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Stanley–Reisner rings and the occurrence of the Steinberg representation in the hit problem

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Abstract. A result of G. Walker and R. Wood states that the space of indecomposable elements in degree $2^n - 1 - n$ of the polynomial algebra $\mathbb{F}_2[x_1, \dots, x_n]$, considered as a module over the mod 2 Steenrod algebra, is isomorphic to the Steinberg representation of $\mathrm{GL}_n(\mathbb{F}_2)$. We generalize this result to all finite fields by studying a family of finite quotient rings $\mathbf{R}_{n,k}$, $k \in \mathbb{N}^*$, of $\mathbb{F}_q[x_1, \dots, x_n]$, where each $\mathbf{R}_{n,k}$ is defined as a quotient of the Stanley–Reisner ring of a matroid complex. By considering a variant of $\mathbf{R}_{n,k}$, we also show that the space of indecomposable elements of $\mathbb{F}_q[x_1, \dots, x_n]$ in degree $q^{n-1} - n$ has dimension equal to that of a complex cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, that is $(q-1)(q^2-1)\cdots(q^{n-1}-1)$.

Over the field $\mathbb{F}_2$, we also establish a decomposition of the Steinberg summand of $\mathbf{R}_{n,2}$ into a direct sum of suspensions of Brown–Gitler modules. The module $\mathbf{R}_{n,2}$ can be realized as the mod 2 cohomology of a topological space and the result suggests that the Steinberg summand of this space admits a stable decomposition into a wedge of suspensions of Brown–Gitler spectra.

Résumé. Un résultat de G. Walker et R. Wood dit que l'espace des indécomposables en degré $2^n - 1 - n$ de l'algèbre polynômiale $\mathbb{F}_2[x_1, \dots, x_n]$, considérée comme module sur l'algèbre de Steenrod modulo 2, est isomorphe à la représentation de Steinberg de $\mathrm{GL}_n(\mathbb{F}_2)$. Dans ce travail, on cherche à généraliser ce résultat à tous les corps finis. Pour ce faire, on étudie une famille d'anneaux quotients finis $\mathbf{R}_{n,k}$, $k \in \mathbb{N}^*$, de $\mathbb{F}_q[x_1, \dots, x_n]$, où chaque $\mathbf{R}_{n,k}$ est défini comme quotient de l'anneau de Stanley–Reisner d'un complexe de matroïde. On montre aussi en utilisant un variant de $\mathbf{R}_{n,k}$ que la dimension de l'espace des indécomposables de $\mathbb{F}_q[x_1, \dots, x_n]$ en degré $q^{n-1} - n$ est égale à celle d'une représentation cuspidale complexe de $\mathrm{GL}_n(\mathbb{F}_q)$, à savoir $(q-1)(q^2-1)\cdots(q^{n-1}-1)$.

Sur le corps $\mathbb{F}_2$, on établit une décomposition du facteur de Steinberg de $\mathbf{R}_{n,2}$ en somme directe de suspensions de modules de Brown–Gitler. Ceci suggère une décomposition du facteur stable de Steinberg de la réalisation topologique de $\mathbf{R}_{n,2}$ en bouquet de suspensions de spectres de Brown–Gitler.

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1. Introduction

Let $\mathbb{F}_q$ denote a finite field of q elements where q is a power of a fixed prime p. Let V be an n-dimensional vector space over $\mathbb{F}_q$. The symmetric power algebra $\mathbf{S}(V^*) = \bigoplus_{k\geq 0}\mathbf{S}^k(V^*)$ on the dual V^* of V is identified with the polynomial algebra $\mathbb{F}_q[x_1,\dots,x_n]$ where $\{x_1,\dots,x_n\}$ is a basis of V^*. We assign to each variable x_i degree one, so $\mathbf{S}^k(V^*)$ has a basis consisting of all monomials of degree k.

The algebra $\mathscr{P}$ of Steenrod reduced powers over the finite field $\mathbb{F}_q$, defined as in [18], acts on $\mathbf{S}(V^*)$ as follows:

$$\mathscr{P}^i\left(x^m\right) = \binom{m}{i} x^{m+(q-1)i}, \quad x \in \mathbf{S}^1\left(V^*\right),$$
$$\mathscr{P}^i\left(f_1 f_2\right) = \sum_{i_1+i_2=i} \mathscr{P}^{i_1}\left(f_1\right)\mathscr{P}^{i_2}\left(f_2\right), \quad f_1, f_2 \in \mathbf{S}\left(V^*\right).$$

This action commutes with the *right* action of $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{F}_q)$ by linear substitutions of the variables:

$$(f\cdot g)\left(x_1,\dots,x_n\right) = f\left(x_1\cdot g,\dots,x_n\cdot g\right),$$

where $f \in \mathbf{S}(V^*)$, $g = (g_{i,j}) \in \mathrm{GL}_n$, and $x_i\cdot g = g_{i,1}x_1+\cdots+g_{i,n}x_n$ for $1\leq i\leq n$.

The *Peterson hit problem* in algebraic topology, focused mainly on the finite field $\mathbb{F}_2$, asks for a determination of a minimal generating set of the polynomial algebra $\mathbf{S}(V^*)$ as a module over the algebra $\mathscr{P}$. This is the same as the problem of determining the graded vector space $\mathrm{Q}(\mathbf{S}(V^*))$. Here given M a $\mathscr{P}$-module, $\mathrm{Q}(M)$ stands for the quotient $M/\mathscr{P}^+M$, $\mathscr{P}^+$ denoting the augmentation ideal of $\mathscr{P}$. Since the actions of $\mathscr{P}$ and GL_n commute, one is also interested in studying $\mathrm{Q}(\mathbf{S}(V^*))$ as a graded module over the group ring $\mathbb{F}_q[\mathrm{GL}_n]$. Though much work has been done for the hit problem, the general answer seems to be out of reach with the present techniques. The reader is referred to the recent volumes by G. Walker and R. M. W. Wood [24, 25] for a thorough exposition of this problem.

The starting point of this work is the following result, which is also due to Walker and Wood [23]:

Theorem 1 (Walker–Wood). *For $q=2$, $\mathrm{Q}^{2^n-1-n}(\mathbf{S}(V^*))$ is isomorphic to the Steinberg representation of $\mathrm{GL}_n(\mathbb{F}_2)$.*

Walker and Wood proved this by establishing a link between the hit problem over $\mathbb{F}_2$ and Young tableaux. They proved that in suitable generic degrees δ, the semistandard tableaux can be used to index a generating set for $\mathrm{Q}^\delta(\mathbf{S}(V^*))$. When $\delta = 2^n-1-n$, the hook formula gives the upper bound $\dim_{\mathbb{F}_2}\mathrm{Q}^\delta(\mathbf{S}(V^*)) \leq 2^{\frac{n(n-1)}{2}}$ and the equality then follows from the first occurrence of the Steinberg module in this degree [7, 13].

The purpose of this paper is to present a new approach to the above result which is valid for all finite fields. The point we discover here is that there is a finite quotient $\mathbf{R}(V^*)$ of $\mathbf{S}(V^*)$ such that its top-degree submodule, $\mathbf{R}^{(q^n-1)/(q-1)-n}(V^*)$, is $\mathscr{P}$-indecomposable and isomorphic to the Steinberg module, and that the natural projection $\mathrm{Q}^i(\mathbf{S}(V^*)) \twoheadrightarrow \mathrm{Q}^i(\mathbf{R}(V^*))$ is an isomorphism in the range $0\leq i\leq (q^n-1)/(q-1)-n$. Apart from classical tools in studying the hit problem, a novel one which we are going to employ in this work is the Stanley–Reisner ring of a simplicial complex [20]. We note that our method also leads to a formula for the dimension of $\mathrm{Q}(\mathbf{S}(V^*))$ in degree $q^{n-1}-n$. This dimension is equal to that of a complex cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$.

We now describe the main results of the paper. For this let us choose a non-zero vector u_ℓ on each line ℓ in V^*. Given a subspace W of V^*, denote by $\mathscr{L}_W$ the set of all lines in W. Put $\mathbf{L}_W =$

$\prod_{\ell\in\mathscr{L}_W} u_\ell$, the "product of all lines" in W. If W' is a subspace of W, put $\mathbf{V}_{W',W} = \prod_{\ell\in\mathscr{L}_W\setminus\mathscr{L}_{W'}} u_\ell$. If $\{x_1,\ldots,x_n\}$ is a basis of V^* then it is well-known that $\mathbf{L}_{V^*}$ is (up to non-zero scalar) the polynomial

$$\mathbf{L}_n = \mathbf{L}(x_1,\ldots,x_n) := \det\left(x_j^{q^{i-1}}\right)_{1\le i,j\le n}$$

introduced by Dickson [5]. Similarly, if H is the hyperplane generated by $x_1,\ldots,x_{n-1}$ then $\mathbf{V}_{H,V^*}$ is (up to non-zero scalar) the polynomial

$$\mathbf{V}_n = \mathbf{V}(x_1,\ldots,x_n) := \prod_{\lambda_i\in\mathbb{F}_q}(\lambda_1x_1+\cdots+\lambda_{n-1}x_{n-1}+x_n)$$

introduced by Mùi [14].

For $k\ge 1$, let $I(V^*,k)$ (or $I_{n,k}$) denote the ideal of $\mathbf{S}(V^*)$ generated by the polynomials $(\mathbf{V}_{H,V^*})^k$, where H runs over the set of all hyperplanes of V^*. Since GL_n acts transitively on the set of hyperplanes of V^*, it is equivalent to say that $I(V^*,k)$ is generated by the orbit of $\mathbf{V}_n^k$ under the action of GL_n. Let $\mathbf{R}(V^*,k)$ (or $\mathbf{R}_{n,k}$) denote the quotient $\mathbf{S}(V^*)/I(V^*,k)$. It is clear that the natural projection $\mathbf{S}(V^*)\twoheadrightarrow\mathbf{R}(V^*,k)$ is GL_n-linear. Since each generator of the ideal $I(V^*,k)$ is a product of elements in $\mathbf{S}^1(V^*)$, it is easy to check that $I(V^*,k)$ is stable under the action of the algebra $\mathscr{P}$, and thus the projection $\mathbf{S}(V^*)\twoheadrightarrow\mathbf{R}(V^*,k)$ is also $\mathscr{P}$-linear.

The Steinberg representation St_n [21] is a projective absolutely irreducible representation of GL_n of dimension $q^{\frac{n(n-1)}{2}}$. It is isomorphic to the right $\mathbb{F}_q[\mathrm{GL}_n]$-module $\mathrm{e}_n\cdot\mathbb{F}_q[\mathrm{GL}_n]$ where e_n is the Steinberg idempotent defined by

$$\mathrm{e}_n = \frac{1}{[\mathrm{GL}_n:U_n]}\sum_{b\in B_n,\sigma\in\Sigma_n}\mathrm{sgn}(\sigma)b\sigma,$$

B_n, U_n, Σ_n denoting respectively the subgroup of upper triangular matrices, the subgroup of upper triangular matrices with 1's on the diagonal, and the symmetric group of permutation matrices. Let $\det^i$, $i\in\mathbb{Z}$, denote the i^{th} power of the determinant representation of GL_n. Then we have $\det^i\cong\det^j$ if $i\equiv j\bmod(q-1)$ and $\det^0,\ldots,\det^{q-2}$ form a complete set of distinct one-dimensional representations of GL_n over $\mathbb{F}_q$. A twisted Steinberg module is defined to be a tensor product $\mathrm{St}_n\otimes\det^i$ and this is again a projective absolutely irreducible representation of GL_n of dimension $q^{\frac{n(n-1)}{2}}$. The idempotent corresponding to $\mathrm{St}_n\otimes\det^i$ is denoted by $\mathrm{e}_n^{(i)}$ and is given by $\mathrm{e}_n^{(i)}=\phi_i(\mathrm{e}_n)$ where ϕ_i is the automorphism of $\mathbb{F}_q[\mathrm{GL}_n]$ given by $\phi_i(g)=\det^i(g^{-1})g$ (see [12, § 2]).

Theorem 2. *For $k\ge 1$ and $n=\dim V$, set $d:=\frac{k(q^n-1)}{q-1}-n$.*

(1) *$\mathbf{R}^d(V^*,k)$ is isomorphic to the twisted Steinberg module $\mathrm{St}_n\otimes\det^{k-1}$ and $\mathbf{R}^i(V^*,k)$ vanishes if $i>d$.*
(2) *If $k=q^sr$ with $s\ge 0$ and $1\le r\le q-1$, the following hold:*
   (a) *For each $0\le i\le d$, the natural projection $\mathbf{S}(V^*)\twoheadrightarrow\mathbf{R}(V^*,k)$ induces an isomorphism of GL_n-modules $\mathrm{Q}^i(\mathbf{S}(V^*))\cong\mathrm{Q}^i(\mathbf{R}(V^*,k))$.*
   (b) *The natural projection $\mathbf{R}^d(V^*,k)\twoheadrightarrow\mathrm{Q}^d(\mathbf{R}(V^*,k))$ is an isomorphism of GL_n-modules.*

As a consequence, we obtain the following

Corollary 3. *If $k=q^sr$ with $s\ge 0$ and $1\le r\le q-1$, there is an isomorphism of GL_n-modules:*

$$\mathrm{Q}^d\left(\mathbf{S}\left(V^*\right)\right)\cong\mathrm{St}_n\otimes\det{}^{k-1},$$

where $d=\frac{k(q^n-1)}{q-1}-n$.

When $q=2$ and $k=1$, this gives Theorem 1 of Walker and Wood mentioned above.

The corollary is deduced from the following sequence of isomorphisms of GL_n-modules:

$$\mathrm{St}_n\otimes\det{}^{k-1}\cong\mathbf{R}^d\left(V^*,k\right)\xrightarrow{\cong}\mathrm{Q}^d\left(\mathbf{R}\left(V^*,k\right)\right)\xleftarrow{\cong}\mathrm{Q}^d\left(\mathbf{S}\left(V^*\right)\right),$$

where the first isomorphism follows from Theorem 2 (1), the second from (2b) and the third from (2a).

The strategy for the proof of Theorem 2 (1) is as follows. We consider a simplicial complex $\Delta(V^*,k)$ whose Alexander dual is homotopy equivalent to the Tits building of GL_n. The complex $\Delta(V^*,k)$ is a matroid complex and so the Stanley–Reisner ring $\mathbb{F}_q[\Delta(V^*,k)]$ is Cohen–Macaulay. The quotient algebra $\mathbf{R}(V^*,k)$ defined above is then shown to be isomorphic to the quotient of $\mathbb{F}_q[\Delta(V^*,k)]$ by the ideal generated by a well-chosen regular sequence of $\mathbb{F}_q[\Delta(V^*,k)]$. The linear structure of $\mathbf{R}(V^*,k)$, in particular of its top-degree submodule, will follow from some classical results in the theory of Stanley–Reisner rings [20]. In order to prove Theorem 2 (2a), we need to show that all elements of degree $\leq d$ in the ideal $I(V^*,k)$ are *hit* (i.e. $\mathscr{P}$-decomposable) in $\mathbf{S}(V^*)$. For example, when $k=1$, this is proved by using the χ-trick [26] and the following identity (Lemma 13):

$$\mathbf{V}_n = (-1)^{n-1}\sum_{i=0}^{n-1}\chi\left(\mathscr{P}^{\frac{q^{n-1}-1}{q-1}-i}\right)(e_i x_n),$$

where $\chi:\mathscr{P}\to\mathscr{P}$ is the canonical anti-automorphism, and e_i is the i^{th} elementary symmetric function of $x_1^{q-1},\ldots,x_{n-1}^{q-1}$. Theorem 2 (2b) will be proved by showing that $\mathbf{R}(V^*,k)$ can be embedded in a direct product of copies of the quotient

$$\mathbb{F}_q[x_1,\ldots,x_n]\Big/\left(x_1^{q^{n-1}k},\ldots,x_{n-1}^{qk},x_n^k\right)$$

(which is example of a Poincaré duality algebra with trivial Wu classes [10]).

Our method can also be used to obtain the following:

Theorem 4. *For $n=\dim V\geq 2$, $\dim_{\mathbb{F}_q}\mathrm{Q}^{q^{n-1}-n}(\mathbf{S}(V^*)) = (q-1)(q^2-1)\cdots(q^{n-1}-1)$.*

This will be proved by considering a simplicial complex whose Alexander dual is homotopy equivalent to a complex employed by G. Lusztig [9] in his work on constructing discrete series (a.k.a. cuspidal representations) of GL_n.

To state the final result of the paper, suppose $q=k=2$ and denote by $\mathscr{A}$ the mod 2 Steenrod algebra. Using the work of Masateru Inoue [6] on the hit problem of the Steinberg summand, we show that the Steinberg summand of $\mathbf{R}(V^*,2)$ is decomposed into a direct sum of suspensions of Brown–Gitler modules [3].

Theorem 5. *Put $d=2(2^n-1)-n=\sum_{j=1}^n(2^j-1)$. There is an isomorphism of $\mathscr{A}$-modules*

$$\bigoplus_{j=0}^{n}\Sigma^{d-(2^j-1)}\mathrm{B}\left(2^j-1\right)\xrightarrow{\cong}\mathbf{R}_{n,2}\cdot \mathrm{e}_n.$$

Here $\mathrm{B}(k)$ denotes the Brown-Gitler module $\mathrm{B}(k):=\mathscr{A}/\mathscr{A}\langle\chi(\mathrm{Sq}^i)\mid 2i>k\rangle$ and $\mathbf{R}_{n,2}\cdot\mathrm{e}_n$ denotes the direct summand of $\mathbf{R}_{n,2}$ associated to the Steinberg idempotent e_n.

We end this introduction by noting that, using the work [4] of M. W. Davis and T. Januszkiewicz, it is possible to interpret $\mathbf{R}_{n,k}$ as the mod 2 cohomology of the orbit space $\rho(n,k)_{\mathrm{reg}}/V$, where $\rho(n,k)$ denotes the direct sum of k copies of the reduced regular representation of V and $\rho(n,k)_{\mathrm{reg}}$ denotes the regular part of the action of V on $\rho(n,k)$. The decomposition in Theorem 5 strongly suggests that there may exist a homotopy equivalence (of 2-completed spectra)

$$\mathrm{e}_n\cdot\Sigma^\infty\left(\rho(n,2)_{\mathrm{reg}}/V\right)_+\simeq\bigvee_{j=0}^{n}\Sigma^{d-(2^j-1)}\mathbf{B}\left(2^j-1\right),$$

$\mathbf{B}(k)$ denoting the Brown–Gitler spectrum whose mod 2 cohomology is $\mathrm{B}(k)$. We intend to go back to this topological problem in a future work.

The remaining of the paper is organized as follows. In section 2, we review some facts about the Stanley–Reisner ring of a simplicial complex. In section 3, we define the module $\mathbf{R}(V^*,k)$ and prove the first part of Theorem 2. In section 4 we study the $\mathscr{P}$-module structure of $\mathbf{R}(V^*, k)$ and prove the second part of Theorem 2. In section 5, we prove Theorem 4 and finally in section 6, we prove Theorem 5.

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2. The Stanley–Reisner ring of a simplicial complex

In this section, we review some facts about the Stanley–Reisner ring of a simplicial complex which will be used in proving Theorem 2 (1). Our main reference is [20].

Let S be a non-empty finite set. A *simplicial complex* Δ on the vertex set S is a collection of subsets of S that is closed under inclusion. A subset F of S is called a *face* of Δ if $F \in \Delta$ and is called a *non-face* of Δ if $F \notin \Delta$. The *dimension* of a face F is $\dim F := |F| - 1$ and the *dimension* of Δ is $\dim \Delta := \max_{F\in\Delta} \dim F$. A maximal face under inclusion is called a *facet* and Δ is said to be *pure* if all facets have the same dimension.

Let $\mathbb{K}$ be a field and let $\mathbb{K}[X_s \mid s \in S]$ denote the polynomial algebra over $\mathbb{K}$ in the variables X_s indexed by the vertex set S. Given a subset F of S, let X_F denote the monomial $\prod_{s\in F} X_s$. The *face ring* (or the *Stanley–Reisner ring*) $\mathbb{K}[\Delta]$ of Δ is then defined to be the quotient ring

$$\mathbb{K}[\Delta] := \mathbb{K}[X_s | s \in S] / I_\Delta,$$

where I_Δ is the *Stanley–Reisner ideal* defined by $I_\Delta := (X_F | F \notin \Delta)$. It is clear that I_Δ can be defined by using only monomials corresponding to minimal non-faces of Δ.

Next recall that the Poincaré series of an $\mathbb{N}$-graded $\mathbb{K}$-vector space M (with $\dim_{\mathbb{K}} M^i$ finite for all i) is defined by $\mathbf{P}(M, t) := \sum_{i\geq 0}(\dim_{\mathbb{K}} M^i)t^i$. Suppose that $\dim \Delta = d - 1$ and regard the variables X_s as being of degree one. The face ring $\mathbb{K}[\Delta]$ is then a graded ring and its Poincaré series is given by:

$$\mathbf{P}(\mathbb{K}[\Delta], t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}, \tag{1}$$

where $f_{-1} = 1$ and $f_i := f_i(\Delta)$, $i \geq 0$, is the number of i-dimensional faces of Δ [20, Theorem 1.4]. The sequence $(f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Writing $\mathbf{P}(\mathbb{K}[\Delta], t)$ in the form

$$\mathbf{P}(\mathbb{K}[\Delta], t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \tag{2}$$

then the sequence $(h_0, \ldots, h_d)$ is called the *h-vector* of Δ. In terms of the f-vector, each $h_k := h_k(\Delta)$ is given by $h_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{k-i} f_{i-1}$. In particular, $h_d = \sum_{i=0}^{d}(-1)^{d-i} f_{i-1}$ and so

$$h_d(\Delta) = (-1)^{d-1}\widetilde{\chi}(\Delta), \tag{3}$$

where $\widetilde{\chi}(\Delta) = \sum_{i\geq -1}(-1)^i f_i = \sum_{i\geq -1}(-1)^i \dim_{\mathbb{K}} \widetilde{\mathrm{H}}_i(\Delta;\mathbb{K})$ is the reduced Euler characteristic of Δ. Here $\widetilde{\mathrm{H}}_i(\Delta;\mathbb{K})$ denotes the reduced homology of Δ with coefficients in $\mathbb{K}$ [20, Definition 3.1]. We will also need to express h_d in terms of the reduced characteristic of the Alexander dual of Δ. Recall that the *Alexander dual* Δ^* of Δ is the simplicial complex with the same vertex set S such that a subset F of S is a face of Δ^* if the complement $S \setminus F$ is not a face of Δ. Since for each $0 \leq i \leq |S|$, the number of subsets of cardinality i of S is equal to $f_{i-1}(\Delta) + f_{|S|-i-1}(\Delta^*)$ and the alternating sum of these numbers is zero, it follows that $\widetilde{\chi}(\Delta^*) + (-1)^{|S|}\widetilde{\chi}(\Delta^*) = 0$, and so

$$h_d(\Delta) = (-1)^{|S|-d}\widetilde{\chi}\left(\Delta^*\right). \tag{4}$$

We review now some commutative algebra. For this let R be an $\mathbb{N}$-graded finitely-generated commutative algebra over $\mathbb{K}$. The *Krull dimension* of R is the maximal number of algebraically independent homogeneous elements of R. Suppose that the Krull dimension of R is d. A sequence $(\theta_1,\dots,\theta_d)$ of homogeneous elements of R is called a *homogeneous system of parameters* (*h.s.o.p.*) if $R/(\theta_1,\dots,\theta_d)$ is a finite-dimensional $\mathbb{K}$-vector space. Equivalently, $(\theta_1,\dots,\theta_d)$ is a h.s.o.p. if $\theta_1,\dots,\theta_d$ are algebraically independent and R is a finitely-generated $\mathbb{K}[\theta_1,\dots,\theta_d]$-module. A h.s.o.p. $(\theta_1,\dots,\theta_d)$ is called a *linear system of parameters* (l.s.o.p.) if each θ_i is of degree one. A h.s.o.p $(\theta_1,\dots,\theta_d)$ of R is called a *regular sequence* if θ_{i+1} is not a zero-divisor in $R/(\theta_1,\dots,\theta_i)$ for $0\le i<d$. Equivalently, $(\theta_1,\dots,\theta_d)$ is a regular sequence if $\theta_1,\dots,\theta_d$ are algebraically independent and R is a finite-dimensional free $\mathbb{K}[\theta_1,\dots,\theta_d]$-module. The $\mathbb{K}$-algebra R is *Cohen–Macaulay* if it admits a regular sequence. It is known that if R is Cohen–Macaulay then any h.s.o.p. is a regular sequence.

The Krull dimension of the face ring $\mathbb{K}[\Delta]$ is $1+\dim\Delta$ [20, Theorem 1.3]. If the face ring $\mathbb{K}[\Delta]$ is Cohen–Macaulay, then we say that Δ is a *Cohen–Macaulay complex* (over $\mathbb{K}$). A fundamental result of G. A. Reisner [17] states that Δ is Cohen–Macaulay over $\mathbb{K}$ if and only if for all $F\in\Delta$ and all $i<\dim(\mathrm{link}F)$, we have $\widetilde{\mathrm{H}}^i(\mathrm{link}F;\mathbb{K})=0$. Here link$F$ is the *link* of F defined by $\mathrm{link}F=\{G\in\Delta|G\cup F\in\Delta,\ G\cap F=\emptyset\}$.

We note that the existence of a l.s.o.p. for $\mathbb{K}[\Delta]$ is assured by Noether's Normalization Lemma when $\mathbb{K}$ is infinite. When $\mathbb{K}$ is finite, one may need to pass to an infinite extension field; this does not affect the Cohen–Macaulay property. We need the following result to recognize a l.s.o.p. in $\mathbb{K}[\Delta]$.

Lemma 6 ([20, Lemma 2.4]).

1. *Let $\mathbb{K}[\Delta]$ be a face ring of Krull dimension d and let $\theta_1,\dots,\theta_d\in\mathbb{K}[\Delta]^1$. Then $(\theta_1,\dots,\theta_d)$ is a l.s.o.p. for $\mathbb{K}[\Delta]$ if and only if for every face F of Δ, the restrictions $\theta_{1|_F},\dots,\theta_{d|_F}$ span a vector space of dimension equal to $|F|$. Here the restriction $\theta_{|_F}$ of an element $\theta=\sum_{s\in S}\alpha_sX_s$ of $\mathbb{K}[\Delta]^1$ to a face F is defined by $\theta_{|_F}=\sum_{s\in F}\alpha_sX_s$.*
2. *If $(\theta_1,\dots,\theta_d)$ is a l.s.o.p. for $\mathbb{K}[\Delta]$, then the quotient ring $\mathbb{K}[\Delta]/(\theta_1,\dots,\theta_d)$ is spanned as a $\mathbb{K}$-vector space by the monomials X_F, $F\in\Delta$.*

Matroid complexes provide a rich source of Cohen–Macaulay complexes. Recall that a simplicial complex Δ on the vertex set S is a *matroid complex* if it satisfies the *exchange property*: given $F,G\in\Delta$ with $|F|<|G|$, there exists $v\in G\setminus F$ such that $F\cup\{v\}\in\Delta$ [15]. It is known that a matroid complex is pure and for every linear order of S, the induced lexicographical order on the facets of the matroid complex Δ make it into a shellable complex [2, Theorem 7.3.3]. Recall that a *shellable complex* is a pure simplicial complex Δ together with an order of its facets $F_1,\dots,F_s$ such that for each $1\le i\le s$, there is a unique minimal subset $r(F_i)$ of F_i not contained in F_j for any $j<i$ [20, Definition 2.1]. A shellable complex is Cohen–Macaulay over any field; furthermore, if $(\theta_1,\dots,\theta_d)$ is a l.s.o.p. for $\mathbb{K}[\Delta]$, then $\mathbb{K}[\Delta]$ is a free $\mathbb{K}[\theta_1,\dots,\theta_d]$-module with basis $\{X_{r(F_i)}:1\le i\le s\}$ (see [1, Theorem 1.7, Corollary 1.8] or [20, Theorem 2.5]).

The following proposition will be used in the next sections. In order to state it, let us identify the polynomial algebra $\mathbb{K}[X_s|s\in S]$ with the symmetric power algebra $\mathbf{S}(\mathbb{K}^S)$ where $\mathbb{K}^S$ denotes the $\mathbb{K}$-vector space of functions from S to $\mathbb{K}$. The dual of $\mathbb{K}^S$ is the space $\mathbb{K}\langle S\rangle$ of formal sums $\sum_{s\in S}\alpha_sY_s$ with $\alpha_s\in\mathbb{K}$ where Y_s is dual to X_s. For each monomorphism of vector spaces $\iota: V\hookrightarrow\mathbb{K}\langle S\rangle$, let $\iota^*(I_\Delta)$ denote the image of the Stanley–Reisner ideal I_Δ under the induced epimorphism $\iota^*:\mathbf{S}(\mathbb{K}^S)\twoheadrightarrow\mathbf{S}(V^*)$. So $\iota^*(I_\Delta)$ is the ideal of $\mathbf{S}(V^*)$ generated by the polynomials $\iota^*(X_F)=\prod_{s\in F}\iota^*(X_s)$, $F\in\Delta$.

Proposition 7. *Let Δ be a simplicial complex on S with $d = 1+\dim\Delta$, V a $\mathbb{K}$-vector space with $\dim_{\mathbb{K}} V = |S| - d$ and $\iota : V \hookrightarrow \mathbb{K}\langle S\rangle$ a monomorphism satisfying the property:*

$$\textit{For each face } F \textit{ of } \Delta\textit{, the intersection of } \iota(V) \textit{ with the subspace } \mathbb{K}\langle F\rangle \textit{ of } \mathbb{K}\langle S\rangle \textit{ is trivial.} \qquad (*)$$

Then $\mathbf{S}(V^*)/\iota^*(I_\Delta)$ is spanned as a $\mathbb{K}$-vector space by the $\iota^*(X_F)$, $F \in \Delta$. In particular, the top-degree subspace $(\mathbf{S}(V^*)/\iota^*(I_\Delta))^d$ is spanned as a $\mathbb{K}$-vector space by the set $\{\iota^*(X_F) | F \text{ is facet of } \Delta\}$. Furthermore, if Δ is Cohen–Macaulay over $\mathbb{K}$ (in particular if Δ is a matroid complex), then the Poincaré series of $\mathbf{S}(V^*)/\iota^*(I_\Delta)$ is $h_0 + h_1 t + \cdots + h_d t^d$, where h_d can be computed by

$$h_d = (-1)^{d-1}\widetilde{\chi}(\Delta) = (-1)^{|S|-d}\widetilde{\chi}\left(\Delta^*\right).$$

Proof. Let $\lambda : \mathbb{K}\langle S\rangle \twoheadrightarrow W$ be the cokernel of ι. So $\dim_{\mathbb{K}} W = d$ and there is a short exact sequence of vector spaces

$$0 \to V \xrightarrow{\iota} \mathbb{K}\langle S\rangle \xrightarrow{\lambda} W \to 0.$$

We now make use of property $(*)$ to produce a l.s.o.p. for $\mathbb{K}[\Delta]$. For each face F of Δ, the triviality of $\iota(V) \cap \mathbb{K}\langle F\rangle$ implies the injectivity of the composition $\mathbb{K}\langle F\rangle \hookrightarrow \mathbb{K}\langle S\rangle \xrightarrow{\lambda} W$. Taking the dual of this we see that the composition $W^* \xrightarrow{\lambda^*} \mathbb{K}^S \twoheadrightarrow \mathbb{K}^F$ is surjective. Note that $\mathbb{K}^S \twoheadrightarrow \mathbb{K}^F$ is the restriction map $\theta \mapsto \theta|_F$ defined in Lemma 6 above. It follows that if we choose a basis $w_1, \ldots, w_d$ of W^* and put $\theta_i = \lambda^*(w_i)$, then $\theta_{1|_F}, \ldots, \theta_{d|_F}$ span $\mathbb{K}^F$, and so by Lemma 6 (1), $(\theta_1, \ldots, \theta_d)$ is a l.s.o.p. for $\mathbb{K}[\Delta]$.

The map ι now induces an isomorphism of $\mathbb{K}$-algebras:

$$\mathbb{K}[\Delta]/(\theta_1, \ldots, \theta_d) \cong \mathbb{K} \otimes_{\mathbf{S}(W^*)} \left(\mathbf{S}\left(\mathbb{K}^S\right)/I_\Delta\right) \cong \mathbf{S}\left(V^*\right)/\iota^*(I_\Delta). \tag{5}$$

By Lemma 6 (2), this isomorphism implies that the $\mathbb{K}$-vector space $\mathbf{S}(V^*)/\iota^*(I_\Delta)$ is spanned by the $\iota^*(X_F)$, $F \in \Delta$. If $\mathbb{K}[\Delta]$ is Cohen–Macaulay, then $\mathbb{K}[\Delta]$ is free over $\mathbb{K}[\theta_1, \ldots, \theta_d]$ and so the Poincaré series of $\mathbb{K}[\Delta]/(\theta_1, \ldots, \theta_d)$ is equal to $(1-t)^d \mathbf{P}(\mathbb{K}[\Delta], t) = h_0 + h_1 t + \cdots + h_d t^d$. □

3. $\mathbf{R}(V^*, k)$ and the Steinberg module

In this section we define the module $\mathbf{R}(V^*, k)$ and prove the first part of Theorem 2.

Let p be a prime and let $\mathbb{F}_q$ denote a finite field of q elements where q is a power of p. Let V be an n-dimensional vector space over $\mathbb{F}_q$. Given a linear vector space W, denote by $\mathscr{L}_W$ the set of lines (i.e. 1-dimensional linear subspaces) in W and by $\mathscr{H}_W$ the set of hyperplanes (i.e. 1-codimensional linear subspaces) in W. A set of lines $\{\ell_1, \ldots, \ell_m\}$ in W is called an *m-frame* of W if $\{\ell_1, \ldots, \ell_m\}$ linearly spans an m-dimensional subspace of W.

Definition 8. $\Delta(V^*)$ *is the simplicial complex on the vertex set $\mathscr{L} := \mathscr{L}_{V^*}$ in which a face is of the form $\mathscr{L} \setminus F$ where F is a set of lines which linearly spans V^*.*

For each $k \geq 1$, in order to introduce the kth powers in the ideal $I(V^*, k)$, we also consider the following generalization of $\Delta(V^*)$:

Definition 9. $\Delta(V^*, k)$ *is the simplicial complex on $\mathscr{L} \sqcup \cdots \sqcup \mathscr{L}$ (the disjoint union of k copies of $\mathscr{L}$) in which a face is of the form $(\mathscr{L} \setminus F_1) \sqcup \cdots \sqcup (\mathscr{L} \setminus F_k)$ where the union $F_1 \cup \cdots \cup F_k$ linearly spans V^*.*

It is straightforward to check that a facet of $\Delta(V^*)$ is of the form $\mathscr{L} \setminus F$ where F is an n-frame of V^* and a minimal non-face of $\Delta(V^*)$ is of the form $\mathscr{L} \setminus \mathscr{L}_H$ where H is a hyperplane of V^*. Similarly a facet of $\Delta(V^*, k)$ is of the form $(\mathscr{L} \setminus F_1) \sqcup \cdots \sqcup (\mathscr{L} \setminus F_k)$ where $F_1, \ldots, F_k$ form a partition of an n-frame of V^* and a minimal non-face of $\Delta(V^*, k)$ is of the form $(\mathscr{L} \setminus \mathscr{L}_H) \sqcup \cdots \sqcup (\mathscr{L} \setminus \mathscr{L}_H)$ where H is a hyperplane of V^*. The cardinality of a facet of $\Delta(V^*, k)$ is thus given by

$$d := k\left(q^n - 1\right)/\left(q-1\right) - n,$$

and so $\Delta(V^*,k)$ is a $(d-1)$-dimensional simplicial complex.

Lemma 10. *$\Delta(V^*,k)$ is a matroid complex.*

Proof. Recall that the *dual* of a matroid complex Δ on S is defined to be the simplicial complex $\Delta^\#$ on S such that F is a facet of $\Delta^\#$ if $S\setminus F$ is a facet of Δ. It is known that the dual of a matroid complex is also a matroid complex [15, Theorem 2.1.1].

Let $\nabla(V^*,k)$ denote the simplicial complex on $\mathscr{L}\sqcup\cdots\sqcup\mathscr{L}$ (the disjoint union of k copies of $\mathscr{L}$) in which $F_1\sqcup\cdots\sqcup F_k$ is a face if $F_1,\ldots,F_k$ form a partition of an m-frame of V^* for some $m\le n$. The exchange property is easily checked for $\nabla(V^*,k)$ and so it is a matroid complex. Since a facet of $\nabla(V^*,k)$ is of the form $F_1\sqcup\cdots\sqcup F_k$ where $F_1,\ldots,F_k$ form a partition of an n-frame of V^*, it follows that the dual of the matroid complex $\nabla(V^*,k)$ is $\Delta(V^*,k)$, and so $\Delta(V^*,k)$ is also a matroid complex. □

The space $\mathbb{F}_q\langle\mathscr{L}\sqcup\cdots\sqcup\mathscr{L}\rangle$ is the space of formal sums $\sum_{1\le i\le k,\ \ell\in\mathscr{L}}\alpha_{\ell,i}Y_{\ell,i}$ with $\alpha_{\ell,i}\in\mathbb{F}_q$, where $Y_{\ell,i}$ corresponds to $\ell\in\mathscr{L}$ with $\mathscr{L}$ being at the i^{th} position in the disjoint union $\mathscr{L}\sqcup\cdots\sqcup\mathscr{L}$. For each line ℓ in V^*, choose a non-zero vector u_ℓ on it. Such a choice gives rise to a homomorphism $\iota\colon V\to\mathbb{F}_q\langle\mathscr{L}\sqcup\cdots\sqcup\mathscr{L}\rangle$ defined by

$$\iota(v)=\sum_{1\le i\le k,\ \ell\in\mathscr{L}}u_\ell(v)Y_{\ell,i}.$$

Lemma 11. *The map ι is a monomorphism satisfying property $(*)$ of Proposition 7 and its dual $\iota^*:\mathbb{F}_q^{\mathscr{L}\sqcup\cdots\sqcup\mathscr{L}}\to V^*$ sends each basis element $X_{\ell,i}$ to u_ℓ. Here $X_{\ell,i}$ is dual to $\{Y_{\ell,i}\}$.*

Proof. That ι is a monomorphism is clear. Given $(\mathscr{L}\setminus F_1)\sqcup\cdots\sqcup(\mathscr{L}\setminus F_k)$ a face of $\Delta(V^*,k)$, if $\iota(v)=\sum_{1\le i\le k,\ \ell\in\mathscr{L}}u_\ell(v)Y_{\ell,i}$ belongs to $\mathbb{F}_q\langle(\mathscr{L}\setminus F_1)\sqcup\cdots\sqcup(\mathscr{L}\setminus F_k)\rangle$, then we must have $u_\ell(v)=0$ for all $\ell\in F_1\cup\cdots\cup F_k$, which implies that $v=0$ since $F_1\cup\cdots\cup F_k$ linearly spans V^*. For the second assertion, we have

$$\iota^*\left(X_{\ell,i}\right)(v)=X_{\ell,i}(\iota(v))=X_{\ell,i}\left(\sum_{1\le i\le k,\ \ell\in\mathscr{L}}u_\ell(v)Y_{\ell,i}\right)=u_\ell(v)$$

for all $v\in V$. □

We now prove the following which is the first part of Theorem 2. For this recall from the introduction that $\mathbf{R}(V^*,k)$ (or $\mathbf{R}_{n,k}$) denote the quotient $\mathbf{S}(V^*)/I(V^*,k)$, where $I(V^*,k)$ (or $I_{n,k}$) is the ideal of $\mathbf{S}(V^*)$ generated by the polynomials $(\prod_{\ell\in\mathscr{L}\setminus\mathscr{L}_H}u_\ell)^k$, H running over the set of all hyperplanes of V^*.

Proposition 12. *For $k\ge 1$ and $n=\dim V$, set*

$$d:=\frac{k(q^n-1)}{q-1}-n.$$

Then $\mathbf{R}^d(V^*,k)$ is isomorphic to the twisted Steinberg module $\mathrm{St}_n\otimes\det^{k-1}$ and $\mathbf{R}^i(V^*,k)$ vanishes if $i>d$.

Proof. Recall that a minimal non-face of $\Delta(V^*,k)$ is of the form $(\mathscr{L}\setminus\mathscr{L}_H)\sqcup\cdots\sqcup(\mathscr{L}\setminus\mathscr{L}_H)$ where H is a hyperplane of V^*. The generator of the Stanley-Reisner ideal $I_{\Delta(V^*,k)}$ of $\mathbb{F}_q[X_{\ell,i}\mid\ell\in\mathscr{L},\ 1\le i\le k]$ corresponding to such a minimal non-face is $\prod_{1\le i\le k,\ \ell\in\mathscr{L}\setminus\mathscr{L}_H}X_{\ell,i}$. By Lemma 11, this is sent by ι^* to $(\prod_{\ell\in\mathscr{L}\setminus\mathscr{L}_H}u_\ell)^k$ in $\mathbf{S}(V^*)$. It follows that the ideal $\iota^*(I_{\Delta(V^*,k)})$ of $\mathbf{S}(V^*)$ is generated by the polynomials $(\prod_{\ell\in\mathscr{L}\setminus\mathscr{L}_H}u_\ell)^k$ where H runs over the set of hyperplanes of V^*. The quotient ring $\mathbf{S}(V^*)/\iota^*(I_{\Delta(V^*,k)})$ is thus exactly the ring $\mathbf{R}(V^*,k)$ we are considering.

Since $\Delta(V^*,k)$ is a matroid complex (Lemma 10) and ι satisfies property $(*)$ (Lemma 11), we can apply Proposition 7 to analyze the top-degree module $\mathbf{R}^d(V^*,k)$. Given a facet of $\Delta(V^*,k)$ of the form $F := (\mathscr{L} \setminus F_1) \sqcup \cdots \sqcup (\mathscr{L} \setminus F_k)$ where $F_1, \dots, F_k$ form a partition of an n-frame $\{\ell_1, \dots, \ell_n\}$ of V^*, the corresponding generator $\iota^*(X_F)$ of $\mathbf{R}^d(V^*, k)$ is given by

$$\iota^*(X_F) = \frac{\left(\prod_{\ell \in \mathscr{L}} u_\ell\right)^k}{u_{\ell_1} \cdots u_{\ell_n}}.$$

It is well-known [5] that the "product of lines" $\prod_{\ell \in \mathscr{L}} u_\ell$ is up to non-zero scalar equal to $\mathbf{L}_n$ where

$$\mathbf{L}_n = \mathbf{L}(x_1, \dots, x_n) := \det\left(x_j^{q^{i-1}}\right)_{1 \le i,j \le n}.$$

Since GL_n acts transitively on the set of n-frames of V^*, it follows that if we put $\gamma := \frac{\mathbf{L}_n^k}{x_1 \cdots x_n}$ then by Proposition 7, $\{\gamma \cdot g \mid g \in \mathrm{GL}_n\}$ is a spanning set of $\mathbf{R}^d(V^*, k)$.

But it was proved by S. Mitchell [12, Corollary A.7] that γ is fixed by $\mathrm{e}_n^{(k-1)}$, where $\mathrm{e}_n^{(k-1)}$ is the idempotent corresponding to the twisted Steinberg representation $\mathrm{St}_n \otimes \det^{k-1}$. It follows that the GL_n-linear map

$$\mathrm{St}_n \otimes \det^{k-1} \cong \mathrm{e}_n^{(k-1)} \cdot \mathbb{F}_q[\mathrm{GL}_n] \to \mathbf{R}^d\left(V^*, k\right), \quad \mathrm{e}_n^{(k-1)} \cdot g \mapsto \gamma \cdot g,$$

is an epimorphism.

To conclude that this is an isomorphism, we need to show that the dimension of $\mathbf{R}^d(V^*, k)$ is $q^{\frac{n(n-1)}{2}}$. For this we use the formula $h_d = (-1)^n \widetilde{\chi}(\Delta(V^*, k)^*)$ (see (4)) and proceed as follows. Recall [22] first that to every poset (partially ordered set) P, one can associate a simplicial complex $\mathbf{c}(P)$, called the *order complex* of P, whose faces are the chains (i.e. totally ordered subsets) of P. The (reduced) homology of P is defined to be the (reduced) homology of $\mathbf{c}(P)$. Inversely, to every simplicial complex Δ, one can associate a poset $\mathbf{p}(\Delta)$, called the *face poset* of Δ, which is defined to be the poset of nonempty faces ordered by inclusion. It is known that $\mathbf{c}(\mathbf{p}(\Delta))$, called the *barycentric subdivision* of Δ, and Δ have the same geometric realizations. Now let $\mathscr{T}$ denote the Tits building which is defined by

$$\mathscr{T} := \text{partially ordered set of proper subspaces of } V^*.$$

It is well-known [19] that $\widetilde{\mathrm{H}}_*(\mathscr{T}; \mathbb{F}_q)$ is only non-trivial in degree $n-2$, and in this degree, $\widetilde{\mathrm{H}}_{n-2}(\mathscr{T}; \mathbb{F}_q) \cong \mathrm{St}_n$. Note that the dimension of $\widetilde{\mathrm{H}}_{n-2}(\mathscr{T}; \mathbb{F}_q)$, which is $q^{\frac{n(n-1)}{2}}$, can be computed by induction using an exact sequence as in [9, Theorem 1.14]. The Alexander dual $\Delta^* := \Delta(V^*, k)^*$ is related to the Tits building as follows. By definition, Δ^* is the simplicial complex on $\mathscr{L} \sqcup \cdots \sqcup \mathscr{L}$ in which a face is of the form $F_1 \sqcup \cdots \sqcup F_k$ where the union $F_1 \cup \cdots \cup F_k$ does not linearly spans V^*. Consider the map of posets $f : \mathbf{p}(\Delta^*) \to \mathscr{T}$ which sends a face $F_1 \sqcup \cdots \sqcup F_k$ to the subspace spanned by $F_1 \cup \cdots \cup F_k$. For each proper subspace W of V^*, the poset $f^{-1}(\mathscr{T}_{\le W})$ is contractible because it has $\mathscr{L}_W \sqcup \cdots \sqcup \mathscr{L}_W$ as its unique maximal element. The Quillen Fiber Lemma [16] then implies that f is a homotopy equivalence. The identity $h_d = q^{\frac{q(q-1)}{2}}$ follows. □

4. $\mathbf{R}(V^*, k)$ as a $\mathscr{P}$-module

In this section we prove the second part of Theorem 2. For this we first review some facts about the action of the algebra of Steenrod reduced powers $\mathscr{P}$ on $\mathbf{S}(V^*) \cong \mathbb{F}_q[x_1, \dots, x_n]$. Larry Smith [18] defined the algebra $\mathscr{P} := \mathscr{P}(\mathbb{F}_q)$ as the $\mathbb{F}_q$-subalgebra of the endomorphism algebra of the functor $V \mapsto \mathbf{S}(V^*)$, generated by certains natural transformations $1 = \mathscr{P}^0, \mathscr{P}^1, \mathscr{P}^2, \dots$. It is sufficient for us to know that the action of the operations $\mathscr{P}^i$ on $\mathbf{S}(V^*)$ satisfies:

(1) *the unstable condition*: $\mathscr{P}^i(f) = f^q$ if $i = \deg f$ and $\mathscr{P}^i(f) = 0$ if $i > \deg f$, $\forall\, f \in \mathbf{S}(V^*)$, and
(2) *the Cartan formula*: $\mathscr{P}^i(f_1 f_2) = \sum_{i_1+i_2=i} \mathscr{P}^{i_1}(f_1)\mathscr{P}^{i_2}(f_2), \forall\, f_1, f_2 \in \mathbf{S}(V^*)$.

Denote by $\overline{\mathscr{P}}$ the total Steenrod reduced powers $1+\mathscr{P}^1+\mathscr{P}^2+\cdots$. These properties then implies that the map

$$\overline{\mathscr{P}}:\mathbb{F}_q[x_1,\ldots,x_n]\to\mathbb{F}_q[x_1,\ldots,x_n],\quad f\mapsto\sum_{i\ge0}\mathscr{P}^i(f),$$

is a homomorphism of algebras which satisfies $\overline{\mathscr{P}}(x)=x+x^q$ for all $x\in\mathbf{S}^1(V^*)$.

The algebra $\mathscr{P}$ is also a Hopf algebra with the coproduct $\Delta:\mathscr{P}\to\mathscr{P}\otimes\mathscr{P}$ defined by

$$\Delta\left(\mathscr{P}^i\right)=\sum_{i_1+i_2=i}\mathscr{P}^{i_1}\otimes\mathscr{P}^{i_2},\quad i\ge1.$$

The canonical anti-automorphism (the antipode) $\chi:\mathscr{P}\to\mathscr{P}$ then satisfies the relations:

$$\sum_{i_1+i_2=i}\mathscr{P}^{i_1}\chi\left(\mathscr{P}^{i_2}\right)=\sum_{i_1+i_2=i}\chi\left(\mathscr{P}^{i_1}\right)\mathscr{P}^{i_2}=0,\quad i\ge1.$$

This can be used to show that

$$\chi(\mathscr{P}^i)(x)=\begin{cases}(-1)^i x^{q^r} & \text{if } i=\frac{q^r-1}{q-1},\\ 0 & \text{otherwise.}\end{cases}$$

Since χ is a map of coalgebras, the Cartan formula also holds for $\chi(\mathscr{P}^i)$:

$$\chi\left(\mathscr{P}^i\right)(f_1f_2)=\sum_{i_1+i_2=k}\chi\left(\mathscr{P}^{i_1}\right)(f_1)\cdot\chi\left(\mathscr{P}^{i_2}\right)(f_2),\quad\forall f_1,f_2\in\mathbf{S}(V^*).$$

In order to simplify signs, we write $\widehat{\mathscr{P}}^i$ for $(-1)^i\chi(\mathscr{P}^i)$ and let $\widehat{\mathscr{P}}=1+\widehat{\mathscr{P}}^1+\widehat{\mathscr{P}}^2+\cdots$ denote the (signed) total conjugate Steenrod reduced powers. The above formulae for χ then implies that the map

$$\widehat{\mathscr{P}}:\mathbb{F}_q[x_1,\ldots,x_n]\to\mathbb{F}_q[[x_1,\ldots,x_n]],\quad f\mapsto\sum_{i\ge0}\widehat{\mathscr{P}}^i(f),$$

is a homomorphism of algebras which satisfies $\widehat{\mathscr{P}}(x)=x+x^q+x^{q^2}+\cdots$ for all $x\in\mathbf{S}^1(V^*)$. Here $\mathbb{F}_q[[x_1,\ldots,x_n]]$ denotes the $\mathbb{F}_q$-algebra of power series in $x_1,\ldots,x_n$.

Finally, for all $k\ge1$ and $f,g\in\mathbf{S}(V^*)$, we have the following congruence

$$\mathscr{P}^k(f)\cdot g\equiv f\cdot\chi\left(\mathscr{P}^k\right)(g)\left(\operatorname{mod}\mathscr{P}^+\mathbf{S}(V^*)\right),$$

which is known as the the χ-trick [26]. Recall that this congruence follows from the identity

$$\begin{aligned}f\cdot\chi\left(\mathscr{P}^k\right)(g)+\sum_{i=1}^k\mathscr{P}^i\left(f\cdot\chi\left(\mathscr{P}^{k-i}\right)(g)\right)&=\sum_{i=0}^k\sum_{j=0}^i\mathscr{P}^j(f)\cdot\mathscr{P}^{i-j}\left(\chi\left(\mathscr{P}^{k-i}\right)(g)\right)\\&=\sum_{j=0}^k\left(\mathscr{P}^j(f)\cdot\sum_{a+b=k-j}\mathscr{P}^a\left(\chi\left(\mathscr{P}^b\right)(g)\right)\right)\\&=\mathscr{P}^k(f)\cdot g.\end{aligned}$$

4.1. *Proof of Theorem 2* (2a)

We suppose $k=q^sr$ with $s\ge0$, $1\le r\le q-1$, and as in the previous section, we set

$$d:=\frac{k\left(q^n-1\right)}{q-1}-n.$$

We need the following

Lemma 13. *Given r elements $y_1,\ldots,y_r$ in $\mathbf{S}^1(V^*)$, the following identity holds*

$$\mathbf{V}\left(x_1,\ldots,x_{n-1},y_1\right)^{q^s}\cdots\mathbf{V}\left(x_1,\ldots,x_{n-1},y_r\right)^{q^s}=(-1)^{n-1}\sum_{i=0}^{n-1}\chi\left(\mathscr{P}^{\frac{(q^{n-1}-1)q^sr}{q-1}-i}\right)\left(e_i\cdot y_1^{q^s}\cdots y_r^{q^s}\right),$$

where $e_i:=e_i(x_1^{q-1},\ldots,x_{n-1}^{q-1})$ is the i^{th} elementary symmetric function of $x_1^{q-1},\ldots,x_{n-1}^{q-1}$.

Proof. Recall that the map $\widehat{\mathscr{P}}:\mathbb{F}_q[x_1,\dots,x_n]\to\mathbb{F}_q[[x_1,\dots,x_n]]$, $f\mapsto\sum_{i\geq 0}\widehat{\mathscr{P}^i}(f)$, is a homomorphism of algebras and $\widehat{\mathscr{P}}(x)=x+x^q+x^{q^2}+\cdots$ for $x\in\mathbf{S}^1(V^*)$. Here $\widehat{\mathscr{P}^i}=(-1)^i\chi(\mathscr{P}^i)$ and $\widehat{\mathscr{P}}=1+\widehat{\mathscr{P}^1}+\widehat{\mathscr{P}^2}+\cdots$.

Set

$$P\left(x_1,\dots,x_{n-1};y_1,\dots,y_r\right):=\widehat{\mathscr{P}}\left(\prod_{i=1}^{n-1}\left(1-x_i^{q-1}\right)\cdot y_1^{q^s}\cdots y_r^{q^s}\right)$$

and let $R(x_1,\dots,x_{n-1};y_1,\dots,y_r)$ denote its homogeneous component of degree $q^{n-1+s}r$. Since

$$\prod_{i=1}^{n-1}\left(1-x_i^{q-1}\right)=\sum_{i=0}^{n-1}(-1)^i e_i,$$

the lemma will follow from the identity

$$R\left(x_1,\dots,x_{n-1};y_1,\dots,y_r\right)=\mathbf{V}\left(x_1,\dots,x_{n-1},y_1\right)^{q^s}\cdots\mathbf{V}\left(x_1,\dots,x_{n-1},y_r\right)^{q^s}.$$

For this it is sufficient to prove that $R(x_1,\dots,x_{n-1};y_1,\dots,y_r)$ is null whenever each y_j is a linear combination of $x_1,\dots,x_{n-1}$. (This is because each $\mathbf{V}(x_1,\dots,x_{n-1},y_j)$ is equal to

$$\prod_{x\in\mathbb{F}_q\langle x_1,\dots,x_{n-1}\rangle}(y_j+x)\quad\text{and}\quad R\left(x_1,\dots,x_{n-1};y_1,\dots,y_r\right)$$

is clearly divisible by $y_1^{q^s}\cdots y_r^{q^s}$.)

Now since $P(x_1,\dots,x_{n-1};y_1,\dots,y_r)$ is linear in each y_j and symmetric in $x_1,\dots,x_{n-1}$ as well as in $y_1,\dots,y_r$, we need only to prove that $R(x_1,\dots,x_{n-1};y_1,\dots,y_r)$ is null if $y_1=x_1$. But since

$$\widehat{\mathscr{P}}\left(x_1\left(1-x_1^{q-1}\right)\right)=\widehat{\mathscr{P}}\left(x_1-x_1^q\right)=x_1,$$

it follows that

$$\begin{aligned}
&P\left(x_1,\dots,x_{n-1};x_1,y_2,\dots,y_r\right)\\
&\qquad=x_1\left(x_1+x_1^q+\cdots\right)^{q^s-1}\prod_{i=2}^{n-1}\left(1-\left(x_i+x_i^q+\cdots\right)^{q-1}\right)\prod_{j=2}^{r}\left(y_j+y_j^q+\cdots\right)^{q^s}\\
&\qquad=x_1\prod_{k=0}^{s-1}\left(x_1^{q^k}+x_1^{q^{k+1}}+\cdots\right)^{q-1}\prod_{i=2}^{n-1}\left(1-\left(x_i+x_i^q+\cdots\right)^{q-1}\right)\prod_{j=2}^{r}\left(y_j^{q^s}+y_j^{q^{s+1}}+\cdots\right).
\end{aligned}$$

It is clear that a monomial occurring in this product is of the form $x_1^{1+i_1}x_2^{i_2}\cdots x_{n-1}^{i_{n-1}}y_2^{j_2}\cdots y_r^{j_r}$ where $\alpha(i_1)\leq s(q-1)$, $\alpha(i_\ell)\in\{0,q-1\}$ for $2\leq\ell\leq n-1$, and $\alpha(j_\ell)=1$ for $2\leq\ell\leq r$. Here the function α is defined by $\alpha(a):=a_0+a_1+\cdots$ where $a=a_0+a_1q+a_2q^2+\cdots$ is the q-adic expansion of a. It follows that if the degree of such a monomial is $q^{n-1+s}r$ then we must have

$$\begin{aligned}
\alpha\left(q^{n-1+s}r-1\right)&=\alpha\left(i_1+i_2+\cdots+i_{n-1}+j_2+\cdots+j_r\right)\\
&\leq\alpha\left(i_1\right)+\alpha\left(i_2\right)+\cdots+\alpha\left(i_{n-1}\right)+\alpha\left(j_2\right)+\cdots+\alpha\left(j_r\right)\\
&\leq(n-2+s)\left(q-1\right)+(r-1).
\end{aligned}$$

But this is a contradiction since we have $q^{n-1+s}r-1=(q-1)(1+q+\cdots+q^{n-2+s})+(r-1)q^{n-1+s}$, which implies that $\alpha(q^{n-1+s}r-1)=(n-1+s)(q-1)+(r-1)$. Here we use the hypothesis $1\leq r\leq q-1$. The Lemma 13 follows. □

Proof of Theorem 2 (2a)**.** We need to prove that, for each $0\leq i\leq d$, the natural projection $\mathbf{S}(V^*)\twoheadrightarrow\mathbf{R}(V^*,k)$ induces an isomorphism of GL_n-modules

$$\mathrm{Q}^i\left(\mathbf{S}\left(V^*\right)\right)\cong\mathrm{Q}^i\left(\mathbf{R}\left(V^*,k\right)\right).$$

It is sufficient to prove that $f\cdot\mathbf{V}_n^k$ is $\mathscr{P}$-decomposable in $\mathbf{S}(V^*)$ if

$$\deg\left(f\right)\leq d-\deg\left(\mathbf{V}_n^k\right)=d-kq^{n-1}=\frac{k\left(q^{n-1}-1\right)}{q-1}-n.$$

By Lemma 13 and the χ-trick, we have

$$\begin{aligned} f\cdot \mathbf{V}_n^k &= (-1)^{n-1}\sum_{i=0}^{n-1} f\cdot\chi\left(\mathscr{P}^{\frac{(q^{n-1}-1)k}{q-1}-i}\right)\left(e_i\cdot x_n^k\right)\\ &\equiv (-1)^{n-1}\sum_{i=0}^{n-1} e_i\cdot x_n^k\cdot\mathscr{P}^{\frac{(q^{n-1}-1)k}{q-1}-i}(f) \bmod \left(\mathscr{P}^+\mathbf{S}(V^*)\right). \end{aligned}$$

By instability, $\mathscr{P}^{\frac{(q^{n-1}-1)k}{q-1}-i}(f)=0$ for all $0\le i\le n-1$. □

Remark 14. Lemma 13 can be seen a generalization of the formula

$$\mathbf{Q}_{m,0}=\chi\left(\mathscr{P}^{\frac{q^m-1}{q-1}-m}\right)(e_m)$$

in [11, Theorem 1.1]. Indeed, by putting $n=m+1$, in the simplest case where $r=1$ and $s=0$, Lemma 13 gives:

$$\mathbf{V}(x_1,\dots,x_m,x)=(-1)^m\sum_{i=0}^{m}\chi\left(\mathscr{P}^{\frac{q^m-1}{q-1}-i}\right)(xe_i),$$

Comparing this with the formula defining the Dickson invariants $\mathbf{Q}_{m,0},\dots,\mathbf{Q}_{m,m-1}$ [5]:

$$\mathbf{V}(x_1,\dots,x_m,x)=\sum_{j=0}^{m-1}(-1)^{m-j}\mathbf{Q}_{m,j}x^{q^j},$$

we obtain

$$\mathbf{Q}_{m,j}=\sum_{i=0}^{m}\chi\left(\mathscr{P}^{\frac{q^m-q^j}{q-1}-i}\right)(e_i).$$

Note that [11, Lemma 2.4] if f is of degree s and $\alpha(r(q-1)+s)>s$ then $\widehat{\mathscr{P}}^r(f)=0$ (this is proved easily by letting $\widehat{\mathscr{P}}$ act on a product of s elements in $\mathbf{S}^1(V^*)$, and then using the definition of the function α). Applying this to each term in the above expression of $\mathbf{Q}_{m,j}$, we have

$$\alpha\left(\left(\frac{q^m-q^j}{q-1}-i\right)(q-1)+i(q-1)\right)=\alpha\left(q^m-q^j\right)=(m-j)(q-1;)>i(q-1)\Longleftrightarrow m-j>i.$$

It follows that the expression above for $\mathbf{Q}_{m,j}$ can be simplified to:

$$\mathbf{Q}_{m,j}=\sum_{i=m-j}^{m}\chi\left(\mathscr{P}^{\frac{q^m-q^j}{q-1}-i}\right)(e_i).$$

For $j=0$, we recover the above formula for the top Dickson invariant $\mathbf{Q}_{m,0}$.

4.2. *Proof of Theorem 2* (2b)

Recall that $k=q^s r$ with $1\le r\le r-1$. We need the following

Lemma 15. *For each $0\ne y\in V^*$, $I(V^*,k)+(y^k)=I(W,qk)+(y^k)$ where W is linear complement of $\mathbb{F}_q\langle y\rangle$ in V^*.*

Proof. Let W be a linear complement of $\mathbb{F}_q\langle y\rangle$ in V^*. If H is a hyperplane of V^* not passing through y, then $\mathbf{V}_{H,V^*}=\mathbf{L}_{V^*}/\mathbf{L}_H$ is divisible by y. It follows that $I(V^*,k)+(y^k)$ is generated by (y^k) and the $\mathbf{V}_{H,V^*}^k$'s where H is a hyperplane passing through y. Let H be such a hyperplane. Then $W':=H\cap W$ is a hyperplane in W and $H=\mathbb{F}_q\langle y\rangle\oplus W'$. Choose a basis $\{y_2,\dots,y_n\}$ of W such that $\{y_2,\dots,y_{n-1}\}$ is a basis of W'. Then, up to non-zero scalars, we have

$$\mathbf{L}_{V^*}=\mathbf{L}(y,y_2,\dots,y_n),\quad \mathbf{L}_H=\mathbf{L}(y,y_2,\dots,y_{n-1}),\quad \mathbf{L}_W=\mathbf{L}(y_2,\dots,y_n),\quad \mathbf{L}_{W'}=\mathbf{L}(y_2,\dots,y_{n-1}),$$

where, recalling from the introduction, $\mathbf{L}(x_1,\dots,x_n)$ is given by $\mathbf{L}(x_1,\dots,x_n) := \det(x_j^{q^{i-1}})_{1\le i,j\le n}$. By definition, $\mathbf{V}_{H,V^*} = \mathbf{L}_{V^*}/\mathbf{L}_H$, and so

$$\begin{vmatrix} y & y_2 & \cdots & y_{n-1} \\ y^q & y_2^q & \cdots & y_{n-1}^q \\ \vdots & \vdots & \ddots & \vdots \\ y^{q^{n-2}} & y_2^{q^{n-2}} & \cdots & y_{n-1}^{q^{n-2}} \end{vmatrix} \mathbf{V}_{H,V^*} = \begin{vmatrix} y & y_2 & \cdots & y_n \\ y^q & y_2^q & \cdots & y_n^q \\ \vdots & \vdots & \ddots & \vdots \\ y^{q^{n-1}} & y_2^{q^{n-1}} & \cdots & y_n^{q^{n-1}} \end{vmatrix}.$$

Expanding along the first column of each determinant gives

$$y\mathbf{L}_{W'}^q \mathbf{V}_{H,V^*} \equiv y\mathbf{L}_W^q \bmod \big(y^q\big),$$

and so

$$\mathbf{L}_{W'}^q \mathbf{V}_{H,V^*} \equiv \mathbf{L}_W^q \bmod \big(y^{q-1}\big),$$

which, using $\mathbf{V}_{W',W} = \mathbf{L}_W/\mathbf{L}_{W'}$, in turn gives

$$\mathbf{V}_{H,V^*} \equiv \mathbf{V}_{W',W}^q \bmod \big(y^{q-1}\big).$$

Taking the q^sth power of this congruence then gives

$$\mathbf{V}_{H,V^*}^{q^s} \equiv \mathbf{V}_{W',W}^{q^{s+1}} \bmod \left(y^{q^s(q-1)}\right)$$

and so

$$\mathbf{V}_{H,V^*}^{q^s r} \equiv \mathbf{V}_{W',W}^{q^{s+1}r} \bmod \left(y^{q^s(q-1)}\right).$$

But since $k = q^s r$ with $1 \le r \le q-1$, this implies that

$$\mathbf{V}_{H,V^*}^{k} \equiv \mathbf{V}_{W',W}^{qk} \bmod \left(y^{k}\right).$$

We have thus proved that $I(V^*,k)+(y^k)$ is generated by y^k and the $\mathbf{V}_{W',W}^{qk}$'s where $W' = H\cap W$ with H a hyperplane of V^* passing through y. The identity $I(V^*,k)+(y^k) = I(W,qk)+(y^k)$ follows. □

Proposition 16.

(1) $\mathbf{R}_{n,k}$ *can be embedded as a* $\mathscr{P}$*-submodule into a direct product of* $\frac{q^n-1}{q-1}$ *copies of* $\mathbf{R}_{n-1,qk}\otimes \mathbf{R}_{1,k}$.

(2) *For all* $u\in\mathbf{R}_{n,k}$, *we have* $\chi(\mathscr{P}^i)(u) = 0$ *whenever* $|u|+qi > d$.

Proof. The kernel of the natural map of $\mathscr{P}$-modules $\mathbf{S}(V^*)\to\prod_{\ell\in\mathscr{L}_{V^*}}\mathbf{S}(V^*)/(u_\ell^k)$ is the principal ideal of $\mathbf{S}(V^*)$ generated by $\mathbf{L}_n^k$. Since $\mathbf{L}_n^k$ belongs to the ideal $I_{V^*,k}$, this map induces an inclusion of $\mathscr{P}$-modules

$$\mathbf{S}\big(V^*\big)/I\big(V^*,k\big) \hookrightarrow \prod_{\ell\in\mathscr{L}_{V^*}} \mathbf{S}\big(V^*\big)/\Big(I\big(V^*,k\big)+\Big(u_\ell^k\Big)\Big).$$

The part (1) of the proposition now follows from Lemma 15.

For the part (2), by induction we see that $\mathbf{R}_{n,k}$ is embedded in a direct product of copies of $\mathbb{F}_q[x]/(x^{q^{n-1}k})\otimes\cdots\otimes\mathbb{F}_q[x]/(x^k)$. By the Cartan formula, it suffices now to prove that, for all $m\ge 0$, we have $\chi(\mathscr{P}^i)(x^t) = 0$ in $\mathbb{F}_q[x]/(x^{q^m r})$ whenever $t+qi > q^m r-1$. Suppose in contrary that $\chi(\mathscr{P}^i)(x^t) = \lambda x^{t+(q-1)i}$ with $\lambda\in\mathbb{F}_q^\times$, then using χ-trick and instability, we have

$$\lambda x^{q^m r-1} = x^{q^m r-1-(q-1)i-t}\cdot\chi\left(\mathscr{P}^i\right)\left(x^t\right) \equiv x^t\cdot\mathscr{P}^i\left(x^{q^m r-1-(q-1)i-t}\right) = 0$$

modulo $\mathscr{P}$-decomposable elements in $\mathbb{F}_q[x]/(x^{q^m r})$. This contradicts the fact that the monomial $x^{q^m r-1}$ is $\mathscr{P}$-indecomposable in $\mathbb{F}_q[x]/(x^{q^m r})$. Indeed if $x^{q^m r-1}$ were $\mathscr{P}$-decomposable, then up to non-zero scalar it would be the image under some operation $\mathscr{P}^{p^a}$ of $x^{q^m r-1-p^a(q-1)}$ (noting that $\mathscr{P}$ is multiplicatively generated by all the $\mathscr{P}^{p^a}$, $a \geq 0$, and not by all the $\mathscr{P}^{q^a}$, $a \geq 0$ [18, page 338]). But we have

$$\mathscr{P}^{p^a}\left(x^{q^m r-1-p^a(q-1)}\right) = \binom{q^m r - 1 - p^a(q-1)}{p^a} x^{q^m r-1} = \binom{(q^m r - p^a q) + (p^a - 1)}{p^a} x^{q^m r-1}.$$

Since $q^m r - p^a q > 0$, we get $q^m > p^a q/r > p^a$, and so $(q^m r - p^a q)$ is divisible by p^{a+1}. It follows that p^a does not appear in the p-adic expansion of $(q^m r - p^a q) + (p^a - 1)$ and so the binomial coefficient $\binom{(q^m r - p^a q)+(p^a-1)}{p^a}$ is zero, completing the proof that $x^{q^m r-1}$ is $\mathscr{P}$-indecomposable. □

Proof of Theorem 2 (2b)**.** We need to prove that the natural projection $\mathbf{R}^d(V^*,k) \twoheadrightarrow \mathrm{Q}^d(\mathbf{R}(V^*,k))$ is an isomorphism of GL_n-modules. For this it is sufficient to prove that $\mathscr{P}^+\mathbf{R}(V^*,k)$ vanishes in degree d. Using the antipode χ, we need to prove that $\chi(\mathscr{P}^i)(u) = 0$ for $u \in \mathbf{R}(V^*,k)$ and $i > 0$ such that $|u| + (q-1)i = d$. But the identity $|u| + (q-1)i = d$ implies $|u| + qi = d + i > d$, and we can apply Proposition 16 (2) to conclude $\chi(\mathscr{P}^i)(u) = 0$. □

Remark 17. The second part of Proposition 16 is equivalent to the fact that the $\mathscr{P}$-module $\Sigma^d\mathbb{D}(\mathbf{R}_{n,k})$ is unstable. Here given M a $\mathscr{P}$-module, $\mathbb{D}(M)$ is the Spanier–Whitehead dual of M, which is defined by

$$\begin{cases} (\mathbb{D}M)^{-i} = & \mathrm{Hom}_{\mathbb{F}_q}\left(M^i, \mathbb{F}_q\right), \qquad i \in \mathbb{Z}, \\ \quad \theta(f) = & f \circ \left(\chi(\theta)\right), \qquad\qquad f \in \mathbb{D}M, \theta \in \mathscr{P}. \end{cases}$$

5. Proof of Theorem 4

In this section we prove the following:

Theorem 18 (Theorem 4). *For $n = \dim V \geq 2$, $\dim_{\mathbb{F}_q} \mathrm{Q}^{q^{n-1}-n}(\mathbf{S}(V^*)) = (q-1)(q^2-1)\cdots(q^{n-1}-1)$.*

This is proved by considering a matroid complex K which can be seen as an affine version of the complex $\Delta(V^*)$. To define K, let us fix a hyperplane W in V^* and let E denote an affine space not containing the origin and parallel to W. Let K be the simplicial complex on the vertex set E in which a face is of the form $E \setminus F$ where F affinely spans E, or equivalently, F linearly spans V^*.

We check easily that the facets of K are complements in E of affine bases of E and a minimal non-face of K is of the form $E \setminus H$ where H is a hyperplane in the affine space E. Since a facet of K has cardinality $d := q^{n-1} - 1$, K is a $(d-1)$-dimensional simplicial complex.

The Alexander dual K^* of K is the simplicial complex on the vertex set E in which F is a face if F does not affinely span E. A facet of K^* is thus a hyperplane in the affine space E.

Lemma 19. *K is a matroid complex.*

Proof. Let B be the simplicial complex on E whose faces are affinely independent subsets of E. It is clear that E is a matroid complex. The dual of this matroid complex is K and so K is also a matroid complex. □

Consider the map $\iota : V \to \mathbb{F}_q\langle E\rangle$ defined by $\iota(v) = \sum_{\alpha\in E}\alpha(v)Y_\alpha$ where $\mathbb{F}_q\langle E\rangle$ is the space of formal sums $\Sigma_{\alpha\in E}Y_\alpha$.

Lemma 20. *The map $\iota : V \to \mathbb{F}_q\langle E\rangle$ is a monomorphism satisfying property $(*)$ of Proposition 7 and its dual $\iota^* : \mathbb{F}_q^E \to V^*$ sends each basis element $X_\alpha \in \mathbb{F}_q^E$ to $\alpha \in V^*$.Here X_α is dual to Y_α.*

Proof. It is clear that ι is a monomorphism. Now given $E \setminus F$ a face of K, if $\iota(v) = \sum_{\alpha\in E}\alpha(v)Y_\alpha$ belongs to $\mathbb{F}_q\langle E \setminus F\rangle$, then one must have $\alpha(v) = 0$ for all $\alpha \in F$, which implies that $v = 0$ since F is a spanning set of V^*. □

Now we can apply Proposition 7 to study $\mathbf{S}(V^*)/\iota^*(I_K)$. The Stanley-Reisner ideal I_K is generated by the monomials $\prod_{\alpha\in E\setminus H} X_\alpha$ where H is running on the set of hyperplanes in the affine space E. It follows that the polynomials $\Phi_H := \prod_{\alpha\in E\setminus H}\alpha$ generates the ideal $I := \iota^*(I_K)$ of $\mathbf{S}(V^*)$. Note that I is stable under the action of the affine subgroup $\mathrm{Aff}(E)$ of $\mathrm{GL}(V^*)$ and so the natural projection $\mathbf{S}(V^*) \twoheadrightarrow \mathbf{S}(V^*)/I$ is a map of $\mathrm{Aff}(E)$-modules.

Proposition 21. *For each* $0 \le i \le q^{n-1}-n$*, the natural projection* $\mathbf{S}(V^*) \twoheadrightarrow \mathbf{S}(V^*)/I$ *induces an isomorphism of* $\mathrm{Aff}(E)$*-modules* $\mathrm{Q}^i(\mathbf{S}(V^*)) \cong \mathrm{Q}^i(\mathbf{S}(V^*)/I)$.

Proof. Let H be a hyperplane in the affine space E. We need to prove that $f\cdot\Phi_H$ is $\mathscr{P}$-decomposable whenever

$$\deg(f) \le \left(q^{n-1}-n\right) - \deg(\Phi_H) = \left(q^{n-1}-n\right) - \left(q^{n-1}-q^{n-2}\right) = q^{n-2}-n.$$

Suppose $E = y+W$ and $H = y+W'$ where $y\in H$ and W' is a hyperplane in W. Choose a basis $\{y_1,\dots,y_{n-1}\}$ of W such that $\{y_1,\dots,y_{n-2}\}$ is a basis of W'. We have

$$\Phi_H = \prod_{w'\in W',a\in\mathbb{F}_q^\times}\left(w'+ay_{n-1}+y\right) = \prod_{a\in\mathbb{F}_q^\times}\mathbf{V}\left(y_1,\dots,y_{n-2},ay_{n-1}+y\right),$$

and so by Lemma 13,

$$\Phi_H = (-1)^{n-2}\sum_{i=0}^{n-2}\chi\left(\mathscr{P}^{q^{n-2}-1-i}\right)\left(e_i\cdot Y\right),$$

where $e_i := e_i(y_1^{q-1},\dots,y_{n-2}^{q-1})$ and $Y = \prod_{a\in\mathbb{F}_q^\times}(ay_{n-1}+y)$. By instability, $\mathscr{P}^{q^{n-2}-1-i}(f) = 0$ for all $0\le i\le n-2$ and so the result follows from the χ-trick. □

Proposition 22. $\mathbf{S}(V^*)/I$ *can be embedded as a* $\mathscr{P}$*-submodule into a direct product of* q^{n-1} *copies of* $\mathbf{R}(W,q-1)$.

Proof. The kernel of the natural map $\mathbf{S}(V^*) \to \prod_{y\in E}\mathbf{S}(V^*)/(y)$ is the principal ideal of $\mathbf{S}(V^*)$ generated by $\prod_{y\in E} y$. Since this generator belongs to I, this map induces an inclusion of $\mathscr{P}$-modules:

$$\mathbf{S}\left(V^*\right)/I \hookrightarrow \prod_{y\in E}\mathbf{S}\left(V^*\right)/I+(y).$$

It suffices now to prove that, for each $y\in E$, $I+(y) = I(W,q-1)+(y)$, and so $\mathbf{S}(V^*)/I+(y)\cong\mathbf{R}(W,q-1)$. Indeed, given H a hyperplane in E, if $y\notin H$ then Φ_H is divisible by y. If $y\in H$ then $E = y+W$ and $H = y+W'$ where W' is a hyperplane in W. Choose a basis $\{y_1,\dots,y_{n-1}\}$ of W such that $\{y_1,\dots,y_{n-2}\}$ is a basis of W'. We have then

$$\begin{aligned}\Phi_H = \prod_{w'\in W',a\in\mathbb{F}_q^\times}\left(w'+ay_{n-1}+y\right) &\equiv \prod_{a\in\mathbb{F}_q^\times}\mathbf{V}\left(y_1,\dots,y_{n-2},ay_{n-1}\right)\left((y)\right)\\ &= -\mathbf{V}\left(y_1,\dots,y_{n-2},y_{n-1}\right)^{q-1}\left((y)\right)\\ &= -\mathbf{V}_{W',W}^{q-1}\left((y)\right).\end{aligned}$$

The result follows. □

By Proposition 16, we know that the top-degree module $\mathbf{R}^{q^{n-1}-n}(W,q-1)$ is $\mathscr{P}$-indecomposable, and so the above proposition implies that the top-degree module $(\mathbf{S}(V^*)/I)^{q^{n-1}-n}$ is $\mathscr{P}$-indecomposable. Combining this with Proposition 21 yields an isomorphism:

$$\mathrm{Q}^{q^{n-1}-n}\left(\mathbf{S}\left(V^*\right)\right) \cong \left(\mathbf{S}\left(V^*\right)/I\right)^{q^{n-1}-n}.$$

To compute the dimension of $(\mathbf{S}(V^*)/I)^{q^{n-1}-n}$, we need to compute the reduced Euler characteristic $\widetilde{\chi}(K^*)$. Recall that the Alexander dual K^* of K is the simplicial complex on the vertex set E in which a face is a set of points F such that F does not affinely span E. Folowing Lusztig [9], let

$\mathscr{S}$ be the poset of all affine subspaces of E other than E (which is denoted by $S_I(E)$ in [9]). Then as in the linear case, the map of posets $f : \mathbf{p}(K^*) \to \mathscr{S}$ sending a face F of K^* to the affine subspace spanned by F is a homotopy equivalence. This is because for each affine subspace E' of E other than E, the poset $f^{-1}(S_{\le E'})$ has E' as its unique maximal element. The homotopy equivalence then again follows from the Quillen Fiber lemma. The reduced homology $\widetilde{\mathrm{H}}_*(\mathscr{S};\mathbb{F}_q)$ is only non-trivial in degree $n-2$, and in this degree, $\widetilde{\mathrm{H}}_{n-2}(\mathscr{S};\mathbb{F}_q)$, known as the *affine Steinberg module* associated to E [9, 1.14], is of dimension $(q-1)(q^2-1)\cdots(q^{n-1}-1)$. It follows that $\widetilde{\chi}(K^*)$ is up to sign equal to this dimension and so the proof of Theorem 4 is complete.

Remark 23. By Proposition 7, the elements $\prod_{y\in E\setminus B}\alpha$, where B is an affine basis of E, span the $\mathbb{F}_q$-vector space $(\mathbf{S}(V^*)/I)^{q^{n-1}-n}$. It would be interesting to figure out a basis of this space from a shelling of the matroid complex K.

Remark 24. The poset $\mathscr{S}$ above appeared in the study of discrete series (a.k.a. cuspidal representations) of $\mathrm{GL}_n(\mathbb{F}_q)$ by G. Lusztig [9]. Lusztig constructed in this work a distinguished discrete series, $\mathrm{D}(V)$, which is a free module of rank $(q-1)(q^2-1)\cdots(q^{n-1}-1)$ over the ring $W_{\mathbb{F}_q}$ of Witt vectors of $\mathbb{F}_q$. He showed that the restriction of $\mathrm{D}(V)$ to the affine subgroup $\mathrm{Aff}(E)$ is isomorphic to $\widetilde{\mathrm{H}}_{n-2}(\mathscr{S};W_{\mathbb{F}_q})$. The $\mathbb{F}_q$-reduction $\mathbb{F}_q\otimes_{W_{\mathbb{F}_q}}\mathrm{D}(V)$ is also shown by Lusztig to be isomorphic to $\widetilde{\mathrm{H}}_{n-2}(\mathscr{T}(V);\mathscr{G})$ where $\mathscr{T}(V)$ is the Tits building of V and $\mathscr{G}$ the coefficient system over $\mathscr{T}(V)$ which to any flag $\sigma=(V_{i_0}\subset V_{i_1}\subset\cdots\subset V_{i_m})$ associates the vector space $\mathscr{G}_\sigma = V_{i_0}$. We conjecture that the $\mathbb{F}_q$-reduction of $\mathrm{D}(V)$ and $\mathrm{Q}^{q^{n-1}-n}(\mathbf{S}(V^*))$ are isomorphic as $\mathbb{F}_q[\mathrm{GL}_n]$-modules.

6. The Steinberg summand of $\mathbf{R}_{n,2}$ and Brown–Gitler modules

In this section we take $q=2$ and we denote by $\mathscr{A}$ the mod 2 Steenrod algebra. Since $\mathscr{A}\cong\mathscr{P}(\mathbb{F}_2)$, each reduced power $\mathscr{P}^i$ of $\mathscr{P}(\mathbb{F}_2)$ is now denoted by Sq^i.

Recall that $\mathbf{R}_{n,2}=\mathbb{F}_2[x_1,\dots,x_n]/I_{n,2}$, where $I_{n,2}$ is the ideal generated by the orbit of the action of $\mathrm{GL}_n(\mathbb{F}_2)$ on $\mathbf{V}_n^2=\prod_{\lambda_i\in\mathbb{F}_2}(\lambda_1x_1+\cdots+\lambda_{n-1}x_{n-1}+x_n)^2$.

Put $d=2(2^n-1)-n=\sum_{j=1}^n(2^j-1)$.

We now prove Theorem 5 by establishing an isomorphism of $\mathscr{A}$-modules

$$\bigoplus_{j=0}^{n}\Sigma^{d-(2^j-1)}\mathrm{B}\left(2^j-1\right)\xrightarrow{\cong}\mathbf{R}_{n,2}\cdot\mathrm{e}_n.$$

Here $\mathrm{B}(k)$ denotes the Brown–Gitler module $\mathrm{B}(k):=\mathscr{A}/\mathrm{J}_k$ where J_k is the left ideal

$$\mathscr{A}\left\langle\chi\left(\mathrm{Sq}^i\right)\middle|2i>k\right\rangle.$$

We need the following lemmas:

Lemma 25. *There is an $\mathscr{A}$-generating set $\{\alpha_0,\alpha_1,\dots,\alpha_n\}$ with $|\alpha_j|=d-(2^j-1)$ for $\mathbf{R}_{n,2}\cdot\mathrm{e}_n$.*

Proof. Recall that $\mathbf{R}_{n,2}$ is a quotient of the polynomial algebra $\mathbf{S}_n:=\mathbb{F}_2[x_1,\dots,x_n]$ which is trivial in degree greater than d. In [6], M. Inoue proved that the Steinberg summand $\mathbb{F}_2[x_1,\dots,x_n]\cdot\mathrm{e}_n$ is minimally $\mathscr{A}$-generated by the classes

$$\mathrm{Sq}^{2^{i_1}}\cdots\mathrm{Sq}^{2^{i_n}}\left(\frac{1}{x_1\dots x_n}\right)$$

where $i_1>i_2>\cdots>i_n\ge 0$. In degrees less than or equal to d, these corresponds to the sequences $(i_1,\dots,i_n)=(n,\dots,\widehat{j},\dots,0)$ with $0\le j\le n$. Projecting these generators to $\mathbf{R}_{n,2}\cdot\mathrm{e}_n$ gives the desired result. □

Remark 26. The above generating set for $\mathbf{R}_{n,2}$ is *minimal* since we have $\mathrm{Q}^i(\mathbf{S}_n)\cong\mathrm{Q}^i(\mathbf{R}_{n,2})$ in the range $0\le i\le d$ (Theorem 2 (2a)). However we do not need this fact to produce the surjective map in the following lemma.

Lemma 27. *Evaluating on the generators $\alpha_0, \alpha_1, \ldots, \alpha_n$ yields an epimorphism of $\mathscr{A}$-modules:*

$$\bigoplus_{j=0}^{n} \Sigma^{|\alpha_j|}\mathscr{A} \twoheadrightarrow \mathbf{R}_{n,2} \cdot \mathrm{e}_n, \qquad \left(\Sigma^{|\alpha_j|}\theta_j\right)_{0 \le j \le n} \mapsto \sum_{j=0}^{n} \theta_j\left(\alpha_j\right),$$

which factors through the canonical projection $\bigoplus_{j=0}^{n} \Sigma^{|\alpha_j|}\mathscr{A} \twoheadrightarrow \bigoplus_{j=0}^{n} \Sigma^{|\alpha_j|}\mathscr{A}/\mathrm{J}_{d-|\alpha_j|}$. As a consequence, we have an epimorphism of $\mathscr{A}$-modules:

$$\bigoplus_{j=0}^{n} \Sigma^{d-(2^j-1)}\mathrm{B}\left(2^j-1\right) \twoheadrightarrow \mathbf{R}_{n,2} \cdot \mathrm{e}_n.$$

Proof. We have $\chi(\mathrm{Sq}^i)(\alpha_j) = 0$ in $\mathbf{R}_{n,2}$ whenever $2i > d - |\alpha_j|$ (Proposition 16). □

Remark 28. Note that [8, Proposition 5.4.4] $\Sigma^k\mathbb{D}\mathrm{B}(k)$ is isomorphic to the unstable Brown–Gitler module $\mathrm{J}(k)$ ($\mathbb{D}$ denoting the Spanier–Whitehead dual). Taking the Spanier–Whitehead dual of the epimorphism $\bigoplus_{j=0}^{n} \Sigma^{d-(2^j-1)}\mathrm{B}(2^j-1) \twoheadrightarrow \mathbf{R}_{n,2} \cdot \mathrm{e}_n$ yields a monomorphism:

$$\Sigma^d\mathbb{D}\left(\mathbf{R}_{n,2} \cdot \mathrm{e}_n\right) \hookrightarrow \bigoplus_{j=0}^{n} \mathrm{J}\left(2^j-1\right),$$

which is a monomorphism of unstable $\mathscr{A}$-modules since $\bigoplus_{j=0}^{n} \mathrm{J}(2^j-1)$ is unstable (cf. Remark 17).

Lemma 29. *In each degree, the dimension of $\mathbf{R}_{n,2} \cdot \mathrm{e}_n$ is greater than or equal to the dimension of $\bigoplus_{j=0}^{n} \Sigma^{d-(2^j-1)}\mathrm{B}(2^j-1)$.*

Proof. Using the exact sequences (known as Mahowald's exact sequences)

$$0 \to \Sigma^{2^{j-1}}\mathrm{B}\left(2^{j-1}\right) \to \mathrm{B}\left(2^j\right) \to \mathrm{B}\left(2^j-1\right) \to 0,$$

we see that the Poincaré series of $\bigoplus_{j=0}^{n} \Sigma^{d-(2^j-1)}\mathrm{B}(2^j-1)$ is the same as the Poincaré series of $\Sigma^{2^n-1-n}\mathrm{B}(2^n)$. Since $\mathrm{B}(2^n)$ has a basis $\{\chi(\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_n}) \mid 2^{n-1} \ge i_1 \ge 2i_2 \ge \cdots \ge 2^{n-1}i_n \ge 0\}$ [3], it follows that

$$\begin{aligned}
\mathbf{P}\left(\bigoplus_{j=0}^{n} \Sigma^{d-(2^j-1)}\mathrm{B}\left(2^j-1\right), t\right) &= \sum_{2^{n-1} \ge i_1 \ge 2i_2 \ge \cdots \ge 2^{n-1} i_n \ge 0} t^{2^n-1-n+i_1+i_2+\cdots+i_n} \\
&= \sum_{2^{n-1} \ge i_1 \ge 2i_2 \ge \cdots \ge 2^{n-1} i_n \ge 0} t^{(2^{n-1}+i_1)+(2^{n-2}+i_2)+\cdots+(1+i_n)-n} \\
&= \sum_{2^n \ge j_1 \ge 2j_2 \ge \cdots \ge 2^{n-1} j_n > 0} t^{j_1+j_2+\cdots+j_n-n}.
\end{aligned}$$

For the Poincaré series of $\mathbf{R}_{n,2} \cdot \mathrm{e}_n$, we prove that the set

$$\left\{\mathrm{Sq}^{j_1}\cdots\mathrm{Sq}^{j_n}\left(\frac{1}{x_1\cdots x_n}\right) \middle| \, 2^n \ge j_1 \ge 2j_2 \ge \cdots \ge 2^{n-1}j_n > 0\right\}$$

is linearly independent in $\mathbf{R}_{n,2}$. Note that each element in this set is fixed by the Steinberg idempotent e_n [13, Lemma 5.9]. By Lemma 15, we see that $I_{n,2}$ is contained in the ideal $(x_1^{2^n}, \ldots, x_n^2)$. The linear independence then follows from the natural projection

$$\mathbf{R}_{n,2} \twoheadrightarrow \mathbb{F}_2\left[x_1, \ldots, x_n\right] / \left(x_1^{2^n}, \ldots, x_n^2\right)$$

and the formula [13, Lemma 3.6]

$$\mathrm{Sq}^{j_1}\cdots\mathrm{Sq}^{j_n}\left(\frac{1}{x_1\cdots x_n}\right) = x_1^{j_1-1}\cdots x_n^{j_n-1} + \text{monomials of lower order}$$

(in the lexicographical order starting at the left). The proposition is proved. □

Proof of Theorem 5. The epimorphism $\bigoplus_{j=0}^{n}\Sigma^{d-(2^j-1)}\mathrm{B}(2^j-1) \twoheadrightarrow \mathbf{R}_{n,2}\cdot \mathrm{e}_n$ is an isomorphism since in each degree the dimension of $\mathbf{R}_{n,2}\cdot \mathrm{e}_n$ is greater than or equal to the dimension of $\bigoplus_{j=0}^{n}\Sigma^{d-(2^j-1)}\mathrm{B}(2^j-1)$. □

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