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Persistence problems for fractional processes

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Abstract

In this thesis, we deal with several persistence problems for fractional processes. Persistence concerns the event that a stochastic process has a long excursion staying below or above a certain barrier. A central question in this context is the analysis of the probability of this event – the so-called persistence probability.

We first consider the persistence probabilities of integrated fractional Brownian motion and fractionally integrated Brownian motion. While it is well-known that these persistence probabilities decay asymptotically polynomially, their polynomial rates are unknown except for the special cases of Brownian motion and integrated Brownian motion. We show that for both processes, the polynomial rate is a continuous function in the Hurst parameter and determine its asymptotic behaviour at the boundaries of the respective parameter domain.

Subsequently, we study persistence probabilities of mixed processes, such as mixed fractional Brownian motion. Precisely, we consider the sum of two self-similar centred Gaussian processes with different self-similarity indices and show that, under non-negativity assumptions of covariance functions and some further minor conditions, the persistence probability of the sum decays asymptotically polynomially with the same polynomial rate as for the single process with the greater self-similarity index. In particular, this determines the polynomial rate of the persistence probability of mixed fractional Brownian motion.

Lastly, we give estimates for the persistence probabilities of further fractional processes of interest, namely the bifractional Brownian motion and the fractional Ornstein-Uhlenbeck process.

Zusammenfassung

In dieser Dissertation befassen wir uns mit verschiedenen Persistence-Problemen für fraktionale Prozesse. Mit Persistence ist das Ereignis einer langen Exkursion eines stochastischen Prozesses gemeint, bei der dieser unter- oder oberhalb einer bestimmten Schranke bleibt. Eine zentrale Fragestellung in diesem Kontext ist die Analyse der Wahrscheinlichkeit jenes Ereignisses – die sogenannte Persistence-Wahrscheinlichkeit.

Zunächst betrachten wir die Persistence-Wahrscheinlichkeiten der integrierten fraktionalen Brownschen Bewegung sowie der fraktional integrierten Brownschen Bewegung. Während es wohlbekannt ist, dass diese Persistence-Wahrscheinlichkeiten asymptotisch polynomiell abfallen, ist die polynomielle Rate nur in den Spezialfällen der Brownschen Bewegung und der integrierten Brownschen Bewegung bekannt. Wir zeigen, dass bei beiden Prozessen die polynomielle Rate eine stetige Funktion vom Hurst-Parameter ist, und bestimmen ihr asymptotisches Verhalten an den Rändern des jeweiligen Definitionsbereiches des Parameters.

Anschließend beschäftigen wir uns mit den Persistence-Wahrscheinlichkeiten von gemischten Prozessen wie der gemischten fraktionalen Brownschen Bewegung. Um genau zu sein, betrachten wir die Summe zweier selbstähnlicher zentrierter Gaußprozesse mit unterschiedlichen Selbstähnlichkeitsindizes und zeigen, dass, unter der Annahme von nicht-negativen Kovarianzfunktionen und einiger weiterer unwesentlicher Bedingungen, die Persistence-Wahrscheinlichkeit der Summe asymptotisch polynomiell abfällt, und zwar mit der gleichen polynomiellen Rate wie bei demjenigen Einzelprozess, der den größeren Selbstähnlichkeitsindex besitzt. Insbesondere wird damit die polynomielle Rate der Persistence-Wahrscheinlichkeit der gemischten fraktionalen Brownschen Bewegung bestimmt.

Abschließend geben wir noch Abschätzungen für die Persistence-Wahrscheinlichkeiten weiterer relevanter fraktionaler Prozesse, nämlich der bifraktionalen Brownschen Bewegung und des fraktionalen Ornstein-Uhlenbeck-Prozesses.

Contents

Chapter 1

Introduction

This thesis deals with so-called persistence problems for stochastic processes. Persistence concerns the event that a real-valued stochastic process stays below or above a fixed barrier for an untypically long time. While this is a classical topic in probability theory for Markov processes, such as Brownian motion, random walks and Lévy processes, research on this type of problem for fractional processes has just begun, as many powerful tools of the Markovian setting are not applicable anymore. The aim of this thesis is to make further contributions to fill the gap regarding persistence results for fractional Brownian motion and related processes.

In this chapter, we state the general persistence problem we are concerned with, we motivate where the interest for these persistence problems comes from and we give an overview of known results and related literature.

1.1 The general problem

For a real-valued stochastic process $X = (X_t)_{t\geq 0}$ and a barrier $x \in \mathbb{R}$, the so-called (one-sided) *persistence probabilities* of X are given by

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq x\right), \qquad T > 0,$$

or, alternatively, by

$$\mathbb{P}\left(\inf_{t\in[0,T]} X_t \geq -x\right), \qquad T > 0.$$

Typically, these probabilities tend to zero for $T \to \infty$ and the goal is to determine the asymptotic rate of decay. In this thesis, we mainly deal with self-similar centred Gaussian processes, i.e., we consider processes X whose finite-dimensional distributions are centred Gaussian and which satisfy $(X_{ct}) \stackrel{\mathrm{d}}{=} (c^H X_t)$ for some $H > 0$ and all $c > 0$. Thus, due to the fact that centred Gaussian distributions are symmetric, the above two ways of defining persistence probabilities are equivalent. Further, due to self-similarity, we have $X_0 = 0$ a.s., implying $\mathbb{P}\big(\sup_{t\in[0,T]} X_t \le x\big) = 0$ for $x < 0$ and all $T > 0$ (which also holds for $x = 0$, if we exclude degenerate cases where the persistence probability for $x = 0$ does not tend to zero), whereas, for $x > 0$, self-similarity yields $\mathbb{P}\big(\sup_{t\in[0,T]} X_t \le x\big) = \mathbb{P}\big(\sup_{t\in[0,x^{-1/H}T]} X_t \le 1\big)$ for $T > 0$. Hence, w.l.o.g., we can restrict ourselves to determining the asymptotic rate of the persistence probabilities for the barrier $x = 1$. For self-similar processes, one expects asymptotically polynomial decay, i.e.,

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t \le 1\right) = T^{-\theta+o(1)}, \qquad T \to \infty, \tag{1.1}$$

where $\theta = \theta(X) \in (0,\infty)$ is the so-called *persistence exponent* of X.

1.2 Motivation

Persistence probabilities appear in many fields of applied probability. In physics, the first interest for persistence came from the so-called droplet condensation problem: In the context of the formation of dew, the condensation of water vapor on a substrate can be modelled in a way that the expected fraction of the surface of the substrate which was never covered by water is given by a persistence probability. This was discovered due to the fact that this fraction decays asymptotically polynomially in time just like the persistence probability in (1.1), see [48]. Motivated by this observation, persistence properties of various dynamical systems in non-equilibrium states were studied by theoretical physicists and in many cases, persistence probabilities turned out to be related to relevant properties of the system. In particular, persistence exponents often serve as a simple measure of how fast the corresponding system returns to its equilibrium. Models involving fractional Brownian motion and related processes have received considerable attention in recent years. For instance, the inviscid Burgers equation, where the initial velocity is given by a fractional Brownian motion, was studied extensively in the context of persistence, see [75], [78], [58] and [56]. For an overview of persistence problems and results from the perspective of theoretical physics, we refer to the surveys [72], [22] and [47] as well as the monographs [51] and [68].

As a further motivation, we want to mention applications in finance and insurance. An investor, for example, who considers selling a stock performing badly might be interested in the probability that the stock price returns to his input price in a given time, which is clearly the counterpart to a persistence probability. Further, when dealing with limit orders to buy or sell a stock, persistence probabilities occur, as the event that the order will not be executed in a given time is clearly a persistence event. Note in this context that due to long memory, especially fractional and mixed fractional Brownian motion play a major role in finance, see e.g. [27] and [3]. In insurance, one deals with so-called ruin probabilities, where ruin concerns the event that the costs of an insurance portfolio exceed the sum of initial capital and received insurance premiums. If one considers this event on a finite time frame, the ruin probability represents again the counterpart to a persistence probability. Note, however, that in the typical setting, the probability that the ruin never occurs is positive so that one is rather interested in the asymptotic behaviour of the ruin probability (on an infinite time frame) when the initial capital tends to infinity, see [34] for classical results and [7] for results in the context of a modified notion of ruin.

1.3 Known results

In this section, we give an overview of the existing results for (one-sided) persistence probabilties of (time-continuous Gaussian) fractional processes which form the basis of our results in Chapters 3 to 5. Further related results for the two-sided persistence problem, discrete-time processes or non-Gaussian processes will be summarized in Section 1.4. For a recent overview of mathematical results for persistence probabilities in general, we refer to the survey [16].

Fractional Brownian motion. Recall that the fractional Brownian motion (FBM) B^H with Hurst parameter $H \in (0,1)$ is the unique normalized centred Gaussian process with a.s. continuous sample paths which is H-self-similar and which has stationary increments, i.e., B^H satisfies – additionally to the property of self-similarity we have already mentioned – the condition $(B^H_{t+h} - B^H_h) \overset{\mathrm{d}}{=} (B^H_t)$

for all $h > 0$. For $H = 1/2$, these properties imply independent increments so that $W := B^{1/2}$ is a usual Brownian motion.

Let us thus recall shortly the classical result for Brownian motion. The strong Markov property of Brownian motion implies the reflection principle, which states that $(\tilde{W}_t) := (W_t \mathbb{1}_{t\le\tau} + (2W_\tau - W_t)\mathbb{1}_{t>\tau})$, the Brownian motion reflected at a stopping time τ, has the same law as the original Brownian motion W, see e.g. [61, Theorem 2.19]. Setting $\tau := \inf\{t \ge 0 \colon W_t = 1\}$ then determines the persistence probabilities of W explicitly, namely

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} W_t \le 1\right) = 1 - \mathbb{P}\left(\sup_{t\in[0,T]} W_t > 1\right) &= 1 - \mathbb{P}(W_T > 1) - \mathbb{P}(\tau < T, W_T \le 1)\\ &= 1 - \mathbb{P}(W_T > 1) - \mathbb{P}(\tilde{W}_T \ge 1)\\ &= \mathbb{P}(|W_T| \le 1) \sim \sqrt{\frac{2}{\pi}}\, T^{-1/2}, \quad T \to \infty,\end{aligned} \tag{1.2}$$

which in particular shows the asymptotic behaviour as in (1.1) with persistence exponent $\theta(W) = 1/2$. Here and elsewhere, $f(x) \sim g(x)$ stands for $\lim f(x)/g(x) = 1$.

For $H \neq 1/2$, however, these powerful Markov techniques are not applicable anymore. For $H < 1/2$, the increments of B^H are negatively correlated, whereas for $H > 1/2$, the increments are positively correlated and even exhibit long-range dependence, i.e., $\sum_{n=1}^{\infty} \mathbb{E}[B_1^H(B_{n+1}^H - B_n^H)] = \infty$. While these properties are nice for applications, the intrinsical non-Markovian structure makes it hard to derive persistence results. Even for the few fractional processes where results are available, they are typically in the form (1.1) and it remains an open problem to determine the exact asymptotic order as in (1.2). For the FBM, Molchan could derive the result

$$T^{-(1-H)} e^{-c\sqrt{\log T}} \le \mathbb{P}\left(\sup_{t\in[0,T]} B_t^H \le 1\right) \le T^{-(1-H)} e^{c\sqrt{\log T}} \tag{1.3}$$

for some $c > 0$ and T large enough, see [59], which yields the persistence exponent $\theta(B^H) = 1 - H$. The crucial part in the proof was to show that the persistence probability of FBM has – up to terms of lower order – the same asymptotic order as the expectation

$$I(T) := \mathbb{E}\left[\left(\int_0^T e^{B_t^H}\, \mathrm{d}t\right)^{-1}\right] \tag{1.4}$$

of the functional $\left(\int_0^T e^{B_t^H}\,\mathrm{d}t\right)^{-1}$, which is some sort of a smoothed out counterpart of the rough indicator $\mathbb{1}_{\sup_{t\in[0,T]} B_t^H \le 1}$. Heuristically, there is the following connection between the functional and the indicator: The typical paths of B_H contributing to the persistence event, i.e., which satisfy $B_t^H \le 1$ for all $t \in [0,T]$, rather tend to escape to $-\infty$ than to oscillate around the origin. But these are exactly those paths for which the functional is large and which contribute to $I(T)$ the most, consequently.

For $I(T)$, one was able to determine the exact asymptotic order: In [59, Statement 1], it was shown that

$$I(T) \sim H\,\mathbb{E}\left[\max_{t\in[0,1]} B_t^H\right] T^{-(1-H)}, \qquad T \to \infty, \tag{1.5}$$

by using an argument which goes back to Kawazu and Tanaka, see [41, Section 2.2]. They considered the Brownian motion, but basically used only self-similarity and stationary increments (and the existence of the moment-generating function of the maximum) so that Molchan could easily adapt the argument for the FBM. It seems plausible that in fact, the persistence probability of B^H and $I(T)$ have even the same exact asymptotic order. But still up to now, unfortunately, the result in (1.3) could only be slightly improved: For $H > 1/2$, there exists $c > 0$ such that

$$c^{-1} T^{-(1-H)} (\log T)^{-1/(2H)} \le \mathbb{P}\left(\sup_{t\in[0,T]} B_t^H \le 1\right) \le c\, T^{-(1-H)}$$

for T large enough, see [11, Theorem 12], while for $H < 1/2$, there exists $c > 0$ such that

$$c^{-1} T^{-(1-H)} (\log T)^{-1/(2H)} \le \mathbb{P}\left(\sup_{t\in[0,T]} B_t^H \le 1\right) \le T^{-(1-H)} (\log T)^{c}$$

for T large enough, see [11, Theorem 12] and [5, Theorem 1]. The result in [5] was achieved by finding another way to relate $I(T)$ to the persistence probabilities of B^H, whereas the result in [11] was deduced from stronger results for the persistence probabilities of discrete-time analogues of FBM, which we will discuss a little bit more detailed in Section 1.4.

Integrated FBM. Now, we consider the (one-sided) integrated version of B^H, given by $I_t^H := \int_0^t B_s^H\,\mathrm{d}s$, $t \ge 0$. Again, we first discuss the Brownian case $H = 1/2$, which is the integrated Brownian motion $V_t := \int_0^t W_s\,\mathrm{d}s$, $t \ge 0$. The process V itself is non-Markovian, but in contrast to B^H and I^H for $H \ne 1/2$, this is not intrinsical: As W is a Markov process, the pair (V, W) also is, and consequently, a similarly strong result as in (1.2) could be shown for the persistence probabilities of integrated Brownian motion via Markov techniques. In 1962, using the transition density of (V, W), McKean obtained among other results the joint distribution of the time when integrated Brownian motion first returns to zero, if the corresponding Brownian motion is started at 1, together with the absolute value of the Brownian motion at this time, see [50]. Based on this formula, in 1971, Goldman deduced an explicit expression for the density of the distribution of the first hitting time of integrated Brownian motion at 1, if the corresponding Brownian motion is started at 0, see [37]. In particular, he showed the following asymptotics for this density:

$$\frac{\mathbb{P}(\tau_1 \in \mathrm{d}t)}{\mathrm{d}t} \sim c\,t^{-5/4}, \qquad t \to \infty,$$

for some $c > 0$, where $\tau_1 := \inf\{t \geq 0\colon V_t = 1\}$. Since the persistence probabilities of V can be written as $\mathbb{P}\big(\sup_{t\in[0,T]} V_t \leq 1\big) = \mathbb{P}(\tau_1 \geq T)$ for $T > 0$, this implies

$$\mathbb{P}\left(\sup_{t\in[0,T]} V_t \leq 1\right) = \int_T^\infty \frac{\mathbb{P}(\tau_1 \in \mathrm{d}t)}{\mathrm{d}t}\,\mathrm{d}t \sim 4cT^{-1/4}, \qquad T \to \infty, \tag{1.6}$$

yielding $\theta(V) = 1/4$. Later, Sinai (see [77]; refinements of Isozaki and Watanabe in [40]) generalized the result to straight line boundaries (instead of the constant boundary 1) by considering approximating discrete-time processes and using the techniques he developed to deduce the same persistence exponent 1/4 for the integrated simple random walk.

However, as for the persistence probabilities of FBM, the proofs for the Brownian case cannot be adapted to the general case, since Markov tools, such as transition densities or the reflection principle, are not available anymore. Due to the fact that I^H is a self-similar centred Gaussian process with a non-negative covariance function, one knows that the persistence probabilities of integrated FBM (IFBM) behave asymptotically as in (1.1) with some persistence exponent $\theta_I(H) := \theta(I^H) \in (0, \infty)$ (cf. Corollary 2.6). Except for the Brownian case $H = 1/2$, though, where we have already seen that $\theta_I(1/2) = 1/4$, the exact value of $\theta_I(H)$ is unknown.

Based on numerical simulations, Molchan and Khokhlov stated in 2004 the conjecture

$$\theta_I(H) = H(1-H), \qquad H \in (0,1), \tag{1.7}$$

see [58], which was surprising due to its symmetry w.r.t. $H = 1/2$. As already seen in the context of the correlation of the increments, the processes B^H and thus also I^H are very different processes for $H < 1/2$ and $H > 1/2$. Further note that there evidentially is no symmetry in the two-sided persistence problem of IFBM (cf. Section 1.4 for more details). For the one-sided problem nevertheless, the symmetric conjecture has not been disproven for almost 20 years now, and over the years, more and more evidence in favour of the conjecture could be gained.

The first analytical estimate was deduced in [54], where it was shown that there exists $\rho \in (0, 1/2)$ such that $\rho H(1-H) \leq \theta_I(H) \leq 1-H$ for $H \in (0,1)$. The upper bound is due to the fact that the one-sided persistence exponent is always bounded from above by the two-sided persistence exponent of a process (in the case that both exponents exist) and that the two-sided persistence exponent of IFBM is bounded from above by $1-H$ (in fact given by $1-H$; cf. Section 1.4 for more details). The lower bound was proven by estimating the auto-covariance function of the Lamperti transform of IFBM to an auto-covariance function, where results for the persistence exponent of the corresponding stationary Gaussian process are available, and applying Slepian's lemma (cf. Chapter 2 for more details). This is a standard technique in the context of persistence of self-similar Gaussian processes, which could be further exploited to improve both upper and lower bound. In [55], resulting from an inequality of the auto-covariance functions of the Lamperti transform of IFBM with Hurst parameter H and $1-H$, respectively, the relation

$$\theta_I(1-H) \leq \theta_I(H) \text{ for } H < 1/2, \tag{1.8}$$

was used, together with another estimate by Slepian's lemma, to improve the lower bound to

$$\theta_I(H) \geq \frac{1}{2}\min(H, 1-H) \text{ for } H \in (0,1). \tag{1.9}$$

Note that (1.8) itself is of interest, as it represents one direction of the proof that θ_I has indeed the point of symmetry $H = 1/2$, which would be $\theta_I(1-H) = \theta_I(H)$. In [57], further relations of the auto-covariance functions of the Lamperti transforms of FBM and IFBM were used, by applying again Slepian's lemma, to improve the upper bound to

$$\begin{aligned}
&\theta_I(H) \leq \min(H, 1-H) && \text{for } H \in \left(0, \frac{1}{\sqrt{13}}\right] \cup \left(\frac{3}{4}, 1\right), \\
&\theta_I(H) \leq \sqrt{\frac{1-H^2}{12}} && \text{for } H \in \left(\frac{1}{\sqrt{13}}, \frac{1}{2}\right], \text{ and} \\
&\theta_I(H) \leq 1/4 && \text{for } H \in \left(\frac{1}{2}, \frac{3}{4}\right].
\end{aligned} \tag{1.10}$$

Figure 1.1 illustrates the conjecture (1.7) as well as the proven possible range of the persistence exponent of IFBM, given by the bounds (1.9) and (1.10). In Theorem 3.1, we will strengthen this analytical evidence for the conjecture (1.7) by showing that θ_I is a continuous function, which is asymptotically equivalent to the conjecture (1.7) at the boundaries 0 and 1 of the domain of the Hurst parameter $H \in (0,1)$.

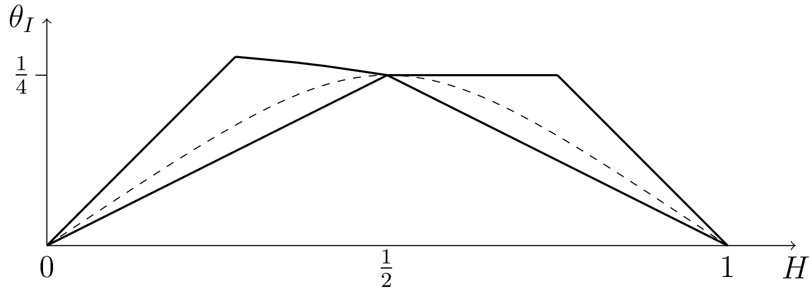

Figure 1.1: Conjecture of Molchan and Khokhlov (dashed line) and upper and lower bounds proven by Molchan (solid lines) for the (one-sided) persistence exponent of IFBM.

Fractionally integrated Brownian motion. For a Brownian motion W and $H > 0$, let us now consider the fractionally integrated Brownian motion (FIBM) with Hurst parameter H, given by

$$R_t^H := \int_0^t (t-s)^{H-\frac{1}{2}} \, \mathrm{d}W_s, \qquad t \geq 0.$$

Note that this is well-defined due to the fact that $\mathbb{E}[\int_0^\infty (t-s)^{2H-1} \, \mathrm{d}\langle W \rangle_{s \wedge t}] = \int_0^t s^{2H-1} \, \mathrm{d}s < \infty$ for $H > 0$. For $H > 1/2$, the stochastic Fubini theorem, see e.g. [67, Theorem IV.65], yields the following alternative representation for R^H:

$$\begin{aligned}
\frac{1}{\Gamma(H+1/2)} R_t^H &= \frac{H-1/2}{\Gamma(H+1/2)} \int_0^t \int_0^t \mathbb{1}_{u \geq s} (t-u)^{H-3/2} \, \mathrm{d}u \, \mathrm{d}W_s \\
&= \frac{1}{\Gamma(H-1/2)} \int_0^t (t-u)^{H-3/2} \int_0^t \mathbb{1}_{u \geq s} \, \mathrm{d}W_s \, \mathrm{d}u \\
&= \frac{1}{\Gamma(H-1/2)} \int_0^t W_u (t-u)^{H-1/2-1} \, \mathrm{d}u, \qquad (1.11)
\end{aligned}$$

which is the Riemann-Liouville fractional integral of W of order $H - 1/2$. For this reason, the FIBM is also called Riemann-Liouville process. Further note that the Riemann-Liouville fractional integral of W for integer orders $n \in \mathbb{N}$ is given by the

n-times integrated version of the Brownian motion W, since

$$\begin{aligned}\int_0^t W_{s_1} \frac{(t-s_1)^{n-1}}{(n-1)!}\,\mathrm{d}s_1 &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s_1} \int_0^{s_1} W_{s_2} \frac{(s_1-s_2)^{n-1}}{(n-1)!}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \int_0^t \int_0^{s_1} W_{s_2} \frac{(s_1-s_2)^{n-2}}{(n-2)!}\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= \cdots = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} W_{s_n}\,\mathrm{d}s_n \ldots \mathrm{d}s_1 \end{aligned} \tag{1.12}$$

by the Leibniz integral rule. For $H \in (0,1)$, the FIBM R^H is closely related to the FBM B^H via the Mandelbrot-van Ness integral representation, which states that

$$R_t^H + \int_{-\infty}^0 (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\,\mathrm{d}W_s, \qquad t \geq 0,$$

is an independent decomposition of FBM (with a non-normalized variance), see e.g. [53, Theorem 1.3.1]. It is well-known that R^H shares many properties with B^H, such as H-self-similarity, continuous sample paths and, for $H \in (1/2, 1)$, long-range dependence of the increments (which are non-stationary in the case of R^H). In some literature, especially in econometrics, R^H is introduced as an alternative type of FBM, see [49]. Regarding persistence, however, the two processes behave quite differently.

As for IFBM, due to the fact that R^H is a self-similar centred Gaussian process with a non-negative covariance function, one knows that the persistence probabilities of R^H decay asymptotically polynomially as in (1.1) with some persistence exponent $\theta_R(H) := \theta(R^H) \in (0,\infty)$. Note that $R^{1/2}$ equals the Brownian motion W by definition and that $R^{3/2}$ equals the integrated Brownian motion V by (1.11). Thus, we have already seen that $\theta_R(1/2) = 1/2$ and that $\theta_R(3/2) = 1/4$. Except for these two Brownian cases, though, the exact value of $\theta_R(H)$ is unknown. Nevertheless, similarly to the case of IFBM, there are some estimates for θ_R which in particular show that the persistence exponents of FIBM and FBM do not coincide, as we will outline now.

By using the fact that fractionally integrating twice with orders $\alpha_1 > 0$ and $\alpha_2 > 0$ is equivalent to fractionally integrating once with the order $\alpha_1 + \alpha_2$, it was shown in [9] that θ_R is non-increasing on $[1/2, \infty)$. Together with the identity $\theta_R(3/2) = 1/4$, this implies $\theta_R(H) \geq 1/4$ for $H \in [1/2, 3/2]$, which shows that $\theta_R(H)$ and $\theta(B^H) = 1 - H$ cannot coincide for $H \in (3/4, 1)$. Furthermore, by again using the technique of estimating the auto-covariance function of the Lamperti transform and applying Slepian's lemma, it was deduced in [9, Corollary 4.1] that it holds

$\theta_R(H) \geq \theta_R(\infty)$ for $H \geq 1/2$, where $\theta_R(\infty) := -\lim_{T\to\infty} \frac{1}{T}\mathbb{P}(\sup_{t\in[0,T]} Z_t \leq 0)$ denotes the persistence exponent (of exponential decay, in contrast to polynomial decay in (1.1); cf. Chapter 2 for more details) of the centred Gaussian process Z with covariance function $(s,t) \mapsto 1/\cosh(|t-s|/2)$. In fact, continuity lemmas for persistence exponents, which were proven in the years that followed (cf. Lemma 3.3), imply even $\theta_R(H) \to \theta_R(\infty)$ for $H \to \infty$. In [55, Proposition 2.3], it was shown that $\theta_R(\infty) \geq 1/(4\sqrt{3})$, whereas [66] gives evidence in favour of the equality $\theta_R(\infty) = 3/16$.

In Chapter 3, we will be concerned with the behaviour of $\theta_R(H)$ for $H \to 0$: We will show in Theorem 3.2 that θ_R tends to ∞ and that the asymptotic behaviour is in the range H^{-1} to H^{-2}, which is again quite contrary to the behaviour of the persistence exponent $\theta(B^H) = 1 - H$ of FBM.

Fractional Ornstein-Uhlenbeck process. Now, at first sight, we leave our basical setting of self-similar centred Gaussian processes. Consider the stochastic differential equation (SDE)

$$\begin{aligned} X_0 &= \xi, \\ \mathrm{d}X_t &= -\lambda X_t \,\mathrm{d}t + \mathrm{d}W_t, \end{aligned} \tag{1.13}$$

where $\lambda > 0$, W is a Brownian motion and ξ is a random variable independent of $(W_t)_{t\geq 0}$ with $\mathbb{E}[\xi^2] < \infty$. The Ornstein-Uhlenbeck process (OU process) is defined as the unique strong solution U^λ of the SDE (1.13) for the initial condition $\xi := \int_{-\infty}^0 e^{\lambda s}\,\mathrm{d}W_s$, which is given by

$$U_t^\lambda = \int_{-\infty}^t e^{-\lambda(t-s)}\,\mathrm{d}W_s, \qquad t \geq 0. \tag{1.14}$$

Due to the the fact that W is a centred Gaussian process with stationary increments, the process U^λ is a stationary centred Gaussian process, where stationarity means $(U_{t+h}^\lambda)_{t\geq 0} \stackrel{\mathrm{d}}{=} (U_t^\lambda)_{t\geq 0}$ for all $h > 0$. For stationary processes Z, one expects asymptotically exponential decay of the persistence probabilities, i.e.,

$$\mathbb{P}\left(\sup_{t\in[0,T]} Z_t \leq x\right) = e^{-T(\theta_x + o(1))}, \qquad T \to \infty, \tag{1.15}$$

for $x \in \mathbb{R}$, where $\theta_x = \theta_x(Z) \in (0,\infty)$ is also called persistence exponent and, in contrast to the case of self-similar processes, usually depends on the barrier x.

Nevertheless, this has a deep connection to persistence problems of self-similar processes. As U^λ is a stationary centred Gaussian process, its distribution is characterized uniquely by the auto-covariance function, which is given by

$$\begin{aligned}\mathbb{E}[U_0^\lambda U_t^\lambda] = e^{-\lambda t}\,\mathbb{E}\left[\left(\int_{-\infty}^0 e^{\lambda s}\,\mathrm{d}W_s\right)^2\right] &= e^{-\lambda t}\,\mathbb{E}\left[\left(\int_0^\infty e^{-\lambda s}\,\mathrm{d}W_s\right)^2\right] \\ &= e^{-\lambda t}\int_0^\infty e^{-2\lambda s}\,\mathrm{d}s = \frac{1}{2\lambda}\,e^{-\lambda t} \qquad (1.16)\end{aligned}$$

for $t \geq 0$, where we used centredness of W as well as the independence of $(W_t)_{t<0}$ and $(W_t)_{t>0}$ in the first, the identity $(W_{-t}) \overset{\mathrm{d}}{=} (W_t)$ in the second and the Itô isometry in the third step. This is the auto-covariance function of the stationary centred Gaussian process

$$\frac{1}{\sqrt{2\lambda}}\,e^{-\lambda t}\,W_{e^{2\lambda t}}, \qquad t \geq 0, \tag{1.17}$$

which is thus an alternative representation of the OU process U^λ. This is the so-called Lamperti transform of the scaled Brownian motion $(W_{t^{2\lambda}}/\sqrt{2\lambda})$, which is a λ-self-similar process (cf. Chapter 2). In general, the Lamperti transformation provides a bijective way to transform self-similar processes into stationary processes and vice versa.

Using this alternative representation and the powerful Markov tools available for Brownian motion yields the classical persistence result for the OU process in the case of the barrier $x = 0$, which was first deduced in [76], but which can also be found in [69, Section IX] and in [79]. One has

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} U_t^\lambda \leq 0\right) &= \mathbb{P}\left(\sup_{t\in[1,e^{2\lambda T}]} W_t \leq 0\right) \\ &= \int_{-\infty}^0 \mathbb{P}\left(\sup_{t\in[1,e^{2\lambda T}]} W_t - W_1 \leq -x \;\middle|\; W_1 \in \mathrm{d}x\right)\mathbb{P}(W_1 \in \mathrm{d}x) \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathbb{P}\left(\sup_{t\in[0,e^{2\lambda T}-1]} W_t \leq x\right) e^{-x^2/2}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathbb{P}(|W_{e^{2\lambda T}-1}| \leq x)\,e^{-x^2/2}\,\mathrm{d}x \\ &= \frac{1}{2} - \frac{1}{\pi}\arctan\left(\sqrt{e^{2\lambda T}-1}\right) \\ &= \frac{1}{\pi}\arcsin\left(e^{-\lambda T}\right) \sim \frac{1}{\pi}\,e^{-\lambda T}, \qquad T \to \infty, \qquad (1.18)\end{aligned}$$

where we used the Markov property as well as the stationary increments of W in the third, the reflection principle in the fourth, the integral formula [38, eq. 6.285.1] in the fifth and the functional equation $\arctan x = \pi/2 - \arctan\frac{1}{x}$ for $x > 1$ as well as the identity $\arcsin x = \arctan(x/\sqrt{1-x^2})$ for $|x| < 1$ in the sixth step. This yields the persistence exponent $\theta_0(U^\lambda) = \lambda$. For general barriers $x \neq 0$, tools for stationary centred Gaussian processes imply that the persistence exponent $\theta_x(U^\lambda) \in (0,\infty)$ exists (by non-negativity and integrability of the auto-covariance function, see Proposition 2.2) and that the function $x \mapsto \theta_x(U^\lambda)$ is continuous (cf. [35, Lemma 1.1]). Further, clearly, $x \mapsto \theta_x(U^\lambda)$ is non-increasing since the persistence probability is non-decreasing in x. For $x > 0$, one additionally knows that $\theta_x(U^\lambda)$ is given as a solution of some explicit equation and that $\lim_{x\to\infty}\theta_x(U^\lambda) = 0$, see [74], where this is contained as a result for persistence probabilities of W with square root boundaries, due to the representation (1.17).

Due to the two representations (1.14) and (1.17) of the OU process, there are two ways to define fractional analogues of U^λ. For $H \in (0,1)$, let B^H be an FBM with Hurst parameter H. We first consider the fractional generalization of the representation (1.17). As B^H is H-self-similar, the scaled FBM $(B^H_{t^{2\lambda}}/\sqrt{2\lambda})$ is $2\lambda H$-self-similar and its Lamperti transform is given by

$$Z_t^{H,\lambda} := \frac{1}{\sqrt{2\lambda}}\, e^{-2\lambda H t}\, B^H_{e^{2\lambda t}}, \qquad t \geq 0, \tag{1.19}$$

cf. Chapter 2. Note that the persistence probabilities of $Z^{H,\lambda}$ for the barrier $x = 0$ can be written as

$$\mathbb{P}\left(\sup_{t\in[0,T]} Z_t^{H,\lambda} \leq 0\right) = \mathbb{P}\left(\sup_{t\in[1,e^{2\lambda T}]} B_t^H \leq 0\right), \qquad T > 0,$$

and it is well-known that $\mathbb{P}\big(\sup_{t\in[1,T]} B_t^H \leq 0\big)$ and $\mathbb{P}\big(\sup_{t\in[0,T]} B_t^H \leq 1\big)$ have – up to terms of lower order – the same asymptotic behaviour for $T \to \infty$ (cf. Corollary 2.6). Thus, using the persistence result (1.3) for B^H, these persistence probabilities of Z^H behave asymptotically as in (1.15) with persistence exponent $\theta_0(Z^{H,\lambda}) = 2\lambda\,\theta(B^H) = 2\lambda(1-H)$.

In Chapter 5, we will be concerned with the fractional generalization of the representation (1.14), which was introduced in [26]. For $\lambda > 0$, the so-called fractional OU process is given by

$$U_t^{H,\lambda} := \int_{-\infty}^t e^{-\lambda(t-s)}\,\mathrm{d}B_s^H, \qquad t \geq 0,$$

which was shown to be well-defined for all $H \in (0,1)$ as a Riemann-Stieltjes integral, by using that $\int_{-\infty}^t B_s^H e^{\lambda s}\,\mathrm{d}s$ is well-defined (due to the Hölder continuity of the sample paths of B^H and the invariance of B^H under time inversion) and applying an integration by parts formula for Riemann-Stieltjes integrals (cf. [26, Proposition A.1]). Further, it was shown in [26] that this is the unique stationary solution with continuous sample paths of the SDE $\mathrm{d}X_t = -\lambda X_t\,\mathrm{d}t + \mathrm{d}B_t^H$, which is the fractional analogue of the SDE (1.13). Except for the Brownian case $H = 1/2$, the distributions of $U^{H,\lambda}$ and $Z^{H,\lambda}$ are quite different. In [26, Theorem 2.3], for example, it was proven that the auto-covariance function of $U^{H,\lambda}$ decays polynomially for $H \neq 1/2$. On the contrary, the auto-covariance function of $Z^{H,\lambda}$ decays exponentially, as

$$\begin{aligned} 4\lambda\,\mathbb{E}[Z_0^{H,\lambda} Z_t^{H,\lambda}] &= e^{-2\lambda Ht} + e^{2\lambda Ht} - \left(e^{\lambda t} - e^{-\lambda t}\right)^{2H} \\ &= e^{-2\lambda Ht} - \sum_{n=1}^{\infty} (-1)^n \binom{2H}{n} e^{2\lambda t(H-n)} \\ &= e^{-2\lambda Ht} + 2H\, e^{-2\lambda(1-H)t} + o(e^{-2\lambda t}), \qquad t \to \infty, \end{aligned}$$

by the binomial theorem. Also regarding persistence, they behave very differently. We will show in Proposition 5.5 that for $H > 1/2$, it holds $\theta_x(U^{H,\lambda}) = 0$ for every $x \in \mathbb{R}$, i.e., that in this case, the persistence probability of the fractional OU process does not have a true exponential decay as in (1.15).

Figure 1.2 illustrates the results presented in this section.

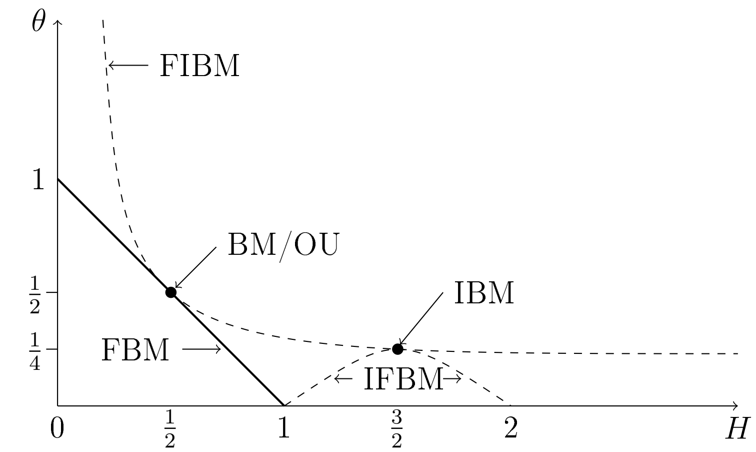

Figure 1.2: Relation of the persistence exponents of FBM, Brownian motion (BM), integrated Brownian motion (IBM), IFBM, FIBM and OU process for $\lambda = 1/2$ (OU). For IFBM with parameter $H \in (0, 1)$, we shift the function by 1 because H-IFBM corresponds to $(H + 1)$-FIBM.

1.4 Related work

In this section, we mention further results for persistence probabilities of fractional processes which do not fit directly into the setting of this thesis.

Two-sided persistence problem. All the processes $X = (X_t)_{t\geq 0}$ considered in the previous section can be extended to two-sided processes $(X_t)_{t\in\mathbb{R}}$. Thus, one can also consider the two-sided persistence probabilities of X given by

$$\mathbb{P}\left(\sup_{t\in[-T,T]} X_t \leq 1\right), \qquad T > 0,$$

and for self-similar processes X, where in the two-sided case, H-self-similarity means $(X_{ct}) \overset{\mathrm{d}}{=} (|c|^H X_t)$ for all $c \in \mathbb{R}$, one expects again

$$\mathbb{P}\left(\sup_{t\in[-T,T]} X_t \leq 1\right) = T^{-\theta'+o(1)}, \qquad T \to \infty,$$

for some persistence exponent $\theta' = \theta'(X) \in (0,\infty)$. Note that this is no substantial new problem for processes X for which $(X_t)_{t\geq 0}$ and $(X_t)_{t<0}$ are independent, as in this case

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[-T,T]} X_t \leq 1\right) &= \mathbb{P}\left(\sup_{t\in[-T,0]} X_t \leq 1\right) \cdot \mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right)\\ &= \mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right)^2,\end{aligned}$$

where we used $(X_{-t}) \overset{\mathrm{d}}{=} (|-1|^H X_t) = (X_t)$. This is fulfilled for Markov processes and integrated Markov processes, where $\mathbb{P}_{X_0}$ is trivial, so that we can infer from the one-sided persistence results that $\theta'(W) = 2\,\theta(W) = 1$ for the Brownian motion and $\theta'(V) = 2\,\theta(V) = 1/2$ for the integrated Brownian motion.

For fractional processes X, however, this is typically not the case and one has to solve the one-sided and two-sided case completely separately. A priori, one only has – in the case that both persistence exponents exist – the estimate $\theta'(X) \geq \theta(X)$, since trivially $\mathbb{P}\big(\sup_{t\in[-T,T]} X_t \leq 1\big) \leq \mathbb{P}\big(\sup_{t\in[0,T]} X_t \leq 1\big)$. For the FBM, it was shown by Molchan in 1999 that $\theta'(B^H) = 1$, independent of H, see [59, Theorem 3]. This is a consequence of the fact that the distribution of the position of the maximum of B^H on a symmetric interval w.r.t. the origin has a finite density. Furthermore, in contrast to the one-sided case, the two-sided persistence exponent could also be determined for the IFBM. The upper bound $\theta'(I^H) \leq 1-H$ for $H \in (0,1)$ is due to a relation of the two-sided persistence probability of IFBM and the inviscid Burgers equation with FBM initial velocity that was established in [58], although up to 2017, the existence of $\theta'(I^H)$ had only been known for $H \geq 1/2$. In 2017, by discretizing the problem and considering the expectation of a useful functional, Molchan could derive the lower bound $\theta'(I^H) \geq 1-H$ for $H \in (0,1)$, see [56], which proved the existence of $\theta'(I^H)$ for $H < 1/2$ and simultaneously showed that $\theta'(I^H)$ equals $1-H$ for all $H \in (0,1)$.

Discrete-time processes. Similarly, one can consider one-sided and two-sided persistence probabilities of fractional discrete-time processes $(X_n)_{n\in\mathbb{Z}}$, given by $\mathbb{P}(\max_{n=0,\dots,N} X_n \leq x)$ and $\mathbb{P}(\max_{n=-N,\dots,N} X_n \leq x)$, respectively, for $N \in \mathbb{N}$ and $x \in \mathbb{R}$. For discrete-time analogues of the processes discussed in the previous section, one expects that the persistence probabilities have the same asymptotic behaviour in N as the persistence probabilities of the corresponding continuous-time process in T, in particular that persistence exponents coincide in the case they exist. By Donsker's theorem, discrete-time analogues of Brownian motion are given by centred random walks with finite variance. By classical results going back to Sparre Andersen and Rogozin, see [80], [81] and [70], one knows that, for $x \geq 0$,

$$\mathbb{P}\left(\max_{n=0,\dots,N} S_n \leq x\right) \sim c_x N^{-1/2}, \qquad N \to \infty,$$

where (S_n) is a centred random walk with finite variance and $c_x > 0$ is a constant dependent on x and the distribution of S_1. Note that due to the Markov property of (S_n), this directly implies a similar result for the two-sided persistence probabilities, where the persistence exponent doubles to 1.

A fractional analogue to Donsker's theorem, the functional central limit theorem for strong dependence and light tails, see e.g. [87, Theorem 4.6.1], provides the discrete-time analogues of FBM. These are given by sums of stationary centred sequences which are either assumed to be Gaussian or to have a representation as a so-called linear process and to fulfill a certain moment condition, and for which the variance of the sum increases asymptotically as cn^{2H} for some $c > 0$, where $H \in (0,1)$ is the Hurst parameter of the corresponding FBM. This in particular includes the trivial discrete-time analogue of FBM, fractional Gaussian noise (B_n^H), but in fact defines a much larger class of discrete-time processes. For this reason, starting with

Molchan's result for the FBM from 1999, it took a long time to deduce a similar result for discrete-time analogues of FBM. In particular, one had to find new proof techniques which do not rely on self-similarity anymore, as this property does not make sense for discrete-time processes. In [11, Theorem 11], it was shown that in the case of sums (S_n^H) of stationary centred *Gaussian* sequences which additionally satisfy $\inf_{n\in\mathbb{N}} \mathbb{E}[S_n^H S_1^H] > 0$, it holds, for $x \geq 0$,

$$c_x^{-1} N^{-(1-H)} (\log N)^{-1/2} \leq \mathbb{P}\left(\max_{n=0,\dots,N} S_n^H \leq x \right) \leq N^{-(1-H)} e^{c_x \sqrt{\log N}} \tag{1.20}$$

for some $c_x > 0$ and N large enough. Note that the additional assumption $\inf_{n\in\mathbb{N}} \mathbb{E}[S_n^H S_1^H] > 0$ is only needed for the upper bound to pass over from the persistence probability for the barrier -1 to the persistence probability for a barrier $x > 0$. In [11], this is done for $x = 1$, but the argument works for any $x > 0$. Further note that exactly this argument (which is a change of measure using functions in the reproducing kernel Hilbert space; cf. Proposition 2.3) adds the factor $e^{c\sqrt{\log N}}$. In the case $H \geq 1/2$, under the stronger assumption of non-negative covariances of the stationary centred Gaussian sequence, one could derive by Slepian's lemma (cf. Proposition 2.1) the optimal upper bound $c_x N^{-(1-H)}$. Later, in [6, Corollary 8], the strong assumption $\inf_{n\in\mathbb{N}} \mathbb{E}[S_n^H S_1^H] > 0$ could be replaced by a rather moderate assumption on the spectral measure of the stationary centred Gaussian sequence to deduce the same result as in (1.20).

In [6], it was further shown that – analogously to the continuous-time case – the two-sided persistence exponent of sums of stationary centred Gaussian sequences, for which the variance of the sum increases asymptotically as cn^{2H} and which satisfy the assumption on the spectral measure, is given by 1, independent of H ([6, Theorem 1]), and that the two-sided persistence exponent of integrals of these sums (which are discrete-time analogues of IFBM) is given by $1 - H$ ([6, Theorem 2]). Other results for discrete-time fractional processes include weighted sums of stationary centred Gaussian sequences (cf. [14]), random walks in random scenery (cf. [11]), random walks in random environment (cf. [10]), and branching processes in correlated random environment (cf. [24] and [10]).

Non-Gaussian fractional processes. Although most of the proof techniques for the results that we discussed previously crucially rely on Gaussianity (cf. Chapter 2), there are also a few results for persistence probabilities of non-Gaussian fractional processes. In [15], persistence probabilties of so-called Hermite processes, which generalize the FBM to non-Gaussian H-self-similar processes with stationary increments, were studied. One expects that these persistence probabilities all decay asymptotically polynomially with persistence exponent $1 - H$, just as for the FBM. This could be shown for the Hermite process of order 2, the so-called Rosenblatt process, by establishing a decorrelation inequality similar to Slepian's inequality (which requires Gaussianity and thus cannot be applied; cf. Proposition 2.1). For the Hermite processes of higher order, upper and lower bounds for persistence probabilities with non-matching barriers were proven.

Furthermore, the proofs for the presented results for discrete-time fractional processes only partially rely on Gaussianity and thus also yield (weaker) results for non-Gaussian fractional processes. In particular, one can deduce the following result for non-Gaussian discrete-time analogues of FBM, which is not published in the corresponding paper: As already mentioned, the non-Gaussian discrete-time analogues for the FBM with Hurst parameter $H \in (0,1)$, given by the functional central limit theorem [87, Theorem 4.6.1], are sums $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$, of stationary centred sequences $(X_i)_{i\in\mathbb{N}}$, where $\mathbb{V}[S_n] \sim cn^{2H}$ for some $c > 0$ and $n \to \infty$, and additionally, (X_i) has a representation as a linear process and fulfills some moment condition. In this context, linear processes are of the form $\sum_{j\in\mathbb{Z}} c_{i-j}\xi_j$, $i \in \mathbb{N}$, where (ξ_j) is the so-called innovation process – a centred i.i.d. sequence with finite variance – and (c_j) is a square-summable sequence of constants. If one further assumes $c_j \geq 0$ for all $j \in \mathbb{Z}$ and $\mathbb{E}|\xi_0|^p < \infty$ for all $p > 2$, the results of [11] imply

$$N^{-(1-H)+o(1)} \leq \mathbb{P}\left(\max_{n=0,\dots,N} S_n \leq 1\right) \leq cN^{-(1-H)}$$

for some $c > 0$ and $N \to \infty$. The upper bound is due to the fact that, by [31, p. 63], the assumption $c_j \geq 0$ for all $j \in \mathbb{Z}$ implies that, for every $i \in \mathbb{N}$, the family $X_1, \dots, X_i$ of random variables is positively associated. Then, one applies [11, Theorem 8] together with [11, Proposition 9]. Note that the condition of [11, Theorem 8] is fulfilled for $B := \mathbb{E}[\sup_{t\in[0,1]} B_t^H]$ due to the fact that the functional central limit theorem holds for (S_n) and that, by the association of $X_1, \dots, X_i$ and [62, Theorem 2], one has $\mathbb{E}[(\max_{n=0,\dots,N} S_n)^2] \leq \mathbb{V}[S_N]$ for $N \in \mathbb{N}$ so that the renormalized maximum is bounded in L^2. The lower bound follows from [11, Theorem 5], together with the fact that, by [73, Theorem 1.4.1], the assumption $\mathbb{E}|\xi_0|^p < \infty$ for all $p > 2$ implies $\mathbb{E}[(S_1)_-^\beta] = \mathbb{E}[(X_1)_-^\beta] \leq \mathbb{E}|X_1|^\beta < \infty$ for all $\beta > 2$. Note that [11, Theorem 5] even demands $\mathbb{E}[(S_1)_-^\beta] < \infty$ for all $\beta > 0$, but the assumption is only used in the proof for large β.

1.5 Outline

In Chapter 2, we will introduce the central tools for dealing with persistence probabilities of self-similar centred Gaussian processes, which we will need in the proofs of the results of Chapters 3 to 5. In particular, in Proposition 2.4, we will deduce a new result for the reproducing kernel Hilbert space (RKHS) of self-similar centred Gaussian processes, which could be of independent interest, and which, together with Slepian's lemma, provides an easy way to derive the equality of the persistence exponents of a self-similar centred Gaussian process and its Lamperti transform. Later, in Chapter 4, this result on the RKHS will be crucial to prove the persistence result for general mixed processes.

In Chapter 3, we will then consider the persistence probabilities of integrated FBM (IFBM) and fractionally integrated Brownian motion (FIBM). In both cases, we will show that the persistence exponent is a continuous function in the respective Hurst parameter. Further, for the IFBM with Hurst parameter H, we will determine the asymptotic behaviour of the persistence exponent at the boundaries 0 and 1 of the domain of H, which is in accordance with the conjecture (1.7) of Molchan and Khokhlov (Theorem 3.1). For the FIBM with Hurst paramter H, we will determine the asymptotic behaviour of the persistence exponent for $H \to 0$ (Theorem 3.2).

Thereafter, in Chapter 4, we will study persistence probabilities of mixed processes, which are given by sums of self-similar centred Gaussian processes with different indices of self-similarity. We will first prove our main result, which states that under the assumptions of non-negative covariances and some further minor conditions, the persistence probability of the sum decays asymptotically polynomially with the persistence exponent of the single process with the greater index of self-similarity (Theorem 4.1). Afterwards, we will use this result to deduce corollaries for the mixed FBM, the mixed IFBM and the mixed FIBM.

Lastly, in Chapter 5, we will consider the persistence probabilities of bifractional Brownian motion (biFBM), which is a generalization of FBM, and of the fractional Ornstein-Uhlenbeck (fOU) process. For the biFBM, we will prove lower and upper bounds for subsets of the parameter domain which unfortunately do not intersect (Proposition 5.1). For the fOU process with Hurst parameter H, we will show that for $H > 1/2$, the persistence probability does not have a true exponential decay, i.e., the persistence exponent equals zero (Proposition 5.5).

Remark. This thesis is mainly based on the articles [12] and [13]; see also the remarks at the end of each chapter.

Chapter 2

Fundamental tools

In this chapter, we present the general tools which will help us in Chapters 3 to 5 to deal with persistence probabilities of self-similar centred Gaussian processes. First, in Section 2.1, we recall Slepian's lemma, which turns an inequality for the covariances of two centred Gaussian random vectors with same variances into an inequality for their persistence probabilities. In Section 2.2, we will then explain how self-similar processes can be transformed via the so-called Lamperti transformation into stationary processes and what this means for their respective persistence probabilities. This is relevant since one consequence of Slepian's lemma is that for stationary centred Gaussian processes with a non-negative covariance function (and some additional condition on the sample paths), the persistence exponent as in (1.15) exists. Subsequently, in Section 2.3, we will introduce the so-called reproducing kernel Hilbert space (RKHS), for which it is well-known that shifts by elements of this function space do not change the persistence exponent in the case it exists. Afterwards, we will show a new result for the RKHS of self-similar centred Gaussian processes, which allows us to state general conditions under which the persistence exponents of a self-similar process and its Lamperti transform are identical, and which will help us in Chapter 4 to prove the main theorem.

2.1 Slepian's lemma

As the name suggests, Slepian's lemma in its original version was proven in [79, Lemma 1] by Slepian, where the probabilities of two centred Gaussian random vectors being non-negative, respectively, were estimated. In the usual, more general form, see e.g. [42, Corollary 3.12], the statement is as follows:

For $n \in \mathbb{N}$, let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be centred Gaussian random vectors in $\mathbb{R}^n$, respectively, such that $\mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2]$ and $\mathbb{E}[X_i X_j] \le \mathbb{E}[Y_i Y_j]$ for all $i, j = 1, \dots, n$. Then, for all $(x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\mathbb{P}(X_i \le x_i \,\forall i = 1, \dots, n) \le \mathbb{P}(Y_i \le x_i \,\forall i = 1, \dots, n). \tag{2.1}$$

By setting $x_i :\equiv x$, one directly concludes that in the discrete-time case, an inequality in the covariance functions of centred Gaussian processes with identical variances implies the same inequality in the persistence probabilities. Due to the continuity of $\mathbb{P}$, the inequality (2.1) also holds for $n = \infty$, i.e., for countable families X and Y of random variables. To be able to go over to continuous-time processes X and Y, the natural additional assumption is separability, since in this case, their suprema are determined by countable subsets of time indices.

Recall that a real-valued stochastic process $(X_t)_{t\in\mathbb{T}}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with $(\mathbb{T}, d)$ being a separable metric space (e.g. $[0, T]$ or $[0, \infty)$ with the Euclidean metric), is called *separable* if there exists a countable subset $D \subseteq \mathbb{T}$ and a set $\Omega_0 \in \mathscr{F}$ of probability $\mathbb{P}(\Omega_0) = 0$ such that, for every $t \in \mathbb{T}$, $\varepsilon > 0$ and $\omega \in \Omega_0^c$,

$$X_t(\omega) \in \overline{\{X_s(\omega) \colon s \in D,\ d(s,t) < \varepsilon\}},$$

where the closure is taken in $\mathbb{R} \cup \{\infty\}$, see e.g. [42, p. 45]. In view of the fact that we want to consider processes $(X_t)_{t\ge 0}$ with a.s. càdlàg paths in Chapter 4, note that a.s. right-continuous sample paths imply separability, by taking e.g. $D := \mathbb{Q} \cap [0, \infty)$.

Then, the continuous-time analogue of (2.1) is given as follows, which is [19, Lemma 1.2.5].

Proposition 2.1. *Let $(\mathbb{T}, d)$ be a separable metric space, $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ be two real-valued separable centred Gaussian processes, and $f \colon \mathbb{T} \to \mathbb{R}$ be a measurable function whose set of discontinuity points is at most countable.*

(a) If $\mathbb{E}[X_t^2] = \mathbb{E}[Y_t^2]$ and $\mathbb{E}[X_s X_t] \le \mathbb{E}[Y_s Y_t]$ for all $s, t \in \mathbb{T}$, then

$$\mathbb{P}(X_t \le f(t) \,\forall t \in \mathbb{T}) \le \mathbb{P}(Y_t \le f(t) \,\forall t \in \mathbb{T}).$$

(b) If $\mathbb{T} = [0, \infty)$ and $\mathbb{E}[X_s X_t] \ge 0$ for all $s, t \ge 0$, then, for all $T_1, T_2 > 0$,

$$\begin{aligned}&\mathbb{P}(X_t \le f(t) \,\forall t \in [0, T_1]) \cdot \mathbb{P}(X_t \le f(t) \,\forall t \in [T_1, T_1 + T_2])\\ &\le \mathbb{P}(X_t \le f(t) \,\forall t \in [0, T_1 + T_2]).\end{aligned}$$

Note that part (b) directly follows from part (a) by considering the process $(\tilde{X}_t)_{t\geq 0}$ given by Kolmogorov's existence theorem, for which $(\tilde{X}_t)_{t<T_1} \stackrel{\mathrm{d}}{=} (X_t)_{t<T_1}$ as well as $(\tilde{X}_t)_{t\geq T_1} \stackrel{\mathrm{d}}{=} (X_t)_{t\geq T_1}$, and for which $(\tilde{X}_t)_{t<T_1}$ and $(\tilde{X}_t)_{t\geq T_1}$ are independent, and taking a separable modification, which is possible by construction and due to the fact that X is separable.

2.2 Lamperti transformation

For $H > 0$, let $X = (X_t)_{t>0}$ be an H-self-similar process, i.e., for all $c > 0$, it holds $(X_{ct}) \stackrel{\mathrm{d}}{=} (c^H X_t)$. It is easy to see that then, the so-called *Lamperti transform* of X, given by

$$Z_\tau := e^{-\tau H} X_{e^\tau}, \qquad \tau \in \mathbb{R},$$

is a stationary process, i.e., $(Z_{\tau+h}) \stackrel{\mathrm{d}}{=} (Z_\tau)$ for all $h \in \mathbb{R}$. Also conversely, if $(Z_\tau)_{\tau\in\mathbb{R}}$ is a stationary process, setting $X_t := t^H Z_{\log t}$, $t > 0$, yields an H-self-similar process so that there is a bijective relation between H-self-similar processes on $(0, \infty)$ and stationary processes on $\mathbb{R}$.

Persistence probabilities of a self-similar process X and its Lamperti transform Z are related as follows. One has, by definition of the Lamperti transform, for the persistence probability of Z with barrier $x \in \mathbb{R}$,

$$\mathbb{P}\left(\sup_{\tau\in[0,T]} Z_\tau \leq x\right) = \mathbb{P}\left(\sup_{t\in[1,e^T]} X_t - x\,t^H \leq 0\right), \qquad T > 0. \tag{2.2}$$

For $x := 0$, the right hand side looks quite similar to the persistence probability of the self-similar X as defined in (1.1), where we explained that changing the (constant) barrier does not affect the asymptotic polynomial order of the persistence probability in the self-similar setup. Thus, if indeed

$$\mathbb{P}\left(\sup_{\tau\in[0,T]} Z_\tau \leq 0\right) = e^{-T(\theta+o(1))}, \qquad T \to \infty, \tag{2.3}$$

for some persistence exponent $\theta \in (0, \infty)$, it seems plausible that also

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) = T^{-\theta+o(1)}, \qquad T \to \infty, \tag{2.4}$$

with the same exponent θ, which we have already seen for fractional Brownian motion and its Lamperti transform. In the following, we want to develop sufficient conditions under which this is true for general self-similar centred Gaussian processes.

We start with a result stating sufficient conditions under which (2.3) holds, which essentially is a consequence of Slepian's lemma.

Proposition 2.2. *Let $(Z_\tau)_{\tau\geq 0}$ be a separable stationary centred Gaussian process with a non-negative auto-covariance function $r(\tau) := \mathbb{E}[Z_0 Z_\tau]$, $\tau \geq 0$. Then, for every $x \in \mathbb{R}$, the persistence exponent*

$$\theta_x(Z) := -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{\tau\in[0,T]} Z_\tau \leq x \right) \in [0,\infty]$$

exists. If, moreover, Z is a.s. right-continuous at 0 and r is not the zero function, then

$$\theta_x(Z) \in [0,\infty) \qquad \text{for all } x \in \mathbb{R}.$$

If, moreover, Z has a.s. càdlàg sample paths and $\int_0^\infty r(\tau)\,\mathrm{d}\tau < \infty$, then

$$\theta_x(Z) \in (0,\infty) \qquad \text{for all } x \in \mathbb{R}.$$

Proof. The first statement (existence in $[0,\infty]$) is a direct consequence from Proposition 2.1 (b), implying due to stationarity that $-\log \mathbb{P}\big(\sup_{\tau\in[0,T]} Z_\tau \leq x\big)$, $T>0$, is subadditive, and Fekete's lemma (see e.g. [82, Lemma 1.2.1]). In particular, Fekete's lemma yields

$$\theta_x(Z) = \inf_{T>0} \frac{-\log \mathbb{P}\big(\sup_{\tau\in[0,T]} Z_\tau \leq x\big)}{T},$$

which is clearly in $[0,\infty]$ and equals ∞ if and only if $\mathbb{P}\big(\sup_{\tau\in[0,T]} Z_\tau \leq x\big) = 0$ for all $T>0$. In this case, continuity of $\mathbb{P}$ implies

$$0 \geq \lim_{T\to 0} \mathbb{P}\left(\sup_{\tau\in[0,T]} Z_\tau < x \right) = \mathbb{P}\left(\bigcup_{T>0,\, T\in\mathbb{Q}} \left\{ \sup_{\tau\in[0,T]} Z_\tau < x \right\} \right),$$

which equals $\mathbb{P}(Z_0 < x)$ if Z is a.s. right-continuous at 0. If r is not the zero function, Z_0 is non-trivially normal distributed so that $\mathbb{P}(Z_0 < x) > 0$, implying the second statement.

The third statement can be proven by basically performing exactly the steps in the proof of [14, Lemma 3.2(a)]. But since the result there assumes a.s. continuous sample paths and the proof is not too long, we include it here.

Fix $M > 1$ and set $s_i := Mi$ for $i \in \mathbb{N}_0$. Then, for $T > M$ and $N := \lfloor \frac{T}{M} \rfloor$, one estimates

$$\mathbb{P}\bigg(\sup_{\tau \in [0,T]} Z_\tau \le x \bigg) \le \mathbb{P}\bigg(\max_{i=0,\dots,N-1} \sup_{\tau \in [s_i, s_i+1]} Z_\tau \le x \bigg) \le \mathbb{P}\bigg(\max_{i=0,\dots,N-1} \int_{s_i}^{s_i+1} Z_\tau \,\mathrm{d}\tau \le x \bigg), \tag{2.5}$$

where the integral in the last probability is well-defined, since it is a.s. an integral of a càdlàg function over a finite interval. Let us set $\zeta^2 := \int_0^1 \int_0^1 r(|\tau_1 - \tau_2|) \,\mathrm{d}\tau_1 \,\mathrm{d}\tau_2$ and $X_i := \zeta^{-1} \int_{s_i}^{s_i+1} Z_\tau \,\mathrm{d}\tau$ for $i \in \mathbb{N}_0$. Then, the right-hand side of (2.5) equals $\mathbb{P}(\max_{i=0,\dots,N-1} X_i \le \zeta^{-1} x)$ and $(X_i)_{i \in \mathbb{N}_0}$ is a stationary centred Gaussian process due to the fact that Z is. Consequently, $(X_0, \dots, X_{N-1})$ is a centred Gaussian vector with covariance matrix

$$B(i,j) = B(0, |i-j|) = \zeta^{-2} \int_0^1 \int_{s_{|i-j|}}^{s_{|i-j|}+1} r(|\tau_1 - \tau_2|) \,\mathrm{d}\tau_1 \,\mathrm{d}\tau_2, \qquad i,j = 0, \dots, N-1.$$

We estimate, by using the non-negativity of r,

$$\max_{i=0,\dots,N-1} \sum_{j \ne i} B(0, |i-j|) \le 2 \sum_{i=1}^{N-1} B(0,i) \le 2\,\zeta^{-2} \int_0^1 \int_M^\infty r(\tau_1 - \tau_2) \,\mathrm{d}\tau_1 \,\mathrm{d}\tau_2 \le 2\,\zeta^{-2} \int_{M-1}^\infty r(\tau) \,\mathrm{d}\tau =: \varepsilon_M,$$

where $\lim_{M \to \infty} \varepsilon_M = 0$, as $\int_0^\infty r(\tau) \,\mathrm{d}\tau < \infty$ by assumption. Thus, the Gershgorin discs (cf. [36]) are centred at $B(0,0) = 1$ and have radius at most ε_M. Since B is real and symmetric, all eigenvalues of B are real and Gershgorin's circle theorem yields that all eigenvalues lie within $[1 - \varepsilon_M, 1 + \varepsilon_M]$. We get

$$\mathbb{P}\bigg(\max_{i=0,\dots,N-1} X_i \le \zeta^{-1} x \bigg) = \int_{(-\infty,\, \zeta^{-1} x)^N} (2\pi)^{-N/2} (\det B)^{-1/2} e^{-\mathbf{z}^T B^{-1} \mathbf{z}/2} \,\mathrm{d}\mathbf{z} \le (2\pi(1 - \varepsilon_M))^{-N/2}\, \mathbb{P}\bigg(\mathcal{N}(0,1) \le \frac{\zeta^{-1} x}{\sqrt{1 - \varepsilon_M}} \bigg)^N =: \beta_M^N,$$

where $\lim_{M \to \infty} \beta_M = \frac{1}{\sqrt{2\pi}} \mathbb{P}(\mathcal{N}(0,1) \le \zeta^{-1} x) < 1$. Thus, there exists $M_0 > 1$ such that $\beta_{M_0} < 1$. Together with (2.5), we get, for $T > M_0$, the estimate $\mathbb{P}\big(\sup_{\tau \in [0,T]} Z_\tau \le x \big) \le \beta_{M_0}^N$. Taking logarithms, dividing by T and letting $T \to \infty$, this yields $-\theta_x(Z) \le \log(\beta_{M_0})/M_0 < 0$ and thus the assertion. □

For a long time, the comparatively strong assumption of non-negative covariances had been the only condition for which the existence of the persistence exponent of general stationary centred Gaussian processes had been known. Recently, in [35], this assumption could be replaced by a rather moderate assumption on the spectral measure of the stationary process. In this thesis, though, the existence result as in Proposition 2.2 is completely adequate for our purposes.

Suppose $(X_t)_{t\geq 0}$ is a separable self-similar centred Gaussian process with a non-negative covariance function. Then, its Lamperti transform $(Z_\tau)_{\tau\in\mathbb{R}}$ is a separable stationary centred Gaussian process with a non-negative auto-covariance function and Proposition 2.2 yields (2.3) for a persistence exponent $\theta \in [0,\infty]$. By (2.2), this implies $\mathbb{P}\big(\sup_{t\in[1,T]} X_t \leq 0\big) = T^{-\theta+o(1)}$ for $T\to\infty$. Now, (2.4) follows, if one is able to show that $\mathbb{P}\big(\sup_{t\in[1,T]} X_t \leq 0\big)$ and $\mathbb{P}\big(\sup_{t\in[0,T]} X_t \leq 1\big)$ have the same asymptotic polynomial order for $T\to\infty$. Let us now explain how this can be done.

The typical approach is to show inequalities in both directions. One direction is a direct consequence of Proposition 2.1(b). Due to non-negative covariances,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) &\geq \mathbb{P}\left(\sup_{t\in[0,1]} X_t \leq 1\right)\cdot\mathbb{P}\left(\sup_{t\in[1,T]} X_t \leq 1\right)\\ &\geq \mathbb{P}\left(\sup_{t\in[0,1]} X_t \leq 1\right)\cdot\mathbb{P}\left(\sup_{t\in[1,T]} X_t \leq 0\right) = T^{-\theta+o(1)}, \qquad T\to\infty.\end{aligned} \tag{2.6}$$

Note that the last equality is trivial for $\theta=\infty$, whereas for $\theta<\infty$, the fact that $\mathbb{P}(\sup_{t\in[x,1]} X_t \leq 0) = \mathbb{P}(\sup_{t\in[1,1/x]} X_t \leq 0) = x^{\theta+o(1)}$ for $x\to 0$ implies that

$$\mathbb{P}\left(\sup_{t\in[0,1]} X_t \leq 1\right) \geq \mathbb{P}\left(\sup_{t\in[0,x]} X_t \leq 1\right)\cdot\mathbb{P}\left(\sup_{t\in[x,1]} X_t \leq 0\right) > 0$$

for some $x>0$.

For the inverse direction, it is useful to see that

$$\mathbb{P}\left(\sup_{t\in[1,T]} X_t \leq 0\right) \geq \mathbb{P}\left(\sup_{t\in[0,T]} X_t + h(t) \leq 1\right) \tag{2.7}$$

holds for any (deterministic) measurable function $h\colon [0,\infty)\to\mathbb{R}$ satisfying $h(t)\geq 1$ for all $t\geq 1$. In view of this estimate, it suffices to show that there exists such a function h for which additionally holds that the persistence probabilities of X and of the shifted process $X+h$ have the same asymptotic polynomial order. As we will see in the next section, this is the case, if X has a.s. càdlàg sample paths and if there exists such h in the so-called reproducing kernel Hilbert space of X.

2.3 Reproducing kernel Hilbert space

Recall that for a non-empty set $\mathbb{T}$ and a kernel $K\colon \mathbb{T}\times\mathbb{T}\to\mathbb{R}$, a Hilbert space $\mathcal{H}\subseteq\{f\colon\mathbb{T}\to\mathbb{R}\}$ with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is called *reproducing kernel Hilbert space* (RKHS) with reproducing kernel K, if

$$k_t := K(\cdot,t)\in\mathcal{H} \qquad \text{for all } t\in\mathbb{T}, \text{ and} \tag{2.8}$$

$$h(t) = \langle h, k_t\rangle_{\mathcal{H}} \qquad \text{for all } t\in\mathbb{T} \text{ and all } h\in\mathcal{H}. \tag{2.9}$$

Further recall that such an RKHS exists if and only if K is symmetric and positive definite and the RKHS is unique in this case, see e.g. [71, Theorem 2.2]. Thus, there is a bijective relation between centred Gaussian processes and RKHSs, by considering covariance functions as reproducing kernels.

For a centred Gaussian process $X=(X_t)_{t\in\mathbb{T}}$, let us denote by $\mathcal{H}_X$ the RKHS of X, i.e., the RKHS with reproducing kernel $K(s,t):=\mathbb{E}[X_sX_t]$, $s,t\in\mathbb{T}$. It is easy to see that

$$\mathcal{H}_X = \left\{ t\mapsto \mathbb{E}[\xi X_t] \;\middle|\; \xi\in\mathbb{H}_X := \overline{\operatorname{span}\{X_t\colon t\in\mathbb{T}\}} \right\}, \tag{2.10}$$

where the closure is in L^2. This is due to the fact that $\xi\in\operatorname{span}\{X_t\colon t\in\mathbb{T}\}$ is mapped linearly to $h:=\mathbb{E}[\xi X_\cdot]\in\operatorname{span}\{k_t\colon t\in\mathbb{T}\}$ with $\langle h_1,h_2\rangle_{\mathcal{H}_X}=\langle\xi_1,\xi_2\rangle_{L^2}$, that $\operatorname{span}\{X_t\colon t\in\mathbb{T}\}$ is dense in $\mathbb{H}_X$, and that $\operatorname{span}\{k_t\colon t\in\mathbb{T}\}$ is dense in $\mathcal{H}_X$. Note here that property (2.8) ensures that $\operatorname{span}\{k_t\colon t\in\mathbb{T}\}\subseteq\mathcal{H}_X$, whereas property (2.9) implies that $0\in\mathcal{H}_X$ is the only element orthogonal to $\operatorname{span}\{k_t\colon t\in\mathbb{T}\}$, so that this is indeed a dense subspace of $\mathcal{H}_X$.

As already mentioned, the importance of the RKHS in the context of persistence is that, if some condition on the sample paths of X is satisfied, shifts by functions in the RKHS of X do not change the asymptotic polynomial order of the persistence probability of X. This is a consequence of the fact that functions in the RKHS of X are so-called admissible shifts, i.e., functions h for which the distribution of the shifted process $X+h$ is absolutely continuous w.r.t. the distribution of the process X. Let us recall this result, which is the so-called Cameron-Martin formula, see e.g. [44, Theorems 9.3/9.4].

Let X be a centred Gaussian process taking values in some topological space E such that the distribution of X on $\mathscr{E}$ is a Radon measure, where $\mathscr{E}$ is the Borel-σ-algebra of E, i.e.,

$$\mathbb{P}(X\in A) = \sup\{\mathbb{P}(X\in B)\colon B\subseteq A,\, B\in\mathscr{E},\, B \text{ compact}\} \tag{2.11}$$

for all $A \in \mathscr{E}$. Then, for all $A \in \mathscr{E}$ and $h := \mathbb{E}[\xi X.] \in \mathcal{H}_X =: \mathcal{H}$, where $\xi \in \mathbb{H}_X$, it holds

$$\mathbb{P}(X + h \in A) = \mathbb{E}\Big[e^{\xi - \|h\|_{\mathcal{H}}^2/2}\, \mathbb{1}_{X \in A}\Big]. \tag{2.12}$$

Note that this result is quite elementary in the finite-dimensional case: For $n \in \mathbb{N}$, let $E := \mathbb{R}^n$ with the Euclidean topology and $X \sim \mathcal{N}(0, K)$, where $K \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite matrix. Then, we have

$$\mathcal{H}_X \subseteq \mathbb{R}^n = \{K\lambda \colon \lambda \in \mathbb{R}^n\} = \operatorname{span}\{k_t \colon t = 1, \dots, n\} \subseteq \mathcal{H}_X,$$

i.e., all shifts are admissible, and $\mathbb{H}_X = \{\lambda^T X \colon \lambda \in \mathbb{R}^n\}$. One easily deduces that, for $h = K\lambda \in \mathbb{R}^n$, $\xi = \lambda^T X$ and $A \in \mathscr{B}(\mathbb{R}^n)$, it indeed holds

$$\begin{aligned}
\sqrt{(2\pi)^n \det K}\; \mathbb{P}(X + h \in A) &= \int_{A-h} e^{-\mathbf{z}^T K^{-1} \mathbf{z}/2}\, \mathrm{d}\mathbf{z} \\
&= \int_A e^{-(\mathbf{z}-h)^T K^{-1} (\mathbf{z}-h)/2}\, \mathrm{d}\mathbf{z} \\
&= \int_A e^{-\mathbf{z}^T K^{-1} \mathbf{z}/2 + \lambda^T \mathbf{z} - \lambda^T K \lambda/2}\, \mathrm{d}\mathbf{z} \\
&= \sqrt{(2\pi)^n \det K}\; \mathbb{E}\Big[e^{\xi - \|h\|_{\mathcal{H}}^2/2}\, \mathbb{1}_{X \in A}\Big],
\end{aligned}$$

where we used that $\|h\|_{\mathcal{H}}^2 = \|\xi\|_{L^2}^2 = \mathbb{E}[(\lambda^T X)^2] = \lambda^T K \lambda$.

In the infinite-dimensional case, however, this is highly non-trivial and we refer to [44] for the proof. By applying Hölder's inequality and the reverse Hölder inequality, respectively, for arbitrary $p > 1$ to the right-hand side of (2.12) and then optimizing in p, the Cameron-Martin formula implies the following upper and lower bounds for the quotient $\mathbb{P}(X + h \in A)/\mathbb{P}(X \in A)$, which is [9, Proposition 1.6].

Proposition 2.3. *Let X be a centred Gaussian process taking values in some topological space E such that the distribution of X on $\mathscr{E}$ is a Radon measure, where $\mathscr{E}$ is the Borel-σ-algebra of E. Then, for all $A \in \mathscr{E}$ with $\mathbb{P}(X \in A) \in (0,1)$ and all $h \in \mathcal{H}_X =: \mathcal{H}$, it holds*

$$\frac{\mathbb{P}(X + h \in A)}{\mathbb{P}(X \in A)} \geq e^{-\sqrt{2\|h\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X \in A))} - \|h\|_{\mathcal{H}}^2/2}.$$

If additionally $\|h\|_{\mathcal{H}}^2 < 2\log(1/\mathbb{P}(X \in A))$, then also

$$\frac{\mathbb{P}(X + h \in A)}{\mathbb{P}(X \in A)} \leq e^{\sqrt{2\|h\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X \in A))} - \|h\|_{\mathcal{H}}^2/2}.$$

Remark. (i) The assumption (2.11) is always satisfied if E is a separable complete metric space, since in this case, every probability measure on $\mathscr{E}$ is a Radon measure, see [20, Theorems 1.1/1.4].

(ii) In [9, Proposition 1.6], it is assumed that E is a Banach space, but this is used in the proof only to be able to apply the Cameron-Martin formula. As we want to consider processes $(X_t)_{t\geq 0}$ with a.s. càdlàg sample paths and thus take $E := D[0,\infty)$ with the Skorokhod topology, we need the more general assumptions made in [44, Theorem 9.3].

(iii) In [9, Proposition 1.6], the upper bound is stated without the additional assumption. However, it is proven by applying Hölder's inequality for $p = \left(1 - \sqrt{\|h\|_{\mathcal{H}}^2/(2\log(1/\mathbb{P}(X \in A)))}\right)^{-1}$, which is not well-defined without the additional assumption, see also [6, Proposition 3]. Nevertheless, note that in the context of persistence, the typical application is to consider $A := A_T \in \mathscr{E}$ such that $\lim_{T\to\infty} \mathbb{P}(X \in A_T) = 0$, and that in this case, the condition is always fulfilled for T large enough.

Before we present the mentioned result for the RKHS of self-similar centred Gaussian processes, let us consider the RKHS $\mathcal{H}_W$ of a Brownian motion $W = (W_t)_{t\geq 0}$, which is given by

$$\mathcal{H}_W = \left\{ h\colon [0,\infty) \to \mathbb{R} \;\middle|\; h(0) = 0,\ h \text{ is differentiable a.e.}, \int_0^\infty (h'(t))^2 \,\mathrm{d}t < \infty \right\} \tag{2.13}$$

with inner product $\langle h_1, h_2\rangle_{\mathcal{H}_W} = \int_0^\infty h_1'(t)h_2'(t)\,\mathrm{d}t$. This is due to the fact that in this case, we have $k_t(s) = s \wedge t$ for $s,t \geq 0$. Note that $\operatorname{span}\{k_t\colon t \geq 0\}$ is a subspace of the right-hand side of (2.13), and that by property (2.9) of $\mathcal{H}_W$, it holds

$$\langle k_s, k_t\rangle_{\mathcal{H}_W} = s \wedge t = \int_0^{s\wedge t} 1\,\mathrm{d}u = \int_0^\infty (s\wedge u)'(t \wedge u)'\,\mathrm{d}u = \int_0^\infty k_s'(u)k_t'(u)\,\mathrm{d}u$$

for $s,t \geq 0$. Together with the fact that the derivatives of elements of $\operatorname{span}\{k_t\colon t \geq 0\}$ – piecewise constant functions with compact support – are dense in the space of continuous functions with compact support, which again are dense in $L^2([0,\infty))$, this implies the identity (2.13).

Thus, there are functions in the RKHS of W growing faster than t^γ for $t \to \infty$ if and only if $\gamma < 1/2$. Recall here that we have already seen in Section 1.3 that the persistence exponent of the OU process depends on the barrier x, which in view of (2.2) means that $\mathcal{H}_W$ cannot contain any function growing as $t^{1/2}$. It seems plausible that for general H-self-similar centred Gaussian processes, this transfers in the sense that one can take $\gamma < H$. Still, this has only been known for specific processes, such as fractional Brownian motion (cf. [18, Section 4] for $H > 1/2$

and [65, Section 6] for general $H \in (0,1)$), integrated fractional Brownian motion (by using the result for fractional Brownian motion and representation (2.10)) or fractionally integrated Brownian motion (by using representation (2.10) and the seen result (2.13) for Brownian motion).

In the following, we will show that for general H-self-similar centred Gaussian processes fulfilling some additional assumptions, there exists a function h in the RKHS growing faster than t^γ for any $\gamma < H$, which will be crucial in the proof of Theorem 4.1. Secondary, we will show that this function can be chosen in a way such that $h(t) \geq 1$ for $t \geq 1$. Thus, it will also be suitable for the estimate (2.7) to deduce the equality of the persistence exponents of the self-similar process and its Lamperti transform.

This will be done by going over to the Lamperti transform and using a representation of the RKHS of Gaussian stationary processes (GSPs) with a continuous auto-covariance function via the spectral measure. Recall that for a GSP $Z = (Z_\tau)_{\tau\in\mathbb{R}}$ with a continuous auto-covariance function $r(\tau) := \operatorname{cov}(Z_0, Z_\tau)$, $\tau \in \mathbb{R}$, Bochner's theorem provides a unique finite measure μ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, satisfying

$$r(\tau) = \int_{\mathbb{R}} e^{i\tau x}\,\mathrm{d}\mu(x), \qquad \tau \in \mathbb{R}, \tag{2.14}$$

which is called *spectral measure* of Z, see e.g. [4, Theorem 1.2.7]. The RKHS of Z can then be written in the form

$$\mathcal{H}_Z = \left\{ \tau \mapsto \int_{\mathbb{R}} \varphi(x) e^{-i\tau x}\,\mathrm{d}\mu(x) \;\middle|\; \varphi \in L^2(\mu) \right\}. \tag{2.15}$$

This is due to the fact that, by (2.14), it holds $\int_{\mathbb{R}} e^{itx}e^{-i\tau x}\,\mathrm{d}\mu(x) = k_t(\tau)$ for $t, \tau \in \mathbb{R}$. Thus, $\varphi \in \operatorname{span}\{e^{it\,\cdot} \colon t \in \mathbb{R}\}$ is mapped linearly to $h := \int_{\mathbb{R}} \varphi(x) e^{-ix\,\cdot}\,\mathrm{d}\mu(x) \in \operatorname{span}\{k_t \colon t \in \mathbb{R}\}$ with $\langle h_1, h_2\rangle_{\mathcal{H}_Z} = \langle \varphi_1, \varphi_2\rangle_{L^2(\mu)}$. Further, since μ is finite, the subspace $\operatorname{span}\{e^{it\,\cdot} \colon t \in \mathbb{R}\}$ is dense in $L^2(\mu)$.

Using this representation, we first get the following result for the RKHS of Z.

Proposition 2.4. *Let $Z = (Z_\tau)_{\tau\in\mathbb{R}}$ be a real-valued GSP with a continuous auto-covariance function. If the spectral measure of Z is abolutely continuous w.r.t. the Lebesgue measure in some neighbourhood of the origin and the corresponding spectral density is bounded away from zero, then for every $\alpha \in (0, 1/2)$, there exists $h \in \mathcal{H}_Z$ satisfying $h(\tau) \sim c\,\tau^{\alpha-1}$ for $\tau \to \infty$ and some $c > 0$ as well as $h(\tau) > 0$ for $\tau \geq 0$.*

Remark. Note that, by (2.15), if the spectral density p exists on whole $\mathbb{R}$, any function $h \in \mathcal{H}_Z$ in the RKHS of Z is the Fourier transform of a function of the form $\varphi \cdot p$, where $\varphi \in L^2(\mu)$. In view of Tauberian theorems for Fourier transforms, see e.g. [21, Theorem 4.10.3], one would expect that $h(\tau) \sim c\,\tau^{\alpha-1}$ holds for $\tau \to \infty$ and some $\alpha \in (0,1)$ if and only if $\varphi(x)p(x) \sim c'\,x^{-\alpha}$ for $x \to 0$, which is in accordance with the condition $\alpha \in (0,1/2)$.

Proof. Let $\alpha \in (0,1/2)$. Similary to the proof of [6, Proposition 5], we will first construct a function $h_1 \in \mathcal{H}_Z$ with the desired asymptotic behaviour, which unfortunately may attain non-positive values up to some $\tau_0 > 0$. Afterwards, we will show the existence of another function $h_2 \in \mathcal{H}_Z$ which is non-negative on $[0,\infty)$, even positive on $[0,\tau_0]$, and decays faster than h_1. Then, the function $h := h_1 + 2\max_{\tau\in[0,\tau_0]} |h_1(\tau)|/\min_{\tau\in[0,\tau_0]} h_2(\tau) \cdot h_2$ yields the assertion.

By assumption, there exists a spectral density $p\colon (-x_0,x_0) \to [c_0,\infty)$ of $\mu|_{(-x_0,x_0)}$ for some $c_0, x_0 > 0$, where μ denotes the spectral measure of Z.

Construction of h_1: We set $\varphi_1(x) := \mathbb{1}_{|x|<x_0}\,|x|^{-\alpha}/p(x)$. Then, we have

$$\int_{\mathbb{R}} \varphi_1^2(x)\,\mathrm{d}\mu(x) = 2\int_0^{x_0} x^{-2\alpha}/p(x)\,\mathrm{d}x \le 2\,c_0^{-1}\int_0^{x_0} x^{-2\alpha}\,\mathrm{d}x < \infty$$

as $\alpha < 1/2$. Thus $\varphi_1 \in L^2(\mu)$. For the h_1 corresponding to φ_1 (as in (2.15)), we get

$$\begin{aligned} h_1(\tau) &= \int_{\mathbb{R}} \varphi_1(x)e^{-i\tau x}\,\mathrm{d}\mu(x) = \int_{\mathbb{R}} \varphi_1(x)\cos(\tau x)\,\mathrm{d}\mu(x) \\ &= 2\int_0^{x_0} x^{-\alpha}\cos(\tau x)\,\mathrm{d}x = 2\tau^{\alpha-1}\int_0^{\tau x_0} y^{-\alpha}\cos(y)\,\mathrm{d}y \sim c\,\tau^{\alpha-1} \end{aligned}$$

for $c := 2\int_0^\infty y^{-\alpha}\cos(y)\,\mathrm{d}y$ and $\tau \to \infty$. Note that due to $\alpha < 1$, the fact that $\cdot^{-\alpha}$ is decreasing and fulfills $\lim_{x\to\infty} x^{-\alpha} = 0$, the fact that the integrals of $\cos(\cdot)$ over any interval are uniformly bounded, and Dirichlet's test, the integral in the definition of c exists and is positive. Further note that this fails for $\alpha \le 0$.

Construction of h_2: Due to the asymptotic behaviour of h_1, there exists $\tau_0 > \pi/x_0$ such that $h_1(\tau) > 0$ for $\tau \ge \tau_0$. Let $g\colon \mathbb{R} \to [0,\infty)$ be a smooth even function with $g(x) > 0$ for $|x| < \pi/(2\tau_0)$ and $g(x) = 0$ otherwise.

We set $f := g * g$ and $\varphi_2(x) := \mathbb{1}_{|x|<\pi/\tau_0}\,f(x)/p(x)$. Then $\varphi_2 \in L^2(\mu)$ as

$$\int_{\mathbb{R}} \varphi_2^2(x)\,\mathrm{d}\mu(x) = \int_{-\pi/\tau_0}^{\pi/\tau_0} f^2(x)/p(x)\,\mathrm{d}x \le \frac{2\pi\max_{x\in[-\pi/\tau_0,\pi/\tau_0]} f^2(x)}{c_0\tau_0} < \infty,$$

where we used that $\pi/\tau_0 < x_0$. Note that by definition of f and g, we have $f(x) = 0$

for $|x| \geq \pi/\tau_0$. Thus, the h_2 corresponding to φ_2 fulfills

$$\begin{aligned} h_2(\tau) &= \int_{\mathbb{R}} \varphi_2(x) e^{-i\tau x}\, \mathrm{d}\mu(x) = \int_{\mathbb{R}} f(x) e^{-i\tau x}\, \mathrm{d}x \\ &= \Big(\int_{\mathbb{R}} g(x) e^{-i\tau x}\, \mathrm{d}x \Big)^2 = \Big(\int_{\mathbb{R}} g(x) \cos(\tau x)\, \mathrm{d}x \Big)^2 \begin{cases} > 0, & \text{if } |\tau| \leq \tau_0, \\ \geq 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where we used in the second line that the Fourier transform of a convolution is given by the product of the Fourier transforms of the convoluted functions as well as that g vanishes outside of $(-\pi/(2\tau_0), \pi/(2\tau_0))$ by definition. Furthermore, by integration by parts, we have

$$\begin{aligned} h_2(\tau) = \int_{\mathbb{R}} f(x) e^{-i\tau x}\, \mathrm{d}x &= \frac{1}{(i\tau)^2} \int_{\mathbb{R}} f''(x) e^{-i\tau x}\, \mathrm{d}x \\ &\leq \frac{2\pi \max_{x \in [-\pi/\tau_0, \pi/\tau_0]} |f''(x)|}{\tau_0} \cdot \tau^{-2}. \end{aligned}$$ □

Applying this to the Lamperti transform of a self-similar process gives the following corollary.

Corollary 2.5. *For $H > 0$, let $X = (X_t)_{t \geq 0}$ be an H-self-similar process such that the Lamperti transform of X satisfies the conditions of Proposition 2.4. Then, for every $\alpha \in (0, 1/2)$, there exists $h \in \mathcal{H}_X$ satisfying $h(t) \sim c\, t^H (\log t)^{\alpha - 1}$ for $t \to \infty$ and some $c > 0$ as well as $h(t) \geq 1$ for $t \geq 1$.*

Proof. Let $\alpha \in (0, 1/2)$ and $Z_\tau := e^{-\tau H} X_{e^\tau},\ \tau \in \mathbb{R}$, be the Lamperti transform of X. Proposition 2.4 yields the existence of a function $\tilde{h} \in \mathcal{H}_Z$ and $c_0 > 0$ such that $\tilde{h}(\tau) \sim c_0\, \tau^{\alpha - 1}$ for $\tau \to \infty$ and $\tilde{h}(\tau) > 0$ for all $\tau \geq 0$.

By representation (2.10), this implies that there exists a random variable $\xi \in \mathbb{H}_Z$ such that $\tilde{h}(\tau) = \mathbb{E}[\xi Z_\tau]$, $\tau \in \mathbb{R}$. Plugging in the definition of Z, this gives $e^{\tau H} \tilde{h}(\tau) = \mathbb{E}[\xi X_{e^\tau}]$ for $\tau \in \mathbb{R}$ and $h_0(t) := t^H \tilde{h}(\log t) = \mathbb{E}[\xi X_t]$ for $t > 0$. Since $\mathrm{span}\{Z_\tau \colon \tau \in \mathbb{R}\} = \mathrm{span}\{X_t \colon t > 0\}$ and thus $\mathbb{H}_Z = \mathbb{H}_X$, we get $h_0 \in \mathcal{H}_X$, by using again (2.10). Further, h_0 is a continuous function (by the continuity of the covariance function) and satisfies $h_0(t) \sim c_0\, t^H (\log t)^{\alpha - 1}$ for $t \to \infty$ as well as $h_0(t) > 0$ for all $t \geq 1$. In particular, we have $h_0(t) \to \infty$ for $t \to \infty$. Thus, there exists $t_0 > 1$ such that $h_0(t) \geq 1$ for $t \geq t_0$. Setting $h := h_0 / (\min_{t \in [1, t_0]} h_0(t) \wedge 1)$ yields the assertion for $c := c_0 / (\min_{t \in [1, t_0]} h_0(t) \wedge 1)$. □

Finally, we deduce the result giving sufficient conditions under which the persistence exponents of X and Z coincide.

Corollary 2.6. *For $H > 0$, let $(X_t)_{t\geq 0}$ be an H-self-similar centred Gaussian process with a.s. càdlàg sample paths and $Z_\tau := e^{-\tau H} X_{e^\tau}$, $\tau \in \mathbb{R}$, be its Lamperti transform. Let us further assume that the auto-covariance function of Z is continuous, non-negative, integrable and not the zero function. Then, it holds*

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t \leq 1\right) = T^{-\theta+o(1)}, \qquad T \to \infty,$$

where $\theta := -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\big(\sup_{\tau\in[0,T]} Z_\tau \leq 0\big) \in (0,\infty)$.

Proof. The existence of $\theta = \theta_0(Z) \in (0,\infty)$ follows from Proposition 2.2. Further, we have already seen in (2.6) that the non-negativity of the covariance function implies $\mathbb{P}\big(\sup_{t\in[0,T]} X_t \leq 1\big) \geq T^{-\theta+o(1)}$.

For the upper bound, we want to apply Proposition 2.3 for the process X and the function $h \in \mathcal{H}_X$ given by Corollary 2.5. Note that, since $r(\tau) := \mathbb{E}[Z_0 Z_\tau]$, $\tau \in \mathbb{R}$, is assumed to be integrable and – by definition of the spectral measure μ of Z (cf. (2.14)) – represents the characteristic function of the finite measure μ, the inversion theorem for characteristic functions yields $\mathrm{d}\mu(x) = p(x)\,\mathrm{d}x$ on $\mathbb{R}$ with the density

$$p(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau x} r(\tau)\,\mathrm{d}\tau \to \frac{1}{2\pi}\int_{\mathbb{R}} r(\tau)\,\mathrm{d}\tau \in (0,\infty), \qquad x \to 0,$$

so that the conditions of Corollary 2.5 are fulfilled.

By assumption, X takes values in $D[0,\infty)$, the space of càdlàg functions on $[0,\infty)$. In [86, Theorem 2.6], it is shown that $D[0,\infty)$ together with the Skorokhod topology is metrizable as a separable complete metric space so that, as already mentioned in the remark after Proposition 2.3, every probability measure on this space is a Radon measure. In particular, the distribution of X on the Borel-σ-algebra w.r.t. the Skorokhod topology fulfills the assumption (2.11) and Proposition 2.3 is applicable.

We take $A := A_T := \{f \in D[0,\infty) \colon \sup_{t\in[0,T)} f(t) \leq 1\}$, which is closed w.r.t. the Skorokhod topology and thus element of the Borel-σ-algebra. Indeed, if $(f_n) \subseteq A_T$ converges to $f_0 \in D[0,\infty)$, then the Skorokhod convergence implies pointwise convergence for all continuity points of f_0. Consequently, it holds $f_0(t) \leq 1$ for all continuity points $t \in [0,T)$. By choice of the interval $[0,T)$ and due to right-continuity of f_0, this already implies $f_0(t) \leq 1$ for all $t \in [0,T)$ and thus $f_0 \in A_T$.

Hence, Proposition 2.3 yields

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} X_t \le 1\right) &\le \mathbb{P}(X \in A_T)\\
&\le \mathbb{P}(X+h \in A_T)\, e^{\sqrt{2\|h\|_{\mathcal{H}}^2 \log(1/\mathbb{P}(X\in A_T))}+\|h\|_{\mathcal{H}}^2/2}\\
&\le \mathbb{P}(X+h \in A_T)\, e^{\sqrt{2\|h\|_{\mathcal{H}}^2 (\theta+o(1))\log T}+\|h\|_{\mathcal{H}}^2/2}\\
&\le \mathbb{P}\left(\sup_{t\in[1,T]} X_t \le 0\right) e^{(1+o(1))\sqrt{2\|h\|_{\mathcal{H}}^2\theta \log T}} = T^{-\theta+o(1)}, \qquad T\to\infty,
\end{aligned}$$

where we used that $\sup_{t\in[0,T]} X_t \le 1$ implies $\sup_{t\in[0,T)} X_t \le 1$ in the first, the lower bound of Proposition 2.3 in the second, (2.6) in the third and (2.7) in the fourth step. Recall that here we use the property $h(t) \ge 1$ for $t \ge 1$, in the way that in this case, $X_t + h(t) \le 1$ for $t \in [0,T)$ implies $X_t \le 1 - h(t) \le 0$ for $t \in [1,T)$. To deduce the fourth step, we further use that $\sup_{t\in[1,T)} X_t = \sup_{t\in[1,T]} X_t$ holds a.s., since X is even a.s. continuous at any fixed deterministic time. □

Remark. Proposition 2.4 and Corollary 2.5 appeared in the journal *Journal of Physics A: Mathematical and Theoretical* in the article *Persistence probabilities of mixed FBM and other mixed processes*, see [13, Lemma 7 / Corollary 8].

Chapter 3

Persistence probabilities of integrated fractional Brownian motion and fractionally integrated Brownian motion

In this chapter, we are concerned with the persistence probabilities of two related processes – the integrated fractional Brownian motion (IFBM) and the fractionally integrated Brownian motion (FIBM) – and determine the asymptotic behaviour of the persistence exponents at the boundaries of their respective domain of the Hurst parameter.

3.1 Introduction and main results

Recall that for $H \in (0,1)$, the IFBM $I^H = (I_t^H)_{t\geq 0}$ is given by

$$I_t^H := \int_0^t B_s^H \,\mathrm{d}s, \qquad t \geq 0,$$

where $(B_t^H)_{t\geq 0}$ is an FBM with Hurst parameter H. Note that the H-self-similarity of B^H implies that I^H is $(H+1)$-self-similar. As we will see, the auto-covariance function of the Lamperti transform of I^H is continuous, non-negative and integrable so that Corollary 2.6 implies the existence of the persistence exponent

$$\theta_I(H) := -\lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\left(\sup_{t\in[0,T]} I_t^H \leq 1 \right) \in (0,\infty).$$

However, as already mentioned in Section 1.3, its value is unknown unless $H = 1/2$. Recall that $I^{1/2}$ is integrated Brownian motion, where it was shown via Markov techniques that $\theta_I(1/2) = 1/4$ (cf. (1.6)). Further recall that for the general case $H \in (0,1)$, there are some analytical estimates (see (1.9), (1.10) and Figure 1.1) and the conjecture (1.7).

We show that θ_I is continuous and determine the asymptotic behaviour of $\theta_I(H)$ for $H \to 0$ and $H \to 1$, which is in accordance with the conjecture (1.7). This is our first main result in this chapter.

Theorem 3.1. *The function $H \mapsto \theta_I(H)$ is continuous on $(0,1)$. Further, $\theta_I(H) \sim H$ as $H \to 0$ and $\theta_I(H) \sim 1 - H$ as $H \to 1$.*

For the second result, recall that for $H > 0$, the FIBM $R^H = (R_t^H)_{t\geq 0}$, also known as Riemann-Liouville process, is given by

$$R_t^H := \int_0^t (t-s)^{H-\frac{1}{2}}\, \mathrm{d}W_s, \qquad t \geq 0,$$

where $W = (W_t)_{t\geq 0}$ is a Brownian motion. Note that R^H is H-self-similar due to the fact that W is 1/2-self-similar. Again, we will see that the auto-covariance function of the Lamperti transform of R^H is continuous, non-negative and integrable so that, by Corollary 2.6, also the persistence exponent

$$\theta_R(H) := -\lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\left(\sup_{t\in[0,T]} R_t^H \leq 1 \right) \in (0,\infty)$$

exists. However, similarly to the IFBM, its value is unknown except for the Brownian cases. Recall that a Fubini argument shows that, for $n \in \mathbb{N}_0$, the FIBM $R^{n+1/2}$ is the n-times integrated version of the Brownian motion W (cf. (1.11) and (1.12)). Thus, it holds $\theta_R(1/2) = 1/2$ (Brownian motion) and $\theta_R(3/2) = 1/4$ (integrated Brownian motion). Further recall that θ_R is non-increasing on $[1/2,\infty)$ (cf. [9]).

We show that θ_R is continuous and that, for $H \to 0$, the exponent $\theta_R(H)$ tends to infinity and is in the range H^{-1} to H^{-2}. This is our second main result.

Theorem 3.2. *The function $H \mapsto \theta_R(H)$ is continuous on $(0,\infty)$. Further,* *(a)* $\liminf_{H\to 0} \theta_R(H)H > 0$ *and*

(b) $\theta_R(H)H^2 \leq 14^2$ *for* $H \in (0, 1/2)$.

For $H \to \infty$, we will see that the auto-covariance function of the Lamperti transform of R^H converges to the auto-covariance function $\tau \mapsto 1/\cosh(\tau/2)$ (see (3.12) below), which is non-negative and integrable. Further, it was shown in [30, eq. (1.4)] that the corresponding stationary centred Gaussian process Z has a representation as an Itô integral w.r.t. Brownian motion so that there exists a modification with continuous sample paths. Thus, by Proposition 2.2, the corresponding stationary centred Gaussian process Z has a persistence exponent $\theta_R(\infty) := \theta_0(Z) \in (0, \infty)$. Now, it is an easy consequence of continuity theorems for persistence exponents (see [28, Theorem 1.6], [29, Lemma 3.1], or [14, Lemma 3.6]; these results are summarized in Lemma 3.3 below in a way suitable for our purposes) that $\theta_R(H) \to \theta_R(\infty)$ as $H \to \infty$. Moreover, recall that one knows that $\theta_R(\infty) \geq 1/(4\sqrt{3})$ (cf. [55]) and that there is evidence in favour of the equality $\theta_R(\infty) = 3/16$ (cf. [66]).

The rest of this chapter is organized as follows. We first sketch the general proof technique in the next subsection. Section 3.2 then contains the proofs related to Theorem 3.1, while Section 3.3 is devoted to the proofs related to Theorem 3.2.

Ideas of the proofs

The first step in our proofs is to go over to Gaussian stationary processes (GSPs) by considering the Lamperti transforms of the self-similar processes I^H and R^H. Consequently, we consider the Lamperti transform of I^H defined by

$$U_\tau^H := \sqrt{2(1+H)}\, e^{-(1+H)\tau}\, I_{e^\tau}^H, \qquad \tau \in \mathbb{R},$$

where the normalization constant is given in order to have a unit variance process. Similarly, we consider the normalized Lamperti transform of R^H defined by

$$V_\tau^H := \sqrt{2H}\, e^{-\tau H} R_{e^\tau}^H, \qquad \tau \in \mathbb{R}.$$

The basic idea of our proofs is then as follows. First, we convince ourselves that the auto-covariance functions of U^H and V^H are indeed continuous, non-negative and integrable so that Corollary 2.6 is applicable, yielding that the persistence exponents of the Lamperti transform and the corresponding self-similar process coincide.

The next step in proving Theorem 3.1 (Theorem 3.2 is proven similarly, but the argument is much more technical) is to consider the GSP $(U^H_{\tau/H})_{\tau\in\mathbb{R}}$ as $H \to 0$ and the GSP $(U^H_{\tau/(1-H)})_{\tau\in\mathbb{R}}$ as $H \to 1$. Their persistence exponents are given by $\theta_I(H)/H$ and $\theta_I(H)/(1-H)$, respectively, as a quick computation shows. We will show that in both of these cases, the respective auto-covariance function of that GSP tends to the auto-covariance function $\tau \mapsto e^{-\tau}$, which is the (normalized) auto-covariance function of the Ornstein-Uhlenbeck process (OU process) for $\lambda := 1$ (cf. (1.16)). Recall that this OU process has persistence exponent 1 (cf. (1.18)). Then, we use the following lemma, which is [14, Lemma 3.6] together with [14, Remark 3.8], [28, Theorem 1.6], and [14, Lemma 3.10], to conclude the convergence of the persistence exponents $\theta_I(H)/H \to 1$ as $H \to 0$ and, respectively, $\theta_I(H)/(1-H) \to 1$ as $H \to 1$.

Lemma 3.3. *For $k \in \mathbb{N}$, let $(Z^{(k)}_\tau)_{\tau\geq 0}$ be a centered GSP with non-negative auto-covariance function $A_k(\tau)$, $\tau \geq 0$, satisfying $A_k(0) = 1$. Suppose that $A_k(\tau) \to A(\tau)$ for $k \to \infty$ and all $\tau \geq 0$, where $A\colon [0,\infty) \to [0,1]$ is the auto-covariance function of a centered GSP $(Z_\tau)_{\tau\geq 0}$.*

(a) If $Z^{(k)}$ and Z have continuous sample paths, and the conditions

$$\lim_{L\to\infty} \limsup_{k\to\infty} \sum_{\tau=L}^{\infty} A_k\left(\frac{\tau}{\ell}\right) = 0 \text{ for every } \ell \in \mathbb{N}, \tag{3.1}$$

$$\limsup_{\varepsilon\downarrow 0} |\log\varepsilon|^\eta \sup_{k\in\mathbb{N},\, \tau\in[0,\varepsilon]} (1 - A_k(\tau)) < \infty \text{ for some } \eta > 1, \tag{3.2}$$

$$\limsup_{\tau\to\infty} \frac{\log A(\tau)}{\log \tau} < -1 \tag{3.3}$$

are fulfilled, then

$$\lim_{k,T\to\infty} \frac{1}{T} \log \mathbb{P}\bigg(\sup_{\tau\in[0,T]} Z^{(k)}_\tau \leq 0 \bigg) = \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\bigg(\sup_{\tau\in[0,T]} Z_\tau \leq 0 \bigg). \tag{3.4}$$

(b) If $A(\tau) = 0$ for all $\tau > 0$ and (3.1) is fulfilled, then

$$-\lim_{k,T\to\infty} \frac{1}{T} \log \mathbb{P}\bigg(\sup_{\tau\in[0,T]} Z^{(k)}_\tau \leq 0 \bigg) = \infty.$$

Remark. Note that the statements in [14] and [28] concern the probabilities $\mathbb{P}(\sup_{\tau\in[0,T]} Z^{(k)}_\tau < 0)$ and $\mathbb{P}(\sup_{\tau\in[0,T]} Z_\tau < 0)$ (instead of $\leq$). However, due to Gaussianity, stationarity and continuous sample paths, the distributions of $\sup_{\tau\in[0,T]} Z^{(k)}_\tau$ and $\sup_{\tau\in[0,T]} Z_\tau$ are, for every $k \in \mathbb{N}$ and $T \geq 0$, absolutely continuous w.r.t. the Lebesgue measure, see [84, Theorem 3]. Thus, it holds $\mathbb{P}(\sup_{\tau\in[0,T]} Z^{(k)}_\tau = 0) = \mathbb{P}(\sup_{\tau\in[0,T]} Z_\tau = 0) = 0$ for every $k \in \mathbb{N}$ and $T \geq 0$.

The lemma says that if the auto-covariance functions of the processes $Z^{(k)}$ converge pointwise to the auto-covariance function of the process Z and the technical conditions (3.1)–(3.3) are satisfied, then the persistence exponents of the processes $Z^{(k)}$ converge to the persistence exponent of the process Z. Here, the existence of the persistence exponents, i.e., the existence of the (negative) limits in (3.4), follows from Proposition 2.2.

3.2 Proofs for the case of IFBM

In this section, we prove Theorem 3.1. We start with a lemma giving important properties of the auto-covariance function of U^H. In [54, Lemma 2], the auto-covariance function $\rho_H(\tau) = \mathbb{E}\big[U_0^H U_\tau^H\big]$ was found to be

$$\rho_H(\tau) = \frac{(1+H)\left(e^{-H\tau}+e^{H\tau}\right)}{1+2H} + \frac{\left(e^{\tau/2}-e^{-\tau/2}\right)^{2\,(1+H)}}{2\,(1+2H)} - \frac{e^{(1+H)\tau}+e^{-(1+H)\tau}}{2\,(1+2H)}$$

for $\tau \geq 0$ and shown to be non-increasing on $(0,\infty)$.

By the binomial theorem, we also have the following useful representation:

$$\begin{aligned}\rho_H(\tau) &= \frac{(1+H)\left(e^{-H\tau}+e^{H\tau}\right)}{1+2H} \\ &\quad + \frac{\sum_{k=0}^{\infty}(-1)^k\binom{2+2H}{k}e^{-\tau(k-1-H)}}{2\,(1+2H)} - \frac{e^{(1+H)\tau}+e^{-(1+H)\tau}}{2\,(1+2H)} \\ &= \frac{(1+H)\,e^{-H\tau}}{1+2H} + \frac{\sum_{k=2}^{\infty}(-1)^k\binom{2+2H}{k}e^{-\tau(k-1-H)}}{2\,(1+2H)} - \frac{e^{-(1+H)\tau}}{2\,(1+2H)},\end{aligned} \tag{3.5}$$

which even holds for $\tau = 0$ due to the fact that the exponent $2+2H$ is positive, see e.g. [1]. This gives the following asymptotics.

Lemma 3.4. *For all $\tau \geq 0$,*

$$\lim_{H\to 0}\rho_H\left(\frac{\tau}{H}\right) = \lim_{H\to 1}\rho_H\left(\frac{\tau}{1-H}\right) = e^{-\tau}.$$

Proof. Considering (3.5) for the argument τ/H yields

$$\rho_H\left(\frac{\tau}{H}\right) = \frac{(1+H)\,e^{-\tau}}{1+2H} + \frac{\sum_{k=2}^{\infty}(-1)^k\binom{2+2H}{k}e^{-\tau(k-1-H)/H}}{2\,(1+2H)} - \frac{e^{-\tau(1+H)/H}}{2\,(1+2H)} \tag{3.6}$$

As $H < 1$, we estimate, for all $k \geq 2$,

$$\left|\binom{2+2H}{k}\right| = \frac{2H+2}{1} \cdot \frac{2H+1}{2} \cdots \frac{|2H+4-k|}{k-1} \cdot \frac{|2H+3-k|}{k} \leq 4 \cdot \frac{3}{2} \cdot 1 \cdots 1 = 6. \tag{3.7}$$

Therefore, we can majorize $(-1)^k \binom{2+2H}{k} e^{-\tau(k-1-H)/H}$, $k \geq 2$, by the sequence $6\,e^{-\tau(k-2)}$, $k \geq 2$, which is summable for $\tau > 0$, to conclude with the dominated convergence theorem that, for all $\tau > 0$,

$$\rho_H\Big(\frac{\tau}{H}\Big) = \frac{(1+H)\,e^{-\tau}}{1+2H} + o(1) \to e^{-\tau}, \qquad H \to 0.$$

Similarly, by considering (3.5) for the argument $\tau/(1-H)$, we get

$$\begin{aligned}\rho_H\Big(\frac{\tau}{1-H}\Big) &= \frac{(1+H)\,e^{-\tau H/(1-H)}}{1+2H} \\ &\quad + \frac{\sum_{k=2}^{\infty}(-1)^k\binom{2+2H}{k}e^{-\tau(k-1-H)/(1-H)}}{2\,(1+2H)} - \frac{e^{-\tau(1+H)/(1-H)}}{2\,(1+2H)} \\ &= \frac{(1+H)\,e^{-\tau}}{2} + \frac{\sum_{k=3}^{\infty}(-1)^k\binom{2+2H}{k}e^{-\tau(k-1-H)/(1-H)}}{2\,(1+2H)} + o(1) \\ &= \frac{(1+H)\,e^{-\tau}}{2} + o(1) \to e^{-\tau}, \qquad H \to 1,\end{aligned}$$

for all $\tau > 0$, where we again used (3.7) to be able to apply the dominated convergence theorem. Noting that $\rho_H(\tau/H) = \rho_H(\tau/(1-H)) = 1 = e^{-\tau}$ for $\tau = 0$ finishes the proof. □

Proof of Theorem 3.1. First observe that ρ_H is continuous, non-negative and integrable. For the non-negativity, note that

$$\mathbb{E}\big[I_s^H I_t^H\big] = \int_0^s \int_0^t \mathbb{E}\big[B_r^H B_u^H\big]\,\mathrm{d}r\,\mathrm{d}u, \qquad s,t \geq 0,$$

and that the covariance function $(s,t) \mapsto \mathbb{E}\big[B_s^H B_t^H\big] = (s^{2H} + t^{2H} - |t-s|^{2H})/2$ is non-negative. For the integrability, consider (3.5) to deduce that $\rho_H(\tau) \sim \frac{1+H}{1+2H}\,e^{-H\tau}$ for $\tau \to \infty$ and $H < 1/2$, whereas $\rho_H(\tau) \sim \frac{1+H}{2}\,e^{-(1-H)\tau}$ for $\tau \to \infty$ and $H > 1/2$.

Thus, Corollary 2.6 is applicable and it holds

$$\theta_I(H) = -\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\bigg(\sup_{\tau\in[0,T]} U_\tau^H \leq 0\bigg).$$

The case $H \to 0$. Observe that

$$\frac{\theta_I(H)}{H} = -\frac{1}{H}\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\sup_{\tau\in[0,T]} U_\tau^H \le 0\right)$$
$$= -\frac{1}{H}\lim_{T\to\infty}\frac{1}{T/H}\log\mathbb{P}\left(\sup_{\tau\in[0,T/H]} U_\tau^H \le 0\right)$$
$$= -\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left(\sup_{\tau\in[0,T]} U_{\tau/H}^H \le 0\right). \tag{3.8}$$

By Lemma 3.4, the auto-covariance function $\tau \mapsto \rho_H(\tau/H)$ of $(U_{\tau/H}^H)_{\tau\ge 0}$ converges pointwise as $H \to 0$ to $\tau \mapsto e^{-\tau}$. As already mentioned, this is the auto-covariance function of the OU process for $\lambda := 1$, which has persistence exponent 1 (cf. (1.18)).

So, as soon as we have also proven that the persistence exponents converge, the desired convergence $\theta_I(H)/H \to 1$ as $H \to 0$ follows. In order to achieve this, we want to apply Lemma 3.3(a), i.e., we check the conditions (3.1)–(3.3) for the process $(U_{\tau/H}^H)_{\tau\ge 0}$ with the auto-covariance function $\tau \mapsto \rho_H(\tau/H)$. Obviously, (3.3) is fulfilled for the limiting auto-covariance function $\tau \mapsto e^{-\tau}$.

To check (3.1), note that for $H < 1/2$ and every $k \ge 4$, one has

$$(-1)^k\binom{2+2H}{k}$$
$$= -\frac{2H+2}{1}\cdot\frac{2H+1}{2}\cdot\frac{2H}{3}\cdot\frac{1-2H}{4}\cdot\frac{2-2H}{5}\cdots\frac{k-3-2H}{k} < 0 \tag{3.9}$$

and also $(-1)^k\binom{2+2H}{k} = -\frac{2H+2}{1}\cdot\frac{2H+1}{2}\cdot\frac{2H}{3} < 0$ for $k = 3$. Thus, considering (3.6) and estimating all negative terms by zero, we get

$$\rho_H\Big(\frac{\tau}{H}\Big) \le \frac{(1+H)\,e^{-\tau}}{1+2H} + \frac{(1+H)\,e^{-(1-H)\tau/H}}{2} \le \frac{7\,e^{-\tau}}{4}$$

for $H < 1/2$, where the right-hand side is independent of H and integrable in τ.

Similarly, for (3.2), we write, for every $H < 1/2$ and $\varepsilon > 0$, using the non-increasing character of ρ_H and the fact that $1 - e^{-x} \le x$,

$$\sup_{\tau\in[0,\varepsilon]}\left(1 - \rho_H\Big(\frac{\tau}{H}\Big)\right) = 1 - \rho_H\Big(\frac{\varepsilon}{H}\Big) = \rho_H\Big(\frac{0}{H}\Big) - \rho_H\Big(\frac{\varepsilon}{H}\Big)$$
$$= \frac{(1+H)\,(1-e^{-\varepsilon})}{1+2H} - \frac{1-e^{-(1+H)\varepsilon/H}}{2\,(1+2H)}$$
$$+ \frac{\sum_{k=2}^{\infty}(-1)^k\binom{2+2H}{k}\left(1-e^{-\varepsilon(k-1-H)/H}\right)}{2\,(1+2H)}$$
$$\le \varepsilon + \left(\frac{1+H}{2} - \frac{1}{2(1+2H)}\right)\left(1-e^{-\varepsilon/H}\right) \le \varepsilon + \frac{(3+2H)\,\varepsilon}{2(1+2H)} \le 3\,\varepsilon. \tag{3.10}$$

The case $H \to 1$. Similarly to (3.8),

$$\frac{\theta_I(H)}{1-H} = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{\tau\in[0,T]} U^H_{\tau/(1-H)} \le 0 \right),$$

and by Lemma 3.4, the auto-covariance function $\tau \mapsto \rho_H(\tau/(1-H))$ of $(U^H_{\tau/(1-H)})$ converges pointwise as $H \to 1$ to $\tau \mapsto e^{-\tau}$. Again, this is the auto-covariance function of the OU process with persistence exponent 1. Applying Lemma 3.3(a) for the process $(U^H_{\tau/(1-H)})_{\tau\ge 0}$ completes the proof of the asymptotics, subject to checking the technical conditions (3.1) and (3.2).

Considering (3.9) for $H > 1/2$, we see that $(-1)^k\binom{2+2H}{k} < 0$ for $k = 3$ and $(-1)^k\binom{2+2H}{k} > 0$ for $k \ge 4$. So, estimating again the negative terms by 0, we get

$$\begin{aligned}
\rho_H\left(\frac{\tau}{1-H}\right) &\le \frac{(1+H)\, e^{-H\tau/(1-H)}}{1+2H} + \frac{(1+H)\, e^{-\tau}}{2} \\
&\quad + \frac{1}{2(1+2H)} \sum_{k=4}^{\infty} (-1)^k \binom{2+2H}{k} e^{-\tau(k-1-H)/(1-H)} \\
&\le \left(2 + \frac{1}{2\,(1+2H)} \sum_{k=4}^{\infty} (-1)^k \binom{2+2H}{k} \right) e^{-\tau} \\
&= \left(\frac{5}{2} + (1+H)\left(\frac{H}{3} - \frac{1}{2} \right) \right) e^{-\tau} \le \frac{13\, e^{-\tau}}{6}
\end{aligned}$$

for $H \in (1/2, 1)$, where in the last equality, again by the binomial theorem, we used the fact that $\sum_{k=0}^{\infty}(-1)^k\binom{2+2H}{k} = (1-1)^{2+2H} = 0$. This shows (3.1).

Condition (3.2) can be verified similarly to (3.10), since in this case

$$\begin{aligned}
&\sup_{\tau\in[0,\varepsilon]} \left(1 - \rho_H\left(\frac{\tau}{1-H} \right) \right) = \rho_H\left(\frac{0}{1-H} \right) - \rho_H\left(\frac{\varepsilon}{1-H} \right) \\
&= \frac{(1+H)\,(1-e^{-\varepsilon})}{2} + \frac{(1+H)\left(1 - e^{-H\varepsilon/(1-H)}\right)}{1+2H} - \frac{1 - e^{-(1+H)\varepsilon/(1-H)}}{2\,(1+2H)} \\
&\quad + \frac{1}{2\,(1+2H)} \sum_{k=3}^{\infty} (-1)^k \binom{2+2H}{k} \left(1 - e^{-\varepsilon(k-1-H)/(1-H)}\right) \\
&\le \varepsilon + \left(\frac{1+H}{1+2H} - \frac{H(1+H)}{3} \right) \frac{\varepsilon}{1-H} \\
&\quad + \frac{H(1+H)(2H-1)}{12} \left(1 - e^{-(3-H)\varepsilon/(1-H)}\right) \\
&\quad - \frac{1}{2\,(1+2H)} \left(1 - e^{-(1+H)\varepsilon/(1-H)}\right) + \sum_{k=5}^{\infty} (-1)^k \binom{2+2H}{k} \frac{(k-1-H)\,\varepsilon}{1-H},
\end{aligned}$$

where we used again $1 - e^{-x} \le x$ and estimated $\varepsilon\, H/(1-H) \le \varepsilon/(1-H)$ as well as $\varepsilon\,(2-H)/(1-H) \ge \varepsilon/(1-H)$ in the last step. Note that

$$\frac{1+H}{1+2H} - \frac{H(1+H)}{3} = \frac{(3+2H)(1+H)(1-H)}{3(1+2H)};$$

that

$$\begin{aligned}
&\frac{H(1+H)(2H-1)}{12}\left(1-e^{-(3-H)\varepsilon/(1-H)}\right) - \frac{1}{2\,(1+2H)}\left(1-e^{-(1+H)\varepsilon/(1-H)}\right)\\
&\le \left(\frac{H(2H-1)(3-H)}{12} - \frac{1}{2\,(1+2H)}\right)\left(1-e^{-(1+H)\varepsilon/(1-H)}\right)\\
&= \frac{H(4H^2-1)(3-H)-6}{12\,(1+2H)}\left(1-e^{-(1+H)\varepsilon/(1-H)}\right) < 0,
\end{aligned}$$

for $H \in (0,1)$ and $\varepsilon > 0$ since $(1-x^{3-H})/(1-x^{1+H}) < (3-H)/(1+H)$ for $x \in (0,1)$; and that, for $k \ge 5$,

$$\begin{aligned}
(-1)^k \binom{2+2H}{k}\frac{k-1-H}{1-H} &= \frac{2H+2}{k-2}\cdot\frac{(k-1-H)(2H+1)}{k-1}\cdot\frac{2H}{k}\cdot\frac{2H-1}{1}\\
&\quad\times\frac{2-2H}{2(1-H)}\cdot\frac{3-2H}{3}\cdots\frac{k-3-2H}{k-3}\\
&\le \frac{4}{k-2}\cdot 3\cdot\frac{2}{k}\cdot 1\cdots 1 = \frac{24}{k(k-2)},
\end{aligned}$$

which is summable in k. Putting these facts together, we get, for every $\eta > 1$,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} |\log\varepsilon|^\eta \sup_{H\in(1/2,1),\tau\in[0,\varepsilon]}\left(1-\rho_H\left(\frac{\tau}{1-H}\right)\right)\\
&\le \limsup_{\varepsilon\to 0} |\log\varepsilon|^\eta\,\varepsilon \sup_{H\in(1/2,1)}\left(1+\frac{(3+2H)(1+H)}{3(1+2H)}+\sum_{k=5}^{\infty}\frac{24}{k(k-2)}\right)\\
&= 0 < \infty.
\end{aligned}$$

Finally, the continuity of θ_I follows from that of $H \mapsto \rho_H(\tau)$ and Lemma 3.3(a), since it is easily seen that conditions (3.1)–(3.3) are satisfied for the sequence $\tau \mapsto \rho_H(\tau)$, $H \in [H_0-\delta, H_0+\delta]$, with fixed $H_0 \in (0,1)$, small $\delta > 0$, and $H \to H_0$. □

3.3 Proofs for the case of FIBM

In this section, we prove Theorem 3.2. For this purpose, we first need the following two lemmas on the auto-covariance function of V^H. The auto-covariance function

$r_H(\tau) = \mathbb{E}\big[V_0^H V_\tau^H\big]$ can be found, e.g., in [45, eq. (12)] and reads

$$r_H(\tau) = \frac{4H}{1+2H}\, e^{-\tau/2} {}_2F_1\left(1, \frac{1}{2} - H, \frac{3}{2} + H, e^{-\tau}\right) \tag{3.11}$$

with the standard notation for the Gauss hypergeometric function.

Note here that, for $\tau > 0$,

$$\begin{aligned} {}_2F_1\left(1, \frac{1}{2} - H, \frac{3}{2} + H, e^{-\tau}\right) &= \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2-H)}{\Gamma(1/2-H)} \left(\frac{\Gamma(n+3/2+H)}{\Gamma(3/2+H)}\right)^{-1} e^{-n\tau} \\ &= \sum_{n=0}^{\infty} \frac{(n+1/2-H)\cdots(1/2-H)}{(n+3/2+H)\cdots(3/2+H)}\, e^{-n\tau} \\ &\to \sum_{n=0}^{\infty} (-1)^n e^{-n\tau} = \frac{1}{1+e^{-\tau}}, \qquad H \to \infty, \end{aligned} \tag{3.12}$$

where we used that the modulus of the fraction in the second equality is bounded by 1 and we consequently could interchange limit and sum by the dominated convergence theorem. Thus, as stated at the beginning of this chapter, $r_H(\tau) \to 2e^{-\tau/2}/(1+e^{-\tau}) = 1/\cosh(\tau/2)$ for $H \to \infty$.

Moreover, we have the following representation.

Lemma 3.5. *For all $\tau, H > 0$,*

$$e^{-\tau/2} - r_H(\tau) = \frac{1-2H}{1+2H}\, e^{-\tau/2}\,(1-e^{-\tau})^{2H}\, {}_2F_1\left(\frac{1}{2} + H, 2H, \frac{3}{2} + H, e^{-\tau}\right).$$

Proof. The result follows by applying the Euler transform of ${}_2F_1$ and from the formula

$$e^{-\tau/2} - r_H(\tau) = \frac{1-2H}{1+2H}\, e^{-\tau/2}\,(1-e^{-\tau})\, {}_2F_1\left(1, \frac{3}{2} - H, \frac{3}{2} + H, e^{-\tau}\right).$$

To verify this formula note that after plugging in the definiton of r_H, we are left with showing

$$\begin{aligned} &4H\, {}_2F_1\left(1, \frac{1}{2} - H, \frac{3}{2} + H, x\right) \\ &\qquad + (1-2H)(1-x)\, {}_2F_1\left(1, \frac{3}{2} - H, \frac{3}{2} + H, x\right) = 1 + 2H \end{aligned}$$

for every $H > 0$ and $x \in (0,1)$. But this contiguous relationship is easily obtained in equating the coefficients of x^n in the two series. $\square$

Let us now analyse the behaviour of the rescaled auto-covariance function $r_H(\tau/\gamma)$ with $\gamma = \gamma_H \to \infty$ as $H \to 0$.

Lemma 3.6. *Let $\gamma = \gamma_H$ be a function tending to infinity with $H \to 0$. If $\gamma^{-2H} \to c$ for $H \to 0$ and some $c \in [0,1]$, then $r_H(\tau/\gamma) \to 1 - c$ for $H \to 0$ and all $\tau > 0$.*

Proof. By Lemma 3.5,

$$e^{-\tau/(2\gamma)} - r_H\left(\frac{\tau}{\gamma}\right) \sim (1 - e^{-\tau/\gamma})^{2H} {}_2F_1\left(\frac{1}{2} + H, 2H, \frac{3}{2} + H, e^{-\tau/\gamma}\right)$$

as $H \to 0$. Now $(1 - e^{-\tau/\gamma})^{2H} \sim (\tau/\gamma)^{2H} \to c$ as $H \to 0$ and

$$\begin{aligned}
&\left| {}_2F_1\left(\frac{1}{2} + H, 2H, \frac{3}{2} + H, e^{-\tau/\gamma}\right) - 1 \right| \\
&= \frac{\Gamma(3/2 + H)}{\Gamma(1/2 + H)\,\Gamma(2H)} \sum_{n=1}^{\infty} \frac{\Gamma(n + 1/2 + H)\,\Gamma(n + 2H)}{\Gamma(n + 3/2 + H)} \cdot \frac{e^{-n\tau/\gamma}}{n!} \\
&= \frac{1 + 2H}{2\Gamma(2H)} \sum_{n=1}^{\infty} \frac{\Gamma(n + 2H)}{(n-1)!} \cdot \frac{e^{-n\tau/\gamma}}{(n + 1/2 + H)n} \\
&\leq \frac{1}{\Gamma(2H)} \sum_{n=1}^{\infty} n^{-2} \to 0, \qquad H \to 0.
\end{aligned}$$

□

Proof of Theorem 3.2. Again, we first note that r_H is indeed continuous, non-negative and integrable. For the integrability, observe that, by (3.11),

$$\begin{aligned}
\int_0^{\infty} r_H(\tau)\,\mathrm{d}\tau &= \frac{4H\,\Gamma(3/2 + H)}{(1 + 2H)\,\Gamma(1/2 - H)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2 - H + n)}{\Gamma(3/2 + H + n)} \int_0^{\infty} e^{-\tau(n + 1/2)}\,\mathrm{d}\tau \\
&= \frac{2H\,\Gamma(1/2 + H)}{\Gamma(1/2 - H)} \sum_{n=0}^{\infty} \frac{\Gamma(1/2 - H + n)}{\Gamma(1/2 + H + n)\,(n + 1/2)\,(n + 1/2 + H)} < \infty.
\end{aligned} \tag{3.13}$$

Consequently, Corollary 2.6 yields

$$\theta_R(H) = -\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{\tau \in [0,T]} V_\tau^H \leq 0 \right).$$

Now, similarly to (3.8), for every γ, we have

$$\frac{\theta_R(H)}{\gamma} = -\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{\tau \in [0,T]} V_{\tau/\gamma}^H \leq 0 \right).$$

We will show $\theta_R(H)/\gamma \to \infty$ for any function $\gamma = \gamma_H$ with $\gamma \ll H^{-1}$, where $f(x) \ll g(x)$ means $\lim f(x)/g(x) = 0$. This proves part (a).

Let $\gamma = \gamma_H$ be a function satisfying $\gamma \to \infty$ and $\gamma \ll H^{-1}$ as $H \to 0$. Since $\lim_{H\to 0} \gamma^{-2H} \geq \lim_{H\to 0} H^{2H} = 1$ and $\lim_{H\to 0} \gamma^{-2H} \leq \lim_{H\to 0} 1^{-2H} = 1$, it follows from Lemma 3.6 that $r_H(\tau/\gamma) \to 0$ for $H \to 0$ and all $\tau > 0$. To conclude the assertion $\theta_R(H)/\gamma \to \infty$, we want to apply Lemma 3.3(b) and thus have to check (3.1) for the auto-covariance function $\tau \mapsto r_H(\tau/\gamma)$.

Indeed, by (3.13), one has for every $\ell \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty r_H\left(\frac{\tau}{\ell\gamma}\right) \mathrm{d}\tau &= \ell\gamma \int_0^\infty r_H(\tau)\, \mathrm{d}\tau \\ &\sim 2\ell\gamma H \sum_{n=0}^\infty \left(n + \frac{1}{2}\right)^{-2} =: c \cdot \ell\gamma H, \qquad H \to 0,\end{aligned}$$

and thus, for every $\ell, L \in \mathbb{N}$,

$$\begin{aligned}\limsup_{H\to 0} \sum_{\tau=L}^\infty r_H\left(\frac{\tau}{\ell\gamma}\right) &\leq \limsup_{H\to 0} \int_{L-1}^\infty r_H\left(\frac{\tau}{\ell\gamma}\right) \mathrm{d}\tau \\ &\leq \limsup_{H\to 0} \int_0^\infty r_H\left(\frac{\tau}{\ell\gamma}\right) \mathrm{d}\tau = c\,\ell \limsup_{H\to 0} \gamma H = 0,\end{aligned}$$

where we used that r_H is non-negative and non-increasing, which is easily seen by (3.11).

Now, we prove part (b). We will show that

$$r_H(\tau) \geq (1 - |\tau|^H)_+ = \mathbb{E}[S_0^{H/2} S_\tau^{H/2}]$$

for $H \in (0, 1/2)$ and all $\tau \in \mathbb{R}$, where (S_τ^H) is the so-called fractional Slepian's process (see [55, Section 2.3]). Then, Proposition 2.1(a) implies that $\theta_R(H) \leq \theta_S(H/2)$, where $\theta_S(H)$ denotes the persistence exponent of (S_τ^H), and the assertion follows by [55, Proposition 2.9].

We have

$$\begin{aligned}&\frac{1-2H}{1+2H}\, {}_2F_1\left(\frac{1}{2} + H, 2H, \frac{3}{2} + H, e^{-\tau}\right) \leq \frac{1-2H}{1+2H}\, {}_2F_1\left(\frac{1}{2} + H, 2H, \frac{3}{2} + H, 1\right) \\ &= \frac{1-2H}{1+2H} \cdot \frac{\Gamma(3/2+H)\,\Gamma(1-2H)}{\Gamma(3/2-H)} = \frac{\Gamma(1/2+H)\,\Gamma(1-2H)}{\Gamma(1/2-H)} \\ &= \frac{\Gamma(1/2+H)}{\Gamma(1/2)} \cdot \frac{\Gamma(1-H)}{2^{2H}} \leq 1, \qquad H \in \left(0, \frac{1}{2}\right),\end{aligned}$$

where we used the Legendre duplication formula in the last equality and the monotonicity of $\Gamma(\cdot)$ on $(1/2, 1)$ as well as the fact that

$$\Gamma(1-H) = \Gamma\Big(2H \cdot \frac{1}{2} + (1-2H)\cdot 1\Big)$$
$$\leq \left(\Gamma\Big(\frac{1}{2}\Big)\right)^{2H} \cdot (\Gamma(1))^{1-2H} = \pi^H \leq 2^{2H}, \qquad 2H \in (0,1),$$

(due to the logarithmic convexity of $\Gamma(\cdot)$) in the last inequality. Together with Lemma 3.5, this gives

$$r_H(\tau) \geq e^{-\tau/2}\left(1 - \left(1-e^{-\tau}\right)^{2H}\right) \geq e^{-\tau/2}\left(1-\tau^{2H}\right) = \phi(\tau)\left(1-\tau^H\right)$$

for $\tau \geq 0$, where $\phi(\tau) := e^{-\tau/2}(1+\tau^H)$. Now, $\varphi(\tau) := 1+\tau^H - e^{\tau/2}$ satisfies $\varphi(0)=0$, $\varphi(1) = 2-\sqrt{e} > 0$, and

$$\varphi''(\tau) = -H(1-H)\tau^{-(2-H)} - \frac{e^{\tau/2}}{4} < 0, \qquad \tau \geq 0,$$

which implies $\varphi(\tau) \geq 0$ and thus $\phi(\tau) \geq 1$ for $\tau \in [0,1]$. This shows $r_H(\tau) \geq (1-|\tau|^H)_+$ for $\tau \in [0,1]$ and, due to the symmetry and the non-negativity of r_H, even for all $\tau \in \mathbb{R}$.

Finally, similarly to the proof of Theorem 3.1, the continuity of θ_R follows from the continuity of $H \mapsto r_H(\tau)$ and Lemma 3.3(a), since the sequence $\tau \mapsto r_H(\tau)$, $H \in [H_0-\delta, H_0+\delta]$, with fixed $H_0 \in (0,\infty)$, small $\delta > 0$, and $H \to H_0$ fulfills conditions (3.1)–(3.3). One easily checks (3.1) and (3.3), while for checking (3.2), we note that

$$1 - r_H(\varepsilon) = 1 - e^{-\varepsilon/2} + e^{-\varepsilon/2} - r_H(\varepsilon) \leq \frac{\varepsilon}{2} + c_{H_0}\varepsilon^{2(H_0-\delta)}$$

for suitable c_{H_0} and small ε using Lemma 3.5, with τ replaced by ε, and the fact that $1-e^{-x} \leq x$. □

Remark. Most parts of this chapter appeared in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya* as well as in its English translation *Theory of Probability and its Applications* in the article *Asymptotics of the persistence exponent of integrated fractional Brownian motion and fractionally integrated Brownian motion*, see [12].

Chapter 4

Persistence probabilities of mixed processes

In this chapter, we study persistence probabilities of mixed processes, i.e., sums of self-similar processes with different self-similarity indices. In the setting of centred Gaussian processes with non-negative covariances, we will first show a general result and then consider several mixed processes of the literature. In particular, we determine the persistence exponent of mixed fractional Brownian motion.

4.1 Introduction

Mixed fractional Brownian motion (mixed FBM) $M^{H,\alpha}$ is defined as

$$M_t^{H,\alpha} := W_t + \alpha B_t^H, \qquad t \geq 0, \tag{4.1}$$

where $\alpha \in \mathbb{R} \setminus \{0\},\ H \in (0,1),\ B^H$ is an FBM with Hurst parameter H and W is an independent Brownian motion. This process was first introduced by Cheridito in [25] and has turned out to be useful in the modelling of stock prices, as it provides models with long memory and no arbitrage, see e.g. [25, Section 6] and [3]. Note that this process still has stationary increments, but is *not self-similar* itself.

We will derive a persistence result for a more general class of sums of self-similar centred Gaussian processes with different self-similarity indices, covering not only the mixed FBM $M^{H,\alpha}$, but also e.g. the case of completely correlated mixed FBM introduced in [32]. Note that the latter process neither is self-similar nor has stationary increments. Thus, our result contributes to the amount of rather rare persistence results for stochastic processes violating both the properties of self-similarity and stationary increments. As seen in Chapter 2, self-similarity is a valuable property in the context of persistence as in this case, one is able to apply the Lamperti transformation to get a stationary process and concerning persistence, many powerful tools are available for the class of stationary Gaussian processes. In the case that self-similarity is not available, the property of stationary increments turned out to be appropriate as another property that can be used to prove the existence of the persistence exponent, see [11]. Besides, one could derive persistence results even outside of the Gaussian setting if one assumes both self-similarity and stationary increments, see [15] and [60].

The outline of this chapter is as follows. In Section 4.2, we will introduce the class of mixed processes which are suitable for our purposes and present our main result that for these processes, the persistence probability decays asymptotically polynomially with the persistence exponent of the self-similar process with the greater self-similarity index. In Section 4.3, we will then use this result to derive persistence results for the (completely correlated) mixed FBM and other explicit mixed processes of interest. Finally, in Section 4.4, we will prove the main result.

4.2 Main result

Recall that for $H > 0$, a stochastic process $(X_t)_{t\geq 0}$ is called H-self-similar if $(X_{ct}) \stackrel{\mathrm{d}}{=} (c^H X_t)$. We consider the sum of two self-similar centred Gaussian processes with different self-similarity indices, i.e., $X^H + Y^K$, where X^H is an H-self-similar centred Gaussian process, Y^K is a K-self-similar centred Gaussian process and $K < H$. The main result of this paper, which is given in the following theorem, states that under the assumption that X^H and $X^H + Y^K$ have non-negative covariance functions, respectively, and that the conditions of Corollary 2.6 hold for X^H, the persistence probability of $X^H + Y^K$ has – up to terms of lower order – the same asymptotic behaviour as the persistence probability of X^H.

Theorem 4.1. *For $0 < K < H$, let X^H and Y^K be self-similar centred Gaussian processes with a.s. càdlàg sample paths and self-similarity indices H and K, respectively. Let us assume that the covariance functions of the processes X^H and $X^H + Y^K$ are non-negative, respectively, and that the auto-covariance function*

$\tau \mapsto \operatorname{cov}(X_1^H, e^{-\tau H} X_{e^\tau}^H)$ of the Lamperti transform of X^H is continuous, integrable and not the zero function. Then, it holds

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \leq 1 \right) = T^{-\theta_X + o(1)}, \qquad T \to \infty,$$

where

$$\theta_X := - \lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\left(\sup_{t\in[0,T]} X_t^H \leq 1 \right) \in (0,\infty)$$

denotes the persistence exponent of X^H.

The result remains true if one replaces Y^K by a finite sum of self-similar centred Gaussian processes Y^{K_i} with self-similarity indices $K_i < H$, as our proof in Section 4.4 can be easily adapted to this setting. From a mathematical point of view, it would be an interesting open problem to study the persistence probability when Y^K is replaced by a (well-defined) infinite sum of the form $\sum_{i=1}^{\infty} \alpha_i Y^{K_i}$, where e.g. $\sup_i K_i = H$, as in this case, our proof technique does not work anymore.

Note that the assumptions on X^H together with Corollary 2.6 guarantee the existence of $\theta_X \in (0,\infty)$. For the mixed process $X^H + Y^K$ on the contrary, the condition of non-negative covariances does not yield the existence of a persistence exponent a priori, since the mixed process is not self-similar anymore. Further note that we do not need any direct assumption on the covariance function of Y^K or on the correlation of X^H and Y^K. Thus, in particular, X^H and Y^K do not need to be independent and a persistence exponent of Y^K does not necessarily have to exist.

4.3 Mixed FBM and further corollaries

Mixed FBM. Let us now come back to the case of mixed FBM, which we defined in (4.1). Note that this is a special case of the so-called fractional mixed FBM, which covers all linear combinations of independent FBMs with different Hurst parameters, see [33] and [52]. Recall that the FBM B^H has the covariance function $(s,t) \mapsto \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$, which is non-negative. Due to the independence of the underlying FBMs, this directly implies also the non-negativity of the covariance function of the (fractional) mixed FBM. Note that the continuous and integrable function $\tau \mapsto \frac{1}{2}\left(e^{H\tau} + e^{-H\tau} - |e^{\tau/2} - e^{-\tau/2}|^{2H}\right)$ is the auto-covariance function of the Lamperti transform of B^H. Further recall that we have seen in (1.3) that

$$\mathbb{P}\left(\sup_{t\in[0,T]} B_t^H \le 1 \right) = T^{-(1-H)+o(1)}, \qquad T \to \infty.$$

This yields the following corollary of Theorem 4.1 for the (fractional) mixed FBM.

Corollary 4.2. *For $0 < K < H < 1$, let B^H and B^K be independent FBMs with Hurst parameters H and K, respectively, and $a, b \in \mathbb{R}$ with $ab \neq 0$. Then*

$$\mathbb{P}\left(\sup_{t\in[0,T]} aB_t^H + bB_t^K \le 1 \right) = T^{-(1-H)+o(1)}, \qquad T \to \infty.$$

In particular, for the mixed FBM as defined in (4.1), we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} M_t^{H,\alpha} \le 1 \right) = T^{-(1-\max\{\frac{1}{2},H\})+o(1)}, \qquad T \to \infty.$$

Note that the local behaviour of fractional mixed FBM is completely different: In [85], it was shown that $aB^H + bB^K$ is locally equivalent to bB^K if and only if $H - K > 1/4$.

Completely correlated mixed FBM. Recall that Corollary 4.2 assumes the independence of B^H and B^K. As already mentioned, Theorem 4.1 also covers the case of completely correlated mixed FBM. Under this term, it was introduced recently in [32], while the process itself had already been studied as the driving process of an SDE in [53, Section 3.2.3]. The definition is as follows. Let B^H be an FBM with Hurst parameter $H \in (0,1)$. Then, there exists a Brownian motion W such that

$$B_t^H = \int_0^t k_H(t,s)\,\mathrm{d}W_s, \qquad t \ge 0, \tag{4.2}$$

where k_H is the so-called Molchan-Golosov kernel, see [63, Section 5.1.3] and (4.5) and (4.6) below. Completely correlated mixed FBM (ccmFBM) $X^{H,a,b}$ is given by

$$X_t^{H,a,b} := aW_t + bB_t^H, \qquad t \ge 0, \tag{4.3}$$

where $a, b \in \mathbb{R}$ with $ab \neq 0$ and W is the same Brownian motion as in (4.2). Similarly to the fractional mixed FBM, as $k_{1/2} \equiv 1$ (see (4.6)), one can generalize $X^{H,a,b}$ to linear combinations $aB^H + bB^K$ of fractional Brownian motions generated by the same Brownian motion W via the Molchan-Golosov kernels k_H and k_K with different Hurst parameters H and K, which were discussed recently in [64] and which we want to call fractional ccmFBM. Using the Itô-isometry, the fractional ccmFBM has the covariance function

$$(s,t) \mapsto a^2\mathbb{E}[B_s^H B_t^H] + b^2\mathbb{E}[B_s^K B_t^K] \\ + ab\int_0^{s\wedge t} (k_H(t,u)k_K(s,u) + k_H(s,u)k_K(t,u))\,\mathrm{d}u. \tag{4.4}$$

Set $C(H) := \sqrt{\frac{2H\,\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma(2-2H)}}$. Then, for $H > 1/2$ and $0 < s < t$, we have

$$k_H(t,s) = \frac{C(H)}{\Gamma\left(H-\frac{1}{2}\right)}\, s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,\mathrm{d}u \geq 0, \tag{4.5}$$

whereas for $H \leq 1/2$ and $0 < s < t$, it holds

$$k_H(t,s) \\ = \frac{C(H)}{\Gamma\left(H+\frac{1}{2}\right)}\left((\frac{t^2}{s}-t)^{H-\frac{1}{2}} + (\frac{1}{2}-H)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u \right) \geq 0. \tag{4.6}$$

Thus, the covariance function of the (fractional) ccmFBM is non-negative, if $ab > 0$, and Theorem 4.1 together with (1.3) gives the following corollary.

Corollary 4.3. *For $0 < K < H < 1$ and a Brownian motion W, define $B_t^H := \int_0^t k_H(t,s)\,\mathrm{d}W_s$ and $B_t^K := \int_0^t k_K(t,s)\,\mathrm{d}W_s$. Further let $a, b \in \mathbb{R}$ with $ab > 0$. Then*

$$\mathbb{P}\left(\sup_{t\in[0,T]} aB_t^H + bB_t^K \leq 1 \right) = T^{-(1-H)+o(1)}, \qquad T \to \infty.$$

In particular, for the ccmFBM as defined in (4.3)*, we have*

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t^{H,a,b} \leq 1 \right) = T^{-(1-\max\{\frac{1}{2},H\})+o(1)}, \qquad T \to \infty.$$

Mixed integrated FBM. Recall that for $H \in (0,1)$, the integrated FBM (IFBM) I^H is the $(H+1)$-self-similar centred Gaussian process given by

$$I_t^H := \int_0^t B_s^H\,\mathrm{d}s, \qquad t \geq 0.$$

Further recall our earlier discussions in Section 1.3 and in Chapter 3 about the persistence results for IFBM. One knows that the persistence exponent $\theta_I(H) \in (0,\infty)$ of I^H exists, that the inequalities (1.9) and (1.10) are fulfilled and that it holds $\theta_I(1/2) = 1/4$ in the Brownian case. Moreover, we have seen in Theorem 3.1 that θ_I is continuous and satisfies $\theta_I(H) \sim H$ as $H \to 0$ as well as $\theta_I(H) \sim 1 - H$ as $H \to 1$, which supports the conjecture (1.7).

As we have already convinced ourselves at the beginning of the proof of Theorem 3.1 that the auto-covariance function of the Lamperti transform of I^H is continuous and integrable, Theorem 4.1 yields the following corollary for mixed integrated FBM.

Corollary 4.4. *For $0 < K < H < 1$, let B^H and B^K be independent FBMs with Hurst parameters H and K, respectively, and $a, b \in \mathbb{R}$ with $ab \neq 0$. Let $I_t^H = \int_0^t B_s^H \,\mathrm{d}s$ and $I_t^K = \int_0^t B_s^K \,\mathrm{d}s$. Further let $\theta_I : (0,1) \to (0,\infty)$ denote the persistence exponent of IFBM depending on the Hurst parameter. Then*

$$\mathbb{P}\left(\sup_{t\in[0,T]} aI_t^H + bI_t^K \leq 1 \right) = T^{-\theta_I(H)+o(1)}, \qquad T \to \infty.$$

Of course, the same result also holds for the integral of (fractional) ccmFBM, again in the case $ab > 0$, as the only difference in verifying the assumptions of Theorem 4.1 is that the covariance function of the mixed process has additional summands. But these are given as the double integral of the additional summands in (4.4), which is again non-negative if $ab > 0$.

Mixed fractionally integrated Brownian motion. As a last example, we want to consider mixed fractionally integrated Brownian motion (mixed FIBM), which was introduced as mixed Riemann-Liouville process in [23, Section 8]. Recall that for $H > 0$, the FIBM R^H is the H-self-similar process given by

$$R_t^H := \int_0^t (t-s)^{H-\frac{1}{2}} \,\mathrm{d}W_s, \qquad t \geq 0.$$

Again, recall our earlier discussions in Section 1.3 and in Chapter 3 about the persistence results of FIBM. Similarly to IFBM, one knows that the persistence exponent $\theta_R(H) \in (0,\infty)$ of R^H exists, but its exact value is unknown except for the Brownian cases, where $\theta_R(1/2) = 1/2$ (Brownian motion) and $\theta_R(3/2) = 1/4$ (integrated Brownian motion). Further, one knows that θ_R is non-increasing on $[1/2,\infty)$. In Theorem 3.2, we have seen that θ_R is continuous and that $\theta_R(H) \to \infty$ for $H \to 0$, where the asymptotic behaviour is in the range H^{-1} to H^{-2}.

Since we have already convinced ourselves at the beginning of the proof of Theorem 3.2 that the auto-covariance function of the Lamperti transform of R^H is continuous and integrable, Theorem 4.1 yields the following corollary.

Corollary 4.5. *For $0 < K < H$ and independent Brownian motions $W^{(1)}$ and $W^{(2)}$, define $R_t^H := \int_0^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}W_s^{(1)}$ and $R_t^K := \int_0^t (t-s)^{K-\frac{1}{2}}\,\mathrm{d}W_s^{(2)}$. Let $a, b \in \mathbb{R}$ with $ab \neq 0$ and $\theta_R : (0,\infty) \to (0,\infty)$ denote the persistence exponent of FIBM depending on the Hurst parameter. Then*

$$\mathbb{P}\left(\sup_{t\in[0,T]} aR_t^H + bR_t^K \leq 1\right) = T^{-\theta_R(H)+o(1)}, \qquad T \to \infty.$$

Again, in the case $ab > 0$, the same result also holds for the completely correlated mixed FIBM, where R^H and R^K are generated by the same Brownian motion (instead of two independent Brownian motions), as the covariance function of the mixed process gets additional summands which are non-negative.

4.4 Proof of the main result

In this section, we give the proof of Theorem 4.1. The main idea is as follows. We restrict the interval $[0,T]$ of persistence to an interval $[a(T),T]$, where $a(T)$ has to be small enough such that the asymptotic polynomial order of the persistence probability does not change and large enough such that we are able to control the range of the process Y^K on the interval $[a(T),T]$. It turns out that $a(T) := (\log T)^p$ for p large enough is a suitable choice. The following lemma shows that the probability that Y_t^K exceeds t^γ for $\gamma > K$ on the interval $[a(T),T]$ is of neglectable order.

Lemma 4.6. *Let Y^K be as in Theorem 4.1, $\theta \geq 0$, $\gamma > K$ and $\delta > 0$. Then there exists $p \geq \mathrm{e}^2$ such that for T large enough, it holds*

$$\mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \big|Y_t^K\big| > t^\gamma\big) \leq T^{-\theta-\delta}.$$

Proof. We estimate

$$\begin{aligned}\mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \big|Y_t^K\big| > t^\gamma\big) &\leq \sum_{s=\lfloor(\log T)^p\rfloor}^{\lfloor T\rfloor} \mathbb{P}\big(\exists t \in [s, s+1] \,:\, \big|Y_t^K\big| > t^\gamma\big)\\ &\leq \sum_{s=\lfloor(\log T)^p\rfloor}^{\lfloor T\rfloor} \mathbb{P}\left(\sup_{t\in[s,s+1]} \big|Y_t^K\big| > s^\gamma\right). \qquad (4.7)\end{aligned}$$

For $s = \lfloor (\log T)^p \rfloor, \dots, \lfloor T \rfloor$ and $\sigma_K^2 := \mathbb{V}[Y_1^K] \vee \sup_{t\in[1,2]} \mathbb{V}[Y_t^K - Y_1^K]$, we may further estimate

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[s,s+1]} \big|Y_t^K\big| > s^\gamma \Big) \\
&\le \mathbb{P}\Big(\big|Y_s^K\big| > \frac{s^\gamma}{2} \Big) + \mathbb{P}\Big(\sup_{t\in[s,s+1]} \big|Y_t^K\big| > s^\gamma,\ \big|Y_s^K\big| \le \frac{s^\gamma}{2} \Big) \\
&\le \mathbb{P}\Big(|\mathcal{N}(0,1)| > \frac{s^{\gamma-K}}{2\,\sigma_K} \Big) + \mathbb{P}\Big(\sup_{t\in[s,s+1]} \big|Y_t^K\big| - \big|Y_s^K\big| > \frac{s^\gamma}{2} \Big) \\
&\le c_1\, e^{-s^{2(\gamma-K)}/(8\,\sigma_K^2)} + \mathbb{P}\Big(\sup_{t\in[s,s+1]} \big|Y_t^K - Y_s^K\big| > \frac{s^\gamma}{2} \Big) \\
&= c_1\, e^{-s^{2(\gamma-K)}/(8\,\sigma_K^2)} + \mathbb{P}\Big(\sup_{t'\in[1,1+s^{-1}]} \big|Y_{t'}^K - Y_1^K\big| > \frac{s^{\gamma-K}}{2} \Big) \qquad (4.8)
\end{aligned}$$

for some constant $c_1 > 0$ and T large enough, where we used self-similarity of Y^K in the second, the reverse triangle inequality in the third, and again self-similarity in the fourth step.

Now, we estimate the probability in (4.8) as follows:

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t'\in[1,1+s^{-1}]} \big|Y_{t'}^K - Y_1^K\big| > \frac{s^{\gamma-K}}{2} \Big) \le \mathbb{P}\Big(\sup_{t'\in[1,2]} \big|Y_{t'}^K - Y_1^K\big| > \frac{s^{\gamma-K}}{2} \Big) \\
&\le \mathbb{P}\Big(\sup_{t'\in[1,2]} \big(Y_{t'}^K - Y_1^K\big) > \frac{s^{\gamma-K}}{2} \Big) + \mathbb{P}\Big(\sup_{t'\in[1,2]} \big(Y_1^K - Y_{t'}^K\big) > \frac{s^{\gamma-K}}{2} \Big) \\
&= 2\,\mathbb{P}\Big(\sup_{t\in[1,2]} \big(Y_t^K - Y_1^K\big) > \frac{s^{\gamma-K}}{2} \Big). \qquad (4.9)
\end{aligned}$$

The last probability is a probability of large deviation of a bounded Gaussian random function and can therefore be estimated by the tail of a one-dimensional Gaussian distribution.

More precisely, by e.g. [44, Theorem 12.1], there exist constants $c_2 > 0$ and $d \in \mathbb{R}$ such that

$$\mathbb{P}\Big(\sup_{t\in[1,2]} \big(Y_t^K - Y_1^K\big) > \frac{s^{\gamma-K}}{2} \Big) \le c_2\, e^{s^{\gamma-K}/2 - \left(s^{\gamma-K}/2+d\right)^2/(2\sigma_K^2)}.$$

Together with (4.9) and (4.8), this yields for $s = \lfloor (\log T)^p \rfloor, \dots, \lfloor T \rfloor$:

$$\begin{aligned}
\mathbb{P}\Big(\sup_{t\in[s,s+1]} \big|Y_t^K\big| > s^\gamma \Big) &\le e^{-s^{2(\gamma-K)}/(8\,\sigma_K^2) + c_3 s^{\gamma-K}} \\
&\le e^{-(\log T)^{2(\gamma-K)p}/(8\,\sigma_K^2) + c_0 (\log T)^{(\gamma-K)p}}
\end{aligned}$$

for constants $c_3, c_0 > 0$. Combining this with (4.7), we get

$$\begin{aligned}&\mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \big|Y_t^K\big| > t^\gamma\big)\\&\le (T - (\log T)^p + 2)\, e^{-(\log T)^{2(\gamma-K)p}/(8\,\sigma_K^2) + c_0 (\log T)^{(\gamma-K)p}}.\end{aligned}$$

Taking e.g. $p = \max\{1/(\gamma - K), e^2\}$, the right-hand-side decays faster than any polynomial, which shows the assertion. □

Thus, we can estimate the persistence probability of $X^H + Y^K$ on $[a(T), T]$ by the persistence probability of X^H shifted by $t \mapsto t^\gamma$ on $[a(T), T]$. As seen in Proposition 2.3, shifting a Gaussian process by a deterministic function does not change the asymptotic polynomial order of the persistence probability if the function belongs to the reproducing kernel Hilbert space (RKHS) of the process. Therefore, we have to estimate $t \mapsto t^\gamma$ on $[a(T), T]$ by a function in the RKHS of X^H. The assumptions of Theorem 4.1 together with Corollary 2.5 guarantee that this is possible if $\gamma < H$, so that we are ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. First observe that the assumptions of Corollary 2.6 are fulfilled so that the persistence exponent $\theta_X \in (0, \infty)$ indeed exists. Further note that we have seen in the proof of Corollary 2.6 that the integrability of the autocovariance function of the Lamperti transform of X^H implies that the Lamperti transform of X^H satisfies the condition on the spectral measure of Corollary 2.5.

Thus, by taking e.g. $\alpha := 1/4$ in Corollary 2.5, there exists a function $h \in \mathcal{H}_{X^H}$ such that $h(t) \sim c\, t^H (\log t)^{-3/4}$ for $t \to \infty$ and some $c > 0$. We additionally choose γ with $K < \gamma < H$, $\delta > 0$ and p according to Lemma 4.6 for $\theta := \theta_X$.

Similarly as in the proof of Corollary 2.6, the a.s. càdlàg sample paths of X^H imply that the distribution of X^H on the Borel-σ-algebra of $D[0, \infty)$ w.r.t. the Skorokhod topology is a Radon measure, so that Proposition 2.3 is applicable. Further, the set $A := A_T := \{f \in D[0, \infty) \colon \sup_{t \in [(\log T)^p, T)} f(t) \le 1\}$ is closed w.r.t. the Skorokhod topology and thus element of the Borel-σ-algebra of $D[0, \infty)$. Proposition 2.3 together with the fact that $\theta_X \in (0, \infty)$ exists consequently yields

$$\frac{\mathbb{P}\big(\sup_{t \in [(\log T)^p, T]} X_t^H \pm h(t) \le 1\big)}{\mathbb{P}\big(\sup_{t \in [(\log T)^p, T]} X_t^H \le 1\big)} = T^{o(1)}, \qquad T \to \infty, \tag{4.10}$$

where we additionally used that the suprema of X^H and $X^H \pm h$, respectively, on $[(\log T)^p, T)$ and $[(\log T)^p, T]$ coincide a.s. due to the fact that h is continuous (by construction in Corollary 2.5 and the continuity of the covariance function) and that X^H is even a.s. continuous at any fixed deterministic time.

Upper bound: It holds

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \le 1\right)\\
&\le \mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \left|Y_t^K\right| > h(t)\big) + \mathbb{P}\left(\sup_{t\in[(\log T)^p,T]} X_t^H - h(t) \le 1\right)\\
&\le \mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \left|Y_t^K\right| > t^\gamma\big) + \mathbb{P}\left(\sup_{t\in[(\log T)^p,T]} X_t^H \le 1\right) T^{o(1)}\\
&\le T^{-\theta_X-\delta} + \frac{\mathbb{P}\big(\sup_{t\in[0,T]} X_t^H \le 1\big)}{\mathbb{P}\big(\sup_{t\in[0,(\log T)^p]} X_t^H \le 1\big)}\, T^{o(1)}\\
&\le T^{-\theta_X-\delta} + T^{-\theta_X+o(1)} (\log T)^{p\,\theta_X+o(1)}\, T^{o(1)}\\
&\le T^{-\theta_X+o(1)}
\end{aligned}$$

for T large enough. Here, the second inequality uses (4.10) and the property of h that $h(t) \sim c\,t^H (\log t)^{-3/4} > t^\gamma$ for t large enough, while the third inequality is Lemma 4.6 together with Proposition 2.1(b).

Lower bound: The opposite reasoning gives

$$\begin{aligned}
T^{-\theta_X+o(1)} &= \mathbb{P}\left(\sup_{t\in[0,T]} X_t^H \le 1\right)\\
&\le \mathbb{P}\left(\sup_{t\in[(\log T)^p,T]} X_t^H + h(t) \le 1\right) T^{o(1)}\\
&\le \mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \left|Y_t^K\right| > h(t)\big)\, T^{o(1)}\\
&\qquad + \mathbb{P}\left(\sup_{t\in[(\log T)^p,T]} X_t^H + Y_t^K \le 1\right) T^{o(1)}\\
&\le \mathbb{P}\big(\exists t \in [(\log T)^p, T] \,:\, \left|Y_t^K\right| > t^\gamma\big)\, T^{o(1)}\\
&\qquad + \mathbb{P}\left(\sup_{t\in[(\log T)^p,T]} X_t^H + Y_t^K \le 1\right) T^{o(1)}\\
&\le T^{-\theta_X-\delta} + \frac{\mathbb{P}\big(\sup_{t\in[0,T]} X_t^H + Y_t^K \le 1\big)}{\mathbb{P}\big(\sup_{t\in[0,(\log T)^p]} X_t^H + Y_t^K \le 1\big)}\, T^{o(1)},
\end{aligned}$$

where we used (4.10) in the second and Lemma 4.6 as well as Proposition 2.1(b) in the fifth step. Precisely here, we use the assumption of non-negative covariances of

$X^H + Y^K$. So we have

$$\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \le 1\right) \ge T^{-\theta_X+o(1)}\,\mathbb{P}\left(\sup_{t\in[0,(\log T)^p]} X_t^H + Y_t^K \le 1\right). \tag{4.11}$$

We then further estimate the right-hand side of (4.11) by replacing T in (4.11) by $(\log T)^p$ and get

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \le 1\right)\\ &\ge T^{-\theta_X+o(1)}(\log T)^{-p\,\theta_X+o(1)}\,\mathbb{P}\left(\sup_{t\in[0,(p\log\log T)^p]} X_t^H + Y_t^K \le 1\right).\end{aligned} \tag{4.12}$$

We set $f_0(T) := \log\log T$ and $f_N(T) := \log p + \log f_{N-1}(T)$ for $N \ge 1$. Using (4.11) iteratively then gives

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \le 1\right)\\ &\ge T^{-\theta_X+o(1)}(\log T)^{(-p\,\theta_X+o(1))(N+1)}\,\mathbb{P}\left(\sup_{t\in[0,(pf_N(T))^p]} X_t^H + Y_t^K \le 1\right)\\ &= T^{-\theta_X+o(1)+\frac{(\log\log T)(N+1)}{\log T}(-p\,\theta_X+o(1))}\,\mathbb{P}\left(\sup_{t\in[0,(pf_N(T))^p]} X_t^H + Y_t^K \le 1\right),\end{aligned} \tag{4.13}$$

for $N \in \mathbb{N}$. This can be seen by induction: The induction base is (4.12), while for the induction step, one has to note that

$$(\log((pf_{N-1}(T))^p))^p = (p(\log p + \log f_{N-1}(T)))^p = (pf_N(T))^p\,.$$

Now we consider the function

$$\varphi_p(x) := \log p + \log x, \qquad x \in [2,\infty).$$

This is a contraction with Lipschitz constant 1/2. The Lipschitz constant can be computed by the fact that $\varphi_p'(x) = 1/x \le 1/2$ for $x \ge 2$, while the self-map property of φ_p is deduced from the fact that $\log p \ge 2$ holds by Lemma 4.6. Thus, the Banach fixed-point theorem yields a unique fixed-point $a_p \ge 2$ of φ_p, which does not depend on T. Further, as $f_N(T) = \varphi_p(f_{N-1}(T))$, we can estimate

$$\begin{aligned}|f_N(T) - a_p| &\le \frac{2^{-N}}{1-\frac{1}{2}}\,|f_1(T) - f_0(T)|\\ &= 2^{1-N}\,|\log p + \log\log\log T - \log\log T| \le 2^{1-N}\cdot 3\log\log T\end{aligned}$$

for $N \in \mathbb{N}$ and T large enough, see e.g. [2, Theorem 1.1(iii)]. For $N_T := \lceil(\log\log\log T + \log 6)/\log 2\rceil$, this implies

$$|f_{N_T}(T) - a_p| \leq 2^{1-N_T} \cdot 3\log\log T \leq 1.$$

Considering (4.13) for $N := N_T$ consequently yields

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} X_t^H + Y_t^K \leq 1\right) &\geq T^{-\theta_X + o(1)}\,\mathbb{P}\left(\sup_{t\in[0,(p\,f_{N_T}(T))^p]} X_t^H + Y_t^K \leq 1\right) \\
&\geq T^{-\theta_X + o(1)}\,\mathbb{P}\left(\sup_{t\in[0,(p\,(1+a_p))^p]} X_t^H + Y_t^K \leq 1\right) \\
&= T^{-\theta_X + o(1)},
\end{aligned}$$

which finishes the proof. □

Remark. Most parts of this chapter appeared in the journal *Journal of Physics A: Mathematical and Theoretical* in the article *Persistence probabilities of mixed FBM and other mixed processes*, see [13].

Chapter 5

Persistence probabilities of further fractional processes

In this chapter, we are concerned with the persistence probabilities of two further fractional processes of the literature – the so-called bifractional Brownian motion, which is a generalization of fractional Brownian motion, and the fractional Ornstein-Uhlenbeck process, which we have already seen in Section 1.3.

5.1 Bifractional Brownian motion

For $H > 0$ and $K \in (0, 2)$ with either $H \leq 1$, $HK < 1$ or $H > 1$, $2HK \leq 1$, the bifractional Brownian motion (biFBM) $B^{H,K} = (B_t^{H,K})_{t \geq 0}$ is defined as the centred Gaussian process with the covariance function

$$\mathbb{E}\Big[B_t^{H,K} B_s^{H,K}\Big] := \frac{1}{2^K} \left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK} \right), \qquad t, s \geq 0. \tag{5.1}$$

Note that $B^{H,K}$ generalizes the fractional Brownian motion (FBM) B^H since for $H \in (0, 1)$ and $K = 1$, (5.1) becomes the covariance function of B^H. This process was introduced in [39] for $H \in (0, 1]$ and $K \in (0, 1]$, motivated by the fact that the property of stationary increments of FBM turned out to be appropriate for applications, when one is interested in small increments of a process, whereas it appeared inadequate for modelling large increments. In particular, it was shown in [39] that for $H \in (0, 1]$ and $K \in (0, 1]$, the function (5.1) is indeed a covariance function, i.e., that it is positive definite. Later, in [17], it was proven that this is also the case for $K \in (1,2)$ if $HK < 1$. By [43], the conditions $K \leq 2$ and $HK \leq 1$ are even necessary for the existence of $B^{H,K}$. Recently, in [83], it was shown that $B^{H,K}$ exists also for $H > 1$ if $2HK \leq 1$.

We will see that our results below hold whenever the process $B^{H,K}$ exists and additionally, slightly stronger assumptions than the necessary conditions are fulfilled, namely $K < 2$ and $HK < 1$. So any potential extension of the range of (H,K) where existence of $B^{H,K}$ is proven would extend also the validity of our results.

The biFBM $B^{H,K}$ is an HK-self-similar process, as is easily seen by (5.1). Further important properties transfer from HK-FBM to $B^{H,K}$, such as long-range dependent increments for $HK > 1/2$ (cf. [46, Corollary 4.3] for $K \leq 1$ and [17, Proposition 2.6] for $K > 1$) and the existence of a locally Hölder continuous modification for any index of $(0, HK)$ (cf. [39, Proposition 3.1] for $H \leq 1, K \leq 1$; [17, Proposition 2.5] for $H < 1, K > 1$; and [83, Proposition 1.5] for $H > 1$). Nevertheless, since it is well-known that (up to normalization) the only self-similar centred Gaussian process with stationary increments is FBM, $B^{H,K}$ has *no stationary increments* unless $K = 1$.

Using self-similarity, as presented in Chapter 2, we can transform $B^{H,K}$ into a stationary centred Gaussian process by considering its Lamperti transform $Z^{H,K}$, which has the auto-covariance function

$$\mathbb{E}\Big[Z_0^{H,K} Z_\tau^{H,K}\Big] = \frac{1}{2^K}\left(\left(e^{H\tau} + e^{-H\tau}\right)^K - \left(e^{\tau/2} - e^{-\tau/2}\right)^{2HK}\right)\right), \qquad \tau \geq 0. \tag{5.2}$$

Note that the monotonicity of $\cdot^{2H}$ implies $e^{H\tau} + e^{-H\tau} \geq (e^{\tau/2})^{2H} \geq (e^{\tau/2} - e^{-\tau/2})^{2H}$, which, by the montonicity of $\cdot^K$, again implies $(e^{H\tau} + e^{-H\tau})^K \geq (e^{\tau/2} - e^{-\tau/2})^{2HK}$ so that the auto-covariance function (5.2) is non-negative and, clearly, continuous. Further, it holds, by the binomial theorem,

$$\begin{aligned} 2^K\,\mathbb{E}\Big[Z_0^{H,K} Z_\tau^{H,K}\Big] &= \sum_{k=1}^{\infty} \binom{K}{k} e^{H\tau(K-2k)} - \sum_{n=1}^{\infty} (-1)^n \binom{2HK}{n} e^{\tau(HK-n)} \\ &= K e^{-H\tau(2-K)} + 2HK e^{-\tau(1-HK)} + o(e^{-(1\wedge 2H)\tau}), \qquad \tau \to \infty, \end{aligned}$$

where we estimated all terms of the first sum with $k \geq 2$ by $o(e^{-2H\tau})$ and all terms of the second sum with $n \geq 2$ by $o(e^{-\tau})$. As $K < 2$ and $HK < 1$, this shows that the auto-covariance function (5.2) is also integrable. Thus, Corollary 2.6 yields

$$\mathbb{P}\left(\sup_{t\in[0,T]} B_t^{H,K} \leq 1\right) = T^{-\theta_B(H,K)+o(1)}, \qquad T \to \infty, \tag{5.3}$$

where $\theta_B(H,K) := -\lim_{T\to\infty} \frac{1}{T}\log \mathbb{P}\big(\sup_{\tau\in[0,T]} Z_\tau^{H,K} \le 0\big) \in (0,\infty)$.

As already mentioned, the process $B^{H,1}$ for $H<1$ is just the FBM so that we have seen in (1.3) that it holds $\theta_B(H,1) = 1-H$ for $H<1$. For the general persistence exponent of biFBM, we have the following upper and lower bounds, which unfortunately hold on disjoint parameter domains (except for the FBM case $K=1$).

Proposition 5.1. *For $H>0$ and $K\in(0,2)$ with either $H\le 1$, $HK<1$ or $H>1$, $2HK\le 1$, let*

$$\theta_B(H,K) = -\lim_{T\to\infty} \frac{1}{\log T}\log \mathbb{P}\left(\sup_{t\in[0,T]} B_t^{H,K} \le 1\right)$$

denote the persistence exponent of $B^{H,K}$. Then, it holds

(a) $\theta_B(H,K) \le 1-HK$ *for* $K\ge 1$, *and*

(b) $\theta_B(H,K) \ge 1-HK$ *for* $K\le 1$ *and* $2HK\le 1$.

Recall that $1-HK = \theta_B(HK,1)$. Using this identity, part(a) in the case $2HK\le 1$ as well as part(b) of Proposition 5.1 are an immediate consequence of Proposition 2.1(a) and the following lemma.

Lemma 5.2. *Let $R_{H,K}(t,s) := \mathbb{E}[B_s^{H,K}B_t^{H,K}]$ be the covariance function of $B^{H,K}$ as given in* (5.1)*. Then, it holds, for every $t,s\ge 0$,*

(a) $R_{H,K}(t,s) \ge R_{HK,1}(t,s)$ *for* $K\ge 1$ *and* $2HK\le 1$, *and*

(b) $R_{H,K}(t,s) \le R_{HK,1}(t,s)$ *for* $K\le 1$ *and* $2HK\le 1$.

Proof. Let w.l.o.g. $t\ge s\ge 0$. If $s=0$, it holds $B_s^{H,K} = B_s^{HK,1} = 0$ a.s. by self-similarity so that the statement is trivial. For $s>0$, we have

$$\begin{aligned}
&s^{-2HK}(R_{H,K}(t,s) - R_{HK,1}(t,s)) \\
&= 2^{-K}\left(\left(\frac{t}{s}\right)^{2H}+1\right)^{K} - 2^{-1}\left(\left(\frac{t}{s}\right)^{2HK}+1\right) + \left(2^{-1}-2^{-K}\right)\left(\frac{t}{s}-1\right)^{2HK} \\
&=: f\left(\frac{t}{s}\right).
\end{aligned}$$

Now, we have to show $f(u) \geq 0$ for $K \geq 1$ and $2HK \leq 1$ as well as $f(u) \leq 0$ for $K \leq 1$ and $2HK \leq 1$. The function $f\colon [1,\infty) \to \mathbb{R}$ fulfills $f(1) = 0$ and is differentiable with derivative

$$\begin{aligned} f'(u) &= 2^{-K} K \left(u^{2H} + 1\right)^{K-1} \cdot 2Hu^{2H-1} - HKu^{2HK-1} \\ &\qquad + \left(2^{-1} - 2^{-K}\right) \cdot 2HK \left(u-1\right)^{2HK-1} \\ &= HKu^{2HK-1} \Bigg(2^{-K+1} \left(1 + u^{-2H}\right)^{K-1} - 1 \\ &\qquad\qquad + \left(1 - 2^{-K+1}\right) \left(1 - u^{-1}\right)^{2HK-1} \Bigg), \qquad u \geq 1. \end{aligned}$$

As $1 - u^{-1} \leq 1$ and $2HK \leq 1$, we have $(1-u^{-1})^{2HK-1} \geq 1$. Now, depending on whether $K \geq 1$ or $K \leq 1$, we have further $(1+u^{-2H})^{K-1} \geq 1$ or $(1+u^{-2H})^{K-1} \leq 1$, respectively, and $1 - 2^{-K+1} \geq 0$ or $1 - 2^{-K+1} \leq 0$, respectively. This leads to

$$f'(u) \geq HKu^{2HK-1} \left(2^{-K+1} \cdot 1 - 1 + \left(1 - 2^{-K+1}\right) \cdot 1\right) = 0$$

for all $u \geq 1$, if $K \geq 1$, and

$$f'(u) \leq HKu^{2HK-1} \left(2^{-K+1} \cdot 1 - 1 + \left(1 - 2^{-K+1}\right) \cdot 1\right) = 0$$

for all $u \geq 1$, if $K \leq 1$, implying the assertion. □

The proof of Proposition 5.1(a) in the case $2HK > 1$ is based on the fact that the increment process $(B^{H,K}_{t+T} - B^{H,K}_T)_{t \geq 0}$ of $B^{H,K}$ at time T converges for $T \to \infty$ in the sense of finite-dimensional distributions to a (non-normalized) FBM with Hurst parameter HK. This was proven in [46] for $K < 1$, by using a decomposition which is only available for $K < 1$, but by just considering the covariances of the centred Gaussian processes, the result also follows in the general case, which is the following proposition.

Proposition 5.3. *For $H > 0$ and $K \in (0, 2)$ with either $H \leq 1$, $HK < 1$ or $H > 1$, $2HK \leq 1$, it holds*

$$\left(B^{H,K}_{t+T} - B^{H,K}_T\right)_{t \geq 0} \overset{\text{fdd}}{\to} \left(2^{(1-K)/2} B^{HK,1}_t\right)_{t \geq 0}, \qquad T \to \infty.$$

Proof. As both sequence and limit are centred Gaussian processes, it suffices to show convergence of the covariance functions. It holds, for $s, t \geq 0$, $T > 0$,

$$
\begin{aligned}
f_{H,K}(s,t,T) &:= 2^K\, \mathbb{E}\Big[\Big(B^{H,K}_{s+T} - B^{H,K}_T\Big)\Big(B^{H,K}_{t+T} - B^{H,K}_T\Big)\Big] - 2\,\mathbb{E}\Big[B^{HK,1}_s B^{HK,1}_t\Big] \\
&= \big((t+T)^{2H} + (s+T)^{2H}\big)^K - \big((t+T)^{2H} + T^{2H}\big)^K \\
&\qquad - \big((s+T)^{2H} + T^{2H}\big)^K + 2^K T^{2HK} \\
&= T^{2HK}\left(\left(\left(\frac{t}{T}+1\right)^{2H} + \left(\frac{s}{T}+1\right)^{2H}\right)^K - \left(\left(\frac{t}{T}+1\right)^{2H} + 1\right)^K\right. \\
&\qquad \left. - \left(\left(\frac{s}{T}+1\right)^{2H} + 1\right)^K + 2^K\right). \qquad (5.4)
\end{aligned}
$$

Using the Taylor expansion $(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)x^2/2 + o(x^2)$ for $\alpha = 2H$ and $\alpha = K$, respectively, and $x \to 0$, we get

$$
\begin{aligned}
&T^{-2HK}\, f_{H,K}(s,t,T) \\
&= \left(1 + 2H\,\frac{t}{T} + H(2H-1)\,\frac{t^2}{T^2} + 1 + 2H\,\frac{s}{T} + H(2H-1)\,\frac{s^2}{T^2} + o(T^{-2})\right)^K \\
&\quad - \left(1 + 2H\,\frac{t}{T} + H(2H-1)\,\frac{t^2}{T^2} + o(T^{-2}) + 1\right)^K \\
&\quad - \left(1 + 2H\,\frac{s}{T} + H(2H-1)\,\frac{s^2}{T^2} + o(T^{-2}) + 1\right)^K + 2^K \\
&= 2^K\left(\left(1 + H\,\frac{t+s}{T} + \frac{H(2H-1)}{2}\,\frac{t^2+s^2}{T^2} + o(T^{-2})\right)^K\right. \\
&\qquad - \left(1 + H\,\frac{t}{T} + \frac{H(2H-1)}{2}\,\frac{t^2}{T^2} + o(T^{-2})\right)^K \\
&\qquad \left. - \left(1 + H\,\frac{s}{T} + \frac{H(2H-1)}{2}\,\frac{s^2}{T^2} + o(T^{-2})\right)^K + 1\right) \\
&= 2^K\left(\frac{K(K-1)}{2}\left(H\frac{t+s}{T}\right)^2 - \frac{K(K-1)}{2}\left(H\frac{t}{T}\right)^2\right. \\
&\qquad \left. - \frac{K(K-1)}{2}\left(H\frac{s}{T}\right)^2 + o(T^{-2})\right) \\
&= 2^K\,\frac{K(K-1)}{2}\,H^2\,\frac{2\,ts}{T^2} + o(T^{-2}), \qquad T \to \infty. \qquad (5.5)
\end{aligned}
$$

This implies $\lim_{T\to\infty} f_{H,K}(s,t,T) = 0$ for every $s, t \geq 0$, whenever we take $H, K > 0$ with $HK < 1$. □

In view of this convergence result, it seems plausible that $\theta_B(H,K) = \theta_B(HK,1) = 1 - HK$. However, our technique to use this result is based on applying Proposition

2.1(b) for the increment process of $B^{H,K}$, which only gives an estimate in one direction and additionally requires non-negatively correlated increments. The following lemma shows that this approach is possible for $K \geq 1$ and $2HK > 1$.

Lemma 5.4. *If $K \geq 1$, it holds, for every $t, s \geq 0$, $T > 0$,*

$$\mathbb{E}\Big[\left(B^{H,K}_{s+T} - B^{H,K}_{T}\right)\left(B^{H,K}_{t+T} - B^{H,K}_{T}\right)\Big] \geq 0. \tag{5.6}$$

If moreover $2HK > 1$, it also holds, for every $t \geq 0$, $T > 0$,

$$\mathbb{E}\Big[B^{H,K}_{T}\left(B^{H,K}_{t+T} - B^{H,K}_{T}\right)\Big] \geq 0. \tag{5.7}$$

Proof. Since the covariance function of $B^{HK,1}$ is non-negative, it suffices for (5.6) to show that $f_{H,K}$ as defined in (5.4) is non-negative for $K \geq 1$. Note that $f_{H,K}(s, 0, T) = 0$ for all $s \geq 0$, $T > 0$, and that

$$\begin{aligned}&\frac{\partial}{\partial t} f_{H,K}(s,t,T)\\ &= 2HK(t+T)^{2H-1}\left(\left((t+T)^{2H} + (s+T)^{2H}\right)^{K-1} - \left((t+T)^{2H} + T^{2H}\right)^{K-1}\right) \geq 0\end{aligned} \tag{5.8}$$

for all $s, t \geq 0$, $T > 0$, if $K \geq 1$.

For (5.7), note that it holds, for $t \geq 0$, $T > 0$,

$$\begin{aligned}\mathbb{E}\Big[B^{H,K}_{T}\left(B^{H,K}_{t+T} - B^{H,K}_{T}\right)\Big] &= \left(\frac{T^{2H}}{2}\right)^{K}\left(\left(1 + \left(1 + \frac{t}{T}\right)^{2H}\right)^{K} - \left(\frac{t}{T}\right)^{2HK} - 2^{K}\right)\\ &= \left(\frac{T^{2H}}{2}\right)^{K}\varphi\left(1 + \frac{t}{T}\right)\end{aligned}$$

for $\varphi(u) := (1 + u^{2H})^{K} - (u-1)^{2HK} - 2^{K}$, $u \geq 1$. The function φ satisfies $\varphi(1) = 0$ and is differentiable with derivative

$$\begin{aligned}\varphi'(u) &= K\left(1 + u^{2H}\right)^{K-1} \cdot 2Hu^{2H-1} - 2HK(u-1)^{2HK-1}\\ &\geq 2HK\left(u^{2H(K-1)} \cdot u^{2H-1} - (u-1)^{2HK-1}\right) \geq 0, \qquad u \geq 1,\end{aligned} \tag{5.9}$$

where we used $K \geq 1$ for the first and $2HK > 1$ for the second inequality. □

Remark. (i) Note that for $K \leq 1$ and $2HK < 1$, one gets the opposite inequality in (5.9). Hence, (5.7) is violated in this case. Furthermore, numerical calculations show that (5.7) is also violated for $K < 1$ and $2HK > 1$ if $2HK$ is close to 1, as well as for $K > 1$ and $2HK < 1$ if $2HK$ is sufficiently far away from 1.

(ii) For $K < 1$, one gets the opposite inequality in (5.8). Thus, $f_{H,K}$ is non-positive in this case. Nevertheless, it seems plausible numerically that $f_{H,K}$ could be dominated by the covariance of $B^{HK,1}$ so that (5.6) could still be fulfilled in this case.

Now, we are ready to give the proof of Proposition 5.1.

Proof of Proposition 5.1. Lemma 5.2 together with Proposition 2.1(a) implies part (b) as well as part (a) under the additional assumption $2HK \leq 1$. Thus, it remains to show (a) in the case $2HK > 1$.

Recall that we have seen in (5.3) that the persistence exponents of $B^{H,K}$ and its Lamperti transform coincide, i.e.,

$$\theta_B(H,K) = -\lim_{T\to\infty} \frac{1}{\log T} \log \mathbb{P}\left(\sup_{t\in[1,T]} B_t^{H,K} \leq 0 \right).$$

Further recall that $\theta_B(HK,1) = 1 - HK$.

Step 1: Relating the persistence probability of $B^{H,K}$ to persistence probabilities of increments of $B^{HK,1}$.

Let $T > 0$, $N > 1$. Setting $c_T := \mathbb{P}(\sup_{t\in[1,T+1]} B_t^{H,K} \leq 0)/2$, we estimate

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[1,T+N]} B_t^{H,K} \leq 0 \right) &\geq \mathbb{P}\left(\sup_{t\in[1,T+1]} B_t^{H,K} \leq 0 \right) \cdot \mathbb{P}\left(\sup_{t\in[T+1,T+N]} B_t^{H,K} \leq 0 \right) \\
&\geq 2\, c_T\, \mathbb{P}\left(B_T^{H,K} \leq 0, \sup_{t\in[T+1,T+N]} B_t^{H,K} - B_T^{H,K} \leq 0 \right) \\
&\geq 2\, c_T\, \mathbb{P}\left(B_T^{H,K} \leq 0 \right) \cdot \mathbb{P}\left(\sup_{t\in[T+1,T+N]} B_t^{H,K} - B_T^{H,K} \leq 0 \right) \\
&= c_T\, \mathbb{P}\left(\sup_{t\in[1,N]} B_{t+T}^{H,K} - B_T^{H,K} \leq 0 \right) \\
&= c_T\, \mathbb{P}\left(\sup_{\tau\in[0,\log N]} \sigma_{\tau,T}^{-1} Y_{e^\tau}^T \leq 0 \right), \qquad (5.10)
\end{aligned}$$

where we set $Y_t^T := B_{t+T}^{H,K} - B_T^{H,K}$ and $\sigma_{\tau,T}^2 := \mathbb{V}[Y_{e^\tau}^T]$. In the first and the third step, we applied Proposition 2.1(b) using non-negative covariances of $B^{H,K}$ and (5.7), respectively.

For $s, \tau \geq 0$, Proposition 5.3 yields

$$\begin{aligned} A_T(s, s+\tau) &:= \operatorname{cov}\big(\sigma_{s,T}^{-1}\, Y_{e^s}^T,\, \sigma_{s+\tau,T}^{-1}\, Y_{e^{s+\tau}}^T\big) \\ &= \frac{\mathbb{E}\big[Y_{e^s}^T Y_{e^{s+\tau}}^T\big]}{\sqrt{\mathbb{V}[Y_{e^s}^T] \cdot \mathbb{V}\big[Y_{e^{s+\tau}}^T\big]}} \\ &\to \frac{R_{HK,1}(e^s, e^{s+\tau})}{\sqrt{R_{HK,1}(e^s, e^s) \cdot R_{HK,1}(e^{s+\tau}, e^{s+\tau})}} \\ &= \frac{1}{2}\, e^{-HK\tau} \Big(1 + e^{2HK\tau} - (e^\tau - 1)^{2HK}\Big) =: A_\infty(0, \tau), \qquad T \to \infty, \end{aligned} \tag{5.11}$$

which is the auto-covariance function of the Lamperti transform of $B^{HK,1}$. To get an estimate of the corresponding persistence probabilities, we proceed analogously to the proof of the lower bound of (1.18) in [28]. We first show

$$\liminf_{T\to\infty} \inf_{s\geq 0} \mathbb{P}\left(\sup_{\tau\in[0,M]} \sigma_{\tau,T}^{-1}\, Y_{e^{s+\tau}}^T \leq 0 \right) \geq \mathbb{P}\left(\sup_{\tau\in[0,M]} e^{-HK\tau} B_{e^\tau}^{HK,1} \leq 0 \right) \tag{5.12}$$

for any $M > 0$.

Step 2: The crucial inequality for the proof of (5.12).

As in the proof of Proposition 5.3, we set, for $s, t \geq 0$, $T > 0$,

$$\begin{aligned} f(s,t,T) := f_{H,K}(s,t,T) &:= \big((t+T)^{2H} + (s+T)^{2H}\big)^K - \big((t+T)^{2H} + T^{2H}\big)^K \\ &\quad - \big((s+T)^{2H} + T^{2H}\big)^K + 2^K T^{2HK}. \end{aligned}$$

Then, we have

$$\begin{aligned} &A_T(s, s+\tau) \\ &= \frac{2^{1-K} R_{HK,1}(e^s, e^{s+\tau}) + 2^{-K} f(e^s, e^{s+\tau}, T)}{\sqrt{2^{1-K} e^{2HKs} + 2^{-K} f(e^s, e^s, T)} \cdot \sqrt{2^{1-K} e^{2HK(s+\tau)} + 2^{-K} f(e^{s+\tau}, e^{s+\tau}, T)}} \\ &= \frac{A_\infty(0,\tau) + 2^{-1} f(1, e^\tau, Te^{-s})\, e^{-HK\tau}}{\sqrt{1 + 2^{-1} f(1,1,Te^{-s})} \cdot \sqrt{1 + 2^{-1} f(1,1,Te^{-s}e^{-\tau})}} \\ &\geq \frac{A_\infty(0,\tau) + 2^{-1} f(1, 1, Te^{-s})\, e^{-HK\tau}}{\sqrt{1 + 2^{-1} f(1,1,Te^{-s})} \cdot \sqrt{1 + 2^{-1} f(1,1,Te^{-s}e^{-\tau})}} \\ &= \left(1 - \frac{\sqrt{1 + 2^{-1} f(1,1,Te^{-s})} \cdot \sqrt{1 + 2^{-1} f(1,1,Te^{-s}e^{-\tau})} - 1}{\sqrt{1 + 2^{-1} f(1,1,Te^{-s})} \cdot \sqrt{1 + 2^{-1} f(1,1,Te^{-s}e^{-\tau})}}\right) A_\infty(0,\tau) \\ &\qquad + \frac{2^{-1} e^{(2-2HK)\tau} f(1,1,Te^{-s})}{\sqrt{1 + 2^{-1} f(1,1,Te^{-s})} \cdot \sqrt{1 + 2^{-1} f(1,1,Te^{-s}e^{-\tau})}} \cdot e^{-(2-HK)\tau}, \end{aligned} \tag{5.13}$$

where we used the identities $R_{HK,1}(e^s, e^{s+\tau}) = e^{2HKs}e^{HK\tau}A_\infty(0,\tau)$ (cf. (5.11)) as well as $f(\alpha\, s', \alpha\, t', \alpha\, T') = \alpha^{2HK} f(s',t',T')$ for $\alpha, s', t', T' > 0$ (cf. (5.4)) in the second equality and the monotonicity of $f(s,t,T)$ in t for $K \geq 1$ (cf. (5.8)) in the inequality.

Now, we want to estimate (5.13) to get an inequality of the form

$$A_T(s, s+\tau) \geq (1-\varepsilon_T)A_\infty(0,\tau) + \varepsilon_T\, D(0,\tau) \tag{5.14}$$

for all $M > 0$, $\tau \in [0, M]$ and $s \in [0, t_T^*]$, where $(\varepsilon_T)_{T>0}$ is a null sequence which does not depend on s or τ, $(t_T^*)_{T>0}$ is a sequence tending to infinity and D is the covariance function of a stationary centred Gaussian process with continuous sample paths satisfying $D(0,0) = 1$.

Step 3: The derivation of (5.14).

We set $t_T^* := \log\log T$ and $D(0,\tau) := e^{-(2-HK)\tau}$, which is (up to normalization) the auto-covariance function of the Ornstein-Uhlenbeck process for $\lambda := 2 - HK$ (cf. (1.16)). Let $M > 0$. We will show that there exists $T_0 > 0$ independent of τ such that

$$f(1,1,T'e^{-\tau}) \leq e^{(2-2HK)\tau} f(1,1,T') \text{ for all } T' \geq T_0 \text{ and } \tau \in [0, M]. \tag{5.15}$$

Then, (5.15) for $T' := Te^{-s}$, together with (5.13) and the fact that $1 \leq e^{(2-2HK)\tau}$ for $HK \leq 1$ and all $\tau \geq 0$, implies

$$\begin{aligned}
&A_T(s, s+\tau) \\
&\geq \left(1 - \frac{\sqrt{1+2^{-1}f(1,1,Te^{-s})}\cdot\sqrt{1+2^{-1}e^{(2-2HK)\tau}f(1,1,Te^{-s})} - 1}{\sqrt{1+2^{-1}f(1,1,Te^{-s})}\cdot\sqrt{1+2^{-1}f(1,1,Te^{-s}e^{-\tau})}}\right) A_\infty(0,\tau) \\
&\qquad + \frac{2^{-1}e^{(2-2HK)\tau}f(1,1,Te^{-s})}{\sqrt{1+2^{-1}f(1,1,Te^{-s})}\cdot\sqrt{1+2^{-1}f(1,1,Te^{-s}e^{-\tau})}} \cdot e^{-(2-HK)\tau} \\
&\geq A_\infty(0,\tau) \\
&\qquad + \frac{2^{-1}e^{(2-2HK)\tau}f(1,1,Te^{-s})}{\sqrt{1+2^{-1}f(1,1,Te^{-s})}\cdot\sqrt{1+2^{-1}f(1,1,Te^{-s}e^{-\tau})}} \left(e^{-(2-HK)\tau} - A_\infty(0,\tau)\right)
\end{aligned} \tag{5.16}$$

for all T such that $T(\log T)^{-1} \geq T_0$, all $s \in [0, \log\log T]$ and $\tau \in [0, M]$.

Using the non-negativity of f for $K \geq 1$ (cf. (5.8)) together with the convergence result (5.5) (in $T' := Te^{-s} \to \infty$ uniformly in $s \in [0, \log\log T]$), we have

$$\begin{aligned}
&\frac{2^{-1}e^{(2-2HK)\tau} f(1,1,Te^{-s})}{\sqrt{1+2^{-1}f(1,1,Te^{-s})}\cdot\sqrt{1+2^{-1}f(1,1,Te^{-s}e^{-\tau})}}\\
&\le 2^{-1}e^{(2-2HK)\tau} f(1,1,Te^{-s})\\
&= 2^{K-1}e^{(2-2HK)\tau}\, T^{2HK} e^{-2HKs}\left(K(K-1)H^2T^{-2}e^{2s} + o(T^{-2}e^{2s})\right)\\
&\le 2^K K(K-1)H^2 e^{(2-2HK)M}\left(\frac{\log T}{T}\right)^{2-2HK} =: \varepsilon_T \to 0, \qquad T\to\infty,
\end{aligned}$$

where the second inequality holds for T large enough. Noting that $e^{2HK\tau} - (e^\tau - 1)^{2HK} \ge 1$ for $2HK > 1$ and thus $A_\infty(0,\tau) \ge e^{-HK\tau} \ge e^{-(2-HK)\tau}$ (as $2 - HK > HK$), (5.16) leads to

$$A_T(s, s+\tau) \ge A_\infty(0,\tau) + \varepsilon_T\left(e^{-(2-HK)\tau} - A_\infty(0,\tau)\right),$$

which is (5.14).

Step 4: Showing (5.15).

Adding the next order in the Taylor expansions in (5.5) yields

$$\begin{aligned}
&2^{-K}T^{-2HK} f(1,1,T)\\
&= \left(1 + \frac{2H}{T} + \frac{H(2H-1)}{T^2} + \frac{H(2H-1)(2H-2)}{3T^3} + o(T^{-3})\right)^K\\
&\quad - 2\left(1+\frac{H}{T} + \frac{H(2H-1)}{2T^2} + \frac{H(2H-1)(2H-2)}{6T^3} + o(T^{-3})\right)^K + 1\\
&= \frac{H^2K(K-1)}{T^2} + \frac{K(K-1)}{2}\left(2\cdot\frac{2H}{T}\cdot\frac{H(2H-1)}{T^2} - 4\cdot\frac{H}{T}\cdot\frac{H(2H-1)}{2T^2}\right)\\
&\quad + \frac{K(K-1)(K-2)}{6}\left(\frac{8H^3}{T^3} - 2\cdot\frac{H^3}{T^3}\right) + o(T^{-3})\\
&= \frac{H^2K(K-1)}{T^2} - \frac{H^2K(K-1)(1-HK)}{T^3} + o(T^{-3}), \qquad T\to\infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
&2^{-K}\left(Te^{-\tau}\right)^{-2HK}\left(e^{(2-2HK)\tau}f(1,1,T) - f(1,1,Te^{-\tau})\right)\\
&= \frac{H^2K(K-1)(1-HK)}{T^3}\left(e^{3\tau}-e^{2\tau}\right) + o\big(T^{-3}\left(e^{3\tau}-e^{2\tau}\right)\big), \qquad T\to\infty,
\end{aligned}$$

where the convergence of $o(T^{-3}(e^{3\tau} - e^{2\tau}))$ is uniform in $\tau \in (0, M]$. This implies that there exists $T_0 > 0$ independent of τ such that $e^{(2-2HK)\tau} f(1,1,T) - f(1,1,Te^{-\tau}) \ge 0$ for all $T \ge T_0$ and $\tau \in [0, M]$, as desired.

Step 5: Concluding (5.12) with (5.14) and Slepian's lemma.

Let $(U_\tau)_{\tau\geq 0}$ be a version with continuous sample paths of the Ornstein-Uhlenbeck process for $\lambda := 2 - HK$, independent of Y^T. Then, (5.14) and Proposition 2.1(a) lead to

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\tau\in[0,M]} \sigma_{\tau,T}^{-1} Y^T_{e^{s+\tau}} \leq 0\right)\\
&\geq \mathbb{P}\left(\sup_{\tau\in[0,M]} \left(\sqrt{1-\varepsilon_T}\, e^{-HK\tau} B^{HK,1}_{e^\tau} + \sqrt{\varepsilon_T}\, U_\tau\right) \leq 0\right)\\
&\geq \mathbb{P}\left(\sup_{\tau\in[0,M]} e^{-HK\tau} B^{HK,1}_{e^\tau} \leq -2\,\varepsilon_T^{1/4},\ \sup_{\tau\in[0,M]} U_\tau < \varepsilon_T^{-1/4}\right)\\
&\geq \mathbb{P}\left(\sup_{\tau\in[0,M]} e^{-HK\tau} B^{HK,1}_{e^\tau} \leq -2\,\varepsilon_T^{1/4}\right) - \mathbb{P}\left(\sup_{\tau\in[0,M]} U_\tau \geq \varepsilon_T^{-1/4}\right)
\end{aligned}$$

for all $M > 0$, $s \in [0, \log\log T]$ and T large enough such that $\varepsilon_T \leq 3/4$, thus in fact for all $s \geq 0$. Here, we used that if $e^{-HK\tau} B^{HK,1}_{e^\tau} \leq -2\,\varepsilon_T^{1/4}$, $U_\tau < \varepsilon_T^{-1/4}$, and $\varepsilon_T \leq 3/4$, then it holds $\sqrt{1-\varepsilon_T}\, e^{-HK\tau} B^{HK,1}_{e^\tau} + \sqrt{\varepsilon_T}\, U_\tau \leq (1 - 2\sqrt{1-\varepsilon_T})\varepsilon_T^{1/4} \leq 0$. Noting that $\sup_{\tau\in[0,M]} U_\tau$ is finite a.s. for any $M > 0$, this shows (5.12).

Final Step: Let $M \in (0, \log N)$. Considering (5.10), we deduce

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[1,T+N]} B^{H,K}_t \leq 0\right) &\geq c_T\, \mathbb{P}\left(\sup_{\tau\in[0,M]} \sigma_{\tau,T}^{-1} Y^T_{e^\tau} \leq 0\right) \cdot \mathbb{P}\left(\sup_{\tau\in[M,2M]} \sigma_{\tau,T}^{-1} Y^T_{e^\tau} \leq 0\right)\\
&\qquad \cdots \mathbb{P}\left(\sup_{\tau\in\left[\left(\lceil \frac{\log N}{M}\rceil -1\right)M,\lceil \frac{\log N}{M}\rceil\cdot M\right]} \sigma_{\tau,T}^{-1} Y^T_{e^\tau} \leq 0\right)\\
&\geq c_T \left(\mathbb{P}\left(\sup_{\tau\in[0,M]} e^{-HK\tau} B^{HK,1}_{e^\tau} \leq 0\right) - \xi_T^{(M)}\right)^{\lceil \frac{\log N}{M}\rceil},
\end{aligned}$$

where $(\xi_T^{(M)})_{T>0}$ is a suitable sequence converging to 0 for $T \to \infty$. Here, we used (5.6) together with Proposition 2.1(b) in the first and (5.12) $\lceil \frac{\log N}{M}\rceil$-times in the second step. Taking the logarithm, dividing by $\log N$ and letting $N \to \infty$ then gives

$$-\theta_B(H,K) \geq \frac{1}{M} \log\left(\mathbb{P}\left(\sup_{\tau\in[0,M]} e^{-HK\tau} B^{HK,1}_{e^\tau} \leq 0\right) - \xi_T^{(M)}\right).$$

By letting $T \to \infty$ and then $M \to \infty$, we conclude $-\theta_B(H,K) \geq -\theta_B(HK,1)$. $\square$

5.2 Fractional Ornstein-Uhlenbeck process

Recall that for $\lambda > 0$ and $H \in (0,1)$, the fractional Ornstein-Uhlenbeck process (fOU process) is given by

$$U_t^{H,\lambda} := \int_{-\infty}^{t} e^{-\lambda(t-s)}\,\mathrm{d}B_s^H, \qquad t \geq 0,$$

where B^H is a fractional Brownian motion (FBM) with Hurst parameter H. Further recall that for $H = 1/2$, this is the OU process, which equals in distribution the Lamperti transform of (scaled) Brownian motion (cf. (1.17)). For the Lamperti transform $Z^{H,\lambda}$ of (scaled) FBM, as defined in (1.19), it is a consequence of (1.3) and Corollary 2.6 that

$$\theta_0(Z^{H,\lambda}) = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{t\in[0,T]} Z_t^{H,\lambda} \leq 0 \right) = 2\lambda(1-H).$$

The fOU process for $H \neq 1/2$ behaves very differently regarding persistence. We show that, for $H > 1/2$, the persistence probabilities of $U^{H,\lambda}$ do not have a true exponential decay, i.e., that the persistence exponent equals zero.

Proposition 5.5. *For $\lambda > 0$, $H \in (1/2, 1)$ and $x \in \mathbb{R}$, it holds*

$$\theta_x(U^{H,\lambda}) = -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left(\sup_{t\in[0,T]} U_t^{H,\lambda} \leq x \right) = 0.$$

Proof. For $H \neq 1/2$, the auto-covariance function of $U^{H,\lambda}$ is given by

$$\begin{aligned}
\mathbb{E}\Big[U_0^{H,\lambda} U_t^{H,\lambda}\Big] &= e^{-\lambda t}\,\mathbb{E}\Big[\int_{-\infty}^{0} e^{\lambda u}\,\mathrm{d}B_u^H \Big(\int_{-\infty}^{0} e^{\lambda v}\,\mathrm{d}B_v^H + \int_0^t e^{\lambda v}\,\mathrm{d}B_v^H \Big)\Big] \\
&= e^{-\lambda t} \left(\mathbb{E}\left[\Big(\int_{-\infty}^{0} e^{\lambda u}\,\mathrm{d}B_u^H \Big)^2 \right] \right. \\
&\qquad\qquad \left. + H(2H-1) \int_{-\infty}^{0} e^{\lambda u} \int_0^t e^{\lambda v} (v-u)^{2H-2}\,\mathrm{d}v\,\mathrm{d}u \right), \; t \geq 0,
\end{aligned}$$

where we used [26, Lemma 2.1] in the second step. This function is non-negative for $H > 1/2$. Thus, Proposition 2.2 yields the existence of $\theta_x(U^{H,\lambda})$.

Further, [26, Theorem 2.3] states $\mathbb{E}[U_0^{H,\lambda} U_t^{H,\lambda}] \sim \lambda^{-2} H(2H-1) t^{2H-2}$ for $t \to \infty$ and $H \neq 1/2$. Thus, for $H > 1/2$,

$$\int_0^\infty \mathbb{E}\Big[U_0^{H,\lambda} U_t^{H,\lambda}\Big]\,\mathrm{d}t \geq \frac{1}{2}\lambda^{-2} H(2H-1) \int_{t_0}^\infty t^{2H-2}\,\mathrm{d}t = \infty.$$

Together with continuous sample paths, this implies $\theta_x(U^{H,\lambda}) = 0$ for $H > 1/2$ and every $x \in \mathbb{R}$, see [14, Lemma 3.2]. $\square$

Remark. For $H < 1/2$, the asymptotics $\mathbb{E}[U_0^{H,\lambda} U_t^{H,\lambda}] \sim \lambda^{-2} H(2H-1) t^{2H-2}$ for $t \to \infty$ imply that the auto-covariance function of $U^{H,\lambda}$ is *not* non-negative so that Proposition 2.2 cannot be applied in this case to deduce the existence of the persistence exponent.

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