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Existence of positive solutions for third-order semipositone boundary value problems on time scales*

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Abstract. In this paper, we consider the existence of positive solutions for a semipositone third-order nonlinear ordinary differential equation on time scales. In suitable growth conditions, by considering the properties on time scales and establishing a special cone, some new results on the existence of positive solutions are established when the nonlinearity is semipositone.

Keywords: existence of positive solution, semipositone, time scales, fixed point theorem, integral boundary conditions.

1 Introduction

In this paper, we focus on the existence of positive solutions for the following third-order nonlinear differential equation on time scales:

$$\begin{aligned} &-\left(x^{\Delta\Delta}\right)^{\nabla}(t) + f\left(t, x\left(\sigma(t)\right)\right) + q(t) = 0, \quad t \in [0,1]_{\mathbb{T}}, \\ &x(0) = x^{\Delta}(t_1) = \int\limits_{t_2}^{1} p(s) x^{\Delta\Delta}(s)\,\nabla s = 0, \end{aligned} \tag{1}$$

where $1/2 < t_1 < t_2 < 1$ are two constants, $[0,1]_{\mathbb{T}} =: [0,1] \cap \mathbb{T}$ denotes the time-scale interval, $f : (0,1)_{\mathbb{T}} \times [0,\infty) \to \mathbb{R}$ is continuous, $q \in L(0,1)_{\mathbb{T}}$, and $p : [t_2, 1] \to [0,\infty)$ is a continuous function.

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Nonlinear differential equations have extensive applications in theory as well as in practice such as in engineering, applied mathematics, gas dynamics, and the physical and biological sciences. Some rich sources of nonlinear differential equations can be found in [4, 5, 7, 12, 13, 17, 19–21, 25–27, 29, 31, 32, 34–36] and singular semipositone boundary value problems in [6, 15, 28, 33, 37]. In particular, Graef and Yang [7] considered the existence and nonexistence of positive solutions for a nonlocal boundary value problem of third-order differential equation

$$\begin{aligned} &u'''(t) = g(t)f\bigl(u(t)\bigr), \quad t \in [0,1], \\ &u(0) = u'(p) = \int_q^1 \omega(t)u''(t)\,\mathrm{d}t = 0, \end{aligned}$$

where $1/2 < p < q < 1$ are constants, $g : [0,1] \to [0,\infty)$ is a continuous function such that $g(t) \not\equiv 0$ on $[0,1]$, $f : [0,\infty) \to [0,\infty)$ is continuous. By using Guo–Krasnosel'skii fixed point theorem, some sufficient conditions for the existence and nonexistence of positive solutions were established. In [6], Graef and Kong established the existence of positive solutions for the following third-order semipositone boundary value problem:

$$\begin{aligned} &u'''(t) = \lambda f(t,u) + e(t), \quad t \in [0,1], \\ &u(0) = u'(p) = \int_q^1 \omega(s)u''(s)\,\mathrm{d}s = 0, \end{aligned}$$

where $\lambda > 0$ is a parameter, $1/2 < p < q < 1$ are constants, $f : (0,1) \times [0,\infty) \to \mathbb{R}$, $e : (0,1) \to \mathbb{R}$ and $\omega : [q,1] \to [0,\infty)$ are continuous functions, and $e \in L(0,1)$. For more details about multiple point boundary value problems and integral boundary value problems, we refer the reader to the survey of [22, 36, 38] and [11, 23, 24, 26, 30, 31].

On the other hand, in nature, there exist many time scales such as the Cantor set, the set of harmonic numbers $\{\sum_{k=1}^{n} 1/k,\ n \in \mathbb{Z}\}$ and $h\mathbb{Z}$, $h > 0$, and so on. An example is a population of a species where all of the adults die out before the babies are born, which leads to a union of disjoint closed intervals, i.e., a time scale. In [9], Hao et al. dealt with the following boundary value problem of singular nonlinear dynamic equation on time scales:

$$\begin{aligned} &\bigl(\varphi(t)x^{\Delta}\bigr)^{\nabla}(t) + \lambda m(t)f\bigl(t, x\bigl(\sigma(t)\bigr)\bigr) = 0, \quad t \in (a,b), \\ &\alpha x(a) - \beta x(a) = 0, \\ &\gamma z\bigl(\sigma(b)\bigr) + \delta z^{\Delta}\bigl(\sigma(b)\bigr) = 0, \end{aligned}$$

where $f \in C([a,\sigma(b)] \times [0,+\infty), (0,+\infty))$. By employing the Krasnosel'skii fixed point theorem, an existence theorem of positive solutions was established. For other specific examples and research on related problems, we refer the reader to [2, 3, 10, 16, 18].

However, to the best of our knowledge, few results have been reported for semipositone nonlinear dynamic equation on time scales, thus motivated by the above works, in this paper, we focus on the existence of positive solutions for the third-order nonlinear differential equation on time scales (1) when the nonlinearity is semipositone. By introducing suitable growth conditions and constructing a special cone, some new results on the existence of positive solutions are established under the case where nonlinearity is semipositone.

This paper is organized as follows. In Section 2, we give some preliminaries and lemmas, which will be used to prove our main results. In Section 3, we discuss the existence of positive solutions of the boundary value problems by using the fixed point theorem.

2 Preliminaries and lemmas

To understand so-called time scale (measure chain), in this section, we firstly start with some preliminaries of time scales from recent literatures [1, 9, 14].

Definition 1. Define the forward jump and backward jump operators at t for $t < \sup \mathbb{T}$ and $t > \inf \mathbb{T}$, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T}\colon\ s > t\} \in \mathbb{T},$$
$$\rho(t) := \sup\{s \in \mathbb{T}\colon\ s < t\} \in \mathbb{T}.$$

The point $t \in \mathbb{T}$ is left dense, left scattered, right dense, and right scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. The set $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise, it is $\mathbb{T}$ without this left-scattered maximum.

Definition 2. Assume that $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $x^{\Delta}(t)$ to be the number with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\bigl|x\bigl(\sigma(t)\bigr) - x(s) - x^{\Delta}(t)\bigl[\sigma(t) - s\bigr]\bigr| \leqslant \varepsilon\bigl|\sigma(t) - s\bigr|$$

for all $s \in U$, $t \in \mathbb{T}$. The second derivative of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\nabla}(t)$.

In order to obtain our main results, we give some assumptions that will be used in the rest paper.

- (G1) $q \in L((0,1)_{\mathbb{T}}, (-\infty,+\infty))$, and $p : [t_2, 1]_{\mathbb{T}} \to [0,\infty)$ is a continuous and nondecreasing function.
- (G2) There exist functions $M, N \in L((0,1)_{\mathbb{T}}, (0,+\infty))$ and $g \in C([0,+\infty), (0,+\infty))$ such that $M(t) \leqslant f(t,x) \leqslant N(t)g(x)$, $(t,x) \in (0,1)_{\mathbb{T}} \times [0,\infty)$.
- (G3) $\lim_{x\to+\infty} f(t,x)/x = +\infty$ uniformly on any compact subinterval of $(0,1)_{\mathbb{T}}$.
- (G4) $\lim_{x\to+\infty} g(x)/x = 0$.

Let $I \subseteq \mathbb{R}$ be an interval, denote the characteristic function X of I as

$$X_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Then it follows from [7] that the Green's function for the equation $(x^{\Delta\Delta})^{\nabla} = 0$ subject to the boundary condition

$$x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s)x^{\Delta\Delta}(s)\nabla s = 0$$

is

$$\begin{aligned} G(t,s) &= -t(t_1 - s)X_{[0,t_1]_{\mathbb{T}}}(s) + \frac{(t-s)^2}{2}X_{[0,t]_{\mathbb{T}}}(s) \\ &\quad + \frac{t(2t_1 - t)}{2}P(s)X_{[t_2,1]_{\mathbb{T}}}(s) + \frac{t(2t_1 - t)}{2}X_{[0,t_2]_{\mathbb{T}}}(s), \end{aligned}$$

where

$$P(s) = \left(\int_{t_2}^{1} p(v)\nabla v\right)^{-1} \int_{s}^{1} p(v)\,\nabla v, \quad s \in [t_2, 1]_{\mathbb{T}}.$$

Clearly, $G(t,s) > 0$, $t, s \in (0,1)_{\mathbb{T}}$.

Lemma 1. *Assume that* (G1) *holds, then for any* $t, s \in [0,1]_{\mathbb{T}}$, *the Green's function satisfies*

$$G(t,s) \leqslant c(t)d(s),$$

where

$$c(t) = \frac{2t}{t_1} - \frac{t^2}{t_1^2}, \qquad d(s) = \frac{t_1^2}{2}\left[\frac{2t_1}{2t_1 - 1} + P(s)\right].$$

Proof. Firstly, we consider the case $s \geqslant t_1$.

$$\begin{aligned} G(t,s) &= \frac{t(2t_1 - t)}{t_1^2}\left\{\frac{t_1^2}{2}\left[P(s)X_{[t_2,1]_{\mathbb{T}}}(s) + X_{[0,t_2]_{\mathbb{T}}}(s) + \frac{(t-s)^2}{t(2t_1 - t)}X_{[0,t]_{\mathbb{T}}}(s)\right]\right\} \\ &\leqslant c(t)\left\{\frac{t_1^2}{2}\left[P(s)X_{[t_2,1]_{\mathbb{T}}}(s) + X_{[0,t_2]_{\mathbb{T}}}(s) + \frac{1}{2t_1 - 1}X_{[0,t]_{\mathbb{T}}}(s)\right]\right\} \\ &\leqslant c(t)\left\{\frac{t_1^2}{2}\left[P(s) + 1 + \frac{1}{2t_1 - 1}\right]\right\} = c(t)\left\{\frac{t_1^2}{2}\left[P(s) + \frac{2t_1}{2t_1 - 1}\right]\right\} \\ &= c(t)d(s). \end{aligned}$$

Next, we consider the case $s \leqslant t_1$.
If $s \geqslant t$, we have

$$G(t,s) = \frac{t(2s - t)}{2} = \frac{t(2t_1 - t)}{t_1^2}\left[\frac{t_1^2(2s - t)}{2(2t_1 - t)}\right] \leqslant c(t)\frac{t_1^2 s}{2t_1 - 1} \leqslant c(t)d(s).$$

If $s \leqslant t$, we still have

$$G(t,s) = \frac{s^2}{2} = \frac{t(2t_1 - t)}{t_1^2}\left[\frac{t_1^2 s^2}{2t(2t_1 - t)}\right] \leqslant c(t)\frac{t_1^2 s}{2(2t_1 - 1)} \leqslant c(t)d(s). \qquad \square$$

Remark 1. Noticing $1/2 < t_1 < t_2 < 1$ and $c(t) = 2t/t_1 - t^2/t_1^2$, one has $0 \leqslant c(t) \leqslant 1$ for any $t \in [0,1]_{\mathbb{T}}$.

Now let

$$q_+(t) := \max\{q(t), 0\}, \qquad q_-(t) = \max\{-q(t), 0\}.$$

Consider the following linear equation of third-order boundary value problem on time scales:

$$\begin{gathered}\left(x^{\Delta\Delta}\right)^{\nabla}(t) = q_-(t) + M(t), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s) x^{\Delta\Delta}(s) \nabla s = 0.\end{gathered} \tag{2}$$

Lemma 2. *The linear equation of third-order boundary value problem on time scales* (2) *has a unique solution $\omega(t)$, and there exists a constant $\rho > 0$ such that $\omega(t) \leqslant \rho c(t)$, where*

$$\rho = \|d\| \int_0^1 \left[q_-(s) + M(s)\right] \nabla s.$$

Proof. Obviously, $\omega(t) = \int_0^1 G(t,s)[q_-(s) + M(s)]\,\mathrm{d}s$ is a unique solution of the BVP (2). It follows from Lemma 1 that

$$\omega(t) \leqslant c(t) \int_0^1 d(s)\left[q_-(s) + M(s)\right] \nabla s \leqslant c(t)\|d\| \int_0^1 \left[q_-(s) + M(s)\right] \nabla s.$$

Let $\rho = \|d\| \int_0^1 [q_-(s) + M(s)] \nabla s$, then we have

$$\omega(t) \leqslant \rho c(t), \quad t \in [0,1]_{\mathbb{T}}.$$

In particular, it follows from Lemma 2 that the following boundary value problem

$$\begin{gathered}\left(x^{\Delta\Delta}\right)^{\nabla}(t) = M(t), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s) x^{\Delta\Delta}(s) \nabla s = 0.\end{gathered}$$

has a unique solution $\gamma(t)$ in the form

$$\gamma(t) = \int_0^1 G(t,s) M(s) \nabla s, \quad t, s \in [0,1]_{\mathbb{T}}.$$

Remark 2. Given $\gamma(t) \leqslant \omega(t)$ and Remark 1, we have $\gamma(t) \leqslant \rho$.

Now for any $t \in [0,1]_{\mathbb{T}}$, let us define a star function as follows:

$$[y]^*(t) = \begin{cases} y(t), & y(t) \geqslant 0, \\ 0, & y(t) < 0, \end{cases}$$

and then consider the following boundary value problem:

$$\begin{aligned} &\left(x^{\Delta\Delta}\right)^{\nabla}(t) = f\bigl(t, \bigl([x-\omega]^* + \gamma\bigr)\bigl(\sigma(t)\bigr)\bigr) + q_+(t), \quad t \in [0,1]_{\mathbb{T}}, \\ &x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s) x^{\Delta\Delta}(s)\, \nabla s = 0. \end{aligned} \tag{3}$$

The proof is complete. □

Lemma 3. *Let the nonlinear boundary value problem* (3) *has a solution* $x(t)$ *such that* $x(t) \geqslant \omega(t)$. *Then* $y(t) = x(t) - \omega(t) + \gamma(t)$ *is a positive solution of equation* (1) *with* $y(t) \geqslant \gamma(t)$, $t \in [0,1]_{\mathbb{T}}$.

Proof. It follows from the fact that $x(t)$ is a positive solution of nonlinear boundary value problem satisfying $x(t) \geqslant \omega(t)$ that

$$\begin{aligned} &\left(x^{\Delta\Delta}\right)^{\nabla}(t) = f\bigl(t, \bigl((x-\omega) + \gamma\bigr)\bigl(\sigma(t)\bigr)\bigr) + q_+(t), \quad t \in [0,1]_{\mathbb{T}}, \\ &x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s) x^{\Delta\Delta}(s)\, \nabla s = 0. \end{aligned}$$

Thus,

$$\begin{aligned} \left(y^{\Delta\Delta}\right)^{\nabla}(t) &= \left(x^{\Delta\Delta}\right)^{\nabla}(t) - \left(\omega^{\Delta\Delta}\right)^{\nabla}(t) + \left(\gamma^{\Delta\Delta}\right)^{\nabla}(t) \\ &= f\bigl(t, \bigl((x-\omega) + \gamma\bigr)\bigl(\sigma(t)\bigr)\bigr) + q_+(t) - q_-(t) - M(t) + M(t) \\ &= f\bigl(t, y\bigl(\sigma(t)\bigr)\bigr) + q(t), \end{aligned}$$

and boundary condition $y(0) = y^{\Delta}(t_1) = \int_{t_2}^{1} p(s) y^{\Delta\Delta}(s)\, \nabla s = 0$ also holds. Thus, the proof of Lemma 3 is completed. □

From Lemma 2 and the strategy of [7] we have the following lemma.

Lemma 4. *If* $x \in C^3[0,1]_{\mathbb{T}}$ *satisfies*

$$\begin{aligned} &\left(x^{\Delta\Delta}\right)^{\nabla}(t) \geqslant 0, \quad t \in [0,1]_{\mathbb{T}}, \\ &x(0) = x^{\Delta}(t_1) = \int_{t_2}^{1} p(s) x^{\Delta\Delta}(s)\, \nabla s = 0, \end{aligned} \tag{4}$$

then $x(t) \geqslant c(t)\|x\| \geqslant \min\{t,\, 1-t\}\|x\| \geqslant 0$, $t \in [0,1]_{\mathbb{T}}$.

Proof. In order to prove Lemma 4, we firstly show that x attains its maximum at t_1, i.e., $\|x\| = x(t_1)$. In fact, it follows from (4) that

$$\int_{t_2}^{1} p(s)x^{\Delta\Delta}(s)\,\nabla s = 0,$$

which implies that there exists $t_0 \in (t_2, 1)_{\mathbb{T}}$ such that $p(t_0)x^{\Delta\Delta}(t_0) = 0$. By (G1), we have $x^{\Delta\Delta}(t_0) = 0$. Since $(x^{\Delta\Delta})^{\nabla}(t) \geqslant 0$, we have that $x^{\Delta\Delta}(t)$ is nondecreasing on $[0,1]_{\mathbb{T}}$, it follows from $x^{\Delta\Delta}(t_0) = 0$ that

$$x^{\Delta\Delta}(t) \leqslant 0, \quad t \in [0, t_0]_{\mathbb{T}}; \qquad x^{\Delta\Delta}(t) \geqslant 0, \quad t \in [t_0, 1]_{\mathbb{T}}, \tag{5}$$

which implies that $x^{\Delta}(t)$ is nonincreasing on $[0, t_0]_{\mathbb{T}}$. Notice $[t_1, t_2] \subset [0, t_0]_{\mathbb{T}}$ and $x^{\Delta}(t_1) = 0$, then one has $x^{\Delta}(t_2) \leqslant 0$ and $x^{\Delta}(0) \geqslant 0$.

On the other hand, by (5) and monotonicity of p, for $t \in [t_2, 1]_{\mathbb{T}}$, whether $t \in [t_2, t_0,]_{\mathbb{T}}$ or $t \in [t_0, 1]_{\mathbb{T}}$, we always have

$$\big(p(t) - p(t_0)\big)x^{\Delta\Delta}(t) \geqslant 0.$$

Therefore,

$$\begin{aligned}
0 &= \int_{t_2}^{1} p(s)x^{\Delta\Delta}(s)\nabla s \\
&= \int_{t_2}^{1} p(t_0)x^{\Delta\Delta}(s)\nabla s + \int_{t_2}^{1} \big(p(s) - p(t_0)\big)x^{\Delta\Delta}(s)\nabla s \\
&\geqslant \int_{t_2}^{1} p(t_0)x^{\Delta\Delta}(s)\nabla s = p(t_0)\big(x^{\Delta}(1) - x^{\Delta}(t_2)\big),
\end{aligned}$$

which implies that $x^{\Delta}(1) - x^{\Delta}(t_2) \leqslant 0$. Thus, since $x^{\Delta}(t)$ is concave on $[0,1]_{\mathbb{T}}$,

$$x^{\Delta}(t) \geqslant 0, \quad t \in [0, t_1]_{\mathbb{T}}; \qquad x^{\Delta}(t) \leqslant 0, \quad t \in [t_1, 1]_{\mathbb{T}},$$

that is, $x(t)$ attains its maximum at t_1.

Now let $\|x\| = x(t_1)$ and

$$y(t) = x(t) - c(t)\|x\| = x(t) - \left(\frac{2t}{t_1} - \frac{t^2}{t_1^2}\right)\|x\|,$$

then

$$y^{\Delta}(t) = x^{\Delta}(t) - \left(\frac{2}{t_1} - \frac{2t}{t_1^2}\right)\|x\|, \qquad y^{\Delta\Delta}(t) = x^{\Delta\Delta}(t) + \frac{2}{t_1^2\|x\|},$$
$$\big(y^{\Delta\Delta}\big)^{\nabla}(t) = \big(x^{\Delta\Delta}\big)^{\nabla}(t) \geqslant 0.$$

Clearly, we have

$$y(0) = 0, \qquad y(t_1) = 0, \qquad y^{\Delta}(t_1) = 0.$$

By the mean value theorem, there exists $l_1 \in (0, t_1)_{\mathbb{T}}$ such that $y^{\Delta}(l_1) = 0$. Since $y^{\Delta}(t)$ is concave, we have

$$y^{\Delta}(t) \geqslant 0, \quad t \in [0, l_1]_{\mathbb{T}}; \qquad y^{\Delta}(t) \leqslant 0, \quad t \in [l_1, t_1]_{\mathbb{T}};$$
$$y^{\Delta}(t) \geqslant 0, \quad t \in [t_1, 1]_{\mathbb{T}}.$$

It follows from $y(0) = y(t_1) = 0$ that $y(t) \geqslant 0$, $t \in [0, 1]_{\mathbb{T}}$, i.e.,

$$x(t) \geqslant c(t)\|x\| \geqslant \min\{t,\, 1 - t\}\|x\| \geqslant 0, \quad t \in [0, 1]_{\mathbb{T}}. \qquad \square$$

In this paper, our working space is the Banach space $E = C[0, 1]_{\mathbb{T}}$, which equips the usual maximum normal $\|x(t)\| = \max_{t \in [0,1]_{\mathbb{T}}} |x(t)|$. Now define a cone $K \subset E$ as

$$K = \bigl\{x \in E\colon\, x(t_1) \geqslant 0,\ c(t)\|x\| \leqslant x(t) \leqslant x(t_1),\ t \in [0, 1]_{\mathbb{T}}\bigr\},$$

where $c(t)$ is given by Lemma 1, and then define an operator $T : K \to E$ as follows:

$$(Tx)(t) = \int_0^1 G(t, s)\bigl[f\bigl(s, \bigl([x - \omega]^* + \gamma)(\sigma(s))\bigr) + q_+(s)\bigr]\, \nabla s.$$

According to Lemma 3, the solution of equation (1) is equivalent to the fixed point of the operator T.

Lemma 5. (*See* [8].) *Let E be a real Banach space, K is a cone of E. Assume that Ω_1, Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ satisfies one of the following conditions*:

(i) $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_1$; *and* $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_1$; *and* $\|Tx\| \leqslant \|x\|$, $x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

Theorem 1. *Suppose that* (G1)–(G3) *are satisfied, and*

$$\frac{\int_0^1 [q_-(s) + M(s)]\nabla s}{\int_0^1 [q_+(s) + N(s)]\nabla s} \geqslant \max_{x \in [0, 2\rho]} g(x) + 1, \tag{6}$$

where ρ is defined by Lemma 2*. Then the boundary value problem* (1) *has at least one positive solution $y(t)$ satisfying $y(t) \geqslant \gamma(t)$ on* $[0, 1]_{\mathbb{T}}$.

Proof. In view of Lemma 3, it is sufficient to prove that the boundary value problem (3) has a solution $x(t)$ satisfying $x(t) \geqslant \omega(t)$, i.e., we only need to prove that T has a fixed point $x(t) \geqslant \omega(t)$, $t \in [0,1]_{\mathbb{T}}$. To do this, we firstly show that the operator T is well defined and $T(K) \subset K$ is completely continuous.

In fact, for any $x \in K$, there is a positive constant L such that $\|x\| \leqslant L$ and

$$[x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(t)\bigr) \leqslant \|x\| + \|\gamma\| \leqslant L + \rho.$$

Thus, for any $t \in [0,1]_{\mathbb{T}}$, let

$$\overline{N} = \max_{(t,v)\in[0,1]_{\mathbb{T}}\times[0,L+\rho]} f(t,v),$$

where ρ is defined by Lemma 2. Then it follows from Lemma 2 and (G1)–(G2) that

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)\bigl[f\bigl(s, \bigl([x-\omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr]\nabla s \\
&\leqslant c(t)\int_0^1 d(s)\bigl[f\bigl(s, [x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr]\,\nabla s \\
&\leqslant c(t)\int_0^1 d(s)\bigl(\overline{N} + q_+(s)\bigr)\,\nabla s \leqslant \int_0^1 d(s)\bigl(\overline{N} + q_+(s)\bigr)\,\nabla s \\
&< +\infty,
\end{aligned}$$

which implies that T is well defined and uniformly bounded on K.

Next, we show that the operator $T : K \to K$. In fact, for any $x \in K$, $t \in [0,1]_{\mathbb{T}}$, we have

$$\bigl((Tx)^{\Delta\Delta}\bigr)^{\nabla}(t) = f\bigl(s, [x-\omega]^*\bigl(\sigma(t)\bigr) + \gamma\bigl(\sigma(t)\bigr)\bigr) + q_+(t) \geqslant 0, \quad t \in [0,1]_T,$$

and by simple computation, we also have the corresponding boundary conditions

$$(Tx)(0) = (Tx)^{\Delta}(t_1) = \int_{t_2}^1 p(s)(Tx)^{\Delta\Delta}(s)\,\nabla s = 0.$$

So the same type of arguments as those used in Lemma 4 shows the operator $T : K \to K$. Thus, the operator $T : K \to K$ is well defined, and $T(K) \subset (K)$.

Now we shall prove that $T(K)$ is equicontinuous. For any $\varepsilon > 0$, $m_1, m_2 \in [0,1]_{\mathbb{T}}$, a fixed $s \in [0,1]_{\mathbb{T}}$, there exists $\delta > 0$ and $|m_1 - m_2| < \delta$ such that

$$\bigl|G(m_1,s) - G(m_2,s)\bigr| < \epsilon\left[(N+1)\|d\|\int_0^1 \bigl(q_+(s)+1\bigr)\nabla s\right]^{-1}$$

and

$$\begin{aligned}
&\bigl|(Tx)(m_1) - (Tx)(m_2)\bigr| \\
&\quad \leqslant \int_0^1 \bigl|G(m_1, s) - G(m_2, s)\bigr| \bigl[f\bigl(s, [x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr] \nabla s \\
&\quad \leqslant \bigl|G(m_1, s) - G(m_2, s)\bigr| \Biggl[(N+1)\|d\| \int_0^1 \bigl[q_+(s) + 1\bigr] \nabla s\Biggr] < \epsilon,
\end{aligned}$$

i.e., $T(K)$ is equicontinuous. According to the Ascoli–Arzela Theorem, $T(K)$ is a relatively compact set, and then $T: K \to K$ is a completely continuous operator.

Let $\Omega_1 = \{x \in E\colon \|x\| < \rho\}$ and $\partial\Omega_1 = \{x \in E\colon \|x\| = \rho\}$. We shall show that $\|Tx\| \leqslant \|x\|$ for any $x \in K \cap \partial\Omega_1$. In fact, for any $x \in K \cap \partial\Omega_1$, it follows from Remarks 1 and 2 that

$$[x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr) \leqslant x\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr) \leqslant \|x\| + \rho = 2\rho,$$

thus we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t, s)\bigl[f\bigl(s, \bigl([x-\omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr] \nabla s \\
&\leqslant \int_0^1 d(s)\bigl[f\bigl(s, [x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr] \nabla s \\
&\leqslant \int_0^1 d(s)\bigl[N(s) g\bigl([x-\omega]^*\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr] \nabla s \\
&\leqslant \|d\| \Bigl(\max_{x \in [0, 2\rho]} g(x) + 1\Bigr) \int_0^1 \bigl[N(s) + q_+(s)\bigr] \nabla s \\
&\leqslant \|d\| \int_0^1 \bigl[M(s) + q_-(s)\bigr] \nabla s = \rho = \|x\|.
\end{aligned}$$

Therefore, we have $\|Tx\| \leqslant \|x\|$ for any $x \in K \cap \partial\Omega_1$.

On the other hand, take $0 < \alpha < \beta < t_1$ and choose

$$\lambda = \Biggl\{\frac{2\alpha t_1 - \alpha^2}{2t_1^2} \sup_{t \in [0,1]_{\mathbb{T}}} \int_\alpha^\beta G(t, s) \nabla s\Biggr\}^{-1}. \tag{7}$$

By (G3), there exists $R_1 > \rho$ such that for any $x > R_1$ and $t \in [\alpha, \beta]_{\mathbb{T}}$,

$$f(t, x) > \lambda x. \tag{8}$$

Let

$$R > \frac{2R_1 t_1^2}{2\alpha t_1 - \alpha^2} + \rho,$$

it follows from $0 < \alpha < t_1$ that

$$R > \frac{2R_1 t_1^2}{2\alpha t_1 - \alpha^2} + \rho > \frac{2R_1 t_1^2}{2\alpha t_1 - \alpha^2} > 2R_1 > 2\rho.$$

Thus, let $\Omega_2 = \{x \in E\colon \|x\| < R\}$ and $\partial\Omega_2 = \{x \in E\colon \|x\| = R\}$. We shall show that $\|Tx\| \geqslant \|x\|$ for $x \in K \cap \partial\Omega_2$.

Firstly, for any $x \in K \cap \partial\Omega_2$ and $t \in [\alpha, \beta]_{\mathbb{T}}$, we have

$$\begin{aligned} x(t) - \omega(t) + \gamma(t) &\geqslant x(t) - \omega(t) \geqslant x(t) - \rho c(t) \\ &\geqslant x(t) - \frac{\rho x(t)}{\|x\|} = \left(1 - \frac{\rho}{R}\right) x(t) \geqslant \frac{1}{2} x(t) \\ &\geqslant \frac{1}{2} R c(t) \geqslant \frac{1}{2} R \cdot \frac{2\alpha t_1 - \alpha^2}{t_1^2} \geqslant R_1 > 0. \end{aligned} \tag{9}$$

It follows from (7)–(9) that

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]_{\mathbb{T}}} \bigl|(Tx)(t)\bigr| \\ &= \max_{t\in[0,1]_{\mathbb{T}}} \int_0^1 G(t,s)\bigl[f\bigl(s, \bigl([x-\omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr]\nabla s \\ &\geqslant \max_{t\in[0,1]_{\mathbb{T}}} \int_\alpha^\beta G(t,s) f\bigl(s, \bigl([x-\omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr)\,\nabla s \\ &\geqslant \max_{t\in[0,1]_{\mathbb{T}}} \int_\alpha^\beta G(t,s)\lambda\bigl(x\bigl(\sigma(s)\bigr) - \omega\bigl(\sigma(s)\bigr) + \gamma\bigl(\sigma(s)\bigr)\bigr)\,\nabla s \\ &\geqslant \max_{t\in[0,1]_{\mathbb{T}}} \int_\alpha^\beta G(t,s) \frac{\lambda R}{2} \cdot \frac{2\alpha t_1 - \alpha^2}{t_1^2}\,\nabla s \\ &\geqslant \frac{\lambda R}{2} \cdot \frac{2\alpha t_1 - \alpha^2}{t_1^2} \max_{t\in[0,1]_{\mathbb{T}}} \int_\alpha^\beta G(t,s)\,\nabla s \geqslant R = \|x\|, \end{aligned}$$

i.e., $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_2$. By Lemma 5, T has a fixed point $x \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ satisfying $\rho \leqslant \|x\| \leqslant R$. In addition, notice that

$$x(t) \geqslant c(t)\|x\| \geqslant \rho c(t) \geqslant \omega(t),$$

i.e., $x(t) \geqslant \omega(t)$. Let $y(t) = x(t) - \omega(t) + \gamma(t)$, then from Lemma 3 the BVP (1) has at least a positive solution $y(t)$ satisfying $y(t) \geqslant \gamma(t)$ on $[0,1]_{\mathbb{T}}$. □

Theorem 2. *Suppose that* (G1)–(G2) *and* (G4) *are satisfied and*

(G5) *There exist constants* $0 < \alpha < \beta < t_1 < 1$ *such that for any* $(t, x) \in [\alpha, \beta]_{\mathbb{T}} \times [\kappa\rho, 3\rho]$, $f(t, x) \leqslant 2\rho/\theta$, *where* $\kappa = (2\alpha t_1 - \alpha^2)/t_1^2$, $\theta = \int_\alpha^\beta G(\alpha, s)\,\nabla s$.

Then the boundary value problem (1) *has at least one positive solution* $y(t)$ *satisfying* $y(t) \geqslant \gamma(t)$ *on* $[0, 1]_{\mathbb{T}}$.

Proof. It follows from Theorem 1 that $T(K) \subset K$ is completely continuous.

Now let $\Omega_3 = \{x \in K\colon \|x\| < 2\rho\}$ and $\partial\Omega_3 = \{x \in K\colon \|x\| = 2\rho\}$. Then for any $x \in \partial\Omega_3$, $t \in [0, 1]_{\mathbb{T}}$, we have

$$\begin{aligned} x(t) - \omega(t) + \gamma(t) &\geqslant x(t) - \omega(t) \geqslant x(t) - \rho c(t) \geqslant x(t) - \frac{\rho x(t)}{\|x\|} = \frac{1}{2}x(t) \\ &\geqslant \rho c(t) \geqslant 0. \end{aligned} \tag{10}$$

So from (10) for any $x \in \partial\Omega_3$, $t \in [\alpha, \beta]_{\mathbb{T}}$, one gets

$$\kappa\rho = \frac{2\alpha t_1 - \alpha^2}{t_1^2}\rho \leqslant x(t) - \omega(t) + \gamma(t) \leqslant 3\rho. \tag{11}$$

Consequently, for any $x \in K \cap \partial\Omega_3$, it follows from (11) and (G5) that

$$\begin{aligned} \|Tx\| &\geqslant \int_0^1 G(\alpha, s)\bigl[f\bigl(s, \bigl([x - \omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr]\nabla s \\ &\geqslant \int_\alpha^\beta G(\alpha, s) f\bigl(s, \bigl([x - \omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr)\,\nabla s \geqslant \int_\alpha^\beta G(\alpha, s)\frac{2\rho}{\theta}\nabla s \\ &= 2\rho = \|x\|, \end{aligned}$$

i.e., $\|Tx\| \geqslant \|x\|$, $x \in K \cap \partial\Omega_3$.

Next, choose $\varepsilon > 0$ such that $\epsilon\|d\| \int_0^1 N(s)\,\nabla s < 1$. Then for the above ε, by (G4), there exists $\widetilde{N} > 2\rho > 0$ such that $g(x) \leqslant \varepsilon x$ if $x > \widetilde{N}$.

Take

$$\begin{aligned} R_4 = {}& \frac{\|d\|(\max_{x\in[0,\widetilde{N}]} g(x) + 1)\int_0^1 [N(s) + q_+(s)]\nabla s + \|d\|(\epsilon\rho + 1)\int_0^1 [N(s) + q_+(s)]\nabla s}{1 - \epsilon\|d\|\int_0^1 N(s)\nabla s} \\ &+ \widetilde{N}, \end{aligned}$$

then $R_4 > \widetilde{N} > 2\rho$.

Now let $\Omega_4 = \{x \in K\colon \|x\| < R_4\}$ and $\partial\Omega_4 = \{x \in K\colon \|x\| = R_4\}$. Then for any $x \in K \cap \partial\Omega_4$, we have

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]_{\mathbb{T}}} \bigl|(Tx)(t)\bigr| \\ &= \max_{t\in[0,1]_{\mathbb{T}}} \int_0^1 G(t, s)\bigl[f\bigl(s, \bigl([x - \omega]^* + \gamma\bigr)\bigl(\sigma(s)\bigr)\bigr) + q_+(s)\bigr]\,\nabla s \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int\limits_0^1 d(s)\left[N(s)g\left([x-\omega]^*(\sigma(s)) + \gamma(\sigma(s))\right) + q_+(s)\right] \nabla s \\
&\leqslant \|d\| \Big(\max_{x\in[0,\widetilde{N}]} g(x) + 1\Big) \int\limits_0^1 \left[N(s) + q_+(s)\right] \nabla s \\
&\quad + \|d\| \int\limits_0^1 \left[N(s)\epsilon\left(\|x\| + \rho\right) + q_+(s)\right] \nabla s \\
&\leqslant \|d\| \Big(\max_{x\in[0,\widetilde{N}]} g(x) + 1\Big) \int\limits_0^1 \left[N(s) + q_+(s)\right] \nabla s \\
&\quad + \|d\|(\epsilon\rho + 1) \int\limits_0^1 \left[N(s) + q_+(s)\right] \nabla s + \epsilon\|d\| \int\limits_0^1 N(s)\, \nabla s R_4 \\
&\leqslant R_4 = \|x\|,
\end{aligned}
$$

which implies that

$$\|Tx\| \leqslant \|x\|, \quad x \in K \cap \partial\Omega_4.$$

By Lemma 5, T has a fixed point $x \in K \cap (\overline{\Omega_4} \setminus \Omega_3)$ satisfying $2\rho \leqslant \|x\| \leqslant R_4$.

It follows from (10) that

$$x(t) - \omega(t) + \gamma(t) \geqslant \rho c(t) + \gamma(t) \geqslant \gamma(t).$$

Let $y(t) = x(t) - \omega(t) + \gamma(t)$, then from Lemma 3 the BVP (1) has at least a positive solution $y(t)$ satisfying $y(t) \geqslant \gamma(t)$ on $[0,1]_{\mathbb{T}}$. □

4 Numerical examples

In this section, we present two examples to illustrate our main results.

Example 1. Let $\mathbb{T} = \{1/2^n\}_{n=0}^{\infty} \cup \{0,1\}$. Consider the following third-order boundary value problem on time scales:

$$
\begin{aligned}
&-\left(x^{\Delta\Delta}\right)^{\nabla}(t) + \left(\frac{1}{2000}x^2\left(\sigma(t)\right) + 1\right)\left(\mathrm{e}^{x(\sigma(t))/100} + 1\right)(t+1) \\
&\quad - \frac{2}{\sqrt{t}} = 0, \quad t \in [0,1]_{\mathbb{T}}, \\
&x(0) = x^{\Delta}\left(\frac{2}{3}\right) = \int\limits_{3/4}^1 s x^{\Delta\Delta}(s)\, \nabla s = 0.
\end{aligned}
\tag{12}
$$

We have $t_1 = 2/3$, $t_2 = 3/4$, $p(t) = t$, $q(t) = -2/\sqrt{t}$ and

$$f(t,x) = \left(\frac{1}{2000}x^2 + 1\right)\left(\mathrm{e}^{x/100} + 1\right)(t+1).$$

Obviously, (G1) holds.

In addition, according to Lemmas 1 and 2, we have $\|d\| = 10/9$ and

$$\begin{aligned}
\rho &= \|d\| \int_0^1 \left[q_-(s) + M(s)\right] \nabla s = \frac{10}{9} \int_0^1 \left[\frac{2}{\sqrt{s}} + 2(s+1)\right] \nabla s \\
&= \frac{10}{9}\left[\int_0^1 \frac{2}{\sqrt{s}} \nabla s + 2\int_0^1 (s+1)\nabla s\right] = \frac{20}{9}\int_0^1 \frac{1}{\sqrt{s}}\,\nabla s + \frac{20}{9}\int_0^1 s\,\nabla s + \frac{20}{9}\int_0^1 \nabla s \\
&= \frac{20}{9}\left[1\cdot\frac{1}{2} + \frac{1}{\sqrt{\frac{1}{2}}}\cdot\frac{1}{4} + \frac{1}{\sqrt{\frac{1}{4}}}\cdot\frac{1}{8} + \cdots\right] \\
&\quad + \frac{20}{9}\left[1\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4}\cdot\left(\frac{1}{4} - \frac{1}{8}\right) + \cdots\right] + \frac{20}{9} \\
&= \frac{20}{9}\left(\frac{1}{2-\sqrt{2}} + \frac{2}{3} + 1\right) \approx \frac{200}{27}.
\end{aligned}$$

Let $q_+(t) = 0$, $q_-(t) = 2/\sqrt{t}$, $M(t) = 2(t+1)$, $N(t) = t+1$, $g(x) = (x^2/1500 + 1)\times(\mathrm{e}^{x/100} + 1)$, then we have

$$M(t) = 2(t+1) \leqslant f(t,x) \leqslant \left(\frac{1}{1500}x^2 + 1\right)\left(\mathrm{e}^{x/100} + 1\right)(t+1) = N(t)g(x).$$

Thus, (G2) is satisfied.

Next, for any $t \in [0,1]_{\mathbb{T}}$,

$$\lim_{x\to+\infty} \frac{f(t,x)}{x} = \lim_{x\to+\infty} \frac{(\frac{1}{2000}x^2 + 1)(\mathrm{e}^{x/100} + 1)(t+1)}{x} = +\infty, \tag{13}$$

i.e., (13) holds uniformly on any compact subinterval of $(0,1)_{\mathbb{T}}$, and then (G3) is also satisfied.

In the following, we verify condition (6). Firstly,

$$\begin{aligned}
\int_0^1 \left[q_+(s) + N(s)\right] \nabla s &= \int_0^1 (s+1)\,\nabla s = \int_0^1 s\,\nabla s + \int_0^1 \nabla s \\
&= \left[1\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4}\cdot\left(\frac{1}{4} - \frac{1}{8}\right) + \cdots\right] + 1 \\
&= \frac{2}{3} + 1 = \frac{5}{3}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \left[q_-(s) + M(s)\right] \nabla s &= \int_0^1 \frac{2}{\sqrt{t}} + 2(s+1)\,\nabla s = 2\int_0^1 \frac{1}{\sqrt{t}} \nabla s + 2\int_0^1 (s+1)\,\nabla s \\
&= 2\left[1 \cdot \frac{1}{2} + \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{4} + \frac{1}{\sqrt{\frac{1}{4}}} \cdot \frac{1}{8} + \cdots\right] \\
&\quad + 2\left[1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4} \cdot \left(\frac{1}{4} - \frac{1}{8}\right) + \cdots\right] + 2 \\
&= \frac{2}{2-\sqrt{2}} + \frac{4}{3} + 2 \approx \frac{20}{3}.
\end{aligned}$$

Moreover, for any $x \in [0, 2\rho] = [0,\ (20/9)(2/(2-\sqrt{2}) + 10/3)]$, we have

$$\begin{aligned}
&\max_{x\in[0,\,(20/9)(2/(2-\sqrt{2})+10/3)]} g(x) + 1 \\
&\quad = \max_{x\in[0,\,(20/9)(2/(2-\sqrt{2})+10/3)]} \left(\frac{1}{1500}x^2 + 1\right)\left(\mathrm{e}^{x/100} + 1\right) + 1 \approx 3.484.
\end{aligned}$$

Thus, one has

$$\frac{\int_0^1 [q_-(s) + M(s)]\nabla s}{\int_0^1 [q_+(s) + N(s)]\nabla s} \approx 4 \geqslant 3.484,$$

which implies that condition (6) holds.

It follows from Theorem 1 that the boundary value problem (12) has at least one positive solution $y(t)$ satisfying $y(t) > \gamma(t) = \int_0^1 G(t,s)(s+1)\,\nabla s$, $t \in [0,1]_{\mathbb{T}}$.

Example 2. Let $\mathbb{T} = \{1/2^n\}_{n=0}^{\infty} \cup \{0,1\}$. Consider the following third-order boundary value problem on time scales:

$$\begin{gathered}
-\left(x^{\Delta\Delta}\right)^{\nabla}(t) + \frac{5x(\sigma(t))(t+1)}{2(x(\sigma(t))+1)} - \frac{2}{\sqrt{t}} = 0, \quad t \in [0,1]_{\mathbb{T}}, \\
x(0) = x^{\Delta}\left(\frac{2}{3}\right) = \int_{3/4}^{1} s x^{\Delta\Delta}(s) \nabla s = 0.
\end{gathered}$$

Taking $t_1 = 2/3$, $t_2 = 3/4$, $f(t,x) = 5x(t+1)/(2(x+1))$, $q(t) = -2/\sqrt{t}$, $p(t) = t$, $M(t) = 2(t+1)$, $N(t) = t+1$, $g(x) = 4x/(x+1)$, and letting $q_+(t) = 0$, $q_-(t) = 2/\sqrt{t}$, it is obvious that (G1) and (G2) hold.

Moreover,

$$\lim_{x\to+\infty} \frac{g(x)}{x} = \lim_{x\to+\infty} \frac{4x}{x(x+1)} = 0,$$

which implies that (G4) is also satisfied.

Now take $\alpha = 1/4$, $\beta = 1/2$. We have $\|d\| = 10/9$,

$$\rho = \|d\| \int\limits_0^1 \left[q_-(s) + M(s)\right] \nabla s = \frac{10}{9} \int\limits_0^1 \frac{2}{\sqrt{t}} + 2(s+1)\nabla s \approx \frac{200}{27},$$

$$\begin{aligned}
\theta &= \int\limits_\alpha^\beta G(\alpha, s)\nabla s = \int\limits_{1/4}^{1/2} G\left(\frac{1}{4}, s\right) \nabla s = \int\limits_{1/4}^{1/2} \left(\frac{1}{4}s - \frac{1}{32}\right) \nabla s \\
&= \frac{1}{4} \int\limits_0^{1/2} s\,\nabla s - \frac{1}{4} \int\limits_0^{1/4} s\,\nabla s - \frac{1}{32 \cdot 4} \\
&= \frac{1}{4}\left[\frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{4} \cdot \left(\frac{1}{4} - \frac{1}{8}\right) + \frac{1}{8} \cdot \left(\frac{1}{8} - \frac{1}{16}\right) + \cdots\right] \\
&\quad - \frac{1}{4}\left[\frac{1}{4} \cdot \left(\frac{1}{4} - \frac{1}{8}\right) + \frac{1}{8} \cdot \left(\frac{1}{8} - \frac{1}{16}\right) + \frac{1}{16} \cdot \left(\frac{1}{16} - \frac{1}{32}\right) + \cdots\right] - \frac{1}{32 \cdot 4} \\
&\approx \frac{1}{4} \cdot \frac{1}{6} - \frac{1}{4} \cdot \frac{1}{24} - \frac{1}{32 \cdot 4} \approx \frac{3}{128},
\end{aligned}$$

and

$$\kappa = \frac{2\alpha t_1 - \alpha^2}{t_1^2} = \frac{117}{192}.$$

For any $(t, x) \in [1/4, 1/2]_\mathbb{T} \times [2925/648, 200/9]$,

$$f(t, x) \leqslant \max_{(t,x)\in[1/4,1/2]_\mathbb{T}\times[2925/648,200/9]} f(t, x) \approx 3.589 < \frac{2\rho}{\theta} \approx 632.1,$$

which implies that (G5) is satisfied.

Thus, Theorem 2 guarantees that the boundary value problem (13) has at least one positive solution $y(t)$ such that $y(t) > \gamma(t) = \int_0^1 G(t, s)(s + 1)\,\nabla s$, $t \in [0, 1]_\mathbb{T}$.

References

1. B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, in *Nonlinear Dynamics and Quantum Dynamical Systems: Contributions to the International Seminar ISAM-90, held in Gaussig (GDR), March 19–23, 1990*, De Gruyter, Berlin, Boston, 1990, pp. 9–20, `https://doi.org/10.1515/9783112581445-002`.
2. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Springer, New York, 2001, `https://doi.org/10.1007/978-1-4612-0201-1`.
3. M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Springer, New York, 2003, `https://doi.org/10.1007/978-0-8176-8230-9`.
4. D. Chang, Z. Fu, D.n Yang, S. Yang, Real-variable characterizations of Musielak–Orlicz–Hardy spaces associated with Schrödinger operators on domains, *Math. Methods Appl. Sci.*, **39**(3):533–569, 2016, `https://doi.org/10.1002/mma.3501`.

5. B. Dong, Z. Fu, J. Xu, Riesz-Kolmogorov theorem in variable exponent Lebesgue spaces and its applications to Riemann-Liouville fractional differential equations, *Sci. China, Math.*, **61**(10):1807–1824, 2018, `https://doi.org/10.1007/s11425-017-9274-0`.
6. J. Graef, L. Kong, Positive solutions for third order semipositone boundary value problems, *Appl. Math. Lett.*, **22**(8):1154–1160, 2009, `https://doi.org/10.1016/j.aml.2008.11.008`.
7. J.R. Graef, B. Yang, Positive solutions of a third order nonlocal boundary value problem, *Discrete Contin. Dyn. Syst.*, **1**(1):89–97, 2008, `https://doi.org/10.3934/dcdss.2008.1.89`.
8. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes Rep. Math. Sci. Eng., Vol. 5, Academic Press, Boston, 2014, `https://doi.org/10.1016/B978-0-12-293475-9.50001-7`.
9. Z. Hao, T. Xiao, J. Liang, Existence of positive solutions for singular boundary value problem on time scales, *J. Math. Anal. Appl.*, **325**(1):517–528, 2007.
10. J. He, X. Zhang, L. Liu, Y. Wu, Y. Cui, Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions, *Bound. Value Probl.*, **2018**:189, 2018, `https://doi.org/10.1186/s13661-018-1109-5`.
11. J. He, X. Zhang, L. Liu, Y. Wu, Y. Cui, A singular fractional Kelvin–Voigt model involving a nonlinear operator and their convergence properties, *Bound. Value Probl.*, **2019**(1):112, 2019, `https://doi.org/10.1186/s13661-019-1228-7`.
12. S. Shi, J. Xiao, On fractional capacities relative to bounded open Lipschitz sets, *Potential Anal.*, **45**(2):261–298, 2016, `https://doi.org/10.1007/s11118-016-9545-2`.
13. S. Shi, Q. Xue, K. Yabuta, On the boundedness of multilinear Littlewood–Paley g_λ^* function, *J. Math. Pures Appl. (9)*, **101**(3):394–413, 2014, `https://doi.org/10.1016/j.matpur.2013.06.007`.
14. H. Tian, X. Zhang, Y. Wu, B. Wiwatanapataphee, Existence of positive solutions for a singular second-order changing-sign differential equation on time scales, *Fractal and Fractional*, **6**(6): 315, 2022. 10.3390/fractalfract6060315
15. Y. Wang, L. Liu, X. Zhang, Y. Wu, Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection, *Appl. Math. Comput.*, **258**:312–324, 2015, `https://doi.org/10.1016/j.amc.2015.01.080`.
16. J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity, *Bound. Value Probl.*, **2018**:82, 2018, `https://doi.org/10.1186/s13661-018-1003-1`.
17. J. Wu, X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, Iterative algorithm and estimation of solution for a fractional order differential equation, *Bound. Value Probl.*, **2016**:116, 2016, `https://doi.org/10.1186/s13661-016-0608-5`.
18. J. Xu, D. O'Regan, Positive solutions for a second order boundary value problem on time scales, *J. Appl. Math. Computing*, **51**(1–2):127–144, 2016, `https://doi.org/10.1007/s12190-015-0895-5`.
19. M. Yang, Z. Fu, S. Liu, Analyticity and existence of the Keller–Segel–Navier–Stokes equations in critical Besov spaces, *Adv. Nonlinear Stud.*, **18**(3):517–535, 2018, `https://doi.org/10.1515/ans-2017-6046`.

20. M. Yang, Z. Fu, J. Sun, Existence and large time behavior to coupled chemotaxis-fluid equations in Besov–Morrey spaces, *J. Differ. Equations*, **266**(9):5867–5894, 2019, https://doi.org/10.1016/j.jde.2018.10.050.

21. S. Yang, D. Chang, D. Yang, Z. Fu, Gradient estimates via rearrangements for solutions of some Schrödinger equations, *Anal. Appl.*, **16**(03):339–361, 2018, https://doi.org/10.1142/S0219530517500142.

22. X. Zhang, J. Jiang, L.n Liu, Y. Wu, Extremal solutions for a class of tempered fractional turbulent flow equations in a porous medium, *Math. Probl. Eng.*, **2020**:2492193, 2020, https://doi.org/10.1155/2020/2492193.

23. X. Zhang, J. Jiang, Y. Wu, Y. Cui, The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach, *J. Math. Anal. Appl.*, **464**(2):1089–1106, 2018, https://doi.org/10.1016/j.jmaa.2018.04.040.

24. X. Zhang, J. Jiang, Y. Wu, Y. Cui, Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows, *Appl. Math. Lett.*, **90**:229–237, 2019, https://doi.org/10.1016/j.aml.2018.11.011.

25. X. Zhang, J. Jiang, Y. Wu, B. Wiwatanapataphee, Iterative properties of solution for a general singular n-Hessian equation with decreasing nonlinearity, *Appl. Math. Lett.*, **112**:106826, 2021, https://doi.org/10.1016/j.aml.2020.106826.

26. X. Zhang, D. Kong, H. Tian, Y. Wu, B. Wiwatanapataphee, An upper-lower solution method for the eigenvalue problem of Hadamard-type singular fractional differential equation, *Nonlinear Anal. Model. Control*, **27**(4):789–202, 2022, https://doi.org/10.15388/namc.2022.27.27491.

27. X. Zhang, L. Liu, Y. Wu, Y. Cui, A sufficient and necessary condition of existence of blow-up radial solutions for a k-Hessian equation with a nonlinear operator, *Nonlinear Anal. Model. Control*, **25**(1):126–143, 2020, https://doi.org/10.15388/namc.2020.25.15736.

28. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, The spectral analysis for a singular fractional differential equation with a signed measure, *Applied Math. Comput.*, **257**:252–263, 2015, https://doi.org/10.1016/j.amc.2014.12.068.

29. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, Multiple solutions for a modified quasilinear Schrödinger elliptic equation with a nonsquare diffusion term, *Nonlinear Anal. Model. Control*, **26**(4):702–717, 2021, https://doi.org/10.15388/namc.2021.26.22520.

30. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, Y. Cui, Solvability and asymptotic properties for an elliptic geophysical fluid flows model in a planar exterior domain, *Nonlinear Anal. Model. Control*, **26**(2):315–333, 2021, https://doi.org/10.15388/namc.2021.26.21202.

31. X. Zhang, C. Mao, L. Liu, Y. Wu, Exact iterative solution for an abstract fractional dynamic system model for bioprocess, *Qual. Theory Dyn. Syst.*, **16**(1):205–222, 2017, https://doi.org/10.1007/s12346-015-0162-z.

32. X. Zhang, H. Tain, Y. Wu, B. Wiwatanapataphee, The radial solution for an eigenvalue problem of singular augmented Hessian equation, *Appl. Math. Lett.*, **134**:108330, 2022, https://doi.org/10.1016/j.aml.2022.108330.

33. X. Zhang, Y. Wu, L. Caccetta, Nonlocal fractional order differential equations with changing-sign singular perturbation, *Appl. Math. Modelling*, **39**(21):6543–6552, 2015, https://doi.org/10.1016/j.apm.2015.02.005.

34. X. Zhang, J. Xu, J. Jiang, Y. Wu, Y. Cui, The convergence analysis and uniqueness of blow-up solutions for a Dirichlet problem of the general k-Hessian equations, *Appl. Math. Lett.*, **102**: 106124, 2020, https://doi.org/10.1016/j.aml.2019.106124.

35. X. Zhang, P. Xu, Y. Wu, The eigenvalue problem of a singular k-Hessian equation, *Appl. Math. Lett.*, **124**:107666, 2022, https://doi.org/10.1016/j.aml.2021.107666.

36. X. Zhang, P. Xu, Y. Wu, B. Wiwatanapataphee, The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model, *Nonlinear Anal. Model. Control*, **27**(3):428–444, 2022, https://doi.org/10.15388/namc.2022.27.25473.

37. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Positive solutions for a weakly singular Hadamard-type fractional differential equation with changing-sign nonlinearity, *J. Funct. Spaces*, **2020**: 5623589, 2020, https://doi.org/10.1155/2020/5623589.

38. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Solutions for a singular Hadamard-type fractional differential equation by the spectral construct analysis, *J. Funct. Spaces*, **2020**:8392397, 2020, https://doi.org/10.1155/2020/8392397.