AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **55**, 112201 (2014); doi: 10.1063/1.4901009
View online: http://dx.doi.org/10.1063/1.4901009
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/55/11?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **55**, 112201 (2014)

Decoherence free subspaces of a quantum Markov semigroup

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(Received 16 August 2013; accepted 22 October 2014; published online 10 November 2014)

We give a full characterisation of decoherence free subspaces of a given quantum Markov semigroup with generator in a generalised Lindbald form which is valid also for infinite-dimensional systems. Our results, extending those available in the literature concerning finite-dimensional systems, are illustrated by some examples. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4901009]

I. INTRODUCTION

Decoherence occurs when a quantum system interacts with its environment in an irreversible way. Decoherence and noise (Refs. 3, 16, 17, 23, 24, and 26 and references therein) typically affect quantum features of a state over its time evolution, however it may be possible to find states with a unitary evolution in some of the "good" portion of a system.

Two main approaches to decoherence of open quantum systems have been proposed in the literature; both are based on quantum Markov semigroups (QMS).

Blanchard and Olkiewicz,[2] starting from an algebraic setting, defined environment induced decoherence and found many physical models where the system algebra decomposes as the direct sum of two pieces: a subalgebra, called the decoherence-free algebra, where the semigroup acts homomorphically, a Banach subspace where the semigroup action is purely dissipative (see, e.g., Refs. 3 and 7 and the references therein) and vanishing as time tends to infinity. The decoherence-free subalgebra was later characterised in Refs. 10 and 13 as the commutant (or generalised commutant for unbounded operators) of certain families of operators arising form the GKSL (Gorini-Kossakowski-Sudarshan-Lindblad) representation of the generator.

In the approach to decoherence proposed by Lidar *et al.*[17,18] registers of a quantum computer are modeled by a quantum open system on a finite-dimensional Hilbert space h. The time evolution of states is described by a semigroup $\mathcal{T}_*$ on the Banach space of trace class operators on h which is the predual of a QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$, the algebra of all bounded operators on h.

A subspace h_{f} of h is *decoherence-free* if the time evolution of states ω supported in h_{f} is given by $\omega \to \mathrm{e}^{-\mathrm{i}tK}\omega\mathrm{e}^{\mathrm{i}tK}$ for some self-adjoint operator K on h_{f}.

Decoherence-free subspaces were identified in Ref. 18 (see also Ref. 25) under some physical (somewhat implicit) assumptions, we refer to Ref. 17 for an introduction to the theory of decoherence-free subspaces with a lot of examples and applications to protection of quantum information.

The papers of Lidar *et al.*,[17,18] however, are concerned only with *finite-dimensional* systems and focus on explicit physical models. Moreover, the method essentially depends on the choice of an orthonormal basis at the outset. This basis is determined by the spectral analysis of the

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coefficients of the GKSL generator of the quantum Markov semigroup. As a result, this method cannot be extended to infinite dimensions, or to the case of continuous spectra and unbounded coefficients of the GKSL-generator. In this paper, we look at the decoherence-free subspace issue from a mathematical point of view and study the following problem: given a quantum Markov semigroup on the algebra $\mathcal{B}(\mathsf{h})$ with generator represented in a generalised GKSL form, identify its decoherence-free subspaces. Our contribution comes from a general, basis independent, algebraic and operator-theoretic approach inspired by our previous work characterising the decoherence-free subalgebra,[10,13] developed independently of Lidar's[17,18] research. Our strategy is to reduce a problem on a QMS or, equivalently, on its generator, which is a map on operators, to a simpler problem on operators on h appearing in the GKSL representation of the generator. As a result, we give a full characterisation of decoherence-free subspaces for norm-continuous QMS on $\mathcal{B}(\mathsf{h})$ in terms of eigenspaces of operators on h for a possibly *infinite dimensional* Hilbert space h (Theorem 4). Moreover, under some natural assumptions on domains of unbounded operators involved, we extend this characterisation to weak*-continuous QMS with generator represented in a generalised GKSL form (Theorem 8). The key step allowing us to cope with domain problems is Proposition 7 where, applying our previous results[12] on the characterisation of subharmonic projections for QMS with unbounded generators, we establish a relationship between the domain of the dissipative operator arising in a GKSL representation of the generator and the self-adjoint operator K associated with a decoherence-free subspace h_f. Indeed, as we prove in Lemma 5, the orthogonal projection onto a decoherence-free subspace is subharmonic and, therefore, determines an invariant subspace for the operators G, L_ℓ in a GKSL representation of the generator. In this way, we can immediately identify a common essential domain for the self-adjoint K on h_f and the restriction to h_f of the dissipative operator G, L_ℓ.

An important feature of our analysis, is its validity for semigroup generators defined by operators with arbitrary (not only pure point) spectra (see, Sec. VI).

The structure of the paper is as follows. Section II contains the definition of decoherence-free subspaces and some preliminary remarks. Our characterisation of decoherence-free subspaces for norm-continuous QMSs (Theorem 4) is presented in Sec. III and illustrated by a couple of examples in Sec. IV. Weakly* continuous QMS are considered in Sec. V, our main result (Theorem 8) is proved and an application to a system with a Boson Fock space is discussed in Sec. VI. Our conclusions are collected in Sec. VII.

II. DECOHERENCE-FREE SUBSPACES

Let h be a complex separable Hilbert space and let $\mathcal{B}(\mathsf{h})$ be the von Neumann algebra of all bounded operators on h. A QMS on h is a weak* continuous family $(\mathcal{T}_t)_{t\geq 0}$ of completely positive, normal, identity preserving linear maps on $\mathcal{B}(\mathsf{h})$. The predual semigroup on trace-class operators on h will be denoted by $\mathcal{T}_*$.

A *state* ω on $\mathcal{B}(\mathsf{h})$ is a positive, trace-one, operator on h. A normal linear functional on $\mathcal{B}(\mathsf{h})$ will be identified with its density, which is a trace-class operator on h.

The support supp(ω) of ω is the closed subspace of h generated by eigenvectors with strictly positive eigenvalues. Due to dissipativity, the support of states $\mathcal{T}_{*t}(\omega)$ usually spreads over a wide range (see, e.g., Ref. 14, Sec. 6) and the initial state ω loses its quantum features (see Refs. 2, 6, 7, 10, 13, 19, 23, 24, and 26 for a dual approach based on observables rather than states and references therein).

Definition 1. A subspace h_f of h is called decoherence-free(DF) if there exists a self-adjoint operator K on h_f such that for all state ω with support in h_f we have

$$\mathcal{T}_{*t}(\omega) = \mathrm{e}^{-\mathrm{i}tK}\,\omega\,\mathrm{e}^{\mathrm{i}tK} \tag{1}$$

for all $t \geq 0$.

Note that a self-adjoint operator K on h_f can always be extended to the whole Hilbert space h, therefore DF subspaces could be defined in an equivalent way with a self-adjoint operator K on h leaving the subspace h_f invariant. In a more precise way, for an unbounded K, this means that

$e^{-itK}(\mathsf{h}_\mathsf{f}) \subseteq \mathsf{h}_\mathsf{f}$ for all $t \in \mathbb{R}$. Moreover, note that a DF subspace is necessarily closed with respect to the norm topology of h.

Lemma 2. If h_f is a DF subspace and K, K' are two self-adjoint operators on h satisfying (1) for all state $\omega$with support in h_f, then there exists a real constant c such that $K'u = Ku + cu$ for all $u \in \mathsf{h}_\mathsf{f}$.

Proof. If K and K' are two self-adjoint operators satisfying (1), then for all non-zero $u \in \mathsf{h}_\mathsf{f}$, we have $|e^{-itK}u\rangle\langle e^{itK}u| = |e^{-itK'}u\rangle\langle e^{itK'}u|$. It follows that there exists a complex constant $z_t(u)$ such that $e^{-itK'}u = z_t(u)e^{-itK}u$ and, since both e^{-itK} and $e^{-itK'}$ are unitaries, $|z_t(u)| = 1$. Clearly, for all $\lambda \in \mathbb{C}$ we have

$$\lambda z_t(u)e^{-itK}u = e^{-itK'}(\lambda u) = z_t(\lambda u)e^{-itK}(\lambda u) = \lambda z_t(\lambda u)e^{-itK}u$$

so that $z_t(u) = z_t(\lambda u)$ for all non-zero $u \in \mathsf{h}_\mathsf{f}$. In addition, if $u, v \in \mathsf{h}_\mathsf{f}$ are linearly independent, we have

$$\begin{aligned} z_t(u)e^{-itK}u + z_t(v)e^{-itK}v &= e^{-itK'}(u+v) = z_t(u+v)e^{-itK}(u+v) \\ &= z_t(u+v)e^{-itK}u + z_t(u+v)e^{-itK}v \end{aligned}$$

and so, by the linear independence of $e^{-itK}u$ and $e^{-itK}v$, constants $z_t(u)$ turn out to be independent of u. Dropping u, we also have

$$z_{t+s}e^{itK}e^{isK}u = z_{t+s}e^{i(t+s)K}u = e^{i(t+s)K'}u = e^{itK'}e^{isK'}u = z_s e^{itK'}e^{isK}u = z_s z_t\, e^{itK}e^{isK}u,$$

namely, $z_{t+s} = z_t z_s$ for all $s, t \in \mathbb{R}$. Since the map $t \to z_t$ is continuous, by a well-known fact on multiplicative functions, there exists a real constant c such that $z_t = e^{-ict}$. The conclusion is now immediate. □

From now on, we will call a self-adjoint operator K *associated with a* DF *subspace* h_f, if (1) holds for all state ω with support in h_f.

Note that, if h_f is a DF subspace with associated self-adjoint operator K, by the weak* density in $\mathcal{B}(\mathsf{h}_\mathsf{f})$ of trace class operators on h_f, the predual semigroup $\mathcal{T}_*$ can be extended to the subalgebra $\mathcal{B}(\mathsf{h}_\mathsf{f})$ and its action on $\mathcal{B}(\mathsf{h}_\mathsf{f})$ is also given by (1).

III. NORM-CONTINUOUS QMS

In this section, we consider norm-continuous quantum Markov semigroups. The generator can be represented in the well-known GKSL form

$$\mathcal{L}(x) = i[H, x] + \frac{1}{2}\sum_{\ell\geq 1}\left(-L_\ell^* L_\ell x + 2L_\ell^* x L_\ell - x L_\ell^* L_\ell\right), \tag{2}$$

where (see, e.g., Ref. 21, Theorem 30.16, p. 271) $L_\ell, H \in \mathcal{B}(\mathsf{h})$ with H self-adjoint, $(L_\ell)_{\ell\geq 1}$ is a finite or infinite sequence and the series $\sum_{\ell\geq 1} L_\ell^* L_\ell$ is strongly convergent. Writing $G = -\frac{1}{2}\sum_{\ell\geq 1} L_\ell^* L_\ell - iH$ we also have

$$\mathcal{L}(x) = G^* x + \sum_{\ell\geq 1} L_\ell^* x L_\ell + xG. \tag{3}$$

Recall that the operators $L_\ell, H \in \mathcal{B}(\mathsf{h})$ in a GKSL representation of $\mathcal{L}$ are not unique, we may, for instance, translate each L_ℓ by adding multiples $z_\ell \mathbb{1}$ of the identity operator $\mathbb{1}$, with $\sum_\ell |z_\ell|^2 < \infty$. In this way, we obtain another GKSL representation of $\mathcal{L}$ with $L'_\ell = L_\ell + z_\ell \mathbb{1}$ and $H' = H + (2i)^{-1}\sum_{\ell\geq 1}(\bar{z}_\ell L_\ell - z_\ell L_\ell^*)$. We refer to Ref. 21, pp. 272–273 for a detailed discussion on this subject.

The operator K in Definition 1 in this section will be assumed to be bounded.

First, we prove the following result.

Proposition 3. If h_f *is a decoherence-free subspace and* K *is a self-adjoint operator associated with* h_f*, then there exists complex numbers* λ_ℓ*and a real number* r *such that* $L_\ell u = \lambda_\ell u$, $\sum_{\ell\geq 1}|\lambda_\ell|^2 < \infty$ *and*$(G + \mathrm{i}K)u = -(\frac{1}{2}\sum_{\ell\geq 1}|\lambda_\ell|^2 + \mathrm{i}r)u$*for all* $u \in \mathsf{h}_\mathsf{f}$.

Proof. By the well-known polarisation identity (1) also holds for $\omega = |u\rangle\langle v|$ with $u, v \in \mathsf{h}_\mathsf{f}$. Differentiating we have

$$-\mathrm{i}[K, |u\rangle\langle v|] = G|u\rangle\langle v| + \sum_\ell |L_\ell u\rangle\langle L_\ell v| + |u\rangle\langle v|G^*. \tag{4}$$

If $v = u$, for all $w \in \mathsf{h}$ orthogonal to u we find

$$\sum_\ell |\langle w, L_\ell u\rangle|^2 = 0, \tag{5}$$

it follows that u is an eigenvector of all L_ℓ, i.e., $L_\ell u = \lambda_\ell(u)u$ for $\lambda_\ell(u) \in \mathbb{C}$.

The identity (5) for $w \in \mathsf{h}_\mathsf{f}$ also yields

$$\lambda_\ell(u)\langle w, u\rangle = \langle w, p_\mathsf{f} L_\ell p_\mathsf{f} u\rangle = \langle p_\mathsf{f} L_\ell^* p_\mathsf{f} w, u\rangle,$$

i.e., $p_\mathsf{f} L_\ell^* p_\mathsf{f} w = 0$ if $\langle w, u\rangle = 0$ and $p_\mathsf{f} L_\ell^* p_\mathsf{f} u = \overline{\lambda_\ell}(u)u$ otherwise, showing that the operator $p_\mathsf{f} L_\ell p_\mathsf{f}$ is normal.

We now prove that eigenvalues $\lambda_\ell(u)$ do not depend on the choice of the vector $u \in \mathsf{h}_\mathsf{f}$. Note, first of all, that eigenvectors u, v in h_f of $p_\mathsf{f} L_\ell p_\mathsf{f}$ with different eigenvalues $\lambda_\ell(u) \neq \lambda_\ell(v)$ are orthogonal since

$$\lambda_\ell(v)\langle v, u\rangle = \langle p_\mathsf{f} L_\ell^* p_\mathsf{f} v, u\rangle = \langle v, p_\mathsf{f} L_\ell p_\mathsf{f} u\rangle = \lambda_\ell(u)\langle v, u\rangle.$$

Thus, the Hilbert space h being separable, the spectrum of $p_\mathsf{f} L_\ell p_\mathsf{f}$ is at most countable, hence totally disconnected. The function on the unit sphere of h_f

$$u \to \langle u, L_\ell u\rangle = \lambda_\ell(u)$$

is continuous and so its range must be connected. It follows that the function $u \to \lambda_\ell(u)$ is constant.

Now, rewriting (4) as

$$|(G + \mathrm{i}K)u\rangle\langle v| + |u\rangle\langle (G + \mathrm{i}K)v| + \sum_\ell |\lambda_\ell|^2\ |u\rangle\langle v| = 0, \tag{6}$$

we see that u and v are also eigenvectors for $G + \mathrm{i}K$. The eigenvalues $z(u)$ and $z(v)$ fulfill the identity

$$\left(z(u) + \overline{z(v)} + \sum_\ell |\lambda_\ell|^2\right)|u\rangle\langle v| = 0, \tag{7}$$

hence $z(u) + \overline{z(v)} + \sum_\ell |\lambda_\ell|^2 = 0$ for all $u, v \in \mathsf{h}_\mathsf{f}$. Taking $u = v$ we see that

$$z(u) = -\mathrm{i}r(u) - \frac{1}{2}\sum_\ell |\lambda_\ell|^2$$

for some $r(u) \in \mathbb{R}$. Finally, replacing this in (7), we see that $r(u)$ must be independent of $u \in \mathsf{h}_\mathsf{f}$.

This completes the proof. □

Remark. Note that Proposition 3 holds even if h_f is defined as a subspace such that all states ω supported in h_f evolve $\omega \to \mathrm{e}^{-\mathrm{i}tK}\omega\,\mathrm{e}^{\mathrm{i}tK}$ for some self-adjoint K on h (i.e., if h_f is not K-invariant). If h_f is as in Definition 1 we also have the following:

Theorem 4. *A subspace* h_f *is a DF subspace with associated self-adjoint operator* K*, if and only if in any GKSL representation of*$\mathcal{L}$ *by means of operators* L_ℓ, G *there exist complex numbers* λ_ℓ $(\ell \geq 1)$ *and a real numberr such that* $\sum_{\ell\geq 1}|\lambda_\ell|^2 < \infty$ *and*

1. $L_\ell\, u = \lambda_\ell\, u$ *for all* $u \in \mathsf{h}_\mathsf{f}$ *and* $\ell \geq 1$,
2. $(G + \mathrm{i}K)u = -(\frac{1}{2}\sum_{\ell\geq 1}|\lambda_\ell|^2 + \mathrm{i}r)u$ *for all* $u \in \mathsf{h}_\mathsf{f}$.

Proof. Consider the GKSL representation (2) of the generator $\mathcal{L}$. If $\mathsf{h_f}$ is a DF subspace, the above conditions hold by Proposition 3.

Conversely, suppose that 1 and 2 hold, then we compute immediately

$$-\mathrm{i}\,[K, |u\rangle\langle v|\,] = \mathcal{L}_*(|u\rangle\langle v|)$$

for all $u, v \in \mathsf{h_f}$. Since $\mathsf{h_f}$ is K-invariant, replacing u, v by $\mathrm{e}^{-\mathrm{i}(t-s)K}u, \mathrm{e}^{-\mathrm{i}(t-s)K}v \in \mathsf{h_f}$ the above relationship also holds for $\left|\mathrm{e}^{-\mathrm{i}(t-s)K}u\rangle\langle\mathrm{e}^{-\mathrm{i}(t-s)K}v\right|$ and we have,

$$\frac{d}{ds}\mathcal{T}_{*s}\left(\mathrm{e}^{-\mathrm{i}(t-s)K}|u\rangle\langle v|\mathrm{e}^{\mathrm{i}(t-s)K}\right) = \mathcal{T}_{*s}\left((\mathcal{L}_*+\mathrm{i}[K,\cdot])(\mathrm{e}^{-\mathrm{i}(t-s)K}|u\rangle\langle v|\mathrm{e}^{\mathrm{i}(t-s)K}\right) = 0.$$

Therefore,

$$\mathcal{T}_{*t}\left(|u\rangle\langle v|\right) = \mathrm{e}^{-\mathrm{i}tK}\,|u\rangle\langle v|\,\mathrm{e}^{\mathrm{i}tK}$$

and $\mathsf{h_f}$ is decoherence-free. □

Remark. The above result shows that, translating the operators L_ℓ by $-\lambda_\ell$, we find another GKSL representation of $\mathcal{L}$ with $L'_\ell = L_\ell - \lambda_\ell\mathbb{1}$ and $H' = H + (2\mathrm{i})^{-1}\sum_\ell(\bar{z}_\ell L_\ell - z_\ell L^*_\ell)$. In this way, since $\sum_{\ell\geq 1}(L'_\ell)^* L'_\ell$ vanishes on $\mathsf{h_f}$, we find as self-adjoint operator K associated with $\mathsf{h_f}$ the generator of the one-parameter group originating from the action of the semigroup in the new GKSL representation of $\mathcal{L}$.

Theorem 4 provides a recipe for finding DF subspaces. First of all, look for common eigenspaces for all the operators L_ℓ, then, translate L_ℓ to $L_\ell - \lambda_\ell\mathbb{1}$ with eigenvalues λ_ℓ finding a new GKSL representation of the generator $\mathcal{L}$. The intersection of common eigenspaces of all the operators L_ℓ is now the common kernel of all the operators $L_\ell - \lambda_\ell\mathbb{1}$. Finally, check that the operator G found in the new GSKL representation of $\mathcal{L}$ leaves the common kernel invariant and is anti self-adjoint on this subspace.

Theorem 4 also expresses in a simple and direct way the relationship between a property of the QMS $\mathcal{T}$ and the structure of operators G and L_ℓ in a GKSL representation of its generator (see, e.g., Refs. 10, 12, and 15 for results of the same flavour).

Remark. A K-invariant subspace of a DF subspace is itself a DF subspace, therefore we will be interested in *maximal* DF subspaces.

IV. EXAMPLES I

In this section, we give a couple of examples of DF subspaces. The first one shows, in particular, that vectors in a DF subspace may not be eigenvectors for L^*_ℓ (but they are for $p_\mathsf{f} L^*_\ell p_\mathsf{f} = (p_\mathsf{f} L_\ell p_\mathsf{f})^*$).

A. Two coupled two-level systems

This example is inspired by the two two-level system interacting with a common squeezed bath studied by Mundarain and Orszag in Ref. 20 (see also Ref. 17, Sec. III B). Let $\mathsf{h} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and let L be the operator on h

$$L = S \otimes \mathbb{1} + \mathbb{1} \otimes S,$$

where S is a 2×2 matrix of the form

$$\begin{bmatrix} 0 & z^2 \\ w^2 & 0 \end{bmatrix}$$

and z, w are two non-zero complex numbers such that $|z|^2 + |w|^2 = 1$ to ease the notation. We consider the QMS generated by the operator $\mathcal{L}$ in (3) with $L_1 = L$ and $G = -\frac{1}{2}L^*L$. The operator S

has two opposite eigenvalues zw and $-zw$ with normalised eigenvectors

$$f_+ = \begin{bmatrix} z \\ w \end{bmatrix}, \qquad f_- = \begin{bmatrix} -z \\ w \end{bmatrix}.$$

It follows that

$$L(f_\pm \otimes f_\pm) = \pm 2zw f_\pm \otimes f_\pm, \qquad L(f_+ \otimes f_-) = L(f_- \otimes f_+) = 0.$$

Thus, $L^*L(f_+ \otimes f_-) = L^*L(f_- \otimes f_+) = 0$ and the two-dimensional linear space generated by vectors $f_- \otimes f_+$ and $f_+ \otimes f_-$ is a DF subspace for the QMS generated by $\mathcal{L}$.

Indeed, any state ω supported in this space is an invariant state because $\mathcal{L}_*(\omega) = 0$. Moreover, there are no further invariant states if $z^2 \neq \overline{w}^2$ (i.e., L is not self-adjoint) because the support projection of an invariant state is subharmonic (see Ref. 12), hence it determines an invariant subspace for L and L^*L and we can easily check that the linear span of $f_- \otimes f_+$ and $f_+ \otimes f_-$ is the only common invariant subspace for L and L^*L.

B. Generic QMS

Generic QMS arise in the stochastic limit of a open discrete quantum system with generic Hamiltonian, interacting with Gaussian fields through a dipole type interaction (see Refs. 1 and 5). The system space is $\mathsf{h} = \ell^2(I)$, the Hilbert space of square-summable, complex-valued sequences, indexed by a discrete (finite or infinite) set I. Let $(e_i)_{i\geq 0}$ be the canonical orthonormal basis and let L_ℓ be the operators, in this case labeled by a double index (i, j) with $i \neq j$, are

$$L_{ij} = \gamma_{ij}^{1/2} \left| e_j \right\rangle \left\langle e_i \right|,$$

where $\gamma_{ij} \geq 0$ are positive constants and the Hamiltonian H is a self-adjoint operator diagonal in the given basis $H = \sum_{i\geq 0} \kappa_i |e_i\rangle\langle e_i|$. Suppose, for simplicity, that

$$\sup_i |\kappa_i| < \infty \qquad \text{and} \qquad \sup_i \sum_j \gamma_{ij} < \infty.$$

Thus, the generator $\mathcal{L}$ of the generic QMS is bounded (see Ref. 5, Proposition 1) and

$$\mathcal{L}(x) = \mathrm{i}[H, x] + \frac{1}{2} \sum_{i\neq j} \left(-L_{ij}^* L_{ij} x + 2L_{ij}^* x L_{ij} - x L_{ij}^* L_{ij} \right). \tag{8}$$

Note that it can be written in the form (3) with

$$G = -\frac{1}{2} \sum_{i\neq j} L_{ij}^* L_{ij} - \mathrm{i}H = -\frac{1}{2} \sum_i \left(\sum_{\{j \mid j\neq i\}} \gamma_{ij} \right) |e_i\rangle \langle e_i| - \mathrm{i}H.$$

The restriction of $\mathcal{L}$ to the algebra of diagonal matrices coincides with the generator of a time continuous Markov chain with states I and jump rates γ_{ij}. Let

$$I_0 = \left\{ i \in I \mid \gamma_{ij} = 0 \quad \forall j \neq i \right\}$$

be the set of trap states of the classical Markov chain. We claim that the closed subspace h_f generated by vectors e_i with $i \in I_0$ is a decoherence-free subspace, with $K = H$, for the generic QMS.

First note that the only eigenvalue of an operator L_{ij} with $i \neq j$ and $\gamma_{ij} \neq 0$ is 0 and its eigenspace is clearly the orthogonal space of e_i. Indeed, if $u = \sum_k u_k e_k$ $(u \neq 0)$ is an eigenvector with eigenvalue $\lambda \neq 0$, we have $\lambda u = L_{ij} u = \gamma_{ij}^{1/2} u_i e_j$, then $u_k = 0$ for all $k \neq j$ so that $\lambda u_j e_j = \gamma_{ij}^{1/2} u_i e_j = 0$. Thus, also u_j is zero contradicting the assumption $u \neq 0$. Second, note that

$$\cap_{i,j\in I, i\neq j} \ker\left(L_{ij}\right) = \cap_{i\in I-I_0,\, j\in I} \ker\left(L_{ij}\right) = \cap_{i\in I-I_0} \{e_i\}^\perp = \overline{\mathrm{Lin}\,\{e_i \mid i \in I_0\}}.$$

Finally, for all u in this subspace we have $L_{ij}^* L_{ij} u = 0$ for all i, j and so $(G + \mathrm{i}K)u = \mathrm{i}(-H + K)u = 0$. The conclusion follows applying Theorem 4.

Note that the approach first proposed in Ref. 18 in both the above examples fails because both generators $\mathcal{L}$ have an Abelian invariant subalgebra (see Ref. 5).

V. WEAK*-CONTINUOUS QMS

In this section, we will be concerned with QMS on $\mathcal{B}(\mathsf{h})$ with a formal generator represented in a generalised GKSL form by means of operators G, L_ℓ ($\ell \geq 1$) on h with the following property:

(**H-min**) *the operator G is the generator of a strongly continuous semigroup $(P_t)_{t\geq 0}$ on h, the domain* $\mathrm{Dom}(L_\ell)$ *of each operator L_ℓ is contained in* $\mathrm{Dom}(G)$ *and*

$$\langle Gv, u\rangle + \sum_{\ell\geq 1}\langle L_\ell v, L_\ell u\rangle + \langle v, Gu\rangle = 0 \tag{9}$$

for all $u, v \in$ $\mathrm{Dom}(G)$.

For each $x \in \mathcal{B}(\mathsf{h})$, we can consider the quadratic form $\pounds(x)$ with domain $\mathrm{Dom}(G)\times\mathrm{Dom}(G)$ defined by

$$\pounds(x)[v, u] = \langle Gv, xu\rangle + \sum_{\ell\geq 1}\langle L_\ell v, xL_\ell u\rangle + \langle v, xGu\rangle.$$

The hypothesis (**H-min**) allows us to construct the minimal semigroup on $\mathcal{B}(\mathsf{h})$ associated with the operators G, L_ℓ (see, e.g., Refs. 8, 9, and 11). This is the weak*-continuous semigroup $(\mathcal{T}_t)_{t\geq 0}$ of completely positive maps on $\mathcal{B}(\mathsf{h})$ satisfying

$$\langle v, \mathcal{T}_t(x)u\rangle = \langle v, xu\rangle + \int_0^t \pounds(\mathcal{T}_s(x))\,[v, u]ds. \tag{10}$$

It is well-known that, in spite of (9), meaning that $\pounds(\mathbb{1}) = 0$, the minimal semigroup may not be unital, i.e., $\mathcal{T}_t(\mathbb{1}) < \mathbb{1}$ (see, e.g., Davies,[9] Example 3.3, p. 174 and Fagnola,[11] Example 3.4, p. 58). In this case, it is not the unique weak*-continuous semigroup of completely positive maps on $\mathcal{B}(\mathsf{h})$ satisfying (10) (see, e.g., Ref. 11, Theorem 3.22, p. 52, Corollary 3.23, p. 53).

Throughout we will assume

(**H-Markov**) The minimal QMS $\mathcal{T}$ associated with operators G, L_ℓ is Markov.

We refer to Chebotarev and Fagnola,[8] Theorem 4.4, p. 394, for useful conditions allowing us to check the above hypothesis.

Recall that, since we assume that h_f is $\mathrm{e}^{-\mathrm{i}tK}$ invariant, for a self-adjoint K, we have $\mathrm{e}^{-\mathrm{i}tK}p_\mathsf{f}\mathrm{e}^{\mathrm{i}tK} = p_\mathsf{f}$. Moreover, $\mathsf{h}_\mathsf{f}\cap\mathrm{Dom}(K)$ dense in h_f and $(\lambda + \mathrm{i}K)(\mathsf{h}_\mathsf{f}\cap\mathrm{Dom}(K)) = \mathsf{h}_\mathsf{f}$ for all λ in the resolvent of K (see Ref. 22, Sec. 4.5 with a slightly different language).

Lemma 5. Assume (**H-min**) *and* (**H-Markov**). *If h_f is a DF subspace, then the orthogonal projection p_f onto h_f is $\mathcal{T}$-subharmonic, namely, $\mathcal{T}_t(p_\mathsf{f}) \geq p_\mathsf{f}$ for all $t \geq 0$.*

Proof. Let p_f be the orthogonal projection onto h_f. By Lemma 2, and the invariance of h_f, for all state η with support in h_f, we have

$$\mathrm{tr}\,(\mathcal{T}_t(p_\mathsf{f})\eta) = \mathrm{tr}\,(p_\mathsf{f}\mathcal{T}_{*t}(\eta)) = \mathrm{tr}\,(p_\mathsf{f}\eta)\,,$$

for all $t \geq 0$. The above identity implies $\mathrm{tr}\left(\mathcal{T}_t(p_\mathsf{f}^\perp)\eta\right) = \mathrm{tr}\left(p_\mathsf{f}^\perp\eta\right) = 0$. Thus, $p_\mathsf{f}\mathcal{T}_t(p_\mathsf{f}^\perp)p_\mathsf{f} = 0$, and positivity of $\mathcal{T}_t(p_\mathsf{f}^\perp)$ implies $\mathcal{T}_t(p_\mathsf{f}^\perp) = p_\mathsf{f}^\perp\mathcal{T}_t(p_\mathsf{f}^\perp)p_\mathsf{f}^\perp$. It follows that $\mathcal{T}_t(p_\mathsf{f}^\perp) \leq p_\mathsf{f}^\perp$, namely, $\mathcal{T}_t(p_\mathsf{f}) \geq p_\mathsf{f}$ for all $t \geq 0$. □

As a consequence, from Theorem 3.1 in Ref. 12, we have immediately

Lemma 6. Assume (**H-min**) *and* (**H-Markov**). *If h_f is a DF subspace, then h_f is an invariant subspace for the operators P_t for all $t \geq 0$, $\mathsf{h}_\mathsf{f}\cap\mathrm{Dom}(G)$ is dense in h_f and $L_\ell(\mathsf{h}_\mathsf{f}\cap\mathrm{Dom}(G)) \subseteq$ $\mathrm{Dom}(G)$ for all $\ell \geq 1$.*

We can now prove the technical result allowing us to compare the domains of G and K.

Proposition 7. Assume (**H-min**) *and* (**H-Markov**). *Then*

$$\mathsf{h_f} \cap \mathrm{Dom}(G) \subseteq \mathsf{h_f} \cap \mathrm{Dom}(K).$$

Proof. For all $u, v \in \mathsf{h_f} \cap \mathrm{Dom}(G)$ and $g, f \in \mathrm{Dom}(K)$ we have

$$\left\langle \mathrm{e}^{\mathrm{i}tK} g, v\rangle\langle u, \mathrm{e}^{\mathrm{i}tK} f\right\rangle = \langle g, \mathcal{T}_{*t}(|v\rangle\langle u|) f\rangle .$$

The derivative of both sides at $t = 0$ yields

$$\langle \mathrm{i}Kg, v\rangle\langle u, f\rangle + \langle g, v\rangle\langle u, \mathrm{i}Kf\rangle = \langle g, \mathcal{L}_*(|v\rangle\langle u|) f\rangle .$$

By the density of $\mathrm{Dom}(K)$ we can choose, and fix, a $g \in \mathrm{Dom}(G)$ such that $\langle g, v\rangle \neq 0$ and so find the identity

$$\langle u, Kf\rangle = -\mathrm{i}\left(\langle g, \mathcal{L}_*(|v\rangle\langle u|) f\rangle - \langle \mathrm{i}Kg, v\rangle\langle u, f\rangle\right) \cdot \langle g, v\rangle^{-1},$$

where $\mathcal{L}_*(|v\rangle\langle u|)$ is a trace class operator on h. It follows that the linear form on $\mathrm{Dom}(K)$ given by $f \to \langle u, Kf\rangle$ can be continuously extended to h, thus u belongs to the domain of K because K is self-adjoint. □

Having fixed the domain problems we can now extend Theorem 4 to QMS with generators in a generalised GKSL form.

Theorem 8. *Suppose that the minimal semigroup $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ associated with operators G, L_ℓ satisfies* (**H-min**) *and* (**H-Markov**). *Moreover, assume that*

(a) *the operators L_ℓ are closed,*
(b) $\mathrm{Dom}(G)$ *is contained in*$\mathrm{Dom}(L_\ell^*)$ *for all* $\ell \geq 1$,
(c) *K is a self-adjoint operator on h such that*$\mathrm{Dom}(G)$ *is* $\mathrm{e}^{-\mathrm{i}tK}$ *invariant for all* $t \geq 0$.

A subspace $\mathsf{h_f}$ is a DF subspace with associated self-adjoint operator K if and only if there exists complex numbersλ_ℓ $(\ell \geq 1)$ and a real number r such that $\sum_{\ell \geq 1} |\lambda_\ell|^2 < \infty$and conditions 1 and 2 of Theorem 4 hold for all $u \in \mathsf{h_f} \cap \mathrm{Dom}(G)$.

Proof. Arguing as in Proposition 3 with $\eta = |u\rangle\langle v|$ and $u, v \in \mathsf{h_f} \cap \mathrm{Dom}(G)$ and $w \in \mathsf{h} \cap \mathrm{Dom}(G)$ orthogonal to u we find Eqs. (4)(5) and deduce that u is an eigenvector of each L_ℓ, namely, $L_\ell u = \lambda_\ell(u) u$ for some $\lambda_\ell(u) \in \mathbb{C}$.

Since $\mathsf{h_f} \cap \mathrm{Dom}(G)$ is dense in $\mathsf{h_f}$ by Lemma 6, we can find an increasing sequence $(\mathsf{s}_n)_{n\geq 1}$ of finite-dimensional subspaces $\mathsf{h_f} \cap \mathrm{Dom}(G)$ invading $\mathsf{h_f}$, i.e., such that the closure of $\cup_{n\geq 1}\mathsf{s}_n$ coincides with $\mathsf{h_f}$. Denote by p_n the orthogonal projection onto s_n. For every $n \geq 1$, the identity (5), for $u, w \in \mathsf{s_n} \subset \mathrm{Dom}(G) \subset \mathrm{Dom}(L_\ell^*)$, yields

$$\lambda_\ell(u)\langle w, u\rangle = \langle w, p_n L_\ell p_n u\rangle = \langle p_n L_\ell^* p_n w, u\rangle,$$

i.e., $p_n L_\ell p_n w = 0$ if $\langle w, u\rangle = 0$ and $p_n L_\ell p_n u = \overline{\lambda_\ell}(u) u$ otherwise, showing that the operator $p_n L_\ell p_n$ is a multiplication operator and so it is normal.

Moreover, since s_n is finite-dimensional, its spectrum is finite, the same argument of Proposition 3, based on orthogonality of eigenvectors of a normal operators corresponding to different eigenvalues, now shows that functions $\mathsf{s}_n \ni u \to \lambda_\ell(u)$ are constant on s_n.

For every $u \in \mathsf{h_f}$, and every sequence $(u_n)_{n \geq 1}$ with $u_n \in \mathsf{s}_n$ converging to u, we have

$$\lim_{n\to\infty} L_\ell u_n = \lambda_\ell \lim_{n\to\infty} u_n = \lambda_\ell u.$$

Since the operator L_ℓ is closed by the assumption (a), u belongs to its domain and $L_\ell u = \lambda_\ell u$. This shows that condition 1 of Theorem 4 holds.

Following again the same line of argument as used in the proof of Proposition 3 we can write down Eqs. (6)(7) for $u, v \in \mathsf{h_f} \cap \mathrm{Dom}(G)$ and show that condition 2 of Theorem 4 also holds.

Conversely, if conditions 1 and 2 of Theorem 4 hold, since $\mathrm{Dom}(G)$ is $\mathrm{e}^{-\mathrm{i}tK}$ invariant for all $t \geq 0$, we can show by the same argument of Theorem 4 that $\mathcal{T}_{*t}(|u\rangle\langle v|) = \mathrm{e}^{-\mathrm{i}tK}|u\rangle\langle v|\,\mathrm{e}^{\mathrm{i}tK}$ for all $t \geq 0$ and $u, v \in \mathsf{h}_{\mathsf{f}} \cap \mathrm{Dom}(G)$. The conclusion follows from the density of $\mathsf{h}_{\mathsf{f}} \cap \mathrm{Dom}(G)$ in h_{f}. □

VI. EXAMPLES II

In this section, we exhibit an example of a QMS with generator $\mathcal{L}$ in a generalised GKSL form and "big" DF subspaces.

Let $f_1, \ldots, f_d$ be linearly independent vectors in $\mathbb{C}^k$ $(d \leq k)$ and let $\mathsf{h} = \Gamma(\mathbb{C}^k)$ be the Boson Fock space over the one-particle space $\mathbb{C}^k$. This is the direct sum of the n-fold symmetric tensor products $\left(\mathbb{C}^k\right)^{\otimes n}$ and every vector $u \in \Gamma(\mathbb{C}^k)$ has a chaos decomposition $u = \sum_{n\geq 0} u_n$. Exponential vectors $\mathrm{e}(g)$ are defined by

$$\mathrm{e}(g) = \sum_{n\geq 0} \frac{g^{\otimes n}}{\sqrt{n!}},$$

where $g^{\otimes n}$ denotes the symmetric tensor product of n copies of g. As in Bratteli and Robinson,[4] Sec. 5.2.1, we define the number operator N by

$$\mathrm{Dom}\,(N) = \left\{ u \in \mathsf{h} \mid \sum_{n\geq 0} n^2 \|u_n\|^2 < \infty \right\}, \qquad Nu = \sum_{n\geq 0} n u_n.$$

Let $a(f_\ell)$ be the annihilation operators defined by

$$\mathrm{Dom}\,(a(f_\ell)) = \mathrm{Dom}\left(N^{1/2}\right), \qquad a(f_\ell)\, g^{\otimes n} = \sqrt{n}\,\langle f_\ell, g\rangle\, g^{\otimes(n-1)}.$$

Let L_ℓ be the closure of $a(f_\ell)$ and let $G = -\frac{1}{2}\sum_{\ell=1}^{d} L_\ell^* L_\ell - i\omega N$ with $\omega \in \mathbb{R}$.

We consider the QMS $\mathcal{T}$ with generator represented in a generalised GKSL form by means of operators G, L_ℓ $(\ell \geq 1)$.

The assumptions (**H-min**) and (**H-Markov**) can be checked by the same methods of Ref. 11, Sec. 4.3 while hypotheses (a), (b), (c) of Theorem 8 will follow from standard properties of creation, annihilation, and number operators (see Ref. 4).

The spectrum of annihilation operators $a(f)$ is the whole complex plane since $a(f)\mathrm{e}(u) = \langle f, u\rangle \mathrm{e}(u)$. As a consequence, the QMS $\mathcal{T}$ admits non-trivial decoherence-free subspaces. Indeed, let S be the subspace of $\mathbb{C}^k$ spanned by vectors $f_1, \ldots, f_d$ and let V be the orthogonal subspace in $\mathbb{C}^k$. Clearly,

$$\mathsf{h} = \Gamma(S \oplus V) = \Gamma(S) \otimes \Gamma(V)$$

and letting 0_S denoting the 0 vector in S we can think of $\Gamma(V)$ as a subspace of h via the natural embedding $\mathrm{e}(v) \to \mathrm{e}(0_S) \otimes \mathrm{e}(v)$. Now $\Gamma(V)$ is contained into the kernel of all L_ℓ because $a(f_\ell)e(v) = 0$ for all $v \in V$ and is clearly $\mathrm{e}^{-\mathrm{i}tN}$ invariant for all $t \geq 0$. Therefore, $\Gamma(V)$ is a decoherence-free subspace with $K = \omega N$.

VII. OUTLOOK

In this paper, we completely characterised decoherence-free subspaces of a given QMS with generator in a generalised GKSL form in terms of common eigenspaces of operators G, L_ℓ. It is worth noticing here that the case of semigroups which are only w^*-continuous is always difficult to deal with due to technical problems on domains of unbounded operators on infinite dimensional Hilbert spaces. In particular, conservativity and Markovianity are not guaranteed. This is a common feature of several semigroups appearing in physical models, for instance, in Quantum Optics. While previous methods[17] heavily rely on finite-dimensionality, our method is well suited for analysing decoherence-free issues of open quantum dynamics in general as in the algebraic approach to open quantum systems. Thus, the current paper also establishes a bridge between decoherence-free subspaces and decoherence-free subalgebras and an extension of the first to a more general framework.

ACKNOWLEDGMENTS

This work has been partially supported by Grant Nos. FONDECYT 1120063 and PIA-ACT 1112 "Stochastic Analysis Research Network." F.F. gratefully acknowledges financial support from Centro Vito Volterra. J.A. and F.F. would like to thank S. Attal, A. Joye, and C.-A. Pillet for the invitation to the summer school and workshop "Advances in Open Quantum Systems" where a part of this work was done. We would like to thank an anonymous referee for comments and remarks that improved an earlier version of this paper.

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