representing $S_{i,j}$. The down main gadget $dMG_{i,j}$ is surrounded by two connector gadgets: $CG_{i,j}$ lying to its left and $CG_{i+1,j}$ lying to its right. We identify each right vertex of $CG_{i,j}$ with the left vertex of $dMG_{i,j}$ of the same index, and each left vertex of $CG_{i+1,j}$ with the right vertex of $dMG_{i,j}$ of the same index. Furthermore, for every $j \in [k - 1]$, $dMG_{i,j}$ lies above $dMG_{i,j+1}$ and we identify each bottom vertex of $dMG_{i,j}$ with the top vertex of $dMG_{i,j+1}$ of the same index.
- We introduce k terminals $t_1, \ldots, t_k$ and add the following edges of weight 0: for every odd $i \in [k]$, we add an edge from each top vertex of $uMG_{i,1}$ to t_i and for every even $i \in [k]$,

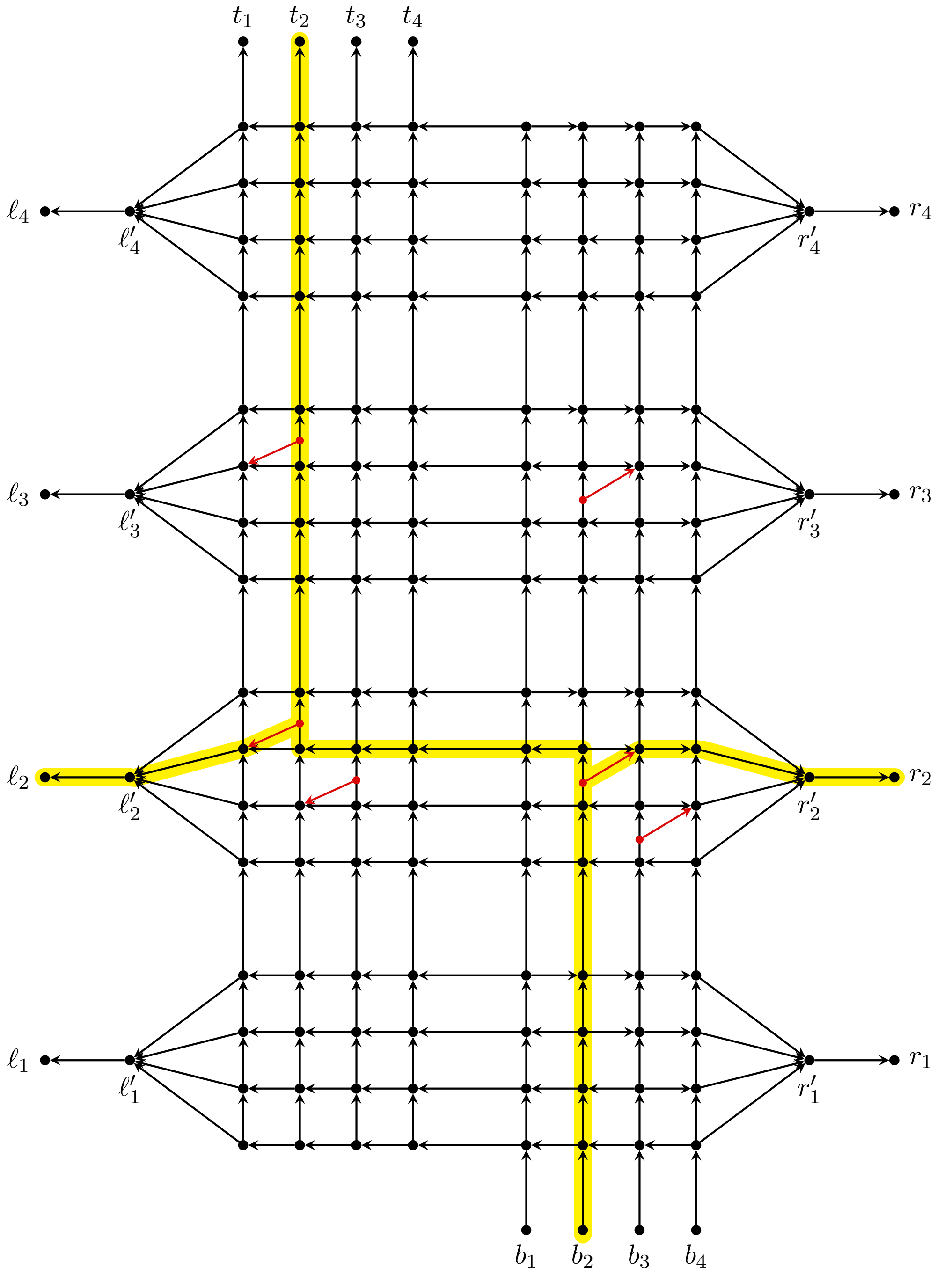

Figure 15: The up main gadget uMG_S with $n = 4$ representing $S = \{(2,2),(2,3),(3,2)\}$ (the red edges are the shortcut edges). A set of edges representing $(2,2)$ is highlighted.

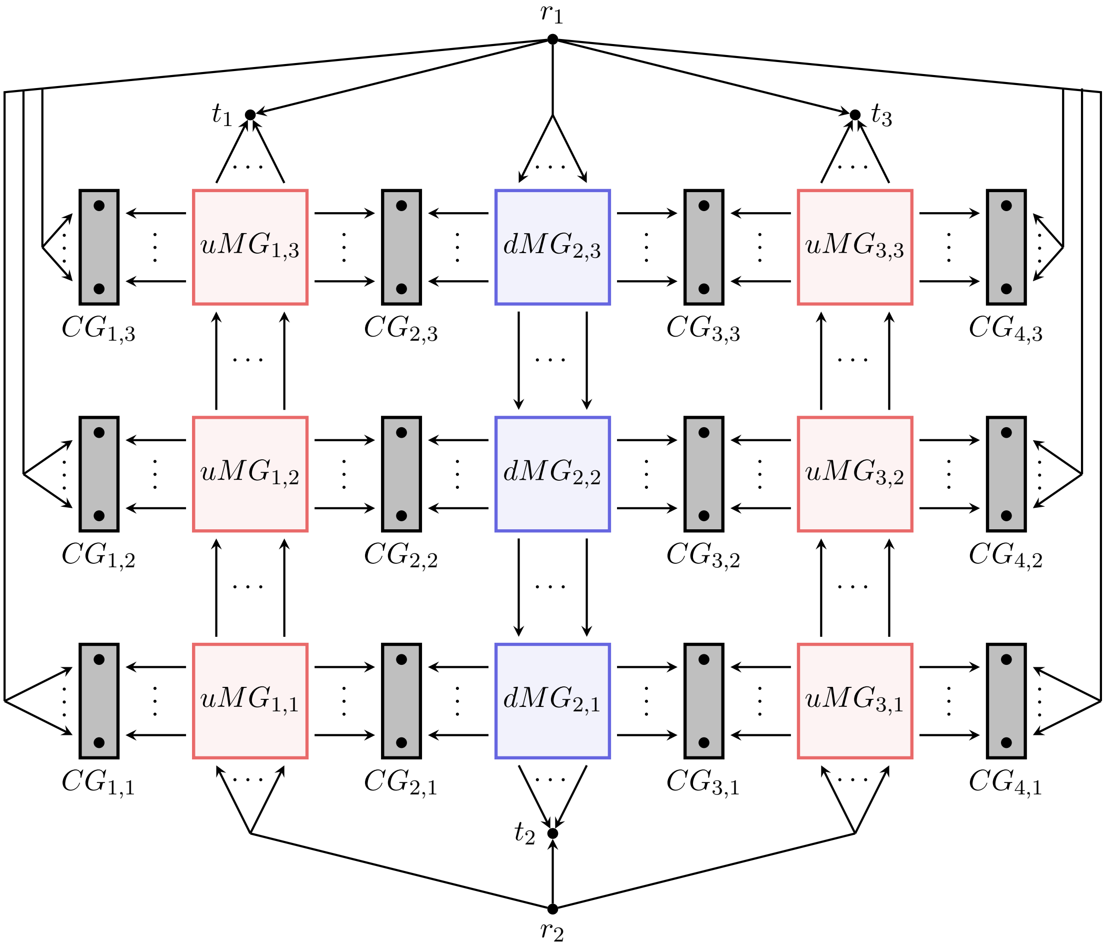

Figure 16: An illustration of the reduction from Grid Tiling to Planar $\mathcal{A}_1$-Steiner Network with $k = 3$ (the black vertices are the terminals).

we add an edge from each bottom vertex of $dMG_{i,k}$ to t_i.

- We introduce two vertices r_1 and r_2 and add the following edges of weight 0: there is an edge from r_1 to every terminal t_i such that $i \in [k]$ is odd; for every even $i \in [k]$, there is an edge from r_1 to every top vertex of $dMG_{i,1}$; for every $j \in [k]$, there is an edge from r_1 to every left vertex of $CG_{1,j}$; and if k is odd then for every $j \in [k]$, there is an edge from r_1 to every right vertex of $CG_{k+1,j}$. Similarly, there is an edge from r_2 to every terminal t_i such that i is even; for every odd $i \in [k]$, there is an edge from r_1 to every bottom vertex of $dMG_{i,k}$; and if k is even then for every $j \in [k]$, there is an edge from r_1 to every right vertex of $CG_{k+1,j}$.

This concludes the construction of G. The set T of terminals consists of the union of the two terminal vertices in each connector gadget and $\{r_1, r_2\} \cup \{t_i \mid i \in [k]\}$ (note that $|T| = 2k(k+1) + k + 2$). The demand graph D is the pure out-diamond on vertex set T where r_1 and r_2 are the two vertices of in-degree 0. In the following, we let

$$W_n^* = k^2 M_n^* + k(k+1)C_n^*.$$

Lemma 5.25. *The* Grid Tiling *instance* $(k, n, \{S_{i,j} \mid i, j \in [k]\})$ *has a solution if and only if the (edge-weighted)* Planar $\mathcal{A}_1$-Steiner Network *instance* (G, T, D) *has a solution of weight at most* W_n^*.

Proof. Assume first that the instance $(k, n, \{S_{i,j} \mid i, j \in [k]\})$ of Grid Tiling has a solution, that is, for every $i, j \in [k]$, there is an entry $(x_{i,j}, y_{i,j}) \in S_{i,j}$ such that

- for every $i \in [k]$, $x_{i,1} = x_{i,2} = \ldots = x_{i,k} = \alpha_i$ and
- for every $j \in [k]$, $y_{1,j} = y_{2,j} = \ldots = y_{k,j} = \beta_j$.

We construct a solution E for the Planar $\mathcal{A}_1$-Steiner Network instance (G, T, D) of weight at most W_n^* as follows. Include in E

- for every odd $i \in [k]$, the edge (r_1, t_i) of weight 0 and for every even $i \in [k]$, the edge (r_2, t_i) of weight 0;
- for every even $i \in [k]$, the edge from r_1 to the top vertex of index α_i in $dMG_{i,1}$ of weight 0 and for every odd $i \in [k]$, the edge from r_2 to the bottom vertex of index α_i in $dMG_{i,k}$ of weight 0;
- for every $j \in [k]$, the edge from r_1 to the left vertex of index β_j in $CG_{1,j}$ of weight 0;
- if k is odd then for every $j \in [k]$, the edge from r_1 to the right vertex of index β_j in $CG_{k+1,j}$ of weight 0 and if k is even then for every $j \in [k]$, the edge from r_2 to the right vertex of β_j in $CG_{k+1,j}$ of weight 0;
- for every $j \in [k]$ and every $i \in [k]$, the set $E^C_{i,j} \subseteq E(CG_{i,j})$ of weight C_n^* representing β_j whose existence is guaranteed by Lemma 5.2(1);
- for every $j \in [k]$ and every odd $i \in [k]$, the set $E^M_{i,j} \subseteq E(uMG_{i,j})$ of weight M_n^* representing (α_i, β_j) whose existence is guaranteed by Lemma 5.24(1); and
- for every $j \in [k]$ and every even $i \in [k]$, the set $E^M_{i,j} \subseteq E(dMG_{i,j})$ of weight M_n^* representing (α_i, β_j) whose existence is guaranteed by Lemma 5.7(1).

It is not difficult to see that the weight of E is $k^2 M_n^* + k(k+1)C_n^* = W_n^*$ and that by the connectedness of the sets $E^C_{p,j}$ and $E^M_{i,j}$ for every $i, j \in [k]$ and $p \in [k+1]$, r_1 and r_2 can reach every terminal in $T \setminus \{r_1, r_2\}$ in E.

Conversely, assume that the Planar $\mathcal{A}_1$-Steiner Network instance (G, T, D) has a solution E of weight at most W_n^*. We contend that for every $i \in [k+1]$ and every $j \in [k]$, $E \cap E(CG_{i,j})$ satisfies the connectedness property. Indeed, if this is not the case for some $i \in [k+1]$ and $j \in [k]$, say no left vertex of $V(CG_{i,j})$ can reach the terminal $p \in V(CG_{i,j})$ in E (the other cases are symmetric) then either i is even in which case r_2 cannot reach p in E, or i is odd in which case r_1 cannot reach p in E, a contradiction in both cases. Similarly, the restriction of E to any (down/up) main gadget satisfies the connectedness property: indeed, if for some $j \in [k]$ and for some even $i \in [k]$ (we argue similarly if i is odd), $E \cap E(dMG_{i,j})$ does not satisfy the connectedness property, then either no top vertex of $dMG_{i,j}$ can reach a bottom vertex of $dMG_{i,j}$ in E in which case r_2 cannot reach t_i in E; or no top vertex of $dMG_{i,j}$ can reach a left (or right) vertex of $dMG_{i,j}$ in E in which case r_2 (or r_1) cannot reach the terminal vertices of $CG_{i,j}$ in E, a contradiction in both cases.

Next, we argue that the weight of the restriction of E to any connector gadget is C_n^* and that the weight of the restriction of E to any (down/up) main gadget is M_n^*. To this end, let c and C be the number of connector gadgets whose weight in E is at most C_n^* and greater than C_n^*, respectively. Then $c + C = k(k+1)$ and by Lemma 5.2(2), any connector gadget whose weight in E is at most C_n^* has in fact a weight of exactly C_n^* in E. Similarly, let m and M be the number of (down/up) main gadgets whose weight in E is at most M_n^* and greater than M_n^* respectively. Then $m + M = k^2$ and by Lemmas 5.7 and 5.24, any (down/up) main gadget whose weight in E is at most M_n^* has in fact a weight of exactly M_n^* in E. Now by definition of W_n^*,

$$\begin{aligned} W_n^* &= k^2 M_n^* + k(k+1)C_n^* \\ &\geq mM_n^* + M(M_n^* + 1) + cC_n^* + C(C_n^* + 1) \\ &= k^2 M_n^* + M + k(k+1)C_n^* + C \end{aligned}$$

which implies that $M = C = 0$. Thus, every connector gadget has weight C^*_n in E and every (down/up) main gadget has weight M^*_n in E. From Lemmas 5.2(2), 5.7(2) and 5.24(2), it then follows that

- for every $j \in [k]$ and every $i \in [k+1]$, the restriction of E to the connector gadget $CG_{i,j}$ represents an integer $\beta'_{i,j} \in [n]$;
- for every $j \in [k]$ and every even $i \in [k]$, the restriction of E to the down main gadget $dMG_{i,j}$ represents a pair $(\alpha_{i,j}, \beta_{i,j}) \in [n] \times [n]$; and
- for every $j \in [k]$ and every odd $i \in [k]$, the restriction of E to the up main gadget $uMG_{i,j}$ represents a pair $(\alpha_{i,j}, \beta_{i,j}) \in [n] \times [n]$.

Let us show that for every $i, j \in [k]$ the entries $(\alpha_{i,j}, \beta_{i,j}) \in S_{i,j}$ form a solution to the Grid Tiling instance $(k, n, \{S_{i,j} \mid i, j \in [k]\})$ which if true, would conclude the proof. To this end, we first prove that for every $i, j \in [k]$, $\beta'_{i,j} = \beta_{i,j}$. Consider an even $i \in [k]$. Then by Lemma 5.7(2), the only left sink edge in E incident to a left vertex of $dMG_{i,j}$ is the one incident to the left vertex of index $\beta_{i,j}$; and by Lemma 5.2(2), the only right source edge in E incident to a right vertex of $CG_{i,j}$ is the one incident to the right vertex of index $\beta'_{i,j}$. Thus, if $\beta'_{i,j} \neq \beta_{i,j}$ then r_1 cannot reach the terminal vertices of $CG_{i,j}$ in E, a contradiction. We conclude similarly if i is odd.

Second, we show that for every $i, j \in [k]$, $\beta'_{i+1,j} = \beta_{i,j}$. Consider an even $i \in [k]$. Then by Lemma 5.7(2), the only right sink edge in E incident to a right vertex of $dMG_{i,j}$ is the one incident to the right vertex of index $\beta_{i,j}$; and by Lemma 5.2(2), the only left source edge in E incident to a left vertex of $CG_{i+1,j}$ is the one incident to the left vertex of index $\beta'_{i,j}$. Thus, if $\beta'_{i,j} \neq \beta_{i,j}$ then r_2 cannot reach the terminal vertices of $CG_{i+1,j}$ in E, a contradiction. We conclude similarly if i is odd.

It follows from the above that for every $i \in [k-1]$ and every $j \in [k]$, $\beta_{i+1,j} = \beta'_{i+1,j} = \beta_{i,j}$; we next show that for every $i \in [k]$ and every $j \in [k-1]$, $\alpha_{i,j} = \alpha_{i,j+1}$. Consider an even $i \in [k]$. Then by Lemma 5.7(2), the only bottom sink edge in E incident to a bottom vertex of $dMG_{i,j}$ is the one incident to the bottom vertex of index $\alpha_{i,j}$; and by Lemma 5.7(2), the only source edge in E incident to a top vertex of $dMG_{i,j+1}$ is the one incident to the top vertex of index $\alpha_{i,j+1}$. Thus, if $\alpha_{i,j} \neq \alpha_{i,j+1}$ then r_1 cannot reach the terminal t_i in E, a contradiction. We conclude similarly if i is odd. Therefore, the entries $(\alpha_{i,j}, \beta_{i,j}) \in S_{i,j}$ form a solution to the Grid Tiling instance $(k, n, \{S_{i,j} \mid i, j \in [k]\})$ as claimed. □

Let us finally explain how to get rid of the edge-weights (we use the same trick as in [8]). We replace every edge (x, y) of weight w in the instance (G, T, D) of (edge-weighted) Planar $\mathcal{C}_1$-Steiner Network constructed above, with a directed path from x to y of length $w \cdot n + 1$ where $n = |V(G)|$. We let G' be the resulting graph. Then similarly to [8, Theorem A.1], we can show that the instance (G, T, D) of edge-weighted Planar $\mathcal{C}_1$-Steiner Network has a solution of weight at most W if and only if the instance (G', T, D) of Planar $\mathcal{C}_1$-Steiner Network has a solution of size at most $Wn + n$.

5.3 Hard patterns

The aim of this section is to prove that for every $\ell \in [8104]$, Planar $\mathcal{C}_\ell$-Steiner Network is W[1]-hard parameterized by the number k of terminals and does not admit a $f(k) \cdot n^{o(k)}$ algorithm for any computable function f, unless ETH fails. For each $\ell \in [8104]$, we give a reduction which transforms an instance of $k \times k$-Grid Tiling (see Section 5.2 for a definition of this problem) into an instance of (edge-weighted)[6] Planar $\mathcal{C}_\ell$-Steiner Network with $O(k)$ terminals. The constructed instances in each case are very similar and are based on a construction developed in the proof of [8, Theorem 1.4], which we describe below as the *main gadget*.

[6]We then use the same trick as in Section 5.2 to get rid of the edge-weights.

We then show how to built upon this construction to handle the hard matching patterns (see Section 5.3.1) and the hard biclique patterns (see Section 5.3.2).

Main Gadget. As mentioned above, we use the same construction as in the proof of [8, Theorem 1.4]. More precisely, given an integer $n > 0$ and a subset $S \subseteq [n] \times [n]$[7], we first construct an edge-weighted planar digraph $G(S)$ as follows. The graph $G(S)$ consists of an $n \times n$ grid where the horizontal edges are oriented towards the right and the vertical edges are oriented towards the bottom, that is, denoting by $x_{i,j}$ the vertex lying at the intersection of column i and row j, there is an edge

- $(x_{i+1,j}, x_{i,j})$ (of weight 2) for every $i \in [n-1]$ and $j \in [n]$, and
- $(x_{i,j}, x_{i,j+1})$ (of weight 2) for every $i \in [n]$ and $j \in [n-1]$.

Then for every $(i,j) \in S$, we subdivide the edge $(x_{i-1,j}, x_{i,j})$, by adding a vertex $y_{i,j}$ and the edges $(x_{i-1,j}, y_{i,j})$ and $(y_{i,j}, x_{i,j})$ (both of weight 1), and further add the edge $(x_{i,j-1}, y_{i,j})$ (of weight 1). This concludes the construction of $G(S)$. In the following, we call the vertices $x_{1,1}, x_{1,2}, \dots, x_{1,n}$ the *left vertices*, the vertices $x_{n,1}, x_{n,2} \dots, x_{n,n}$ the *right vertices*, the vertices $x_{1,n}, x_{2,n}, \dots, x_{n,n}$ the *top vertices* and the vertices $x_{1,1}, x_{2,1}, \dots, x_{n,1}$ the *bottom vertices*.

Now given a collection $\mathcal{S} = \{S_{i,j} \mid i,j \in [k]\}$ of k^2 subsets of $[n] \times [n]$, the *main gadget* $MG(\mathcal{S})$ for $\mathcal{S}$ is constructed as follows.

- For every set $S_{i,j} \in \mathcal{S}$, we introduce a copy of the graph $G(S_{i,j})$ as constructed above. For every $i \in [n]$ and $j \in [n-1]$, the graph $G(S_{i,j+1})$ lies below the $G(S_{i,j})$; we add an edge (of weight 2) from each top vertex of $G(S_{i,j+1})$ to the bottom vertex of $G(S_{i,j})$ of the same index. Similarly, for every $i \in [n-1]$ and $j \in [n]$, the graph $G(S_{i,j})$ lies to the left of the graph $G(S_{i+1,j})$; we add an edge (of weight 2) from each right vertex of $G(S_{i,j})$ to the left vertex of $G(S_{i+1,j})$ of the same index.
- We introduce $4k$ additional vertices $a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_k, d_1, \dots, d_k$ and the following edges (we fix $\Delta = 5n^2$).
  - For every $j \in [k]$ and every $i \in [n]$, there is an edge from a_j to the left vertex of $G(S_{1,j})$ of index i of weight $\Delta(n+1-i)$.
  - For every $j \in [k]$ and every $i \in [n]$, there is an edge from the right vertex of $G(S_{n,j})$ of index i to b_j of weight Δi.
  - For every $j \in [k]$ and every $i \in [n]$, there is an edge from c_j to the top vertex of $G(S_{j,1})$ of index i of weight $\Delta(n+1-i)$.
  - For every $j \in [k]$ and every $i \in [n]$, there is an edge from the bottom vertex of $G(S_{j,n})$ of index i to d_j of weight Δi.

This concludes the construction of the main gadget $MG(\mathcal{S})$ for $\mathcal{S}$ (see Figure 17 for an illustration of the main gadget $MG(\mathcal{S})$ with $n = 4$ for $\mathcal{S} = \{S_{i,j} \mid i,j \in [3]\}$ where $S_{1,1} = \{(2,2),(2,3),(3,3)\}$ and $S_{i,j} = \emptyset$ for every $(i,j) \neq (1,1)$). We now set $I = \{a_i, b_i, c_i, d_i \mid i \in [k]\}$ and let M be the induced matching $\{(a_i, b_i), (c_i, d_i) \mid i \in [k]\}$. We further set

$$B^* = 2k(\Delta(n+1) + 2(k+1) + 2k(n-1)).$$

Lemma 5.26 ([8]). *The $k \times k$-Grid Tiling instance $(k, n, \mathcal{S})$ has a solution if and only if the (edge-weighted) Planar M-Steiner Network instance $(MG(\mathcal{S}), I, M)$ has a solution of weight at most $B^* - k^2$.*

5.3.1 Hard matching patterns

The aim of this section is to prove hardness for the hard matching patterns. We restate here the definition of these graphs for the reader's convenience.

[7]Recall that, by assumption, $1 < x, y < n$ holds for every $(x,y) \in S$.

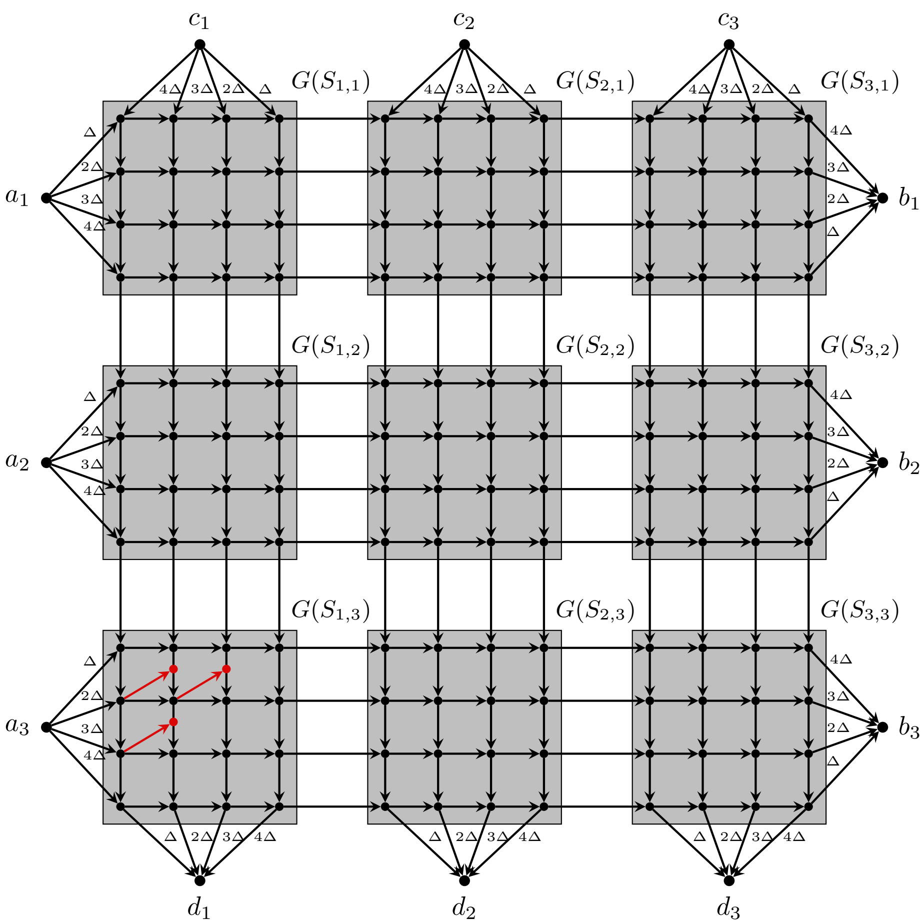

Figure 17: The main gadget $MG(\mathcal{S})$ with $n = 4$ for $\mathcal{S} = \{S_{i,j} \mid i, j \in [3]\}$ where $S_{1,1} = \{(2, 2), (2, 3), (3, 3)\}$ and $S_{i,j} = \emptyset$ for every $(i, j) \neq (1, 1)$.

Definition 5.27 (t-hard matching pattern)**.** A *t-hard matching pattern* is an (acyclic) digraph G constructed the following way. We start with disjoint vertex sets $A = \{a_1, \ldots, a_t\}$, $B = \{b_1, \ldots, b_t\}$, $C = \{c_1, \ldots, c_t\}$ and $D = \{d_1, \ldots, d_t\}$ and introduce the edges (a_i, b_i) and (c_i, d_i) for every $i \in [t]$. Furthermore, we introduce into G any combination of the following items:

1. either the directed path $a_1 \to a_2 \to \ldots \to a_t \to d_1 \to d_2 \to \ldots \to d_t$, or any of the directed paths $a_1 \to a_2 \to \ldots \to a_t$ and $d_1 \to d_2 \to \ldots \to d_t$;
2. either the directed path $c_1 \to c_2 \to \ldots \to c_t \to b_1 \to b_2 \to \ldots \to b_t$, or any of the directed paths $b_1 \to b_2 \to \ldots \to b_t$ and $c_1 \to c_2 \to \ldots \to c_t$;
3. an S-source for exactly one $S \in \{A, B, C, D, A \cup C, B \cup D, B \cup C, A \cup D\}$;
4. an S-sink for exactly one $S \in \{A, B, C, D, A \cup C, B \cup D, B \cup C, A \cup d\}$;
5. a vertex r_{AD} such that $N^-(r_{AD}) = A$ and $N^+(r_{AD}) = D$;
6. a vertex r_{CB} such that $N^-(r_{CB}) = Y$ and $N^+(r_{CB}) = B$.

In particular, there are $5 \cdot 5 \cdot 9 \cdot 9 \cdot 2 \cdot 2$ types of t-hard matching patterns: we let $\mathcal{C}_5, \ldots, \mathcal{C}_{8104}$ be the 8100 classes that each contain all the t-hard matching patterns of a specific type for every t.

Formally, we aim to prove the following.

Lemma 5.27. *For every $\ell \in [5, 8104]$,* Planar $\mathcal{C}_\ell$-Steiner Network *is* W[1]*-hard parameterized by the number k of terminals and does not admit a $f(k) \cdot n^{o(k)}$ algorithm for any computable function f, unless* ETH *fails.*

To prove the lemma, we use the construction described above and further add vertices and edges so as to take into account the specific combination of the items in Definition 1.4 for each fixed $\ell \in [5, 8104]$.

Reduction. Consider $\ell \in [7, 2031]$. Given an instance $(k, n, \mathcal{S} = \{S_{i,j} \mid i, j \in [k]\})$ of $k \times k$-Grid Tiling, we construct an equivalent instance $(G(\mathcal{S}), T, D)$ of (edge-weighted) Planar $\mathcal{C}_\ell$-Steiner Network as follows. Let $MG(\mathcal{S})$ be the main gadget for $\mathcal{S}$ as constructed above. Then T contains $\{a_i, b_i, c_i, d_i \mid i \in [k]\}$ (and possibly, as described below, some additional vertices depending on the class $\mathcal{C}\ell$) where $\{(a_i, b_i) \mid i \in [k]\}$ and $\{(c_i, d_i) \mid i \in [k]\}$ are the two perfect matchings contained in D. The graph $G(\mathcal{S})$ is then obtained from $MG(\mathcal{S})$ by introducing the directed paths

$$a_1 \to \ldots \to a_k, d_1 \to \ldots \to d_k, c_1 \to \ldots \to c_k \text{ and } b_1 \to \ldots \to b_k$$

where the weight of each newly added edge is set to 0. Furthermore, if the patterns in the class C_ℓ contain

- a source (that is, a vertex of item 3 in Definition 1.4) then we add a vertex s to $G(\mathcal{S})$ and T, and the edges (s, a_1) and (s, c_1), both of weight 0;
- a sink (that is, a vertex of item 4 in Definition 1.4) then we add a vertex t to $G(\mathcal{S})$ and T, and the edges (b_k, t) and (d_k, t), both of weight 0;
- the vertex of item 5 in Definition 1.4, then we add a vertex r_{AD} to $G(\mathcal{S})$ and T, and the edges (a_k, r_{AD}) and (r_{AD}, d_1), both of weight 0;
- the vertex of item 6 in Definition 1.4, then we add a vertex r_{CB} to $G(\mathcal{S})$ and T, and the edges (c_k, r_{CB}) and (r_{CB}, b_1), both of weight 0.

This concludes the construction of $G(\mathcal{S})$. We let D be the corresponding k-hard matching pattern of $\mathcal{C}_\ell$ on vertex set T. Lemma 5.27 then follows from Lemma 5.26 and the lemma below.

Lemma 5.28. *The (edge-weighted)* Planar $\mathcal{C}_\ell$-Steiner Network *instance* $(G(\mathcal{S}), T, D)$ *has a solution of weight at most $B^* - k^2$ if and only if the (edge-weighted)* Planar M-Steiner Network *instance* $(MG(\mathcal{S}), I, M)$ *has a solution of weight at most $B^* - k^2$.*

Proof. If E is a solution of $(MG(\mathcal{S}), I, M)$ of weight at most $B^* - k^2$ then it is easy to see that $E \cup (E(G(\mathcal{S})) \setminus E(MG(\mathcal{S})))$ is a solution of $(G(\mathcal{S}), T, D)$ of weight at most $B^* - k^2$. Conversely, if E is a solution of $(G(\mathcal{S}), T, D)$ of weight at most $B^* - k^2$ then the restriction of E to $MG(\mathcal{S})$ is readily seen to be a solution of $(MG(\mathcal{S}), I, M)$ of weight at most $B^* - k^2$. □

5.3.2 Hard biclique patterns

The aim of this section is to prove hardness for the hard biclique patterns. We restate here the definition of these graphs for the reader's convenience.

Definition 5.29 (t-hard biclique pattern)**.** A *t-hard biclique pattern* is an (acyclic) digraph D constructed the following way. We start with two disjoint sets A and B with $|A| = |B| = t$ and introduce every edge from A to B. Furthermore, we introduce into D any combination of the following items (see Figure 1):

1. an A-source;
2. a B-sink.

In particular, there are $2 \cdot 2$ types of t-hard biclique patterns: we let $\mathcal{C}_1, \ldots, \mathcal{C}_4$ be the 4 classes that each contain all the t-hard biclique patterns of a specific type for every t.

Formally, we aim to prove the following.

Lemma 5.29. *For every $\ell \in [4]$,* Planar $\mathcal{C}_\ell$-Steiner Network *is* W[1]*-hard parameterized by the number k of terminals and does not admit a $f(k) \cdot n^{o(k)}$ algorithm for any computable function f, unless* ETH *fails.*

We only formally prove the statement for the class of all hard biclique patterns containing no further vertices as it will become clear from the proof that to handle the classes of all hard biclique patterns containing

- the sink vertex, it suffices to add a vertex t and an edge (t_i, t) for each $i \in [2k+1]$ in the construction below; or
- the source vertex, it suffices to add a vertex s and an edge (s, s_i) for each $i \in [2k+1]$ in the construction below.

In the following, we assume that the class of all hard biclique patterns with no further vertices is $\mathcal{C}_1$.

Reduction. Given an instance $(k, n, \mathcal{S} = \{S_{i,j} \mid i, j \in [k]\})$ of $k \times k$-Grid Tiling, we construct an equivalent instance $(G(\mathcal{S}), T, D)$ of (edge-weighted) Planar $\mathcal{C}_1$-Steiner Network as follows. We start by constructing an auxiliary planar digraph H consisting of $2(2k+1)$ distinguished vertices $s_1, \ldots, s_{2k+1}, t_1, \ldots, t_{2k+1}$, and $2(2k+1)$ edge-disjoint directed paths $P_{i,j}$ with $i \in [2k+1]$ and $j \in \{i, i+1\}$ (where indices are taken modulo $2k+1$ henceforth), defined as follows.

- For every $i \in [2k+1]$, $P_{i,i} = s_i u_i^1 \ldots u_i^{2k-1} t_i$ is a directed path from s_i to t_i of length $2k$.
- For every $i \in [2k+1]$, $P_{i,i+1}$ is the directed path $s_i u_{i-1}^1 u_{i-2}^2 \ldots u_{i-j}^j \ldots u_{i+2}^{2k-1} t_{i+1}$ from s_i to t_{i+1} of length $2k$.

(For $k = 3$, the graph H can be obtained from the graph depicted in Figure 18 by ignoring the blue vertices and contracting every grey box into a single vertex.) Note that by construction, the following holds.

Observation 5.30. *For every $i \in [2k+1]$, there is a unique path from s_i to t_i (t_{i+1}, respectively) in H, namely $P_{i,i}$ ($P_{i,i+1}$, respectively).*

The graph $G(\mathcal{S})$ is then obtained from H as follows (see Figure 18).

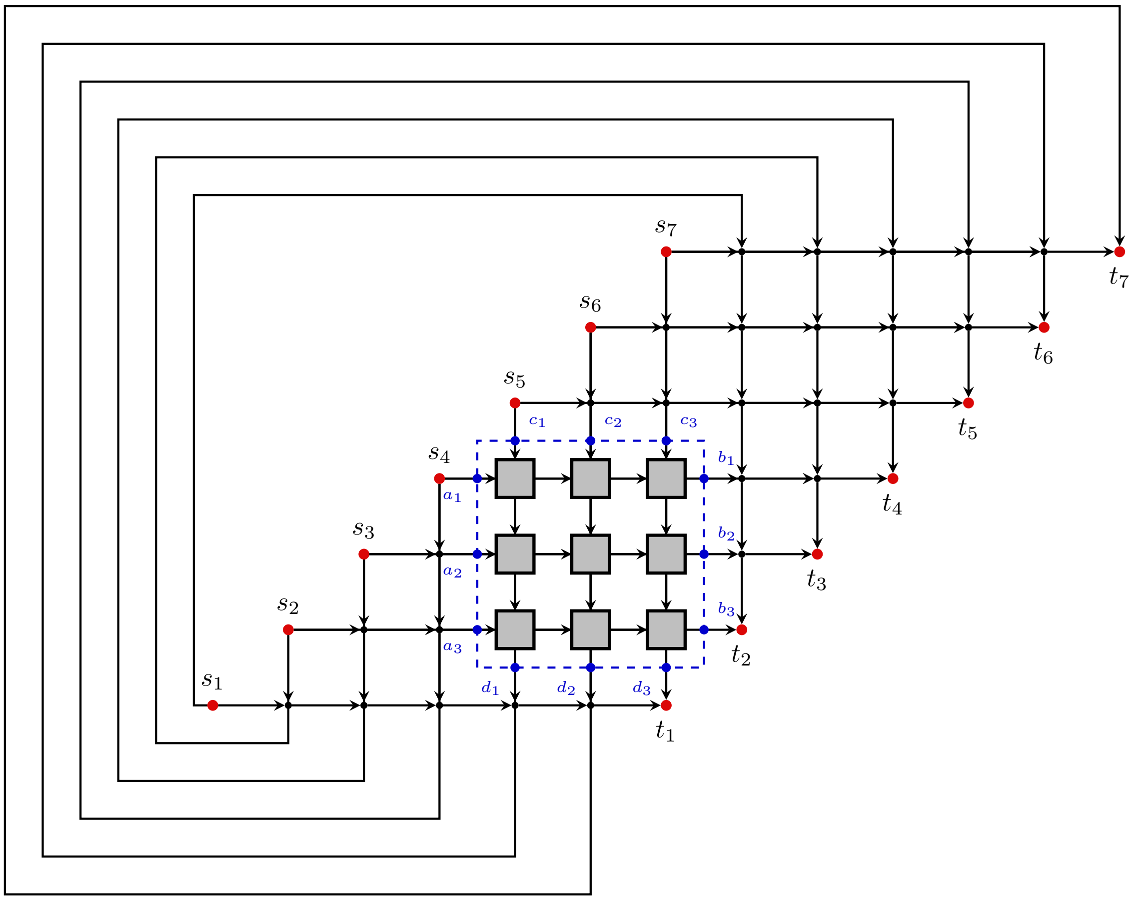

Figure 18: An illustration of the reduction from $k \times k$-GRID TILING to PLANAR $\mathcal{C}_1$-STEINER NETWORK with $k = 3$ (the red vertices are the terminals and the dashed blue square together with the grey boxes represent the main gadget).

- We subdivide the edge (s_{k+1}, u_1^{k+1}) by adding the vertex a_1 and the edges (s_{k+1}, a_1) and (a_1, u_1^{k+1}); and for every $2 \le i \le k$, we subdivide the edge $(u_{k+1-i}^i, u_{k+2-i}^i)$ by adding the vertex a_{k+2-i} and the edges (u_{k+1-i}^i, a_{k+2-i}) and (a_{k+2-i}, u_{k-i}^i).
- We subdivide the edge (u_{2k-1}^2, t_2) by adding the vertex b_k and the edges (u_{2k-1}^2, b_k) and (b_k, t_2); and for every $3 \le i \le k+1$, we subdivide the edge $(u_{2k+1-i}^i, u_{2k+2-i}^i)$ by adding the vertex b_{k+2-i} and the edges $(u_{2k+1-i}^i, b_{k+2-i})$ and $(b_{k+2-i}, u_{2k+2-i}^i)$.
- We subdivide the edge (s_{k+2}, u_1^{k+1}) by adding the vertex c_1 and the edges (s_{k+2}, c_1) and (c_1, u_1^{k+1}); and for every $k+3 \le i \le 2k+1$, we subdivide the edge $(u_{i-k-2}^{k+2}, u_{i-k-2}^{k+1})$ by adding the vertex c_{i-k-1} and the edges $(u_{i-k-2}^{k+2}, c_{i-k-1})$ and $(c_{i-k-1}, u_{i-k-2}^{k+1})$.
- For every $k+2 \le i \le 2k+1$, we subdivide the edge $(u_{i-k-2}^2, u_{i-k-2}^1)$ by adding the vertex d_{i-k-1} and the edges (u_{i-k-2}^2, d_{i-k-1}) and (d_{i-k-1}, u_{i-k-2}^1).
- For every $1 \le i, j \le k$, we replace the vertex u_{k-j+i}^{j+1} with a copy of the graph $G(S_{i,k+1-j})$ and add the necessary edges so that the subgraph of G induced by $\{a_i, b_i, c_i, d_i \mid 1 \le i \le k\} \cup \bigcup_{1 \le i,j \le k} V(G(S_{i,j}))$ is isomorphic to $MG(\mathcal{S})$ (and has the same edge-weight function).

The weight of each edge outside the copy of the main gadget $MG(\mathcal{S})$ is set to 0. This concludes the construction of $G(\mathcal{S})$. We set $T = \{s_i, t_i \mid i \in [2k+1]\}$ and let the demand graph D be the corresponding $(\{s_i \mid i \in [2k+1]\}, \{t_i \mid i \in [2k+1]\})$-biclique. Lemma 5.29 for $\ell = 1$ then follows from Lemma 5.26 and the lemma below.

Lemma 5.31. *The (edge-weighted)* PLANAR $\mathcal{C}_1$-STEINER NETWORK *instance* $(G(\mathcal{S}), T, D)$ *has* *a solution of weight at most* $B^* - k^2$ *if and only if the (edge-weighted)* Planar M-Steiner Network *instance* $(MG(\mathcal{S}), I, M)$ *has a solution of weight at most* $B^* - k^2$.

Proof. If E is a solution of $(MG(\mathcal{S}), I, M)$ of weight at most $B^* - k^2$ then it is not difficult to see that $E \cup (E(G(\mathcal{S})) \setminus E(MG(\mathcal{S})))$ is a solution of $(G(\mathcal{S}), T, D)$ of weight at most $B^* - k^2$. Conversely, let E be a solution of $(G(\mathcal{S}), T, D)$ of weight at most $B^* - k^2$. Since for every $2 \leq i \leq k+1$, there is a unique path in H from s_i to t_i, namely $P_{i,i}$, and for every $k+2 \leq i \leq 2k+1$, there is a unique path in H from s_i to t_{i+1}, namely $P_{i,i+1}$, the restriction E' of E to $MG(\mathcal{S})$ is a solution of $(MG(\mathcal{S}), I, M)$; and since every edge in $E(G(\mathcal{S})) \setminus E(MG(\mathcal{S}))$ has weight 0, the weight of E' is that of E, that is, E' has weight at most $B^* - k^2$. □

References

[1] M. Bateni, C. Chekuri, A. Ene, M. T. Hajiaghayi, N. Korula, and D. Marx, *Prize-collecting Steiner Problems on Planar Graphs*, in Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2011, San Francisco, California, USA, January 23-25, 2011, 2011, pp. 1028–1049.

[2] M. Bateni, M. T. Hajiaghayi, and D. Marx, *Approximation Schemes for Steiner Forest on Planar Graphs and Graphs of Bounded Treewidth*, J. ACM, 58 (2011), pp. 21:1–21:37.

[3] P. Berman, A. Bhattacharyya, K. Makarychev, S. Raskhodnikova, and G. Yaroslavtsev, *Approximation algorithms for spanner problems and directed steiner forest*, Inf. Comput., 222 (2013), pp. 93–107.

[4] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto, *Fourier meets möbius: fast subset convolution*, in Proceedings of the 39th Annual ACM Symposium on Theory of Computing, San Diego, California, USA, June 11-13, 2007, D. S. Johnson and U. Feige, eds., ACM, 2007, pp. 67–74.

[5] M. Charikar, C. Chekuri, T. Cheung, Z. Dai, A. Goel, S. Guha, and M. Li, *Approximation Algorithms for Directed Steiner Problems*, J. Algorithms, 33 (1999), pp. 73–91.

[6] R. Chitnis, H. Esfandiari, M. T. Hajiaghayi, R. Khandekar, G. Kortsarz, and S. Seddighin, *A Tight Algorithm for Strongly Connected Steiner Subgraph on Two Terminals with Demands*, Algorithmica, 77 (2017), pp. 1216–1239.

[7] R. Chitnis, A. E. Feldmann, and P. Manurangsi, *Parameterized Approximation Algorithms for Bidirected Steiner Network Problems*, in 26th Annual European Symposium on Algorithms, ESA 2018, August 20-22, 2018, Helsinki, Finland, 2018, pp. 20:1–20:16.

[8] R. H. Chitnis, A. E. Feldmann, M. T. Hajiaghayi, and D. Marx, *Tight bounds for planar strongly connected steiner subgraph with fixed number of terminals (and extensions)*, SIAM J. Comput., 49 (2020), pp. 318–364.

[9] M. Cygan, F. V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, *Parameterized Algorithms*, Springer Publishing Company, Incorporated, 1st ed., 2015.

[10] É. C. de Verdière, *Multicuts in planar and bounded-genus graphs with bounded number of terminals*, Algorithmica, 78 (2017), pp. 1206–1224.

[11] P. Dvorák, A. E. Feldmann, D. Knop, T. Masarík, T. Toufar, and P. Veselý, *Parameterized Approximation Schemes for Steiner Trees with Small Number of Steiner Vertices*, in 35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France, 2018, pp. 26:1–26:15.

[12] E. Eiben, D. Knop, F. Panolan, and O. Suchý, *Complexity of the Steiner Network Problem with Respect to the Number of Terminals*, in 36th International Symposium on Theoretical Aspects of Computer Science (STACS 2019), R. Niedermeier and C. Paul, eds., vol. 126 of Leibniz International Proceedings in Informatics (LIPIcs), Dagstuhl, Germany, 2019, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, pp. 25:1–25:17.

[13] J. Feldman and M. Ruhl, *The Directed Steiner Network Problem is Tractable for a Constant Number of Terminals*, SIAM J. Comput., 36 (2006), pp. 543–561.

[14] A. E. Feldmann and D. Marx, *The complexity landscape of fixed-parameter directed steiner network problems*, in 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, vol. 55 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2016, pp. 27:1–27:14.

[15] F. V. Fomin, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, *Subexponential parameterized algorithms for planar and apex-minor-free graphs via low treewidth pattern covering*, in IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA, I. Dinur, ed., IEEE Computer Society, 2016, pp. 515–524.

[16] J. Guo, R. Niedermeier, and O. Suchý, *Parameterized complexity of arc-weighted directed steiner problems*, SIAM J. Discrete Math., 25 (2011), pp. 583–599.

[17] S. L. Hakimi, *Steiner's problem in graphs and its implications*, Networks, 1 (1971), pp. 113–133.

[18] R. M. Karp, *Reducibility Among Combinatorial Problems*, in Proceedings of a Symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA, 1972, pp. 85–103.

[19] P. N. Klein and D. Marx, *Solving planar k -terminal cut in $o(n^{c \sqrt{k}})$ time*, in Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part I, A. Czumaj, K. Mehlhorn, A. M. Pitts, and R. Wattenhofer, eds., vol. 7391 of Lecture Notes in Computer Science, Springer, 2012, pp. 569–580.

[20] ——, *A subexponential parameterized algorithm for subset TSP on planar graphs*, in Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, C. Chekuri, ed., SIAM, 2014, pp. 1812–1830.

[21] A. Levin, *Algorithm for the shortest connection of a group of graph vertices*, Soviet Math. Dokl., 12 (1971), pp. 1477–1481.

[22] C. Li, S. T. McCormick, and D. Simchi-Levi, *The point-to-point delivery and connection problems: complexity and algorithms*, Discrete Applied Mathematics, 36 (1992), pp. 267–292.

[23] D. Lokshtanov, S. Saurabh, and M. Wahlström, *Subexponential parameterized odd cycle transversal on planar graphs*, in IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2012, December 15-17, 2012, Hyderabad, India, D. D'Souza, T. Kavitha, and J. Radhakrishnan, eds., vol. 18 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2012, pp. 424–434.

[24] D. Marx, *On the Optimality of Planar and Geometric Approximation Schemes*, in 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2007), October 20-23, 2007, Providence, RI, USA, Proceedings, 2007, pp. 338–348.

[25] ——, *A tight lower bound for planar multiway cut with fixed number of terminals*, in Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part I, A. Czumaj, K. Mehlhorn, A. M. Pitts, and R. Wattenhofer, eds., vol. 7391 of Lecture Notes in Computer Science, Springer, 2012, pp. 677–688.

[26] D. Marx and M. Pilipczuk, *Optimal parameterized algorithms for planar facility location problems using voronoi diagrams*, ACM Trans. Algorithms, 18 (2022), pp. 13:1–13:64.

[27] D. Marx, M. Pilipczuk, and M. Pilipczuk, *On subexponential parameterized algorithms for steiner tree and directed subset TSP on planar graphs*, in 59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018, M. Thorup, ed., IEEE Computer Society, 2018, pp. 474–484.

[28] M. Natu and S. Fang, *The Point-to-point Connection Problem - Analysis and Algorithms*, Discrete Applied Mathematics, 78 (1997), pp. 207–226.

[29] J. Nederlof, *Detecting and counting small patterns in planar graphs in subexponential parameterized time*, in Proccedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020, K. Makarychev, Y. Makarychev, M. Tulsiani, G. Kamath, and J. Chuzhoy, eds., ACM, 2020, pp. 1293–1306.

[30] S. Ramanathan, *Multicast tree generation in networks with asymmetric links*, IEEE/ACM Trans. Netw., 4 (1996), pp. 558–568.

[31] N. Robertson, P. D. Seymour, and R. Thomas, *Quickly excluding a planar graph*, J. Comb. Theory, Ser. B, 62 (1994), pp. 323–348.

[32] H. F. Salama, D. S. Reeves, and Y. Viniotis, *Evaluation of Multicast Routing Algorithms for Real-Time Communication on High-Speed Networks*, IEEE Journal on Selected Areas in Communications, 15 (1997), pp. 332–345.

[33] P. Winter, *Steiner problem in networks: A survey*, Networks, 17 (1987), pp. 129–167.