FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 37, No 5 (2022), 943–950
https://doi.org/10.22190/FUMI220128065P
Original Scientific Paper

INCLUSION THEOREMS DOUBLE DEFERRED CESÀRO MEANS III

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Abstract. R. P. Agnew presents a definition for Deferred Cesàro mean. Using this definition R. P. Agnew presents inclusion theorems for the deferred and none Deferred Cesàro means. This paper is the part III of a series of papers that present extensions to the notion of double Deferred Cesàro means. Similar to the part I [11] and part II [12], this paper uses these definitions and the notion of regularity for four-dimensional matrices, to present a multidimensional inclusion theorem and a multidimensional equivalent theorem, which are multidimensional analog of R. P. Agnew's results in [2].
Key words: Cesàro mean, inclusion theorems, four-dimensional matrices.

1. Introduction

This paper is the part III of a series of papers that characterize the inclusion between Cesàro means and double Deferred Cesàro means. In the part I [11] and the part II [12] we presented the notion of double Deferred Cesàro means which is a multi-dimensional analog of Agnew's Deferred Cesàro means in [2]. By using these notions and results in [11] and [12], this paper continue the progression with the presentation of a multidimensional inclusion theorem and the following equivalent theorem:

Let α and β be two positive real constant, and let Ω_i and Λ_j be the greatest integers

Received January 28, 2022, accepted: September 05, 2022
Communicated by Dijana Mosić
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2010 *Mathematics Subject Classification.* Primary 40b05; Secondary 40C05

$\leq \alpha^i$ and $\leq \beta^j$, respectively, and chose $R > 1$ and $S > 1$ such that $\Omega_{i+1} \geq \Omega_i + 1$ and $\Lambda_{j+1} \geq \Lambda_j + 1$ whenever $i > R$ and $j > S$ respectively. If

$$p_m^{(\alpha)} = \begin{cases} \Omega_{R+1}, & \text{if} \quad n < \Omega_R + 1; \\ \Omega_i, & \text{if} \quad i > R+1 \text{ and } \Omega_{i-2} + 1 < m \leq \Omega_{i-1} + 1 \end{cases}$$

and

$$q_n^{(\beta)} = \begin{cases} \Lambda_{S+1}, & \text{if} \quad n < \Lambda_S + 1; \\ \Lambda_j, & \text{if} \quad j > S+1 \text{ and } \Lambda_{j-2} + 1 < m \leq \Lambda_{j-1} + 1 \end{cases}$$

then $D_{m-1,n-1,p_m^{(\alpha)},q_n^{(\beta)}}$ is equivalent to the double Cesàro mean.

2. Definitions, Notations, and Preliminary Results

The definitions, notations, and preliminary results are similar to those in Part I [11] which are restated here for the purpose of completeness.

Definition 2.1. [Pringsheim, 1900] A double sequence $x = \{x_{k,l}\}$ has a **Pringsheim limit** L (denoted by P-$\lim x = L$) provided that, given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. Such an $\{x\}$ is described more briefly as "P-convergent".

Definition 2.2. [Patterson, 2000] A double sequence $\{y\}$ is a **double subsequence** of $\{x\}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then {y} is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [14] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix A is said to be **RH-regular** if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

The assumption of bounded was added because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [3] and [14].

Theorem 2.1. *(Hamilton* [3]*, Robison* [14]*) The four-dimensional matrix A is RH-regular if and only if*

RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l;

RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3: $P\text{-}\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each* l;

RH_4: $P\text{-}\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each* k;

RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*

RH_6: *there exist finite positive integers* Δ *and* Γ *such that*

$$\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta.$$

The main goals of this paper includes the comparison of double Cesàro mean transformation

$$(C,1,1)_{m,n,k,l} := \begin{cases} \frac{1}{mn}, & \text{if} \quad k \leq m \text{ and } l \leq n \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

with the double Deferred Cesàro mean

$$D_{m,n,k,l} := \begin{cases} \frac{1}{(\alpha_m-\beta_m)(q_n-p_n)}, & \text{if} \quad \beta_m < k \leq \alpha_m \text{ and } p_n < l \leq q_n, \\ 0, & \text{if} \quad \text{otherwise} \end{cases}$$

where $[p_n]$ $[q_n]$ $[\alpha_m]$, and $[\beta_m]$ are sequences of nonnegative integers satisfying

$$\alpha_m < \beta_m, \text{ and } p_n < q_n \text{ for } m,n = 1,2\ldots; \tag{2.1}$$

and

$$\lim_m \beta_m = +\infty \text{ and } \lim_n q_n = +\infty. \tag{2.2}$$

Using these four dimensional transformations we shall present an equivalent theorem and the following inclusion theorem. *In order that double Cesàro transformation include a* $D_{m-1,n-1,p_m,q_n}$ *where* $\{p_m\}$ *and* $\{q_n\}$ *are monotonically increasing sequences and the characteristic sequences satisfy the condition*

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \alpha_3 \leq \beta_3 < \cdots$$

and

$$\gamma_1 \leq \Gamma_1 < \gamma_2 \leq \Gamma_2 < \gamma_3 \leq \Gamma_3 < \cdots$$

it is necessary and sufficient that $\left\{\frac{p_m}{m}\right\}$ *and* $\left\{\frac{q_n}{n}\right\}$ *be bounded for all* m *and* n.

3. Main Results

The following is the first of our main results.

Theorem 3.1. *In order that double Cesàro transformation include a $D_{m-1,n-1,p_m,q_n}$ where $\{p_m\}$ and $\{q_n\}$ are monotonically increasing sequences and the characteristic sequences satisfy the condition*

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \alpha_3 \leq \beta_3 < \cdots$$

and

$$\gamma_1 \leq \Gamma_1 < \gamma_2 \leq \Gamma_2 < \gamma_3 \leq \Gamma_3 < \cdots$$

it is necessary and sufficient that $\left\{\frac{p_m}{m}\right\}$ and $\left\{\frac{q_n}{n}\right\}$ be bounded for all m and n.

Proof. For each order pair (m, n) choose $i = i_m$ and $j = j_n$ such that

$$\alpha_{i-1} \leq m < \alpha_i$$

and

$$\gamma_{j-1} \leq n < \gamma_j.$$

Then for any double sequences $\{s_{k,l}\}$ we can rewrite the corresponding double Cesàro mean using the following sums which are denoted by A^1, A^2, A^3, $\ldots$, A^{i-1}, A^i, respectively

$$\begin{array}{ccccccc}
s_{1,1} & + & s_{1,2} & + & \cdots & + & s_{1,\beta_1} \\
s_{2,1} & + & s_{2,2} & + & \cdots & + & s_{2,\beta_1} \\
s_{3,1} & + & s_{3,2} & + & \cdots & + & s_{3,\beta_1} \\
\vdots & + & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_1,1} & + & s_{\Gamma_1,2} & + & \cdots & + & s_{\Gamma_1,\beta_1}
\end{array},$$

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_{1,\beta_1+1} & + & \cdots & + & s_{1,\beta_2} \\
 & & & & & & & & s_{2,\beta_1+1} & + & \cdots & + & s_{2,\beta_2} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & & & s_{\Gamma_1,\beta_1+1} & + & \cdots & + & s_{\Gamma_1,\beta_2} \\
s_{\Gamma_1+1,1} & + & s_{\Gamma_1+1,2} & + & \cdots & + & s_{\Gamma_1+1,\beta_1} & + & s_{\Gamma_1+1,\beta_1+1} & + & \cdots & + & s_{\Gamma_1+1,\beta_2} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_2,1} & + & s_{\Gamma_2,2} & + & \cdots & + & s_{\Gamma_2,\beta_1} & + & s_{\Gamma_2,\beta_1+1} & + & \cdots & + & s_{\Gamma_2,\beta_2}
\end{array},$$

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_{1,\beta_2+1} & + & \cdots & + & s_{1,\beta_3} \\
 & & & & & & & & s_{2,\beta_2+1} & + & \cdots & + & s_{2,\beta_3} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & & & s_{\Gamma_2,\beta_1+1} & + & \cdots & + & s_{\Gamma_2,\beta_2} \\
s_{\Gamma_2+1,1} & + & s_{\Gamma_2+1,2} & + & \cdots & + & s_{\Gamma_2+1,\beta_2} & + & s_{\Gamma_2+1,\beta_2+1} & + & \cdots & + & s_{\Gamma_2+1,\beta_3} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_3,1} & + & s_{\Gamma_3,2} & + & \cdots & + & s_{\Gamma_3,\beta_2} & + & s_{\Gamma_3,\beta_2+1} & + & \cdots & + & s_{\Gamma_3,\beta_3}
\end{array},$$

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_{1,\beta_3+1} & + & \cdots & + & s_{1,\beta_4} \\
 & & & & & & & & s_{2,\beta_3+1} & + & \cdots & + & s_{2,\beta_4} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & & & s_{\Gamma_3,\beta_1+1} & + & \cdots & + & s_{\Gamma_3,\beta_3} \\
s_{\Gamma_3+1,1} & + & s_{\Gamma_3+1,2} & + & \cdots & + & s_{\Gamma_3+1,\beta_2} & + & s_{\Gamma_3+1,\beta_3+1} & + & \cdots & + & s_{\Gamma_2+1,\beta_3} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_4,1} & + & s_{\Gamma_4,2} & + & \cdots & + & s_{\Gamma_4,\beta_3} & + & s_{\Gamma_4,\beta_3+1} & + & \cdots & + & s_{\Gamma_4,\beta_4}
\end{array},$$

we continue in a similar manner and obtain the following A^{i-1} and A^i

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_{1,\beta_{i-1}+1} & + & \cdots & + & s_{1,\beta_i} \\
 & & & & & & & & s_{2,\beta_{i-1}+1} & + & \cdots & + & s_{2,\beta_i} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & & & s_{\Gamma_2,\beta_1+1} & + & \cdots & + & s_{\Gamma_{j-1},\beta_i} \\
s_{\Gamma_{j-1}+1,1} & + & s_{\Gamma_{j-1}+1,2} & + & \cdots & + & s_{\Gamma_{j-1}+1,\beta_{i-1}} & + & s_{\Gamma_{j-1}+1,\beta_{i-1}+1} & + & \cdots & + & s_{\Gamma_{j-1}+1,\beta_i} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_j,1} & + & s_{\Gamma_j,2} & + & \cdots & + & s_{\Gamma_j,\beta_{i-1}} & + & s_{\Gamma_j,\beta_{i-1}+1} & + & \cdots & + & s_{\Gamma_j,\beta_i}
\end{array},$$

$$\begin{array}{ccccccccccccc}
 & & & & & & & & s_{1,\beta_i+1} & + & \cdots & + & s_{1,\beta_n} \\
 & & & & & & & & s_{2,\beta_i+1} & + & \cdots & + & s_{2,\beta_n} \\
 & & & & & & & & \vdots & + & \cdots & + & \vdots \\
 & & & & & & & & s_{\Gamma_j,\beta_1+1} & + & \cdots & + & s_{\Gamma_{j-1},\beta_n} \\
s_{\Gamma_j+1,1} & + & s_{\Gamma_j+1,2} & + & \cdots & + & s_{\Gamma_j+1,\beta_i} & + & s_{\Gamma_j+1,\beta_i+1} & + & \cdots & + & s_{\Gamma_j+1,\beta_n} \\
\vdots & + & \vdots & + & \cdots & + & \vdots & \vdots & \vdots & + & \cdots & + & \vdots \\
s_{\Gamma_m,1} & + & s_{\Gamma_m,2} & + & \cdots & + & s_{\Gamma_m,\beta_i} & + & s_{\Gamma_m,\beta_i+1} & + & \cdots & + & s_{\Gamma_m,\beta_n}
\end{array}.$$

Thus

$$M_{m,n} = \frac{A^1 + A^2 + \cdots + A^i}{mn}$$

which a sure us that the double Cesàro mean can be rewritten in terms of the double Deferred Cesàro mean as follows:

$$M_{m,n} = \left\{ \begin{array}{ccccc}
\frac{\Gamma_1\beta_1}{mn} D_{1,1} & + & \frac{(\Gamma_2-\Gamma_1)(\beta_2-\beta_1)}{mn} D_{\Gamma_1+1,\beta_1+1} & + & \frac{(\Gamma_3-\Gamma_2)(\beta_3-\beta_2)}{mn} D_{\Gamma_2+1,\beta_2+1} \\
\vdots & + & \vdots & + & \vdots \\
+ & + & \frac{(\Gamma_j-\Gamma_{j-1})(\beta_i-\beta_{i-1})}{mn} D_{\Gamma_{j-1}+1,\beta_{i-1}+1} & + & \frac{(m-\Gamma_j)(n-\beta_i)}{mn} D_{\Gamma_j+1,\beta_i+1}
\end{array} \right. .$$

The above rewriting of the double Deferred Cesàro mean in terms of double Cesàro mean grants us the following RH_1, RH_3, RH_4, RH_5, and RH_6, and condition

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| = 1$$

holds only if $\left\{\frac{p_m}{m}\right\}$ and $\left\{\frac{q_n}{n}\right\}$ be bounded for all m and n. Thus the transformation is RH-regular when and only if $\left\{\frac{p_m}{m}\right\}$ and $\left\{\frac{q_n}{n}\right\}$ be bounded for all m and n.

The final part of this theorem is establish by showing that each D's satisfy the hypotheses of this theorem and has an inverse. To establish that the D's has inverse it is sufficient to show that the system of equations

$$E_{m,n} = \left\{ \begin{array}{cccccccc}
s_{m,n} & + & s_{m,n+1} & + & \cdots & + & s_{m,q_n} & + \\
s_{m+1,n} & + & s_{m+1,n+11} & + & \cdots & + & s_{m+1,q_n} & + \\
 & + & \vdots & + & \ldots & + & \vdots & + \\
s_{p_m,n} & + & s_{p_m,n+1} & + & \cdots & + & s_{p_m,q_n} &
\end{array} \right\}$$

where $m, n = 1, 2, 3, \ldots$ has corresponding to each double sequence $\{E_{m,n}\}$ of constant has a unique solution whose elements are $\{s_{m,n}\}$. For each ordered pair (m, n) with the corresponding elements $m \neq \alpha_1, \alpha_2, \alpha_3, \ldots$ and $n \neq \gamma_1, \gamma_2, \gamma_3, \ldots$ chosen such that $i = i_m$ and $j = j_n$ with

$$\alpha_{i-1} < m < \alpha_i$$

and

$$\gamma_{j-1} < n < \gamma_j.$$

Now with the properties of $\{p_m\}$, $\{q_n\}$, and the following definition of $E_{m,n}$ and $E_{m+1,n+1}$

$$E_{m,n} = \left\{ \begin{array}{ccccccccc} s_{m,n} & + & s_{m,n+1} & + & \cdots & + & s_{m,\beta_j} & + \\ s_{m+1,n} & + & s_{m+1,n+1} & + & \cdots & + & s_{m+1,\beta_j} & + \\ & + & \vdots & + & \ldots & + & \vdots & + \\ s_{\alpha_i,n} & + & s_{\alpha_i,n+1} & + & \cdots & + & s_{\alpha_i,\beta_j} & \end{array} \right\}$$

and

$$E_{m+1,n+1} = \left\{ \begin{array}{ccccccccc} s_{m+1,n+1} & + & s_{m+1,n+2} & + & \cdots & + & s_{m+1,\beta_j} & + \\ s_{m+2,n+1} & + & s_{m+2,n+2} & + & \cdots & + & s_{m+2,\beta_j} & + \\ & + & \vdots & + & \ldots & + & \vdots & + \\ s_{\alpha_i,n+1} & + & s_{\alpha_i,n+2} & + & \cdots & + & s_{\alpha_i,\beta_j} & \end{array} \right\}$$

we are granted

$$s_{m,n} + s_{m,n+1} + \cdots + s_{m,\beta_j} + s_{m+1,n} + s_{m+2,n} + \cdots + s_{\alpha_i,n} = E_{m,n} - E_{m+1,n+1}$$

where $m \neq \alpha_1, \alpha_2, \alpha_3, \ldots$ and $n \neq \gamma_1, \gamma_2, \gamma_3, \ldots$ Therefore for each ordered pair (r, s) we obtain the following:

$$E_{\alpha_r,\gamma_s} = \left\{ \begin{array}{ccccccccc} s_{\alpha_r,\gamma_s} & + & s_{\alpha_r,\gamma_s+1} & + & \cdots & + & s_{\alpha_r,\Gamma_s} & + \\ s_{\alpha_r+1,\gamma_s} & + & s_{\alpha_r+1,\gamma_s+1} & + & \cdots & + & s_{\alpha_r+1,\Gamma_s} & + \\ & + & \vdots & + & \ldots & + & \vdots & + \\ s_{\beta_r,\gamma_s} & + & s_{\beta_r,\gamma_s+1} & + & \cdots & + & s_{\beta_r,\Gamma_s} & \end{array} \right\}$$

and since $\beta_r < \alpha_{r+1}$ and $\Gamma_r < \gamma_{r+1}$ except for the terms in the first column and first row. Also note that

$$s_{\alpha_r,\gamma_s} + s_{\alpha_r,\gamma_s+1} + \cdots + s_{\alpha_r,\Gamma_s} + s_{\alpha_r+1,\gamma_s} + \cdots + s_{\beta_r,\gamma_s} = E_{\alpha_r,\gamma_s} - E_{\alpha_r+1,\gamma_s+1}.$$

Now since the $s_{\alpha_r,\gamma_s} + s_{\alpha_r,\gamma_s+1} + \cdots + s_{\alpha_r,\Gamma_s} + s_{\alpha_r+1,\gamma_s} + \cdots + s_{\beta_r,\gamma_s}$ are uniquely determine in terms of the elements of $\{E_{m,n}\}$ an inverse exists. Thus the existence of a inverse of D is proven. $\square$

We will now establish equivalency between double Cesàro mean and special double Deferred Cesàro means.

Theorem 3.2. *Let α and β be two positive real constant, and let Ω_i and Λ_j be the greatest integers $\leq \alpha^i$ and $\leq \beta^j$, respectively, and chose $R > 1$ and $S > 1$ such that $\Omega_{i+1} \geq \Omega_i + 1$ and $\Lambda_{j+1} \geq \Lambda_j + 1$ whenever $i > R$ and $j > S$ respectively. If*

$$p_m^{(\alpha)} = \begin{cases} \Omega_{R+1}, & \text{if} \quad n < \Omega_R + 1; \\ \Omega_i, & \text{if} \quad i > R+1 \text{ and } \Omega_{i-2} + 1 < m \leq \Omega_{i-1} + 1 \end{cases}$$

and

$$q_n^{(\beta)} = \begin{cases} \Lambda_{S+1}, & \text{if} \quad n < \Lambda_S + 1; \\ \Lambda_j, & \text{if} \quad j > S+1 \text{ and } \Lambda_{j-2} + 1 < m \leq \Lambda_{j-1} + 1 \end{cases}.$$

then $D_{m-1,n-1,p_m^{(\alpha)},q_n^{(\beta)}}$ is equivalent to the double Cesàro mean.

Proof. It is clear that $p_m^{(\alpha)}$ and $q_n^{(\beta)}$ monotonically increasing sequence and the sequences $\{p_m^{(\alpha)}\}$ and $\{q_n^{(\beta)}\}$ are given by $\alpha_i = \Omega_{R+i-1} + 1$, $\beta_i = \Omega_{R+i}$ and $\gamma_j = \Lambda_{S+j-1} + 1$, $\Gamma_i = \Lambda_{S+j}$, respectively. Thus using the notation from Theorem 3.1 we are granted the following

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \alpha_3 \leq \beta_3 < \cdots$$

and

$$\gamma_1 \leq \Gamma_1 < \gamma_2 \leq \Gamma_2 < \gamma_3 \leq \Gamma_3 < \cdots.$$

It is also clear that $\left\{\frac{p_m^{(\alpha)}}{m}\right\}$ and $\left\{\frac{q_n^{(\beta)}}{n}\right\}$ are bounded for all m and n. Thus the double Deferred Cesàro means are contain in the double Cesàro means. Now let us consider the final part. It is also clear that $\left\{\frac{m-1}{p_m - m + 1}\right\}$ and $\left\{\frac{n-1}{q_n - n + 1}\right\}$ are bounded double sequences. Thus Theorem 3.1 of [11] insure that double Cesàro means are contain in the double Deferred Cesàro means. Therefore double Cesàro means and the double Deferred Cesàro means are equivalent. □

If α and β are two positive integers then the above equivalent theorem can be stated as follows:

Theorem 3.3. *If α and β be two positive integers and*

$$p_m = \begin{cases} \alpha^2, & \text{if} \quad m < \alpha + 1; \\ \alpha^i, & \text{if} \quad i > 2 \text{ and } \alpha_{i-2} + 1 < m \leq \alpha_{i-1} + 1 \end{cases}$$

and

$$q_n = \begin{cases} \beta^2, & \text{if} \quad n < \beta + 1; \\ \beta^j, & \text{if} \quad j > 2 \text{ and } \beta_{j-2} + 1 < n \leq \beta_{j-1} + 1 \end{cases}$$

then $D_{m-1,n-1,p_m,q_n}$ is equivalent to the double Cesàro mean.

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