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Fractional uncertain differential equations with general memory effects: Existences and α-path solutions

Cheng Luo[a,b], **Guo-Cheng Wu**[b,1], **Lan-Lan Huang**[b]

[a]School of Mathematics and Statistics, Southwest University,
Chongqing 400715, China

[b]Data Recovery Key Laboratory of Sichuan Province,
College of Mathematics and Information Science,
Neijiang Normal University,
Neijiang 641100, Sichuan Province, China
wuguocheng@gmail.com

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Abstract. General fractional calculus is popular recently. Fractional uncertain differential equations (FUDEs) with general memory effects are proposed in this paper. Firstly, existence and uniqueness theorems of solutions for general fractional uncertain differential equations (GFUDEs) are presented, and an exact solution of a linear one is given. Then the concept of α-path is introduced, and relationship between solution of GFUDE and corresponding α-path is also discussed. In addition, a theorem is proved to obtain the expected value of a monotonic function related to solutions of GFUDEs. Finally, a numerical example is given to better understand the significance of general memory effects. This paper provides more types of FUDEs to better describe some phenomena in uncertain environments.

Keywords: general fractional calculus, uncertainty theory, α-path.

1 Introduction

Recently, in order to model the time evolution of indeterministic phenomena, a class of differential equations driven by the Wiener process, stochastic differential equations were proposed in [6, 7]. They were widely applied in various fields such as option pricing problems [1], multiagent systems [29] and so on. Then the Wiener process plays an important role in dynamical systems with continuous-time noise. However, in real-life applications, it may be impractical and unreasonable that the noise term $\mathrm{d}W_t/\mathrm{d}_t$ is regarded as a normal random variable with expected value 0 and variance ∞ for any fixed time. Moreover, the stochastic calculus is carried out in the framework of probability

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theory, which needs an assumption that estimated probability distribution is close to the real frequency. But there is usually not enough, or even no samples available to estimate probability distribution via statistics due to some privacy or technological reasons, for example, aftershock frequency data during earthquake and nuclear test data. So, experts are invited to provide some empirical data or evaluate the belief degree that each event will happen. Limited by personal professional knowledge and experiences, the belief degree may be subjective and even far away from the real frequency. In this case, it is not appropriate to treat belief degree as the probability distribution when one describes indeterministic phenomena by probability theory.

In order to better deal with human beliefs, the uncertainty theory was proposed by use of uncertain measure in 2007 [12] and refined successively [14, 15]. Especially, to describe time evolution with uncertain environment, the uncertain differential equations (UDEs) driven by Liu process were firstly introduced in [13]. Subsequently, existence and uniqueness theorems of solutions of UDEs were provided in [2]. Since many UDEs usually cannot be solved analytically, some numerical methods were suggested for numerical solutions [26,31,32]. Besides, the concept of stability was introduced in [14], and stability theorems of UDEs were discussed in [33]. Later, various stabilities for UDEs were investigated such as stability in moment [25], stability in mean [34] and stability in distribution [30].

In the past decades, the fractional calculus holds memory effects, and fractional differential equations (FDEs) become a popular modeling approach in various fields such as material viscoelastic mechanics [20], financial time series [19], anomalous diffusion phenomena [21] and so on. Fractional uncertain differential equations (FUDEs) driven by Liu process were proposed, and the existence and uniqueness of FUDEs were given in [35]. FUDEs can be transformed into a family of FDEs, and a numerical algorithm was designed [18]. For the discrete-time case, the fractional uncertain difference equations were also given [17]. Moreover, Hadamard FDEs also has been paid much attention. A Caputo–Hadamard FUDE was also proposed to model the evolution process with historical dependence and uncertain factors [16].

Many fractional operators have been proposed to meet the demand of real-life applications, for example, Riemann–Liouville, Caputo, Hadamard, Exp and Katugampola fractional integrals or derivatives [5, 9, 10]. There is still an interesting and challenging question: which fractional derivative is the best? In fact, a general fractional integral was discussed in [10, 22, 23]. The fractional integral with respect to a general kernel function $g(t)$ is defined by

$$ {}_aI_t^{\nu,g}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)x(s)\,\mathrm{d}s. $$

The kernel function $g(t)$ can be assumed to be unknown and the general fractional integral can be reduced as some well-known definitions:

$$ {}_aI_t^{\nu,t}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t (t-s)^{\nu-1}x(s)\,\mathrm{d}s \quad \text{(Riemann–Liouville integral [10]),} $$

$$
{}_aI_t^{\nu,\ln t}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t \ln\left(\frac{t}{s}\right)^{\nu-1}\frac{x(s)}{s}\,\mathrm{d}s \quad \text{(Hadamard integrals [10])},
$$

$$
{}_aI_t^{\nu,\mathrm{e}^t}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t \mathrm{e}^s\big(\mathrm{e}^t-\mathrm{e}^s\big)^{\nu-1}x(s)\,\mathrm{d}s \quad \text{(Exp integral [5])},
$$

$$
{}_aI_t^{\nu,t^\beta/\beta}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t \left(\frac{t^\beta-s^\beta}{\beta}\right)^{\nu-1}\frac{x(s)}{s^{1-\beta}}\mathrm{d}s \quad \text{(Katumapola integral [9])}
$$

etc. Recently, some scholars have paid much attention to theories and applications such as Laplace transform [8] and predictor–corrector method [27]. Moreover, the physical meaning of the kernel function $g(t)$ are explained by use of continuous time random walk theory [4], and the boundedness condition is provided to guarantee that the general fractional integral is well defined [3].

In order to better understand uncertain phenomena with different memory effects, general fractional uncertain systems can be suggested. So, we need to introduce the uncertainty into general fractional differential equations (GFDEs). The formation and existence should be discussed now, which is the main purpose of this study.

It is arranged in the following sections. Section 2 revisits some concepts and properties of fractional calculus and uncertainty theory. The definition of GFUDE is proposed and analytical solution of a linear GFUDE is also given in Section 3. Then the existence and uniqueness theorems of solutions of GFUDE are proved in Section 4. Next, Section 5 introduces the concept of α-path of GFUDE, and the relationship between one and solution of GFUDE is discussed subsequently. In Section 6, a theorem is presented to obtain the expected value of a monotonic function of solution of GFUDE and a numerical example is given to demonstrate the efficiency of general fractional operators under the uncertainty theory.

2 Preliminaries

Let us revisit some definitions and properties of general fractional calculus and uncertainty theory.

2.1 General fractional calculus

Definition 1. (See [3].) Suppose $x \in L^1[a,b]$ and $g \in C^1[a,b]$ is a strictly monotone increasing function with $g(a) \geqslant 0$. The ν-order general fractional integral of $x(t)$ with respect to $g(t)$ is defined as

$$
{}_aI_t^{\nu,g}x(t) := \frac{1}{\Gamma(\nu)}\int_a^t \big(g(t)-g(s)\big)^{\nu-1}g'(s)x(s)\,\mathrm{d}s, \quad \nu > 0.
$$

The function space and boundedness theorem of general fractional integral were discussed in [3]. Next, we introduce the space $AC_\delta^n[a,b]$ defined by the use of the δ derivative, namely,

$$AC_\delta^n[a,b] := \left\{x\colon [a,b] \to \mathbb{C} : x^{[n-1]} \in AC[a,b],\ x^{[n-1]} = \delta^{n-1}x,\ \delta := \frac{1}{g'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right\}.$$

Then the general fractional derivatives can be given in the space $AC_\delta^n[a,b]$ as follows.

Definition 2. (See [8].) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = [\nu] + 1$, $x \in AC_\delta^n[a,b]$, and let $g \in C^1[a,b]$ be a strictly monotone function with $g(a) \geqslant 0$ and $g'(t) > 0$. The left (L.H.S.) Riemann–Liouville general fractional derivative of arbitrary order ν can be defined as

$$\begin{aligned} {}_aD_t^{\nu,g}x(t) &:= \delta^n {}_aI_t^{n-\nu,g}x(t) \\ &= \frac{(\frac{1}{g'(t)}\frac{\mathrm{d}}{\mathrm{d}t})^n}{\Gamma(n-\nu)} \int\limits_a^t \bigl(g(t)-g(s)\bigr)^{n-\nu-1} g'(s)x(s)\,\mathrm{d}s. \end{aligned}$$

For $\nu = n$, ${}_aD_t^{n,g}x(t) = \delta^n x(t)$.

Definition 3. (See [27].) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = [\nu] + 1$, $x \in AC_\delta^n[a,b]$, and let $g \in C^1[a,b]$ be a strictly monotone function with $g(a) \geqslant 0$ and $g'(t) > 0$. The left (L.H.S.) Caputo general fractional derivative of arbitrary order ν can be defined as

$$\begin{aligned} {}_a^CD_t^{\nu,g}x(t) &:= {}_aI_t^{n-\nu,g}x^{[n]}(t) \\ &= \frac{1}{\Gamma(n-\nu)} \int\limits_a^t \bigl(g(t)-g(s)\bigr)^{n-\nu-1} g'(s)x^{[n]}(s)\,\mathrm{d}s. \end{aligned}$$

For $\nu = n$, ${}_a^CD_t^{n,g}x(t) = \delta^n x(t)$.

Lemma 1. (*See* [8].) *Let* $n-1 < \nu \leqslant n$ *and* $x \in AC_\delta^n[a,b]$. *Then the Leibniz integral law holds*:

$$ {}_aI_t^{\nu,g}\,{}_aD_t^{\nu,g}x(t) = x(t) - \sum_{j=1}^n \frac{{}_aI_t^{n-\nu,g}x(a)}{\Gamma(\nu-j+1)}\bigl(g(t)-g(a)\bigr)^{\nu-j}. \tag{1}$$

Lemma 2. (*See* [4].) *Let* $n-1 < \nu \leqslant n$ *and* $x \in AC_\delta^n[a,b]$. *Then it holds*

$$ {}_aI_t^{\nu,g}\,{}_a^CD_t^{\nu,g}x(t) = x(t) - \sum_{j=1}^n \frac{\delta^{n-j}x(a)}{\Gamma(n-j+1)}\bigl(g(t)-g(a)\bigr)^{n-j}. \tag{2}$$

Lemma 3 [Generalized Gronwall inequality]. *Suppose* $\nu > 0$, $u(t)$ *and* $\psi(t)$ *are nonnegative functions and locally integrable defined on* $t \in [a,T)$ $(a \geqslant 0, T \leqslant +\infty)$, *and let* $p : [a,T) \to [0,M]$ (M *is a constant*) *be a nondecreasing and continuous function.*

If the inequality

$$\psi(t) \leqslant u(t) + \frac{p(t)}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\psi(s)\,\mathrm{d}s, \quad a \leqslant t < T,$$

holds, then

$$\psi(t) \leqslant u(t) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(p(t))^n}{\Gamma(n\nu)} \big(g(t) - g(s)\big)^{n\nu-1} u(s)\right] g'(s)\,\mathrm{d}s, \quad a \leqslant t < T.$$

Proof. For locally integrable function χ, let

$$B\chi(t) = \frac{p(t)}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\chi(s)\,\mathrm{d}s.$$

Then we have

$$\psi(t) \leqslant u(t) + B\psi(t),$$

which implies

$$\psi(t) \leqslant \sum_{k=0}^{n-1} B^k u(t) + B^n \psi(t).$$

Next, we aim to prove that

$$B^n \psi(t) \leqslant \int_a^t \frac{(p(t))^n}{\Gamma(n\nu)} \big(g(t) - g(s)\big)^{n\nu-1} g'(s)\psi(s)\,\mathrm{d}s \tag{3}$$

and $B^n\psi(t) \to 0$ as $n \to \infty$ for any $t \in [a, T)$.

Specifically, inequality (3) is true for $n = 1$. Now assume that it is still true for $n = k$. If $n = k + 1$, then one obtains

$$\begin{aligned}
&B^{k+1}\psi(t) \\
&\quad = B\big(B^k\psi(t)\big) \\
&\quad \leqslant \frac{p(t)}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} \left[\int_a^s \frac{(p(s))^k}{\Gamma(k\nu)} \big(g(s)-g(\tau)\big)^{k\nu-1} g'(\tau)\psi(\tau)\,\mathrm{d}\tau\right] g'(s)\,\mathrm{d}s \\
&\quad \leqslant \frac{(p(t))^{k+1}}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} \left[\int_a^s \frac{1}{\Gamma(k\nu)} \big(g(s)-g(\tau)\big)^{k\nu-1} g'(\tau)\psi(\tau)\,\mathrm{d}\tau\right] g'(s)\,\mathrm{d}s.
\end{aligned}$$

Interchanging the order of integration and beta function, we can derive

$$B^{k+1}\psi(t) \leqslant \int_a^t \frac{(p(t))^{k+1}}{\Gamma((k+1)\nu)} \big(g(t)-g(s)\big)^{(k+1)\nu-1} \psi(s) g'(s)\,\mathrm{d}s.$$

So, inequality (3) is proved. In addition, from (3) we obtain

$$B^n\psi(t) \leqslant \int_a^t \frac{M^n}{\Gamma(n\nu)} \big(g(t)-g(s)\big)^{n\nu-1} g'(s)\psi(s)\,\mathrm{d}s \to 0$$

as $n \to +\infty$ for any $t \in [a, T)$, which completes the proof. □

2.2 Uncertainty theory

Definition 4. (See [12].) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 [Normality axiom]. $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 [Duality axiom]. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 [Subadditivity axiom]. For every countable sequence of events $\Lambda_1, \Lambda_2, \dots,$ we have

$$\mathcal{M}\Bigg\{\bigcup_{i=1}^{\infty} \Lambda_i\Bigg\} \leqslant \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Axiom 4 [Product axiom]. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$. Then the product uncertain measure $\mathcal{M}$ satisfies

$$\mathcal{M}\Bigg\{\prod_{k=1}^{\infty} \Lambda_k\Bigg\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen event from $\mathcal{L}_k$ for $k = 1, 2, \dots$, respectively.

Definition 5. (See [12].) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\Phi \leqslant x\}$$

for any real number x.

Definition 6. (See [12].) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \qquad \lim_{x\to+\infty} \Phi(x) = 1.$$

Definition 7. (See [12].) Let ξ be an uncertain variable on an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then its expected value $\mathbf{E}[\xi]$ is

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geqslant x\}\,\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leqslant x\}\,\mathrm{d}x,$$

provided that at least one of the two integrals exists, and its variance $\mathbf{V}[\xi]$ is

$$\mathbf{V}[\xi] = \mathbf{E}\bigl[\bigl(\xi - \mathbf{E}[\xi]\bigr)^2\bigr].$$

An uncertain process is essentially a sequence of uncertain variable indexed by time.

Definition 8. (See [13].) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space, and let T be a totally ordered set (e.g., time). An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t.

Definition 9. (See [14].) An uncertain process C_t is called a Liu process if

(i) $C_0 = 0$ and almost all simple paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) the increment $C_{s+t} - C_s$ has a normal uncertain distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \mathbb{R}.$$

Definition 10. (See [14].) Let X_t be an uncertain process, and let C_t be a Liu process. For arbitrary partition of interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leqslant i \leqslant k} |t_{i+1} - t_i|.$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t\,\mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Lemma 4. (*See* [15].) *Let $f(t)$ be an integrable function with respect to t. Then the Liu integral $\int_0^s f(t)\,\mathrm{d}C_t$ is a normal uncertain variable at each time s, and*

$$\int_0^s f(t)\,\mathrm{d}C_t \sim \mathcal{N}\left(0, \int_0^s \bigl|f(t)\bigr|\,\mathrm{d}t\right).$$

Lemma 5. (*See* [2].) *Suppose that C_t is a Liu process, and let X_t is an integrable uncertain process on $[c,d]$ with respect to t. Then the following inequality holds*:

$$\left|\int_c^d X_t(\gamma)\,\mathrm{d}C_t(\gamma)\right| \leqslant K(\gamma)\int_c^d \left|X_t(\gamma)\right|\mathrm{d}t,$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

Lemma 6. (*See* [35].) *Let $\xi_1, \xi_2, \dots,$ be uncertain variables and $\lim_{i\to\infty}\xi_i = \xi$ almost surely. Then ξ is an uncertain variable.*

Lemma 7. (*See* [15].) *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_n$, respectively. If $f(x_1, x_2, \dots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \dots, x_n$. Then $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\bigl(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{(m+1)}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)\bigr).$$

Lemma 8. (*See* [15].) *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. If the expected value exists, then*

$$\mathbf{E}[\xi] = \int_0^1 \Phi^{-1}(\alpha)\,\mathrm{d}\alpha.$$

3 General fractional uncertain differential equations

In this section, we present the definitions and properties of GFUDEs.

Definition 11. Let $n-1 < \nu \leqslant n$, $n \in \mathbb{N}_1$, $g \in C^1[a,b]$, and let $X \in AC_\delta^n[a,b]$ be an uncertain process and C_t be a Liu process. Assume that the functions $f, h : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous. Then the Riemann–Liouville GFUDE with initial conditions is defined as

$$\begin{aligned} &{}_aD_t^{\nu,g}X_t = f(t,X_t) + h(t,X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t},\\ &{}_aD_t^{\nu-i,g}X_t\big|_{t=a} = \kappa_i, \quad i = 1,2,\dots,n. \end{aligned} \tag{4}$$

Definition 12. Let $n-1 < \nu \leqslant n$, $n \in \mathbb{N}_1$, $g \in C^1[a,b]$, and let $X \in AC_\delta^n[a,b]$ be an uncertain process and C_t be a Liu process. Assume that the functions $f, h : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous. Then the Caputo FUDE with initial conditions is defined as

$$\begin{aligned} &{}_a^CD_t^{\nu,g}X_t = f(t,X_t) + h(t,X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t},\\ &\delta^i X_t\big|_{t=a} = \kappa_i, \quad i = 0,1,\dots,n-1. \end{aligned} \tag{5}$$

Remark 1. All kinds of GFUDEs with initial conditions are equivalent to a class of general fractional uncertain integral equations by Leibniz integral laws (1) and (2). That is, the Riemann–Liouville GFUDE (4) and Caputo one (5) are equivalent to

$$\begin{aligned} X_t &= \sum_{i=1}^{n} \frac{(g(t)-g(a))^{\nu-i}\kappa_i}{\Gamma(\nu-i+1)} + {}_aI_t^{\nu,g} f(t,X_t) +_a I_t^{\nu,g} h(t,X_t) \\ &= \sum_{i=1}^{n} \frac{(g(t)-g(a))^{\nu-i}\kappa_i}{\Gamma(\nu-i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) f(s,X_s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) h(s,X_s)\,\mathrm{d}C_s \end{aligned}$$

and

$$\begin{aligned} X_t &= \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^{i}\kappa_i}{\Gamma(i+1)} + {}_aI_t^{\nu,g} f(t,X_t) +_a I_t^{\nu,g} h(t,X_t) \\ &= \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^{i}\kappa_i}{\Gamma(i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) f(s,X_s)\,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) h(s,X_s)\,\mathrm{d}C_s, \end{aligned}$$

respectively.

Generally, since the initial value condition of Caputo-type fractional differential equations is similar to one of integer order, which can more conveniently model physical phenomena in practice. In the next section, we only focus on discussing the properties of Caputo GFUDE. Certainly, for Riemann–Liouville GFUDE, the similar results also can be obtained in the same way.

Theorem 1. *Let $\nu > 0$ be a real number with $n-1 < \nu < n$, $n \in \mathbb{N}_1$. Assume that the functions $\mu(t)$ and $\sigma(t)$ defined on $[a,T]$ are continuous, λ is a real number and C_t is a Liu process. Then the Caputo GFUDE with initial conditions*

$$\begin{aligned} {}_a^C D_t^{\nu,g} X_t &= \lambda X_t + \mu(t) + \sigma(t)\frac{\mathrm{d}C_t}{\mathrm{d}t}, \\ \delta^i X_t\big|_{t=a} &= \kappa_i, \quad i = 0,1,\dots,n-1, \end{aligned} \tag{6}$$

has a solution

$$\begin{aligned} X_t &= \sum_{i=0}^{n-1} \big(g(t)-g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(t)-g(a)\big)^\nu\big) \\ &\quad + \int_a^t \big(g(t)-g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(s)\big)^\nu\big)\mu(s) g'(s)\,\mathrm{d}s + P_t, \end{aligned} \tag{7}$$

where

$$P_t = \int_a^t \big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t) - g(s)\big)^{\nu}\big) \sigma(s) g'(s) \, \mathrm{d}C_s$$

is a normal uncertain variable for each time t, and

$$P_t \sim \mathcal{N}\left(0, \int_a^t \big|\big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}(\lambda, \big(g(t) - g(a)\big)^{\nu}\big) \sigma(s) \big| g'(s) \, \mathrm{d}s\right).$$

Proof. Obviously, according to Definition 12, the solution of (6) should satisfy

$$X_t = \sum_{i=0}^{n-1} \frac{(g(t) - g(a))^i \kappa_i}{\Gamma(i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s) \big(\lambda X_s + \mu(s)\big) \, \mathrm{d}s$$
$$+ \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s) \sigma(s) \, \mathrm{d}C_s. \tag{8}$$

Therefore, for X_t provided by (7), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s) \lambda X_s \, \mathrm{d}s \\
&\quad = \sum_{i=0}^{n-1} \big(g(t) - g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(t) - g(a)\big)^{\nu}\big) - \sum_{i=0}^{n-1} \frac{(g(t) - g(a))^i}{\Gamma(i+1)} \kappa_i \\
&\qquad + \int_a^t \big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t) - g(s)\big)^{\nu}\big) \mu(s) g'(s) \, \mathrm{d}s \\
&\qquad - \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} \mu(s) g'(s) \, \mathrm{d}s \\
&\qquad + \int_a^t \big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t) - g(s)\big)^{\nu}\big) \sigma(s) g'(s) \, \mathrm{d}s \\
&\qquad - \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} \sigma(s) g'(s) \, \mathrm{d}s.
\end{aligned} \tag{9}$$

The the proof is given in Appendix A.

Therefore, from Eqs. (8) and (9), the solution of (6) is equivalent to

$$\begin{aligned} X_t = {} & \sum_{i=0}^{n-1} \big(g(t)-g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(t)-g(a)\big)^\nu\big) \\ & + \int_a^t \big(g(t)-g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(s)\big)^\nu\big)\mu(s)g'(s)\,\mathrm{d}s \\ & + \int_a^t \big(g(t)-g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(s)\big)^\nu\big)\sigma(s)g'(s)\,\mathrm{d}C_s. \end{aligned}$$

Then let

$$P_t = \int_a^t \big(g(t)-g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(s)\big)^\nu\big)\sigma(s)g'(s)\,\mathrm{d}C_s.$$

According to Lemma 4, we can conclude

$$P_t \sim \mathcal{N}\left(0, \int_a^t \big|\big(g(t)-g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(a)\big)^\nu\big)\sigma(s)\big|g'(s)\,\mathrm{d}s\right),$$

which completes the proof. □

4 Existence and uniqueness

In this section, we present the existence and uniqueness theorems of solution of Caputo GFUDE and prove that unique solution is sample-continuous.

Theorem 2. *The GFUDE* (5) *has a unique solution if the coefficient functions f and $h : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the Lipschitz condition*

$$\big|f(t,x)-f(t,y)\big| + \big|h(t,x)-h(t,y)\big| \leqslant L|x-y| \quad \forall x,y \in \mathbb{R},\ a \leqslant t,$$

and the linear growth condition

$$\big|f(t,x)\big| + \big|h(t,x)\big| \leqslant L\big(1+|x|\big) \quad \forall x \in \mathbb{R},\ a \leqslant t,$$

where L is a positive constant. Furthermore, the unique solution is sample-continuous.

Proof. First, the successive approximation method is employed to prove the existence of solution of GFUDE (5) with initial conditions. We adopt Picard's method as follows:

$$X_t^{(0)} = \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}$$

and

$$X_t^{(n+1)} = \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) f\big(s, X_s^{(n)}\big)\,\mathrm{d}s$$
$$+ \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) h\big(s, X_s^{(n)}\big)\,\mathrm{d}C_s, \quad n = 0,1,2,\dots .$$

For any sample γ, we can set

$$S_t^{(n)}(\gamma) = \max_{a\leqslant u\leqslant t} \big|X_u^{(n+1)}(\gamma) - X_u^{(n)}(\gamma)\big|, \quad n = 0,1,2,\dots,$$

which satisfies

$$S_t^{(n)}(\gamma) \leqslant \frac{(L(1+K(\gamma))(g(t)-g(a))^\nu)^n}{\Gamma(n\nu+1)} \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right), \tag{10}$$

where $a \leqslant t \leqslant T$, T is an arbitrary constant. The proof can be seen in Appendix B.

Therefore, for any γ, we construct function term series

$$\sum_{n=0}^{\infty} \frac{((1+K(\gamma))L(g(t)-g(a))^\nu)^n}{\Gamma(n\nu+1)} \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right)$$
$$\leqslant \sum_{n=0}^{\infty} \frac{((1+K(\gamma))L(g(T)-g(a))^\nu)^n}{\Gamma(n\nu+1)} \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(T)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right)$$
$$= E_{\nu,1}\big(\big(1+K(\gamma)\big)L\big(g(T)-g(a)\big)^\nu\big) \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(T)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right).$$

Since the Mittag-Leffler function $E_{\nu,1}((1+K(\gamma))L(g(T)-g(a))^\nu)$ is a convergent positive series, the sequence of functions $X_t^{(n)}$, $n = 0,1,\dots$, converges uniformly in $[a,T]$ by the Weierstrass discriminance. Hence, for any γ, we have

$$X_t(\gamma) = \lim_{n\to\infty} X_t^{(n)}(\gamma), \quad t \in [a,T],$$

and

$$\max_{a\leqslant u\leqslant t} \big|X_u(\gamma)\big| \leqslant \sum_{n=0}^{\infty} \max_{a\leqslant u\leqslant t} \big|X_u^{(n+1)}(\gamma) - X_u^{(n)}(\gamma)\big| + \max_{a\leqslant u\leqslant t} \big|X_u^{(0)}(\gamma)\big|$$
$$\leqslant E_{\nu,1}\big(\big(1+K(\gamma)\big)L\big(g(t)-g(a)\big)^\nu\big)$$
$$\times \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right) + \left|\sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|. \tag{11}$$

Since $X_t^{(n)}$, $n = 0,1,\dots,$ are uncertain variables for each time $t \in [a,T]$, then the solution of (5) is an uncertain process by Lemma 6.

Next, we present the uniqueness of solution of GFUDE (5). For each γ, suppose $X_t(\gamma)$ and $X_t^*(\gamma)$ are the solutions of GFUDE (5) with the same initial value. We give

$$\begin{aligned}
&\left|X_t(\gamma) - X_t^*(\gamma)\right| \\
&\quad = \left|\frac{1}{\Gamma(\nu)}\int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big(f\big(s,X_s\big) - f(s,X_s^*)\big)\,\mathrm{d}s\right. \\
&\qquad \left.+ \frac{1}{\Gamma(\nu)}\int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big(h\big(s,X_s\big) - h(s,X_s^*)\big)\,\mathrm{d}C_s\right| \quad \text{(by Lemma 5)} \\
&\quad \leqslant \frac{(1+K(\gamma))L}{\Gamma(\nu)}\int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\left|X_s(\gamma) - X_s^*(\gamma)\right|\mathrm{d}s.
\end{aligned}$$

From Lemma 3 (generalized Gronwall inequality) it follows $|X_t(\gamma) - X_t^*(\gamma)| \leqslant 0$, which means the solution of GFUDE (5) is unique.

Finally, we claim the unique solution is sample-continuous. Assume $a \leqslant \omega \leqslant t$. For any γ, we get

$$\begin{aligned}
&\left|X_t(\gamma) - X_\omega(\gamma)\right| \\
&\quad = \left|\sum_{i=0}^{n-1}\frac{(g(t)-g(a))^i - (g(\omega)-g(a))^i}{\Gamma(i+1)}\kappa_i\right. \\
&\qquad + \frac{1}{\Gamma(\nu)}\int_\omega^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) f\big(s,X_s(\gamma)\big)\,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\nu)}\int_\omega^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) h\big(s,X_s(\gamma)\big)\,\mathrm{d}C_s \\
&\qquad + \frac{1}{\Gamma(\nu)}\int_a^\omega \Big(\big(g(t)-g(s)\big)^{\nu-1} - (g(\omega)-g(s))^{\nu-1}\Big) g'(s) f\big(s,X_s(\gamma)\big)\,\mathrm{d}s \\
&\qquad \left.+ \frac{1}{\Gamma(\nu)}\int_a^\omega \Big(\big(g(t)-g(s)\big)^{\nu-1} - \big(g(\omega)-g(s)\big)^{\nu-1}\Big) g'(s) h\big(s,X_s(\gamma)\big)\,\mathrm{d}C_s\right| \\
&\quad \leqslant \left|\sum_{i=0}^{n-1}\frac{(g(t)-g(a))^i - (g(\omega)-g(a))^i}{\Gamma(i+1)}\kappa_i\right| \\
&\qquad + \frac{(1+K(\gamma))L}{\Gamma(\nu)}\int_\omega^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big(1+\left|X_s(\gamma)\right|\big)\,\mathrm{d}s \\
&\qquad + \frac{(1+K(\gamma))L}{\Gamma(\nu)}\int_a^\omega \Big(\big(g(t)-g(s)\big)^{\nu-1} - \big(g(\omega)-g(s)\big)^{\nu-1}\Big) g'(s)\big(1+\left|X_s(\gamma)\right|\big)\,\mathrm{d}s.
\end{aligned}$$

From (11) we have

$$
\begin{aligned}
&\bigl|X_t(\gamma) - X_\omega(\gamma)\bigr| \\
&\quad \leqslant \left| \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i - (g(\omega)-g(a))^i}{\Gamma(i+1)} \kappa_i \right| \\
&\qquad + \frac{(1+K(\gamma))L}{\Gamma(\nu)} \bigl(1 + E_{\nu,1}\bigl((1+K(\gamma))L\bigl(g(t)-g(a)\bigr)^\nu\bigr)\bigr) \\
&\qquad \times \left(1 + \left|\sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)}\right|\right) \left(\int_\omega^t \bigl(g(t)-g(s)\bigr)^{\nu-1} g'(s)\,\mathrm{d}s \right. \\
&\qquad \left. + \int_a^\omega \bigl(\bigl(g(t)-g(s)\bigr)^{\nu-1} - \bigl(g(\omega)-g(s)\bigr)^{\nu-1}\bigr) g'(s)\,\mathrm{d}s \right) \\
&\quad \leqslant \left| \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i - (g(\omega)-g(a))^i}{\Gamma(i+1)} \kappa_i \right| \\
&\qquad + \frac{(1+K(\gamma))L}{\Gamma(\nu+1)} \bigl(1 + E_{\nu,1}\bigl((1+K(\gamma))L\bigl(g(t)-g(a)\bigr)^\nu\bigr)\bigr) \\
&\qquad \times \left(1 + \left|\sum_{i=0}^{n-1} \frac{\bigl(g(t)-g(a)\bigr)^i \kappa_i}{\Gamma(i+1)}\right|\right) \bigl(\bigl(g(t)-g(a)\bigr)^\nu - \bigl(g(\omega)-g(a)\bigr)^\nu\bigr).
\end{aligned}
$$

Hence, $|X_t(\gamma) - X_\omega(\gamma)| \to 0$ as $\omega \to t$, and then X_t is sample-continuous. As a result, the existence and uniqueness theorems are proved. □

5 α-path of general fractional uncertain differential equations

In this section, the definition of α-path of the Caputo GFUDE is given, and relationship between α-path and solution of Caputo GFUDE is discussed.

Lemma 9 [General first comparison principle]. *Suppose that continuous functions $q(t,x)$ and $Q(t,x)$ defined on $\Omega = [a,T] \times \mathbb{R}$ satisfy Lipschitz condition with respect to x. Let $x = \phi(t)$ and $\varphi(t)$ be the solutions of the following Cauchy problems* ($0 \leqslant n-1 < \nu \leqslant n$):

$$
\begin{aligned}
&{}_a^C D_t^{\nu,g} x(t) = q\bigl(t, x(t)\bigr), \\
&\delta^i x(t)|_{t=a} = \kappa_i, \quad i = 0, 1, \dots, n-1,
\end{aligned} \tag{E$_1$}
$$

$$
\begin{aligned}
&{}_a^C D_t^{\nu,g} x(t) = Q\bigl(t, x(t)\bigr), \\
&\delta^i x(t)|_{t=a} = \kappa_i, \quad i = 0, 1, \dots, n-1,
\end{aligned} \tag{E$_2$}
$$

respectively. If the inequality $q(t,x) < Q(t,x)$ holds, then we get $\phi(t) < \varphi(t)$, $t \in (a,T]$.

Proof. For the continuous functions $q(t,x)$ and $Q(t,x)$, the Cauchy problems (E_1) and (E_2) are equivalent to the following Volterra integral equations:

$$\phi(t) = \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) q\big(s,\phi(s)\big)\,\mathrm{d}s$$

and

$$\varphi(t) = \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)} + \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) Q\big(s,\varphi(s)\big)\,\mathrm{d}s,$$

respectively. Hence, we obtain

$$\varphi(t)-\phi(t) = \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big[Q\big(s,\varphi(s)\big) - q\big(s,\phi(s)\big)\big]\,\mathrm{d}s.$$

Then, for any $t \in (a,T]$, there always exists a small enough number $\epsilon > 0$ that satisfies

$$\begin{aligned}
\varphi(t)-\phi(t) &= \frac{1}{\Gamma(\nu)} \int_a^{t-\epsilon} \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big[Q\big(s,\varphi(s)\big) - q\big(s,\phi(s)\big)\big]\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)} \int_{t-\epsilon}^{t} \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big[Q\big(s,\varphi(s)\big) - q\big(s,\phi(s)\big)\big]\,\mathrm{d}s \\
&= \frac{Q(s^*,\varphi(s^*)) - q(s^*,\phi(s^*))}{\Gamma(\nu)} \int_a^{t-\epsilon} \big(g(t)-g(s)\big)^{\nu-1} g'(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)} \int_{t-\epsilon}^{t} \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big[Q\big(s,\varphi(s)\big) - q\big(s,\phi(s)\big)\big]\,\mathrm{d}s \\
&\geqslant \frac{Q(s^*,\varphi(s^*)) - q(s^*,\phi(s^*))}{\Gamma(\nu+1)} \big[\big(g(t)-g(a)\big)^{\nu} - \big(g(t)-g(t-\epsilon)^{\nu}\big)\big] \\
&\quad + \frac{\varrho_\epsilon(t)}{\Gamma(\nu+1)} \big(g(t)-g(t-\epsilon)\big)^{\nu} \\
&> 0,
\end{aligned}$$

where $s^* \in [a,\, t-\epsilon]$ and $\varrho_\epsilon(t) = \inf_{s\in[t-\epsilon,\, T]}\{Q(s,\varphi(s)) - q(s,\phi(s))\}$. When $t = a$, $\varphi(t)-\phi(t) = 0$ holds. This proof has been completed. □

Lemma 10 [General second comparison principle]. *Suppose that continuous functions $q(t,x)$ and $Q(t,x)$ defined on $\Omega = [a,T]\times\mathbb{R}$ satisfy Lipschitz condition with respect to x. Let $x = \phi(t)$ and $\varphi(t)$ be the solutions of the Cauchy problems* (E_1) *and* (E_2)*, respectively. If the inequality $q(t,x) \leqslant Q(t,x)$ holds, then we have $\phi(t) \leqslant \varphi(t)$.*

Proof. First, an initial value problem is constructed as follows:

$$
\begin{aligned}
&{}_a^C D_t^{\nu,g} x(t) = Q\bigl(t, x(t)\bigr) + \theta, \\
&\delta^i x(t)|_{t=a} = \kappa_i, \quad i = 0, 1, \dots, n-1,
\end{aligned}
\tag{$\widehat{\mathrm{E}}_2$}
$$

where $\theta > 0$ is a small constant. There exists a positive finite constant θ_0. When $0 < \theta \leqslant \theta_0$, the function $Q(t, x(t)) + \theta$ is still continuous and satisfies Lipschitz condition with respect to x. Let $x = \varphi_\theta(t)$ be the unique solution of Cauchy problem ($\widehat{\mathrm{E}}_2$).

Since $q(t, x) \leqslant Q(t, x)$, then it always holds that $q(t, x) < Q(t, x) + \theta$ for any constant $\theta > 0$. Hence, from Lemma 9 we have $\phi(t) < \varphi_\theta(t)$, $t \in (a, T]$.

While the constant $\theta \to 0$, we can obtain $\phi(t) \leqslant \varphi(t)$, which finishes the proof. $\square$

Definition 13. Suppose $\alpha \in (0, 1)$. The α-path of a Caputo GFUDE (5) with initial value conditions

$$
\begin{aligned}
&{}_a^C D_t^{\nu,g} X_t = f(t, X_t) + h(t, X_t) \frac{\mathrm{d}C_t}{\mathrm{d}t}, \\
&\delta^i X_t|_{t=a} = \kappa_i, \quad i = 0, 1, \dots, n-1,
\end{aligned}
$$

is a deterministic function X_t^α of t, which solves the corresponding GFDE with the same initial value conditions

$$
{}_a^C D_t^{\nu,g} X_t = f(t, X_t) + \bigl|h(t, X_t)\bigr| \Phi^{-1}(\alpha),
$$

where $\Phi^{-1}(\alpha)$ denotes the inverse standard normal uncertainty distribution, namely,

$$
\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1.
$$

Theorem 3. *Suppose functions $f, h : [a, T] \times \mathbb{R} \to \mathbb{R}$ are continuous. If X_t and X_t^α are the solution and α-path of Caputo GFUDE* (5) *with initial value conditions, respectively, then we can conclude*

$$
\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha \ \forall t \in [a, T]\bigr\} = \alpha
$$

and

$$
\mathcal{M}\bigl\{X_t > X_t^\alpha \ \forall t \in [a, T]\bigr\} = 1 - \alpha.
$$

Proof. First, the time interval $[a, T]$ for each $X(t) \in \mathbb{R}$ can be divided into

$$
\mathcal{T}^+ = \bigl\{t \in [a, T] \bigm| h\bigl(t, X(t)\bigr) \geqslant 0\bigr\}
$$

and

$$
\mathcal{T}^- = \bigl\{t \in [a, T] \bigm| h\bigl(t, X(t)\bigr) < 0\bigr\},
$$

which holds $\mathcal{T}^+ \cap \mathcal{T}^- = \emptyset$ and $\mathcal{T}^+ \cup \mathcal{T}^- = [a, T]$.

Denote two sets

$$S_1^+ = \left\{\gamma \;\middle|\; \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \leqslant \Phi^{-1}(\alpha) \text{ for any } t \in \mathcal{T}^+\right\}$$

and

$$S_1^- = \left\{\gamma \;\middle|\; \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \geqslant \Phi^{-1}(\alpha) \text{ for any } t \in \mathcal{T}^-\right\},$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution. Obviously, $\mathcal{T}^+$ and $\mathcal{T}^-$ are disjoint, and Liu process C_t has independent increments. So, we can obtain

$$\mathcal{M}\{S_1^+\} = \alpha, \qquad \mathcal{M}\{S_1^-\} = \alpha, \qquad \mathcal{M}\{S_1^+ \cap S_1^-\} = \alpha.$$

For any $\gamma \in S_1^+ \cap S_1^-$, it always satisfies that

$$h\bigl(t, X(t)\bigr)\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \leqslant \bigl|h\bigl(t, X(t)\bigr)\bigr|\Phi^{-1}(\alpha) \quad \forall t \in [a, T].$$

According to Lemma 10, X_t and X_t^α are the unique solutions of

$${}_a^C D_t^{\nu,g} X(t) = f(t, X(t)) + h\bigl(t, X(t)\bigr)\frac{\mathrm{d}C_t}{\mathrm{d}t} \quad \forall t \in [a, T]$$

and

$${}_a^C D_t^{\nu,g} X(t) = f\bigl(t, X(t)\bigr) + \bigl|h\bigl(t, X(t)\bigr)\bigr|\Phi^{-1}(\alpha) \quad \forall t \in [a, T]$$

with the same initial value conditions, respectively, so that $X_t \leqslant X_t^\alpha$ always holds. Since $S_1^+ \cap S_1^- \subset \{X_t \leqslant X_t^\alpha\ \forall t \in [a, T]\}$, from the monotonicity theorem we can get

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha\ \forall t \in [a, T]\bigr\} \geqslant \mathcal{M}\bigl\{S_1^+ \cap S_1^-\bigr\} = \alpha. \tag{12}$$

Next, we can also define the sets

$$S_2^+ = \left\{\gamma \;\middle|\; \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > \Phi^{-1}(\alpha) \text{ for any } t \in \mathcal{T}^+\right\}$$

and

$$S_2^- = \left\{\gamma \;\middle|\; \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} < \Phi^{-1}(\alpha) \text{ for any } t \in \mathcal{T}^-\right\}.$$

Similarly, since Liu process C_t has independent increments, we have

$$\mathcal{M}\{S_2^+\} = 1 - \alpha, \qquad \mathcal{M}\{S_2^-\} = 1 - \alpha, \qquad \mathcal{M}\{S_2^+ \cap S_2^-\} = 1 - \alpha.$$

For any $\gamma \in S_2^+ \cap S_2^-$, it obviously satisfies that

$$h\bigl(t, X(t)\bigr)\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > \bigl|h\bigl(t, X(t)\bigr)\bigr|\Phi^{-1}(\alpha) \quad \forall t \in [a, T].$$

From Lemma 9 we can derive $X_t > X_t^\alpha$. Since $S_2^+ \cap S_2^- \subset \{X_t > X_t^\alpha \ \forall t \in [a,T]\}$ always holds, then we get

$$\mathcal{M}\bigl\{X_t > X_t^\alpha \ \forall t \in [a,T]\bigr\} \geqslant \mathcal{M}\bigl\{S_2^+ \cap S_2^-\bigr\} = 1 - \alpha. \tag{13}$$

Obviously, notice that $\{X_t \leqslant X_t^\alpha \ \forall t \in [a,T]\}$ and $\{X_t \nleqslant X_t^\alpha \ \forall t \in [a,T]\}$ are opposite events, which satisfy

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha \ \forall t \in [a,T]\bigr\} + \mathcal{M}\{X_t \nleqslant X_t^\alpha \ \forall t \in [a,T]\bigr\} = 1.$$

Here $\{X_t > X_t^\alpha \ \forall t \in [a,T]\} \subset \{X_t \ngeqslant X_t^\alpha \ \forall t \in [a,T]\}$ is true. Therefore, by use of monotonicity theorem we conclude

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha \ \forall t \in [a,T]\bigr\} + \mathcal{M}\bigl\{X_t > X_t^\alpha \ \forall t \in [a,T]\bigr\} \leqslant 1. \tag{14}$$

From inequalities (12), (13) and (14) we can derive

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha \ \forall t \in [a,T]\bigr\} = \alpha$$

and

$$\mathcal{M}\bigl\{X_t > X_t^\alpha \ \forall t \in [a,T]\bigr\} = 1 - \alpha,$$

which completes the proof. □

Theorem 4. *Suppose functions $f, h : [a,T] \times \mathbb{R} \to \mathbb{R}$ are continuous. Let X_t and X_t^α by the solution and α-path of Caputo GFUDE* (5) *with initial value conditions, respectively. Then an inverse uncertainty distribution of X_t can be given as*

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Proof. Obviously, it always holds that $\{X_t \leqslant X_t^\alpha\} \supset \{X_m \leqslant X_m^\alpha \ \forall m\}$ and $\{X_t > X_t^\alpha\} \supset \{X_m > X_m^\alpha, \forall m\}$ for each time $t \in [a,T]$. According to monotonicity and Theorem 3, we can obtain

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha\bigr\} \geqslant \mathcal{M}\bigl\{X_m \leqslant X_m^\alpha \ \forall m\bigr\} = \alpha$$
$$\mathcal{M}\bigl\{X_t > X_t^\alpha\bigr\} \geqslant \mathcal{M}\bigl\{X_m > X_m^\alpha \ \forall m\bigr\} = 1 - \alpha.$$

Since $\{X_t \leqslant X_t^\alpha\}$ and $\{X_t > X_t^\alpha\}$ satisfy $\mathcal{M}\{X_t \leqslant X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1$, we have

$$\mathcal{M}\bigl\{X_t \leqslant X_t^\alpha\bigr\} = \alpha$$

for each time $t \in [a,T]$, which finishes the proof. □

6 Expected value of a monotone function

Theorem 5. *Let $n-1<\nu\leqslant n$, $n\in\mathbb{N}_1$, $X\in AC^n_\delta[a,b]$ be an uncertain process and C_t be a Liu process. Assume that the functions f and $h:[a,\infty)\times\mathbb{R}\to\mathbb{R}$ are continuous. Assume that X_t and X^α_t are the solution and α-path of the Caputo GFUDE* (5) *with initial conditions*

$$\begin{aligned}
&{}^C_aD^{\nu,g}_t X_t = f(t,X_t)+h(t,X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t},\\
&\delta^i X_t|_{t=a}=\kappa_i,\quad i=0,1,\dots,n-1,
\end{aligned}$$

respectively. Then a monotone function $J(x)$ is strictly increasing (*decreasing*)*. So, we have*

$$\mathbf{E}\bigl[J(X_t)\bigr]=\int_0^1 J(X^\alpha_t)\,\mathrm{d}\alpha\left(\int_0^1 J\bigl(X^{1-\alpha}_t\bigr)\,\mathrm{d}\alpha\right).$$

Proof. According to Theorem 4, one means that the solution X_t has an inverse uncertainty distribution ${\Psi_t}^{-1}(\alpha)=X^\alpha_t$. Due to Lemma 7, for the strictly increasing or decreasing monotone function $J(X_t)$, the corresponding inverse uncertainty distributions are $\Upsilon^{-1}_t(\alpha)=J(X^\alpha_t)$ and $\hat{\Upsilon}^{-1}_t(\alpha)=J(X^{1-\alpha}_t)$, respectively.

Naturally, using Lemma 8, we can derive

$$\mathbf{E}\bigl[J(X_t)\bigr]=\int_0^1\Upsilon^{-1}_t(\alpha)\,\mathrm{d}\alpha=\int_0^1 J\bigl(X^\alpha_t\bigr)\,\mathrm{d}\alpha$$

and

$$\mathbf{E}\bigl[J(X_t)\bigr]=\int_0^1\hat{\Upsilon}^{-1}_t(1-\alpha)\,\mathrm{d}\alpha=\int_0^1 J\bigl(X^{1-\alpha}_t\bigr)\,\mathrm{d}\alpha,$$

which finishes the proof. □

Remark 2. If $J(X_t)=X_t$, then the expected value of uncertain variable X_t for each $t\in[a,T]$ can be obtained by $\mathbf{E}[X_t]=\int_0^1 X^\alpha_t\,\mathrm{d}\alpha$.

Example. In this subsection, we first introduce a fractional model to describe aftershock frequency in real life, that is, temporal decay feature of aftershocks is characterized by the fractional reactive equation [11, 24]

$$\begin{aligned}
&{}^C_aD^{\nu}_t X_t=\lambda X_t,\quad 0<\nu\leqslant 1,\\
&X_t|_{t=a}=\kappa_0.
\end{aligned}$$

Based on the uncertainty theory, the aftershock frequency is assumed to follow the FUDE

$$\begin{aligned}
&{}^C_aD^{\nu}_t X_t=\lambda X_t+\sigma\frac{\mathrm{d}C_t}{\mathrm{d}t},\quad 0<\nu\leqslant 1,\\
&X_t|_{t=a}=\kappa_0,
\end{aligned}$$

where the parameters λ and $\sigma>0$ are constant.

To reveal the significance of general memory effects in real world, we consider that temporal decay feature of aftershocks frequency obey the following linear Caputo GFUDE with initial condition:

$$\begin{aligned} &{}_a^C D_t^{\nu,g} X_t = \lambda X_t + \sigma \frac{\mathrm{d}C_t}{\mathrm{d}t}, \quad 0 < \nu \leqslant 1, \\ &X_t|_{t=a} = \kappa_0. \end{aligned} \tag{15}$$

Obviously, the coefficient functions $f(t, X_t) = \lambda X_t$ and $h(t, X_t) = \sigma$ satisfy the Lipschitz and linear growth conditions. Using Theorems 1 and 2, the linear Caputo GFUDE (6) with initial condition has a unique solution, that is,

$$\begin{aligned} X_t = {}& \kappa_0 E_{\nu,1}\big(\lambda, \big(g(t) - g(a)\big)^\nu\big) \\ &+ \sigma \int_a^t \big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t) - g(s)\big)^\nu\big) g'(s)\, \mathrm{d}C_s. \end{aligned}$$

Furthermore, according to Definition 13, the corresponding α-path (X_t^α) of the Caputo GFUDE (15) should satisfy the following GFDE with the same initial value condition:

$$\begin{aligned} &{}_a^C D_t^{\nu,g} X_t^\alpha = \lambda X_t^\alpha + \sigma \Phi^{-1}(\alpha), \\ &X_t^\alpha|_{t=a} = \kappa_0, \end{aligned} \tag{16}$$

where $\Phi^{-1}(\alpha)$ denotes the inverse standard normal uncertainty distribution, namely,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}, \quad 0 < \alpha < 1.$$

Hence, the analytical form of the solution of GFDE (16) can be solved as

$$\begin{aligned} X_t^\alpha = {}& \kappa_0 E_{\nu,1}\big(\lambda, \big(g(t) - g(a)\big)^\nu\big) \\ &+ \sigma \int_a^t \big(g(t) - g(s)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t) - g(s)\big)^\nu\big) \Phi^{-1}(\alpha) g'(s)\, \mathrm{d}s. \end{aligned}$$

Especially, let parameters $\lambda = -0.4$ and $\sigma = 1$, and set the initial conditions $a = 1$, $\kappa_0 = 10$ and $\nu = 0.85$. Note that when the general kernel function $g(t)$ is given a special function (for example, $g(t) = t$, $\ln t$, e^t and t^β/β) shown in Fig. 1, the uncertain variable for time $t_1 = 2$ of GFUDE (15) may have different uncertainty distributions.

7 Conclusions

In this study, a class of GFDEs driven by Liu process (GFUDE) is proposed. Existence and uniqueness theorems are given, respectively. Then the relation between α-path and

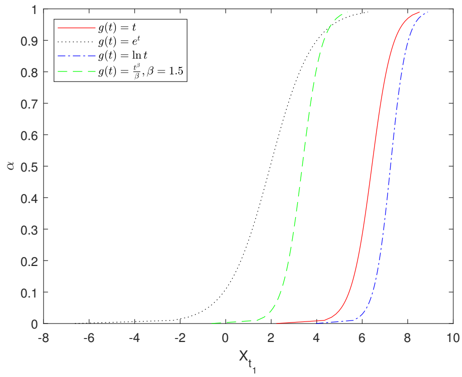

Figure 1. Uncertainty distributions of X_{t_1} with different memory effects.

solution of GFUDE is proved that the known GFUDE can be transformed into a class of GFDEs with the same initial conditions. In addition, a theorem is given to calculate the expected value of a monotone function. A numerical example is provided to declare that GFUDE can be reduced to various fractional uncertain ones within some specific kernel functions, for instance,

- For $g(t) = t$, the GFUDEs (4) and (5) can be reduced as the Riemann–Liouville FUDEs and Caputo FUDEs, respectively [35].
- For $g(t) = \ln t$, the GFUDE (5) can be reduced as the Caputo–Hadamard FUDEs [16].
- For $g(t) = \mathrm{e}^t$, an Exp FUDE can be newly proposed (see Fig. 1).

On the other hand, as an inverse problem, parameter estimation of FUDEs based on observed data is popular very recently, which is also demonstrated to be more accurate than UDEs [28]. In the nearest future, we will consider the parameter estimation of GFUDEs. The study will determine which fractional derivative operator with memory effects is more accurate under uncertain assumptions.

Appendix A

For X_t provided by (7), we obtain

$$\begin{aligned} &\frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s) \lambda X_s \, \mathrm{d}s \\ &\quad = \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s) \lambda \\ &\qquad \times \left(\sum_{i=0}^{n-1} \big(g(s) - g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(s) - g(a)\big)^\nu\big) \right) \mathrm{d}s \end{aligned}$$

$$+ \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\lambda \times \left(\int_a^s \big(g(s) - g(\tau)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(s) - g(\tau)\big)^{\nu}\big) \mu(\tau) g'(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s$$

$$+ \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\lambda \times \left(\int_a^s \big(g(s) - g(\tau)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(s) - g(\tau)\big)^{\nu}\big) \sigma(\tau) g'(\tau)\,\mathrm{d}C_\tau \right) \mathrm{d}s$$

$$:= D_1 + D_2 + D_3. \tag{A.17}$$

In order to simplify (A.17), we have

$$\begin{aligned}
D_1 &= \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\lambda \times \left(\sum_{i=0}^{n-1} \big(g(s) - g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(s) - g(a)\big)^{\nu}\big) \right) \mathrm{d}s \\
&= \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t) - g(s)\big)^{\nu-1} g'(s)\lambda \times \left(\sum_{i=0}^{n-1} \big(g(s) - g(a)\big)^i \kappa_i \sum_{m=0}^{\infty} \frac{\lambda^m (g(s) - g(a))^{m\nu}}{\Gamma(m\nu + i + 1)} \right) \mathrm{d}s \\
&= \frac{1}{\Gamma(\nu)} \sum_{i=0}^{n-1} \kappa_i \left(\sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{\Gamma(m\nu+i+1)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) \big(g(s)-g(a)\big)^{m\nu+i}\,\mathrm{d}s \right) \\
&\quad \left(\text{by } u = \frac{g(s) - g(a)}{g(t) - g(a)} \right) \\
&= \frac{1}{\Gamma(\nu)} \sum_{i=0}^{n-1} \kappa_i \left(\sum_{m=0}^{\infty} \frac{\lambda^{m+1} (g(t) - g(a))^{(m+1)\nu+i}}{\Gamma(m\nu + i + 1)} \int_0^1 (1 - u)^{\nu-1} u^{m\nu+i}\,\mathrm{d}u \right) \\
&= \sum_{i=0}^{n-1} \big(g(t) - g(a)\big)^i \kappa_i \sum_{m=0}^{\infty} \frac{\lambda^{m+1} (g(t) - g(a))^{(m+1)\nu}}{\Gamma((m + 1)\nu + i + 1)} \\
&= \sum_{i=0}^{n-1} \big(g(t) - g(a)\big)^i \kappa_i E_{\nu,i+1}\big(\lambda, \big(g(t) - g(a)\big)^{\nu}\big) - \sum_{i=0}^{n-1} \frac{(g(t) - g(a))^i}{\Gamma(i + 1)} \kappa_i,
\end{aligned}$$

$$
\begin{aligned}
D_2 &= \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\lambda \\
&\qquad \times \left(\int_a^s \big(g(s)-g(\tau)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(s)-g(\tau)\big)^{\nu}\big)\mu(\tau)g'(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s \\
&= \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\lambda \\
&\qquad \times \left(\int_a^s \big(g(s)-g(\tau)\big)^{\nu-1} \sum_{m=0}^{\infty} \frac{\lambda^m (g(s)-g(\tau))^{m\nu}}{\Gamma(m\nu+\nu)} \mu(\tau)g'(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s \\
&= \frac{1}{\Gamma(\nu)} \int_a^t \mu(\tau) \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{\Gamma(m\nu+\nu)} g'(\tau)\,\mathrm{d}\tau \\
&\quad \times \int_\tau^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\big(g(s)-g(\tau)\big)^{m\nu+\nu-1}\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\nu)} \int_a^t \mu(\tau) \sum_{m=0}^{\infty} \frac{\lambda^{m+1}(g(t)-g(\tau))^{m\nu+2\nu-1}}{\Gamma(m\nu+\nu)} g'(\tau)\,\mathrm{d}\tau \\
&\quad \times \int_0^1 (1-\eta)^{\nu-1}\eta^{m\nu+\nu-1}\,\mathrm{d}\eta \\
&\quad \left(\text{by } \eta = \frac{g(s)-g(\tau)}{g(t)-g(\tau)}\right) \\
&= \int_a^t \big(g(t)-g(\tau)\big)^{\nu-1} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}(g(t)-g(\tau))^{(m+1)\nu}}{\Gamma((m+1)\nu+\nu)} \mu(\tau)g'(\tau)\,\mathrm{d}\tau \\
&= \int_a^t \big(g(t)-g(\tau)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(\tau)\big)^{\nu}\big)\mu(\tau)g'(\tau)\,\mathrm{d}\tau \\
&\quad - \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(\tau)\big)^{\nu-1}\mu(\tau)g'(\tau)\,\mathrm{d}\tau, \\
D_3 &= \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\lambda \\
&\qquad \times \left(\int_a^s \big(g(s)-g(\tau)\big)^{\nu-1} E_{\nu,\nu}(\lambda, \big(g(s)-g(\tau)\big)^{\nu}\big)\sigma(\tau)g'(\tau)\,\mathrm{d}C_\tau \right) \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^t \big(g(t)-g(\tau)\big)^{\nu-1} E_{\nu,\nu}\big(\lambda, \big(g(t)-g(\tau)\big)^{\nu}\big)\sigma(\tau)g'(\tau)\,\mathrm{d}\tau \\
&\quad - \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(\tau)\big)^{\nu-1}\sigma(\tau)g'(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Hence, Eq. (9) can be obtained.

Appendix B

According to mathematical induction, for $n = 0$, we get

$$
\begin{aligned}
S_t^{(0)}(\gamma) &= \max_{a\leqslant u\leqslant t} \big|X_u^{(1)}(\gamma) - X_u^{(0)}(\gamma)\big| \\
&= \max_{a\leqslant u\leqslant t} \Bigg| \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) f\big(s, X_s^{(0)}(\gamma)\big)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) h\big(s, X_s^{(0)}(\gamma)\big)\,\mathrm{d}C_s \Bigg| \\
&\leqslant \max_{a\leqslant u\leqslant t} \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big|f\big(s, X_s^{(0)}(\gamma)\big)\big|\,\mathrm{d}s \\
&\quad + \max_{a\leqslant u\leqslant t} \frac{K(\gamma)}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big|h\big(s, X_s^{(0)}(\gamma)\big)\big|\,\mathrm{d}s \\
&\quad \text{(by Lemma 5)} \\
&\leqslant \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) \big|f\big(s, X_s^{(0)}(\gamma)\big)\big|\,\mathrm{d}s \\
&\quad + \frac{K(\gamma)}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) \big|h\big(s, X_s^{(0)}(\gamma)\big)\big|\,\mathrm{d}s \\
&\leqslant \frac{(1+K(\gamma))L(1+|X_s^0|)}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s)\,\mathrm{d}s \\
&\quad \text{(by linear growth condition)} \\
&= \frac{(1+K(\gamma))L\big(g(t)-g(a)\big)^{\nu}}{\Gamma(\nu+1)} \left(1 + \left|\sum_{i=0}^{n-1} \frac{\big(g(t)-g(a)\big)^{i}\kappa_i}{\Gamma(i+1)}\right|\right).
\end{aligned}
$$

Then we assume that inequality (10) for $n-1$ is still valid, that is,

$$S_t^{(n-1)}(\gamma) = \max_{a \leqslant u \leqslant t} \left| X_u^{(n)} - X_u^{(n-1)} \right| \leqslant \frac{[L(1+K(\gamma))(g(t)-g(a))^\nu]^{n-1}}{\Gamma((n-1)\nu+1)} \left(1 + \left| \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)} \right| \right).$$

Then we can prove

$$\begin{aligned}
S_t^{(n)}(\gamma) &= \max_{a \leqslant u \leqslant t} \left| X_u^{(n+1)}(\gamma) - X_u^{(n)}(\gamma) \right| \\
&= \max_{a \leqslant u \leqslant t} \left| \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big(f\big(s, X_s^{(n)}(\gamma)\big) - f\big(s, X_s^{(n-1)}(\gamma)\big)\big)\, \mathrm{d}s \right. \\
&\quad \left. + \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big(h\big(s, X_s^{(n)}(\gamma)\big) - h\big(s, X_s^{(n-1)}(\gamma)\big)\big)\, \mathrm{d}C_s \right| \\
&\leqslant \max_{a \leqslant u \leqslant t} \frac{1}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big| f\big(s, X_s^{(n)}(\gamma)\big) - f\big(s, X_s^{(n-1)}(\gamma)\big) \big|\, \mathrm{d}s \\
&\quad + \max_{a \leqslant u \leqslant t} \frac{K(\gamma)}{\Gamma(\nu)} \int_a^u \big(g(u)-g(s)\big)^{\nu-1} g'(s) \big| h\big(s, X_s^{(n)}(\gamma)\big) - h\big(s, X_s^{(n-1)}(\gamma)\big) \big|\, \mathrm{d}s \\
&\leqslant \frac{1}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) \big| f\big(s, X_s^{(n)}(\gamma)\big) - f\big(s, X_s^{(n-1)}(\gamma)\big) \big|\, \mathrm{d}s \\
&\quad + \frac{K(\gamma)}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) \big| h\big(s, X_s^{(n)}(\gamma)\big) - h\big(s, X_s^{(n-1)}(\gamma)\big) \big|\, \mathrm{d}s \\
&\leqslant \frac{(1+K(\gamma))}{\Gamma(\nu)} \int_a^t \big(g(t)-g(s)\big)^{\nu-1} g'(s) L \big| X_s^{(n)}(\gamma) - X_s^{(n-1)}(\gamma) \big|\, \mathrm{d}s \\
&\leqslant \frac{(1+K(\gamma))^n L^n \big(g(t)-g(a)\big)^{n\nu}}{\Gamma(\nu)\Gamma((n-1)\nu+1)} \left(1 + \left| \sum_{i=0}^{n-1} \frac{(g(t)-g(a))^i \kappa_i}{\Gamma(i+1)} \right| \right) \\
&\quad \times \int_0^1 (1-\eta)^{\nu-1} \eta^{(n-1)\nu}\, \mathrm{d}\eta,
\end{aligned}$$

$$\eta = \frac{g(s)-g(a)}{g(t)-g(a)} = \frac{((1+K(\gamma))L(g(t)-g(a))^\nu)^n}{\Gamma(n\nu+1)} \left(1 + \left| \sum_{i=0}^{n-1} \frac{\big(g(t)-g(a)\big)^i \kappa_i}{\Gamma(i+1)} \right| \right),$$

which implies that inequality (10) is true.

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