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Upper bounds for superquantiles of martingales

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Abstract. Let $(M_n)_n$ be a discrete martingale in L^p for p in $]1,2]$ or $p = 3$. In this note, we give upper bounds on the superquantiles of M_n and the quantiles and superquantiles of $M_n^* = \max(M_0, M_1, \ldots, M_n)$.

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1. Introduction

Throughout this note, we consider a nondecreasing filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and a real-valued martingale $(M_n)_{n\in\mathbb{N}}$ adapted to this filtration. We use the notations $X_n = M_n - M_{n-1}$ and $M_n^* = \max(M_0, M_1, \ldots, M_n)$ for any positive integer n.

The tail and tail-quantile functions of a real-valued random variable X are defined by

$$H_X(x) = \mathbb{P}(X > x) \text{ for } x \in \mathbb{R},\ Q_X(u) = \inf\{x \in \mathbb{R} : H_X(x) \le u\} \text{ for } u \in]0,1]. \tag{1}$$

Recall that H_X is cadlag and nonincreasing and Q_X is the cadlag generalized inverse function of H_X. From the definition of Q_X, if U has the uniform law over $[0,1]$, then $Q_X(U)$ has the same law as X. The tail-quantile function Q_X is often called Value at Risk (VaR). The Conditional Value at Risk or superquantile $\widetilde{Q}_X$ of X is defined by

$$\widetilde{Q}_X(u) = u^{-1}\int_0^u Q_X(t)\,dt = \int_0^1 Q_X(us)\,ds, \text{ for any } u \in]0,1]. \tag{2}$$

Since Q_X is nonincreasing, $\widetilde{Q}_X \ge Q_X$. From a result which goes back to [2],

$$Q_{M_n^*}(u) \le \widetilde{Q}_{M_n}(u) \text{ for any } u \in]0,1]. \tag{3}$$

We also refer to [6] for a proof of this result. Consequently any upper bound on the superquantiles of M_n provides the same upper bound on the tail-quantiles of M_n^*. Furthermore (3) cannot be improved without additional conditions, as proved by [5]. These facts motivate this note.

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Our approach to bound up $\widetilde{Q}_{M_n}$ is based on the p-risks $Q_p(X,.)$ introduced in [11]. Let z_+ and z_- denote respectively the positive and the negative part of the real z. The p-risk $Q_p(X,.)$ of a real-valued random variable X or its law P_X is defined in [11, Theorem 2.3] for p in $]0,\infty[$ by

$$Q_p(X,u) = Q_p(P_X,u) = \inf\{-t + u^{-1/p}\|(X+t)_+\|_p : t \in \mathbb{R}\} \text{ for any } u \in]0,1]. \tag{4}$$

These p-risks are nondecreasing with respect to p. The main feature is that they are easier to bound up than the quantiles or superquantiles. Furthermore, in the case $p=1$,

$$Q_1(X,u) = Q_X(u) + u^{-1}\mathbb{E}\big((X - Q_X(u))_+\big) = \widetilde{Q}_X(u) \text{ for any } u \in]0,1]. \tag{5}$$

Hence $Q_1(X,.)$ is exactly the superquantile of X. Therefrom

$$\widetilde{Q}_X(u) \le Q_p(X,u) \text{ for any } u \in]0,1] \text{ and any } p \ge 1. \tag{6}$$

We refer to [11] for more about the properties of the p-risks.

In order to bound up $\widetilde{Q}_{M_n^*}$, we will introduce supersuperquantiles. Let U be a random variable with uniform law over $[0,1]$. For a real-valued random variable X, the supersuperquantile $Q_{1,1}(X,.)$ of X is defined by

$$Q_{1,1}(X,u) = Q_1\left(Q_1(X,U),u\right) = \widetilde{Q}_{Q_1(X,U)}(u) = \widetilde{Q}_{\tilde{Q}_X(U)}(u) \text{ for any } u \in]0,1]. \tag{7}$$

Then, from (3),

$$\widetilde{Q}_{M_n^*}(u) \le Q_{1,1}(M_n,u) \text{ for any } u \in]0,1], \tag{8}$$

so that any upper bound on the supersuperquantile of M_n yields the same upper bound on $\widetilde{Q}_{M_n^*}$. Therefore the upper bounds on $\widetilde{Q}_{M_n^*}$ will be derived from the inequality below, proved in Section 2: for any $p>1$ and any u in $]0,1]$,

$$Q_{1,1}(X,u) \le Q_p\left(X, \left(\Pi(q)\right)^{1-p} u\right) \text{ where } q = p/(p-1),\ \Pi(q) = \int_0^\infty t^q e^{-t}\,dt. \tag{9}$$

According to the above inequalities, it is enough to bound up the p-risks of M_n. For martingales in L^p for some p in $]1,2]$, these upper bounds will be derived from one-sided von Bahr–Esseen type inequalities stated in Section 3. In the case of martingales in L^2 satisfying an additional condition of order 3, these upper bounds will be derived from Inequality (11) below. For random variables Y and Z such that $\mathbb{E}(Y_+^p) < \infty$ and $\mathbb{E}(Z_+^p) < \infty$, let

$$D_p^+(Y,Z) = \sup\left\{\mathbb{E}\left((Z+t)_+^p - (Y+t)_+^p\right) : t \in \mathbb{R}\right\}. \tag{10}$$

Then, from the definition (4) of the p-risks, it is immediate that, for any u in $]0,1]$,

$$Q_p(Z,u) \le \inf\left\{-t + u^{-1/p}\left(\mathbb{E}(Y+t)_+^p + D_p^+(Y,Z)\right)^{1/p} : t \in \mathbb{R}\right\} \le Q_p(Y,u) + u^{-1/p}\left(D_p^+(Y,Z)\right)^{1/p}. \tag{11}$$

This inequality will be used in Section 4 to provide upper bounds on the superquantiles of martingales under additional assumptions on the conditional variances of the increments and the moments of order 3 of their positive parts.

2. Comparison inequalities for risks

In this section we prove the comparison inequality (9) and we give applications of this inequality to upper bounds on the superquantiles of M_n^*. We now state the main results of this section.

Proposition 1. *Let p in $]1,\infty[$ and X be an integrable real-valued random variable such that $\mathbb{E}(X_+^p) < \infty$. Then $Q_{1,1}(X,u) \le Q_p(X,(\Pi(q))^{1-p}u)$ for any u in $]0,1]$, where $q = p/(p-1)$ and $\Pi(q) = \int_0^\infty t^q e^{-t}dt$.*

From Proposition 1 and (8), we immediately get the result below.

Corollary 2. *Let $(M_n)_n$ be a martingale such that $\mathbb{E}(M_{n+}^p) < \infty$ for some $p > 1$. Set $M_n^* = \max(M_0, M_1, \dots, M_n)$. Then $Q_1(M_n^*, u) \le Q_p(M_n, (\Pi(q))^{1-p}\, u)$ for any u in $]0,1]$.*

Proof of Proposition 1. From the fact that $\widetilde{Q}_X$ is nonincreasing, $Q_{\widetilde{Q}_X(U)}(t) = \widetilde{Q}_X(t)$ for any t in $]0,1]$. Integrating this equality, we get from (2) and (7) that

$$Q_{1,1}(X,u) = \int_0^1 \widetilde{Q}_X(us)ds = \int_0^1\int_0^1 Q_X(t)(us)^{-1}\mathbb{1}_{t\le us}ds dt = u^{-1}\int_0^u Q_X(t)\log(u/t)dt \tag{12}$$

by the Fubini theorem, where log denotes the Neper logarithm. Now, V be a random variable with uniform law over $[0,1]$. Using the change of variable $v = t/u$ in the above integral, we get that

$$Q_{1,1}(X,u) = \int_0^1 Q_X(uv)\log(1/v)dv = \mathbb{E}\big(Q_X(uV)\log(1/V)\big). \tag{13}$$

Next, since $\mathbb{E}\log(1/V) = 1$,

$$Q_{1,1}(X,u) = -t + \mathbb{E}\big(\log(1/V)\,(Q_X(uV)+t)\big) \le -t + \mathbb{E}\big(\log(1/V)\,(Q_X(uV)+t)_+\big). \tag{14}$$

Now, applying the Hölder inequality, with exponents $q = p/(p-1)$ and p,

$$\mathbb{E}\big(\log(1/V)\,(Q_X(uV)+t)_+\big) \le \big\|\log(1/V)\big\|_q\ \big\|(Q_X(uV)+t)_+\big\|_p.$$

Since $\log(1/V)$ has the exponential law $\mathcal{E}(1)$, $\|\log(1/V)\|_q = (\Pi(q))^{1/q}$ and, setting $w = uv$,

$$\int_0^1 (Q_X(uv)+t)_+^p\,dv = u^{-1}\int_0^u (Q_X(w)+t)_+^p\,dw \le u^{-1}\int_0^1 (Q_X(w)+t)_+^p\,dw.$$

Hence

$$\mathbb{E}\big(\log(1/V)\,(Q_X(uV)+t)_+\big) \le \big(\Pi(q)\big)^{1/q}\,u^{-1/p}\,\|(X+t)_+\|_p. \tag{15}$$

Combining (14) and (15), we now get that, for any real t,

$$Q_{1,1}(X,u) \le -t + \Big(\big(\Pi(q)\big)^{1-p}\,u\Big)^{-1/p}\,\|(X+t)_+\|_p, \tag{16}$$

which implies Proposition 1. □

Remark 3. From (4), $Q_p(M_n,u) \le u^{-1/p}\|M_{n+}\|_p$. Hence, if $M_0 = 0$, Corollary 2 applied with $u = 1$ implies the known inequality $\|M_n^*\|_1 \le (\Pi(q))^{1/q}\|M_{n+}\|_p$. The constant $(\Pi(q))^{1/q}$ in this inequality is sharp, which proves that our constant is also sharp. We refer to [9, Theorem 7.8] for more about this.

We now discuss Corollary 2. If the martingale $(M_n)_n$ is conditionally symmetric, then, by the Lévy symmetrization inequality, $H_{M_n^*}(x) \le 2H_{M_n}(x)$ for any real x, which implies that $Q_p(M_n^*,u) \le Q_p(M_n,u/2)$ for $p \ge 1$ and u in $]0,1]$. Therefrom, for conditionally symmetric martingales,

$$Q_1\left(M_n^*,u\right) \le Q_p\left(M_n,u/2\right) \ \text{ for any } p \ge 1. \tag{17}$$

If $p = 2$, Corollary 2 also yields $Q_1(M_n^*,u) \le Q_2(M_n,u/2)$. Recall now that $\Pi(q) = \mathbb{E}(\tau^q)$, if τ is a random variable with law $\mathcal{E}(1)$. Thus, if $p > 2$, then $1 < q < 2$ and $\Pi(q) = \mathbb{E}(\tau^q) < (\mathbb{E}\tau)^{2-q}(\mathbb{E}\tau^2)^{q-1} = 2^{q-1}$, which implies that $(\Pi(q))^{1-p} > 1/2$, since $(q-1)(1-p) = -1$. Consequently, for $p > 2$ Corollary 2 is more efficient than (17), because $Q_p(X,u)$ is nonincreasing in u for u in $]0,1]$. For example, if $p = 3$, $Q_1(M_n^*,u) \le Q_3(M_n, 16u/(9\pi))$ by Corollary 2, and $16/(9\pi) = 0.565\ldots > 1/2$.

3. Martingales in L^p for p in $]1,2]$

In this section, p is any real in $]1,2]$ and $(M_n)_n$ is a martingale in L^p. Our aim is to obtain upper bounds on the risks of M_n and M_n^*. From (4), these upper bounds can be derived from upper bounds on the moments of order p of $(M_n+t)_+$. At the present time, moment inequalities with sharp constants are only available for the absolute value of M_n. More precisely, by [12, Proposition 1.8],

$$\mathbb{E}\left(|M_n|^p\right) \le \mathbb{E}\left(|M_0|^p\right) + K_p \mathbb{E}\left(|X_1|^p + \cdots + |X_n|^p\right), \quad \text{where } K_p = \sup_{x\in[0,1]} \left(px^{p-1} + (1-x)^p - x^p\right). \tag{18}$$

As shown in [12], the constant K_p is sharp. The constant K_p is decreasing with respect to p, $K_2 = 1$ and $\lim_{p\searrow 1} K_p = 2$. However, for conditionally symmetric martingales, it is known since a long time that the constant in the above inequality is equal to 1 for any p in $]1,2]$. So it seems clear that the constants in the one-sided case are smaller than K_p. Below we give a new inequality.

Theorem 4. *Let p be any real in $]1,2]$ and $(M_n)_n$ be a martingale in L^p. Then*

$$\mathbb{E}\left(M_{n+}^p\right) \le \mathbb{E}\left(M_{0+}^p\right) + \Delta_p, \ \text{ with } \Delta_p = \mathbb{E}\left(X_{1+}^p + \cdots + X_{n+}^p\right) + (p-1)^{p-1}\mathbb{E}\left(X_{1-}^p + \cdots + X_{n-}^p\right). \tag{19}$$

Before proving Theorem 4, we give an application to risks.

Corollary 5. *Let p be any real in $]1,2]$ and $(M_n)_n$ be a martingale in L^p such that $M_0 = 0$. Set $q = p/(p-1)$. Then $Q_p(M_n,u) \le \Delta_p^{1/p}(u^{1-q}-1)^{1/q}$ and $Q_1(M_n^*,u) \le \Delta_p^{1/p}(\Pi(q)u^{1-q}-1)^{1/q}$ for any u in $]0,1]$.*

Remark 6. If $p = 2$, $q = 2$ and $\Pi(q) = 2$. Then we get from Corollary 5 that

$$Q_2\left(M_n,u\right) \le \sqrt{\mathbb{E}\left(M_n^2\right)(1/u-1)}, \quad Q_1\left(M_n^*,u\right) \le \sqrt{\mathbb{E}\left(M_n^2\right)(2/u-1)}. \tag{20}$$

The first inequality is a version of an inequality of Tchebichef [16], often called Cantelli's inequality. For $p < 2$, $(p-1)^{p-1} < 1$. In that case the results are new.

Proof of Corollary 5. We start by the first inequality. Let u be any real in $]0,1[$. From Theorem 4 applied to $(t+M_n)_n$, we get $Q_p(M_n,u) \le -t + u^{-1/p}(t^p+\Delta_p)^{1/p}$. Now the function $f: t \mapsto -t + u^{-1/p}(\Delta_p + t^p)^{1/p}$ has a unique minimum at point $t = t_u = \Delta_p^{1/p}(u^{1-q}-1)^{-1/p}$ and $f(t_u) = \Delta_p^{1/p}(u^{1-q}-1)^{1/q}$, which completes the proof of the first inequality in the case $u < 1$. Since $Q_p(M_n,.)$ is nonincreasing, the case $u = 1$ follows by taking the limit as $u \uparrow 1$. The second part follows from the first part, Corollary 2 and the fact that $(1-p)(1-q) = 1$. □

Proof of Theorem 4. Theorem 4 follows immediately from the Lemma below by induction on n. □

Lemma 7. *Let Z and X be real-valued random variables in L^p for some p in $]1,2]$. If $\mathbb{E}(X \mid Z) = 0$, then $\mathbb{E}((Z+X)_+^p) \le \mathbb{E}(Z_+^p) + \mathbb{E}(X_+^p) + (p-1)^{p-1}\mathbb{E}(X_-^p)$.*

Proof of Lemma 7. Define the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi(z,x) = (z+x)_+^p - z_+^p - pz_+^{p-1}x. \tag{21}$$

From the assumption $\mathbb{E}(X \mid Z) = 0$, $\mathbb{E}((Z+X)_+^p) - \mathbb{E}(Z_+^p) = \mathbb{E}(\varphi(Z,X))$.

Consequently, Lemma 7 follows immediately from the upper bound

$$\varphi(z,x) \le x_+^p + (p-1)^{p-1}x_-^p \ \text{ for any } (x,z) \in \mathbb{R}\times\mathbb{R}. \tag{22}$$

It only remains to prove (22). If $z \le 0$, then $\varphi(z,x) = (z+x)_+^p \le x_+^p$, which proves (22) for $z \le 0$.

Value of u	0.999	0.990	0.900	0.75	0.50	0.25	0.10	0.010	0.001
(24)	0.186	**0.387**	**0.808**	**1.18**	**1.90**	3.53	7.27	38.34	185.8
(25)	**0.180**	0.390	0.881	1.31	2.06	**3.52**	**6.61**	**30.77**	**142.8**

If $z \geq 0$, let the function η_x be defined by $\eta_x(z) = \varphi(z,x)$. The function η_x is continuous on $[0,\infty[$, differentiable on $]0,\infty[$, and $\eta'_x(z) = p((z+x)_+^{p-1} - z^{p-1} - (p-1)z^{p-2}x)$ for $z > 0$. If $z \geq x_-$, $z + x \geq x_+ + x \geq 0$, which implies that $(z+x)_+^{p-1} = (z+x)^{p-1}$. Then the concavity of $t \mapsto t^{p-1}$ ensures that $\eta'_x(z) \leq 0$. It follows that η_x is nonincreasing on $[x_-,\infty[$. If $x \geq 0$, then $x_- = 0$ and $\eta_x(z) \leq \eta_x(0) = x_+^p$, which proves (22) for $z \geq 0$ and $x \geq 0$.

Finally, if $z \geq 0$ and $x < 0$, $z + x \leq 0$ for z in $[0, x_-]$. Thus $\eta'_x(z) = pz^{p-2}(-z + (p-1)x_-)$ for z in $]0, x_-]$. Since η_x is nonincreasing on $[x_-,\infty[$, it follows that η_x has a unique maximum at point $z = (p-1)x_-$ and, subsequently,

$$\eta_x(z) \leq \eta_x\big((p-1)x_-\big) = \big(-(p-1)^p + p(p-1)^{p-1}\big)\,x_-^p = (p-1)^{p-1}x_-^p, \tag{23}$$

which proves (22) for $z \geq 0$ and $x < 0$, therefore completing the proof of (22). □

3.1. *Numerical comparisons*

To conclude this section, we compare the upper bounds given by Corollary 5 with the inequality below, derived from (18) and [15, Theorem 4.1]:

$$Q_1(M_n,u) \leq \Sigma_p^{1/p} u^{-1/p}\big(1 + (1-u)^{1-p}u^{p-1}\big)^{-1/p}, \ \text{ with } \Sigma_p = K_p \mathbb{E}\big(|X_1|^p + \cdot + |X_n|^p\big). \tag{24}$$

For the numerical comparisons we assume that

$$\sum_{k=1}^n \mathbb{E}\big(X_{k+}^p\big) = \sum_{k=1}^n \mathbb{E}\big(X_{k-}^p\big) = 1.$$

Then Corollary 5 yields

$$Q_1(M_n,u) \leq \big(1 + (p-1)^{p-1}\big)^{1/p} u^{-1/p}\big(1 - u^{q-1}\big)^{1/q}, \text{ with } q = p/(p-1), \tag{25}$$

and $\Sigma_p = 2K_p$ in (24). The table below gives values of the upper bounds (24) and (25) for $p = 3/2$, in which case $2K_p = 2(1+1/\sqrt{2})^{1/2}$ and $1 + (p-1)^{p-1} = 1 + 1/\sqrt{2}$. Here (25) provides better bounds for $u \leq 0.25$ and $u \geq 0.9922$.

4. The case $p = 3$

In this section, $(M_n)_n$ is a martingale in L^2 such that $M_0 = 0$. We assume that, for some sequence $(\sigma_k)_{k>0}$ of nonrandom positive reals,

$$\mathbb{E}\big(X_{k+}^3\big) < \infty \ \text{ and } \ \mathbb{E}\big(X_k^2\big|\mathcal{F}_{k-1}\big) \leq \sigma_k^2 \ \text{ almost surely, for any positive } k. \tag{26}$$

Although the above condition on the conditional variances is very strong, is is sometimes fulfilled. For example, the second part of (26) holds for martingale decompositions associated to dynamical systems or suprema of empirical processes. We refer to [3, Inequality (4.9), page 861], for dynamical systems and to [8] for empirical processes. The main result of this section is the following upper bound for $\mathbb{E}((M_n + t)_+^3)$.

Theorem 8. *Let Y be a random variable with law $N(0,1)$ and $(M_n)_n$ be a martingale such that $M_0 = 0$, satisfying (26). Set $V_n = \sigma_1^2 + \cdots + \sigma_n^2$. Then*

$$\mathbb{E}\big((M_n+t)_+^3\big) \leq \mathbb{E}\left(\Big(Y\sqrt{V_n} + t\Big)_+^3\right) + \sum_{k=1}^n \mathbb{E}\big(X_{k+}^3\big)$$

for any real t.

Remark 9. From Theorem 8 with $t = 0$, $\mathbb{E}(M_{n+}^3) \le (2/\pi)^{1/2} V_n^{3/2} + \sum_{k=1}^n \mathbb{E}(X_{k+}^3)$, which is is a one-sided version of the Rosenthal inequality, with the optimal constants. We refer to [13] and the references therein for more about the constants in the Rosenthal inequalities.

Proof of Theorem 8. Let $(Y_k)_{k>0}$ be a sequence of independent random variables with law $N(0,1)$, independent of the sequence $(M_n)_n$. Define the random variables T_k^n and the reals D_k^n for k in $[1,n]$ by

$$T_k^n = t + M_{k-1} + (\sigma_{k+1} Y_{k+1} + \cdots + \sigma_n Y_n), \quad D_k^n = \mathbb{E}\Big(\big(T_k^n + X_k\big)_+^3 - \big(T_k^n + \sigma_k Y_k\big)_+^3\Big), \tag{27}$$

with the convention that $T_n^n = t + M_{n-1}$. Then

$$\mathbb{E}\left((M_n + t)_+^3 - \Big(Y\sqrt{V_n} + t\Big)_+^3\right) = D_1^n + \cdots + D_n^n. \tag{28}$$

Now the function φ defined by $\varphi(x) = x_+^3$ for x in $\mathbb{R}$ is two times continuously differentiable and $\varphi'(x) = 3x_+^2$, $\varphi''(x) = 6x_+$. Hence, applying the Taylor integral formula at order 2 to the function φ at point T_k^n,

$$D_k^n = 3\,\mathbb{E}\Big(\big(T_{k+}^n\big)^2 (X_k - \sigma_k Y_k)\Big) + 3\,\mathbb{E}\big(T_{k+}^n \big(X_k^2 - \sigma_k^2 Y_k^2\big)\big) + 6\int_0^1 (1-s) R_{k,n}(s)\,ds, \tag{29}$$

$$\text{with } R_{k,n}(s) = \mathbb{E}\Big(\big(\big(T_k^n + sX_k\big)_+ - T_{k+}^n\big) X_k^2 - \big(\big(T_k^n + s\sigma_k Y_k\big)_+ - T_{k+}^n\big)\sigma_k^2 Y_k^2\Big). \tag{30}$$

From the martingale assumption, the first term on right hand in (29) is equal to 0. Next

$$\mathbb{E}\big(T_{k+}^n \big(X_k^2 - \sigma_k^2 Y_k^2\big)\big) = \mathbb{E}\big(T_{k+}^n \big(\mathbb{E}\big(X_k^2 \big| \mathcal{F}_{k-1}\big) - \sigma_k^2\big)\big) \le 0,$$

since $T_{k+}^n \ge 0$ and $\mathbb{E}(X_k^2 \mid \mathcal{F}_{k-1}) - \sigma_k^2 \le 0$ almost surely.

From the above inequalities, the two first terms in (29) are nonpositive. It remains to bound up the integral term in (29). First $(T_{k,n} + sX_k)_+ - T_{k,n+} \le sX_k^+$ for any s in $[0,1]$, which implies that

$$\mathbb{E}\big(\big(\big(T_k^n + sX_k\big)_+ - T_{k+}^n\big) X_k^2\big) \le s\mathbb{E}\big(X_{k+}^3\big). \tag{31}$$

And second the normal law is symmetric, whence

$$\mathbb{E}\big(\big(\big(T_k^n + s\sigma_k Y_k\big)_+ - T_{k+}^n\big) Y_k^2\big) = \tfrac{1}{2}\,\mathbb{E}\big(\big(\big(T_k^n + s\sigma_k Y_k\big)_+ + \big(T_k^n - s\sigma_k Y_k\big)_+ - 2T_{k+}^n\big) Y_k^2\big).$$

Since the function $x \mapsto x_+$ is convex, $(T_k^n + s\sigma_k Y_k)_+ + (T_k^n - s\sigma_k Y_k)_+ - 2T_{k+}^n \ge 0$. It follows that

$$\mathbb{E}\big(\big(\big(T_k^n + s\sigma_k Y_k\big)_+ - T_{k+}^n\big) Y_k^2\big) \ge 0. \tag{32}$$

Now (30), (31) and (32) imply that $R_{k,n}(s) \le s\mathbb{E}(X_{k+}^3)$. Finally, putting this inequality in (29) and integrating, we get that $D_k^n \le \mathbb{E}(X_{k+}^3)$, which, by (28), implies Theorem 8. □

Remark 10. If (26) does not hold, the second term in decomposition (29) may fail to be nonpositive. Nevertheless, choosing $\sigma_k^2 = \mathbb{E}(X_k^2)$ in (27) and proceeding as in [4], one can prove that

$$\sum_{k=1}^n \mathbb{E}\left(T_{k+}^n \big(\mathbb{E}\big(X_k^2 \big| \mathcal{F}_{k-1}\big) - \sigma_k^2\big) \le \sum_{j=1}^{n-1} \mathbb{E}\left(\left| X_j \sum_{k=j+1}^n \big(\mathbb{E}\big(X_k^2 \big| \mathcal{F}_j\big) - \sigma_k^2\big)\right|\right)\right), \tag{33}$$

which gives the upper bound

$$\mathbb{E}\big((M_n + t)_+^3\big) \le \mathbb{E}\left(\Big(Y\sqrt{\operatorname{Var} M_n} + t\Big)_+^3\right) + \sum_{k=1}^n \mathbb{E}\big(X_{k+}^3\big) + 3\sum_{j=1}^{n-1} \mathbb{E}\left(\left| X_j \sum_{k=j+1}^n \big(\mathbb{E}\big(X_k^2 \big| \mathcal{F}_j\big) - \sigma_k^2\big)\right|\right). \tag{34}$$

This upper bound may be of interest in the case of dependent sequences, such as absolutely regular Markov chains.

From Theorem 8, (11) and Corollary 2, we immediately get the following asymptotically subGaussian upper bounds on the superquantiles of M_n and M_n^*.

Corollary 11. *Let Y be a random variable with law $N(0,1)$ and $(M_n)_n$ be a martingale such that $M_0 = 0$, satisfying* (26)*. Set $V_n = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$. Then, for any p in* $[1,3]$ *and any u in* $]0,1]$

$$Q_p(M_n,u) \le \inf_{t\in\mathbb{R}} \left\{ -t + u^{-\frac{1}{3}} \left(\mathbb{E}\left(\left(Y\sqrt{V_n} + t \right)_+^3 \right) + \sum_{k=1}^{n} \mathbb{E}\left(X_{k+}^3 \right) \right)^{\frac{1}{3}} \right\}$$

$$\le \sqrt{V_n} Q_3(Y,u) + u^{-\frac{1}{3}} \left(\sum_{k=1}^{n} \mathbb{E}\left(X_{k+}^3 \right) \right)^{\frac{1}{3}}, \tag{a}$$

$$xQ_1\left(M_n^*,u\right) \le \inf\left\{ -t + \left(\frac{9\pi}{16u} \right)^{1/3} \left(\mathbb{E}\left(\left(Y\sqrt{V_n} + t \right)_+^3 \right) + \sum_{k=1}^{n} \mathbb{E}(X_{k+}^3) \right)^{1/3} : t \in \mathbb{R} \right\}. \tag{b}$$

4.1. *Numerical comparisons*

To conclude this section, we compare Corollary 11 (a) with previous results in two different cases. First, we compare Corollary 11 (a) in the independent case under the condition $\mathbb{E}(M_n^3) = 0$ with upper bounds derived from moment inequalities or estimates of the Kantorovich distance in the central limit theorem. And second, we compare Corollary 11 (a) with exponential inequalities in the case of independent and bounded increments.

(a) Independent increments with finite third moments

From Theorem 8 applied with $t = 0$ and the Hölder inequality,

$$Q_1(M_n,u) \le u^{-1/3} \left(\sqrt{2/\pi}\, V_n^{3/2} + \mathbb{E}\left(X_{1+}^3 + \cdots + X_{n+}^3 \right) \right)^{1/3}. \tag{35}$$

Such a result is called weak L^3 concentration inequality in [3]. If furthermore the increments X_k are in L^3, then, by [7, Theorem 1.1], for any 1-Lipschitz function f,

$$\mathbb{E}\left(f(M_n) - f\left(Y\sqrt{V_n} \right) \right) \le V_n^{-1} \mathbb{E}\left(|X_1|^3 + \cdots + |X_n|^3 \right), \text{ with } V_n = \operatorname{Var} M_n. \tag{36}$$

Now, since $\mathbb{E}(M_n) = \mathbb{E}(Y) = 0$, by (36) and the elementary equality $x_+ = (x + |x|)/2$,

$$\mathbb{E}\left((M_n + t)_+ - \left(Y\sqrt{V_n} + t \right)_+ \right) \le \tfrac{1}{2} V_n^{-1} \mathbb{E}\left(|X_1|^3 + \cdots + |X_n|^3 \right)$$

for any real t. Hence, by (11) applied with $p = 1$, for any u in $]0,1]$,

$$Q_1(M_n,u) \le \sqrt{V_n}\, Q_1(Y,u) + (2uV_n)^{-1} \mathbb{E}\left(|X_1|^3 + \cdots + |X_n|^3 \right). \tag{37}$$

The table below gives numerical values for the upper bounds of (37), Corollary 11 (a), their respective limits $Q_1(Y,u)$ and $Q_3(Y,u)$ (as the Liapounov ratio tends to 0) and (35) in the case $V_n = 1$ and $L_3^+ := \mathbb{E}(X_{1+}^3 + \cdots + X_{n+}^3) = \mathbb{E}(X_{1-}^3 + \cdots + X_{n-}^3) := L_3^-$, for $L_3^+ = 10^{-m}$, $m = 1,2,3$ and $u = 2^{k-2}10^{-k}$, $k = 0,1,2$. For sake of completeness, the values of the usual subGaussian bound $\sqrt{2|\log u|}$ (which is larger than $Q_p(Y,u)$, as shown in [11]) are also included. One can observe that the convergence to the limit is much faster in Corollary 11 (a) than in (37). As a by-product, Corollary 11 (a) still provides better bounds for $u \le 1/20$ if $L_3^+ = 10^{-2}$, which is in the range of normal approximation, since the Liapounov ratio $L_3 := L_3^+ + L_3^-$ is equal to 2.10^{-2}. For all the values of the Liapounov ratio in the table, Inequality (35) is of poor quality for $u = 1/20$ and very poor quality for $u = 10^{-2}$, which shows that moment inequalities are not a suitable tool to achieve efficient concentration inequalities if the Liapounov ratio is small.

Value of (k,m)	$(0,1)$	$(1,1)$	$(2,1)$	$(0,2)$	$(1,2)$	$(2,2)$	$(0,3)$	$(1,3)$	$(2,3)$
Value of $Q_1(Y,u)$	1.27	2.06	2.67	1.27	2.06	2.67	1.27	2.06	2.67
Inequality (37)	1.67	4.06	12.67	**1.31**	**2.26**	3.67	**1.274**	**2.08**	**2.77**
Inequality (35)	1.53	2.62	4.48	1.478	2.53	4.32	1.473	2.52	4.31
Corollary 11(a)	**1.50**	**2.48**	**3.67**	1.467	**2.26**	**3.01**	1.463	2.23	2.84
Value of $Q_3(Y,u)$	1.462	2.22	2.81	1.462	2.22	2.81	1.462	2.22	2.81
Value of $\sqrt{2\lvert\log u\rvert}$	1.665	2.447	3.035	1.665	2.447	3.035	1.665	2.447	3.035

(b) Sums of bounded random variables

Let v be a real in $]0,1[$ and $(\xi_k)_{k>0}$ be a sequence of independent random variables such that $\mathbb{P}(\xi_k = 1) = v/(1+v)$ and $\mathbb{P}(\xi_k = -v) = 1/(1+v)$. Let us consider a sequence $(a_k)_{k>0}$ of positive weights. Define the sequence $(M_n)_{n\in\mathbb{N}}$ by $M_0 = 0$ and $M_n = a_1\xi_1 + \cdots + a_n\xi_n$ for $n > 0$. Then

$$\operatorname{Var} M_n = v\sum_{k=1}^n a_k^2,\ \sum_{k=1}^n \mathbb{E}\left(X_{k+}^3\right) = \frac{v}{1+v}\sum_{k=1}^n a_k^3 \text{ and } \sum_{k=1}^n \mathbb{E}\left(|X_k|^3\right) = \frac{v(1+v^2)}{1+v}\sum_{k=1}^n a_k^3. \tag{38}$$

Since the increments of M_n are bounded, M_n has a finite Laplace transform. Let ℓ denote the logarithm of the Laplace transform of M_n, defined by $\ell(t) = \log\mathbb{E}(e^{tM_n})$ for any real t. By [11, Theorem 3.3], for any $p \geq 1$,

$$Q_p(M_n,u) \leq \inf\left\{t^{-1}\left(\left|\log u\right| + \ell(t)\right) : t > 0\right\} \text{ for any } u \in]0,1]. \tag{39}$$

Assume now that

$$\operatorname{Var} M_n = 1 \text{ and } \mathbb{E}\left(X_{1+}^3 + \cdots + X_{n+}^3\right) = L_3^+. \tag{40}$$

Under (40), classical estimates of the subGaussian constant of binary random variables (*see* [1, Section 2.5]) yield the upper bound $\ell(t) \leq (1-v^2)t^2/(4v|\log v|)$. Hence, by (39), for any $p \geq 1$,

$$Q_p\left(M_n,u\right) \leq \sqrt{2\kappa(v)\left|\log u\right|} \text{ for any } u \in]0,1], \text{ with } \kappa(v) = (1-v^2)/(2v|\log v|). \tag{41}$$

The constant $\kappa(v)$ is larger than 1, which induces a loss. For example, $\kappa(v) = 2.0227\ldots$ if $v = 1/9$. In order to avoid this loss on the variance factor, one can use Bennett type inequalities. Define the p-norm $|a|_p$ of $(a_1,\ldots,a_n)$ by

$$|a|_p = \left(|a_1|^p + \cdots + |a_n|^p\right)^{1/p} \text{ for } p \in [1,\infty[\text{ and } |a|_\infty = \sup\left(|a_1|,\ldots,|a_n|\right). \tag{42}$$

Then $|a|_\infty \leq \min(|a|_2,|a|_3)$. Therefrom, under (40), by (38),

$$|a|_\infty \leq \min\left(v^{-1/2}, \left(L_3^+(1+v)/v\right)^{1/3}\right) := K \tag{43}$$

The above inequality cannot be improved under condition (40). From (43),

$$\ell(t) \leq K^{-2}\left(e^{Kt} - 1 - Kt\right) = \left(t^2/2\right) + K\left(t^3/6\right) + \cdots \text{ for any } t > 0 \tag{44}$$

(*see* [1, Section 2.4]). It follows that, for $p \geq 1$ and u in $]0,1]$,

$$Q_p(M_n,u) \leq \inf\left\{t^{-1}\left(\left|\log u\right| + K^{-2}\left(e^{Kt} - 1 - Kt\right)\right) : t > 0\right\} \leq \sqrt{2|\log u|} + K|\log u|/3. \tag{45}$$

In the above inequality, the first order term $\sqrt{2|\log u|}$ is the optimal one. However K is large, which induces a big loss in the second order term. In order to reduce this loss, one can use [14, Theorem 2.1]. Define $\ell_v(t)$ by $\ell_v(t) = \log\mathbb{E}(e^{t\xi_1})$. Then, by [14, Theorem 2.1],

$$\ell(t) \leq v\left(\sum_{k=1}^n a_k^2\right)\frac{t^2}{2} + \gamma(v)\left(\sum_{k=1}^n a_k^3\right)\frac{t^3}{6}, \text{ with } \gamma(v) = 6\sup_{t>0}\left(\frac{\ell_v(t) - vt^2/2}{t^3}\right). \tag{46}$$

For example, if $v = 1/9$, then $\gamma(v) \leq 0.1176$. From (46) and (39), for any $p \geq 1$ and any u in $]0,1]$,

$$Q_p\left(M_n,u\right) \leq \inf\left\{t^{-1}\left(|\log u| + \left(t^2/2\right) + \eta(v)L_3^+\left(t^3/6\right)\right) : t > 0\right\}, \text{ with } \eta(v) = \gamma(v)(1+v)/v. \tag{47}$$

Value of (k,m)	$(0,0)$	$(1,0)$	$(2,0)$	$(0,1)$	$(1,1)$	$(2,1)$
Inequality (37)	3.29	12.18	52.67	**1.47**	3.07	7.73
Corollary 11 (a)	**1.73**	3.30	5.54	1.50	**2.48**	3.67
Inequality (47)	2.04	**3.17**	**4.08**	1.72	2.56	**3.20**
Inequality (45)	2.49	4.15	5.58	2.08	3.32	4.34
Inequality (41)	2.37	3.48	4.32	2.37	3.48	4.32
Value of $\sqrt{2\lvert\log u\rvert}$	1.665	2.447	3.035	1.665	2.447	3.035

The table below gives numerical values for the upper bounds of (37), Corollary 11 (a), (47), (45) and (41) in the case $v = 1/9$ and $V_n = 1$ for $L_3^+ = 10^{-m}$, $m = 0,1$ and $u = 2^{k-2}10^{-k}$, $k = 0,1,2$. For sake of completeness, the values of the usual subGaussian bound $\sqrt{2|\log u|}$ are also included. For all the values of L_3^+ and u in the table, (45) and (41) are of very poor quality. Inequality (37) is also of very poor quality, except in the case $u = 1/4$ and $L_3^+ = 1/10$. One can observe that (47) is more efficient than Corollary 11 (a) for $u = 1/20$ and $u = 10^{-2}$ if $L_3^+ = 1$ and for $u = 10^{-2}$ if $L_3^+ = 1/10$.

5. Concluding remarks and comments

5.1. *About Section 4*

I consider Section 4 as the most relevant of this note. Clearly the assumptions of Corollary 11 cannot be used to provide a rate of convergence in the global central limit theorem, since the negative parts of the increments X_k have only a finite moment of ordre 2. Nevertheless, one can still recover partly the missing factor in the deviation inequalities on the right, by using the techniques introduced in [11]. It would be of interest to obtain lower bounds in the independent and identically distributed case. For example, if $\mathcal{L}_{2,3+}(1,m)$ denotes the class of probability laws on the real line such that

$$\int_{\mathbb{R}} x\,d\mu(x) = 0,\ \int_{\mathbb{R}} x^2\,d\mu(x) = 1,\ \int_{[0,\infty[} x^3\,d\mu(x) \le m,$$

and γ denotes the standard normal law, I conjecture that, for any positive m and any u in $]0,1]$,

$$\liminf_{n\to\infty} \sup_{\mu\in\mathcal{L}_{2,3+}(1,m)} n^{-1/2} Q_1\left(\mu^{*n}, u\right) \ge Q_2(\gamma, u). \tag{48}$$

Such a result would prove that the asymptotic lower bound cannot be equal to the usual superquantile. However I have no idea of an outline of proof for such a result.

5.2. *About the p-risks of the standard normal law.*

For any real-valued random variable Y, let $H_p(Y,.)$ denote the generalized inverse function of $Q_p(Y,.)$. If the tail function of Y is log-concave on $\mathbb{R}$, then, for any real x and any positive p,

$$H_p(Y,x) \le \Pi(p)(e/p)^p \mathbb{P}(Y > x) \tag{49}$$

(we refer to [10, Theorem 1.2] for an available reference). The above inequality shows that the p-risks can be used to partly recover the missing factor. From the above inequality, one immediately gets that, for any positive p and any u in $]0,1]$,

$$Q_p(Y,u) \le Q_Y\left(\left(\Pi(p)\right)^{-1}(p/e)^p u\right). \tag{50}$$

The above inequality holds, in particular, for the standard Gaussian law. However, in the case $p = 3$, this upper bound is significantly larger than the exact value for usual values of u, as shown in the numerical table below.

Value of u	**0.250**	**0.050**	**0.010**
Value of $Q_3(Y,u)$	1.462	2.22	2.81
(50) with $p=3$	1.588	2.283	2.85
Value of $\sqrt{2\lvert\log u\rvert}$	1.665	2.447	3.035

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