

The Complexity of Contracting Bipartite Graphs into Small Cycles ^{*†}

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Abstract

For a positive integer $\ell \geq 3$, the C_ℓ -CONTRACTIBILITY problem takes as input an undirected simple graph G and determines whether G can be transformed into a graph isomorphic to C_ℓ (the induced cycle on ℓ vertices) using only edge contractions. Brouwer and Veldman [JGT 1987] showed that C_4 -CONTRACTIBILITY is NP-complete in general graphs. It is easy to verify that C_3 -CONTRACTIBILITY is polynomial-time solvable. Dabrowski and Paulusma [IPL 2017] showed that C_ℓ -CONTRACTIBILITY is NP-complete on bipartite graphs for $\ell = 6$ and posed as open problems the status of the problem when ℓ is 4 or 5. In this paper, we show that both C_5 -CONTRACTIBILITY and C_4 -CONTRACTIBILITY are NP-complete on bipartite graphs.

1 Introduction

Operations on graphs produce new graphs from existing ones. Elementary editing operations include deleting vertices, deleting and/or adding edges, subdividing edges and contracting edges. Due to the ubiquitous presence of graphs in modeling real-world networks, many problems of practical importance may be posed as editing problems on graphs. In this work, we focus on modifying a graph by only performing edge contractions. Contracting an edge in a graph results in the addition of a new vertex adjacent to the neighbors of its endpoints followed by the deletion of the endpoints. As graphs typically represent binary relationships among a collection of objects, edge contractions naturally correspond to merging two objects into a single entity or to treating two objects as indistinguishable. Contractions can therefore be seen as a way of ‘simplifying’ the graph and they have applications in clustering, compression, sparsification and computer graphics [1, 3, 6, 7, 14, 20]. Edge contractions also play an important role in Hamiltonian graph theory, planar graph theory and graph minor theory [5, 16, 25].

Given graphs G and H , the GRAPH CONTRACTIBILITY problem decides whether G can be transformed into a graph isomorphic to H using only edge contractions. GRAPH CONTRACTIBILITY is known to be NP-complete [11, GT51]. This led to the study of the problem

*A preliminary version of this paper has been accepted in the 48th International Workshop on Graph-Theoretic Concepts in Computer Science (WG2022).

†The third author has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement SYSTEMATICGRAPH (No. 725978).

on special graph classes and for restricted choices of H . When H is a fixed graph, the GRAPH CONTRACTIBILITY problem is called H -CONTRACTIBILITY. Intuitively, this problem of determining whether G is contractible to H may be seen as the task of determining if the ‘underlying structure’ of G is H . One of the related graph parameters in this context is cyclicity. The cyclicity of a graph is the largest integer ℓ for which the graph is contractible to the induced cycle on ℓ vertices (denoted as C_ℓ). This parameter was introduced in the study of another important graph invariant called circularity [4]. Ever since, there have been efforts towards understanding the complexity of computing cyclicity and expressing it in terms of some structural property of the graph. Brouwer and Veldman [5] showed that C_4 -CONTRACTIBILITY is NP-complete, hence proving that determining cyclicity is NP-hard in general. This result led to the study of the problem on special graph classes including bipartite graphs, claw-free graphs and planar graphs [8, 10, 12].

Hammack [12] showed that the cyclicity of planar graphs can be computed in polynomial time and in another work [13], he showed that C_ℓ -CONTRACTIBILITY is NP-complete for every $\ell \geq 5$ in general. Later, Kaminski et al. [19] showed that H -CONTRACTIBILITY is polynomial-time solvable on planar graphs for any H . Levin et al. [22] showed that H -CONTRACTIBILITY is polynomial-time solvable on general graphs if H is a graph on at most 5 vertices containing a universal vertex. However, the presence of a universal vertex in H (on more than 5 vertices) does not guarantee that the H -CONTRACTIBILITY can be solved in polynomial time [17]. Fiala et al. [10] showed that C_ℓ -CONTRACTIBILITY is NP-complete for claw-free graphs for every $\ell \geq 6$. Heggernes et al. [15] proved that P_ℓ -CONTRACTIBILITY is polynomial-time solvable on chordal graphs for every $\ell \geq 1$, where P_ℓ denotes the induced path on ℓ vertices. Later, Belmonte et al. [2] proved that H -CONTRACTIBILITY is polynomial-time solvable on chordal graphs for every H . Dabrowski and Paulusma [8] showed that C_6 -CONTRACTIBILITY is NP-complete for bipartite graphs. It is easy to verify that that C_3 -CONTRACTIBILITY is polynomial-time solvable in general graphs. In this paper, we show that both C_5 -CONTRACTIBILITY and C_4 -CONTRACTIBILITY are NP-complete on bipartite graphs.

Theorem 1. C_5 -CONTRACTIBILITY is NP-complete on bipartite graphs.

Theorem 2. C_4 -CONTRACTIBILITY is NP-complete on bipartite graphs.

Theorems 1 and 2 involve reductions from the POSITIVE NOT ALL EQUAL SAT (POSITIVE NAE-SAT) problem where given a formula ψ in conjunctive normal form with no negative literals, the objective is to determine if there is an assignment of **True** or **False** to each of the variables such that for each clause at least one but not all variables in it are set to **True**. Such an assignment is called a *not-all-equal satisfying assignment*. POSITIVE NAE-SAT (also referred to as MONOTONE NAE-SAT) is known to be NP-complete [26]. Also, a straight-forward reduction from SET SPLITTING or HYPERGRAPH 2-COLORABILITY [11, SP4] to POSITIVE NAE-SAT ascertains this fact.

2 Preliminaries

For a positive integer q , $[q]$ denotes the set $\{1, 2, \dots, q\}$. \mathbb{N} denotes the collection of all non-negative integers. A partition of a set S is a set of disjoint subsets of S whose union is S .

For standard graph-theoretic terminology not stated here, we refer the reader to the book by Diestel [9]. In this work, we only consider simple undirected graphs. Unless otherwise specified, we use n to denote the number of vertices in the graph under consideration G . For an undirected graph G , its sets of vertices and edges, are denoted by $V(G)$ and $E(G)$, respectively. An edge between vertices u and v is denoted as uv . Two vertices u, v in $V(G)$ are *adjacent* if there is

an edge uv in G . The *open neighborhood* of a vertex v , denoted by $N_G(v)$, is the set of vertices adjacent to v and the *closed neighborhood* of v , denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. A vertex u is a *pendant vertex* if $|N_G(v)| = 1$. The notion of neighborhood is extended to a set $S \subseteq V(G)$ of vertices by defining $N_G[S]$ as $\bigcup_{v \in S} N[v]$ and $N_G(S)$ as $N[S] \setminus S$. We omit the subscript in the notation for neighborhood if the graph under consideration is clear.

A set $S \subseteq V(G)$ of vertices is a *dominating set* if $V(G) = N[S]$. For a subset F of edges, $V(F)$ denotes the set of endpoints of edges in F . For a subset S of $V(G)$ (resp. a subset F of $E(G)$), $G - S$ (resp. by $G - F$) denotes the graph obtained by deleting S (resp. deleting F) from G . The subgraph of G induced on the set $S \subseteq V(G)$ is denoted by $G[S]$. For two subsets S_1, S_2 of $V(G)$, $E(S_1, S_2)$ denotes the set of edges with one endpoint in S_1 and the other endpoint in S_2 . With a slight abuse of notation, we use $E(S)$ to denote $E(S, S)$. We say that the sets S_1, S_2 are adjacent if $E(S_1, S_2) \neq \emptyset$.

A *path* P in G is a sequence (v_1, \dots, v_k) of distinct vertices such that for each $i \in [k - 1]$, $v_i v_{i+1} \in E(G)$. A *cycle* C in G is a sequence (v_1, \dots, v_k) of distinct vertices such that (v_1, \dots, v_k) is a path and $v_k v_1 \in E(G)$. A cycle $C = (v_1, \dots, v_k)$ is called an *induced* (or *chordless*) cycle if there is no edge in G that is between two non-consecutive vertices of C with the exception of the edge $v_k v_1$. The length of a path or cycle X is the number of vertices in it and is denoted by $|X|$. An induced cycle of length q is called a q -cycle and denoted by C_q . The *distance* between any two vertices u, v in $V(G)$ is the length of a shortest path from u to v in G . The *diameter* of G is the maximum length of a shortest path between two vertices in G . A graph is *connected* if there is a path between every pair of distinct vertices. A subset S of $V(G)$ is said to be a *connected set* if $G[S]$ is connected. A *spanning tree* of a connected graph is a connected acyclic subgraph which includes all the vertices of the graph. A *spanning forest* of a disconnected graph is a collection of spanning trees of its components.

A set of vertices Y is said to be an *independent set* if no two vertices in Y are adjacent. A graph G is a *bipartite graph* if its vertex set can be partitioned into two sets X and Y such that every edge in the graph has one endpoint in X and the other endpoint in Y . Such a partition $\{X, Y\}$ of a bipartite graph is called a bipartition. The *subdivision* of the edge uv in G results in another graph that is obtained from G by deleting the edge uv and adding a new vertex w adjacent to u and v . Observe that subdividing all edges of an arbitrary graph results in a bipartite graph. A complete bipartite graph with bipartition $\{X, Y\}$ is a bipartite graph where every vertex of X is adjacent to every vertex of Y .

The *contraction* of an edge $e = uv$ in G results in another graph denoted by G/e that is obtained from G by deleting vertices u and v from G , and adding a new vertex which is adjacent to the vertices that are adjacent to either u or v in G . This process does not introduce self-loops or parallel edges. Formally G/e is defined as $V(G/e) = (V(G) \cup \{w\}) \setminus \{u, v\}$ and $E(G/e) = \{xy \mid x, y \in V(G) \setminus \{u, v\}, xy \in E(G)\} \cup \{wx \mid x \in N_G(u) \cup N_G(v)\}$ where w is a new vertex. Observe that contracting an edge reduces the number of vertices in the graph by exactly one and reduces the number of edges by at least one. For a subset F of edges in G , G/F denotes the graph obtained from G by contracting all the edges (in some order) in F . We now formally define the notion of graph contractibility.

Definition 2.1. G is said to be contractible to H if there is a surjective function $\psi : V(G) \rightarrow V(H)$ such that the following properties hold.

1. For each $h \in V(H)$, $\psi^{-1}(h)$, called the witness set corresponding to h , is connected.
2. For each $h, h' \in V(H)$, $hh' \in E(H)$ if and only if $E(\psi^{-1}(h), \psi^{-1}(h')) \neq \emptyset$.

Then, we say that G is contractible to H via the function ψ and that G has a H -witness structure $\mathcal{W} = \{\psi^{-1}(h) \mid h \in V(H)\}$ which is the collection of all witness sets.

In Definition 2.1, a witness set that contains more than one vertex is called a *big witness set* and the one that is a singleton set is called a *small witness set* or *singleton witness set*. Note

that a witness structure \mathcal{W} is a partition of $V(G)$. Also, if a vertex v is in some big witness set W , then at least one neighbor of v is also in W . Recall that the H -CONTRACTIBILITY problem takes as input a graph G and decides whether G is contractible to H or not. Observe that this task is equivalent to determining if G has a H -witness structure or not.

Now, we proceed to proving Theorems 1 and 2 in Sections 3 and 4, respectively.

3 C_5 -Contractibility on Bipartite Graphs

In this section, we prove Theorem 1. It is easy to verify that C_5 -CONTRACTIBILITY is in NP. Given an instance ψ of POSITIVE NAE-SAT with N variables and M clauses, we give a polynomial-time algorithm that outputs a bipartite graph G equivalent to ψ . For the sake of simplicity, we describe the algorithm in two steps. In the first step, the algorithm constructs a non-bipartite graph H equivalent to ψ (Lemmas 3 and 4) and then in the second step, the algorithm constructs a bipartite graph G that is equivalent to H (Lemma 1). We remark that G is obtained from H by dividing some (and not all) of the edges of H .

3.1 Construction of H and G

Let $\{X_1, X_2, \dots, X_N\}$ and $\{C_1, C_2, \dots, C_M\}$ be the sets of variables and clauses, respectively, in ψ . The non-bipartite graph H is constructed as follows. Refer to Figure 1 for an illustration.

1. Add a set $V_\alpha = \{\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ of five vertices that induce the 5-cycle $(\alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4)$. This set forms the “base cycle” in the witness structure.
2. For every $i \in [N]$, add a set of five vertices that induce a 5-cycle $C^i = (x_i^0, x_i^1, x_i^2, x_i^3, x_i^4)$ and two sets of edges $\{x_i^0\alpha^0, x_i^1\alpha^1, x_i^2\alpha^2, x_i^3\alpha^3, x_i^4\alpha^4\}$ and $\{x_i^0\alpha^1, x_i^1\alpha^2, x_i^2\alpha^3, x_i^3\alpha^4, x_i^4\alpha^0\}$. The variable gadget is designed so that there are two choices for C^i to co-exist (in a C_5 -witness structure) with the C_5 induced by V_α . We will associate these two choices with a **True** or **False** assignment to the corresponding variable.
3. For every $j \in [M]$, add vertices c_j and b_j and a set $\{c_j\alpha^0, c_j\alpha^2, b_j\alpha^2, b_j\alpha^4\}$ of edges. The neighbours of c_j and b_j are defined so that c_j will be in the same witness set as α^1 (a non-neighbor of c_j) and b_j will be in the same witness set as α^3 (a non-neighbor of b_j).
4. Finally, for every $i \in [N]$ and $j \in [M]$ such that X_i appears in C_j , add edges $x_i^1c_j$ and $x_i^2b_j$. This step is the one that encodes the clause-variable relationship. Relevant variables are expected to help c_j (and b_j) to be connected to witness sets containing α^1 (and α^3).

This completes the construction of H .

For $p \in \{0, 1, 2, 3, 4\}$, define $X^p := \{x_i^p \mid i \in [N]\}$. Also, define $Y^c := \{c_j \mid j \in [M]\}$ and $Y^b := \{b_j \mid j \in [M]\}$. For an edge $uv \in E(H)$, let $\lambda(u, v)$ denote the new vertex added while subdividing uv in the construction of G . Let $L = \{\alpha^0, \alpha^2, \alpha^4\} \cup X^1 \cup X^3$ and $R = \{\alpha^1, \alpha^3\} \cup X^0 \cup X^2 \cup X^4 \cup Y^c \cup Y^b$. Then, $\{L, R\}$ is a partition of H into two parts where there are certain edges with both endpoints in the same part. We subdivide exactly these edges to obtain G .

5. Subdivide the edge $\alpha^0\alpha^4$.
6. For every $i \in [N]$, subdivide the edges $x_i^0x_i^4, x_i^0\alpha^1, x_i^1\alpha^2, x_i^2\alpha^3, x_i^3\alpha^4$.
7. For every $i \in [N]$ and $j \in [M]$, subdivide the edge $x_i^2b_j$ if it exists.

This completes the construction of G .

We now argue that G is a bipartite graph. Observe that L and R are independent sets in G . We will extend this partition $\{L, R\}$ of H into a bipartition of G as follows: $\lambda(\alpha^0, \alpha^4) \in R$ and for every $i \in [N]$, $\lambda(x_i^0, x_i^4) \in L$, $\lambda(x_i^0, \alpha^1) \in L$, $\lambda(x_i^1, \alpha^2) \in R$, $\lambda(x_i^2, \alpha^3) \in L$ and $\lambda(x_i^3, \alpha^4) \in R$. For every $i \in [N]$, $j \in [M]$, if $x_i^2b_j \in E(H)$, then $\lambda(x_i^2, b_j) \in L$. See Figure 1 for an illustration. It is easy to verify that $\{L, R\}$ is a bipartition of G and hence G is a bipartite graph.

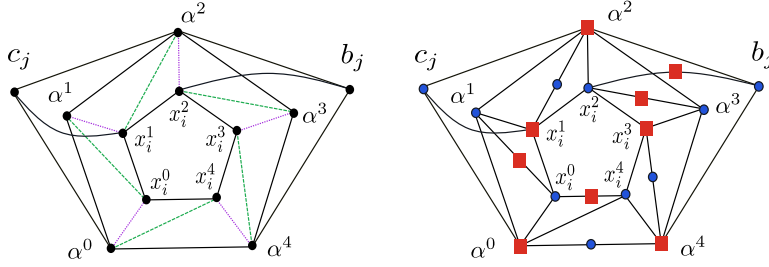


Figure 1: (Left) The graph H with certain edges highlighted as purple (dotted) edges denote setting variable X_i to **True** and as green (dashed) edges denote setting X_i to **False**, respectively. (Right) The bipartite graph G where blue (round) and red (squares) vertices denote a bipartition.

We remark that the natural bipartite graph obtained from H by subdividing all the edges may not be equivalent to H in the context of C_5 -CONTRACTIBILITY. In Lemma 1, we show that the set of edges of H that are subdivided to obtain G are safe (in preserving contractibility to C_5) to subdivide.

3.2 Equivalence of H and G

We show that G and H are equivalent in the context of C_5 -CONTRACTIBILITY. As G is obtained from H by subdividing some edges, one can obtain H from G by contracting some edges. Hence, if one can obtain a C_5 by contracting edges in H , then one can also obtain a C_5 by contracting edges in G by first contracting G to H and then contracting H to C_5 . To prove the converse, we first argue that no vertex in $V(G) \setminus V(H)$ is a singleton witness set in any C_5 -witness structure \mathcal{W} of G . Then, we show that deleting vertices of $V(G) \setminus V(H)$ from \mathcal{W} results in a C_5 -witness structure \mathcal{W}' of H .

Lemma 1. H is a YES-instance of C_5 -CONTRACTIBILITY if and only if G is a YES-instance of C_5 -CONTRACTIBILITY.

Proof. As G is obtained from H by subdividing some edges, one can obtain H from G by contracting some edges. Hence, if one can obtain a C_5 by contracting edges in H , then one can also obtain a C_5 by contracting edges in G by first contracting G to H and then contracting H to C_5 . Therefore, if H is a YES-instance of C_5 -CONTRACTIBILITY then G is a YES-instance of C_5 -CONTRACTIBILITY.

Suppose G is a YES-instance of C_5 -CONTRACTIBILITY and $\mathcal{W} = \{W^i \mid i \in [4] \cup \{0\}\}$ is a C_5 -witness structure of G where $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i+1) \bmod 5$. We first argue that no vertex in $V(G) \setminus V(H)$ is a singleton witness set in \mathcal{W} . Suppose for some $uv \in E(H)$, $\lambda(u, v)$ is a singleton witness set. Without loss of generality, let $W^0 = \{\lambda(u, v)\}$. As $E(W^0, W^1)$ and $E(W^0, W^4)$ are non-empty sets in G and $\lambda(u, v)$ is of degree two, either $u \in W^1$ and $v \in W^4$ or $v \in W^1$ and $u \in W^4$. In either case, as \mathcal{W} is a C_5 -witness structure of G , any path from u to v in $G - \{\lambda(u, v)\}$ is of length at least three. We will now show that for any edge uv in H which is subdivided while constructing G , the length of a shortest path between u and v in $G - \{\lambda(u, v)\}$ is two. This will lead to a contradiction which will enable us to conclude that no vertex in $V(G) \setminus V(H)$ is a singleton witness set in \mathcal{W} .

Consider the following triples: $(\alpha^0, \alpha^4, x_i^4)$, (x_i^0, x_i^4, α^0) , $(x_i^p, \alpha^{p+1}, x_i^{p+1})$, and (x_i^2, b_j, α^2) for every $i \in [N]$, $j \in [M]$, and $p \in \{0, 1, 2, 3\}$. For any triple (u', v', w') , there is a path from u' to v' via w' of length two in G that does not contain $\lambda(u', v')$, i.e., w' is a common neighbor of

u' and v' in G . This implies that the vertex obtained by subdividing the edge $u'v'$ in H while constructing G cannot be a singleton witness set in \mathcal{W} . However, these triples represent all edges of H that were subdivided while constructing G . Hence, there is no vertex in $V(G) \setminus V(H)$ which is a singleton witness set in \mathcal{W} . Equivalently, for any $\lambda(u, v) \in V(G)$, if $\lambda(u, v)$ is contained in $W^* \in \mathcal{W}$, then $u \in W^*$ or $v \in W^*$. That is, for any edge $uv \in E(H)$ which was subdivided while constructing G , the vertices u, v are in the same witness set or in adjacent witness sets.

Let \mathcal{W}' be the partition of $V(H)$ obtained from \mathcal{W} by removing vertices in $V(G) \setminus V(H)$. Formally, $\mathcal{W}' = \{W^* \setminus (V(G) \setminus V(H)) \mid W^* \in \mathcal{W}\}$. Since, no vertex in $V(G) \setminus V(H)$ is a singleton witness set in \mathcal{W} , \mathcal{W}' contains five non-empty sets. Also, the endpoints of any edge $uv \in E(H) \setminus E(G)$ are either in the same witness set or in witness sets that are adjacent in G . In both the cases, each witness set of \mathcal{W} remains connected in H even after deleting the vertices in $V(G) \setminus V(H)$. This implies that \mathcal{W}' is a C_5 -witness structure of H . \square

3.3 Properties of a C_5 -Witness Structure of H

Before we state properties of H , we mention the following observation.

Observation 1. *In any partition $\{X, Y\}$ of the vertices of an induced 5-cycle into 2 non-empty parts, $E(X, Y) \neq \emptyset$.*

Now, we state certain properties of vertex subsets in H that we later use to show properties of a C_5 -witness structure of H .

Observation 2. *$X^0, X^1, X^2, X^3, X^4, Y^c$ and Y^b are independent sets and V_α is a dominating set in H . Further, $X^0 \cup X^4 \cup Y^c \subseteq N(\alpha^0)$, $X^1 \cup X^0 \subseteq N(\alpha^1)$, $X^2 \cup X^1 \cup Y^c \cup Y^b \subseteq N(\alpha^2)$, $X^3 \cup X^2 \subseteq N(\alpha^3)$ and $X^4 \cup X^3 \cup Y^b \subseteq N(\alpha^4)$.*

Next, we show a property of a C_5 -witness structure of H that will be crucial to proving the correctness of the reduction. As we have indicated in the construction of H , we need a handle on the base cycle of the C_5 -witness structure (for YES-instances) which Lemma 2 provides.

Lemma 2. *In any C_5 -witness structure of G , every pair of vertices in V_α are in different witness sets.*

Proof. Suppose $\mathcal{W} = \{W^i \mid i \in [4] \cup \{0\}\}$ is a C_5 -witness structure of H where $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i \pm 1) \pmod{5}$. We argue that V_α has a non-empty intersection with each W^i . Suppose $V_\alpha \subseteq W^i$ for some $0 \leq i \leq 4$. Then, $W^{(i+2) \pmod{5}} = \emptyset$ and $W^{(i+3) \pmod{5}} = \emptyset$ leading to a contradiction. Suppose V_α intersects exactly two witness sets. We will consider the cases when these sets are W^0, W^1 and W^0, W^2 . The other cases are similar to these cases. If V_α intersects only with W^0 and W^1 , then since V_α is a dominating set in H it follows that $W^3 = \emptyset$ and this leads to a contradiction. Suppose V_α intersects only with W^0 and W^2 . From Observation 1, this implies that $E(W^0, W^2) \neq \emptyset$ leading to a contradiction. Suppose V_α intersects exactly four witness sets, say W^0, W^1, W^2 , and W^3 . Without loss of generality, assume $\alpha^0 \in W^0$. As α^1 and α^4 are adjacent to α^0 , we have $\{\alpha^1, \alpha^4\} \subseteq W^0 \cup W^1$. Then, one of α^2 or α^3 is in W^2 and the other is in W^3 . However, as $\alpha^1\alpha^2, \alpha^3\alpha^4 \in E(H)$, neither α^2 nor α^3 can be in W^3 implying that $W^3 = \emptyset$ and leading to a contradiction.

Suppose V_α intersects exactly three witness sets. Without loss of generality, let $\alpha^0 \in W^0$. We consider the following cases.

- Case (i) V_α intersects W^0, W^1 and W^2 .
- Case (ii) V_α intersects W^0, W^1 and W^4 .
- Case (iii) V_α intersects W^0, W^2 and W^3 . This leads to contradiction as Observation 1 implies $E(W^0, W^2 \cup W^3) \neq \emptyset$.

- Case (iv) V_α intersects W^0, W^2 and W^4 . This leads to contradiction as Observation 1 implies $E(W^2, W^0 \cup W^4) \neq \emptyset$.
- Case (v) V_α intersects W^0, W^4 and W^3 . This is similar to Case (i).
- Case (vi) V_α intersects W^0, W^1 and W^3 . This is similar to Case (iv).

Consider Case (i). As α^1 and α^4 are adjacent to α^0 , we have $\{\alpha^1, \alpha^4\} \subseteq W^0 \cup W^1$. Then, at least one of α^2 or α^3 is in W^2 and since $\alpha^2\alpha^3 \in E(H)$, neither α^2 nor α^3 can be in W^0 . Thus, we have $\{\alpha^2, \alpha^3\} \subseteq W^1 \cup W^2$. Since $E(W^0, W^3) = \emptyset$ and $E(W^1, W^3) = \emptyset$, we have $W^3 \cap N(\alpha^0) = \emptyset$, $W^3 \cap N(\alpha^1) = \emptyset$ and $W^3 \cap N(\alpha^4) = \emptyset$. From Observation 2, this implies that $W^3 \subseteq X^2$. Similarly, since $E(W^1, W^4) = \emptyset$ and $E(W^2, W^4) = \emptyset$, we have $W^4 \cap N(\alpha^2) = \emptyset$ and $W^4 \cap N(\alpha^3) = \emptyset$. From Observation 2, this implies $W^4 \subseteq (X^0 \cup X^4)$. However, by the construction, $E(X^2, X^0 \cup X^4) = \emptyset$ implying that $E(W^3, W^4) = \emptyset$ which leads to a contradiction.

Let us now consider Case (ii). Recall that $\alpha^0 \in W^0$. Then, either $\alpha^1 \in W^0 \cup W^1$ or $\alpha^1 \in W^0 \cup W^4$. As both these cases are similar, we consider the case when $\alpha^1 \in W^0 \cup W^1$. Suppose $\alpha^1 \in W^1$. Then, we have $\{\alpha^1, \alpha^2\} \subseteq W^0 \cup W^1$ since $\alpha^1\alpha^2 \in E(H)$. We will show that this leads to a contradiction. At least one of α^3 or α^4 is in W^4 and since $\alpha^3\alpha^4 \in E(H)$, neither α^3 nor α^4 can be in W^1 . Thus, we have $\{\alpha^3, \alpha^4\} \subseteq W^0 \cup W^4$. Since $E(W^0, W^3) = \emptyset$ and $E(W^1, W^3) = \emptyset$, we have $W^3 \cap N(\alpha^0) = \emptyset$, $W^3 \cap N(\alpha^1) = \emptyset$ and $W^3 \cap N(\alpha^2) = \emptyset$. From Observation 2, this implies $W^3 \subseteq X^3$. Similarly, since $E(W^0, W^2) = \emptyset$ and $E(W^4, W^2) = \emptyset$, we have $W^2 \cap N(\alpha^0) = \emptyset$, $W^2 \cap N(\alpha^3) = \emptyset$ and $W^2 \cap N(\alpha^4) = \emptyset$. From Observation 2, this implies $W^2 \subseteq X^1$. However, by construction, $E(X^1, X^3) = \emptyset$ implying that $E(W^2, W^3) = \emptyset$ which leads to a contradiction.

Suppose $\alpha^1 \in W^0$. If $\alpha^2 \in W^0$, then one of α^3 or α^4 is in W^1 and the other is in W^4 resulting in an edge between W^1 and W^4 . Thus, $\alpha^2 \in W^1$ or $\alpha^2 \in W^4$. As these cases are similar, we only consider $\alpha^2 \in W^1$. Then we once again have $\{\alpha^1, \alpha^2\} \subseteq W^0 \cup W^1$ which leads to a contradiction. \square

3.4 Equivalence of H and ψ

Now, we are ready to establish the equivalence of ψ and H .

Lemma 3. *If ψ is a YES-instance of POSITIVE NAE-SAT then H is a YES-instance of C_5 -CONTRACTIBILITY.*

Proof. Suppose $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\text{True}, \text{False}\}$ is a not-all-equal satisfying assignment of ψ . Define the following partition of $V(H)$.

$$\begin{aligned}
W^0 &:= \{\alpha^0\} \cup \{x_i^0 \mid i \in [N], \pi(X_i) = \text{True}\} \cup \{x_i^4 \mid i \in [N], \pi(X_i) = \text{False}\}, \\
W^1 &:= \{\alpha^1\} \cup \{x_i^1 \mid i \in [N], \pi(X_i) = \text{True}\} \cup \{x_i^0 \mid i \in [N], \pi(X_i) = \text{False}\} \\
&\quad \cup \{c_j \mid j \in [M]\}, \\
W^2 &:= \{\alpha^2\} \cup \{x_i^2 \mid i \in [N], \pi(X_i) = \text{True}\} \cup \{x_i^1 \mid i \in [N], \pi(X_i) = \text{False}\}, \\
W^3 &:= \{\alpha^3\} \cup \{x_i^3 \mid i \in [N], \pi(X_i) = \text{True}\} \cup \{x_i^2 \mid i \in [N], \pi(X_i) = \text{False}\} \\
&\quad \cup \{b_j \mid j \in [M]\}, \\
W^4 &:= \{\alpha^4\} \cup \{x_i^4 \mid i \in [N], \pi(X_i) = \text{True}\} \cup \{x_i^3 \mid i \in [N], \pi(X_i) = \text{False}\},
\end{aligned}$$

Clearly W^0, W^2 , and W^4 are connected sets. For any $j \in [M]$, there exists $i \in [N]$ such that $x_i^1 \in W^1$ (since π sets at least one of the variables in C_j to **True**) and $i' \in [N]$ such that $x_{i'}^2 \in W^3$ (since π sets at least one of the variables in C_j to **False**). Also, $c_j x_i^1, b_j x_{i'}^2 \in E(H)$. As for every $i \in [N]$, α^1 is adjacent to x_i^1 and α^3 is adjacent to x_i^2 , it follows that W^1 and W^3 are connected sets. Now, it is easy to verify that $\{W^0, W^1, W^2, W^3, W^4\}$ is a C_5 -witness structure. \square

In the proof of the converse of Lemma 3, we crucially use Lemma 2. That is, if H is contractible to a 5-cycle, then in any C_5 -witness structure $\{W^0, W^1, W^2, W^3, W^4\}$ with $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i \pm 1) \pmod{5}$, each of the five witness sets has a non-empty intersection with V_α . This structure along with a couple of other properties translates to a not-all-equal satisfying assignment of ψ .

Lemma 4. *If H is a YES-instance of C_5 -CONTRACTIBILITY then ψ is a YES-instance of POSITIVE NAE-SAT.*

Proof. Suppose $\mathcal{W} = \{W^0, W^1, W^2, W^3, W^4\}$ is a C_5 -witness structure of H where $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i \pm 1) \pmod{5}$. Then, by Lemma 2, V_α has a non-empty intersection with each W^i . Without loss of generality, let $\alpha^p \in W^p$ for every $p \in \{0, 1, 2, 3, 4\}$. We first argue that for any $i \in [N]$, the set $S_i = \{x_i^0, x_i^1, x_i^2, x_i^3, x_i^4\}$ also has a non-empty intersection with each W^j . Suppose $S_i \cap W^0 = \emptyset$. Then, as α^0 is adjacent to x_i^0, x_i^4 and $x_i^0 x_i^4 \in E(H)$, either $\{x_i^0, x_i^4\} \subseteq W^1$ or $\{x_i^0, x_i^4\} \subseteq W^4$. As $\alpha^4 x_i^4, \alpha^1 x_i^0 \in E(H)$, both these cases contradict the fact that $E(W^1, W^4) = \emptyset$. Using the similar arguments, it follows that S_i has a non-empty intersection with each W^j .

Next, we claim that for each $i \in [N]$ and $0 \leq p \leq 4$, $x_i^p \in W^p \cup W^{(p+1) \pmod{5}}$. This is due to the fact that x_i^p is adjacent with α^p and $\alpha^{p+1 \pmod{5}}$. Now, we show that for each $i \in [N]$ and $0 \leq p \leq 4$, $x_i^p \in W^p$ if and only if $x_i^{(p+1) \pmod{5}} \in W^{(p+1) \pmod{5}}$ and $x_i^p \in W^{(p+1) \pmod{5}}$ if and only if $x_i^{(p+1) \pmod{5}} \in W^{(p+2) \pmod{5}}$. If $x_i^0 \in W^0$ and $x_i^1 \notin W^1$, then $E(W^0, W^2) \cup E(W^0, W^3) \cup E(W^2, W^4) \neq \emptyset$ leading to a contradiction. If $x_i^0 \in W^1$ and $x_i^1 \notin W^2$, then $E(W^1, W^3) \cup E(W^0, W^2) \cup E(W^1, W^4) \neq \emptyset$ leading to a contradiction. Similar arguments hold for x_i^1, x_i^2, x_i^3 and x_i^4 . This is indicated by the collections of purple (dotted) edges and green (dashed) edges in Figure 1. We will associate these two choices with setting X_i to **True** and to **False**, respectively.

We now construct an assignment $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\mathbf{True}, \mathbf{False}\}$. Consider the witness set W^1 . For each $i \in [N]$, if $x_i^1 \in W^1$ then set $\pi(X_i) = \mathbf{True}$, otherwise ($x_i^1 \in W^2$) set $\pi(X_i) = \mathbf{False}$. We argue that π is a not-all-equal satisfying assignment for ψ . We show that for each $j \in [M]$, $c_j \in W^1$ and $b_j \in W^3$, further, the clause C_j has variables X_i and $X_{i'}$ such that $x_i^1 \in W^1$ and $x_{i'}^2 \in W^3$. Observe that c_j (being adjacent with α^0 and α^2) is in the same witness set that has α^1 and b_j (being adjacent with α^2 and α^4) is in the same witness set that has α^3 . Thus, for each $j \in [M]$, $c_j \in W^1$ and $b_j \in W^3$. By the property of witness structures, W^1 and W^3 are connected sets. As the only vertices outside V_α that are adjacent to c_j are vertices x_i^1 corresponding to variables X_i appearing in C_j , it follows that C_j has a variable X_i such that $x_i^1 \in W^1$. Similarly, as the only vertices outside V_α that are adjacent to b_j are vertices $x_{i'}^2$ corresponding to variables $X_{i'}$ appearing in C_j , it follows that C_j has a variable $X_{i'}$ such that $x_{i'}^2 \in W^3$. \square

4 C_4 -Contractibility on Bipartite Graphs

In this section, we prove Theorem 2. It is easy to verify that C_4 -CONTRACTIBILITY is in NP. Given an instance ψ of POSITIVE NAE-SAT with N variables and M clauses, we give a polynomial-time algorithm that outputs a bipartite graph G equivalent to ψ (Lemmas 7 and 8).

4.1 Construction of G

Let $\{X_1, X_2, \dots, X_N\}$ and $\{C_1, C_2, \dots, C_M\}$ be the sets of variables and clauses, respectively, in ψ . The graph G with a partition $\{V, V'\}$ of its vertex set is constructed as follows. See Figure 2 for an illustration.

1. Add vertices t, f to V , vertices t', f' to V' and edges tt', ff' to $E(G)$. This set would eventually form the “base cycle” in the witness structure.
2. For every $i \in [N]$, add vertices x_i, y_i, z_i to V and x'_i, y'_i, z'_i to V' corresponding to the variable X_i . Further, make every vertex in $\{x'_i, y'_i, z'_i\}$ adjacent to every vertex in $\{x_i, t, f\}$ and every vertex in $\{x_i, y_i, z_i\}$ adjacent to every vertex in $\{x'_i, t', f'\}$. Let $X = \{x_i \mid i \in [N]\}$, $X' = \{x'_i \mid i \in [N]\}$, $Y = \{y_i \mid i \in [N]\}$, $Y' = \{y'_i \mid i \in [N]\}$, $Z = \{z_i \mid i \in [N]\}$, $Z' = \{z'_i \mid i \in [N]\}$. The neighborhood of X' is set so that every element of X' is in the witness set containing t or f . This forces every element of X to be respectively in the witness set containing t' or f' . These binary choices would be associated with setting the corresponding variable to **True** or **False**. The sets Y, Y', Z, Z' are added for technical reasons.
3. For every $j \in [M]$, add vertices c_j, b_j to V , c'_j, b'_j to V' and edges $c_j f', b_j f', c'_j t, b'_j t$ to $E(G)$ corresponding to clause C_j . Let $C = \{c_j \mid j \in [M]\}$, $C' = \{c'_j \mid j \in [M]\}$, $B = \{b_j \mid j \in [M]\}$, $B' = \{b'_j \mid j \in [M]\}$. Subsequently, we will add more vertices (sets D and D' defined subsequently) adjacent to vertices in $C \cup B \cup C' \cup B'$ so that no vertex in $B \cup C$ is in a witness set that is non-adjacent to the one containing t and no vertex in $B' \cup C'$ is in a witness set that is non-adjacent to the one containing f' .
4. For every $i \in [N]$ and $j \in [M]$, if X_i appears in C_j then add edges $c_j x'_i, b_j x'_i, x_i c'_j$, and $x_i b'_j$ to $E(G)$. This step is the one that encodes the clause-variable relationship. Relevant variables are expected to help clause vertices to be connected to witness sets containing them.
5. Let \mathcal{D} denote the following collection of pairs of vertices: $\{\{t, f\}, \{t', f'\}\} \cup \{\{t, c_j\}, \{t, b_j\}, \{f', c'_j\}, \{f', b'_j\} \mid j \in [M]\}$. Note that for any pair of vertices in \mathcal{D} , either both elements of the pair are in V or both are in V' . For every pair $\{u, v\}$ of vertices in \mathcal{D} that are in V , add three vertices $d'_{u,v,1}, d'_{u,v,2}, d'_{u,v,3}$ to V' and make them adjacent to both u, v . For every pair $\{u, v\}$ of vertices in \mathcal{D} that are in V' , add three vertices $d_{u,v,1}, d_{u,v,2}, d_{u,v,3}$ to V and make them adjacent to both u, v . The pairs in \mathcal{D} are the ones that should not be in non-adjacent witness sets and the common neighbors are added to achieve this property.

This completes the construction of G .

As the reduction always adds edges with one of its endpoints in V and the other endpoint in V' , G is a bipartite graph with bipartition $\{V, V'\}$. Let $D = \{d_{u,v,p} \mid \{u, v\} \in \mathcal{D}, u, v \in V \text{ and } p \in [3]\}$ and $D' = \{d'_{u,v,p} \mid \{u, v\} \in \mathcal{D}, u, v \in V' \text{ and } p \in [3]\}$.

4.2 Properties of a Nice C_4 -Witness Structure of G

Now, we show that if G is contractible to a 4-cycle, then there is a C_4 -witness structure of G satisfying certain nice properties. For this purpose, we introduce the following notion of a *nice C_4 -witness structure*.

Definition 4.1. *A C_4 -witness structure of G is a nice C_4 -witness structure if the following properties hold.*

(P1) *For every pair $\{u, v\}$ in \mathcal{D} , u and v are in the same or adjacent witness sets.*

(P2) *Every vertex in $D \cup D'$ is in a big witness set. Further, every vertex in D' is in the same witness set as t and every vertex in D is in the same witness set as f' .*

Next, we show the existence of a nice C_4 -witness structure for YES-instances.

Lemma 5. *If G is contractible to a 4-cycle, then there is a nice C_4 -witness structure of G .*

Proof. Let $\mathcal{W} = \{W^0, W^1, W^2, W^3\}$ be a C_4 -witness structure of G such that $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i + 1) \pmod{4}$. We first show that \mathcal{W} satisfies Property (P1). Assume for the sake of contradiction that there is a pair $\{u, v\} \in \mathcal{D}$ such that $u \in W^i$ and $v \in W^{(i+2) \pmod{4}}$.

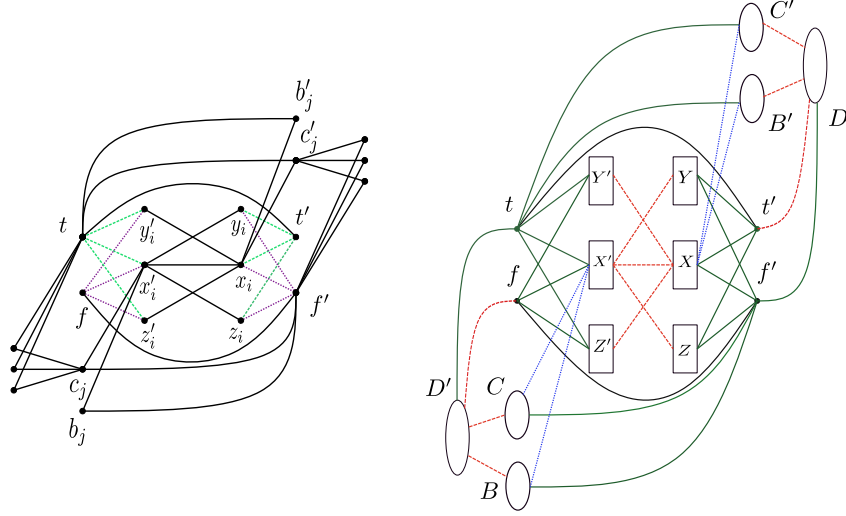


Figure 2: (Left) The graph G where only three vertices each in D and D' shown with purple (dotted) edges denote setting variable X_i to **True** and green (dashed) edges denote setting X_i to **False**. (Right) Adjacency relation between different subsets of vertices.

Recall there are three pairwise non-adjacent vertices $d_{u,v,1}$, $d_{u,v,2}$ and $d_{u,v,3}$ whose neighborhood in G is $\{u, v\}$. It follows that each of these vertices is in $W^{(i+1) \bmod 4}$ or in $W^{(i+3) \bmod 4}$. Then, one of these sets, say W^j , contains $d_{u,v,1}$ and $d_{u,v,2}$. However, W^j has no vertex that is adjacent to either of these vertices contradicting the fact that W^j is a connected set.

Next, we show that every vertex in $D \cup D'$ is in a big witness set. Assume for the sake of contradiction that there is a vertex, say $d_{u,v,1} \in D \cup D'$, which is in a singleton witness set W^i . As W^i is adjacent to $W^{(i+1) \bmod 4}$ and $W^{(i+3) \bmod 4}$ while $d_{u,v,1}$ is adjacent to only u and v , it follows that $u \in W^{(i+3) \bmod 4}$ and $v \in W^{(i+1) \bmod 4}$. Then, we get $d_{u,v,2}, d_{u,v,3} \in W^{(i+2) \bmod 4}$. However, $N(d_{u,v,2}) = N(d_{u,v,3}) = \{u, v\} \subseteq W^{(i+1) \bmod 4} \cup W^{(i+3) \bmod 4}$. This contradicts the fact that $W^{(i+2) \bmod 4}$ is a connected set.

Subsequently, we assume that \mathcal{W} satisfies Property (P1) and every vertex in $D \cup D'$ is in a big witness set. Without loss of generality assume $t \in W^0$. Let $\mu(\mathcal{W})$ be the number of vertices in D' that are not in W^0 . If $\mu(\mathcal{W}) = 0$ then it follows that every vertex in D' is in the same witness set as t . Suppose $\mu(\mathcal{W}) \geq 1$. Let $d'_{t,u,p}$ be a vertex in $D' \setminus W^0$ for some $u \in \{f\} \cup C \cup B$ and $p \in [3]$. As t is adjacent to $d'_{t,u,p}$ and $t \in W^0$, $d'_{t,u,p}$ is either in W^1 or in W^3 . Without loss of generality, suppose $d'_{t,u,p} \in W^1$. As every vertex in $D \cup D'$ is in a big witness set, W^1 is a big witness set. As $d'_{t,u,p}$ is adjacent to only t and u , we have $u \in W^1$. Define $W_\star^0 = W^0 \cup \{d'_{t,u,p}\}$ and $W_\star^1 = W^1 \setminus \{d'_{t,u,p}\}$. It is easy to verify that $\mathcal{W}_\star = \{W_\star^0, W_\star^1, W^2, W^3\}$ is a C_4 -witness structure of G . Moreover, $\mu(\mathcal{W}_\star) < \mu(\mathcal{W})$. Hence, by repeating the this process at most $|D'|$ times, we obtain a C_4 -witness structure of G in which every vertex in D' is in the same witness set as t . Using identical arguments, we can obtain a witness structure which also satisfies the property that every vertex in D is in the same witness set as f' . \square

Now, we show a property of a nice C_4 -witness structure of G that will be crucial to proving the correctness of the reduction.

Lemma 6. *In any nice C_4 -witness structure of G , every pair of vertices in $\{t, t', f, f'\}$ are in different witness sets.*

Proof. We first show that t, f' are in different witness sets. Assume, for the sake of contradiction,

that t and f' are in the same witness set, say W^0 . By construction, every vertex in $V(G) \setminus \{t, f'\}$ is adjacent to either t or f' . That is, $\{t, f'\}$ is a dominating set in G implying that W^2 is empty since every (non-empty) witness set containing any vertex in $V(G) \setminus \{t, f'\}$ is adjacent to W^0 . Hence, we conclude that t and f' are in different witness sets.

Next, we prove that t', f are in different witness sets. Assume, for the sake of contradiction, that t', f are in the same witness set, say W^0 . As W^2 is not adjacent to W^0 , this implies that $W^2 \subseteq C \cup B \cup C' \cup B' \cup D \cup D'$. By Property (P2) of a nice C_4 -witness structure of G , we have $W^2 \subseteq C \cup B \cup C' \cup B'$. As $C \cup B \cup C' \cup B'$ is an independent set and W^2 is a connected set, it follows that W^2 is a singleton set. Suppose $W^2 = \{c_j\}$ for some $j \in [M]$. By construction, $N(c_j) \subseteq \{f'\} \cup X' \cup D'$. As every vertex in $X' \cup \{f'\}$ is adjacent to $f \in W^0$, we have $X' \cup \{f'\} \subseteq W^1 \cup W^3$. Once again by Property (P2) of a nice C_4 -witness structure of G , all the vertices in D are in the same witness set that contains f' . As W^1 and W^3 are adjacent to W^2 , we have $(D' \cup X' \cup \{f'\}) \cap W^1 \neq \emptyset$ and $(D' \cup X' \cup \{f'\}) \cap W^3 \neq \emptyset$. By construction, $N(c_j) \cap (X' \cup \{f'\}) = N(b_j) \cap (X' \cup \{f'\})$. This implies that b_j is in W^0 as $W^2 = \{c_j\}$. By our assumption, $W^0 \setminus \{b_j\}$ contains t', f and hence is non-empty. However, as $N(b_j) \subseteq W^1 \cup W^3$, b_j is not adjacent with any other vertex in W^0 . This contradicts the fact that W^0 is a connected set. A similar argument holds if $W^2 = \{c'_j\}$ or $W^2 = \{b_j\}$ or $W^2 = \{b'_j\}$. Thus, t', f are in different witness sets.

Now, we show that t, t' are in different witness sets and f, f' are in different witness sets. Assume, for the sake of contradiction, that t and t' are in the same witness set, say W^0 . As W^2 is not adjacent with W^0 , this implies that $W^2 \subseteq \{f, f'\} \cup C \cup B \cup D$. Recall that the pairs $\{t, f\}$, $\{t', f'\}$, $\{t, c_j\}$ and $\{t, b_j\}$ are in \mathcal{D} for every $j \in [M]$. As $t, t' \in W^0$, by Property (P1) of a nice C_4 -witness structure of G , we have that $f, f', c_j, b_j \notin W^2$ for any $j \in [M]$. By Property (P2), we have $(D \cup D') \cap W^2 = \emptyset$. This contradicts the fact that W^2 is a nonempty set. Hence, our assumption is wrong, and t, t' are indeed in different set. Using a similar argument, it is easy to see that f, f' are in different witness sets as well.

Finally, we show that t, f are in different witness sets and t', f' are in different witness sets. Assume, for the sake of contradiction, that t, f are in the same witness set, say W^0 . As W^2 is not adjacent to W^0 , we have $W^2 \subseteq X \cup Y \cup Z \cup C \cup B \cup D$. As f' is adjacent to f , we have $f' \in W^1 \cup W^3$. As $\{t, f'\} \cap W^2 = \emptyset$, by Property (P2) of a nice C_4 -witness structure of G , we have $D \cap W^2 = \emptyset$. Hence, $W^2 \subseteq X \cup Y \cup Z \cup C \cup B$. Recall that for any $j \in [M]$, the pair $\{t, c_j\}$ is in \mathcal{D} . As $t \in W^0$, by Property (P1) of a nice C_4 -witness structure of G , we get that $c_j \notin W^2$ for any $j \in [M]$. Using symmetric arguments, we can conclude $b_j \notin W^2$ for any $j \in [M]$ as well. This implies that $W^2 \subseteq X \cup Y \cup Z$. By construction, $X \cup Y \cup Z$ is an independent set in G . As W^2 is a connected set, it follows that it is a singleton witness set. Suppose $W^2 = \{x_i\}$ for some $i \in [N]$. Recall that t', f' are adjacent to t and f , respectively, and are adjacent to x_i . As $t, f \in W^0$ and $x_i \in W^2$, we have $t', f' \in W^1 \cup W^3$. However, the pair $\{t', f'\}$ is in \mathcal{D} . By Property (P1) of a nice C_4 -witness structure of G , either $\{t', f'\} \subseteq W^1$ or $\{t', f'\} \subseteq W^3$. Without loss of generality, suppose $\{t', f'\} \subseteq W^3$. Recall that $N(y'_i) = N(z'_i) = \{t, f, x_i\}$. Then, $t, f \in W^0$ and $\{x_i\} = W^2$ imply that $y'_i, z'_i \in W^1 \cup W^3$. As $N(y'_i) \subseteq W^0 \cup W^2$, if $y'_i \in W^1$, then $W^1 = \{y'_i\}$. A similar statement holds for z'_i . This implies that either y'_i or z'_i is present in W^3 . Suppose $z'_i \in W^3$. Then, $W^3 \setminus \{z'_i\}$ contains t', f' , and hence is non-empty. As $N(z'_i) \subseteq W^0 \cup W^2$, z'_i is not adjacent with any other vertex in W^3 . This contradicts the fact that W^3 is connected. Hence, our assumption is wrong and t, f are in different witness sets. A similar argument holds if $W^2 = \{y_i\}$ or $W^2 = \{z_i\}$. Using a symmetric argument, it follows that t', f' are in different witness sets. \square

4.3 Equivalence of G and ψ

Now, we are ready to establish the equivalence of ψ and G .

Lemma 7. *If ψ is a YES-instance of POSITIVE NAE-SAT then G is a YES-instance of C_4 -CONTRACTIBILITY.*

Proof. Suppose $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\text{True}, \text{False}\}$ is a not-all-equal satisfying assignment of ψ . Define the following partition of $V(G)$.

$$\begin{aligned} W^0 &:= \{t\} \cup \{x'_i, y'_i, z'_i \mid i \in [N] \text{ and } \pi(X_i) = \text{True}\} \cup D', \\ W^1 &:= \{t'\} \cup \{x_i, y_i, z_i \mid i \in [N] \text{ and } \pi(X_i) = \text{True}\} \cup B' \cup C', \\ W^2 &:= \{f'\} \cup \{x_i, y_i, z_i \mid i \in [N] \text{ and } \pi(X_i) = \text{False}\} \cup D, \text{ and} \\ W^3 &:= \{f\} \cup \{x'_i, y'_i, z'_i \mid i \in [N] \text{ and } \pi(X_i) = \text{False}\} \cup B \cup C. \end{aligned}$$

As t is adjacent to every vertex in $X' \cup Y' \cup Z' \cup D'$, and f' is adjacent to every vertex in $X \cup Y \cup Z \cup D$, W^0 and W^2 are connected sets in G . Further, by construction, $E(W^0, W^2) = \emptyset$ and $E(W^1, W^3) = \emptyset$. W^1 is a connected set since $X \cup Y \cup Z \subseteq N(t')$ and for each $j \in [M]$, there exists $i \in [N]$ such that $x_i \in W^1$ (corresponding to a variable in C_j set to **True**) and $c'_j x_i, b'_j x_i \in E(G)$. Similarly, W^3 is also a connected set. The edges tt' and ff' , respectively, ensure that W^0 is adjacent to W^1 and W^3 is adjacent to W^2 . As for any $i \in [N]$, x'_i is adjacent with t and f and $x'_i \in W^0 \cup W^3$, it follows that W^0 and W^3 are adjacent. Similarly, W^1 and W^2 are adjacent. Hence, $\{W^0, W^1, W^2, W^3\}$ is a C_4 -witness structure. \square

Now, we proceed to show the converse of Lemma 7. We crucially use the properties of a nice C_4 -witness structure. This structure along with certain other properties help to obtain a not-all-equal satisfying assignment of ψ .

Lemma 8. *If G is a YES-instance of C_4 -CONTRACTIBILITY then ψ is a YES-instance of POSITIVE NAE-SAT.*

Proof. Suppose $\mathcal{W} = \{W^0, W^1, W^2, W^3\}$ is a C_4 -witness structure of G where $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i \pm 1) \bmod 4$. From Lemmas 5 and 6, we may assume that \mathcal{W} is a nice C_4 -witness structure in which every pair of vertices in $\{t, t', f, f'\}$ are in different witness sets. As $\{t, f\}$ and $\{t', f'\}$ are in \mathcal{D} , by Property (P1) of a nice C_4 -witness structure of G , t and f are in adjacent witness sets and t' and f' are in adjacent witness sets. Hence, without loss of generality, we may assume that $t \in W^0$, $t' \in W^1$, $f' \in W^2$, and $f \in W^3$. Also, by Property (P2) of a nice C_4 -witness structure of G , we have $D' \subseteq W^0$ and $D \subseteq W^2$.

For each $i \in [N]$, x'_i is adjacent to t, f and x_i is adjacent to t', f' . Therefore, $x_i \notin W^0 \cup W^3$, $x'_i \notin W^1 \cup W^2$ and we have $X' \subseteq W^0 \cup W^3$ and $X \subseteq W^1 \cup W^2$. Further, since $x_i x'_i \in E(G)$, it follows that $x_i \in W^1$ if and only if $x'_i \in W^0$ and $x_i \in W^2$ if and only if $x'_i \in W^3$. Refer to Figure 2 for an illustration where these two choices are indicated by the purple (dotted) edges and green (dashed) edges. We will associate these two choices with setting the variable X_i to **True** or **False**, respectively. Consider a vertex $c_j \in C$ for some $j \in [M]$. As $f' \in W^2$ and $f' c_j \in E(G)$, it follows that $c_j \notin W^0$. Also, since $t \in W^0$ and $\{t, c_j\}$ is in \mathcal{D} , by Property (P1) of a nice C_4 -witness structure of G , it follows that c_j is not in W^2 . As $N(c_j) \subseteq W^0 \cup W^2 \cup W^3$ and $t' \in W^1$, if $c_j \in W^1$, then W^1 cannot be a connected set. Hence, $c_j \in W^3$. As c_j is an arbitrary vertex of C in this reasoning, we have $C \subseteq W^3$. Similarly, $B \subseteq W^3$. This implies $C \cup B \subseteq W^3$. By a symmetric argument, we have $C' \cup B' \subseteq W^1$.

We now construct an assignment $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\text{True}, \text{False}\}$ using \mathcal{W} . For every $i \in [N]$, set $\pi(X_i) = \text{True}$ if $x_i \in W^1$ (or equivalently $x'_i \in W^0$) and set $\pi(X_i) = \text{False}$ if $x'_i \in W^3$ (or equivalently $x_i \in W^2$). As mentioned before, $x_i \in W^1$ if and only if $x'_i \in W^0$ and $x'_i \in W^3$ if and only if $x_i \in W^2$. As W^3 is connected and $f, c_j \in W^3$, for every $j \in [M]$, there exists $i \in [N]$, such that $x'_i \in W^3$ and $c_j x_i \in E(G)$. Similarly, as W^1 is connected, for every $j \in [M]$, there exists $i \in [N]$ such that $x_i \in W^1$ and $c'_j x_i \in E(G)$. \square

5 Conclusion and Future Directions

In this work, we showed that C_ℓ -CONTRACTIBILITY is NP-complete on bipartite graphs for $\ell \in \{4, 5\}$ by giving polynomial-time reductions from POSITIVE NAE-SAT.

POSITIVE NAE-SAT (or equivalently, HYPERGRAPH 2-COLORABILITY) has been one of the canonical NP-complete problems in many intractability results on C_ℓ -CONTRACTIBILITY [5, 8, 10]. In general, in most contraction problems, it is a non-trivial task to forbid certain edges from being contracted in a solution. The simultaneous property of requiring a variable to be **True** and a variable to be **False** in every clause of a YES-instance of POSITIVE NAE-SAT helps to encode that certain edges in the output graph of the reduction cannot be contracted, hence, giving a handle on the required structure of the witness sets. This is one of the reasons that makes POSITIVE NAE-SAT an amenable choice in many reductions for graph contractibility problems. However, the sophistication level of the gadgets involved in the reduction increases with the restriction required on the input graph (eg. bipartite graphs, claw-free graphs). In contrast, the sophistication decreases with increase in the size of the target graph, for instance, the gadgets required for the NP-hardness of C_4 -CONTRACTIBILITY are more complex than those needed for C_5 -CONTRACTIBILITY, which are more complex than what are required for C_6 -CONTRACTIBILITY.

Continuing along the direction of solving cycle contractibility in restricted graph classes, we can also show the following result.

Theorem 3. C_4 -CONTRACTIBILITY is NP-complete on K_4 -free graphs of diameter 2.

We postpone the proof of the theorem in Subsection 5.1. In the subsection, we also argue that Theorem 3 can be generalized to show that $K_{p,q}$ -CONTRACTIBILITY (the problem of determining if a graph is contractible to the complete bipartite graph with p vertices in one part and q vertices in the other part) is also NP-complete for each $p, q \geq 2$ on K_4 -free graphs of diameter 2. Our interest in this restricted case stems from its relationship with DISCONNECTED CUT, the problem of determining if a connected graph G contains a subset $U \subseteq V(G)$ such that both $G[U]$ and $G - U$ are disconnected [18, 23, 24]. If the diameter of G is 2, then G has a disconnected cut if and only if G is contractible to $K_{p,q}$ for some $p, q \geq 2$ [18, Proposition 1]. Martin et al. proved that DISCONNECTED CUT is polynomial-time solvable for H -free graphs when $H \neq K_4$ is a graph on at most 4 vertices [24, Theorem 7]. Theorem 3 (and its generalization to $p, q \geq 2$) implies that (p, q) -DISCONNECTED CUT (see [18]) is NP-complete for all $p, q \geq 2$ on K_4 -free graphs. Although this falls short of completing the dichotomy of [24, Theorem 7], we believe that it strongly suggests that there is no polynomial-time algorithm for DISCONNECTED CUT on K_4 -free graphs.

Finally, determining the longest cycle to which an H -free graph (for a fixed H) is contractible is another interesting future direction. A similar study on H -free graphs in the context of longest paths is known [21]. Note that assuming $P \neq NP$, the complexities of contracting to a longest path and longest cycle do not coincide on H -free graphs.

5.1 Proof of Theorem 3

It is easy to verify that the problem is in NP. To show NP-hardness, once again we give a polynomial-time reduction from POSITIVE NAE-SAT. Let $\{X_1, X_2, \dots, X_N\}$ and $\{C_1, C_2, \dots, C_M\}$ be the sets of variables and clauses, respectively, in an instance ψ of POSITIVE NAE-SAT. We construct a K_4 -free diameter 2 graph G with a partition $\{X, S^+, S^-\}$ of its vertex set as follows.

1. Add vertices t, t' to S^+ and vertices f, f' to S^- . Let A denote $\{t, t', f, f'\}$.
2. For every $i \in [N]$, we add a vertex x_i to X corresponding to variable X_i .
3. For every $j \in [M]$, we add vertices c_j^+, b_j^+ to S^+ and c_j^-, b_j^- to S^- corresponding to clause C_j .

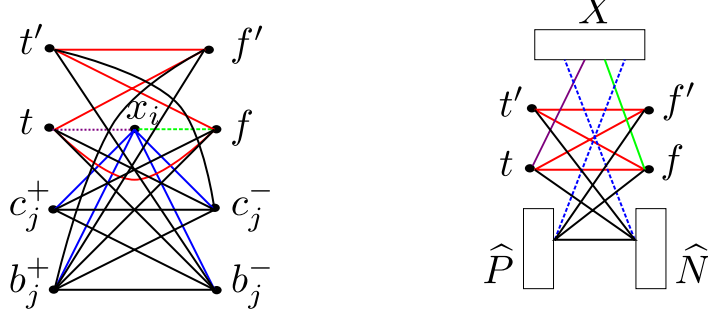


Figure 3: (Left) The graph G with different sets of edges highlighted. Here, purple (dotted) edges denote setting the variable to **True** and the green (dashed) edges denote setting it to **False** in a YES-instance. (Right) Adjacency relation between subset of vertices where $\widehat{P} = S^+ \setminus \{t, t'\}$ and $\widehat{N} = S^- \setminus \{f, f'\}$.

4. Make every vertex in S^+ adjacent to every vertex in S^- .
5. Make every vertex in X adjacent to both t and f .
6. For every $i \in [N]$ and $j \in [M]$ such that X_i appears in C_j , add edges $x_i c_j^+$, $x_i b_j^+$, $x_i c_j^-$, and $x_i b_j^-$.

This completes the construction of G . See Figure 3 for an illustration. It is easy to verify that G is K_4 -free as the three sets S^+ , S^- , X that partition $V(G)$ are independent sets. Further, the diameter of G is two as any pair of non-adjacent vertices have a common neighbor. We now prove the correctness of the reduction. Suppose $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\text{True}, \text{False}\}$ is a not-all-equal satisfying assignment of ψ . Then, let $W^0 = \{t'\}$, $W^1 = \{f'\}$, $W^2 = \{t\} \cup \{x_i \mid i \in [N] \text{ and } \pi(X_i) = \text{True}\} \cup (S^+ \setminus \{t'\})$ and $W^3 = \{f\} \cup \{x_i \mid i \in [N] \text{ and } \pi(X_i) = \text{False}\} \cup (S^- \setminus \{f'\})$. It is easy to verify that $\{W^0, W^1, W^2, W^3\}$ is a C_4 -witness structure of G .

Conversely, suppose $\mathcal{W} = \{W^0, W^1, W^2, W^3\}$ is a C_4 -witness structure of G where $E(W^i, W^j) \neq \emptyset$ if and only if $j = (i+1) \pmod 4$. Then, we claim that every pair of vertices in A are in different witness sets. Assume that the claim holds. Then, without loss of generality, we may assume that $t' \in W^0$, $f' \in W^1$, $t \in W^2$, and $f \in W^3$. Observe that as f and t are adjacent to every vertex in X , it follows that for each $i \in [N]$, $x_i \in W^2 \cup W^3$. We now construct an assignment $\pi : \{X_1, X_2, \dots, X_N\} \mapsto \{\text{True}, \text{False}\}$ using \mathcal{W} . For every $i \in [N]$, set $\pi(X_i) = \text{True}$ if $x_i \in W^2$ and set $\pi(X_i) = \text{False}$ if $x_i \in W^3$. As every vertex in S^+ is adjacent to f and f' , for every $j \in [M]$, we have $c_j^+, b_j^+ \in W^0 \cup W^2$. Similarly, as every vertex in S^- is adjacent to t and t' , for every $j \in [M]$, we have $c_j^-, b_j^- \in W^1 \cup W^3$. If for some $j \in [M]$, $c_j^+ \in W^0$, then as W^0 has no neighbors of c_j^+ , W^0 cannot be a connected set. Then, as $N(b_j^+) = N(c_j^+)$, we have $S^+ \subseteq W^2$. Similarly, we have $S^- \subseteq W^3$. Consider $j \in [M]$. As $t, c_j^+, b_j^+ \in W^2$ and these three vertices form an independent set, it follows that there is an index $i \in [N]$ such that $x_i \in W^2$ satisfying $x_i c_j^+, x_i b_j^+ \in E(G)$. Similarly, for each $j \in [M]$, there is an index $i \in [N]$ such that $x_i \in W^3$ satisfying $x_i c_j^-, x_i b_j^- \in E(G)$. It now follows that for any $j \in [M]$, π sets at least one of the variables in clause C_j to **True** and at least one of variables in C_j to **False**.

It now remains to show that in the C_4 -witness structure \mathcal{W} of G , every pair of vertices in A are in different witness sets. As $S_1 = \{t, f\}$, $S_2 = \{t', f\}$ and $S_3 = \{t, f'\}$ are dominating sets in G , for each $i \in [3]$, the vertices in S_i are in different witness sets. If for some i , the two vertices S_i are in the same witness set W^j , then it follows that $W^{(j+2) \pmod 4}$ is empty leading to a contradiction. Now suppose f and f' are in the same witness set, say W^0 . Then, $W^2 \subseteq S^- \setminus \{f, f'\}$ as every other vertex is adjacent to f or f' . However, as $S^- \setminus \{f, f'\}$ is an independent set, it follows that W^2 is a singleton set, say $\{c_j^-\}$. Then, $N(c_j^-) \cap W^1 \neq \emptyset$ and

$N(c_j^-) \cap W^3 \neq \emptyset$. As $N(c_j^-) = N(b_j^-)$, we have $b_j^- \in W^0$. However, $b_j^-, f, f' \in W^0$ implies that W^0 is not a connected set (as all neighbors of b_j^- are in $W^1 \cup W^3$) leading to a contradiction. A similar argument holds if $W^2 = \{b_j^-\}$. A similar argument shows that t and t' cannot be in the same witness set as well. Finally, we show that t' and f' are in different witness sets. Assume on the contrary that t' and f' are in the same witness set, say W^0 . Then, as t and f cannot be in W^0 , we have $t, f \in W^1 \cup W^3$. However, as t and f are in different witness sets, it follows that $E(W^1, W^3) \neq \emptyset$ leading to a contradiction.

Remark: Observe that the base cycle (t', f', t, f) in the above construction may be viewed as a $K_{2,2}$. Theorem 3 can be generalized to show that $K_{p,q}$ -CONTRACTIBILITY is NP-complete for each $p, q \geq 2$ on K_4 -free graphs of diameter 2 by blowing up the graph G as follows: add $p - 2$ new vertices to S^+ and $q - 2$ vertices to S^- ensuring that every vertex in S^+ is adjacent to every vertex in S^- . Now, we can show that ψ is a YES-instance of POSITIVE NAE-SAT if and only if G is contractible to $K_{p,q}$.

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