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SPECIAL ISSUE ARTICLE

Asymptotic stabilization with group-wise sparse input based on control Lyapunov function approach

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Summary

This study proposes a novel stabilizing controller for nonlinear systems using group-wise sparse inputs. The input variables are divided into several groups. In the situations when the input constraints can be ignored, one input becomes active for each group at each moment. Our method improves energy efficiency, as sparse input vectors often reduce the standby power of inactive actuators. Large-scale systems, such as those consisting of multiple subsystems, often require the manipulation of multiple inputs simultaneously to be controlled. Our method can be applied to such systems due to the group-wise sparsity of the inputs. The proposed controller is based on the control Lyapunov function approach and includes Sontag's universal formula as a special case. The controllers designed in our method have best-effort property, which means even when a restriction for the decreasing rate of the Lyapunov function cannot be fulfilled, the controller minimizes the time derivative of the Lyapunov function within the input constraint. The effectiveness of the proposed method can be confirmed through simulations.

KEYWORDS:
Sparse input, control Lyapunov function, asymptotic stabilization

1 | INTRODUCTION

Many practical controlled systems have redundant inputs. These redundancies are used to make fault-tolerant systems, save energy, and utilize emergency actuators. Some actuators continuously consume a certain amount of energy, even when not in use, which is called the *standby power*. Hence, in redundant-input systems, stopping an inefficient actuator is an effective way of energy conservation. The selection of active actuators should depend on the state variable, and the selection mechanism must be systematically designed. When most actuators are deactivated, the input vector becomes *sparse*. A sparse vector is one with many zero-valued elements. Therefore, sparse input may be effective for saving energy.

In recent years, control methods with sparse input selection have been studied. In particular, the maximum hands-off control approach[1,2,3,4,5,6,7] and control allocation methods with 1-norm[8,9,10,11] were studied extensively. The controllers of these methods generate sparse input vectors, which can reduce the standby power of the actuators.

In the maximum-hands-off control approach, an optimal control problem with cost functional evaluating L_0-norm or CLOT (Combined L-One and Two) norm[6] for the finite or infinite horizon is solved. The solution for the finite horizon setting can be used for the model predictive control (MPC). A terminal constraint for the finite-horizon L_0 optimal control is expected to

[0]**Abbreviations:** CLF, control Lyapunov function; scp, small control property; MPC, model predictive control

contribute to the stabilization of the system under MPC. The maximum hands-off control can be used for both discrete- and continuous-time controls.

The control allocation technique determines the redundant inputs under the restriction on the time derivative of the state. The restriction varies over time and is calculated by other methods. One-norm optimization is often used to select redundant inputs in the control allocation because of its energy efficiency, as explained above. If the redundant inputs are calculated using one-norm optimization, the resulting input vector often becomes sparse. Input restrictions can be considered in the optimization procedure.

We previously proposed sparse-input stabilization of nonlinear systems using control Lyapunov functions (CLFs) [12]. This stabilization method is intermediate between the maximum-hands-off control approach and control allocation methods. Similar to control allocation methods, the CLF method uses myopic cost evaluation without future prediction. The CLF can be considered to be designed including future costs, such as the value function of optimal control. The time-derivative constraint of the state in the control allocation method can be too restrictive. In the CLF-based sparse control method, an inequality constraint on the time derivative of the CLF, rather than the state, is used. CLF-based approaches do not require solving Hamilton-Jacobi partial differential equations as well as complex two-point boundary-value problems, and this point is an advantage of this method over the maximum hands-off control methods.

In situations where input constraints are negligible, the previous method often selects only one nonzero input variable to control the system, at each moment. However, some systems require multiple input variables to be controlled, and the previously proposed CLF-based sparse-input control method often induces frequent switching of active inputs. This behavior is explained in the next section. This study resolves this problem by introducing a new concept of input grouping. Our objective is to choose one adequate input variable, which will be activated, for each group when the input constraints are inactive. Our method is based on our previous result [12] and extended to the problem settings with the grouped input. The key idea of this study is to introduce a new cost function, which is the square of the two-norm of the input-group costs, where each group cost is defined as a weighted 1-norm of the input group.

This paper is organized as follows: Section 2 explains the issues with of previous sparse input control methods when they are applied to large-scale systems and describes the problem setting. The control law for cases without input constraints is presented in section 3. The control law includes the Sontag's universal formula [13,14] as a special case. Section 4 extends the results of section 3 to cases with input constraints. Owing to the input constraints, the obtained control law is generally local, but it is shown to be a best-effort-type controller in terms of the time derivative of the CLF. Section 4 also introduces a lazy-switching algorithm to suppress the chattering phenomenon. It is also proposed a parameter selection method that maintains the load balance between input groups in Section 4. Simulations for a particular example are presented in section 5 to demonstrate the effectiveness of the proposed method. Section 6 summarizes this paper and presents a future plan for this study.

2 | MOTIVATION AND PROBLEM SETTINGS

2.1 | CLF-based Sparse Control Method Suitability for Large-Scale Systems

An advantage of the control method using CLF [13,14] over optimal control methods like the maximum hands-off control methods is that it does not require the solution of Hamilton-Jacobi partial differential equations as well as complicated two-point boundary-value problems. Although the difficulty of designing CLFs remains, several linear matrix inequality (LMI) methods can be used to design CLFs if the system trajectories are included in those for a linear differential inclusion expressing polytopic uncertainty (See Boyd et al. [15], for example). Moreover, for feedback-linearizable systems, the CLFs can be easily obtained. Thus, CLF-based sparse input control methods have excellent applicability to real systems.

However, previously proposed CLF-based sparse control method may inherently cause the chattering phenomenon in large-scale systems. For example, we consider a system with two weakly coupled unstable subsystems (Figure 1) as follows:

$$\dot{x}_1 = f_1(x_1; x_2) + g_1(x_1)u_1$$
$$\dot{x}_2 = f_2(x_2; x_1) + g_2(x_2)u_2.$$

Suppose that both state variable vectors x_1 and x_2 include the physical values to be regulated. Therefore, this system requires at least two inputs to be controlled.

The previously proposed control Lyapunov function (CLF) approach for sparse input control [12] generates an input with at most one nonzero element, when the state is near the origin. In this method, there is a restriction $\dot{V} \leq -W(x)$, where $V(x)$ is a CLF, which is defined in the next section, and $W(x)$ is a positive definite function derived from $V(x)$, $f_i(x)$, and $g_i(x)$. Because

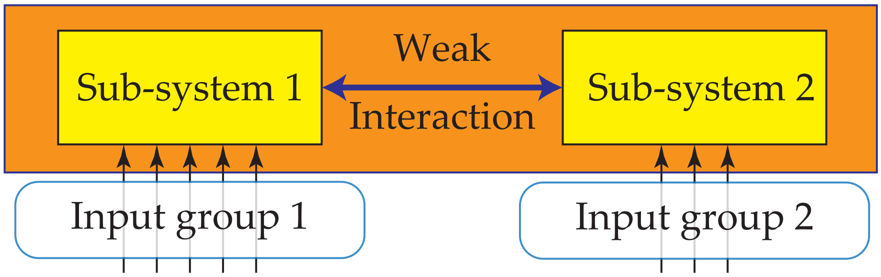

FIGURE 1 System consists of two subsystems.

the value of the function $W(x)$ becomes smaller for smaller state variables, a small input is allowed near the origin, and input restrictions can be ignored. Therefore, a weighted one-norm optimization produces a control input that has at most one non-zero element. The maximum hands-off control considering time-space sparsity[7] may have the same problem if the norm restriction is strict.

As the above system is difficult to be controlled by only one input, the CLF-based method may generate input signals with a chattering phenomenon. Although, the chattering phenomenon can be suppressed by the lazy-switching algorithm[12], a frequently input switching is still expected even when the lazy-switching algorithm is adopted. One important aim of sparse input control methods is the reduction in standby power of actuators for energy conservation. However, a frequently switching input may increase the energy consumption, because extra power may be required for actuator activation and deactivation. Consequently, the previously proposed CLF-based sparse control method may not work well for large-scale systems, which require the manipulations of multiple inputs to be controlled. Herein, an input element having a nonzero value is said to be *active*.

In this study, we propose an idea of input grouping, which allows multiple active inputs near the origin, to resolve this problem. The problem formulation for this study is described in the next subsection.

2.2 | Problem Settings

We consider a nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

as a controlled object to be stabilized, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input of the system, and $f(x)$ and $g(x)$ are smooth vector fields satisfying $f(0) = 0$. We assume that there exists a control Lyapunov function $V(x)$. A function satisfying the following conditions are called a CLF[13,14]:

$$\begin{aligned}
&\text{The sub-level set } \{x \mid V(x) \le a\} \text{ is compact for any } a > 0,\\
&V(0) = 0, \quad V(x) > 0\ (x \neq 0), \text{ and}\\
&L_f V(x) < 0 \text{ for } x \text{ such that } L_g V(x) = 0 \text{ and } x \neq 0,
\end{aligned}$$

where L_f and L_g are Lie derivatives defined by $L_f V = (\partial V/\partial x) f(x)$ and $L_g V = (\partial V/\partial x) g(x)$. Moreover, we assume that $V(x)$ has a small control property (scp)[13,14], i.e., there is a continuous control law $u = \alpha_c(x)$ that makes the value of $V(x)$ decrease near the origin and satisfies $\alpha_c(0) = 0$.

In this study, we divide the input variables into input groups as

$$\begin{aligned}
u &= (u_1^\top, \dots, u_N^\top)^\top,\\
u_i &= (u_{i,1}, \dots, u_{i,m_i})^\top \quad (i = 1, \dots, N),
\end{aligned}$$

where $m_1 + \cdots + m_N = m$. Let $g_{i,j}(x)$ denote the vector field corresponding to the input variable $u_{i,j}$. Therefore, $g(x)$ can be decomposed into N-groups as

$$\begin{aligned}
g(x) &= \left[g_1(x), \dots, g_N(x)\right],\\
g_i(x) &= \left[g_{i,1}(x), \dots, g_{i,m_i}\right] \quad (i = 1, \dots, N).
\end{aligned}$$

We suppose that the input variables in the same input group are responsible for similar actions to the system, and redundant inputs are collected into one group. When no input constraints exist, our objective is to choose a suitable input variable for each group for the stabilization of the system, while actuators for other inputs are kept inactive. Actuator deactivation reduces some standby power and contributes energy conservation.

To realize the group-wise sparsity, we propose a new input cost function, which is different from the weighted 1-norm, in the next section.

3 | STABILIZATION BY SPARSE INPUT WITHOUT INPUT CONSTRAINT

To obtain a stabilizing sparse input vector, we consider the following optimization problem:

$$\text{Find } u \text{ minimizing } J_0(u) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{m_i} k_{i,j}|u_{i,j}|\right)^2, \tag{2}$$

$$\text{subject to } \dot{V}(x,u) = L_f V(x) + L_g V(x)u \le -W(x), \tag{3}$$

where $k_{i,j}$ $(i = 1, \ldots, N;\ j = 1, \ldots, m_i)$ are positive constants; $W(x)$ is a positive-definite function determined later. The input cost function (2) is different from 1-norm. It is the square of the two-norm of the input-group costs, where each group cost is defined as a weighted 1-norm of the input group. Hence, the sparse property of each input group can be expected by the same mechanism as in 1-norm optimization. The input sparsity for the entire cost function (2) will be explained later with a simple example. Introducing this cost function is the key idea of this study.

The constraint (3) ensures the asymptotic stability because $\dot{V} \le 0\,(x \ne 0)$. If $L_f V(x)+W(x) \le 0$, then $u = 0$ is optimal. When $L_f V(x) + W(x) > 0$, the constraint (3) is active for the optimal solution. The conditions for the optimality can be expressed as

$$\left(\sum_{j'=1}^{m_i} k_{i,j'}|u_{i,j'}|\right) k_{i,j}\overline{\mathrm{sgn}}\,(u_{i,j}) + \mu \ell_{i,j}(x) = 0 \quad (i = 1, \ldots, N;\ j = 1, \ldots, m_i), \tag{4}$$

$$L_f V(x) + \ell(x)u \le -W(x), \quad \mu \ge 0, \tag{5}$$

$$\mu(L_f V(x) + \ell(x)u + W(x)) = 0, \tag{6}$$

where μ is a Lagrange multiplier and

$$\begin{aligned}
&\ell_{i,j}(x) = L_{g_{i,j}} V(x) \quad (i = 1, \ldots, N;\ j = 1, \ldots, m_i),\\
&\ell_i(x) = L_{g_i} V(x) = (\ell_{i,1}(x), \ldots, \ell_{i,m_i}(x)) \ \ (i = 1, \ldots, N),\\
&\ell(x) = L_g V(x) = (\ell_1(x), \ldots, \ell_N(x)),\\
&\overline{\mathrm{sgn}}(y) \begin{cases} = 1 & (y > 0)\\ \in [-1, 1] & (y = 0)\\ = -1 & (y < 0). \end{cases}
\end{aligned}$$

We define active indices

$$j_i^*(x) = \operatorname*{argmax}_j \frac{|\ell_{i,j}(x)|}{k_{i,j}} \quad (i = 1, \ldots, N), \tag{7}$$

which refer to active inputs when $\mu > 0$. If multiple indices maximize $|\ell_{i,j}(x)|/k_{i,j}$, then one is selected arbitrarily. A solution of the optimization problem can be expressed as

$$\begin{cases} u_{i,j_i^*} = -\mu \dfrac{\ell_{i,j_i^*}(x)}{k_{i,j_i^*}^2} \quad (i = 1, \ldots, N),\\ u_{i,j} = 0 \quad (i = 1, \ldots, N;\ j = 1, \ldots, m_i;\ j \ne j_i^*), \end{cases} \tag{8}$$

$$\mu = \begin{cases} \dfrac{L_f V(x) + W(x)}{\sum_{i=1}^{N}(\ell_{i,j_i^*}/k_{i,j_i^*})^2} & (L_f V(x) + W(x) > 0)\\ 0 & (L_f V(x) + W(x) \le 0), \end{cases} \tag{9}$$

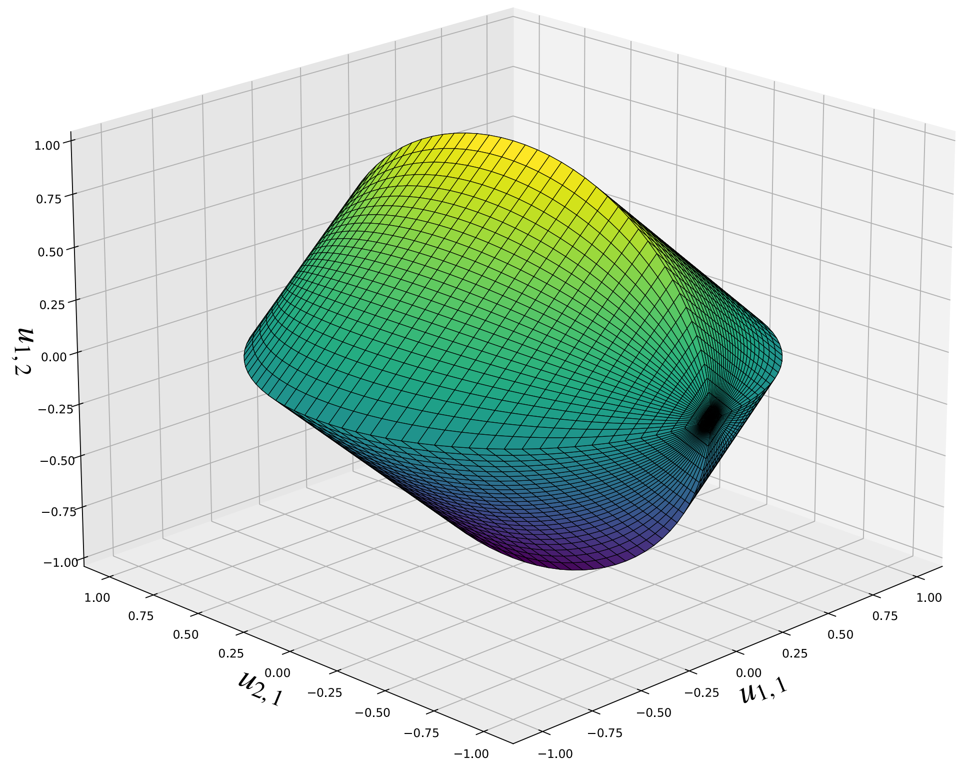

FIGURE 2 Level surface of a cost function.

which satisfy conditions (4), (5), and (6). It will be shown later that (3) does not define an empty set of u for a suitable choice of $W(x)$. Since the cost function (2) is a convex function and the region defined by (3) is a convex set, the stationary point (8) is always the optimal solution of the problem.

The above control input has a sparse property. The group-wise sparse property can be understood graphically. Consider the case of $J_0(u) = (|u_{1,1}| + |u_{1,2}|)^2 + u_{2,1}^2$. The level surface for $J_0(u) = 1$ is illustrated in Figure 2. The surface contacts with a plane at a point on the circles $\{u \mid u_{1,1}^2 + u_{2,1}^2 = 1, u_{1,2} = 0\}$ or $\{u \mid u_{1,2}^2 + u_{2,1}^2 = 1, u_{1,1} = 0\}$, which generates a group-wise sparse input vector. The level surfaces of $J(u)$ are similar to each other. Hence, the surface $J(u) = J^*$ touches the plane that indicates the boundary of the constraint (3), at a point on the circles $\{u \mid u_{1,1}^2 + u_{2,1}^2 = J^*, u_{1,2} = 0\}$ or $\{u \mid u_{1,2}^2 + u_{2,1}^2 = J^*, u_{1,1} = 0\}$, where J^* denotes the optimal value of (2) under (3). In the actual optimization result, as shown in (8), one of the input variables of the first group becomes zero, which is the same result as the discussion by Figure 2. On the other hand, since the second input group consists of only one input variable $u_{2,1}$, the non-zero $u_{2,1}$ will be selected in most cases.

We define

$$
\begin{aligned}
&s_{i,j}(x) = \ell_{i,j}(x)/k_{i,j} \quad (i = 1, \dots, N;\ j = 1, \dots, m_i),\\
&s_i(x) = (s_{i,1}(x), \dots, s_{i,m_i}(x)) \quad (i = 1, \dots, N),\\
&s(x) = (s_1(x), \dots, s_N(x)),\\
&\zeta(s) = (|s| + s)/2 = \max(s, 0),
\end{aligned}
$$

then (9) can be rewritten as

$$
\mu = \frac{\zeta(L_f V(x) + W(x))}{\|s(x)\|_{\infty 2}^2}, \tag{10}
$$

where

$$
\|s(x)\|_{\infty 2} = \left\| (\|s_1(x)\|_\infty, \dots, \|s_N(x)\|_\infty) \right\|_2.
$$

There is no solution when $s(x) = 0$ and $L_f V(x) + W(x) > 0$, because the constraint (3) defines an empty set for such x. Therefore, we should choose $W(x)$ to avoid this case. In this study, we determine $W(x)$ as

$$W(x) = \eta \sqrt{L_f V(x)^2 + c\|s(x)\|_{\infty 2}^4}, \tag{11}$$

where $0 < \eta \leq 1$ and $c > 0$. Obviously, $L_f V(x) + W(x) \leq 0$ when $s(x) = 0$, and we can avoid the cases with no solution by using $W(x)$ above. Moreover, from the scp of $V(x)$, we can show that $\mu(x)$ is locally bounded. The global asymptotical stability of the closed-loop system is clearly shown by the constraint (3).

Remark 1. When each input group has only one input element and $k_{1,1}, \dots, k_{N,1}$ are chosen as one, $\|s(x)\|_{\infty 2} = \|L_g V(x)\|_2$ holds. Therefore, in this case, by setting $c = 1$ and $\eta = 1$, the controller (8) with (10) coincides with Sontag's universal formula[13,14]. Note that $L_f V(x) + W(x)$ is nonnegative when $\eta = 1$.

The controller with $\eta < 1$ is locally bounded, since around the point $x \neq 0$ satisfying $s(x) = 0$, the numerator of (10) becomes zero. The local boundedness of the controller (8) with $\eta = 1$ can also be obtained in the same manner as that for Sontag's universal formula. Moreover, the local boundedness around the origin $x = 0$ can be derived from the scp as in the case of Sontag's controller.

4 | STABILIZATION BY SPARSE INPUT UNDER INPUT CONSTRAINTS

4.1 | Optimizing Problem with Input Constraints

In this section, we consider the stabilization problem using sparse input in the presence of input constraints

$$\underline{u}_{i,j} \leq u_{i,j} \leq \overline{u}_{i,j} \quad (i = 1, \dots, N;\ j = 1, \dots, m_i), \tag{12}$$

where $\underline{u}_{i,j} < 0$ and $\overline{u}_{i,j} > 0$ for $i = 1, \dots, N;\ j = 1, \dots, m_i$.

Under the input constraints (12), there may not be an input that decreases $V(x)$ for some x far from the origin. Hence, the stability property is generally not global. Herein, we consider a *best-effort* type controller, which reduces $\dot{V}$ as much as possible under input constraints (12) when no input exists to make $\dot{V}$ negative.

The proposed control input is derived as the solution of the following optimization problem:

$$\text{Find } u \text{ and } \xi \text{ minimizing } J_1(u, \xi) = \frac{1}{2} \sum_{i=1}^{N} \left(\sum_{j=1}^{m_i} \tilde{k}_{i,j}(x)|u_{i,j}| \right)^2 + B(x)(1 - \xi), \tag{13}$$

$$\text{subject to } \dot{V} = L_f V(x) + L_g V(x) u \leq -\xi W(x), \tag{14}$$

$$\underline{u}_{i,j} \leq u_{i,j} \leq \overline{u}_{i,j} \ (i = 1, \dots, N;\ j = 1, \dots, m_i), \text{ and } \xi \leq 1, \tag{15}$$

where

$$\tilde{k}_{i,j}(x) = \begin{cases} k_{i,j}\{1 - \exp(-\beta_i |s_{i,j}(x)|)\} & (\ell_{i,j} \neq 0) \\ k_{i,j} & (\ell_{i,j} = 0), \end{cases} \tag{16}$$

$$B(x) > \alpha W(x), \tag{17}$$

$$\alpha = \max_i \left(\beta_i \sum_{j=1}^{m_i} k_{i,j} \max(-\underline{u}_{i,j}, \overline{u}_{i,j}) \right). \tag{18}$$

This problem is always feasible because there exists a negative value of ξ such that the constraint (14) is satisfied for $u = 0$. Of course, the constraint (14) with a negative ξ does not guarantee the asymptotic stability of the system origin, and therefore the term $B(x)(1 - \xi)$ is added to the cost function (13) as a penalty for negative ξ. The asymptotic stability of the controlled system will be discussed later in this section. The coefficients β_i are positive values that depend on the state, and $\tilde{k}_{i,j} \approx k_{i,j}$ for sufficiently large β_i. The constraint (14) is weaker than (3) because the upper bound of $\dot{V}$ is multiplied by ξ (≤ 1). When the constraint (3) cannot be maintained, ξ becomes smaller than 1, and the constraint is relaxed. Note that the coefficient of inputs in the new cost function (13) is changed from $k_{i,j}$ to $\tilde{k}_{i,j}(x)$. However, this modification does not affect the choice of nonzero inputs when all input constraints are inactive because the order of magnitude of

$$\frac{|\ell_{i,j}(x)|}{\tilde{k}_{i,j}(x)} = \frac{|s_{i,j}(x)|}{1 - \exp(-\beta_i |s_{i,j}(x)|)}$$

is the same as that of $|s_{i,j}(x)|$. Through a simple calculation, we can show that $h(s) = s/(1 - \exp(-\beta s))$ $(\beta > 0)$ is an increasing function for positive s. Note that the optimization problem (13) with (14) and (15) can be formulated by a usual quadratic programming using some slack variables. Therefore, the optimization can be performed online.

If $L_{g_{i,j}} V(x) = 0$ for some index pair (i, j), the optimal input satisfies $u_{i,j} = 0$. We call such an input element a "trivial input component." We define the set of index pairs of non-trivial inputs as

$$I(x) = \{(i, j) \mid L_{g_{i,j}} V(x) \neq 0\}.$$

We can show that if $\dot{V} \leq -W(x)$ is possible by an input within the constraint (12), then ξ is maintained at 1.

Theorem 1. If the optimal solution includes an unsaturated nontrivial input component, then the optimal ξ becomes 1.

Proof. Let (i^*, j^*) be the index of an unsaturated nontrivial input component. By introducing Lagrange multipliers, we obtain an extended cost function

$$\begin{aligned} J_{1e} = {} & J_1(u, \xi) - \mu_1\{-L_f V(x) - L_g V(x) u - \xi W(x)\} - \mu_2(1 - \xi) \\ & - \mu_3^\top(|\hat{u}_{1,1}(u_{1,1})| - |u_{1,1}|, \ldots, |\hat{u}_{N,m_N}(u_{N,m_N})| - |u_{N,m_N}|) \end{aligned}$$

where μ_1, μ_2 and $\mu_3 = (\mu_{3,1,1}, \ldots, \mu_{3,N,m_n})^\top$ are Lagrange multipliers; and

$$\hat{u}_{N,m_N}(u_{N,m_N}) = \begin{cases} \overline{u}_{N,m_N} & (u_{N,m_N} \geq 0) \\ \underline{u}_{N,m_N} & (u_{N,m_N} < 0). \end{cases}$$

The necessary conditions for the optimality are

$$\frac{\partial J_{1e}}{\partial u_{i,j}} = \mu_1 L_{g_{i,j}} V(x) + (\tilde{k}_{i,j}(x) T_i(x, u_i) + \mu_{3,i,j}) \overline{\mathrm{sgn}}(u_{i,j}) = 0 \tag{19}$$

$$\frac{\partial J_{1e}}{\partial \xi} = -B(x) + \mu_1 W(x) + \mu_2 = 0 \tag{20}$$

with Karush–Kuhn–Tucker conditions

$$\begin{cases} -L_f V(x) - L_g V(x) u - \xi W(x) \geq 0 \\ \mu_1 \geq 0 \\ \mu_1\{-L_f V(x) - L_g V(x) u - \xi W(x)\} = 0 \end{cases} \tag{21}$$

$$\begin{cases} 1 - \xi \geq 0 \\ \mu_2 \geq 0 \\ \mu_2(1 - \xi) = 0 \end{cases} \tag{22}$$

$$\begin{cases} \underline{u}_{i,j} \leq u_{i,j} \leq \overline{u}_{i,j} \\ \mu_{3,i,j} \geq 0 \\ \mu_{3,i,j}(|\hat{u}_{i,j}(u_{i,j})| - |u_{i,j}|) = 0, \end{cases} \tag{23}$$

where

$$\overline{\mathrm{sgn}}(p) = \begin{cases} = 1 & (s > 0) \\ = [-1, 1] & (s = 0) \\ = -1 & (s < 0) \end{cases}$$

$$T_i(x, u_i) = \sum_{j=0}^{m_i} \tilde{k}_{i,j}(x) |u_{i,j}|.$$

From (23), obviously $\mu_{3,i^*,j^*} = 0$ holds. Thus from (19) and (21), we obtain

$$0 \leq \mu_1 \leq \frac{\tilde{k}_{i^*,j^*}(x) T_{i^*}(x, u_{i^*})}{|L_{g_{i^*,j^*}} V(x)|} = \frac{1 - \exp(-\beta_{i^*} s_{i^*,j^*})}{s_{i^*,j^*}} T_{i^*}(x, u_{i^*}) \leq \beta_{i^*} T_{i^*}(x, u_{i^*}) \leq \alpha. \tag{24}$$

In the above, since $\tilde{k}_{i,j} \leq k_{i,j}$ holds, $\beta_i T_i(x, u_i) \leq \alpha$ ($u \in U$, $i = 1, \dots, N$) is satisfied, where U is the set of u satisfying the input constraint (12). Note that $(i^*, j^*) \in I(x)$ implies that the denominators $|L_{g_{i^*,j^*}} V(x)|$ and s_{i^*,j^*} are nonzero. From (17), (20), and (24), the value of μ_2 can be evaluated as

$$\mu_2 = B(x) - \mu_1 W(x) = (B(x) - \alpha W(x)) + (\alpha - \mu_1) W(x) > 0.$$

From (22), we can conclude that $\xi = 1$. □

If it is impossible to make $\xi = 1$, the maximized ξ is chosen because (13) includes the evaluation term of ξ. Therefore, the proposed control input realizes a best-effort type controller. The best-effort property can be shown by the following Corollary.

Corollary 1. Under the control input, which is an optimal solution of (13) with (14) and (15), the value of $\dot{V}$ becomes

$$\dot{V} = \max \left[\min \left\{ L_f V(x), -W(x) \right\}, L_f V(x) + \min_{u' \in U} L_g V(x) u' \right], \tag{25}$$

where U is the set of u that satisfies the input constraint (12).

Proof. First, we consider the case where $\xi < 1$. From theorem 1, when $\xi < 1$, all non-trivial inputs $u_{i,j}$ ($(i, j) \in I(x)$) are saturated. Hence, if $\xi < 1$, the optimal input minimizes $L_g V(x) u$ in U, and $\dot{V} = L_f V(x) + \min_{u \in U} L_g V(x) u = -\xi W(x)$ holds. Obviously, $-\xi W(x) \leq W(x)$ and $L_f V(x) + \min_{u' \in U} L_g V(x) u' \leq L_f V(x)$ hold, and therefore (25) is satisfied when $\xi < 1$.

Next, we assume that $\xi = 1$. If $L_f V(x) + W(x) \leq 0$, zero input $u = 0$ is chosen as the optimal solution, and $\dot{V} = L_f V \leq -W(x)$ holds. Otherwise, if $L_f V(x) + W(x) > 0$, the constraint (14) becomes active, and $\dot{V} = -W(x)$ holds. Hence, when $\xi = 1$, the time derivative of $V(x)$ becomes $\dot{V} = \min \left\{ L_f V(x), -W(x) \right\}$. Since $\dot{V}(x, u) \geq L_f V(x) + \min_{u' \in U} L_g V(x) u'$ ($u \in U$) is always satisfied, (25) also holds when $\xi = 1$. □

The above corollary shows that as long as there is an input that makes $\dot{V}$ negative and satisfies the constraint (12), the proposed control law (13)–(15) will also make $\dot{V}$ negative. The guaranteed domain of attraction for the origin $x = 0$ of the closed system is obtained as a sublevel set

$$X_s = \left\{ x \,\middle|\, V(x) < \inf_{x' \in \overline{X}_s} V(x') \right\}$$

of $V(x)$, where

$$\overline{X}_s = \left\{ x \,\middle|\, L_f V(x) + \min_{u' \in U} L_g V(x) u' \geq 0,\ x \neq 0 \right\}.$$

In the proposed method, there is no conservatism problem because the input that preserves the properties of (25) is always selected. The gain of the controller is determined by the value of c included in the definition of $W(x)$ in (11). Evaluation of the robustness of the proposed control method is a future work. To analyze the robustness, we plan to show the inverse optimality of our controller by using a similar method to the authors' previous paper[16].

4.2 | Lazy-switching algorithm

The control input obtained as the solution of the optimization problem (13), (14), and (15) is discontinuous with respect to x and may cause the chattering phenomenon. As explained in subsection 2.1, frequent changing of the active actuator caused by the chattering phenomenon may increase energy consumption.

Discontinuous change of the input occurs when the index of the active input changes. To prevent the chattering phenomenon, once an input element becomes nonzero, the element must remain active for the immediate non-empty time period. In this section, we introduce a lazy-switching algorithm to suppress chattering, where the values of $k_{i,j}$ change over time. The lazy-switching algorithm is essentially the same as the mechanism proposed in our previous study[12] in the absence of input grouping; however, it will be briefly introduced to assist the reader.

A large $k_{i,j}$ implies a penalty for the activation of the input $u_{i,j}$. Increasing the values of $k_{i,j}$ for the current inactive inputs suppresses the activation of these inputs. We call this mechanism a *lazy-switching algorithm*, which can be described as

$$k_{i,j} = \begin{cases} \sigma k_{i,j,\text{default}} & \text{if } u_{i,j}(t-0) = 0 \\ k_{i,j,\text{default}} & \text{others} \end{cases} \quad (i = 1, \dots, N;\ j = 1, \dots, m_i)$$

where $k_{i,j,\text{default}}$ ($i = 1, \dots, N$; $j = 1, \dots, m_i$) are default values of $k_{i,j}$, $u_{i,j}(t-0)$ is the input just before the current time, and σ is a constant larger than 1. This mechanism can suppress frequent switching when the input constraints are inactive.

The best-effort property shown in Corollary 1 is a characteristic of the time derivative of the CLF at each time, and this property is preserved even if the value of $k_{i,j}$ is varied with respect to time. In other words, the value of $\dot{V}$ in (25) for each x is independent of the choice of $k_{i,j}$. Therefore, as long as it is possible to make the time derivative of CLF negative within the input constraints, the asymptotic stability will not be disturbed by the lazy-switching algorithm proposed here.

4.3 | Load balancing between input groups

Here, we propose a method to choose the values of β_i appropriately depending on the state.

For fixed parameters β_i, the time response of the inputs sometimes exhibits strange behaviors, an example of which will be shown in the next section. Such behaviors appear when $0 < \|s_i\|_\infty \ll \|s\|_\infty$ for some i. The inputs of ith group become large in this situation. This phenomenon occurs because we adopt the modified input weight coefficients $\tilde{k}_{i,j}$ in (16) instead of $k_{i,j}$. The modification does not affect the orders of $s_{i,j}$ within an input group; however, it breaks proper load balancing between the input groups. Solving this phenomenon has been the most difficult part of realizing group-wise sparse-input control. A mechanism adjusting the balance between input groups is necessary.

Hence, we propose a mechanism that selects the values of β_i depending on the state variables x. The proposed mechanism is described as follows:

$$\beta_i(x) = r\frac{\|s\|_\infty}{\|s_i\|_\infty} \quad (i = 1, \dots, N), \tag{26}$$

where r is a positive constant. To analyze the effect of this mechanism, suppose that any input constraints are inactive. Once the indices of the active inputs are determined, the cost function J_1 is equivalent to

$$\hat{J}_1 = \frac{1}{2}\sum_{i=1}^{N}\left\{k_{i,j_i^*}(1-\exp(-r\|s\|_\infty))|u_{i,j_i^*}|\right\}^2 = \frac{1-\exp(-r\|s\|_\infty)}{2}\sum_{i=1}^{N}k_{i,j_i^*}^2u_{i,j_i^*}^2.$$

Note that the coefficient $(1-\exp(-r\|s\|_\infty))/2$ is independent of the choice of u. The above cost $\hat{J}_1$ is essentially equivalent to the no-input-constraint case (2); and therefore, strange behavior does not occur.

For a small $\|s_i\|_\infty$, the value of β_i increases, and the value of α also has a large value. Thus, we propose an additional mechanism that makes the input group with the largest $\|s_i\|_\infty$ merge with other groups whose value of $\|s_i\|_\infty$ is less than $\varphi\|s\|_\infty$, where φ $(0 < \varphi \ll 1)$ is a threshold. The decision to merge and demerge is made at every time point. The resulting values of β_i have an upper bound r/φ, which prevents numerical instability in the optimization problem.

5 | EXAMPLE AND SIMULATIONS

To demonstrate the effectiveness of the proposed method, we performed a simulation for an example.

Consider the following nonlinear system:

$$\begin{aligned}\dot{x} &= f(x) + (g_{1,1}(x), g_{1,2}(x), g_{2,1}(x), g_{2,2}(x))u \\ &= \begin{pmatrix} x_2 \\ x_3 \\ 0.1\sin x_1 - x_2 + 0.1x_4 \\ x_5 \\ -0.2\sin x_2 - x_4 + 0.1x_5 \end{pmatrix} + \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix},\end{aligned}$$

where four dimensional input u includes two input groups $u_1 = (u_{1,1}, u_{1,2})^\top$ and $u_2 = (u_{2,1}, u_{2,2})^\top$. Each input variable has an input constraint $-5 \le u_{i,j} \le 5$. We give a CLF for this system as

$$V(x) = \frac{1}{2}x^\top\begin{bmatrix} 1 & 1/2 & 3/4 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 3/4 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}x.$$

The parameters in the controller design were chosen as $\eta = 0.8$, $c = 2$, $B(x) = 2\alpha W(x) + 0.01$, $\sigma = 3$, $k_{i,j,\mathrm{default}} = 1$, $r = 1$, and $\varphi = 0.0001$. We performed two simulations for the initial state $x(0) = (1, -2, 10, 1, 5)^\top$ One is for the proposed controller,

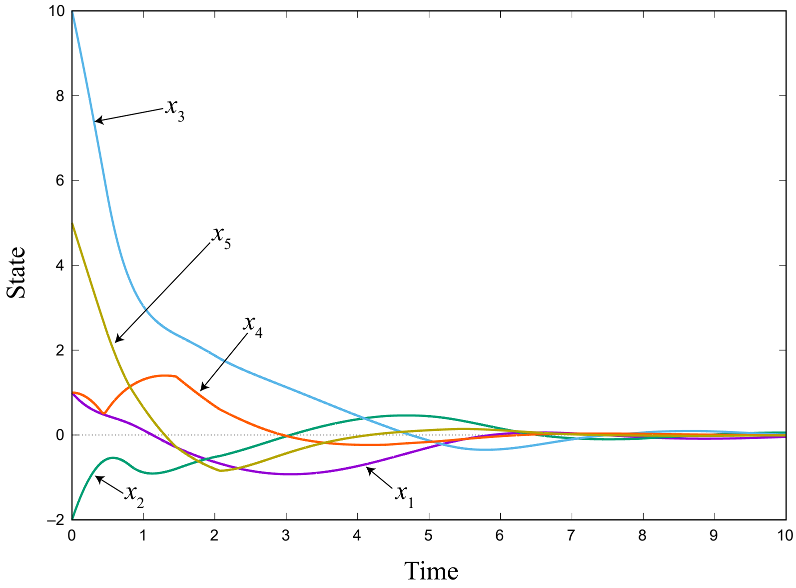

FIGURE 3 Time responses of state variables for the proposed controller.

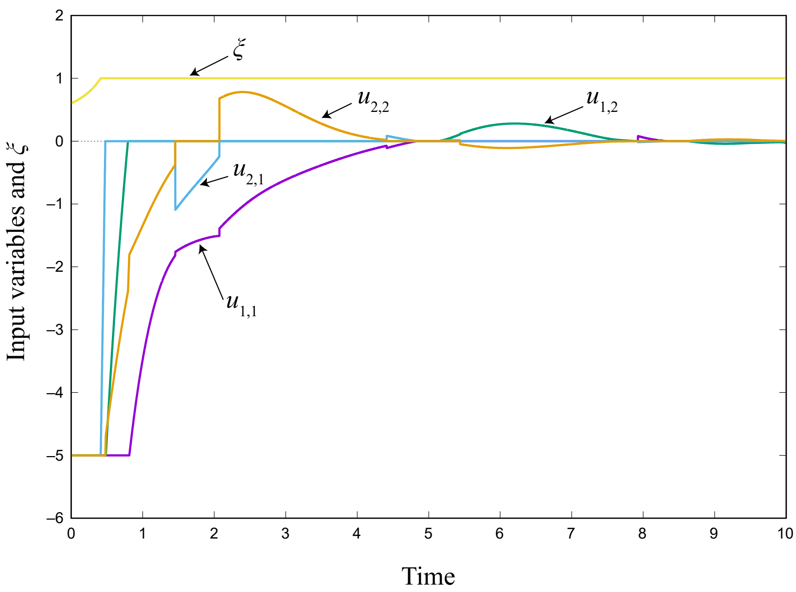

FIGURE 4 Time responses of input variables and ξ for the proposed controller.

and Figures 3 and 4 show the simulation result. For comparison, we performed another simulation where β_1 and β_2 are chosen as constants $\beta_1 = 100$ and $\beta_2 = 70$, and its result is shown in Figures 5 and 6.

Figures 3 and 5 show the time responses of the state variables. In both cases, the state variables converge to zero, as shown in Figures 3 and 5. Figures 4 and 6 display the time responses of the input variables and ξ. In both cases, we can see that the value of ξ is less than 1 while all input variables are saturated, as suggested by theorem 1. During the period when no input is saturated, at most one input element for each input group becomes nonzero. As explained in the previous section, setting β_i to constants may not result in proper load balancing between the input groups. In this case, the inputs belonging to a group can suddenly increase large, as shown in Figure 6. The proposed method resolves this point. The control inputs using β_i proposed in subsection 4.3 smoothly converges to zero, as shown in Figure 4, causing no chattering-like behavior.

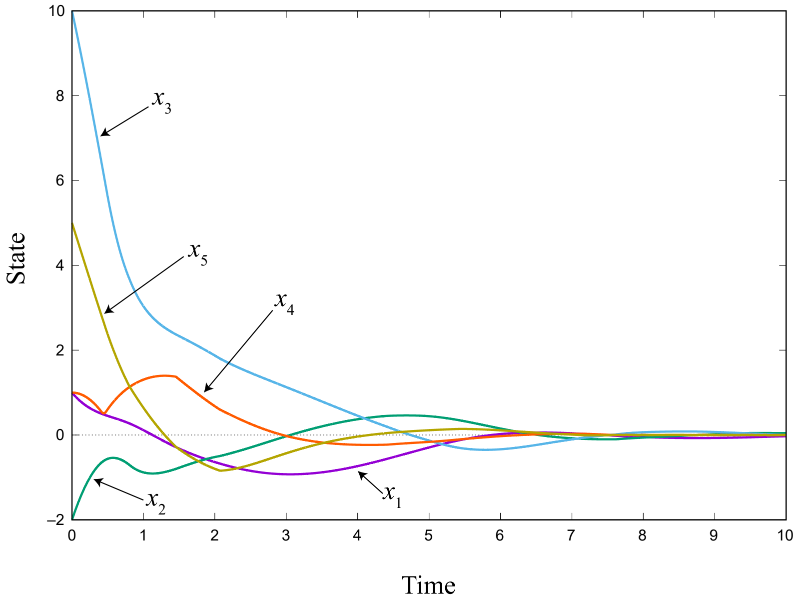

FIGURE 5 Time responses of state variables when the values of β_i are constants.

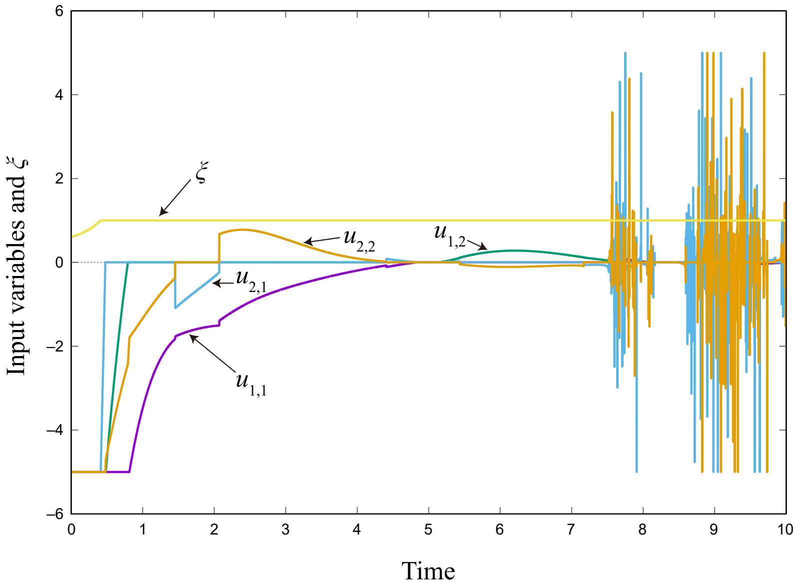

FIGURE 6 Time responses of input variables and ξ when the values of β_i are constants.

Application of the proposed controller to actual large-scale systems is a future task. The proposed controller can be applied to systems with redundant inputs. For example, in some chemical plants, pumps may be redundant for case of high loads or for system safety. The proposed controller automatically activates multiple pumps as needed.

6 | CONCLUSION

We proposed a new stabilizing control method for nonlinear systems using sparse input, where the input variables are divided into groups, and only one input variable in each group becomes active in most cases. The group-wise sparsity is realized by the new input cost function (2) or (13). The sparse input vector is effective for improving energy efficiency by reducing the standby power of inactive actuators. Group-wise sparsity is expected to be useful for the energy-saving control of large-scale system with multiple subsystems.

The controller design is based on the CLF approach. Sontag's universal formula is included in the proposed controllers as a special case. Under the input constraints, the proposed method derives a best-effort type controller. This means that when it is impossible to achieve $\dot{V} \leq -W(x)$, all inputs are saturated, and the controller makes $\dot{V}$ decrease as much as possible.

In future work, the proposed method will be applied to the control of redundant manipulators and more large-scale systems.

Author contributions

Yuh Yamashita presented the basic idea of this paper and wrote this research article. Kiminori Sakano discovered the defect in the constant β_i case described in subsection 4.3, and performed the simulations in section 5. Koichi Kobayashi contributed to the optimal calculations and proofread this paper.

Conflict of interest

The authors declare no potential conflict of interests.

Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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