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Comptes Rendus
Mathématique

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Volume 360 (2022), p. 491-496

<https://doi.org/10.5802/crmath.318>

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Algebra, Combinatorics / *Algèbre, Combinatoire*

On the Hochschild homology of singularity categories

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Abstract. Let k be an algebraically closed field and A a finite-dimensional k-algebra. In this note, we determine complexes which compute the Hochschild homology of the canonical dg enhancement of the bounded derived category of A and of the canonical dg enhancement of the singularity category of A. As an application, we obtain a new approach to the computation of Hochschild homology of Leavitt path algebras.

2020 Mathematics Subject Classification. 16E35, 16E40, 16E45, 18G80.

Funding. The second author thanks the Abel visiting scholar program for financing a visit to the third author in September 2019 during which this collaboration started. The third author thanks the Sherbrooke representation theory team for organizing the VirtARTA 2020, where the material of this note was first presented.

Manuscript received 9 October 2021, revised and accepted 21 December 2021.

1. Reminder on Hochschild homology of algebras and categories

Let k be a field. We write $\otimes$ for $\otimes_k$. Let A be a k-algebra (associative, with 1). We write $\operatorname{Mod} A$ for the category of all (right) A-modules and $\mathscr{D}A = \mathscr{D}(\operatorname{Mod} A)$ for its unbounded derived category. Let $A^e = A \otimes A^{op}$ be the *enveloping algebra* of A so that A^e-modules identify with A-bimodules. The *Hochschild homology* of A is defined by

$$HH_p(A) = \operatorname{Tor}_p^{A^e}(A, A)\,,\ p \in \mathbb{Z}.$$

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Alternatively, we may define it as the pth homology group of the *Hochschild chain complex* $HH(A)$ *of* A, i.e. the complex $C_* A$ concentrated in homological degrees ≥ 0

$$A \longleftarrow A \otimes A \longleftarrow \dots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes (p+1)} \longleftarrow \dots$$

with $C_p A = A^{\otimes(p+1)}$, $p \geq 0$, and differential given by

$$d\left(a_0, \dots, a_p\right) = \sum_{i=0}^{p-1} (-1)^i \left(a_0, \dots, a_i a_{i+1}, \dots, a_p\right) + (-1)^p \left(a_p a_0, \dots, a_{p-1}\right), \tag{1}$$

where we write $(a_0, \dots, a_p)$ for $a_0 \otimes \cdots \otimes a_p$. Notice that the first differential takes $a \otimes b$ to the commutator $ab - ba$.

We see that $HH_0(A)$ is the quotient $A/[A, A]$ of the vector space A by its subspace generated by all commutators and that $HH_p(A)$ and $HH(A) \in \mathcal{D}k$ are functorial in the algebra A. The definitions extend from k-algebras to small k-categories $\mathscr{A}$. For example, the Hochschild complex then becomes the complex

$$\bigoplus \mathscr{A}\left(X_0, X_0\right) \longleftarrow \bigoplus \mathscr{A}\left(X_1, X_0\right) \otimes \mathscr{A}\left(X_0, X_1\right) \longleftarrow \dots$$

whose pth term ($p \geq 0$) is the sum

$$\bigoplus \mathscr{A}\left(X_p, X_0\right) \otimes \mathscr{A}\left(X_{p-1}, X_p\right) \otimes \cdots \otimes \mathscr{A}\left(X_0, X_1\right)$$

taken over all sequences of objects $X_0, X_1, \dots, X_p$ of $\mathscr{A}$ and whose horizontal differential is given by formula (1). One then shows that the inclusion $A \to \operatorname{proj}(A)$ of the one-object category given by A into the category $\operatorname{proj}(A)$ of finitely generated projective right A-modules induces a quasi-isomorphism

$$HH(A) \xrightarrow{\sim} HH(\operatorname{proj} A).$$

In particular, this yields *Morita invariance* of Hochschild homology. The definitions further extend to small differential graded (=dg) categories $\mathscr{A}$, for example the dg category $\mathscr{C}^b_{dg}(\operatorname{proj} A)$ of bounded complexes over $\operatorname{proj}(A)$. We refer the reader to [10] for more information on this example and dg categories in general. The inclusion $\operatorname{proj}(A) \to \mathscr{C}^b_{dg}(\operatorname{proj} A)$ yields an isomorphism

$$HH(\operatorname{proj} A) \xrightarrow{\sim} HH\left(\mathscr{C}^b_{dg}(\operatorname{proj} A)\right)$$

and this yields the invariance of Hochschild homology under *derived equivalences*. We will need the following localization theorem.

Theorem 1 ([9]). *Let*

$$\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$$

be a sequence of dg categories such that the induced sequence of derived categories

$$0 \longrightarrow \mathcal{D}\mathscr{A} \xrightarrow{F^*} \mathcal{D}\mathscr{B} \xrightarrow{G^*} \mathcal{D}\mathscr{C} \longrightarrow 0$$

is exact. Then there is a canonical triangle

$$HH(\mathscr{A}) \xrightarrow{HH(F)} HH(\mathscr{B}) \xrightarrow{HH(G)} HH(\mathscr{C}) \longrightarrow \Sigma HH(\mathscr{A})$$

in $\mathcal{D}k$ *and hence long exact sequences in Hochschild (and cyclic) homology.*

Let Q be a finite quiver and I an admissible ideal in kQ, i.e. a two-sided ideal contained in the square of the ideal generated by the arrows and such that the quotient kQ/I is finite-dimensional. Let R be the quotient of A by its radical. Thus, as an A-module, the algebra R is the direct sum of the simple A-modules. Following [8], we define the Koszul dual of A to be the dg algebra

$$A^! = \operatorname{RHom}_A(R, R).$$

Thus, if P is a projective resolution of the A-module R, then the Koszul dual is quasi-isomorphic to the dg endomorphism algebra $\mathrm{Hom}_A(P,P)$ of P. The following theorem is a special case of Corollary D.2 of Van den Bergh's [2]. We write D for the dual $\mathrm{Hom}_k(?,k)$ over the ground field.

Theorem 2 (Van den Bergh). *We have a canonical isomorphism*

$$HH\left(A^!\right) \xrightarrow{\sim} DHH(A).$$

We refer to [7] for a comparison taking into account much more structure.

2. Hochschild homology of derived categories and singularity categories

Let Q be a finite quiver and I an admissible ideal in kQ. Let $\mathrm{mod}\,A$ be the category of k-finite-dimensional right A-modules. Denote by $\mathcal{D}^b(A) = \mathcal{D}^b(\mathrm{mod}\,A)$ the bounded derived category of A and by $\mathrm{per}(A)$ the perfect derived category, i.e. the thick subcategory generated by the free A-module of rank 1. Following Buchweitz [3] and Orlov [11], one defines the *singularity category of* A as the Verdier quotient

$$\mathrm{sg}(A) = \mathcal{D}^b(A)/\mathrm{per}(A).$$

Using the canonical dg enhancements of $\mathcal{D}^b(A)$ and $\mathrm{per}(A)$, cf. [10], we obtain a canonical exact sequence of dg categories

$$0 \longrightarrow \mathrm{per}_{dg}(A) \longrightarrow \mathcal{D}^b_{dg}(A) \longrightarrow \mathrm{sg}_{dg}(A) \longrightarrow 0\ ,$$

where the dg quotient $\mathrm{sg}_{dg}(A)$ yields a canonical dg enhancement for $\mathrm{sg}(A)$. It is not hard to see that, in the homotopy category of dg categories, it is functorial with respect to bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$ such that X_B is perfect over B and ${}_AX$ is perfect over A. From the localization Theorem 1, we deduce a triangle

$$HH\Big(\mathrm{per}_{dg}(A)\Big) \longrightarrow HH\Big(\mathcal{D}^b_{dg}(A)\Big) \longrightarrow HH\Big(\mathrm{sg}_{dg}(A)\Big) \longrightarrow \Sigma HH\Big(\mathrm{per}_{dg}(A)\Big) \tag{2}$$

in the derived category of vector spaces.

Theorem 3. *We have a canonical isomorphism* $HH(\mathcal{D}^b_{dg}(A)) \xrightarrow{\sim} DHH(A)$.

Proof. Recall that we have defined R to be the quotient of A by its radical and the Koszul dual $A^!$ as $\mathrm{RHom}_A(R,R)$. Since the module R is a classical generator of the bounded derived category $\mathcal{D}^b(A)$, we deduce from the results of [8] that we have a triangle equivalence

$$\mathrm{RHom}_A(R,?) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathrm{per}\left(A^!\right).$$

This lifts to a quasi-equivalence

$$\mathcal{D}^b_{dg}(A) \xrightarrow{\sim} \mathrm{per}_{dg}\left(A^!\right).$$

By Morita invariance of Hochschild homology, we have

$$HH\left(A^!\right) \xrightarrow{\sim} HH\Big(\mathrm{per}_{dg}\left(A^!\right)\Big).$$

By Van den Bergh's Theorem 2, we have

$$HH\left(A^!\right) \xrightarrow{\sim} DHH(A).$$

The claim follows if we combine these isomorphisms. □

Define a linear map $\tau : A \to DA$ by sending an element $a \in A$ to the linear form which takes $b \in A$ to the trace of the linear map

$$\lambda_a \rho_b : A \to A,\ x \mapsto axb,$$

where λ_a is left multiplication by a and ρ_b right multiplication by b. Notice that since A is finite-dimensional, this is well-defined. Moreover, the value of $\langle a,b\rangle = (\tau(a))(b)$ only depends on the classes of a and b in $HH_0(A)$, which is canonically isomorphic to R. It is not hard to check that in the basis formed by the e_i, the matrix of the induced bilinear form

$$HH_0(A) \times HH_0(A) \to k$$

is the Cartan matrix of A, whose (i,j)-entry is the dimension of e_iAe_j. Define the *double Hochschild complex of A* to be the complex

$$\dots \xrightarrow{b} A\otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A\otimes A) \xrightarrow{Db} \dots ,$$

where DA sits in degree 0, the differentials b are those of the Hochschild complex and the Db their duals.

Let us abbreviate $\mathcal{S} = \mathrm{sg}_{dg}(A)$.

Theorem 4. *In $\mathcal{D}k$, we have a canonical isomorphism between $HH(\mathcal{S})$ and the double Hochschild complex of A.*

Notice that this implies in particular that $HH_n(\mathcal{S})$ is finite-dimensional for all n. This is surprising since the singularity category sg(A) is usually not Hom-finite (except if A is Gorenstein), cf. for example [4].

Proof. We use the triangle

$$HH\left(\mathrm{per}_{dg}(A)\right) \longrightarrow HH\left(\mathcal{D}^b_{dg}(A)\right) \longrightarrow HH(\mathcal{S}) \longrightarrow \Sigma HH\left(\mathrm{per}_{dg}(A)\right)$$

obtained from the localization Theorem 1. We have already seen that it is isomorphic to a triangle

$$HH(A) \to HH\left(A^!\right) \to HH(\mathcal{S}) \to \Sigma HH(A),$$

where the first morphism is induced by the inclusion $\mathrm{per}_{dg}(A) \to \mathcal{D}^b_{dg}(A)$. Thus, the complex $HH(\mathcal{S})$ identifies with the mapping cone over the morphism $HH(A) \to HH(A^!)$. Let us determine this morphism explicitly. Recall that the functor HH, considered as a functor on the homotopy category of small dg categories with values in the derived category $\mathcal{D}k$, commutes with tensor products. We have the following commutative square

$$\begin{array}{ccc} \mathrm{per}_{dg}(A^{op})\otimes \mathrm{per}_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \\ \downarrow & & \| \\ \mathrm{per}_{dg}(A)^{op}\otimes \mathcal{D}^b_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \end{array}$$

Here, a pair (P_1,P_2), $P_1 \in \mathrm{proj}(A^{op})$, $P_2 \in \mathrm{proj}(A)$ is taken to $P_2\otimes_A P_1$ by the top arrow and to $(\mathrm{Hom}_A(P_1,A),P_2)$ by the left vertical arrow. It follows from Appendix D in [2] that the lower horizontal arrow induces a non degenerate pairing

$$HH(A)\otimes HH\left(\mathcal{D}^b_{dg}(A)\right) \to HH(k) = k.$$

A direct computation now shows that the morphism

$$HH(A) \to DHH(A)$$

is the composition

$$HH(A) \to HH_0(A) \to DHH_0(A) \to DHH(A)$$

where the middle morphism is induced by the map τ. □

Corollary 5. *For $n \geq 2$, we have canonical isomorphisms*

$$HH_n(\mathscr{S}) \xrightarrow{\sim} HH_{n-1}(A) \xrightarrow{\sim} DHH_{1-n}(\mathscr{S}).$$

Moreover, we have

$$HH_1(\mathscr{S}) \xrightarrow{\sim} \ker\Big(HH_0(A) \xrightarrow{\tau} DHH_0(A)\Big) \xrightarrow{\sim} DHH_0(\mathscr{S}).$$

3. Application: Hochschild homology of dg Leavitt path algebras

Let Q be a finite quiver, for example a quiver with one vertex and a unique loop α. Let A be the associated radical square zero algebra, i.e. the quotient of kQ by the square of the ideal generated by the arrows. So for the one-loop quiver, we have $A = k[\varepsilon]/(\varepsilon^2)$. Let Q^* be the graded quiver obtained from the opposite quiver of Q by assigning each arrow $\alpha^* : j \to i$ corresponding to an arrow $\alpha : i \to j$ of Q the degree $+1$. For each vertex i of Q, consider the arrows $\alpha_s^* : i \to t(\alpha_s^*)$, $1 \leq s \leq t_i$, starting in Q^* at i. Let

$$\varphi_i : P_i \to \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)}$$

be the morphism with components α_s^*, where $P_i = e_i kQ^*$. For example, for the one-loop quiver, we just have $\varphi(1) = \alpha^* : P_1 \to \Sigma P_1$. Note that if i is a sink of Q, then

$$\bigoplus_{s=1}^{t_i} P_{t(\alpha_s^*)} = 0.$$

For each vertex $i \in Q_0$, let

$$\varphi(i)^{-1} = \left[\beta_{i,1}, \ldots, \beta_{i,t_i}\right] : \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)} \to P_i$$

be the formal inverse of $\varphi(i)$. The *graded Leavitt path algebra of* Q is obtained from kQ^* by adjoining all coefficients β_{ij} of all formal inverses $\varphi(i)^{-1}$, $i \in Q_0$. We endow L_Q with the grading inherited from Q^* and with $d = 0$.

Theorem 6 (Smith [12], Chen–Yang [6]). *We have a triangle equivalence* $\mathrm{per}(L_Q) \xrightarrow{\sim} \mathrm{sg}(A)$ *taking* $e_i L_Q$ *to the simple* S_i*.*

Corollary 7. *The Hochschild homology $HH_*(L_Q)$ of the Leavitt path algebra is computed by the double Hochschild complex*

$$\cdots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \cdots,$$

(with DA in degree 0*). In particular, we have*

$$\dim HH_p\left(L_Q\right) = 0 < \infty$$

for all $p \in \mathbb{Z}$.

A different description of the Hochschild homology of Leavitt path algebras is due to Ara–Cortiñas [1].

4. Beyond radical square zero

Let Q be a finite quiver and $A = kQ/I$ the quotient of its path algebra by an admissible ideal. Let J be the radical of A and $R = kQ_0$ so that we have $A = R \oplus J$ as R-bimodules. Let $A_0 = (T_R J)/(J \otimes_R J)$ be the radical square zero algebra associated with A. Thus, we have $A_0 = R \oplus J = A$ as R-bimodules but we have $xy = 0$ in A_0 for any two elements of J. We view A_0 as a degeneration of A and A as a deformation of A_0. As pointed out by Chen–Wang [5], this suggests that the singularity category $\mathrm{sg}(A)$ is a deformation of the singularity category $\mathrm{sg}(A_0)$, which is equivalent to the perfect derived category $\mathrm{per}(L_{A_0})$ of the graded Leavitt path algebra L_{A_0}. Hence we can hope for the existence of a dg algebra L_A obtained from L_{A_0} by deformation such that $\mathrm{per}(L_A)$ is equivalent to $\mathrm{sg}(A)$. We sum up the situation in the following diagram

$$A_0 \xrightarrow[\text{deformation}]{} A$$

$$\begin{array}{ccc} \mathrm{sg}(A_0) & \xrightarrow[\text{deformation}]{} & \mathrm{sg}(A) \\ \big\wr & & \big\wr \quad ? \\ \mathrm{per}(L_{A_0}) & \xrightarrow[\text{deformation?}]{} & \mathrm{per}(L_A) \end{array}$$

$$L_{A_0} \xrightarrow[\text{deformation?}]{} L_A \qquad ?$$

The following theorem confirms this hope.

Theorem 8 (Chen–Wang [5]). *The graded algebra L_{A_0} admits a canonical differential d_A such that for $L_A = (L_{A_0}, d_A)$, we have a triangle equivalence*

$$\mathrm{per}(L_A) \xrightarrow{\sim} \mathrm{sg}(A).$$

Corollary 9. *The Hochschild homology of the dg Leavitt path algebra L_A is computed by the double Hochschild complex of A.*

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