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Paires d'unicité de Heisenberg sur les espaces euclidiens et le groupe des mouvements

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Abstract. In this article, we consider Heisenberg uniqueness pairs corresponding to the exponential curve and surfaces, paraboloid, and sphere. Further, we look for analogous results related to the Heisenberg uniqueness pair on the Euclidean motion group and related product group.

Résumé. Dans cet article, nous considérons des paires d'unicité de Heisenberg correspondant aux courbes et surfaces exponentielles, au paraboloïde, à la sphère. De plus, nous cherchons des résultats analogues reliés à la paire d'unicité de Heisenberg sur le groupe des mouvements euclidiens et le groupe produit apparenté.

1. Introduction

In general, the uncertainty principle states that both function and its Fourier transform cannot be localized simultaneously (see [3,10,14]). As a version of the uncertainty principle, Hedenmalm and Montes-Rodríguez introduced the notion of the Heisenberg uniqueness pair. In [11], Hedenmalm and Montes-Rodríguez propose the following problem: Let Γ be a finite disjoint union of smooth curves in $\mathbb{R}^2$ and Λ be any subset of $\mathbb{R}^2$. Let $X(\Gamma)$ be the space of all finite complex-valued Borel measures in $\mathbb{R}^2$ which are supported on Γ and absolutely continuous with respect to the arc length measure on Γ, and for $(\xi, \eta) \in \mathbb{R}^2$, the Fourier transform of μ is defined by

$$\widehat{\mu}(\xi, \eta) = \int_{\Gamma} e^{-i\pi(x \cdot \xi + y \cdot \eta)} \mathrm{d}\mu(x, y).$$

When it would be possible that any $\mu \in X(\Gamma)$ satisfies $\widehat{\mu}(\xi, \eta) = 0$ for all $(\xi, \eta) \in \Lambda$ implies μ is identically zero? If this is the case, the pair (Γ, Λ) is called a Heisenberg uniqueness pair (or HUP).

The concept of HUP is quite similar to an annihilating pair made of Borel measurable sets of positive measures as described in Havin and Jöricke (see [10]). Consider a pair of Borel measurable sets $\mathscr{S}, \Sigma \subseteq \mathbb{R}$. Then $(\mathscr{S}, \Sigma)$ forms a mutually annihilating pair if for any $\varphi \in L^2(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \mathscr{S}$ and whose Fourier transform $\widehat{\varphi}$ supported on Σ, implies $\varphi = 0$.

Since the Fourier transform is invariant under translation and rotation, one can deduce the following invariance properties about the Heisenberg uniqueness pair (see [11]).

(i) Let $u_o, v_o \in \mathbb{R}^2$. Then the pair (Γ, Λ) is a HUP if and only if the pair $(\Gamma + u_o, \Lambda + v_o)$ is a HUP.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transform whose adjoint is denoted by T^*. Then (Γ, Λ) is a HUP if and only if $\left(T^{-1}\Gamma, T^*\Lambda\right)$ is a HUP.

In [11], Hedenmalm and Montes-Rodríguez have shown that the pair (hyperbola, some lattice-cross) is a Heisenberg uniqueness pair. As a dual problem, a weak-star dense subspace of $L^\infty(\mathbb{R})$ has been constructed to solve the one-dimensional Klein–Gordon equation. Further, they characterize the Heisenberg uniqueness pairs corresponding to any two parallel lines.

Afterward, a considerable amount of work has been done pertaining to the Heisenberg uniqueness pair in the plane as well as in the Euclidean spaces. Next, we briefly summarize the recent progress on Heisenberg uniqueness pair.

Lev [18] and Sjölin [20] have independently shown that circle and certain system of lines are HUP while Γ is the unit circle. Further, Vieli [7] has generalized HUP corresponding to circle in the higher dimension and show that a sphere whose radius does not lie in the zero sets of the Bessel functions $J_{(n+2k-2)/2}$; $k \in \mathbb{Z}_+$, the set of non-negative integers, is a HUP corresponding to the unit sphere S^{n-1}. In [23], Srivastava has shown that a cone is a Heisenberg uniqueness pair corresponding to the sphere as long as the cone does not completely lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$.

Further, Sjölin [21] has investigated some of the Heisenberg uniqueness pairs corresponding to the parabola. In a significant development, Jaming and Kellay [15] have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse, and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$. Further, Gröchenig and Jaming [8] have worked out some of the Heisenberg uniqueness pairs corresponding to the quadratic surface. Subsequently, Babot [2] has given a characterization of the Heisenberg uniqueness pairs corresponding to a certain system of three parallel lines. Thereafter, necessary and sufficient conditions for the Heisenberg uniqueness pairs corresponding to a system of four parallel lines was studied in [6]. For more details on Heisenberg uniqueness pairs, see ([4, 12, 13]).

The rest part of the paper is organized as follows. In Section 2, we study the Heisenberg uniqueness pairs on $\mathbb{R}^n$. In particular, we explore the HUP corresponding to the exponential curve and surfaces, paraboloid, and the sphere. In Section 3, we extend the concept of HUP on the Euclidean motion group and the product group.

2. Heisenberg uniqueness pairs on $\mathbb{R}^n$

2.1. *Heisenberg uniqueness pairs for exponential curve and surfaces.*

Let μ be a finite Borel measure having support on $\Gamma = \{(t, e^{t^2}) : t \in \mathbb{R}\}$ which is absolutely continuous with respect to the arc length on Γ. Then there exists $f_\mu \in L^1(\mathbb{R}, \sqrt{1 + 4t^2 e^{2t^2}}\, \mathrm{d}t)$ such that $\mathrm{d}\mu = g_\mu(t)\mathrm{d}t$, where $g_\mu(t) = f_\mu(t)\sqrt{1 + 4t^2 e^{2t^2}}$. Hence by finiteness of μ, it follows that $g_\mu \in L^1(\mathbb{R})$ and

$$\widehat{\mu}(x, y) = \int_{\mathbb{R}} e^{-i\pi\left(xt + ye^{t^2}\right)} g_\mu(t)\mathrm{d}t. \tag{1}$$

Then $\widehat{\mu}$ satisfies the PDE

$$(1 + \mathscr{D}_x)\,\widehat{\mu} = \beta \partial_y \widehat{\mu} \tag{2}$$

in the sense of distribution, where $\mathscr{D}_x = \sum_{n=0}^{\infty}\left(\sum_{k=1}^{4}\frac{\beta^{4n+k}}{(4n+k)!}\partial_x^{4n+k}\right)$ and $\beta = \frac{i}{\pi}$.

2.1.1. *An exponential type map*

Let $h_c : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_c(t) = \left(t + \frac{1}{2c}\right)^2 + e^{t^2} - t^2,$$

where c is a non-zero constant. It is clear that the range of h_c is $[a^2, \infty)$ for some $a > 0$. By derivative test, there exists $\alpha \in \mathbb{R}$ with $h_c(\alpha) = a^2$ such that h_c is strictly increasing in $[\alpha, \infty)$ and strictly decreasing in $(-\infty, \alpha]$. We denote $\widetilde{h}_c := h_c|_{[\alpha,\infty)}$ and $\bar{h}_c := h_c|_{(-\infty,\alpha]}$. Note that both the functions $\widetilde{h}_c$ and $\bar{h}_c$ are invertible having range $[a^2, \infty)$. Now, consider the reflection type map $\rho_c : [\alpha, \infty) \to (-\infty, \alpha]$ such that $\bar{h}_c \circ \rho_c = \widetilde{h}_c$. Thus, ρ_c is a diffeomorphism and $\rho_c(\alpha) = \alpha$. To describe the dynamic of the function h_c, we need to define an operator $T_c : L^1(\mathbb{R}) \to L^1([a, \infty))$ by

$$T_c[g](u) = \left[g \circ \widetilde{h}_c^{-1}(u^2) - g \circ \rho_c \circ \widetilde{h}_c^{-1}(u^2)\rho_c'(\widetilde{h}_c^{-1}(u^2))\right]\frac{2u}{\widetilde{h}_c'(\widetilde{h}_c^{-1}(u^2))}. \tag{3}$$

Let $\mathscr{N}(T_c)$ denotes the null space of T_c.

Remark 1.

(a) We would like to mention that when $h_c(t) = e^{t^2}$, then $\rho_c(t) = -t$. In this particular case, $\mathscr{N}(T_c)$ is the set of all odd integrable functions.

(b) In the sense of dynamical system and ergodic theory, the operator T_c defined in (3), can be thought as a *transfer-type operator*. For instance, Perron–Frobenius operator (see [19]) is used to study HUP (see [4, 11]). For further details on connection between dynamical system and HUP, we refer [12, 13, 15]. The null space of the operator T_c will be deterministic for our main result about HUP.

Theorem 2. *Let* $\Gamma = \left\{(t, \gamma(t)) : t \in \mathbb{R}\right\}$, *where* $\gamma : \mathbb{R} \to \mathbb{R}_+$ *is defined by* $\gamma(t) = e^{t^2}$.

(a) *Let* Λ *be a straight line and assume that* $\mathscr{N}(T_c) \neq \{0\}$ *for each* $c \neq 0$. *Then* (Γ, Λ) *is a Heisenberg uniqueness pair if and only if* Λ *is parallel to the x-axis.*

(b) *Let* L_j; $j = 1, 2$ *be two parallel lines which are not parallel to either of the axes. Then* $(\Gamma, L_1 \cup L_2)$ *is a Heisenberg uniqueness pair.*

In order to prove Theorem 2, we need the following results. First, we state a result which can be found in Havin and Jöricke [10, p. 36].

Lemma 3 ([10]). *Let* $\varphi \in L^1[0, \infty)$. *If* $\int_{\mathbb{R}} \log|\widehat{\varphi}|\frac{dx}{1+x^2} = -\infty$, *then* $\varphi = 0$.

Next, we state the following form of Radon–Nikodym derivative theorem (see [5, p. 91]).

Proposition 4. *Let* ν *be a* σ-*finite signed measure which is absolutely continuous with respect to a* σ-*finite measure* μ *on the measure space* $(X, \mathscr{M})$. *If* $g \in L^1(\nu)$, *then* $g\frac{d\nu}{d\mu} \in L^1(\mu)$ *and* $\int g d\nu = \int g\frac{d\nu}{d\mu}d\mu$.

As a consequence of Lemma 3 and Proposition 4, we prove the following result. Let $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}$.

Lemma 5. *Let* $g \in L^1(\mathbb{R})$. *Suppose* $E \subset \mathbb{R}$ *and* $|E| > 0$. *Then*

$$\int_{\mathbb{R}} e^{-i\pi cx h_c(t)}g(t)dt = 0 \tag{4}$$

for all $x \in E$ *if and only if* $g \in \mathscr{N}(T_c)$.

Proof. The left-hand side of (4) can be written as

$$I = \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(t)} g(t) \mathrm{d}t + \int_{-\infty}^{\alpha} e^{-i\pi cx\tilde{h}_c(t)} g(t) \mathrm{d}t$$

$$= \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(t)} g(t) \mathrm{d}t - \int_{\alpha}^{\infty} e^{-i\pi cx\tilde{h}_c(s)} g(\rho_c(s)) \cdot \rho_c'(s) \mathrm{d}s,$$

by using the change of variables $s = \rho_c^{-1}(t)$. Further applying the change of variables $\tilde{h}_c(t) = u^2$ we get

$$I = \int_{a}^{\infty} e^{-i\pi cxu^2} \left[g \circ \tilde{h}_c^{-1}(u^2) - g \circ \rho_c \circ \tilde{h}_c^{-1}(u^2) \cdot \rho_c'(\tilde{h}_c^{-1}(u^2)) \right] \frac{2u}{\tilde{h}_c'(\tilde{h}_c^{-1}(u^2))} \mathrm{d}u$$

$$= \int_{a}^{\infty} e^{-i\pi cxu^2} T_c[g](u) \mathrm{d}u.$$

In view of Proposition 4, the function $T_c[g] \in L^1([a, \infty))$, and by the change of variables $u^2 = v$, we have

$$I = \int_{a^2}^{\infty} e^{-i\pi cxv} T_c[g](\sqrt{v}) \frac{\mathrm{d}v}{2\sqrt{v}}. \tag{5}$$

Let $\varphi(v) = T_c[g](\sqrt{v})/2\sqrt{v} \, \chi_{(a^2, \infty)}(v)$. Then $\varphi \in L^1(\mathbb{R})$ and from (5) we obtain $I = \widehat{\varphi}(cx) = 0$ for all $x \in E$. That is, $\widehat{\varphi}$ vanishes on the set cE of positive measure. Thus, by Lemma 3 we conclude that $\varphi = 0$ a.e. Hence, it follows that $T_c[g] = 0$ a.e. on $[a, \infty)$. Conversely, if $T_c[g] = 0$, then (4) trivially holds. $\quad\square$

In view of Remark 1(a) and Lemma 5, we can derive the following result.

Corollary 6. *Let $g \in L^1(\mathbb{R})$ and $\gamma : \mathbb{R} \to \mathbb{R}_+$ be defined by $\gamma(t) = e^{t^2}$. Suppose $E \subset \mathbb{R}$ and $|E| > 0$. Then*

$$\int_{\mathbb{R}} e^{-i\pi x\gamma(t)} g(t) \mathrm{d}t = 0 \tag{6}$$

for all $x \in E$ if and only if g is an odd function.

Proposition 7. *Let $\mu \in X(\Gamma)$ and $g_\mu \in L^1(\mathbb{R})$, as appeared in (1). If $E \subset \mathbb{R}$ with $|E| > 0$, then for $c, d \in \mathbb{R} \setminus \{0\}$ the following holds.*

   (a) $\widehat{\mu}(x, cx) = 0$ *for all $x \in E$ if and only if $g_\mu \in \mathcal{N}(T_c)$.*

   (b) $\widehat{\mu}(x, cx + d) = 0$ *for all $x \in E$ if and only if $\chi_\mu \in \mathcal{N}(T_c)$, where $\chi_\mu(t) = e^{-i\pi de^{t^2}} g_\mu(t)$.*

Proof. (a). From (1) we can express

$$\widehat{\mu}(x, cx) = \int_{\mathbb{R}} e^{-i\pi x\left(t + ce^{t^2}\right)} g_\mu(t) \mathrm{d}t = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cxh_c(t)} g_\mu(t) \mathrm{d}t.$$

By Lemma 5, $T_c[g_\mu] = 0$ if and only if $\widehat{\mu}(x, cx) = 0$ for all $x \in E$.

(b). By a simple computation, we get

$$\widehat{\mu}(x, cx + d) = \int_{\mathbb{R}} e^{-i\pi x\left(t + ce^{t^2}\right)} \chi_\mu(t) \mathrm{d}t = e^{i\pi x/4c} \int_{\mathbb{R}} e^{-i\pi cxh_c(t)} \chi_\mu(t) \mathrm{d}t.$$

As similar to the above case, $T_c[\chi_\mu] = 0$ if and only if $\widehat{\mu}(x, cx + d) = 0$ for all $x \in E$. $\quad\square$

Proof of Theorem 2. (a). In view of the invariance property, we can assume that Λ is the x-axis. Recall that $\widehat{\mu}$ satisfies

$$\widehat{\mu}(x, y) = \int_{\mathbb{R}} e^{-i\pi\left(xt + ye^{t^2}\right)} g_\mu(t) \mathrm{d}t.$$

Hence $\widehat{\mu}|_\Lambda = 0$ implies that $\widehat{g}_\mu(x) = 0$ for all $x \in \mathbb{R}$. Thus, we conclude that $\mu = 0$.

Conversely, suppose Λ is not parallel to the x-axis. If Λ is parallel to the y-axis, then by Corollary 6, it follows that (Γ, Λ) is not a HUP. Hence we can assume that Λ of the form $y = cx$, where $c \neq 0$. Choose a non-zero function $g \in \mathcal{N}(T_c)$, then by Proposition 7, it follows that (Γ, Λ) is not a Heisenberg uniqueness pair.

(b). Let $L_1 = \{(x, cx) : x \in \mathbb{R}\}$ and $L_2 = \{(x, cx + d) : x \in \mathbb{R}\}$, where $c, d \neq 0$. Since $\widehat{\mu}|_{L_j} = 0$; $j = 1, 2$, by Proposition 7 it follows that $T_c[g_\mu] = 0$ and $T_c[\chi_\mu] = 0$. Thus we have

$$\left[e^{i\pi d \left\{ e^{(\widetilde{h}_c^{-1}(u^2))^2} - e^{(\rho_c \circ \widetilde{h}_c^{-1}(u^2))^2} \right\}} - 1 \right] g_\mu \circ \widetilde{h}_c^{-1}(u^2) = 0.$$

For a non-zero c, we have $\rho_c^2(t) \neq t^2$ except at $t = \alpha$. Therefore $g_\mu \circ \widetilde{h}_c^{-1}(u^2) = 0$ a.a. $u \geq a$, that is, $g_\mu = 0$ a.e. on $[\alpha, \infty)$. As $T_c[g_\mu] = 0$ and $\rho_c' \neq 0$ a.e., it follows that $g_\mu = 0$ a.e. Thus, the pair $(\Gamma, L_1 \cup L_2)$ is a Heisenberg uniqueness pair. □

Remark 8.

   (a) Let $\gamma : \mathbb{R} \to \mathbb{R}_+$ be an even smooth function having finitely many local extrema and consider $\Gamma = \{(t, \gamma(t)) : t \in \mathbb{R}\}$. Then the conclusions of Theorem 2 would also hold.
   (b) If we consider $\Gamma = \{(e^{t^2} \cosh t, e^{t^2} \sinh t) : t \in \mathbb{R}\}$ and $\Lambda = L_1 \cup L_2$; where L_j; $j = 1, 2$ are any two straight lines parallel to the X-axis, then (Γ, Λ) is a HUP.

Theorem 9. *Let Γ be the surface $x_{n+1} = e^{x_1^2} + \cdots + e^{x_n^2}$ in $\mathbb{R}^{n+1}$ and Λ an affine hyperplane in $\mathbb{R}^{n+1}$ of dimension n. Assume that $\mathcal{N}(T_c) \neq \{0\}$ for each $c \neq 0$. Then (Γ, Λ) is a Heisenberg uniqueness pair if and only if Λ is parallel to the hyperplane $x_{n+1} = 0$.*

For $u = (u_1, \ldots, u_n)$, denoting $\varphi(u) = e^{u_1^2} + \cdots + e^{u_n^2}$. Let μ be a finite Borel measure which is supported on $\Gamma = \{(u, \varphi(u)) : u \in \mathbb{R}^n\}$ and absolutely continuous with respect to the surface measure on Γ. Then by Radon–Nikodym theorem, there exists a measurable function f_μ on $\mathbb{R}^n$ such that $\mathrm{d}\mu = g_\mu(u)\mathrm{d}u$, where $g_\mu(u) = f_\mu(u)\sqrt{1 + \|\operatorname{grad}\varphi(u)\|^2}$. Then by the finiteness of μ, it follows that $g_\mu \in L^1(\mathbb{R}^n)$. Denote $u' = (u_2, \ldots, u_n)$, $x'' = (x_1, \ldots, x_n)$ and $x' = (x_2, \ldots, x_n)$. Then the Fourier transform of μ can be expressed as

$$\widehat{\mu}(x) = \int_{\mathbb{R}^n} e^{-\pi i (x'' \cdot u + x_{n+1}\varphi(u))} g_\mu(u)\mathrm{d}u \tag{7}$$

for $x \in \mathbb{R}^{n+1}$.

Proof of Theorem 9. Since Λ is an affine hyperplane in $\mathbb{R}^{n+1}$ of dimension n, by the invariance properties of HUP, we can assume that Λ is a linear subspace of $\mathbb{R}^{n+1}$ which can be considered as either $x_{n+1} = cx_1$, where $c \in \mathbb{R}$ or $x_1 = 0$.

If $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, then by the hypothesis, $\widehat{\mu}|_\Lambda = 0$ implies $\widehat{g}_\mu = 0$ on $\mathbb{R}^n$. Thus, it follows that (Γ, Λ) is a HUP.

Conversely, suppose Λ is not parallel to the hyperplane $x_{n+1} = 0$. Consider a non-zero compactly supported function $h \in L^1(\mathbb{R}^{n-1})$. Then we have the following two cases.

Case 1. If $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 = 0\}$, then take a non-zero compactly supported odd function $\psi \in L^1(\mathbb{R})$ and write $g(u) = \psi(u_1)h(u')$. Then we can construct a non-zero measure μ such that for $x \in \Lambda$, we have

$$\begin{aligned}
\widehat{\mu}(x) &= \int_{\mathbb{R}^n} e^{-\pi i (x' \cdot u' + x_{n+1}\varphi(u))} g(u)\mathrm{d}u \\
&= \int_{\mathbb{R}^n} e^{-\pi i (x' \cdot u' + x_{n+1}\varphi(u))} \psi(u_1)h(u')\mathrm{d}u \\
&= \int_{\mathbb{R}^{n-1}} e^{-\pi i \left\{ x' \cdot u' + x_{n+1}\left(\varphi(u) - e^{u_1^2}\right) \right\}} \left(\int_{\mathbb{R}} e^{-\pi i x_{n+1} e^{u_1^2}} \psi(u_1)\mathrm{d}u_1 \right) h(u')\mathrm{d}u' \\
&= 0.
\end{aligned}$$

Thus, (Γ, Λ) is not a Heisenberg uniqueness pair.

Case 2. Suppose $\Lambda = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = cx_1, c \neq 0\}$. Consider a non zero function $\tau \in \mathcal{N}(T_c)$ and write $g(u) = \tau(u_1)h(u')$. Then for $x \in \Lambda$, we have

$$
\begin{aligned}
\widehat{\mu}(x) &= \int_{\mathbb{R}^n} e^{-\pi i (x''.u + cx_1 \varphi(u))} g(u) \mathrm{d}u \\
&= \int_{\mathbb{R}^n} e^{-\pi i (x''.u + cx_1 \varphi(u))} \tau(u_1) h(u') \mathrm{d}u \\
&= \int_{\mathbb{R}^{n-1}} e^{-\pi i \left\{ x'.u' + cx_1 \left(\varphi(u) - e^{u_1^2} \right) \right\}} \left(\int_{\mathbb{R}} e^{-\pi i \left(x_1 u_1 + cx_1 e^{u_1^2} \right)} \tau(u_1) \mathrm{d}u_1 \right) h(u') \mathrm{d}u'.
\end{aligned}
$$

By Lemma 5, it follows that

$$
\int_{\mathbb{R}} e^{-\pi i x_1 (u_1 + c e^{u_1^2})} \tau(u_1) \mathrm{d}u_1 = e^{i\pi x_1/4c} \int_{\mathbb{R}} e^{-\pi i c x_1 h_c(u_1)} \tau(u_1) \mathrm{d}u_1 = 0.
$$

Thus, we conclude that (Γ, Λ) is not a Heisenberg uniqueness pair. $\qquad\square$

2.2. *Heisenberg uniqueness pairs for the paraboloid and the sphere*

2.2.1. *Spherical harmonic*

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $l \in \mathbb{Z}_+$, let P_l denote the space of all homogeneous polynomials of degree l in n variables. Let $H_l = \{P \in P_l : \Delta P = 0\}$, where Δ is the standard Laplacian on $\mathbb{R}^n$. The elements of H_l are called solid spherical harmonics of degree l. It is worthy to mention that H_l is invariant under the natural action of $SO(n)$. A spherical harmonic of degree l is $P|_{S^{n-1}}$, where $P \in H_l$. Let

$$
\mathcal{H}_l = \{P|_{S^{n-1}} : P \in H_l\}.
$$

We write d_l for the dimension of $\mathcal{H}_l$ and let $\{Y_{lj} : 1 \le j \le d_l\}$ be an orthonormal basis of $\mathcal{H}_l$. Let $\mathrm{d}\sigma$ be the natural measure on S^{n-1}, then any two spherical harmonics of different degrees l and k are orthogonal with respect to the usual inner product on $L^2(S^{n-1}, \mathrm{d}\sigma)$. Let the set $\widetilde{\mathcal{H}} = \{Y_{lj} : 1 \le j \le d_l, l \in \mathbb{Z}_+\}$ be an orthonormal basis for $L^2(S^{n-1}, \mathrm{d}\sigma)$. For each fixed $\xi \in S^{n-1}$, define a linear functional on $\mathcal{H}_l$ by $Y_l \mapsto Y_l(\xi)$. Then there exists a unique spherical harmonic, say $Z_\xi^{(l)} \in \mathcal{H}_l$ such that

$$
Y_l(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) Y_l(\eta) \mathrm{d}\sigma(\eta).
$$

The spherical harmonic $Z_\xi^{(l)}$ is called the zonal harmonic of degree l with pole at ξ. Let $f \in L^1(S^{n-1}, \mathrm{d}\sigma)$. For each $l \in \mathbb{Z}_+$, define the l-th spherical harmonic projection of f by

$$
\Pi_l f(\xi) = \int_{S^{n-1}} Z_\xi^{(l)}(\eta) f(\eta) \mathrm{d}\sigma(\eta)
$$

for all $\xi \in S^{n-1}$. The projection $\Pi_l f$ is a spherical harmonic of degree l. For $\delta > (n-2)/2$, write $A_l^m(\delta) = \binom{m-l+\delta}{\delta}\binom{m+\delta}{\delta}^{-1}$. Then the Fourier–Laplace series $\sum_{l=0}^{\infty} \Pi_l f$ is δ-Cesaro summable to f. That is,

$$
f = \lim_{m \to \infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f, \tag{8}
$$

where limit in the right-hand side of (8) exists in $L^1(S^{n-1})$. Also, for $\delta \ge 0$ and $l \in \mathbb{Z}_+$, we have $\lim_{m \to \infty} A_l^m(\delta) = 1$. For more details, see [22].

A set $\mathcal{C}$ in $\mathbb{R}^n$ ($n \ge 2$), which satisfies the scaling condition $\lambda \mathcal{C} \subseteq \mathcal{C}$, for all $\lambda \in \mathbb{R}$, is called a cone. We call a cone to be *non-harmonic* if it is not contained in the zero sets of any homogeneous harmonic polynomial on $\mathbb{R}^n$.

2.2.2. *Heisenberg uniqueness pairs for the paraboloid*

Let Γ be the paraboloid $x_{n+1} = x_1^2 + \cdots + x_n^2$ in $\mathbb{R}^{n+1}$ and $\mu \in X(\Gamma)$. Then there exists $g \in L^1(\mathbb{R}^n)$ such that $\mathrm{d}\mu = g(u)\sqrt{1 + 4\|u\|^2}\mathrm{d}u$. We write $f(u) = g(u)\sqrt{1 + 4\|u\|^2}$, then the Fourier transform of μ can be expressed as

$$\widehat{\mu}(x_1, \ldots, x_{n+1}) = \int_{\mathbb{R}^n} e^{-i(x, x_{n+1}) \cdot (u, \|u\|^2)} f(u)\mathrm{d}u \tag{9}$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x_{n+1} \in \mathbb{R}$.

Proposition 10. *Let Γ be a paraboloid in $\mathbb{R}^{n+1}$ and S^{n-1} be the unit sphere in $\mathbb{R}^n$. Then the following holds.*

(a) *If $\Lambda = S^{n-1} \times \mathbb{R}$, then (Γ, Λ) is a Heisenberg uniqueness pair.*
(b) *If $\Lambda = \mathscr{C} \times \mathbb{R}$, where $\mathscr{C}$ is a cone in $\mathbb{R}^n$, then (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\mathscr{C}$ is non-harmonic.*

In order to prove Proposition 10, we need the following result.

Lemma 11 ([20]). *Let $\varphi \in L^1((0, \infty))$ and E be a measurable subset of $\mathbb{R}$ with $|E| > 0$. If*

$$\int_0^\infty e^{-ixt^2} \varphi(t)\mathrm{d}t = 0 \text{ for all } x \in E,$$

then $\varphi = 0$.

Proof of Proposition 10. (a). In view of (9), $\widehat{\mu}$ vanishes on Λ implies

$$\int_{\mathbb{R}^n} e^{-i(x, x_{n+1}) \cdot (u, \|u\|^2)} f(u)\mathrm{d}u = 0,$$

for all $(x, x_{n+1}) \in S^{n-1} \times \mathbb{R}$. Next, by converting the above integral into the polar coordinates, we get

$$\int_0^\infty e^{-ix_{n+1}r^2} \int_{S_r^{n-1}} e^{-ix \cdot \eta} f(\eta)\mathrm{d}\sigma_r(\eta)\mathrm{d}r = 0.$$

It follows from Lemma 11 that for each $r > 0$,

$$\int_{S_r^{n-1}} e^{-ix \cdot \eta} f(\eta)\mathrm{d}\sigma_r(\eta) = 0 \tag{10}$$

for all $x \in S^{n-1}$. If we write $f_r(\xi) = f(r\xi)$ for $\xi \in S^{n-1}$, then (10) reduce to

$$\int_{S^{n-1}} e^{-iy \cdot \xi} f_r(\xi)\mathrm{d}\sigma(\xi) = 0$$

for all $y \in S_r^{n-1}$ and $r > 0$. In view of ([7, Proposition 1.2]), we have $f_r(\xi) = 0$ for *a.e.* $\xi \in S^{n-1}$ if and only if $J_{d+(n-2)/2}(r) \neq 0$ for all $d \in \mathbb{Z}_+$.

Thus, we infer that $f(\eta) = 0$ for a.e. $\eta \in S_r^{n-1}$ if and only if $J_{d+(n-2)/2}(r) \neq 0$ for all $d \in \mathbb{Z}_+$. Since the set $\{r > 0 : J_{d+(n-2)/2}(r) = 0, \text{ for some } d \in \mathbb{Z}_+\}$ is countable, we conclude that $f = 0$ a.e., and hence $\mu = 0$.

(b). Let $\mathscr{C}$ be a non-harmonic cone. In view of (10), we have

$$\int_{S_r^{n-1}} e^{-ix \cdot \eta} f(\eta)\mathrm{d}\sigma_r(\eta) = 0$$

for all $x \in \mathscr{C}$ and $r > 0$. Since $\mathscr{C}$ is non-harmonic, by ([23, Theorem 3.1]),

$$\int_{S^{n-1}} e^{-iy \cdot \xi} f_r(\xi)\mathrm{d}\sigma(\xi) = 0$$

for all $y \in \mathscr{C}$ and $r > 0$, where $f_r(\xi) = f(r\xi)$ for all $\xi \in S^{n-1}$ implies $f(\eta) = 0$ a.e. $\eta \in S_r^{n-1}$, and for each $r > 0$. Hence $f = 0$ *a.e.* that is, (Γ, Λ) is a Heisenberg uniqueness pair.

Conversely, assume that $\mathscr{C}$ is contained in the zero set of a homogeneous harmonic polynomial $P_l \in H_l$. Define a function f on $\mathbb{R}^n$ by $f(y) = e^{-r^2} Y_l(\xi)$, where $y = r\xi, r > 0$ and $\xi \in S^{n-1}$ and $Y_l \in \mathscr{H}_l$. Then $f \in L^1(\mathbb{R}^n)$. Thus, we can construct a finite complex Borel measure μ in $\mathbb{R}^n$ by $\mathrm{d}\mu = f(u)\mathrm{d}u$. By Funk–Hecke theorem (see [1]), for each $r > 0$ and $x \in \mathscr{C}$,

$$\int_{S_r^{n-1}} e^{-ix\cdot\eta} f(\eta)\mathrm{d}\sigma_r(\eta) = \int_{S^{n-1}} e^{-irx\cdot\xi} e^{-r^2} Y_l(\xi)\mathrm{d}\sigma(\xi)$$

$$= (2\pi)^{n/2} e^{-r^2} i^l \frac{J_{l+(n-2)/2}(rs)}{(rs)^{(n-2)/2}} Y_l(\zeta),$$

where $x = s\zeta$ for some $s > 0$ and $\zeta \in S^{n-1}$. This shows that $\hat{\mu}|_\Lambda = 0$ but μ is non-zero measure supported on Γ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 12. If we consider $\Lambda = S_r^{n-1} \times \mathbb{R}$ for some $r > 0$, then also Proposition 10 (a) remains true.

2.2.3. *Examples of non-uniqueness sets for the sphere*

Next, we extend a result due to Lev (see [18]) to higher dimensions.

Theorem 13. *Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and Λ be the union of two or more spheres in $\mathbb{R}^n$. Then (S^{n-1}, Λ) is not a Heisenberg uniqueness pair if and only if these spheres in Λ are concentric and their radii lay in the zero sets of the same Bessel function of the form $J_{d+(n-2)/2}$ where $d \in \mathbb{Z}_+$.*

Proof. Let Λ be the union of two or more spheres with center at $a \in \mathbb{R}^n$. Further, assume that their radius lies in the zero sets of $J_{d+(n-2)/2}$, for some $d \in \mathbb{Z}_+$. Consider $f(\eta) = e^{ia.\eta} Y_d(\eta)$ and write $\mathrm{d}\mu = f\mathrm{d}\sigma$. For $x \in \Lambda$, there exist $r > 0$ and $\xi \in S^{n-1}$ such that $x = a + r\xi$. Hence the expression

$$\hat{\mu}(x) = \int_{S^{n-1}} e^{-i(a+r\xi).\eta} f(\eta)\mathrm{d}\sigma(\eta)$$

$$= (2\pi)^{n/2} i^d r^{-(n-2)/2} J_{d+(n-2)/2}(r) Y_d(\xi)$$

shows that (S^{n-1}, Λ) is not a Heisenberg uniqueness pair.

Conversely, let Λ be union of two spheres such that (S^{n-1}, Λ) is not a HUP. Due to invariance properties of HUP we can assume that $\Lambda = \Lambda_r \cup \Lambda_\rho$, where Λ_r is the sphere of radius r center at origin and Λ_ρ is the sphere of radius ρ center at $(a_1, 0, \ldots, 0)$ such that $J_{l_1+(n-2)/2}(r) = 0$ and $J_{l_2+(n-2)/2}(\rho) = 0$ for some $l_1, l_2 \in \mathbb{Z}_+$. Since the zero sets of the above two Bessel functions can intersect at most at the origin ([25, p. 484]), $J_{m+(n-2)/2}(r) = 0$ only if $m = l_1$ and similar conclusion holds true for ρ. Now, $\hat{\mu} = 0$ on Λ_r implies

$$(2\pi)^{n/2} i^l r^{-(n-2)/2} J_{l+(n-2)/2}(r) \|\Pi_l f\|_2^2 = 0$$

for all $l \in \mathbb{Z}_+$ (see [7]). It follows that $\Pi_l f = 0$ for all $l \in \mathbb{Z}_+$ except $l = l_1$. Thus from (8) we have $f(\eta) = \Pi_{l_1} f(\eta)$. Further, $\hat{\mu}$ vanishes on Λ_ρ gives $g(\eta) = \Pi_{l_2} g(\eta)$, where $g(\eta) = e^{-ia_1\eta_1} f(\eta)$, that is, $f(\eta) = e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta)$, where η_1 be the first coordinate of η. Therefore

$$\Pi_{l_1} f(\eta) = e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta) \qquad\qquad\qquad\qquad (11)$$

for all $\eta \in S^{n-1}$.

Next, we show that $a_1 = 0$ and $l_1 = l_2$. If $a_1 = 0$, then by the orthogonality of spherical harmonics we get $l_1 = l_2$. Observe from (11) that $e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta)$ is a spherical harmonic of degree l_1. Hence for all $\alpha > 0$, we have

$$\alpha^{l_1} e^{ia_1\eta_1} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta) = e^{i\alpha a_1\eta_1} \Pi_{l_2} e^{-i\alpha a_1\eta_1} f(\alpha\eta) \qquad\qquad (12)$$

$$= e^{i\alpha a_1\eta_1} \alpha^{l_2} \Pi_{l_2} e^{-ia_1\eta_1} f(\eta).$$

We claim that $\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) \neq 0$ for some η such that $\eta_1 \neq 0$. In contrary, if $\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) = 0$ for all η such that $\eta_1 \neq 0$, then

$$\Pi_{l_2} e^{-ia_1\eta_1} f(\eta) = 0$$

for almost all η, which implies $g = 0$ and hence $f = 0$, which is not possible.

Thus by the above claim and (12) we get

$$\alpha^{l_1} e^{ia_1\eta_1} = e^{i\alpha a_1\eta_1} \alpha^{l_2}$$

for all $\alpha > 0$, which is possible only if $l_1 = l_2$ and $a_1 = 0$. This completes the proof while Λ is the union of two spheres.

If Λ is the union of more than two spheres, then by applying the above argument for each pair of spheres in Λ, we can reach to the conclusion. $\square$

3. A connection of Heisenberg uniqueness pair to the Euclidean motion group and the product group

3.1. *Euclidean motion group M(n)*

Let G denotes the Euclidean motion group $M(n)$ is the group of isometries of $\mathbb{R}^n$ that leaves invariant the Laplacian. Since the action of the special orthogonal group $K = SO(n)$ defines a group of automorphisms on $\mathbb{R}^n$ via $y \mapsto ky + x$, where $x \in \mathbb{R}^n$ and $k \in K$, the group $M(n)$ can be identified as the semi-direct product of $\mathbb{R}^n$ and K. Hence the group law on G can be expressed as

$$(x, s) \cdot (y, t) = (x + sy, st).$$

Since a right K-invariant function on G can be thought as a function on $\mathbb{R}^n$, we infer that the Haar measure on G can be written as $dg = dx\,dk$, where dx and dk are the normalized Haar measures on $\mathbb{R}^n$ and K respectively.

Let $\mathbb{R}_+ = (0, \infty)$ and $M = SO(n-1)$ be the subgroup of K that fixes the point $e_n = (0, \ldots, 0, 1)$. Let $\widehat{M}$ be the unitary dual group of M. Given a unitary irreducible representation $\sigma \in \widehat{M}$ realized on the Hilbert space $\mathcal{H}_\sigma$ of dimension d_σ, we consider the space $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ consisting of $d_\sigma \times d_\sigma$ complex matrices valued functions φ on K such that $\varphi(uk) = \sigma(u)\varphi(k)$, where $u \in M$, $k \in K$ and satisfying

$$\int_K \|\varphi(k)\|^2 dk = \int_K \text{tr}(\varphi(k)^* \varphi(k)) dk.$$

It is easy to see that $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is a Hilbert space under the inner product

$$\langle \varphi, \psi \rangle = \int_K \text{tr}(\varphi(k)\psi(k)^*) dk.$$

Each $(a, \sigma) \in \mathbb{R}_+ \times \widehat{M}$, defines a unitary representation $\pi_{a,\sigma}$ of G by

$$\pi_{a,\sigma}(g)(\varphi)(k) = e^{-ia\langle x, k \cdot e_n \rangle} \varphi(s^{-1}k), \tag{13}$$

where $\varphi \in L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$. Let $\varphi = (\varphi_1, \ldots, \varphi_{d_\sigma})$, where φ_j are the column vectors of φ. Then $\varphi_j(uk) = \sigma(u)\varphi_j(k)$. Now, consider the space

$$H(K, \mathbb{C}^{d_\sigma}) = \left\{ \varphi : K \to \mathbb{C}^{d_\sigma}, \int_K |\varphi(k)|^2 dk < \infty, \varphi(uk) = \sigma(u)\varphi(k), u \in M \right\}.$$

It is obvious that $L^2(K, \mathbb{C}^{d_\sigma \times d_\sigma})$ is the direct sum of d_σ copies of the Hilbert space $H(K, \mathbb{C}^{d_\sigma})$ equipped with the inner product

$$\langle \varphi, \psi \rangle = d_\sigma \int_K (\varphi(k), \psi(k)) dk.$$

Now, it can be shown that an infinite-dimensional unitary irreducible representation of G is the restriction of $\pi_{a,\sigma}$ to $H(K, \mathbb{C}^{d_\sigma})$. In other words, each of $(a, \sigma) \in \mathbb{R}_+ \times \widehat{M}$ defines a principal series representation $\pi_{a,\sigma}$ of G via (13).

In addition to the principal series representations, there are finite-dimensional unitary irreducible representations of G, which can be identified with $\widehat{K}$, though these unitary representations do not take part in the Plancherel formula. For more details, we refer to Kumahara [17] and Sugiura [24].

Now, we define the group Fourier transform of a function $f \in L^1(G)$ by

$$\widehat{f}(a,\sigma) = \int_G f(g)\pi_{a,\sigma}(g^{-1})\mathrm{d}g$$

and

$$\widehat{f}(\delta) = \int_G f(x,k)\delta(k^{-1})\mathrm{d}x\mathrm{d}k,$$

where $\delta \in \widehat{K}$. Further, the operator $\widehat{f}(a,\sigma)$ can be explicitly written as

$$\begin{aligned}
(\widehat{f}(a,\sigma)\varphi)(k) &= \int_{\mathbb{R}^n}\int_K f(x,s)e^{-i\langle x,\,ak\cdot e_n\rangle}\varphi(s^{-1}k)\mathrm{d}x\mathrm{d}s,\\
&= \int_K \mathscr{F}_1 f(ak\cdot e_n,s)\varphi(s^{-1}k)\mathrm{d}s,
\end{aligned} \tag{14}$$

where $\mathscr{F}_1$ stands for the usual Fourier transform in the first variable and $\varphi \in H(K,\mathbb{C}^{d_\sigma})$. For more details, we refer to [5, 9, 16].

3.2. *Heisenberg uniqueness pairs on M(n)*

Let Γ be a smooth surface (or a finite union of smooth surfaces) in $\mathbb{R}^n$ and $\Gamma_G = \Gamma \times K$. Let $X(\Gamma_G)$ be the space of all finite complex-valued Borel measures μ in the motion group G, which are supported on Γ_G and absolutely continuous with respect to the surface measure on Γ_G.

We define the Fourier transform of μ on G by

$$(\widehat{\mu}(a,\sigma)\varphi)(k) = \int_\Gamma \int_K e^{-i\langle x,ak\cdot e_n\rangle}\varphi(s^{-1}k)\mathrm{d}\mu(x,s), \tag{15}$$

where $a \in \mathbb{R}^+$ and $\varphi \in H(K,\mathbb{C}^{d_\sigma})$.

Proposition 14. *Let $\Gamma_G = S^{n-1} \times K$ and $\mu \in X(\Gamma_G)$. If $\widehat{\mu}(a_o,\sigma) = 0$ for all $\sigma \in \widehat{M}$ and some $a_o > 0$, then $\mu = 0$ as long as $J_{(n+2l-2)/2}(a_o) \neq 0$ for all $l \in \mathbb{Z}_+$.*

Proof. Since μ is absolutely continuous with respect to the surface measure on Γ_G, by Radon–Nikodym theorem, there exists a function $f \in L^1(\Gamma_G)$ such that $\mathrm{d}\mu = f\mathrm{d}t\mathrm{d}s$. By hypothesis, we have

$$(\widehat{\mu}(a_o,\sigma)\varphi)(k) = \int_{S^{n-1}}\int_K f(t,s)e^{-i\langle t,a_o k\cdot e_n\rangle}\varphi(s^{-1}k)\mathrm{d}t\mathrm{d}s = 0,$$

whenever $\varphi \in C(K,\mathbb{C}^{d_\sigma})$. Now, by Fubini's theorem, we can write

$$\int_K \int_{S^{n-1}} f(t,s)e^{-i\langle t,a_o k\cdot e_n\rangle}\varphi(s^{-1}k)\mathrm{d}t\mathrm{d}s = \int_K \mathscr{F}_1 f(a_o k\cdot e_n,s)\varphi(s^{-1}k)\mathrm{d}s = 0.$$

Hence $\mathscr{F}_1 f(a_o k\cdot e_n,s) = 0$ for almost all $s,k \in K$. Since $S^{n-1} = \{k.e_n : k \in K\}$, it follows that $\mathscr{F}_1 f(y,s) = 0$ for almost all $y \in S_{a_o}^{n-1}(0)$ and $s \in K$. Since $\left(S^{n-1}, S_{a_o}^{n-1}(0)\right)$ is a Heisenberg uniqueness pair as long as $J_{(n+2l-2)/2}(a_o) \neq 0$ for all $l \in \mathbb{Z}_+$ (see [7]), we conclude that $\mu = 0$. $\qquad\square$

3.3. *Product group*

The Haar measure on the product group $G' = \mathbb{R}^n \times K$, where K is a compact group, is given by $\mathrm{d}g = \mathrm{d}x\mathrm{d}k$, where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$ and $\mathrm{d}k$ is normalized Haar measure on K. Since the unitary dual group of G' can be parametrized by $\widehat{G'} = \mathbb{R}^n \times \widehat{K}$, for each $(y,\delta) \in \widehat{G'}$, the map $(x,k) \mapsto e^{-2\pi i x \cdot y}\delta(k)$ is a unitary operator on the Hilbert space $\mathscr{H}_\delta$ of dimension d_δ. Hence, we can define the Fourier transform of the function $f \in L^1(G')$ by

$$\widehat{f}(y,\delta) = \int_{\mathbb{R}^n} \int_K f(x,k)e^{-2\pi i x \cdot y}\delta(k^{-1})\mathrm{d}x\mathrm{d}k. \tag{16}$$

3.4. *Heisenberg uniqueness pairs on the product group*

Let $\Gamma_{G'} = \Gamma \times K$, where Γ is a smooth surface (or a finite union of smooth surfaces) in $\mathbb{R}^n$. Let $X(\Gamma_{G'})$ be the space of all finite complex-valued Borel measure μ in G' which is supported on $\Gamma_{G'}$ and absolutely continuous with respect to the surface measure on $\Gamma_{G'}$. Then by the Radon–Nikodym theorem, there exists a function $f \in L^1(\Gamma_{G'})$ such that $\mathrm{d}\mu = f\mathrm{d}\nu\mathrm{d}k$, where ν is the surface measure on Γ.

Now, the Fourier transform of the measure μ can be defined by

$$\widehat{\mu}(y,\delta) = \int_\Gamma \int_K e^{-2\pi i x \cdot y}\delta(k^{-1})\mathrm{d}\mu(x,k)$$
$$= \int_\Gamma \int_K f(x,k)e^{-2\pi i x \cdot y}\delta(k^{-1})\mathrm{d}\nu(x)\mathrm{d}k. \tag{17}$$

Theorem 15. *The pair (Γ,Λ) is a Heisenberg uniqueness pair on $\mathbb{R}^n$ if and only if $(\Gamma_{G'},\Lambda \times \widehat{K})$ is a Heisenberg uniqueness pair on G'.*

Proof. Suppose (Γ,Λ) is a Heisenberg uniqueness pair on $\mathbb{R}^n$ and $\mu \in X(\Gamma_{G'})$. Then by Fubini's theorem, the map $x \mapsto f(x,k)$ belongs to $L^1(\Gamma,\mathrm{d}\nu)$ for almost all $k \in K$. Hence for $(k,\delta) \in K \times \widehat{K}$, we can define the projection $f_{k,\delta}$ of f by

$$f_{k,\delta}(x) = \int_K f(x,kh^{-1})\chi_\delta(h)\mathrm{d}h, \tag{18}$$

where $\chi_\delta = \mathrm{tr}\,\delta(\cdot)$, the character of the representation δ. Thus, the Euclidean Fourier transform of the projection $f_{k,\delta}$ gives

$$\widehat{f}_{k,\delta}(y) = \int_\Gamma \int_K f(x,kh^{-1})e^{-2\pi i x \cdot y}\chi_\delta(h)\mathrm{d}h\mathrm{d}\nu(x)$$
$$= \mathrm{tr} \int_\Gamma \int_K f(x,kh^{-1})\delta(h)e^{-2\pi i x \cdot y}\mathrm{d}h\mathrm{d}\nu(x)$$
$$= \mathrm{tr} \int_\Gamma \int_K f(x,h)\delta(h^{-1})e^{-2\pi i x \cdot y}\delta(k)\mathrm{d}\nu(x)\mathrm{d}h$$
$$= \mathrm{tr}\left(\widehat{\mu}(y,\delta)\delta(k)\right). \tag{19}$$

Suppose $\widehat{\mu}|_{\Lambda \times \hat{K}} = 0$. Since (Γ,Λ) is a Heisenberg uniqueness pair on $\mathbb{R}^n$, from (19), it follows that $f_{k,\delta} = 0$. Hence by the uniqueness of the Fourier series

$$f(x,k) = \sum_{\delta \in \hat{K}} d_\delta f_{k,\delta}(x),$$

we conclude that $f = 0$.

Conversely, suppose $(\Gamma_{G'},\Lambda \times \widehat{K})$ is a Heisenberg uniqueness pair on G'. Then for $\mu \in X(\Gamma)$, there exists a function $f \in L^1(\Gamma)$ such that $\mathrm{d}\mu = f\mathrm{d}\nu$. If $\widehat{\mu}|_\Lambda = 0$ then

$$\int_\Gamma f(x)e^{-2\pi i x \cdot y}\mathrm{d}\nu(x) = 0$$

for each $y \in \Lambda$. This, in turn, implies

$$\int_\Gamma \int_K f(x) e^{-2\pi i y \cdot x} \delta(k^{-1}) \mathrm{d}k \mathrm{d}\nu(x) = 0. \tag{20}$$

Now, if we write $d\rho = f \mathrm{d}\nu \mathrm{d}k$, then $\rho \in X(\Gamma_{G'})$. Since $(\Gamma_{G'}, \Lambda \times \widehat{K})$ is a HUP, by (20), it follows that $\rho = 0$. Thus, we conclude that the measure $\mu = 0$. $\qquad\square$

Acknowledgements

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