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BV-operators and the secondary Hochschild complex

Mamta Balodi[a], **Abhishek Banerjee**[*, a] **and Anita Naolekar**[b]

[a] Department of Mathematics, Indian Institute of Science, Bangalore

[b] Stat-Math Unit, Indian Statistical Institute, Bangalore

E-mails: mamta.balodi@gmail.com, abhishekbanerjee1313@gmail.com, anita@isibang.ac.in

Abstract. We introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n\geq 0}$ on a homotopy G-algebra $V^\bullet$ such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism. As an application, we produce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra A over a commutative algebra B. In this case, we also show that the operator $\Delta^\bullet$ corresponds to Connes' operator.

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1. Introduction

A Gerstenhaber algebra (see [3]) consists of a graded vector space $W^\bullet = \bigoplus_{n\geq 0} W^n$ equipped with the following two structures:

(a) A dot product $x \cdot y$ of degree zero making $W^\bullet$ into an associative graded commutative algebra.

(b) A bracket $[x, y]$ of degree -1 making $W^\bullet$ into a graded Lie algebra satisfying the compatibility property that

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(\deg(x)-1)\deg(y)} y \cdot [x, z].$$

Gerstenhaber algebra structures appear in a variety of situations, from Hochschild cohomology of algebras to the exterior algebra of a Lie algebra and the algebra of differential forms on a Poisson manifold.

[*] Corresponding author.

An operator $\partial = \{\partial^n : W^n \to W^{n-1}\}_{n\geq 0}$ on $W^\bullet$ of degree -1 is said to generate the Gerstenhaber bracket (see Koszul [7, §2] and also [6, Definition 3.2]) if it satisfies

$$[x,y] = (-1)^{(\deg(x)-1)\deg(y)}(\partial(x)\cdot y + (-1)^{\deg(x)} x\cdot\partial(y) - \partial(x\cdot y))$$

In particular, a Batalin–Vilkovisky algebra (or BV-algebra) consists of a Gerstenhaber algebra along with a generator ∂ for the bracket such that $\partial^2 = 0$.

In [4], [5], Gerstenhaber and Voronov introduced the notion of a homotopy G-algebra, which is a brace algebra equipped with a differential of degree 1 and a dot product of degree 0 satisfying certain conditions. In particular, the cohomology groups $H(V^\bullet)$ of a homotopy G-algebra $V^\bullet$ carry the structure of a Gerstenhaber algebra.

In this paper, we introduce the notion of a BV-operator $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n\geq 0}$ on a homotopy G-algebra $V^\bullet$ such that the Gerstenhaber bracket on $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism. More explicitly, for classes $\overline{f} \in H^n(V^\bullet)$ and $\overline{g} \in H^m(V^\bullet)$, we have

$$[\overline{f},\overline{g}] = (-1)^{(n-1)m}\overline{(\Delta(f)\cdot g + (-1)^n f\cdot\Delta(g) - \Delta(f\cdot g))} \in H^{m+n-1}(V^\bullet)$$

where $f \in Z^n(V^\bullet)$, $g \in Z^m(V^\bullet)$ are cocycles representing $\overline{f}$ and $\overline{g}$ respectively. We note that Δ need not be a morphism of cochain complexes and therefore may not induce any operator on $H(V^\bullet)$. As such, Δ may not descend to a generator for the Gerstenhaber bracket on $H(V^\bullet)$.

Our motivation is to introduce a BV-operator on the cochain complex defining the secondary Hochschild cohomology of a symmetric algebra A over a commutative algebra B. For a datum (A,B,ε) consisting of an algebra A, a commutative algebra B and an extension of rings $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$, the secondary Hochschild cohomology $H^\bullet(A,B,\varepsilon)$ was introduced by Staic [9] in order to study deformations of algebras $A[[t]]$ having a B-algebra structure. More generally, Staic [9] introduced the secondary Hochschild complex $C^\bullet((A,B,\varepsilon);M)$ with coefficients in an A-bimodule M.

In [10], Staic and Stancu showed that the secondary Hochschild complex $C^\bullet(A,B,\varepsilon) := C^\bullet((A,B,\varepsilon);A)$ with coefficients in A is a non-symmetric operad with multiplication, giving it the structure of a homotopy G-algebra. Hence, the secondary cohomology $H^\bullet(A,B,\varepsilon)$ is equipped with a graded commutative cup product and a Lie bracket which makes it a Gerstenhaber algebra. For more on the secondary cohomology, the reader may see, for instance, [1], Corrigan-Salter and Staic [2], Laubacher, Staic and Stancu [8].

Let k be a field. It is well known (see Tradler [11]) that the Hochschild cohomology of a finite dimensional k-algebra A equipped with a symmetric, non-degenerate, invariant bilinear form $\langle\cdot,\cdot\rangle : A\times A \to k$ carries the structure of a BV-algebra. For the terms $C^n((A,B,\varepsilon)) = \mathrm{Hom}_k(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}}, A)$ in the secondary Hochschild complex, we define the BV-operator $\Delta = \sum_{i=1}^{n+1}(-1)^{in}\Delta_i : C^{n+1}(A,B,\varepsilon) \to C^n(A,B,\varepsilon)$ by the condition (see Section 3)

$$\left\langle \Delta_i f\left(\otimes\begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n}\\ 1 & a_2 & b_{2,3} & \dots & b_{2,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & b_{n-1,n}\\ 1 & 1 & 1 & \dots & a_n\end{pmatrix}\right), a_{n+1}\right\rangle$$
$$= \left\langle f\left(\otimes\begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \ldots b_{i,n} & 1 & b_{1,i} & b_{2,i} & \ldots & b_{i-1,i}\\ 1 & a_{i+1} & b_{i+1,i+2} & \ldots b_{i+1,n} & 1 & b_{1,i+1} & b_{2,i+1} & \ldots & b_{i-1,i+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & \ldots & a_n & 1 & b_{1,n} & b_{2,n} & \ldots & b_{i-1,n}\\ 1 & 1 & \ldots & 1 & a_{n+1} & 1 & \ldots & & 1\\ 1 & 1 & \ldots & 1 & 1 & a_1 & b_{1,2} & \ldots & b_{1,i-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & & a_{i-1}\end{pmatrix}\right), 1\right\rangle \quad (1)$$

We then show that the Gerstenhaber bracket on the secondary Hochschild cohomology of (A,B,ε) is determined by Δ in a manner similar to the BV-formalism.

From Tradler [11], we also know that the operator $\Delta^\bullet : C^\bullet(A,A) \to C^{\bullet-1}(A,A)$ on usual Hochschild cochains inducing the BV-structure on $H^\bullet(A,A)$ corresponds to the operator Ns on duals of Hochschild chains, where N is the "norm operator" and s is the "extra degeneracy" (see (16)). The isomorphism between the two complexes is induced by the k-module isomorphism $A^* \cong A$ determined by the non-degenerate bilinear form $\langle\cdot,\cdot\rangle : A \times A \to k$. If we pass to the cohomology and take the normalized Hochschild complex which is a quasi-isomorphic subcomplex of $C^\bullet(A,A)$, it follows that Tradler's $\Delta^\bullet$ operator corresponds to Connes' operator on Hochschild cohomology with coefficients in A.

However, in the case of secondary cohomology, we have mentioned that the operator $\Delta^\bullet$ defined in (1) is not a morphism of complexes and we cannot pass to cohomology. Accordingly, we show that the operator $\Delta^\bullet$ defined in (1) fits into a commutative diagram (see Theorem 10)

$$\begin{array}{ccc} \overline{\overline{C}}^\bullet(A,B,\varepsilon) & \xrightarrow{B} & \overline{\overline{C}}^{\bullet-1}(A,B,\varepsilon) \\ \downarrow & & \downarrow \\ C^\bullet((A,B,\varepsilon);A) & \xrightarrow{\Delta^\bullet} & C^{\bullet-1}((A,B,\varepsilon);A) \end{array} \tag{2}$$

where B is Connes' operator. Here, $\overline{\overline{C}}^\bullet(A,B,\varepsilon)$ is the normalization of the co-simplicial module $\overline{C}^\bullet(A,B,\varepsilon)$ introduced by Laubacher, Staic and Stancu [8], which is used to compute the secondary Hochschild cohomology associated to the triple (A,B,ε). It should be noted (see [8, Remark 4.7]) that despite similar names, the complex $\overline{C}^\bullet(A,B,\varepsilon)$ cannot be expressed as a secondary Hochschild complex with coefficients in some A-bimodule. The vertical morphisms in (2) are induced by composing the canonical morphisms $(A\otimes B^{\otimes n})^* \to A^*$ for each $n \geq 0$, the isomorphism $A^* \cong A$ as well as the inclusion of the quasi-isomorphic subcomplex $\overline{\overline{C}}^\bullet(A,B,\varepsilon) \hookrightarrow \overline{C}^\bullet(A,B,\varepsilon)$.

2. Main Result: BV-operator on homotopy G-algebra

We begin by recalling the notion of a homotopy G-algebra from [5]. A brace algebra (see [5, Definition 1]) is a graded vector space $V = \bigoplus_{n\geq 0} V^n$ with a collection of multilinear operators (braces) $x\{x_1,\dots,x_n\}$ satisfying the following conditions (with $x\{\}$ understood to be x):

(1) $\deg(x\{x_1,\dots,x_n\}) = \deg(x) + \sum_{i=1}^n \deg(x_i) - n$

(2) For homogeneous elements x, $x_1,\dots,x_m$, $y_1,\dots,y_n$, we have

$$\begin{aligned} &x\{x_1,\dots,x_m\}\{y_1,\dots,y_n\} \\ &\quad= \sum_{0\leq i_1\leq j_1\leq i_2\leq\cdots\leq i_m\leq j_m\leq n} (-1)^\epsilon x\big\{y_1,\dots,y_{i_1}, x_1\{y_{i_1+1},\dots,y_{j_1}\}, y_{j_1+1},\dots,y_{i_m}, \\ &\qquad\qquad x_m\{y_{i_m+1},\dots,y_{j_m}\}, y_{j_m+1},\dots,y_n\big\} \end{aligned}$$

where $\epsilon = \sum_{p=1}^m |x_p|\big(\sum_{q=1}^{i_p} |y_q|\big)$ and $|x| := \deg(x) - 1$.

Definition 1 (see [5, Definition 2]). *A homotopy G-algebra consists of the following data:*

(1) *A brace algebra* $V = \bigoplus_{n\geq 0} V^n$.

(2) *A dot product of degree zero*

$$V^m \otimes V^n \longrightarrow V^{m+n} \qquad x\otimes y \longmapsto x\cdot y$$

for all $m, n \geq 0$.

(3) *A differential* $d : V^\bullet \to V^{\bullet+1}$ *of degree one making* V *into a DG-algebra with respect to the dot product.*

(4) *The dot product satisfies the following compatibility conditions*

$$(x_1 \cdot x_2)\{y_1,\dots,y_n\} = \sum_{k=0}^{n} (-1)^{\epsilon_k}(x_1\{y_1,\dots,y_k\}) \cdot (x_2\{y_{k+1},\dots,y_n\})$$

where $\epsilon_k = |x_2|\sum_{p=1}^{k}|y_p|$ *and*

$$\begin{aligned} d(x\{x_1,\dots,x_{n+1}\}) - (dx)\{x_1,\dots,x_{n+1}\} - (-1)^{|x|}\sum_{i=1}^{n+1}(-1)^{|x_1|+\cdots+|x_{i-1}|}x\{x_1,\dots,dx_i,\dots,x_{n+1}\} \\ = (-1)^{|x||x_1|+1}x_1 \cdot x\{x_2,\dots,x_{n+1}\} + (-1)^{|x|} \\ \sum_{i=1}^{n}(-1)^{|x_1|+\cdots+|x_{i-1}|}x\{x_1,\dots,x_i \cdot x_{i+1},\dots,x_{n+1}\} - x\{x_1,\dots,x_n\}\cdot x_{n+1} \end{aligned}$$

In particular, a homotopy G-algebra is equipped with a graded Lie bracket which descends to the cohomology of the corresponding cochain complex $(V^\bullet, d)$ (see [5])

$$[\cdot,\cdot] : H^m(V^\bullet) \otimes H^n(V^\bullet) \longrightarrow H^{m+n-1}(V^\bullet) \tag{3}$$

The dot product also descends to the cohomology and the bracket with an element becomes a graded derivation for the induced dot product on $H(V^\bullet) = \bigoplus_{n\ge 0} H^n(V^\bullet)$. In other words, the cohomology $(H(V^\bullet), [\cdot,\cdot], \cdot)$ of a homotopy G-algebra $V^\bullet$ is canonically equipped with the structure of a Gerstenhaber algebra.

We now introduce the notion of a BV-operator on a homotopy G-algebra.

Definition 2. *Let* $V^\bullet = \bigoplus_{n\ge 0} V^n$ *be a homotopy* G*-algebra, let* $d : V^\bullet \to V^{\bullet+1}$ *be its differential and let* $[\cdot,\cdot] : V^n \otimes V^m \to V^{m+n-1}$ *be its Lie bracket. We will say that a family* $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n\ge 0}$ *is a BV-operator on* $V^\bullet$ *if it satisfies*

$$[f,g] - (-1)^{(n-1)m}(\Delta(f)\cdot g + (-1)^n f \cdot \Delta(g) - \Delta(f\cdot g)) \in d(V^{m+n-2})$$

for any cocycles $f \in Z^n(V^\bullet)$, $g \in Z^m(V^\bullet)$.

If $V^\bullet$ is a homotopy G-algebra equipped with a BV-operator Δ, we now show that the bracket on the Gerstenhaber algebra $H(V^\bullet)$ is determined by Δ in a manner similar to the BV-formalism.

Theorem 3. *Let* $V^\bullet = \bigoplus_{n\ge 0} V^n$ *be a homotopy* G*-algebra equipped with a BV-operator* $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n\ge 0}$. *Consider* $\overline{f} \in H^n(V^\bullet)$ *and* $\overline{g} \in H^m(V^\bullet)$ *and choose cocycles* $f \in Z^n(V^\bullet)$ *and* $g \in Z^m(V^\bullet)$ *corresponding respectively to* $\overline{f}$ *and* $\overline{g}$*. Then, we have*

$$(\Delta(f)\cdot g + (-1)^n f\cdot\Delta(g) - \Delta(f\cdot g)) \in Z^{m+n-1}(V^\bullet)$$

The Gerstenhaber bracket on the cohomology of $V^\bullet$ *is now determined by*

$$[\overline{f},\overline{g}] = (-1)^{(n-1)m}\overline{(\Delta(f)\cdot g + (-1)^n f\cdot\Delta(g) - \Delta(f\cdot g))} \in H^{m+n-1}(V^\bullet)$$

In particular, the right hand side does not depend on the choice of representatives f *and* g.

Proof. We know that $f \in Z^n(V^\bullet)$ and $g \in Z^m(V^\bullet)$. Since the bracket $[\cdot,\cdot] : V^n \otimes V^m \to V^{m+n-1}$ descends to a bracket on the cohomology, it follows that $[f,g] \in Z^{m+n-1}(V^\bullet)$. Since $\Delta = \{\Delta^n : V^n \to V^{n-1}\}_{n\ge 0}$ is a BV-operator, it follows from Definition 2 that

$$[f,g] - (-1)^{(n-1)m}(\Delta(f)\cdot g + (-1)^n f\cdot\Delta(g) - \Delta(f\cdot g)) \in d(V^{m+n-2}) \tag{4}$$

Let us put $z_1 = [f,g]$ and $z_2 = (-1)^{(n-1)m}(\Delta(f)\cdot g + (-1)^n f\cdot\Delta(g) - \Delta(f\cdot g))$. Since $z_1 - z_2 \in d(V^{m+n-2})$, we must have $z_1 - z_2 \in Z^{m+n-1}(V^\bullet)$. We have already seen that $z_1 \in Z^{m+n-1}(V^\bullet)$. Hence, $z_2 \in Z^{m+n-1}(V^\bullet)$. By (4), we know that $z_1 - z_2$ is a coboundary and hence the cohomology classes $\overline{z}_1 = \overline{z}_2$. The result is now clear. □

3. Application : BV-operator on secondary Hochschild cohomology

Let k be a field and A be an algebra over k. Let B be a commutative k-algebra and $\varepsilon : B \to A$ be a morphism of k-algebras such that $\varepsilon(B) \subseteq Z(A)$, where $Z(A)$ denotes the center of A. Let M be an A-bimodule such that $\varepsilon(b)m = m\varepsilon(b)$ for all $b \in B$ and $m \in M$. Following [9, §4.2], we consider the complex $(C^\bullet((A,B,\varepsilon);M),\delta^\bullet)$ whose terms are given by

$$C^n((A,B,\varepsilon);M) = \mathrm{Hom}_k\left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, M\right)$$

An element in $A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}$ will be expressed as a "tensor matrix" of the form

$$\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} \\ 1 & 1 & a_3 & \dots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix}$$

where $a_i \in A$ and $b_{i,j} \in B$. The differentials

$$\delta^n : C^n((A,B,\varepsilon);M) \longrightarrow C^{n+1}((A,B,\varepsilon);M)$$

may be described as follows

$$\begin{aligned}
\delta^n(f)\left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-1} & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-1} & b_{2,n} & b_{2,n+1} \\ 1 & 1 & a_3 & \dots & b_{3,n-1} & b_{3,n} & b_{3,n+1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & a_n & b_{n,n+1} \\ 1 & 1 & 1 & \dots & 1 & 1 & a_{n+1} \end{pmatrix}\right)
&= a_1\varepsilon(b_{1,2}b_{1,3}\dots b_{1,n+1}) f\left(\bigotimes \begin{pmatrix} a_2 & b_{2,2} & \dots & b_{2,n} & b_{2,n+1} \\ 1 & a_3 & \dots & b_{3,n} & b_{3,n+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_n & b_{n,n+1} \\ 1 & 1 & \dots & 1 & a_{n+1} \end{pmatrix}\right) \\
&\quad + \sum_{i=1}^{n} (-1)^i f\left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,i}b_{1,i+1} & \dots & b_{1,n} & b_{1,n+1} \\ 1 & a_2 & \dots & b_{2,i}b_{2,i+1} & \dots & b_{2,n} & b_{2,n+1} \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & \varepsilon(b_{i,i+1})a_i a_{i+1} & \dots & b_{i,n}b_{i+1,n} & b_{i,n+1}b_{i+1,n+1} \\ \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & a_n & b_{n,n+1} \\ 1 & 1 & \dots & \dots & \dots & 1 & a_{n+1} \end{pmatrix}\right) \\
&\quad + (-1)^{n+1} f\left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & 1 & a_n \end{pmatrix}\right) \varepsilon(b_{1,n+1}b_{2,n+1}\dots b_{n,n+1})a_{n+1}
\end{aligned}$$

for $f \in C^n((A,B,\varepsilon);M)$, $a_i \in A$, $b_{i,j} \in B$. The cohomology groups of $(C^\bullet((A,B,\varepsilon);M),\delta^\bullet)$ are known as the secondary Hochschild cohomologies $H^n((A,B,\varepsilon);M)$ of the triple (A,B,ε) with coefficients in M (see [9]).

From [10, Proposition 3.1], we know that the secondary Hochschild complex $C^\bullet(A,B,\varepsilon) := C^\bullet((A,B,\varepsilon);A)$ carries the structure of a homotopy G-algebra. This induces a graded Lie bracket

$$[\cdot,\cdot] : H^m(A,B,\varepsilon) \otimes H^n(A,B,\varepsilon) \longrightarrow H^{m+n-1}(A,B,\varepsilon) \tag{5}$$

on the secondary cohomology. It follows (see [10, Corollary 3.2]) that the secondary cohomology $H^\bullet(A,B,\varepsilon)$ carries the structure of a Gerstenhaber algebra in the sense of [3].

From now onwards, we always let A be a finite dimensional k-algebra equipped with a symmetric, non-degenerate, invariant bilinear form $\langle\cdot,\cdot\rangle : A\times A \to k$. In particular, $\langle a_1,a_2\rangle = \langle a_2,a_1\rangle$, $\langle a_1a_2,a_3\rangle = \langle a_1,a_2a_3\rangle$ for any $a_1,a_2,a_3 \in A$. For $i \in \{1,\dots,n+1\}$, we define the maps $\Delta_i : C^{n+1}(A,B,\varepsilon) \to C^n(A,B,\varepsilon)$ as follows:

$$\left\langle \Delta_i f\left(\otimes\begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n}\\ 1 & a_2 & b_{2,3} & \dots & b_{2,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & b_{n-1,n}\\ 1 & 1 & 1 & \dots & a_n \end{pmatrix}\right), a_{n+1}\right\rangle$$
$$= \left\langle f\left(\otimes\begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i}\\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & \dots & a_n & 1 & b_{1,n} & b_{2,n} & \dots & b_{i-1,n}\\ 1 & 1 & \dots & 1 & a_{n+1} & 1 & \dots & & 1\\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix}\right), 1\right\rangle$$

To clarify the above operator, let us express

$$\bigotimes\begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n}\\ 1 & a_2 & b_{2,3} & \dots & b_{2,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & b_{n-1,n}\\ 1 & 1 & 1 & \dots & a_n \end{pmatrix} = \begin{pmatrix} U(i-1) & X_{12}\\ 1 & U(n-i-1)\end{pmatrix}$$

where $U(k)$ is a square matrix of dimension k. Then, we have

$$\left\langle \Delta_i f\begin{pmatrix} U(i-1) & X_{12}\\ 1 & U(n-i-1)\end{pmatrix}, a_{n+1}\right\rangle = \left\langle f\begin{pmatrix} U(n-i-1) & 1 & X_{12}^t\\ 1 & a_{n+1} & 1\\ 1 & 1 & U(i)\end{pmatrix}, 1\right\rangle$$

where X_{12}^t denotes the transpose of X_{12}. The operator $\Delta : C^{n+1}(A,B,\varepsilon) \to C^n(A,B,\varepsilon)$ is then defined as

$$\Delta := \sum_{i=1}^{n+1} (-1)^{in}\Delta_i.$$

Following [10, §3], we know that the complex $C^\bullet(A,B,\varepsilon)$ carries a dot product of degree 0, i.e., for $f \in C^n(A,B,\varepsilon)$, $g \in C^m(A,B,\varepsilon)$, we have $f\cdot g \in C^{m+n}(A,B,\varepsilon)$. We also consider the operations

$$\circ_i : C^n(A,B,\varepsilon)\otimes C^m(A,B,\varepsilon) \longrightarrow C^{m+n-1}(A,B,\varepsilon)$$

and set $f\circ g := \sum_{i=1}^{n}(-1)^{(i-1)(m-1)} f\circ_i g$ as in [10, §3]. We also set

$$\rho^1,\rho^2 : C^n(A,B,\varepsilon)\otimes C^m(A,B,\varepsilon)\longrightarrow C^{n+m-1}(A,B,\varepsilon)$$

$$\rho^1(f\otimes g) := \sum_{i=1}^{m}(-1)^{i(n+m-1)}\Delta_i(f\cdot g) \qquad \rho^2(f\otimes g) := \sum_{i=m+1}^{m+n}(-1)^{i(n+m-1)}\Delta_i(f\cdot g)$$

for $f\in C^n(A,B,\varepsilon)$, $g\in C^m(A,B,\varepsilon)$. It is clear that $\rho^1(f\otimes g)+\rho^2(f\otimes g)=\Delta(f\cdot g)$.

Lemma 4. *$\rho^1(f\otimes g)=(-1)^{nm}\rho^2(g\otimes f)$ for all $f\in C^n(A,B,\varepsilon)$ and $g\in C^m(A,B,\varepsilon)$.*

Proof. This may be verified by direct computation. □

Lemma 5. *Let $f\in Z^n(A,B,\varepsilon)$, $g\in Z^m(A,B,\varepsilon)$. Then $f\circ g-(-1)^{(n-1)m}\Delta(f)\cdot g+(-1)^{(n-1)m}\rho^2(f\otimes g)$ is a coboundary. In fact, if we define H*

$$H=\sum_{i,j\geq 1,\, i+j\leq n}(-1)^{(j-1)(m-1)+i(n+m)+1}\Delta_i(f\circ_j g),$$

then,

$$\delta H = f\circ g-(-1)^{(n-1)m}\Delta(f)\cdot g+(-1)^{(n-1)m}\rho^2(f\otimes g).$$

Proof. We set, for $k\geq 0$, $p\geq 0$:

$$T^k_{k+p}=\bigotimes\begin{pmatrix} a_{k+1} & \dots & b_{k+1,k+p}\\ \vdots & & \vdots\\ 1 & \dots & a_{k+p}\end{pmatrix}$$

We see that

$$\left\langle \delta(\Delta_i(f\circ_j g))\left(\otimes\begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1}\\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1}\\ 1 & 1 & 1 & \dots & a_{n+m-1}\end{pmatrix}\right), a_{n+m}\right\rangle$$

$$=\left\langle f\begin{pmatrix} a_{i+1} & \dots & b_{i+1,i+j-1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & b_{2,i+1} & \dots & b_{i-1,i+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 1 & \dots & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & b_{2,i+j-1} & \dots & b_{i-1,i+j-1}\\ 1 & \dots & 1 & g(T^{i+j-1}_{i+j+m-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ 1 & \dots & 1 & 1 & 1 & \alpha & 1 & \dots & 1\\ 1 & \dots & 1 & 1 & 1 & 1 & a_2 & \dots & b_{2,i}\\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ 1 & \dots & 1 & 1 & 1 & 1 & 1 & \dots & a_i\end{pmatrix}, 1\right\rangle$$

$$+\sum_{\lambda=1}^{i-1}(-1)^{\lambda}\left\langle f\begin{pmatrix} a_{i+1} & \dots & & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & b_{1,i+1} & b_{\lambda,i+1}b_{\lambda+1,i+1} & b_{i,i+1}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \dots & & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & b_{1,i+j-1} & b_{\lambda,i+j-1}b_{\lambda+1,i+j-1} & b_{i,i+j-1}\\ 1 & \dots & & g(T^{i+j-1}_{i+j+m-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & \dots & \prod_{k=0}^{m-1} b_{1,i+j+k} & \prod_{k=0}^{m-1} b_{\lambda,i+j+k}b_{\lambda+1,i+j+k} & \prod_{k=0}^{m-1} b_{i,i+j+k}\\ \vdots & \vdots & & \vdots & \vdots & 1 & \vdots & \vdots & \vdots\\ 1 & 1 & & \dots & \dots & a_{n+m} & 1 & \dots & 1\\ 1 & 1 & & \dots & \dots & 1 & a_1 & \dots & b_{1.i}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \dots & 1 & 1 & 1 & 1 & \beta_\lambda & \dots b_{\lambda,i}b_{\lambda+1,i}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & a_i\end{pmatrix}, 1\right\rangle$$

$$
+\sum_{\lambda=i}^{i+j-2}(-1)^{\lambda}\left\langle f\begin{pmatrix}
a_i & b_{i,i+1} & b_{i,\lambda}b_{i,\lambda+1} & \prod_{k=0}^{m-1} b_{i,i+j+k} & \dots & 1 & b_{1,i} & \dots & b_{i-1,i}\\
1 & a_{i+1} & b_{i+1,\lambda}b_{i+1,\lambda+1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & b_{1,i+1} & \dots & b_{i-1,i+1}\\
1 & 1 & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
1 & \dots & \beta_\lambda & \prod_{k=0}^{m-1} b_{\lambda,i+j+k}b_{\lambda+1,i+j+k} & \dots & 1 & b_{1,\lambda} & \dots & b_{i-1,\lambda}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
1 & \dots & & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & b_{1,i+j-1} & & b_{i-1,i+j-1}\\
1 & \dots & \dots & g(T^{i+j-1}_{i+j+m-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k}\\
1 & \dots & \dots & 1 & 1 & a_{m+n} & 1 & \dots & 1\\
1 & \dots & \dots & 1 & 1 & 1 & a_1 & \dots & b_{1,i-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots\\
1 & \dots & \dots & 1 & \dots & 1 & 1 & \dots & a_{i-1}
\end{pmatrix},1\right\rangle
$$

$$
+\sum_{\lambda=i+j-1}^{i+j+m-2}(-1)^{\lambda}\left\langle f\begin{pmatrix}
a_i & \dots & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & b_{1,i} & \dots & b_{i-1,i}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & b_{1,i+j-2} & \dots & b_{i-1,i+j-2}\\
1 & 1 & g(\overline{T}^{i+j-2}_{i+j+m-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & 1 & \dots & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k}\\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \vdots\\
1 & 1 & 1 & \dots & a_{n+m} & 1 & 1 & \dots 1\\
1 & 1 & 1 & \dots & 1 & a_1 & b_{1,2} & \dots b_{1,i-1}\\
\vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \\
1 & \dots & 1 & \dots & 1 & 1 & \dots & a_{i-1}
\end{pmatrix},1\right\rangle
$$

$$
+\sum_{\lambda=i+j+m-1}^{n+m-2}(-1)^{\lambda}\left\langle f\begin{pmatrix}
a_i & b_{i,i+1} & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots\\
1 & \dots & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2}\\
1 & \dots & \dots & g(\overline{T}^{i+j-2}_{i+j+m-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k}\\
1 & \dots & \dots & 1 & 1 & \beta_\lambda & 1 & \dots & b_{i-1,\lambda}b_{i-1,\lambda+1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots\\
1 & \dots & \dots & \dots & \dots & \dots & a_{n+m} & 1\dots & 1\\
1 & 1 & 1 & \dots & \dots & \dots & \dots & a_1 & \dots b_{1,i-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 1 & 1 & \dots & \dots & \dots & \dots & 1 & \dots a_{i-1}
\end{pmatrix},1\right\rangle
$$

$$
+(-1)^{n+m-1}\left\langle f\begin{pmatrix}
a_i & b_{i,i+1} & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & 1 & \dots & b_{i-1,i}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots\\
1 & \dots & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & 1 & \dots & b_{i-1,i+j-2}\\
1 & \dots & \dots & g(\overline{T}^{i+j-2}_{i+j+m-2}) & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots\\
1 & \dots & \dots & \dots & \dots & \dots & \gamma & 1 & \dots 1\\
1 & \dots & \dots & \dots & \dots & \dots & \dots & a_1 & \dots b_{1,i-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & \dots & \dots & \dots & \dots & \dots & \dots & 1 & \dots a_{i-1}
\end{pmatrix},1\right\rangle \tag{6}
$$

where $\alpha := \varepsilon(b_{1,2}\dots b_{1,n+m-1})a_{n+m}a_1$, $\gamma := \varepsilon(b_{1,n+m-1}\dots b_{n+m-2,n+m-1})a_{n+m}a_{n+m-1}$, $\beta_\lambda := \varepsilon(b_{\lambda,\lambda+1})a_\lambda a_{\lambda+1}$ for $1\le\lambda\le n+m-2$ and

$$
\overline{T}^{i+j-2}_{i+j+m-2} := \begin{pmatrix}
a_{i+j-1} & b_{i+j-1,i+j} & \dots & b_{i+j-1,\lambda}b_{i+j-1,\lambda+1} & & b_{i+j-1,i+j+m-2}\\
1 & a_{i+j} & \dots & b_{i+j,\lambda}b_{i+j,\lambda+1} & & b_{i+j,i+j+m-2}\\
1 & 1 & \vdots & & & \\
1 & 1 & \dots & \varepsilon(b_{\lambda,\lambda+1})a_\lambda a_{\lambda+1} & \dots & b_{\lambda,i+j+m-2}b_{\lambda+1,i+j+m-2}\\
\vdots & & \vdots & \vdots & \vdots & \vdots\\
\dots & & \dots & \dots & \dots & a_{i+j+m-2}
\end{pmatrix}.
$$

We write the entire expression of (6) as

$$\left\langle \delta(\Delta_i(f\circ_j g))\left(\otimes\begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1}\\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1}\\ 1 & 1 & 1 & \dots & a_{n+m-1}\end{pmatrix}\right), a_{n+m}\right\rangle = E_1+E_2+E_3+E_4+E_5+E_6,$$

where E_k denotes the k-th term in the expression.

We set for $i,\ j\geq 1$ and $i+j\leq n$,

$$A_{i,j} := (-1)^{i+1}\left\langle a_i' f\begin{pmatrix} a_{i+1} & \dots & b_{i+1,i+j-1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \dots & 1 & \dots & \dots & b_{i-1,i+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \dots & \vdots\\ 1 & \dots & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \dots & 1 & \dots & \dots & b_{i-1,i+j-1}\\ 1 & \dots & & g(T^{i+j-1}_{i+j+m-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m}\ \dots & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \dots & \vdots\\ 1 & \dots & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1\\ 1 & 1 & \dots & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 1 & 1 & \dots & \dots & \dots & \dots & 1 & \dots & a_{i-1}\end{pmatrix}, 1\right\rangle$$

$$+E_3 + (-1)^{i+j-1}\left\langle f\begin{pmatrix} a_i & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & & 1 & \dots & \dots & b_{i-1,i}\\ \vdots & \vdots & \vdots & \vdots & \vdots & & 1 & \vdots & \dots & \vdots\\ 1 & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & & 1 & \dots & \dots & b_{i-1,i+j-2}\\ 1 & \dots & \dots & \eta & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} & \dots & 1 & \dots & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \dots & \vdots\\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & \dots & \dots & & a_1 & \dots & b_{1,i-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & \dots & \dots & & \dots & \dots & a_{i-1}\end{pmatrix}, 1\right\rangle$$

where $a_i' = a_i\varepsilon(b_{i,i+1}\dots b_{i,n+m-1}b_{1,i}\dots b_{i-1,i})$,. We also set

$$B_{i,j} := (-1)^{i+j+m-2}\left\langle f\begin{pmatrix} a_i & \dots & b_{i,i+j-2} & \prod_{k=-1}^{m-2} b_{i,i+j+k} & \dots & & 1 & \dots & \dots & b_{i-1,i}\\ \vdots & \vdots & \vdots & \vdots & \vdots & & 1 & \vdots & \dots & \vdots\\ 1 & \dots & a_{i+j-2} & \prod_{k=-1}^{m-2} b_{i+j-2,i+j+k} & \dots & & 1 & \dots & \dots & b_{i-1,i+j-2}\\ 1 & \dots & \dots & \zeta & \prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1} & \dots & 1 & \dots & \dots & \prod_{k=-1}^{m-2} b_{i-1,i+j+k}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \dots & \vdots\\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1\\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & \dots & \dots & & \dots & \dots & a_{i-1}\end{pmatrix}, 1\right\rangle$$

$$+E_5+E_6$$

where

$$\eta = a_{i+j-1}\varepsilon(b_{i+j-1,i+j}\dots b_{i+j-1,n+m-1})g\left(T^{i+j-1}_{i+j+m-1}\right)$$

$$\zeta = g\left(T^{i+j-2}_{i+j+m-2}\right)\varepsilon\left(\prod_{k=-1}^{m-2} b_{i+j+k,i+j+m-1}\right),$$

and

$$C_{i,j} := (-1)^i \left\langle f \begin{pmatrix} a_{i+1} & \dots & b_{i+1,i+j-1} & \prod_{k=0}^{m-1} b_{i+1,i+j+k} & \cdots & 1 & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & a_{i+j-1} & \prod_{k=0}^{m-1} b_{i+j-1,i+j+k} & \cdots & 1 & \dots & b_{i-1,i+j-1} \\ 1 & \dots & \dots & g(T^{i+j-1}_{i+j+m-1}) & \prod_{k=0}^{m-1} b_{i+j+k,i+j+m} \cdots & 1 & \dots & \prod_{k=0}^{m-1} b_{i-1,i+j+k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & a_{n+m} & 1 & 1 \\ 1 & 1 & \dots & \dots & \dots & \dots & a_1 & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & 1 & a_{i-1} \end{pmatrix} a'_i, 1 \right\rangle + E_1 + E_2$$

The first term of $A_{i,j}$ and that of $C_{i,j}$ are the same modulo a sign. Using the fact that $\delta g = 0$, the third term of $A_{i,j}$ and the first term of $B_{i,j}$ add up to give E_4. Thus, we have

$$\left\langle \delta(\Delta_i(f \circ_j g)) \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix} \right), a_{n+m} \right\rangle = A_{i,j} + B_{i,j} + C_{i,j} \tag{7}$$

It may be verified that

$$\begin{aligned} &(-1)^{i+1} A_{i,j-1} + (-1)^{i+m} B_{i,j} + (-1)^{i+n} C_{i-1,j} \\ &\qquad = \left\langle \Delta_i((\delta f) \circ_j g)) \left(\otimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{m+n-1} \end{pmatrix} \right), a_{n+m} \right\rangle = 0 \end{aligned} \tag{8}$$

for $2 \le i \le n$, $2 \le j \le n-1$ and $i + j \le n$. The second equality in (8) uses the fact that $\delta f = 0$.

For $i, j \in \{1, \dots, n+1\}$, we define

$$A_{i,0} := (-1)^{i+1} \left\langle g\left(T^{i-1}_{i+m-1}\right) f \begin{pmatrix} a_{i+m} & b_{i+m,i+m+1} & \dots b_{i+m,n+m-1} & 1 & b_{1,i+m} & b_{2,i+m} \dots & b_{i-1,i+m} \\ 1 & a_{i+m+1} & \dots b_{i+m+1,n+m-1} & 1 & b_{2,i+m+1} & b_{2,i+m+1} \dots & b_{i-1,i+m+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & a_1 & b_{1,2} \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1 \right\rangle,$$

$$C_{0,j} = \left\langle f \begin{pmatrix} a_1 & \dots & b_{1,j-1} & \prod_{k=0}^{m-1} b_{1,j+k} & b_{1,j+m} & \dots & b_{1,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{j-1} & \prod_{k=0}^{m-1} b_{j-1,j+k} & b_{j-1,j+m} & \dots & b_{j-1,n+m-1} \\ 1 & \dots & 1 & g(T^{j-1}_{j+m-1}) & \prod_{k=0}^{m-1} b_{j+k,j+m} & \dots & \prod_{k=0}^{m-1} b_{j+k,n+m-1} \\ 1 & \dots & 1 & 1 & a_{j+m} & \dots & b_{j+m,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m-1} \end{pmatrix} a_{n+m}, 1 \right\rangle,$$

and for $i \in \{1,\dots,n\}$, define

$$B_{i,n-i+1} := (-1)^{n+m+1}\left\langle f\begin{pmatrix} a_i & \dots & b_{i,n-1} & \prod_{k=1}^{m} b_{i,n-1+k} & b_{1,i} & \dots & b_{i-1,i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & a_{n-1} & \prod_{k=1}^{m} b_{n-1,n-1+k} & b_{1,n-1} & \dots & b_{i-1,n-1} \\ 1 & \dots & 1 & g(T^{n-1}_{n+m-1})\cdot a_{n+m} & \prod_{k=0}^{m-1} b_{1,n+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,n+k} \\ 1 & \dots & 1 & 1 & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & \dots & \dots & \dots & \dots & a_{i-1} \end{pmatrix}, 1\right\rangle.$$

Thus, $A_{i,j}$, $B_{i,j+1}$, $C_{i-1,j}$ are defined for all the values of i, j with $i, j \geq 1$ and $i + j \leq n+1$. Moreover, it may be verified that

$$\begin{aligned} &A_{1,j-1} + (-1)^{m+1} B_{1,j} + (-1)^{n+1} C_{0,j} \\ &= \left\langle (\delta f)\begin{pmatrix} a_1 & \dots & b_{1,j-1} & \prod_{k=0}^{m-1} b_{1,j+k} & b_{1,j+m} & \dots & b_{1,n+m-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & \dots & a_{j-1} & \prod_{k=0}^{m-1} b_{j-1,j+k} & b_{j-1,j+m} & \dots & b_{j-1,n+m-1} & 1 \\ 1 & \dots & 1 & g(T^{j-1}_{j+m-1}) & \prod_{k=0}^{m-1} b_{j+k,j+m} & \dots & \prod_{k=0}^{m-1} b_{j+k,n+m-1} & 1 \\ 1 & \dots & 1 & 1 & a_{j+m} & \dots & b_{j+m,n+m-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & \dots & \dots & \dots & \dots & \dots & a_{n+m-1} & 1 \\ 1 & \dots & 1 & 1 & 1 & 1 & 1 & a_{n+m} \end{pmatrix}, 1\right\rangle = 0 \end{aligned}$$

We also have

$$\begin{aligned} &(-1)^{i+1} A_{i,0} + (-1)^{i+m} B_{i,1} + (-1)^{i+n} C_{i-1,1} \\ &= \left\langle (\delta f)\begin{pmatrix} g(T^{i-1}_{i+m-1}) & \prod_{k=0}^{m-1} b_{i+k,i+m} & \dots & \prod_{k=0}^{m-1} b_{i+k,n+m-1} & 1 & \prod_{k=0}^{m-1} b_{1,i+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,i+k} \\ 1 & a_{i+m} & \dots & b_{i+m,n+m-1} & 1 & b_{1,i+m} & \dots & \prod_{k=0}^{m-1} b_{i-1,i+m} \\ \dots & \dots & \dots & \dots & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & a_{n+m-1} & 1 & b_{1,n+m-1} & \dots & b_{i-1,n+m-1} \\ \dots & \dots & \dots & 1 & a_{n+m} & 1 & \dots & 1 \\ \dots & \dots & \dots & 1 & 1 & a_1 & \dots & b_{1,i-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & 1 & 1 & \dots & a_{i-1} \end{pmatrix}, 1\right\rangle = 0 \end{aligned}$$

and

$$\begin{aligned} &(-1)^{i+1} A_{i,n-i} + (-1)^{i+m} B_{i,n-i+1} + (-1)^{i+n} C_{i-1,n-i+1} \\ &= \left\langle (\delta f)\begin{pmatrix} a_i & \dots & \dots & b_{i,n-1} & \prod_{k=0}^{m-1} b_{i,n+k} & 1 & b_{1,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & \dots & b_{i+1,n-1} & \prod_{k=0}^{m-1} b_{i+1,n+k} & 1 & b_{1,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \vdots & \vdots & \\ 1 & \dots & \dots & a_{n-1} & \prod_{k=0}^{m-1} b_{n-1,n+k} & 1 & \dots & \dots & b_{i-1,n-1} \\ 1 & \dots & \dots & & g(T^{n-1}_{n+m-1}) & 1 & \prod_{k=0}^{m-1} b_{1,n+k} & \dots & \prod_{k=0}^{m-1} b_{i-1,n+k} \\ 1 & \dots & \dots & \dots & \dots & a_{n+m} & 1 & \dots & 1 \\ 1 & 1 & \dots & \dots & \dots & \dots & a_1 & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & \dots & \dots & \dots & & \dots & \dots & a_{i-1} \end{pmatrix}, 1\right\rangle \end{aligned}$$

Thus, we obtain

$$0 = \sum_{1\le i\le n, 1\le j\le n-1, i+j\le n+1} (-1)^{(j-1)(m-1)+i(n+m-1)} \left((-1)^{i+1}A_{i,j-1} + (-1)^{i+m}B_{i,j} + (-1)^{i+n}C_{i-1,j}\right) \quad (9)$$

Rearranging the terms in the above sum, and using equation (7), we get,

$$\begin{aligned}
0 &= \sum_{1\le i\le n, 1\le j\le n, i+j\le n} (-1)^{(j-1)(m-1)+i(n+m)+1}(A_{i,j} + B_{i,j} + C_{i,j}) \\
&\quad - \sum_{i=1}^{n} (-1)^{m-1+i(n+m)} A_{i,0} - \sum_{i=1}^{n} (-1)^{n(m+1)+i(n+1)} B_{i,n-i+1} - \sum_{j=1}^{n} (-1)^{(j-1)(m-1)} C_{0,j} \\
&= \left\langle \delta(H) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\
&\quad - (-1)^{m(n+1)} \left\langle (\rho^2(f\otimes g)) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\
&\quad - (-1)^{m(n+1)} \left\langle (\Delta(f)\cdot g) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle \\
&\quad - \left\langle (f\circ g) \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n+m-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & b_{n+m-2,n+m-1} \\ 1 & 1 & 1 & \dots & a_{n+m-1} \end{pmatrix}, a_{n+m} \right\rangle. \qquad \square
\end{aligned}$$

Proposition 6. *The family $\Delta = \{\Delta^\bullet : C^\bullet(A,B,\varepsilon) \to C^{\bullet-1}(A,B,\varepsilon)\}$ determines a BV-operator on the homotopy G-algebra $C^\bullet(A,B,\varepsilon)$.*

Proof. We consider $f \in Z^n(A,B,\varepsilon)$ and $g \in Z^m(A,B,\varepsilon)$. By definition (see [10, §3]), we know that

$$[f,g] = f\circ g - (-1)^{(n-1)(m-1)} g\circ f \in C^{m+n-1}(A,B,\varepsilon) \quad (10)$$

Applying Lemma 5, we know that the cochains

$$f\circ g - (-1)^{(n-1)m}\Delta(f)\cdot g + (-1)^{(n-1)m}\rho^2(f\otimes g)$$
$$g\circ f - (-1)^{(m-1)n}\Delta(g)\cdot f + (-1)^{(m-1)n}\rho^2(g\otimes f)$$

are coboundaries. From (10), it now follows that

$$[f,g] - (-1)^{(n-1)m}\Delta(f)\cdot g + (-1)^{(n-1)m}\rho^2(f\otimes g) + (-1)^{(m-1)}\Delta(g)\cdot f + (-1)^m\rho^2(g\otimes f) \quad (11)$$

is a coboundary. Applying Lemma 4, it follows from (11) that

$$[f,g] - (-1)^{(n-1)m}\Delta(f)\cdot g + (-1)^{m(n-1)}\rho^2(f\otimes g) + (-1)^{(m-1)}\Delta(g)\cdot f + (-1)^{m(n-1)}\rho^1(f\otimes g)$$

is a coboundary. Since $\rho^1(f\otimes g) + \rho^2(f\otimes g) = \Delta(f\cdot g)$, we get

$$[f,g] - (-1)^{(n-1)m}\Delta(f)\cdot g + (-1)^{m(n-1)}\Delta(f\cdot g) + (-1)^{(m-1)}\Delta(g)\cdot f$$

is a coboundary. Using the fact that the dot product is graded commutative, we can put $\Delta(g)\cdot f = (-1)^{n(m-1)} f\cdot\Delta(g)$. The result is now clear. □

Theorem 7. *For secondary cohomology classes $\overline{f}\in H^n(A,B,\varepsilon)$ and $\overline{g}\in H^m(A,B,\varepsilon)$, the Gerstenhaber bracket is determined by*

$$[\overline{f},\overline{g}] = (-1)^{(n-1)m}\overline{(\Delta(f)\cdot g+(-1)^n f\cdot\Delta(g)-\Delta(f\cdot g))}\in H^{m+n-1}(A,B,\varepsilon)$$

Here f and g are any cocycles representing the classes $\overline{f}$ and $\overline{g}$ respectively.

Proof. This follows directly by applying Theorem 3 and Proposition 6. □

It is natural to ask whether the BV-operator determined by $\Delta = \{\Delta^\bullet : C^\bullet(A,B,\varepsilon)\to C^{\bullet-1}(A,B,\varepsilon)\}$ induces a BV-algebra structure on the secondary Hochschild cohomology $H^\bullet(A,B,\varepsilon)$. In the special case of $B=k$, we are reduced to ordinary Hochschild cohomology and hence Δ determines a BV-algebra structure on $H^\bullet(A,A)$. However, this is not true in general because $\Delta^\bullet : C^\bullet(A,B,\varepsilon)\to C^{\bullet-1}(A,B,\varepsilon)$ does not commute with the differentials. For instance, we take $n=2$ and ask when the following diagram commutes:

$$\begin{array}{ccc} C^2(A,B,\varepsilon) & \xrightarrow{\ \delta^2\ } & C^3(A,B,\varepsilon) \\ {\scriptstyle \Delta^2}\big\downarrow & & \big\downarrow{\scriptstyle -\Delta^3} \\ C^1(A,B,\varepsilon) & \xrightarrow{\ \delta^1\ } & C^2(A,B,\varepsilon) \end{array} \tag{12}$$

For any $a_1,a_2,a_3\in A$, $b_{1,2}\in B$ and $f\in C^2(A,B,\varepsilon)$, we have

$$\begin{aligned}
&\left\langle (-\Delta^3\delta^2 f)\begin{pmatrix} a_1 & b_{1,2}\\ & a_2\end{pmatrix}, a_3\right\rangle \\
&= -\left\langle (\delta^2 f)\begin{pmatrix} a_1 & b_{1,2} & 1\\ & a_2 & 1\\ & & a_3\end{pmatrix},1\right\rangle - \left\langle (\delta^2 f)\begin{pmatrix} a_2 & 1 & b_{1,2}\\ & a_3 & 1\\ & & a_1\end{pmatrix},1\right\rangle - \left\langle (\delta^2 f)\begin{pmatrix} a_3 & 1 & 1\\ & a_1 & b_{1,2}\\ & & a_2\end{pmatrix},1\right\rangle \\
&= -\left\langle a_1\varepsilon(b_{1,2})f\begin{pmatrix} a_2 & 1\\ & a_3\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_1a_2\varepsilon(b_{1,2}) & 1\\ & a_3\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_1 & b_{1,2}\\ & a_2a_3\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_1 & b_{1,2}\\ & a_2\end{pmatrix}a_3,1\right\rangle \\
&\quad -\left\langle a_2\varepsilon(b_{1,2})f\begin{pmatrix} a_3 & 1\\ & a_1\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_2a_3 & b_{1,2}\\ & a_1\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_2 & b_{1,2}\\ & a_3a_1\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_2 & 1\\ & a_3\end{pmatrix}a_1\varepsilon(b_{1,2}),1\right\rangle \\
&\quad -\left\langle a_3 f\begin{pmatrix} a_1 & b_{1,2}\\ & a_2\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_3a_1 & b_{1,2}\\ & a_2\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_3 & 1\\ & a_1a_2\varepsilon(b_{1,2})\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_3 & 1\\ & a_1\end{pmatrix}a_2\varepsilon(b_{1,2}),1\right\rangle
\end{aligned}$$

Now, using the properties of the inner product $\langle\cdot,\cdot\rangle$ on A we see that the first term cancels with the eighth, the fourth term cancels with the ninth, the fifth term cancels with the twelfth. Thus, the above expression reduces to

$$\begin{aligned}
&\left\langle f\begin{pmatrix} a_1a_2\varepsilon(b_{1,2}) & 1\\ & a_3\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_1 & b_{1,2}\\ & a_2a_3\end{pmatrix},1\right\rangle + \left\langle f\begin{pmatrix} a_2a_3 & b_{1,2}\\ & a_1\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_2 & b_{1,2}\\ & a_3a_1\end{pmatrix},1\right\rangle \\
&\qquad + \left\langle f\begin{pmatrix} a_3a_1 & b_{1,2}\\ & a_2\end{pmatrix},1\right\rangle - \left\langle f\begin{pmatrix} a_3 & 1\\ & a_1a_2\varepsilon(b_{1,2})\end{pmatrix},1\right\rangle
\end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned}
&\left\langle(\delta^1\Delta^2 f)\begin{pmatrix}a_1 & b_{1,2}\\ & a_2\end{pmatrix}, a_3\right\rangle\\
&\quad= \langle a_1\varepsilon(b_{1,2})(\Delta^2 f)(a_2), a_3\rangle - \langle(\Delta^2 f)(a_1a_2\varepsilon(b_{1,2})), a_3\rangle + \langle(\Delta^2 f)(a_1)a_2\varepsilon(b_{1,2}), a_3\rangle\\
&\quad= \langle(\Delta^2 f)(a_2), a_3a_1\varepsilon(b_{1,2})\rangle - \langle(\Delta^2 f)(a_1a_2\varepsilon(b_{1,2})), a_3\rangle + \langle(\Delta^2 f)(a_1), a_2a_3\varepsilon(b_{1,2})\rangle\\
&\quad= -\left\langle f\begin{pmatrix}a_2 & 1\\ & a_3a_1\varepsilon(b_{1,2})\end{pmatrix}, 1\right\rangle + \left\langle f\begin{pmatrix}a_3a_1\varepsilon(b_{1,2}) & 1\\ & a_2\end{pmatrix}, 1\right\rangle + \left\langle f\begin{pmatrix}a_1a_2\varepsilon(b_{1,2}) & 1\\ & a_3\end{pmatrix}, 1\right\rangle\\
&\qquad -\left\langle f\begin{pmatrix}a_3 & 1\\ & a_1a_2\varepsilon(b_{1,2})\end{pmatrix}, 1\right\rangle - \left\langle f\begin{pmatrix}a_1 & 1\\ & a_2a_3\varepsilon(b_{1,2})\end{pmatrix}, 1\right\rangle + \left\langle f\begin{pmatrix}a_2a_3\varepsilon(b_{1,2}) & 1\\ & a_1\end{pmatrix}, 1\right\rangle \qquad (14)
\end{aligned}$$

From the expressions in (13) and (14), it is clear that the diagram (12) does not commute in general, even if we take A to be commutative and $B = A$.

4. Relation with extra degeneracy and norm operator

We continue with A being a finite dimensional k-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle\cdot,\cdot\rangle : A\times A \to k$ and B a commutative k-algebra with a morphism of k-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. In particular, the non-degenerate pairing on A induces mutually inverse isomorphisms

$$\phi : A^* \xrightarrow{\cong} A \qquad \phi^{-1} : A \xrightarrow{\cong} A^* \tag{15}$$

We let $C^\bullet(A, M)$ denote the ordinary Hochschild complex of A with coefficients in an A-bimodule M and its cohomology by $H^\bullet(A, M)$. In particular, we may set $M = A^*$ equipped with the A-bimodule structure $(a'fa'')(a) := f(a''aa')$ for $f \in A^* = \mathrm{Hom}(A, k)$ and a, a', $a'' \in A$. In that case, the terms $\{C^n(A, A^*)\}_{n\geq 0}$ in the Hochschild complex $C^\bullet(A, A^*)$ may also be written as $C^n(A, A^*) \cong \mathrm{Hom}(A^{\otimes n+1}, k)$. We denote by $\widehat{C}^\bullet(A)$ the corresponding complex defined by setting $\widehat{C}^n(A) := \mathrm{Hom}(A^{\otimes(n+1)}, k)$ for $n \geq 0$.

From Tradler [11], we know that the operator $\Delta^\bullet : C^\bullet(A, A) \to C^{\bullet-1}(A, A)$ on Hochschild cochains inducing the BV-structure on $H^\bullet(A, A)$ fits into the following commutative diagram with the duals of Hochschild chains

$$\begin{array}{ccc}
\widehat{C}^{\bullet+1}(A) & \xrightarrow{Ns} & \widehat{C}^\bullet(A)\\
\cong\downarrow & & \downarrow\cong\\
C^\bullet(A, A^*) & & C^{\bullet-1}(A, A^*)\\
\phi^\bullet\downarrow\cong & & \cong\downarrow\phi^{\bullet-1}\\
C^\bullet(A, A) & \xrightarrow{\Delta^\bullet} & C^{\bullet-1}(A, A)
\end{array} \tag{16}$$

Here, each $\phi^\bullet : C^\bullet(A, A^*) \to C^\bullet(A, A)$ is the isomorphism induced by $\phi : A^* \xrightarrow{\cong} A$, while s and N respectively are the usual extra degeneracy and norm operators given by

$$\begin{aligned}
s : \widehat{C}^{n+1}(A) \longrightarrow \widehat{C}^n(A) \qquad & (sf)(a_1, \dots, a_n) := f(a_1, \dots, a_n, 1)\\
N := 1 + \lambda + \dots + \lambda^n : \widehat{C}^n(A) \longrightarrow \widehat{C}^n(A) \qquad & (\lambda\cdot f)(a_0, \dots, a_n) := (-1)^n f(a_n, a_0, \dots, a_{n-1})
\end{aligned} \tag{17}$$

If we pass to the normalized Hochschild complex which is a quasi-isomorphic subcomplex of $C^\bullet(A, A^*)$, then (16) induces the following commutative diagram

$$\begin{array}{ccc}
H^\bullet(A, A^*) & \xrightarrow{B^\bullet} & H^{\bullet-1}(A, A^*)\\
\phi^\bullet\downarrow\cong & & \cong\downarrow\phi^{\bullet-1}\\
H^\bullet(A, A) & \xrightarrow{\Delta^\bullet} & H^{\bullet-1}(A, A)
\end{array} \tag{18}$$

where $B^\bullet : H^\bullet(A, A^*) \to H^{\bullet-1}(A, A^*)$ is the standard Connes operator.

In the case of secondary Hochschild cohomology, we have shown in Section 3 that $\Delta^\bullet$ is not in general a morphism of complexes, i.e., it does not descend to cohomology. We will now show that the operator $\Delta^\bullet$ on secondary Hochschild cohomology $H^\bullet(A, B, \varepsilon)$ fits into a commutative diagram similar to (18).

In [8], Laubacher, Staic and Stancu have introduced a co-simplicial module $\overline{C}^\bullet(A, B, \varepsilon)$ which is used to compute the secondary Hochschild cohomology associated to the triple (A, B, ε). The terms of this co-simplicial module are given by

$$\overline{C}^\bullet(A,B,\varepsilon) := \left\{ \mathrm{Hom}\left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \mathrm{Hom}(A \otimes B^n, k) \right) \right\}_{n \geq 0} \tag{19}$$

It is important to note (see [8, Remark 4.7]) that despite the similar names, the complex $\overline{C}^\bullet(A, B, \varepsilon)$ cannot be expressed as the secondary Hochschild complex of (A, B, ε) with coefficients in some A-bimodule. This is because the "coefficient module" $\mathrm{Hom}(A \otimes B^n, k)$ appearing in (19) varies with n.

In addition, the cosimplicial module $\overline{C}^\bullet(A, B, \varepsilon)$ is equipped with a cyclic operator, which can be used to compute the secondary cyclic cohomology associated to the triple (A, B, ε). Using the isomorphisms

$$\Psi^n : \mathrm{Hom}\left(A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}, \mathrm{Hom}(A \otimes B^n, k) \right) \xrightarrow{\cong} \mathrm{Hom}\left(A^{\otimes (n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k \right) \tag{20}$$

given by

$$(\Psi^n g)\left(\bigotimes \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & b_{0,n} \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right)$$

$$= g\left(\bigotimes \begin{pmatrix} a_0 & b_{0,1} & \dots & b_{0,n-2} & b_{0,n-1} \\ 1 & a_1 & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) \left(a_n \otimes b_{0,n} \otimes b_{1,n} \otimes \cdots \otimes b_{n-1,n} \right) \tag{21}$$

we first transfer the cyclic operator from [8] to a complex $\widehat{C}^\bullet(A, B, \varepsilon)$ whose terms are given by

$$\left\{ \widehat{C}^n(A,B,\varepsilon) := \mathrm{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k) \right\}_{n \geq 0} \tag{22}$$

Lemma 8. *For each $n \geq 0$, there is an action of the cyclic group $\mathbb{Z}_{n+1} = \langle \lambda \rangle$ on the k-space $\widehat{C}^n(A, B, \varepsilon) = \mathrm{Hom}(A^{\otimes (n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k)$ given by*

$$(\lambda \cdot f)\left(\bigotimes \begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & b_{0,n} \\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & 1 & \dots & 1 & a_n \end{pmatrix} \right) = (-1)^n f\left(\bigotimes \begin{pmatrix} a_n & b_{0,n} & b_{1,n} & \dots & b_{n-2,n} & b_{n-1,n} \\ 1 & a_0 & b_{0,1} & \dots & b_{0,n-2} & b_{0,n-1} \\ 1 & 1 & a_1 & \dots & b_{1,n-2} & b_{1,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right)$$

for any $f \in \widehat{C}^n(A, B, \varepsilon)$, $a_i \in A$ and $b_{i,j} \in B$.

Proof. This is clear from the definition in [8, §4.2] and the isomorphisms in (20). □

The norm operator $N : \widehat{C}^n(A,B,\varepsilon) \to \widehat{C}^n(A,B,\varepsilon)$ is then defined as $N = 1+\lambda+\cdots+\lambda^n$. The extra degeneracy $s : \widehat{C}^n(A,B,\varepsilon) \to \widehat{C}^{n-1}(A,B,\varepsilon)$ is given by

$$(sf)\left(\bigotimes\begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-2} & b_{0,n-1}\\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-2} & b_{1,n-1}\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1}\\ 1 & 1 & 1 & \dots & 1 & a_{n-1}\end{pmatrix}\right) = f\left(\bigotimes\begin{pmatrix} a_0 & b_{0,1} & b_{0,2} & \dots & b_{0,n-1} & 1\\ 1 & a_1 & b_{1,2} & \dots & b_{1,n-1} & 1\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots\\ 1 & 1 & 1 & \dots & a_{n-1} & 1\\ 1 & 1 & 1 & \dots & 1 & 1\end{pmatrix}\right)$$

for any $f \in \widehat{C}^n(A,B,\varepsilon)$, $a_i \in A$ and $b_{i,j} \in B$.

For $n \geq 0$, let $\alpha^n : (A\otimes B^{\otimes n})^* \to A^*$ be the map defined by $p \mapsto \overline{p} := \alpha^n(p)$, where $\overline{p}(a) := p(a\otimes 1_B \otimes\cdots\otimes 1_B)$. We denote by $\alpha^\bullet : \overline{C}^\bullet(A,B,\varepsilon) \to C^\bullet((A,B,\varepsilon);A^*)$ the induced map.

We also let $\Phi^\bullet : C^\bullet((A,B,\varepsilon);A^*) \to C^\bullet((A,B,\varepsilon);A)$ be the map induced by the isomorphism $\phi : A^* \to A$ and $\Phi'^\bullet : C^\bullet((A,B,\varepsilon);A) \to C^\bullet((A,B,\varepsilon);A^*)$ be the map induced by ϕ^{-1}. It may also be verified that the inverse $\Psi'_n : \mathrm{Hom}(A^{\otimes(n+1)}\otimes B^{\otimes\frac{n(n+1)}{2}},k) \to \mathrm{Hom}\left(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}}, \mathrm{Hom}(A\otimes B^n,k)\right)$ of the map in (21) is given by

$$(\Psi'^n f)\left(\bigotimes\begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n}\\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n}\\ \vdots & \vdots & \dots & \vdots & \vdots\\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n}\\ 1 & 1 & \dots & 1 & a_n\end{pmatrix}\right)\begin{pmatrix} a_{n+1}\\ b_{1,n+1}\\ b_{2,n+1}\\ \vdots\\ b_{n-1,n+1}\\ b_{n,n+1}\end{pmatrix} = f\left(\bigotimes\begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} & b_{1,n+1}\\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} & b_{2,n+1}\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} & b_{n-1,n+1}\\ 1 & 1 & \dots & 1 & a_n & b_{n,n+1}\\ 1 & 1 & \dots & 1 & 1 & a_{n+1}\end{pmatrix}\right) \tag{23}$$

Proposition 9. *Let A be a finite dimensional k-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle\cdot,\cdot\rangle : A\times A \to k$ and B be a commutative k-algebra with a morphism of k-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:*

$$\begin{array}{ccc} \overline{C}^\bullet(A,B,\varepsilon) & \xrightarrow{Ns} & \overline{C}^{\bullet-1}(A,B,\varepsilon)\\ \Phi^\bullet\circ\alpha^\bullet\big\downarrow & & \big\downarrow\Phi^{(\bullet-1)}\circ\alpha^{(\bullet-1)}\\ C^\bullet((A,B,\varepsilon);A) & \xrightarrow{\Delta^\bullet} & C^{\bullet-1}((A,B,\varepsilon);A) \end{array} \tag{24}$$

Proof. We will show that for any $n\geq 0$, the following digram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}(A^{\otimes(n+1)}\otimes B^{\otimes\frac{n(n+1)}{2}},k) & \xrightarrow{Ns} & \mathrm{Hom}(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}},k)\\ \Psi'^n\big\downarrow & & \big\downarrow\Psi'^{n-1}\\ \mathrm{Hom}\left(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}},(A\otimes B^{\otimes n})^*\right) & & \mathrm{Hom}\left(A^{\otimes(n-1)}\otimes B^{\otimes\frac{(n-1)(n-2)}{2}},(A\otimes B^{\otimes(n-1)})^*\right)\\ \alpha^n\big\downarrow & & \big\downarrow\alpha^{n-1}\\ \mathrm{Hom}\left(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}},A^*\right) & & \mathrm{Hom}\left(A^{\otimes(n-1)}\otimes B^{\otimes\frac{(n-1)(n-2)}{2}},A^*\right)\\ \Phi^n\big\downarrow & & \big\downarrow\Phi^{n-1}\\ \mathrm{Hom}\left(A^{\otimes n}\otimes B^{\otimes\frac{n(n-1)}{2}},A\right) & \xrightarrow{\Delta} & \mathrm{Hom}\left(A^{\otimes(n-1)}\otimes B^{\otimes\frac{(n-1)(n-2)}{2}},A\right) \end{array} \tag{25}$$

Since Φ'^{n-1} is an isomorphism, it suffices to check that this diagram is commutative when composed with $\Phi'^{n-1} : \operatorname{Hom}\left(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A\right) \to \operatorname{Hom}\left(A^{\otimes(n-1)} \otimes B^{\otimes \frac{(n-1)(n-2)}{2}}, A^*\right)$. Let $f \in \operatorname{Hom}(A^{\otimes(n+1)} \otimes B^{\otimes \frac{n(n+1)}{2}}, k)$. Then, for $a_i \in A$ and $b_{i,j} \in B$, we have

$$
\left\langle (\Delta \circ \Phi^n \circ \alpha^n \circ \Psi'^n(f)) \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right), a_n \right\rangle
$$

$$
= \sum_{i=1}^{n} (-1)^{(n-1)i} \left\langle \Phi^n \alpha^n \Psi'^n(f) \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right), 1 \right\rangle
$$

$$
= \sum_{i=1}^{n} (-1)^{(n-1)i} \phi^{-1} \left(\phi \alpha^n \Psi'^n(f) \right) \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right) (1)
$$

$$
= \sum_{i=1}^{n} (-1)^{(n-1)i} \left(\alpha^n \Psi'^n(f) \right) \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right) (1)
$$

$$
= \sum_{i=1}^{n} (-1)^{(n-1)i} \left(\alpha^n \Psi'^n(f) \right) \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right) (1)
$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} (\Psi'^{n}(f)) \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} \end{pmatrix} \right) (1 \otimes 1 \otimes \cdots \otimes 1)$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} f \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} & 1 \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} & 1 \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} & 1 \\ 1 & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & 1 & 1 \end{pmatrix} \right)$$

On the other hand, let $g := (Ns)(f)$. Then, we have

$$(\alpha^{n-1} \circ \Psi'^{n-1}(g)) \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) (a_n)$$

$$= (\Psi'^{n-1}(g)) \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} \end{pmatrix} \right) (a_n \otimes 1 \otimes \cdots \otimes 1)$$

$$= (Ns)(f) \left(\bigotimes \begin{pmatrix} a_1 & b_{1,2} & b_{1,3} & \dots & b_{1,n-2} & b_{1,n-1} & 1 \\ 1 & a_2 & b_{2,3} & \dots & b_{2,n-2} & b_{2,n-1} & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a_{n-2} & b_{n-2,n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & a_{n-1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a_n \end{pmatrix} \right)$$

$$= \sum_{i=1}^{n} (-1)^{(n-1)i} f \left(\bigotimes \begin{pmatrix} a_i & b_{i,i+1} & b_{i,i+2} & \dots b_{i,n-1} & 1 & b_{1,i} & b_{2,i} & \dots & b_{i-1,i} & 1 \\ 1 & a_{i+1} & b_{i+1,i+2} & \dots b_{i+1,n-1} & 1 & b_{1,i+1} & b_{2,i+1} & \dots & b_{i-1,i+1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & 1 & b_{1,n-1} & b_{2,n-1} & \dots & b_{i-1,n-1} & 1 \\ 1 & 1 & \dots & 1 & a_n & 1 & \dots & & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 & a_1 & b_{1,2} & \dots & b_{1,i-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & \dots & \dots & \dots & \dots & \dots & & a_{i-1} & 1 \\ 1 & 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & 1 & 1 \end{pmatrix} \right)$$

This proves the result. □

We now let $\overline{\overline{C}}^{\bullet}(A,B,\varepsilon)$ denote the normalized complex associated to the cosimplicial module $\overline{C}^{\bullet}(A,B,\varepsilon)$ given in (19). Again, using the isomorphisms in (20), the complex $\overline{\overline{C}}^{\bullet}(A,B,\varepsilon)$ becomes isomorphic to the normalized complex $\overline{\widehat{C}}^{\bullet}(A,B,\varepsilon)$ whose terms are given by

$$\overline{\widehat{C}}^{n}(A,B,\varepsilon) := \operatorname{Ker}\left(\widehat{C}^{n}(A,B,\varepsilon) \xrightarrow{s_j} \bigoplus_{j=0}^{n-1} \widehat{C}^{n-1}(A,B,\varepsilon)\right)$$

where $s_j : \widehat{C}^{n}(A,B,\varepsilon) \to \widehat{C}^{n-1}(A,B,\varepsilon)$ for $0 \le j \le n-1$ is the degeneracy

$$(s_j f)\left(\bigotimes\begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & 1 & a_n \end{pmatrix}\right) = f\left(\bigotimes\begin{pmatrix} a_1 & b_{1,2} & \dots & b_{1,j} & 1 & b_{1,j+1} & \dots & b_{1,n-1} & b_{1,n} \\ 1 & a_2 & \dots & b_{2,j} & 1 & b_{2,j+1} & \dots & b_{2,n-1} & b_{2,n} \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & a_j & 1 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & 1 & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \dots & 1 & a_{j+1} & \dots & b_{j+1,n-1} & b_{j+1,n} \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1 & \dots & \dots & a_{n-1} & b_{n-1,n} \\ 1 & 1 & \dots & \dots & 1 & \dots & \dots & 1 & a_n \end{pmatrix}\right)$$

Theorem 10. *Let A be a finite dimensional k-algebra equipped with a symmetric, non-degenerate and invariant bilinear form $\langle\cdot,\cdot\rangle : A\times A \to k$ and B be a commutative k-algebra with a morphism of k-algebras $\varepsilon : B \to A$ such that $\varepsilon(B) \subseteq Z(A)$. Then, the following diagram commutes:*

$$\begin{array}{ccc} \overline{\overline{C}}^{\bullet}(A,B,\varepsilon) & \xrightarrow{\;B\;} & \overline{\overline{C}}^{\bullet-1}(A,B,\varepsilon) \\ \downarrow & & \downarrow \\ \overline{C}^{\bullet}(A,B,\varepsilon) & \xrightarrow{\;Ns\;} & \overline{C}^{\bullet-1}(A,B,\varepsilon) \\ \Big\downarrow{\scriptstyle \Phi^{\bullet}\circ\alpha^{\bullet}} & & \Big\downarrow{\scriptstyle \Phi^{(\bullet-1)}\circ\alpha^{(\bullet-1)}} \\ C^{\bullet}((A,B,\varepsilon);A) & \xrightarrow{\;\Delta^{\bullet}\;} & C^{\bullet-1}((A,B,\varepsilon);A) \end{array} \qquad (26)$$

where B is Connes' operator.

Proof. The commutativity of the lower square has already been shown in Proposition 9. By definition, Connes' operator on the complex $\overline{C}^{\bullet}(A,B,\varepsilon)$ is given by $B = Ns(1-\lambda)$ which reduces to Ns on the normalized complex $\overline{\overline{C}}^{\bullet}(A,B,\varepsilon)$. Hence, it may be directly verified that the upper diagram commutes. □

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