Nonlinear Analysis: Modelling and Control, Online First, 1–13
https://doi.org/10.15388/namc.2023.28.30517

Global attractive set of neural networks with neutral item*

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Received: July 25, 2022 / **Revised:** December 23, 2022 / **Published online:** January 2, 2023

Abstract. This paper investigates the global attractive set of neural networks with neutral item. To better deal with the neutral terms, different types of activation functions are considered. Based on matrix measures, inequality techniques, and Lyapunov theory, three new types of Lyapunov functions are designed to find the global attractive set of the system. We give out a simulation example to verify the validity of theory results. The result is very inclusive, whether the system has equilibrium or not. As long as the system is stable, we can find its global attractive set.

Keywords: global attractive set, neural networks, neutral, linear matrix inequality.

1 Introduction

The artificial neural network is a highly simplified approximation of human brain neural network through mathematical and physical methods from the perspective of information processing. It has the advantages of fault tolerance, versatility, and adaptability. It has been widely used in speech recognition, visual technology, artificial intelligence, image recognition, associative memory, speech translation, and other fields. The application of the neural networks is related to their dynamic behavior. According to different application requirements, its dynamic characteristics have different requirements. Many scholars have some interesting conclusions in this area [9, 17, 18]. Especially, the global

*This research was funded by the Science and Technology Research Program of Chongqing Municipal Education Commission (Nos. KJQN202001220, KJQN202101220, KJZD-M202001201, KJQN202101212), the Natural Science Foundation Project of Chongqing (Nos. cstc2021jcyj-msxmX0051, cstc2019jcyj-msxmX0381, cstc2021jcyj-msxmX0949), the National Natural Science Foundation of China (No. 11771156), the Science and Technology Innovation Project of Economic Circle Construction in Chengdu-Chongqing Area (No. KJCX2020047), the Youth Fund of Chongqing Three Gorges University (No. 18QN04).

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exponential stability of several types of neural networks is discussed in [2, 4, 5]. By using delayed impulsive control, quantized intermittent control and synchronization control are investigated in [7, 16, 24–26].

The focus of neural network research is the dynamic characteristics of orbits near the equilibrium point of the system. Nonetheless, in many practical problems, the equilibrium of the system is unstable or even nonexistent, so the stability of the system needs to be considered. If the global attractive set of the system is identified, we just need to focus on the dynamics within the set because the equilibrium point of the system only exists in the global attractive set [13, 15].

In real life, the development of many things is affected by some past states, that is, the phenomenon of time delay. Due to the limitation of signal propagation speed, time delay is widespread [14]. For example, when sending emails, pictures, or videos via WeChat, there will be a time delay from the beginning to the success of sending. When the time delay phenomenon occurs, the system will be oscillated or unstable, resulting in network paralysis and other phenomena. There are many results in stability analysis of neural networks with time delays in [6, 8, 19]. In the study of these delays, one kind of delay is worth paying attention to, that is, neutral delay. It not only considers the influence of past state on present state of nonlinear time-delay systems, but also considers the influence of past state changes on present state. The neural networks with neutral items are more complex than the general neural networks of the same dimension. It is a new hot research direction [3, 10, 27]. Liao et al. considered the exponential stability for neutral-type neural networks [12]. Ali et al. discussed finite-time stability of systems with random delays [1].

Based on the above analysis, this paper studies the global attractive set of neural networks with neutral item. The main contributions are presented as follows:

- (I) The neural networks with neutral items are more complex than the general neural networks. We will make some efforts on finding various Lyapunov functions and global attractive sets for system with neutral items.
- (II) External inputs affect the stability of the system. This paper attempts to study the effect of external input with bounded function on the global attractive set and stability.

By matrix measures, inequality techniques, and Lyapunov theory, we can find the global attractive set of the system. We give out simulation examples to verify the validity of theory results.

Notations. In this paper, $\mathbb{R}^n$ represent the n-dimensional real space. $\mathbb{R}^{m\times n}$ represent the set of $m \times n$ real matrix. T and $*$ represent the transposition of matrix and conjugation transposition.

2 Preliminaries

In this paper, we consider the following neural networks model:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Af\bigl(x(t)\bigr) + Bf\bigl(x\bigl(t-h(t)\bigr)\bigr) + Mg\bigl(\dot{x}\bigl(t-\sigma(t)\bigr)\bigr) + U(t), \quad (1)$$

where $x(t) = \{x_1(t), x_2(t), \dots, x_n(t)\}^{\mathrm{T}} \in \mathbb{R}^n$. $C = \operatorname{diag}\{c_1, c_2, \dots, c_n\}$, $c_i > 0$, $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, $B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, and $M = (m_{kj})_{n\times n} \in \mathbb{R}^{n\times n}$ are connection weight matrices. $f(x(t)) \in \mathbb{R}^n$, $f(x(t-h(t))) \in \mathbb{R}^n$, and $g(\dot{x}(t-\sigma(t))) \in \mathbb{R}^n$ are activation functions. $h(t)$ is the delay, $0 \leqslant h(t) \leqslant \bar{h}$. $\sigma(t)$ is the neutral delay with $0 \leqslant \sigma(t) \leqslant \sigma$, and let $\tau = \max\{\bar{h}, \sigma\}$. $U(t) \in \mathbb{R}^n$ is the external input, $U(t) \leqslant U, U = (u_1, u_2, \dots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$.

The initial condition of neural networks (1) is given as

$$x(s) = \varphi(s), \quad \dot{x}(s) = \phi(s), \quad s \in [-\tau, 0].$$

We consider two classes of continuous activation functions:

$$\begin{aligned} \mathfrak{B} &= \bigl\{ f(\cdot) \bigm| \exists\, k_i > 0,\ \bigl|f_i(x_i)\bigr| \leqslant k_i\ \forall x_i \in \mathbb{R},\ i \in \Gamma = \{1, 2, \dots, n\} \bigr\}, \\ &\quad K = \operatorname{diag}\{k_1, k_2, \dots, k_n\} \\ \mathfrak{S} &= \left\{ f(\cdot) \;\middle|\; \exists\, l_i, h_i, l_i \leqslant \frac{f_i(x) - f_i(y)}{x - y} \leqslant h_i,\ x \neq y,\ i \in \Gamma \right\}, \\ &\quad L = \operatorname{diag}\{l_1, l_2, \dots, l_n\}, \quad H = \operatorname{diag}\{h_1, h_2, \dots, h_n\}. \end{aligned}$$

Remark 1. If $f(\cdot) \in \mathfrak{S}$, we can know that $f(\cdot)$ is neither bounded nor monotonously, which can make the results with better applicability. The constants l_i and h_i can be any real value.

Definition 1. (See [22].) The set $\Omega \subset R^n$ is called to be a global attractive set (GAS) of (1) if Ω posses an open neighborhood D for all $\varphi \in D$ and $x(t, t_0, \varphi)$ converges to Ω as $t \to \infty$.

Definition 2. (See [21].) For a real matrix $C = (c_{ij})_{n\times n}$, the matrix measure C is defined as $\mu_p(C) = \lim_{\Delta t \to 0^+} (\|E_n + \Delta t C\|_p - 1)/\Delta t$, where $p = 1, 2, \infty$. The matrix measures are obtained as $\mu_1(C) = \max_j\{c_{jj} + \sum_{i=1,\, i\neq j}^n |c_{ij}|\}$, $\mu_2(C) = \lambda_{\max}(C^{\mathrm{T}} + C)/2$, $\mu_\infty(C) = \max_i\{c_{ii} + \sum_{j=1,\, j\neq i}^n |c_{ij}|\}$.

$\|C\|_p$ is the matrix norm of C, where $\|C\|_1 = \max_j \sum_{i=1}^n |c_{ij}|$, $\|C\|_2 = \sqrt{\lambda_{\max}(C^{\mathrm{T}} C)}$, $\|C\|_\infty = \max_i \sum_{j=1}^n |c_{ij}|$. For $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = \sqrt{x^{\mathrm{T}} x}$, $\|x\|_\infty = \max_{1 \leqslant i \leqslant n} |x_i|$.

Remark 2. Take $C = \left(\begin{smallmatrix} 2 & 2 \\ 1 & -9 \end{smallmatrix}\right)$ for example, therefore $\|-C\|_1 = \|C\|_1 = 11$, $\|-C\|_2 = \|C\|_2 = 9.2364$, $\|C\|_\infty = \|-C\|_\infty = 10$, $\mu_1(C) = 3$, $\mu_2(C) = 2.2009$, $\mu_\infty(C) = 4$, $\mu_1(-C) = 11$, $\mu_2(-C) = 9.2009$, $\mu_\infty(-C) = 10$. Hence, we can get that $\|-C\|_p = \|C\|_p$, $\mu_p(-C) \neq \mu_p(C)$.

It shows that matrix measure is more sensitive to sign, and the results base on matrix measure are more accurate and applicable.

Lemma 1. (*See* [20].) *Let* $r, \xi, \eta, \tau > 0$. *When the function* $V(t) \geqslant 0$ *and* $D^+V(t) \leqslant r - \xi V(t) + \eta \bar{V}(t)$, $t \geqslant t_0$, *there exists* $\sigma > 0$ *such that* $-\xi + \eta \leqslant -\sigma$, *and one has*

$$V(t) \leqslant \begin{cases} \frac{r}{\sigma} + (\bar{V}(t_0) - \frac{r}{\sigma}) \mathrm{e}^{-\mu^*(t - t_0)} & \textit{if}\ \bar{V}(t_0) > \frac{r}{\sigma}, \\ \frac{r}{\sigma} & \textit{if}\ \bar{V}(t_0) \leqslant \frac{r}{\sigma}. \end{cases}$$

Here μ^* *is the unique positive root of* $\mu - \xi + \eta e^{\mu\tau} = 0$, $\bar{V}(t) = \sup_{t-\tau \leqslant s \leqslant t} V(s)$, $D^+ v(t) = \overline{\lim}_{\Delta t \to 0^+} (v(t + \Delta t) - v(t))/\Delta t$.

Lemma 2. (*See* [20].) *For all* $A \in [\underline{A}, \bar{A}]$, *we have*

$$\|A\|_p \leqslant \|A^*\|_p + \|A_*\|_p,$$

where $A_* = (\bar{A} - \underline{A})/2$, $A^* = (\bar{A} + \underline{A})/2$, $p = 1, 2, \infty$.

Lemma 3. (*See* [11].) *Let* $S = \left(\begin{smallmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{smallmatrix}\right) < 0$ *with* $S_{11} = S_{11}^{\mathrm{T}}$, $S_{22} = S_{22}^{\mathrm{T}}$. *Then the following obtained*:

$$\begin{aligned} S_{22} &< 0, & S_{11} - S_{12} S_{22}^{-1} S_{12}^{\mathrm{T}} &< 0, \\ S_{11} &< 0, & S_{22} - S_{12}^{\mathrm{T}} S_{11}^{-1} S_{12} &< 0. \end{aligned}$$

Lemma 4. *For matrices* $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{R}^{n\times n}$ *and reversible matrices* X, Y, Z, *denote*

$$\begin{aligned} \Phi_1 &\triangleq \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} X^{-1} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} Y^{-1} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Z^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^{\mathrm{T}}, \\ \Phi_2 &\triangleq \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 & 0 \\ 0 & Y^{-1} & 0 \\ 0 & 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}^{\mathrm{T}}, \end{aligned}$$

Then $\Phi_1 = \Phi_2$.

Proof. According to the matrix algorithm, we have

$$\begin{aligned} \Phi_1 &= \begin{pmatrix} A_1 X^{-1} A_1 & A_1 X^{-1} A_2 \\ A_2 X^{-1} A_1 & A_2 X^{-1} A_2 \end{pmatrix} + \begin{pmatrix} B_1 Y^{-1} B_1 & B_1 Y^{-1} B_2 \\ B_2 Y^{-1} B_1 & B_2 Y^{-1} B_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} C_1 Z^{-1} C_1 & C_1 Z^{-1} C_2 \\ C_2 Z^{-1} C_1 & C_2 Z^{-1} C_2 \end{pmatrix}, \\ \Phi_2 &= \begin{pmatrix} A_1 X^{-1} & B_1 Y^{-1} & C_1 Z^{-1} \\ A_2 X^{-1} & B_2 Y^{-1} & C_2 Z^{-1} \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \\ C_1 & C_2 \end{pmatrix} = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{11} &= A_1 X^{-1} A_1 + B_1 Y^{-1} B_1 + C_1 Z^{-1} C_1, \\ \Delta_{12} &= A_1 X^{-1} A_2 + B_1 Y^{-1} B_2 + C_1 Z^{-1} C_2, \\ \Delta_{21} &= A_2 X^{-1} A_1 + B_2 Y^{-1} B_1 + C_2 Z^{-1} C_1, \\ \Delta_{22} &= A_2 X^{-1} A_2 + B_2 Y^{-1} B_2 + C_2 Z^{-1} C_2. \end{aligned}$$

Then, by the properties of matrix operations, $\Phi_1 = \Phi_2$. □

Lemma 5. (*See* [23].) *If* P *is a positive definite matrix,* $a, b \in \mathbb{R}^n$, *we have*

$$2a^{\mathrm{T}} b \leqslant a^{\mathrm{T}} P^{-1} a + b^{\mathrm{T}} P b.$$

3 Main results

Theorem 1. *If* $f(\cdot) \in \mathfrak{S}$, $g(\cdot) \in \mathfrak{B}$, *there exists* $\delta_1 > 0$ *such that* $\Phi_1 + \Phi_2 \leqslant -\delta_1 < 0$, *then the set*

$$\Omega_1 = \left\{ x \in \mathbb{R}^n \;\middle|\; \|x\|_p \leqslant \frac{\|\gamma_1\|_p}{\delta_1} \right\}$$

is a GAS of system (1). *Here* $\gamma_1 = \|M\|_p K + \|U\|_p$, $\Phi_1 = \mu_p(-C) + \bar{H}\|A\|_p$, $\Phi_2 = \bar{H}\|B\|_p$, $\bar{H} = \max_{1\leqslant i\leqslant n}\{|l_i|, |h_i|\}$ *with* $p = 1, 2, \infty$.

Proof. Let $V(t) = \|x(t)\|_p$. Since $f(\cdot) \in \mathfrak{S}$, we can get that $\|f(x(t))\|_p \leqslant \bar{H}\|x(t)\|_p$, $\|f(x(t-h(t)))\|_p \leqslant \bar{H}\|x(t-h(t))\|_p$. In the view of Definition 2 and Lemma 2, the following obtained:

$$\begin{aligned}
\mathrm{D}^+V(t) &\leqslant \overline{\lim_{\Delta t\to 0^+}} \frac{\|x(t+\Delta t)\|_p - \|x(t)\|_p}{\Delta t} \\
&= \overline{\lim_{\Delta t\to 0^+}} \frac{\|x(t) + \Delta t\dot{x}(t) + o(\Delta t)\|_p - \|x(t)\|_p}{\Delta t} \\
&\leqslant \overline{\lim_{\Delta t\to 0^+}} \frac{\|x(t) + \Delta t(-C)x(t) - \|x(t)\|_p}{\Delta t} + \overline{\lim_{\Delta t\to 0^+}} \frac{\|Af(x(t))\|_p \Delta t}{\Delta t} \\
&\quad + \overline{\lim_{\Delta t\to 0^+}} \frac{\|Bf(x(t-h(t)))\|_p \Delta t}{\Delta t} + \|M\|_p K + \|U\|_p \\
&\leqslant \overline{\lim_{\Delta t\to 0^+}} \frac{\|E_n + \Delta t(-C)\|_p - 1}{\Delta t} \big\|x(t)\big\|_p + \|A\|_p \bar{H} \big\|x(t)\big\|_p \\
&\quad + \|B\|_p \bar{H} \big\|x\big(t-h(t)\big)\big\|_p + \|M\|_p K + \|U\|_p \\
&\leqslant \Phi_1 \big\|V(t)\big\|_p + \Phi_2 \overline{\big\|V(t)\big\|}_p + \|M\|_p K + \|U\|_p.
\end{aligned}$$

According to Lemma 1, for $\|\varphi(s)\|_p > \|\gamma_1\|_p/\delta_1$, $s \in [-\tau, 0]$, $V(t) \leqslant \|\gamma_1\|_p/\delta_1 + M_1 \mathrm{e}^{-\mu_1 t}$, where $M_1 = \sup(V(s) - \|\gamma_1\|_p/\delta_1)$, $-\tau \leqslant s \leqslant 0$, μ_1 is the root of $\mu + \Phi_1 + \Phi_2 \mathrm{e}^{\mu\tau} = 0$. Then $\Omega_1 = \{x \in \mathbb{R}^n \mid \|x\|_p \leqslant \|\gamma_1\|_p/\delta_1\}$ is a GAS of system (1).

For $\|\varphi(s)\|_p \leqslant \|\gamma_1\|_p/\delta_1$, $s \in [-\tau, 0]$, following Lemma 1, $\|x\|_p \leqslant \|\gamma_1\|_p/\delta_1$, $t \geqslant 0$. The set Ω_1 is also a GAS of system (1). □

Theorem 2. *If* $f(\cdot) \in \mathfrak{S}, g(\cdot) \in \mathfrak{B}$, $c_i > h_i \sum_{j=1}^n |a_{ji}| + (h_i/(1-\mu)) \sum_{j=1}^n |b_{ji}|$, $0 \leqslant \dot{h}(t) \leqslant \mu \leqslant 1$, *then*

$$\Omega_2 = \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n \big|x_i(t)\big| \leqslant \frac{\sum_{j=1}^n |m_{ij}| k_j + \sum_{i=1}^n |u_i|}{\min_{1\leqslant i\leqslant n} \Xi} \right\}$$

is a GAS of system (1), *where* $\Xi = c_i - h_i \sum_{j=1}^n |a_{ji}| - (h_i/(1-\mu)) \sum_{j=1}^n |b_{ji}|$.

Proof. Functional

$$V(t) = \sum_{i=1}^n \left(\big|x_i(t)\big| + \frac{1}{1-\mu} \sum_{j=1}^n |b_{ij}| \int\limits_{t-h(t)}^{t} \big|f_j\big(x_j(s)\big)\big| \,\mathrm{d}s \right)$$

is adopted, then

$$\begin{aligned}
&D^+V(t)\\
&\quad\leqslant \sum_{i=1}^{n}\Biggl(-c_i\bigl|x_i(t)\bigr| + \sum_{j=1}^{n}|a_{ij}|\bigl|f_j\bigl(x_j(t)\bigr)\bigr| + \sum_{j=1}^{n}|b_{ij}|\bigl|f_j\bigl(x_j\bigl(t-h(t)\bigr)\bigr)\bigr|\\
&\qquad + \sum_{j=1}^{n}|m_{ij}|\bigl|g_j\bigl(\dot{x}_j\bigl(t-\sigma(t)\bigr)\bigr)\bigr| + |u_i|\Biggr) + \frac{1}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\bigl|f_j\bigl(x_j(t)\bigr)\bigr|\\
&\qquad - \frac{1}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\bigl(1-\dot{h}(t)\bigr)\bigl|f_j\bigl(x_j\bigl(t-h(t)\bigr)\bigr)\bigr|\\
&\quad\leqslant \sum_{i=1}^{n}\Biggl(-c_i\bigl|x_i(t)\bigr| + \sum_{j=1}^{n}|a_{ij}|h_j|\bigl|x_j(t)\bigr| + \frac{1}{1-\mu}\sum_{j=1}^{n}h_j|b_{ij}|\bigl|x_j(t)\bigr|\\
&\qquad + \sum_{j=1}^{n}|m_{ij}|k_j + |u_i|\Biggr) + \sum_{j=1}^{n}|b_{ij}|\biggl(1-\frac{1}{1-\mu}\bigl(1-\dot{h}(t)\bigr)\biggr)\bigl|f_j\bigl(x_j\bigl(t-h(t)\bigr)\bigr)\bigr|\\
&\quad\leqslant -\sum_{i=1}^{n}\Biggl(c_i - h_i\sum_{j=1}^{n}|a_{ij}| - \frac{h_i}{1-\mu}\sum_{j=1}^{n}|b_{ij}|\Biggr)\bigl|x_j(t)\bigr| + \sum_{j=1}^{n}|m_{ij}|k_j + \sum_{i=1}^{n}|u_i|\\
&\quad\leqslant -\min_{1\leqslant i\leqslant n}\Biggl\{c_i - h_i\sum_{j=1}^{n}|a_{ji}| - \frac{h_i}{1-\mu}\sum_{j=1}^{n}|b_{ji}|\Biggr\}\sum_{i=1}^{n}\bigl|x_i(t)\bigr|\\
&\qquad + \sum_{j=1}^{n}|m_{ij}|k_j + \sum_{i=1}^{n}|u_i|.
\end{aligned}$$

If $x \in \mathbb{R}^n \setminus \Omega_2$, then $D^+V(t)|_{(1)} < 0$, that is, for any $\varphi(s) \in \Omega_2$, $x(t,t_0,\varphi) \subseteq \Omega_2$, $t \geqslant t_0$. For $\varphi(s) \notin \Omega_2$, there exists $T > 0$, $x(t,t_0,\varphi) \subseteq \Omega_2$, where $t \geqslant t_0 + T$. In view of Definition 1, we can know that Ω_2 is a GAS of system (1). $\square$

Theorem 3. *Let $f(\cdot) \in \mathfrak{S}, g(\cdot) \in \mathfrak{B}$, R is a positive definite matrix, D, F, Q, P, S are positive diagonal matrices, and the following LMI holds*:

$$\begin{pmatrix}
\widetilde{\Sigma} & (R-LD)A-CD & (R-LD)B & (R-LD)M & R-LD\\
* & DA+A^{\mathrm{T}}D-P & DB & DM & D\\
* & * & -Q & 0 & 0\\
* & * & 0 & -S & 0\\
* & * & 0 & 0 & -F
\end{pmatrix} < 0, \tag{2}$$

$$WQW \leqslant R. \tag{3}$$

Then

$$\Omega_3 = \biggl\{x(t) \in \mathbb{R}^n \biggm| x^{\mathrm{T}}(t)Rx(t) \leqslant \frac{\gamma_2}{\delta_2}\biggr\}$$

is a GAS of system (1)*, where* $\delta_2 \leqslant \varepsilon$*,* $\gamma_2 = K^{\mathrm{T}} S K + U^{\mathrm{T}} F U$*,* $\widetilde{\Sigma} = (1+\varepsilon)(R + D(H-L) - RC - CR + 2LDC + WPW)$*,* $w_i = \max |l_i|, |h_i|$*,* $W = \operatorname{diag}\{w_1, w_2, \ldots, w_n\}$*,* $i \in \Gamma$*.*

Proof. Let for $f(\cdot) \in \mathfrak{S}$,

$$V(t) = x^{\mathrm{T}}(t) R x(t) + 2 \sum_{i=1}^{n} d_i \int\limits_{0}^{x_i(t)} \bigl(f_i(s) - L_i s\bigr)\, \mathrm{d}s,$$

P is a positive diagonal matrix, and let $\zeta = (x(t), f(x(t))^{\mathrm{T}}$, $\check{R} = R - LD$. Then

$$\begin{aligned}
&2\bigl(x^{\mathrm{T}}(t)R + f^{\mathrm{T}}\bigl(x(t)D - x^{\mathrm{T}}(t)LD\bigr)\bigr)\bigl(-Cx(t) + Af\bigl(x(t)\bigr)\bigr) \\
&\quad = 2x^{\mathrm{T}}(t)(-RC + LDC)x(t) + 2f^{\mathrm{T}}\bigl(x(t)\bigr)(DA)f\bigl(x(t)\bigr) \\
&\qquad + 2x^{\mathrm{T}}(t)(RA - LDA - CD)f\bigl(x(t)\bigr) \\
&\quad \leqslant 2x^{\mathrm{T}}(t)(-RC + LDC)x(t) + 2f^{\mathrm{T}}\bigl(x(t)\bigr)(DA)f\bigl(x(t)\bigr) \\
&\qquad + 2x^{\mathrm{T}}(t)(RA - LDA - CD)f\bigl(x(t)\bigr) \\
&\qquad + x^{\mathrm{T}}(t)WPWx(t) - f^{\mathrm{T}}\bigl(x(t)\bigr)Pf\bigl(x(t)\bigr) \\
&\quad = \zeta^{\mathrm{T}} \Sigma_1 \zeta,
\end{aligned} \tag{4}$$

where

$$\Sigma_1 = \begin{pmatrix} \Lambda & \check{R}A - CD \\ * & DA + A^{\mathrm{T}}D - P \end{pmatrix}, \qquad \Lambda = -RC - CR + 2LDC + WPW.$$

Using Lemma 5, for a given positive diagonal matrix Q, S, F, the following inequality holds*:*

$$\begin{aligned}
&2\bigl(x^{\mathrm{T}}(t)RB + f^{\mathrm{T}}\bigl(x(t)DB - x^{\mathrm{T}}(t)LDB\bigr)\bigr)f\bigl(x\bigl(t - h(t)\bigr)\bigr) \\
&\quad \leqslant \zeta^{\mathrm{T}} \begin{pmatrix} \check{R}B \\ DB \end{pmatrix} Q^{-1} \begin{pmatrix} \check{R}B \\ DB \end{pmatrix}^{\mathrm{T}} \zeta x^{\mathrm{T}}\bigl(t - h(t)\bigr)WQWx\bigl(t - h(t)\bigr),
\end{aligned}$$

$$\begin{aligned}
&2\bigl(x^{\mathrm{T}}(t)RM + f^{\mathrm{T}}\bigl(x(t)DM - x^{\mathrm{T}}(t)LDM\bigr)\bigr)g\bigl(\dot{x}\bigl(t - \sigma(t)\bigr)\bigr) \\
&\quad \leqslant \bigl(x^{\mathrm{T}}(t)RM + f^{\mathrm{T}}\bigl(x(t)DM - x^{\mathrm{T}}(t)LDM\bigr)\bigr)S^{-1}\bigl(x^{\mathrm{T}}(t)RM \\
&\qquad + f^{\mathrm{T}}\bigl(x(t)DM - x^{\mathrm{T}}(t)LDM\bigr)\bigr)^{\mathrm{T}} + g\bigl(\dot{x}\bigl(t - \sigma(t)\bigr)\bigr)^{\mathrm{T}} Sg\bigl(\dot{x}\bigl(t - \sigma(t)\bigr)\bigr) \\
&\quad \leqslant \zeta^{\mathrm{T}} \begin{pmatrix} \check{R}M \\ DM \end{pmatrix} S^{-1} \begin{pmatrix} \check{R}M \\ DM \end{pmatrix}^{\mathrm{T}} \zeta + K^{\mathrm{T}}SK
\end{aligned}$$

and

$$2\bigl(x^{\mathrm{T}}(t)R + f^{\mathrm{T}}\bigl(x(t)D - x^{\mathrm{T}}(t)LD\bigr)\bigr)U \leqslant \zeta^{\mathrm{T}} \begin{pmatrix} \check{R} \\ D \end{pmatrix} F^{-1} \begin{pmatrix} \check{R} \\ D \end{pmatrix}^{\mathrm{T}} \zeta + U^{\mathrm{T}}FU. \tag{5}$$

Based on Lemma 4 and (4) $\sim$ (5), one has

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leqslant x^{\mathrm{T}}\bigl(t - h(t)\bigr)WQWx\bigl(t - h(t)\bigr) \\
&\qquad + \zeta^{\mathrm{T}}\Sigma_1\zeta + \zeta^{\mathrm{T}}\Sigma_2\zeta + K^{\mathrm{T}}SK + U^{\mathrm{T}}FU,
\end{aligned} \tag{6}$$

where

$$\varSigma_2 = \varSigma_3 \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & F^{-1} \end{pmatrix} \varSigma_3^{\mathrm{T}} \quad \text{and} \quad \varSigma_3 = \begin{pmatrix} \check{R}B & \check{R}M & \check{R} \\ DB & DM & D \end{pmatrix}.$$

Using Lemma 3 to (2), let $\widetilde{R} = R + D(H - L)$, then we get

$$\begin{pmatrix} -(1+\varepsilon)(\widetilde{R} + \varLambda) & \check{R}A - CD \\ * & DA + A^{\mathrm{T}}D - P \end{pmatrix} + \varSigma_2 < 0,$$

and it is equal to

$$\varSigma_1 + \varSigma_2 < \begin{pmatrix} -(1+\varepsilon)\widetilde{R} & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Based on (3), (6), and (7), we have

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leqslant -(1+\varepsilon)x^{\mathrm{T}}(t)\widetilde{R}x(t) + x^{\mathrm{T}}\bigl(t - h(t)\bigr)Rx\bigl(t - h(t)\bigr) \\ &\quad + K^{\mathrm{T}}SK + U^{\mathrm{T}}FU. \end{aligned}$$

Since $V(t) \leqslant x^{\mathrm{T}}(t)\widetilde{R}x(t)$,

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant -(1+\varepsilon)V(t) + \bar{V}(t) + K^{\mathrm{T}}SK + U^{\mathrm{T}}FU.$$

According to Lemma 1, if $x^{\mathrm{T}}(t)Rx(t) > \gamma_2/\delta_2$, then $V(t) \leqslant \gamma_2/\delta_2 + M_2\mathrm{e}^{-\mu_2 t}$, and for $x^{\mathrm{T}}(t)Rx(t) \leqslant \gamma_2/\delta_2$, we have $V(t) \leqslant \gamma_2/\delta_2$, where $M_2 = \sup(V(s) - \gamma_2/\delta_2)$, $-\tau \leqslant s \leqslant 0$, μ_2 is the root of $\mu - (1+\varepsilon) + \mathrm{e}^{\mu\tau} = 0$. According to Definition 1, $\varOmega_3 = \{x(t) \in \mathbb{R}^n \mid x^{\mathrm{T}}(t)Px(t) \leqslant \gamma_2/\delta_2\}$ is a GAS of system (1). □

4 Illustrative examples

We will give two examples and simulations to demonstrate the validity of our theory results.

Example 1. Consider the following neural networks model:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -Cx(t) + Af\bigl(x(t)\bigr) + Bf\bigl(x\bigl(t - h(t)\bigr)\bigr) \\ &\quad + Mg\bigl(\dot{x}\bigl(t - \sigma(t)\bigr)\bigr) + U(t), \end{aligned} \tag{8}$$

where $f(x) = (0.05(|x-1| - |x+1|),\ 0.25(|x-1| - |x+1|))^{\mathrm{T}}$, $g(x) = 0.4\tanh x$. $h(t) = (0.25\sin(t) + 0.8,\ 0.25\sin(t) + 0.65)^{\mathrm{T}}$, $\sigma(t) = (0.4\cos(t) + 0.6,\ 0.2\cos(t) + 0.8)^{\mathrm{T}}$, $U(t) = (\sin(3t),\ -\cos(3t))^{\mathrm{T}}$,

$$C = \begin{pmatrix} 5.98 & 0 \\ 0 & 8.85 \end{pmatrix}, \qquad A = \begin{pmatrix} 2.78 & -0.52 \\ 2.28 & 1.59 \end{pmatrix},$$

$$B = \begin{pmatrix} 1.55 & -0.84 \\ 1.06 & 0.77 \end{pmatrix}, \qquad M = \begin{pmatrix} 1.43 & 1.31 \\ -1.36 & 0.87 \end{pmatrix}.$$

Obviously, $U = (1,1)^{\mathrm{T}}$, $\mu = 0.25$, $\sigma = 1$. The activation function $f(x) \in \mathfrak{S}$ with $-l_i = h_i = \operatorname{diag}\{0.1, 0.5\}$, and $g(x) \in \mathfrak{B}$ with $k_i = 0.4$. According to Theorem 2, the GAS of system (8) is $\Omega = \{x(t) \in \mathbb{R}^2 || x(1)| + |x(2)| \leqslant 2.1886\}$. Figure 1 shows the simulation results.

Example 2. Consider the following neural networks model:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= -Cx(t) + Af\bigl(x(t)\bigr) + Bf\bigl(x\bigl(t - h(t)\bigr)\bigr) \\ &\quad + Mg\bigl(\dot{x}\bigl(t - \sigma(t)\bigr)\bigr) + U(t), \end{aligned} \tag{9}$$

where

$$C = \begin{pmatrix} 6.68 & 0 & 0 \\ 0 & 4.76 & 0 \\ 0 & 0 & 5.74 \end{pmatrix}, \qquad A = \begin{pmatrix} 3.05 & 3.22 & 1.65 \\ 2.23 & 1.57 & 2.32 \\ 3.56 & 1.23 & 0.36 \end{pmatrix},$$

$$B = \begin{pmatrix} 3.25 & -0.63 & 1.37 \\ 1.26 & 0.47 & 0.68 \\ 0.26 & 1.26 & 0.77 \end{pmatrix}, \qquad M = \begin{pmatrix} 1.43 & 0.75 & 1.61 \\ 0.42 & 0.63 & 0.38 \\ 0.34 & -1.16 & 0.47 \end{pmatrix},$$

$U(t) = (-2 - \sin(3t)'\,2 - \cos(3t),\ 2 + \cos(3t))^{\mathrm{T}}$, $h(t) = 0.65 + 0.35\sin^2(t)$, $\sigma(t) = 0.6 + 0.4\sin(t)$, $f(x) = 0.1(-2 + |x-1| + |x+1|)$, $g(x) = 0.2\tanh(x)$. Obviously, $U = (-1,1,1)^{\mathrm{T}}$, $\mu = 0.7$, $\sigma = 1$. The activation function $f(x) \in \mathfrak{S}$ with $-l_i = h_i = \operatorname{diag}\{0.2, 0.2, 0.2\}$, and $g(x) \in \mathfrak{B}$ with $k_i = 0.2$.

We have the following results via the LMI control toolbox:

$$P = \begin{pmatrix} 0.3470 & 0.0197 & 0.0662 \\ 0.0197 & 0.4544 & 0.0435 \\ 0.0662 & 0.0435 & 0.2776 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.0095 & 0 & 0 \\ 0 & -0.2049 & 0 \\ 0 & 0 & -0.2473 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1.4901 & 0 & 0 \\ 0 & 1.5397 & 0 \\ 0 & 0 & 0.8676 \end{pmatrix}, \qquad R = \begin{pmatrix} 9.6567 & 0 & 0 \\ 0 & 7.8347 & 0 \\ 0 & 0 & 7.2108 \end{pmatrix},$$

$$S = \begin{pmatrix} 4.3926 & 0 & 0 \\ 0 & 4.9744 & 0 \\ 0 & 0 & 4.4830 \end{pmatrix}, \qquad H = \begin{pmatrix} 4.0371 & 0 & 0 \\ 0 & 4.0986 & 0 \\ 0 & 0 & 4.1039 \end{pmatrix}.$$

According to Theorem 3, when $0 < \delta_2 \leqslant 0.1$, the GAS of the system (9) is $\Omega = \{x(t) \in \mathbb{R}^3 \mid x(t)^{\mathrm{T}} P x(t) \leqslant 12.7936/\delta_2\}$. Figure 2 shows the results of system (9) with $s \in [-2, 0]$. In particular, if the weight matrix are delayed form and the initial value is left unchanged, the value of the external input is set as $U = (0,0,0)^{\mathrm{T}}$. Figure 3 shows the simulation results.

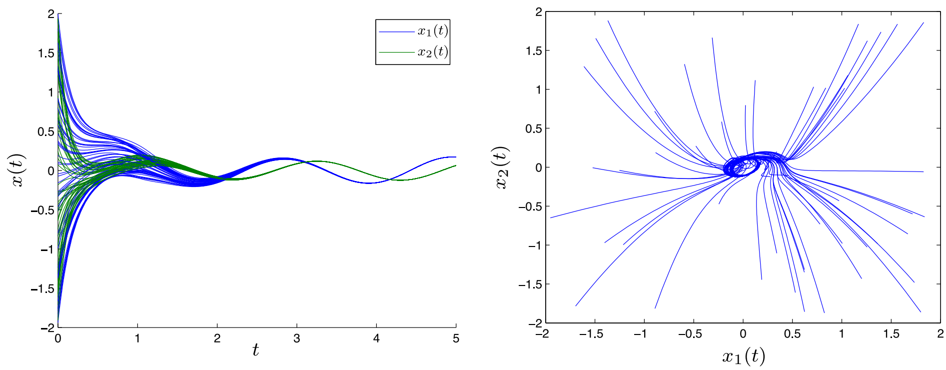

Figure 1. The transient behavior trajectories of the system (8).

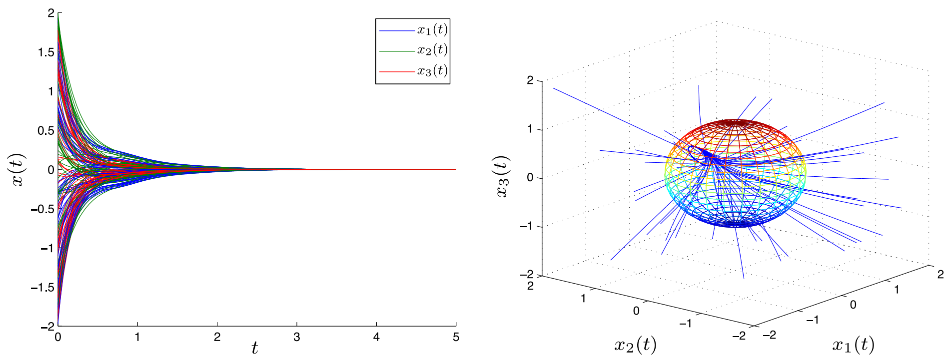

Figure 2. The transient behavior trajectories of the system (9).

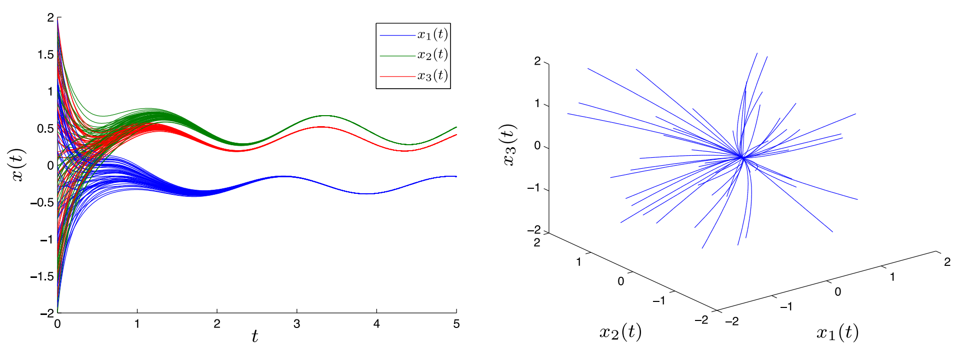

Figure 3. The transient behavior trajectories of the system (9).

Remark 3. From Fig. 2 we can know that system (9) is stable, and we can easily find its global attractive set. The solution converges to the equilibrium point fast when the value of external input is more concise as shown in Fig. 3.

5 Conclusions

This paper discussed the global attractive set of the neural networks with neutral items. We found three types of Lyapunov functions via matrix measures, inequality techniques, and Lyapunov theory. The specific estimations for the GAS of neural networks with neutral item are presented.

In addition, much more work would be considered, such as the dynamical behaviors of system with neutral items in the field of complex-valued or quaternion-valued and how to find the global attractive set of the neural networks when the neutral types activation function with general case.

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