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From Hörmander's L^2-estimates to partial positivity

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Abstract. In this article, using a twisted version of Hörmander's L^2-estimate, we give new characterizations of notions of partial positivity, which are uniform q-positivity and RC-positivity. We also discuss the definition of uniform q-positivity for singular Hermitian metrics.

Keywords. L^2-estimates, q-positivity, RC-positivity.

Mathematical subject classification (2010). 32U05.

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1. Introduction

In this article, we give a new characterization of partial positivity, which is called uniform q-positivity (cf. Definition 4) via Hörmander's L^2-estimate. The statement is the following.

Theorem 1. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n_z$, $\omega = \sqrt{-1}\partial\overline{\partial}|z|^2$ be the standard Kähler metric on D, and $L \to D$ be a line bundle over D. For a smooth Hermitian metric h on L and a non-negative constant $c \geq 0$ on D, the following properties are equivalent for $1 \leq q \leq n$:*

(1) *The summation of any distinct q eigenvalues (counting multiplicity) of the Chern curvature $\sqrt{-1}\Theta_{(L,h)}$ of (L,h) with respect to ω is greater than or equal to c.*
(2) *For any smooth strictly plurisubharmonic function ψ and any smooth $\overline{\partial}$-closed L-valued (n,q)-form f with compact support, there exists L-valued $(n,q-1)$-form u satisfying $\overline{\partial}u = f$ and*

$$\int_D |u|^2_{(\omega,h)} e^{-\psi} \mathrm{d}V_\omega \leq \int_D \left\langle \left(\left[\sqrt{-1}\partial\overline{\partial}\psi, \Lambda_\omega\right] + c\right)^{-1} f, f \right\rangle_{(\omega,h)} e^{-\psi} \mathrm{d}V_\omega.$$

Condition (2) in Theorem 1 allows us to add a weight ψ. Taking an arbitrary weight, we can estimate the curvature $\sqrt{-1}\Theta_{(L,h)}$. This type of condition was firstly introduced in [11], which was called the twisted Hörmander condition. Next, in [7] and [8], Deng et al. generalized this notion and introduced the optimal L^p-estimate condition, which corresponded to the particular case of the twisted Hörmander condition when $p=2$. These studies provide new characterizations of positivity based on the Hörmander-type condition, which was initially observed by Berndtsson in [2].

Theorem 1 is a generalization for partial positivity of the result obtained by the authors in [7].

As a higher-rank analogue, we also establish a characterization of RC-positivity. RC-positivity is a partial positivity notion introduced by Yang in [14] and a higher-rank analogue of $(\dim X-1)$-positivity. We use the same notation as in Theorem 1.

Theorem 2. *Let $E\to D$ be a vector bundle and h be a smooth Hermitian metric on E. Assume that if ψ is a smooth strictly plurisubharmonic function on D and f is a smooth E-valued (n,n)-form with compact support, there exists a solution of $\overline{\partial}u=f$ satisfying*

$$\int_D |u|^2_{(\omega,h)}e^{-\psi}\mathrm{d}V_\omega \le \int_D \left\langle\left(\left[\sqrt{-1}\partial\overline{\partial}\psi\otimes \mathrm{Id}_E,\Lambda_\omega\right]+c\right)^{-1}f,f\right\rangle_{(\omega,h)}e^{-\psi}\mathrm{d}V_\omega.$$

Then we obtain

$$\mathrm{tr}_\omega\left(\sqrt{-1}\Theta_{(E,h)}a,a\right)_h(x)\ge c|a|^2_h(x)$$

for any point $x\in D$ and any element $a\in E_x$. Especially, (E,h) is RC-positive if $c>0$ and RC-semi-positive if $c=0$.

As an application of the characterization in Theorem 1, we prove that uniform q-positivity is preserved with respect to a decreasing sequence (Theorem 9). This property is well-known in the case that $q=0$, that is, it is a sequence of plurisubharmonic functions. We also propose the definition of uniform q-positivity for singular Hermitian metrics (Definition 10).

As a further study, we propose the following problem, which generalizes the Prékopa–Berndtsson theorem (cf. [2, Theorem 1.3]).

Problem 3. *Let U be a bounded domain in $\mathbb{C}^n_z$ and D be a bounded pseudoconvex domain in $\mathbb{C}^m_w$. Let φ be a smooth function on $\overline{U_z\times D_w}\subset\mathbb{C}^n_z\times\mathbb{C}^m_w$. We let ω_0,ω_1 and ω_2 be the standard Kähler metrics on $U\subset\mathbb{C}^n_z$, $D\subset\mathbb{C}^m_w$ and $U\times D\subset\mathbb{C}^n_z\times\mathbb{C}^m_w$, respectively. Assume that*

1. *D is a connected Reinhardt domain and $\varphi(z,w_1,\dots,w_m)$ is independent of $\arg(w_j)$ for $1\le j\le m$.*
2. *The summation of any distinct q eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi$ with respect to ω_2 is greater than or equal to c, where $c\ge 0$ is a non-negative constant and $1\le q\le n$.*

We define the function $\widetilde{\varphi}$ on U by

$$e^{-\widetilde{\varphi}(z)}:=\int_{w\in D}e^{-\varphi(z,w)}\mathrm{d}\omega_1(w).$$

Then the summation of any distinct q eigenvalues of $\sqrt{-1}\partial\overline{\partial}\widetilde{\varphi}$ with respect to ω_0 is greater than or equal to c.

We immediately see that Problem 3 is true in the case that $q=1$, which corresponds to the Prékopa–Berndtsson theorem. The proof of the Prékopa–Berndtsson theorem is based on Hörmander's L^2-estimates, and on a partial converse of these estimates in one variable. In Section 5, we explain the reason why Theorem 1 can be applied to Problem 3.

The organization of this paper is as follows. In Section 2, we introduce definitions of q-positivity, uniform q-positivity and RC-positivity. We also explain the result of Hörmander's L^2-estimate which we use in this article. In Section 3, we characterize partial positivity by using the Hörmander L^2-estimate. We also show the proofs of Theorem 1 and 2. In Section 4, we show applications of the main theorems. We also discuss a definition of uniform q-positivity for singular Hermitian metrics. In Section 5, we propose some problems.

2. Preliminaries

Notation.

- $\mathrm{d}V_\omega := \frac{\omega^n}{n!}$: the volume form determined by ω.
- $C^k_{(p,q)}(X,E) := C^k(X, \wedge^{p,q} T^\star_X \otimes E)$ for $0 \le k \le +\infty$.
- $\mathscr{D}_{(p,q)}(X,E)$: the space of smooth sections of $\wedge^{(p,q)} T^\star_X \otimes E$ with compact support.
- $L^2_{(p,q)}(X,E;\omega,h)$: the space of L^2 sections of $\wedge^{p,q} T^\star_X \otimes E$ with respect to ω and h.
- $\langle\langle \alpha, \beta \rangle\rangle_{(\omega,h)} := \int_X \langle \alpha, \beta \rangle_{(\omega,h)} \mathrm{d}V_\omega$.
- $\|\alpha\|^2_{(\omega,h)} := \langle\langle \alpha, \alpha \rangle\rangle_{(\omega,h)}$.
- $D'^\star_\psi$: the adjoint operator of D'_ψ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, he^{-\psi})}$.
- $\overline{\partial}^\star_\psi$: the adjoint operator of $\overline{\partial}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, he^{-\psi})}$.
- $\Delta'_\psi := D'_\psi D'^\star_\psi + D'^\star_\psi D'_\psi, \Delta''_\psi = \overline{\partial}\overline{\partial}^\star_\psi + \overline{\partial}^\star_\psi \overline{\partial}$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{(\omega, he^{-\psi})}$.
- L_ω : the operator defined by $\omega \wedge \cdot$.
- Λ_ω : the adjoint operator of L_ω.
- $[\cdot,\cdot]$: the graded Lie bracket.
- $\mathbb{B}^n_r := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < r^2\}$.

In [1], Andreotti and Grauert introduced partial positivity notions and studied partially vanishing cohomology. Here, we introduce the notions of q-positivity and uniform q-positivity for smooth Hermitian metrics on line bundles.

Definition 4 (cf. [1], [15, Definition 2.1]). *Let $L \to X$ be a holomorphic line bundle over a complex manifold X with* $\dim X = n$. *Let h be a smooth Hermitian metric on L. We say that*

(1) (L,h) *is* q-(semi-)positive *if the Chern curvature $\sqrt{-1}\Theta_{(L,h)}$ has at least $(n-q)$ (semi-) positive eigenvalues at any point on X. We also say that L is* q-(semi-)positive *if there exists a smooth Hermitian metric h on L such that (L,h) is q-(semi-)positive.*

(2) (L,h) *is* uniformly q-(semi-)positive *if there exists a smooth Hermitian metric ω such that the summation of any distinct $(q+1)$ eigenvalues of the Chern curvature $\sqrt{-1}\Theta_{(L,h)}$ with respect to ω is (semi-)positive at any point on X. We also say that L is* uniformly q-(semi-)positive *if there exist a smooth Hermitian metric h on L and a smooth Hermitian metric ω such that (L,h) is uniformly q-(semi-)positive with respect to ω.*

A simple computation yields that uniform q-(semi-)positivity implies q-(semi-)positivity. Note that usual (semi-)positivity corresponds to 0-(semi-)positivity. Conversely, it is known that the above two positivity notions are equivalent over a compact complex manifold.

Proposition 5 (cf. [15, Proposition 2.2]). *Let X be a compact complex manifold and L be a q-positive line bundle. Then L is a uniformly q-positive line bundle.*

Next, we also give definitions of RC-positivity and weak RC-positivity, which were introduced by Yang in [14].

Definition 6 (cf. [14, Definition 3.3]). *A Hermitian holomorphic vector bundle (E,h) over a complex manifold X is called* RC-positive *(resp.* RC-negative*) if at any point $x \in X$ and for any non-zero element $a \in E_x$, there exists a vector $v \in T_x X$ such that*

$$\left(\sqrt{-1}\Theta_{(E,h)}(v,v)a,a\right)_h > 0 \ (\textit{resp.} < 0).$$

We also call (E,h) weakly RC-positive *if there exists a smooth Hermitian metric h on the tautological line bundle $\mathscr{O}_E(1)$ over $\mathbb{P}(E^\star)$ such that $(\mathscr{O}_E(1),h)$ is* $(\dim X - 1)$*-positive.*

Note that Griffiths positivity implies RC-positivity. Moreover, if $\dim X = 1$, RC-positivity is equivalent to Griffiths positivity. If $\operatorname{rank} E = 1$, RC-positivity is the same concept as $(\dim X - 1)$-positivity.

Finally, we mention the following result, which was initially obtained by Hörmander [10]. Hörmander's L^2-estimate is fundamental and important in several complex variables. In our paper, we use this L^2-estimate to characterize several notions of partial positivity. Here, we adopt the following form.

Theorem 7 (cf. [5], [6, Theorem (5.1)] and [4, Theorem 6.1 in Chapter VIII]). *Let $(X,\widehat{\omega})$ be a complete Kähler manifold, ω be another Kähler metric which is not necessarily complete, and $(E,h)\to X$ be a holomorphic line bundle. We also let $A_{(\omega,h)} = [\sqrt{-1}\Theta_{(E,h)}, \Lambda_\omega]$ be the curvature operator in bidegree (n,q) for $q\geq 1$. Assume that $A_{(\omega,h)}$ is positive definite everywhere on $\wedge^{n,q}T^\star X\otimes E$. Then for any $\overline{\partial}$-closed $f\in L^2_{(n,q)}(X,E;\omega,h)$, there exists $u\in L^2_{(n,q-1)}(X,E;\omega,h)$ such that $\overline{\partial}u = f$ and*

$$\int_X |u|^2_{(\omega,h)}\mathrm{d}V_\omega \leq \int_X \langle A^{-1}_{(\omega,h)} f, f\rangle_{(\omega,h)}\mathrm{d}V_\omega,$$

where we assume that the right-hand side is finite.

3. A characterization of partial positivity via L^2-estimates

3.1. *Uniform q-positivity*

In this subsection, we discuss a characterization of uniform q-positivity in terms of L^2-estimates. In order to prove Theorem 1, we need the following lemma. The proof is a simple computation.

Lemma 8 (cf. [6, (4.10)] and [4, Proposition (5.8) in Chapter VI]). *Let the notation be the same as in Theorem 1. We also let f be any $\overline{\partial}$-closed L-valued (n,q)-form. At a fixed point $p\in X$, we take a coordinate $(z_1,\dots,z_n)$ around p such that*

$$\omega = \sqrt{-1}\sum_{j=1}^n \mathrm{d}z_j\wedge \mathrm{d}\overline{z}_j, \qquad \sqrt{-1}\Theta_{(L,h)} = \sqrt{-1}\sum_{j=1}^n \gamma_j \mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j.$$

We write

$$f = \sum_{1\leq i_1<\cdots<i_q\leq n} f_{i_1\dots i_q}\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\overline{z}_{i_1}\wedge\cdots\wedge\mathrm{d}\overline{z}_{i_q}\otimes e_L$$

for a local holomorphic frame e_L of L around p. Then we get

$$\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right] f = \sum_{1\leq i_1<\cdots<i_q\leq n}\left(\sum_{k=1}^q \gamma_{i_k}\right) f_{i_1\dots i_q}\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\overline{z}_{i_1}\wedge\cdots\wedge\mathrm{d}\overline{z}_{i_q}\otimes e_L.$$

We now give a proof of Theorem 1. The idea for the proof is based on the arguments in [7, Theorem 2.1] and [8, Theorem 3.1].

Proof of Theorem 1. (1) $\Longrightarrow$ (2)**.** We have

$$\left[\sqrt{-1}\Theta_{(L,he^{-\psi})},\Lambda_\omega\right] = \left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right] + \left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]$$

for any smooth strictly plurisubharmonic function ψ. We fix a smooth $\overline{\partial}$-closed L-valued (n,q)-form f with compact support. The assumption of (1) and Lemma 8 implies that

$$\left\langle \left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right] f, f\right\rangle_{(\omega,h)} \geq c|f|^2_{(\omega,h)}.$$

The curvature operator $[\sqrt{-1}\Theta_{(L,he^{-\psi})},\Lambda_\omega]$ is positive definite on $\wedge^{n,q}T^\star D\otimes L$ everywhere. Therefore, by using Theorem 7, we can solve the $\overline{\partial}$-equation $\overline{\partial}u=f$ as follows

$$\begin{aligned}\int_D |u|^2_{(\omega,h)}e^{-\psi}\mathrm{d}V_\omega &\le \int_D \left\langle \left[\sqrt{-1}\Theta_{(L,he^{-\psi})},\Lambda_\omega\right]^{-1} f, f\right\rangle_{(\omega,h)} e^{-\psi}\mathrm{d}V_\omega\\ &\le \int_D \left\langle \left(\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]+c\right)^{-1} f, f\right\rangle_{(\omega,h)} e^{-\psi}\mathrm{d}V_\omega<+\infty\end{aligned}$$

for some $u\in L^2_{(n,q-1)}(D,L;\omega,he^{-\psi})$.

(2) $\Longrightarrow$ (1)**.** For any smooth strictly plurisubharmonic function ψ and any $\overline{\partial}$-closed $f\in\mathscr{D}_{(n,q)}(D,L)$, we get a solution $u\in L^2_{(n,q-1)}(D,L;\omega,he^{-\psi})$ of $\overline{\partial}u=f$ satisfying

$$\|u\|^2_{(\omega,he^{-\psi})}\le \left\langle\!\left\langle \left(\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]+c\right)^{-1} f, f\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}.$$

Set $g:=\left(\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]+c\right)^{-1}f$. We obtain

$$\begin{aligned}|\langle\!\langle g,f\rangle\!\rangle_{(\omega,he^{-\psi})}|^2 &= \left|\left\langle\!\left\langle g,\overline{\partial}u\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}\right|^2\\ &\le \left|\left\langle\!\left\langle \overline{\partial}^\star_\psi g,u\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}\right|^2\\ &\le \left\|\overline{\partial}^\star_\psi g\right\|^2_{(\omega,he^{-\psi})}\|u\|^2_{(\omega,he^{-\psi})}\\ &\le \left\|\overline{\partial}^\star_\psi g\right\|^2_{(\omega,he^{-\psi})}|\langle\!\langle g,f\rangle\!\rangle_{(\omega,he^{-\psi})}|.\end{aligned}$$

Using the Bochner–Kodaira–Nakano identity $\Delta''_\psi=\Delta'_\psi+[\sqrt{-1}\Theta_{(L,he^{-\psi})},\Lambda_\omega]$ (cf. [6, (4.6)]), we have

$$\begin{aligned}\left\|\overline{\partial}^\star_\psi g\right\|^2_{(\omega,he^{-\psi})} &= \left\langle\!\left\langle\left(\Delta''_\psi-\overline{\partial}^\star_\psi\overline{\partial}\right)g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}\\ &= \langle\!\langle\Delta'_\psi g,g\rangle\!\rangle_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left[\sqrt{-1}\Theta_{(L,he^{-\psi})},\Lambda_\omega\right]g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}-\left\|\overline{\partial}g\right\|^2_{(\omega,he^{-\psi})}\\ &\le \left\|D'^\star_\psi g\right\|^2_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\left\langle\!\left\langle g,\left(\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]+c\right)g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}\\ &\qquad\le \left\|D'^\star_\psi g\right\|^2_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left[\sqrt{-1}\partial\overline{\partial}\psi,\Lambda_\omega\right]g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})},\end{aligned}$$

that is,

$$0\le \left\|D'^\star_\psi g\right\|^2_{(\omega,he^{-\psi})}+\left\langle\!\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)g,g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}. \tag{1}$$

We give a proof by contradiction. In other words, we suppose that the summation of some distinct q eigenvalues of $\sqrt{-1}\Theta_{(L,h)}$ with respect to ω is less than c at some point $a\in D$. We can assume that $o=a\in D$, where o is the origin of $\mathbb{C}^n$. Let $\gamma_1,\dots,\gamma_n$ be the eigenvalues of $\sqrt{-1}\Theta_{(L,h)}$ with respect to ω, which are globally defined on D. Changing the coordinate by some unitary transformation, we take a coordinate $(z_1,\dots,z_n)$ centered at o such that

$$\omega=\sqrt{-1}\sum \mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j,$$

on D and

$$\sqrt{-1}\Theta_{(L,h)}=\sqrt{-1}\sum\gamma_j\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j$$

at o. Without any loss of generality, we suppose that

$$\gamma_1(o)+\cdots+\gamma_q(o)-c<0.$$

We fix an open neighborhood U of o and a local holomorphic frame e_L of L on U. We define

$$F:=\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\overline{z}_1\wedge\cdots\wedge\mathrm{d}\overline{z}_q\otimes e_L\quad\in C^\infty_{(n,q)}(U,L).$$

Then we have

$$\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)F,F\right\rangle_{(\omega,h)}(o)=\left\langle\left(\sum_{j=1}^{q}\gamma_j-c\right)F,F\right\rangle_{(\omega,h)}(o)$$
$$=\left(\sum_{j=1}^{q}\gamma_j(o)-c\right)|e_L|_h^2<0.$$

We take a positive constant $\delta>0$ such that

$$\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)F,F\right\rangle_{(\omega,h)}(o)=\left(\sum_{j=1}^{q}\gamma_j(o)-c\right)|e_L|_h^2=-2\delta.$$

Since $\langle([\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega]-c)F,F\rangle_{(\omega,h)}$ has continuous coefficients, we take a sufficiently small $r\in(0,+\infty)$ such that $\mathbb{B}^n_r\subset U\Subset D$ and

$$\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)F,F\right\rangle_{(\omega,h)}<-\delta$$

on $\mathbb{B}^n_r$.

We take a smooth strictly plurisubharmonic function $\psi(z)=|z|^2-\frac{r^2}{4}$ on D. Let χ be a cut-off function on $\mathbb{B}^n_r$ such that χ is smooth, $0\le\chi\le1$, $\operatorname{supp}\chi\Subset\mathbb{B}^n_r$ and $\chi|_{\overline{\mathbb{B}^n_{r/2}}}\equiv1$. We set $v:=(-1)^{n+q-1}\chi\bar{z}_q\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar{z}_1\wedge\cdots\wedge\mathrm{d}\bar{z}_{q-1}\otimes e_L$ and $g:=\bar\partial v$. Then g is a $\bar\partial$-closed L-valued (n,q)-form with compact support and

$$g=\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar{z}_1\wedge\cdots\wedge\mathrm{d}\bar{z}_q\otimes e_L$$

on $\overline{\mathbb{B}^n_{r/2}}$. We remark that $[\sqrt{-1}\partial\bar\partial(m\psi),\Lambda_\omega]f=mqf$ for $f\in\wedge^{n,q}T^\star_D\otimes L$. We define $f_m:=([\sqrt{-1}\partial\bar\partial(m\psi),\Lambda_\omega]+c)g=(mq+c)g$. It clearly holds that f_m is an also $\bar\partial$-closed L-valued (n,q)-form with compact support. Then g satisfies the inequality (1) for every $m\psi$. Considering the commutation relation $\sqrt{-1}[\Lambda_\omega,\bar\partial]=D'^\star_{m\psi}$ (cf. [4, (1.1) in Chapter VII]), we have that

$$D'^\star_{m\psi}g=0$$

on $\mathbb{B}^n_{r/2}$ since ω is the standard Kähler metric and g has constant coefficients on $\mathbb{B}^n_{r/2}$, and

$$\left|D'^\star_{m\psi}g\right|^2_{(\omega,h)}\le C_1$$

for some positive constant $C_1>0$ which is independent of m and ψ on $\mathbb{B}^n_r$.

Since $g=F$ on $\mathbb{B}^n_{r/2}$, we know that $\langle([\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega]-c)g,g\rangle_{(\omega,h)}<-\delta$ on $\mathbb{B}^n_{r/2}$ and $\langle([\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega]-c)g,g\rangle_{(\omega,h)}\le C_2$ for some positive constant $C_2>0$ on $\mathbb{B}^n_r$. Consequently, we can compute the right-hand side of (1) for g and $m\psi$ as follows:

$$\begin{aligned}0&\le\int_D\left|D'^\star_{m\psi}g\right|^2_{(\omega,h)}e^{-m\psi}\mathrm{d}V_\omega+\int_D\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)g,g\right\rangle_{(\omega,h)}e^{-m\psi}\mathrm{d}V_\omega\\&=\int_{\mathbb{B}^n_r\setminus\overline{\mathbb{B}^n_{r/2}}}\left|D'^\star_{m\psi}g\right|^2_{(\omega,h)}e^{-m\psi}\mathrm{d}V_\omega+\int_{\mathbb{B}^n_{r/2}}\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)g,g\right\rangle_{(\omega,h)}e^{-m\psi}\mathrm{d}V_\omega\\&\quad+\int_{\mathbb{B}^n_r\setminus\overline{\mathbb{B}^n_{r/2}}}\left\langle\left(\left[\sqrt{-1}\Theta_{(L,h)},\Lambda_\omega\right]-c\right)g,g\right\rangle_{(\omega,h)}e^{-m\psi}\mathrm{d}V_\omega\\&\le(C_1+C_2)\int_{\mathbb{B}^n_r\setminus\overline{\mathbb{B}^n_{r/2}}}e^{-m\psi}\mathrm{d}V_\omega-\delta\int_{\mathbb{B}^n_{r/2}}e^{-m\psi}\mathrm{d}V_\omega.\end{aligned}$$

Since $\psi>0$ on $\mathbb{B}^n_r\setminus\overline{\mathbb{B}^n_{r/2}}$, the first term goes to zero as $m\to+\infty$ by Lebesgue's dominated convergence theorem. The second term has a negative upper bound

$$-\delta\int_{\mathbb{B}^n_{r/2}}e^{-m\psi}\mathrm{d}V_\omega<-\delta\left|\mathbb{B}^n_{r/2}\right|$$

which is independent of m since $\psi < 0$ on $\mathbb{B}^n_{r/2}$. Taking a sufficiently large $m \gg 1$, we get

$$(C_1 + C_2)\int_{\mathbb{B}^n_r \setminus \overline{\mathbb{B}^n_{r/2}}} e^{-m\psi}\mathrm{d}V_\omega - \delta\int_{\mathbb{B}^n_{r/2}} e^{-m\psi}\mathrm{d}V_\omega < 0,$$

which is a contradiction. □

3.2. *RC-positivity*

In this subsection, we give a characterization of RC-positivity via L^2-estimates. This is a higher-rank analogue of Theorem 1. Although the proof is almost identical to the proof of Theorem 1, we present it for the sake of completeness.

Proof of Theorem 2. We take an arbitrary smooth strictly plurisubharmonic function ψ and an arbitrary $f \in \mathscr{D}_{(n,n)}(D,E)$. Repeating the argument in the proof of Theorem 1 (cf. [8, Theorem 3.1] or [12, Proposition 2.7]), we obtain the following inequality

$$0 \le \|D'^{\star}_{\psi} g\|^2_{(\omega,he^{-\psi})} + \left\langle\!\left\langle\left(\left[\sqrt{-1}\Theta_{(E,h)},\Lambda_\omega\right] - c\right)g, g\right\rangle\!\right\rangle_{(\omega,he^{-\psi})}, \tag{2}$$

where $g = ([\sqrt{-1}\partial\overline{\partial}\psi \otimes \mathrm{Id}_E, \Lambda_\omega] + c)^{-1} f$.

We give a proof by contradiction. We assume that there exists some point $x \in D$ and some element $a \in E_x \setminus \{0\}$ such that

$$\mathrm{tr}_\omega\left(\sqrt{-1}\Theta_{(E,h)}a, a\right)_h(x) < c|a|^2_h(x). \tag{3}$$

We may assume that $x = o \in D$. Since h has smooth coefficients, we can take a sufficiently small $r \in (0,+\infty)$ such that $\mathbb{B}^n_r \Subset D$, $E|_{\mathbb{B}^n_r}$ is trivial, and

$$\mathrm{tr}_\omega\left(\sqrt{-1}\Theta_{(E,h)}a, a\right)_h - c|a|^2_h < -\delta \tag{4}$$

on $\mathbb{B}^n_r$ for some positive constant $\delta > 0$. Here we regard a as a section of E with constant coefficients.

As in the proof of Theorem 1, we take a smooth strictly plurisubharmonic function $\psi(z) = |z|^2 - \frac{r^2}{4}$ and a cut-off function χ such that $\mathrm{supp}\chi \Subset \mathbb{B}^n_r$ and $\chi|_{\mathbb{B}^n_{r/2}} \equiv 1$. We consider the following E-valued (n,n)-form with compact support

$$g = \chi a \mathrm{d}Z \wedge \mathrm{d}\overline{Z}$$

on D. Here we use the notation

$$\mathrm{d}Z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n, \qquad \mathrm{d}\overline{Z} = \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n$$

for simplicity. We also define

$$f_m := \left(\left[\sqrt{-1}\partial\overline{\partial}(m\psi) \otimes \mathrm{Id}_E, \Lambda_\omega\right] + c\right)g = (mn + c)g.$$

Note that $f_m \in \mathscr{D}_{(n,n)}(D,E)$. Hence, we see that g satisfies the inequality (2) for each $m\psi$.

We compute the terms $\langle[\sqrt{-1}\Theta_{(E,h)},\Lambda_\omega](s\mathrm{d}Z\wedge\mathrm{d}\overline{Z}), s\mathrm{d}Z\wedge\mathrm{d}\overline{Z}\rangle_{(\omega,h)}$ and $\mathrm{tr}_\omega(\sqrt{-1}\Theta_{(E,h)}s,s)_h$ for any section s of E. Note that $s\mathrm{d}Z\wedge\mathrm{d}\overline{Z} \in C^\infty_{(n,n)}(D,E)$. We write the curvature tensor $\sqrt{-1}\Theta_{(E,h)}$ as

$$\sqrt{-1}\Theta_{(E,h)} = \sum_{1\le j,k\le n} \Theta_{j\bar{k}}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

where $\Theta_{j\bar{k}}$ are operators on each E_t. Then we get

$$\begin{aligned}\left\langle\left[\sqrt{-1}\Theta_{(E,h)},\Lambda_\omega\right](s\mathrm{d}Z\wedge\mathrm{d}\overline{Z}), s\mathrm{d}Z\wedge\mathrm{d}\overline{Z}\right\rangle_{(\omega,h)} &= \left\langle\left(\sum_{j=1}^n \Theta_{j\bar{j}}s\right)\mathrm{d}Z\wedge\mathrm{d}\overline{Z}, s\mathrm{d}Z\wedge\mathrm{d}\overline{Z}\right\rangle_{(\omega,h)} \\ &= \sum_{j=1}^n (\Theta_{j\bar{j}}s,s)_h\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}_\omega\big(\sqrt{-1}\Theta_{(E,h)}s,s\big)_h &= \operatorname{tr}_\omega\left(\sum_{1\le j,k\le n}(\Theta_{j\bar{k}}s,s)_h \mathrm{d}z_j\wedge \mathrm{d}\bar{z}_k\right)\\ &= \sum_{j=1}^n (\Theta_{j\bar{j}}s,s)_h.\end{aligned}$$

Hence, on $\mathbb{B}^n_{r/2}$, the inequality (4) implies that

$$\big\langle\big(\big[\sqrt{-1}\Theta_{(E,h)},\Lambda_\omega\big]-c\big)g,g\big\rangle_{(\omega,h)} < -\delta.$$

Then, taking a sufficiently large $m \gg 1$ and repeating the argument in the proof of Theorem 1 again, we conclude that the inequality (3) contradicts the inequality (2), which completes the proof. □

4. Applications of a new characterization

In this section, we give some applications of Theorem 1. First, we prove the following theorem. Here we use the same notation as in Theorem 1.

Theorem 9. *Let φ be a smooth function on D. Suppose that there exists a sequence of smooth functions $\{\varphi_j\}_{j=1}^\infty$ decreasing to φ pointwise such that the summation of any distinct q eigenvalues of $\sqrt{-1}\partial\bar{\partial}\varphi_j$ with respect to ω is greater than or equal to some non-negative constant $c \ge 0$. Then the summation of any distinct q eigenvalues of $\sqrt{-1}\partial\bar{\partial}\varphi$ with respect to ω is greater than or equal to c.*

It is well-known that Theorem 9 holds in the case that $q = 1$, that is, φ_j are plurisubharmonic functions.

Proof. We use the characterization in Theorem 1. Since the result is a local property, we may assume that D is pseudoconvex. It is enough to show that for any smooth strictly plurisubharmonic function ψ and any smooth $\bar{\partial}$-closed (n,q)-form f with compact support, there exists a solution of $\bar{\partial}u = f$ satisfying

$$\int_D |u|^2_{\omega_0} e^{-(\varphi+\psi)}\mathrm{d}V_{\omega_0} \le \int_D \big\langle\big(\big[\sqrt{-1}\partial\bar{\partial}\psi,\Lambda_{\omega_0}\big]+c\big)^{-1}f,f\big\rangle_{\omega_0} e^{-(\varphi+\psi)}\mathrm{d}V_{\omega_0}.$$

The assumption of φ_j implies that we get a solution of $\bar{\partial}u_j = f$ satisfying

$$\begin{aligned}\int_D |u_j|^2_{\omega_0} e^{-(\varphi_j+\psi)}\mathrm{d}V_{\omega_0} &\le \int_D \big\langle\big(\big[\sqrt{-1}\partial\bar{\partial}\psi,\Lambda_{\omega_0}\big]+c\big)^{-1}f,f\big\rangle_{\omega_0} e^{-(\varphi_j+\psi)}\mathrm{d}V_{\omega_0}\\ &\le \int_D \big\langle\big(\big[\sqrt{-1}\partial\bar{\partial}\psi,\Lambda_{\omega_0}\big]+c\big)^{-1}f,f\big\rangle_{\omega_0} e^{-(\varphi+\psi)}\mathrm{d}V_{\omega_0}\\ &< +\infty\end{aligned}$$

for each $j \in \mathbb{N}$. Note that the right-hand side of the above inequality has an upper bound independent of j and $\{u_k\}_{k\ge j}$ forms a bounded sequence in $L^2_{(n,q-1)}(D,\mathbb{C};\omega_0,e^{-(\varphi_j+\psi)})$. Therefore, we find a weakly convergent subsequence $\{u_{j_k}\}_{k=1}^\infty$ by using a diagonal argument and monotonicity of $\{\varphi_j\}_{j=1}^\infty$, which is the standard argument of L^2-solutions of $\bar{\partial}$. We have that $\{u_{j_k}\}_{k=1}^\infty$ weakly converges in $L^2_{(n,q-1)}(D,\mathbb{C};\omega_0,e^{-(\varphi_j+\psi)})$ for every j and the weak limit denoted by u_∞ satisfies $\bar{\partial}u_\infty = f$ and

$$\int_D |u_\infty|^2_{\omega_0} e^{-(\varphi+\psi)}\mathrm{d}V_{\omega_0} \le \int_D \big\langle\big(\big[\sqrt{-1}\partial\bar{\partial}\psi,\Lambda_{\omega_0}\big]+c\big)^{-1}f,f\big\rangle_{\omega_0} e^{-(\varphi+\psi)}\mathrm{d}V_{\omega_0}$$

due to the monotone convergence theorem. Then we complete the proof. □

Next, by using the new characterization, we propose the definition of uniform q-positivity for singular Hermitian metrics. Note that we can consider the condition (2) in Theorem 1 without assuming that h is smooth.

Definition 10. *Let L be a holomorphic line bundle over an n-dimensional Kähler manifold (X,ω) and h be a singular Hermitian metric on L such that $-\log h$ is upper semi-continuous. Set $1 \le q \le n$ and $c \ge 0$. We say that (L,h) is* uniformly $(q-1)$-c-positive with respect to ω *if for any point $x \in X$, there exists an open neighborhood U of x such that for any relatively compact pseudoconvex domain D in U, (L,h), ω and c satisfy the condition* (2) *in Theorem 1 on D.*

Thanks to Theorem 1, in the case that h is smooth, the above definition is equivalent to uniform $(q-1)$-positivity. Under this formulation, we can show Theorem 9 without assuming the condition that φ is smooth. The proof remains the same.

Theorem 11 (cf. Theorem 9). *Let $\{\varphi_j\}_{j=1}^{\infty}$ be a sequence of smooth functions decreasing to a locally integrable function $\varphi \not\equiv -\infty$. If the summation of any distinct q eigenvalues of $\sqrt{-1}\partial\overline{\partial}\varphi_j$ with respect to ω is greater than or equal to c, $(\mathbb{C}, e^{-\varphi})$ is uniformly $(q-1)$-c-positive in the sense of Definition 10.*

The argument above has many other applications. For instance, it is known that Nakano semi-positivity can be characterized via L^2-estimates (cf. [8, Theorem 1.1]). By using the same method, we can also show that if a sequence of smooth Nakano semi-positive Hermitian metrics $\{h_\nu\}_{\nu=1}^{\infty}$ increasing to a (possibly singular) Hermitian metric h, h is also Nakano semi-positive (for Nakano semi-positivity of singular Hermitian metrics, see [12, Definition 1.2]).

5. Further study

In this section, we propose some problems. First, we discuss Problem 3. In [2], Berndtsson generalized the Prékopa theorem [13] by assuming that the plurisubharmonic function satisfies some invariant properties.

Theorem 12 (cf. [2, Theorem 1.3]). *Let φ be a plurisubharmonic function on $U_z \times D_w \subset \mathbb{C}_z^n \times \mathbb{C}_w^m$, where D_w is pseudoconvex. Assume that one of the following conditions holds:*

(1) *D is a connected Reinhardt domain and $\varphi(z, w_1, \dots, w_m)$ is independent of* $\arg(z_j)$ *for $1 \le j \le m$.*
(2) *D contains the origin and for any $z \in U$, $w \in D$, and $\theta \in \mathbb{R}$, we have $e^{\sqrt{-1}\theta} w \in D$ and $\varphi(z, e^{\sqrt{-1}\theta} w) = \varphi(z, w)$.*

Then the function $\widetilde{\varphi}$ defined on U by

$$e^{-\widetilde{\varphi}(z)} := \int_{w \in D} e^{-\varphi(z,w)}$$

is plurisubharmonic.

This research has been generalized in a variety of directions (cf. [3,9]). Since Theorem 12 can be applied in the case where D is bounded, Problem 3 is a generalization of Theorem 12 for partial positivity. Here we explain the reason why Theorem 1 is one strategy to prove Problem 3. Let $\pi : U \times D \to U$ be the projection map and $dZ = dz_1 \wedge \cdots \wedge dz_n$, $dW = dw_1 \wedge \cdots \wedge dw_m$.

Thanks to Theorem 1, for any smooth strictly plurisubharmonic function ψ on U and any $\overline{\partial}$-closed $f \in \mathscr{D}_{(n,q)}(U)$, it is enough to show that there exists a solution $\overline{\partial}u = f$ satisfying

$$\int_U |u|^2_{\omega_0} e^{-(\widetilde{\varphi}+\psi)} dV_{\omega_0} \le \int_U \left\langle \left(c + \left[\sqrt{-1}\partial\overline{\partial}\psi, \Lambda_{\omega_0}\right]\right)^{-1} f, f \right\rangle_{\omega_0} e^{-(\widetilde{\varphi}+\psi)} dV_{\omega_0}. \tag{5}$$

Consider the $\overline{\partial}$-closed $(n+m,q)$-form $\pi^\star f \wedge \mathrm{d}W$ on $U \times D$. By assumption of φ, we can get a solution of $\overline{\partial}\widetilde{v} = \pi^\star f \wedge \mathrm{d}W$ satisfying an L^2-estimates. Take the L^2-minimal solution $\widetilde{u}$. "If" $\widetilde{u}$ has the form

$$\widetilde{u} = \sum_{1 \le j_1 < \cdots < j_{q-1} \le n} \widetilde{u}_{j_1 \dots j_{q-1}} \mathrm{d}Z \wedge \mathrm{d}W \wedge \mathrm{d}\overline{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{j_{q-1}}, \tag{6}$$

each coefficient $\widetilde{u}_{j_1 \dots j_{q-1}}$ is holomorphic in w and invariant under the rotation of w due to the uniqueness of the minimal solution. Hence, we have $\widetilde{u} = \pi^\star u \wedge \mathrm{d}W$ for some $u \in C^\infty_{(n,q-1)}(U)$ satisfying $\overline{\partial} u = f$ and the inequality (5) on U.

However, the fact that $\widetilde{u}$ has the form (6) does not immediately follow from that $\widetilde{u}$ is the minimal L^2-solution, which is pointed out by Wang Xu. While there are still technical problems, we believe that Theorem 1 is a valid way to solve Problem 3.

As an application of Theorem 2, we also propose the following problem. The reason why Theorem 2 is useful to Problem 13 is the same reason why Theorem 1 is useful to Problem 3.

Problem 13. *Assume that E is weakly RC-positive. Is $S^k E \otimes \det E$ RC-positive for every $k \ge 1$?*

Weak RC-positivity of E implies that $\mathscr{O}_E(1)$ is uniformly $(\dim X - 1)$-positive, where $\mathscr{O}_E(1)$ is the tautological line bundle over the projectivized bundle $\mathbb{P}(E^\star)$ (cf. Proposition 5). This problem asserts that if $\mathscr{O}_E(1)$ is uniformly $(\dim X - 1)$-positive, $\pi_\star(K_{\mathbb{P}(E^\star)/X} \otimes \mathscr{O}(r+k)) \cong S^k E \otimes \det E$ is RC-positive. This is related to the following conjecture raised by Yang.

Conjecture 14 (cf. [14, Question 7.11]). *Assume that E is weakly RC-positive. Then E is RC-positive.*

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