A Formal Proof of the Optimal Frame Setting for Dynamic-Frame Aloha With Known Population Size

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I. Introduction

COLLISION resolution protocols have played a fundamental role in communication systems starting with the appearance of the Aloha protocol [1]–[3] back in 1970. Since then, a variety of such protocols have been proposed and have influenced satellite, radio and local area networks, being nowadays applied also to radio frequency identification (RFID) systems [4], [5]. In RFID systems a reader interrogates a set of tags in order to identify each one of them [5]. Collisions may occur among the responses of tags, and collision resolution protocols are used to arbitrate transmissions so that all tags can be finally identified. In this environment, the tag population size $\mathcal{N}$ is not a random variable as it happens in multiple access systems, but is a constant n, either known or unknown; nevertheless, the collision resolution problem is quite similar in both environments, and RFID protocols often represent a straightforward derivation of those proposed for multiple access.

Among the different protocols envisaged in past years, Dynamic Frame Aloha (DF-Aloha) is the most popular in RFID [6], [7]. In Frame-Aloha (F-Aloha) time is divided into time slots equal to a packet transmission time, slots are grouped into frames, and a tag is allowed to transmit only a single packet per frame in a randomly chosen slot. In the first frame all tags transmit, but only a part of them avoid collisions with other transmissions and get through. The remaining ones, referred to as the backlog, re-transmit in the subsequent frames until all of them are successful. Although some versions allow the restart of a new frame at any slot, should this be convenient, here we deal with the original one, where the frame is explored in its entirety.

Unfortunately, F-Aloha, like other protocols of the Aloha family [8], [9], is intrinsically unstable and its throughput is very small unless some stabilizing control is used. A way to do this is to dynamically adapt the frame length r according to the backlog size n, hence the name Dynamic Frame Aloha (DF-Aloha). This strategy has been proposed for the first time in [10], in the field of satellite communications, where the author proposes to set the frame length exactly equal to a backlog estimate $\hat{n}$. The reason for adopting this strategy is that the throughput in a slot of a frame of length r:

$$\frac{n}{r}\left(1-\frac{1}{r}\right)^{n-1}, \tag{1}$$

is maximized for $r=n$.

As a matter of fact, the performance figure to be optimized is the overall efficiency

$$\eta=\frac{\mathsf{E}\left[\mathcal{N}\right]}{\mathsf{E}\left[\mathcal{L}\right]}, \tag{2}$$

where $\mathcal{N}$ is the original tag population size and $\mathcal{L}$ is the average length of the identification period (IP), i.e., the average number of slots needed to successfully transmit all the $\mathcal{N}$ tags. In RFID systems $\mathcal{N}$ is usually a constant n and, therefore, the efficiency is maximized by minimizing $L(n)=\mathsf{E}\left[\mathcal{L}\right]$.

A recursive formula is given [10] for the calculation of $L(n)$. By applying this formula with known n, we can numerically show that the strategy that sets $r=n$ at each frame provides the shortest $L(n)$ for any value of n attempted. However, up to now, to the knowledge of authors, none has provided a theoretical verification of the result.

In practical RFID applications n is usually unknown; however, in order to meet optimality conditions, it is usually replaced by an estimate $\hat{n}$ based on the observation of outcomes in a frame or in the entire history (see for example [4], [11]–[13]). Quite often, setting $r=\hat{n}$ has been assumed, never really discussing the optimal strategy when n is unknown and an estimate is needed, with the notable exception of [14]. In that paper the authors have pointed out the non-optimality of the above setting, and suggest a procedure to numerically find the best frame-length choice when the initial backlog size n is known in distribution. This procedure, when applied to known

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n, provides the recursive formula for $L(n)$ cited above, which is still solved only numerically.

In this paper we present an analysis of DF-Aloha with known backlog size n, that definitely proves that local optimization, i.e., maximizing the throughput/efficiency in each frame (i.e., setting $r = n$ at each frame), also maximizes the overall efficiency. We rigorously prove that the optimal asymptotic efficiency is e^{-1}, further providing tight upper and lower bounds to $L(n)$, and showing that its asymptotic behavior is $\sim ne - \gamma \ln(n)$, with $\gamma = -0.5/\ln(1-e^{-1})$.

The proof starts with providing, in Theorem 1, some general properties of $L(n)$ with strategy $r = n$ at each frame, i.e., the strategy we will prove to be optimal. In particular, we show that $L(n)$ is an increasing function upper bounded by ne. We then demonstrate two lemmas about upper and lower bounds for the derivative of $\varepsilon(n) = L(n) - ne$. Subsequent lemmas provide upper and lower bounds to error $\varepsilon(n)$, and together provides Theorem 2, that in turn gives the error's asymptotic behavior. Finally, from all the preceding results, we are able to prove main Theorem 3, that confirms the optimality of the cited strategy. The key lemmas, together with the final theorem, make use of some properties of stochastic dominance of the first order [15], [16]. To this end, Appendix A extends a known result of stochastic dominance, and provides a lemma where the distributions of collided tags in frame (n, r), for different r and n, are ranked in terms of such stochastic dominance. Often the proofs are analytically valid starting from a population size $n = n_0$ somewhat greater than zero, and implying numerical verification up to n_0. To this purpose, in Appendix B, we provide the distribution of the number of successes in a frame (n, r), together with additional proofs of some properties used in theorems and lemmas.

II. Analysis

Let n be the number of tags to be identified and $L(n, r_n)$ the average length of the identification period, where we have made explicit its dependence on r_n, the length of the frame with n tags. The latter can be expressed as [10]

$$L(n, r_n) = r_n + \sum_{s=0}^{m} p_{n,r_n}(s) L(n-s, r_{n-s}), \qquad n \geq 2, \quad (3)$$

where $m = \min\{n-2, r_n - 1\}$, and p_{n,r_n} is the probability distribution of the number of successes $\mathcal{S}_{n,r_n}$ in the first frame, of length r_n. Making the term $L(n, r_n)$ explicit yields

$$L(n, r_n) = \frac{r_n + \sum_{s=1}^{m} p_{n,r_n}(s) L(n-s, r_{n-s})}{1 - p_{n,r_n}(0)}, \qquad n \geq 2. \quad (4)$$

If the sequence $\{r_n\}$ is known, then (4) can be used recursively to get the sequence $L(n, r_n)$ starting from $L(0, r_0) = L(1, r_0) = r_0$.

A recursive expression of $p_{n,r_n}(s, c)$, the probability of having s successes and c collided slots in the first frame, is given in [10]. A closed form expression for $p_{n,r_n}(s)$, given in Appendix B, can be derived from formulas in [17].

Let now call "Selected Strategy" the one that assumes $r_n = n$ at all frames. Later in the paper (Theorem 3) we show that the Selected Strategy is indeed the optimum strategy. In the remainder of the paper, for the sake of compactness, when $r = n$ we use a single subscript in the notation, e.g., $\mathcal{S}_n$ in place of the more general $\mathcal{S}_{n,n}$. In the analysis that follows we also make use of some properties of random variable (RV) $\mathcal{S}_{n,r}$, and the related RV $\mathcal{R}_{n,r} = n - \mathcal{S}_{n,r}$, that are listed in Appendix B.

Expressions (3) and (4) with the Selected Strategy can be rewritten as follows, with the dependence on r_n omitted in the notation:

$$L(n) = n + \sum_{i=1}^{n} \pi_{n,n}(i) L(i), \qquad n \geq 2, \quad (5)$$

$$L(n) = \frac{n + \sum_{i=1}^{n-1} \pi_{n,n}(i) L(i)}{1 - \pi_{n,n}(n)}, \qquad n \geq 2, \quad (6)$$

where $\pi_{n,r}(i) = p_{n,r}(n-i)$, for $i = 0, \ldots, n$, is the probability distribution of RV $\mathcal{R}_{n,r} = n - \mathcal{S}_{n,r}$, the number of collided tags out of the initial n. Note that $\pi_{n,r}(1) = 0$ for any pair (n, r), therefore the summation in (5) can be started from $i = 2$.

We now prove the following:

Theorem *1:* With the Selected Strategy the average identification period in identifying n tags, $L(n)$, presents the following properties:

(a) $L(n)$ is an increasing function of n,
(b) $L(n) < ne$.

Proof: (a) For $n = 1$ the thesis holds because $L(1) = 1$ and $L(2) = 4$. We assume that $L(i) > L(i-1)$ for $i \leq n$ and show that it holds also for $i = n+1$.

We can easily lower bound the difference between (6), evaluated in $n+1$, and (5) as follows

$$L(n+1) - L(n) > 1 + \sum_{i=2}^{n} (\pi'_{n+1}(i) - \pi_n(i)) L(i), \quad (7)$$

where

$$\pi'_n(i) = \frac{\pi_n(i)}{1 - \pi_n(n)}, \qquad 0 \leq i \leq n-1. \quad (8)$$

Lemma 6 part 2, with $r = n$, implies that π'_{n+1} stochastically dominates π_n. Therefore, owing to the fact that $L(i), i \leq n$, is an increasing function of i, by the basic property (78), the summation in (7) can not be negative, and the thesis is proved.

(b) We assume that $L(i) < ie$ for $i \leq n$ and show that it holds also for $i = n+1$ (it is trivially $L(1) = 1 < e$). From (6), the assumption allows to write

$$L(n+1) < \frac{n + 1 + \sum_{2}^{n} \pi_{n+1}(i)\, ie}{1 - \pi_{n+1}(n+1)} \quad (9)$$

$$= \frac{n + 1 + (R_{n+1} - (n+1)\pi_{n+1}(n+1))e}{1 - \pi_{n+1}(n+1)}, \quad (10)$$

where $R_n = \mathsf{E}[\mathcal{R}_n]$. By Property 3b of Appendix B, we have $R_n < n(1-e^{-1})$, which, used in (10), finally provides

$$L(n+1) < \frac{(n+1-(n+1)\cdot \pi_{n+1}(n+1))\, e}{1-\pi_{n+1}(n+1)} = (n+1)e. \tag{11}$$

∎

Denoting by $\Delta f(n) = f(n) - f(n-1)$ the derivative of function $f(n)$, the next lemma provides an upper bound to the derivative of error $\varepsilon(n) = ne - L(n)$.

Lemma *1:* Given the function

$$g(n) = \nu \ln(n) + \frac{\mu}{\sqrt{n}}, \tag{12}$$

the following inequality holds for $\nu = 1.5$ and $\mu = 2$:

$$\Delta\varepsilon(n) \leq \Delta g(n), \qquad n \geq 2. \tag{13}$$

Proof: By (5) we have

$$\varepsilon(n) = ne - L_n = n(e-1) - R_n e + \sum_{2}^{n} \pi_n(i)\varepsilon(i), \qquad n \geq 2. \tag{14}$$

The general term of the difference sequence for $n \geq 2$ can be written as

$$\begin{aligned}\Delta\varepsilon(n+1) &= e - 1 - (R_{n+1} - R_n)\, e \\ &\quad + \sum_{i=2}^{n} \varepsilon(i)\, (\pi'_{n+1}(i) - \pi_n(i)) \\ &\quad + \frac{\pi_{n+1}(n+1)}{1-\pi_{n+1}(n+1)}((n+1)(e-1) - R_{n+1}e) \\ &< \sum_{i=2}^{n} \varepsilon(i)\, (\pi'_{n+1}(i) - \pi_n(i)) + \mathcal{O}(c^n),\end{aligned} \tag{15}$$

where we have exploited inequality $R_{n+1} - R_n > (1-e^{-1})$ derived in Property 3c of Appendix B, and $\mathcal{O}(c^n)$ corresponds to fractional term on the left hand side of (15) with $c = 0.9157$ being derived in Property 2. If we assume that

$$\Delta\varepsilon(i) \leq \Delta g(i), \qquad 2 \leq i \leq n \tag{16}$$

we can show that

$$\Delta\varepsilon(n+1) < \Delta g(n+1), \qquad n \geq 2, \tag{17}$$

proving the theorem by induction. Lemma 6 part 2, with $r = n$, implies that π'_{n+1} stochastically dominates π_n. Therefore, by Lemma 5 of Appendix A and (16), we can write

$$\begin{aligned}\overline{\varepsilon(i)} &= \sum_{i=1}^{n} \varepsilon(i)\, (\pi'_{n+1}(i) - \pi_n(i)) \\ &\leq \sum_{i=1}^{n} g(i)\, (\pi'_{n+1}(i) - \pi_n(i)) = \overline{g(i)} = \nu\overline{\ln(i)} + \overline{\mu/\sqrt{i}}.\end{aligned} \tag{18}$$

In order to upper bound $\overline{g(i)}$ we focus on term $\overline{\ln(i)}$, and make use again of Lemma 5 of Appendix A in the form provided by Corollary 3, by writing

$$\begin{aligned}\overline{\ln(i)} &= \sum_{i=1}^{n} \ln(i)\, (\pi'_{n+1}(i) - \pi_n(i)) \\ &\leq \sum_{i=1}^{n} g_1(i)\, (\pi'_{n+1}(i) - \pi_n(i)),\end{aligned} \tag{19}$$

where

$$g_1(x) = \sum_{i=1}^{4} (-1)^{i+1} \frac{(x-a)^i}{ia^i} + \frac{(x-a)^5}{5a^4}, \tag{20}$$

and

$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} \leq \frac{\mathrm{d}g_1(x)}{\mathrm{d}x} = \sum_{i=0}^{3} (-1)^i \frac{(x-a)^i}{a^{i+1}} + \frac{(x-a)^4}{a^4} \tag{21}$$

for all $x \geq 1$ and $a \geq 1$.

By choosing $a = R_n$ in (20) and expliciting it in (19) the latter becomes

$$\begin{aligned}\overline{\ln(i)} &\leq \sum_{i=1}^{4} (-1)^{i+1} \left(\frac{\mathsf{E}\left[(\mathcal{R}'_{n+1} - R_n)^i\right]}{iR_n^i} - \frac{\mathsf{E}\left[(\mathcal{R}_n - R_n)^i\right]}{iR_n^i} \right) \\ &\quad + \frac{\mathsf{E}\left[(\mathcal{R}'_{n+1} - R_n)^5\right]}{5R_n^4} - \frac{\mathsf{E}\left[(\mathcal{R}_n - R_n)^5\right]}{5R_n^4} \end{aligned} \tag{22}$$

$$= n^{-1} + \frac{(6+3e-e^2)}{2e(e-1)} n^{-2} + \mathcal{O}(n^{-3}), \tag{23}$$

where in the last step moments of RVs $\mathcal{R}'_{n+1}$ and $\mathcal{R}_n$ have been computed as described in Property 5 of Appendix B, and Taylor's expansions have been used. We repeat the same procedure for term $1/\sqrt{n}$ in $g(n)$. We have

$$\frac{\mathrm{d}(1/\sqrt{x})}{\mathrm{d}x} \leq -\frac{1}{2\sqrt{a^3}} + \frac{3(x-a)}{4\sqrt{a^5}} \tag{24}$$

for all $x \geq 1$ and $a \geq 1$, which implies

$$\begin{aligned}\overline{1/\sqrt{i}} &= \sum_{i=1}^{n} (1/\sqrt{i})\, (\pi'_{n+1}(i) - \pi_n(i)) \\ &\leq \sum_{i=1}^{n} g_2(i)\, (\pi'_{n+1}(i) - \pi_n(i)),\end{aligned} \tag{25}$$

being

$$g_2(x) = -\frac{x-a}{2\sqrt{a^3}} + \frac{3(x-a)^2}{8\sqrt{a^5}} \tag{26}$$

for all $x \geq 1$ and $a \geq 1$.

Again, choosing $a = R_n$, inequality (25) becomes

$$\begin{aligned}\overline{1/\sqrt{i}} &\leq -\frac{R'_{n+1} - R_n}{2\sqrt{R_n^3}} + \frac{3\mathsf{E}\left[(\mathcal{R}'_{n+1} - R_n)^2\right]}{8\sqrt{R_n^5}} \\ &\quad - \frac{3\mathsf{E}\left[(\mathcal{R}_n - R_n)^2\right]}{8\sqrt{R_n^5}}\end{aligned} \tag{27}$$

$$= \frac{1}{2\sqrt{1-e^{-1}}}\, n^{-3/2} + \mathcal{O}(n^{-5/2}), \tag{28}$$

and putting together (23) and (28) we finally get

$$\overline{g(i)} = \nu\overline{\ln(i)} + \overline{\mu/\sqrt{i}} \leq \nu n^{-1} + \mu \frac{1}{2\sqrt{1-e^{-1}}}\, n^{-3/2} + \mathcal{O}(n^{-2}). \tag{29}$$

From (15) and (18), we see that, to prove the thesis, we must have

$$\begin{aligned}\overline{g(i)} + \mathcal{O}(c^n) &\leq \Delta g(n+1) \\ &= \nu\left(\ln(n+1) - \ln(n)\right) + \mu\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) \\ &= \nu\left(n^{-1} - \frac{1}{2}n^{-2} + \frac{1}{3}n^{-3} + \ldots\right) - \frac{\mu}{2}n^{-3/2} + \ldots .\end{aligned}$$

Substituting inequality (29), term n^{-1} simplifies and, to prove the thesis, the final condition becomes

$$\frac{\mu}{2\sqrt{1-e^{-1}}}\ n^{-3/2} \leq -\frac{\mu}{2}n^{-3/2} + \mathcal{O}(n^{-2}). \tag{30}$$

Disregarding the asymptotic term, the above inequality is always true for any μ. This means that there exists an n_0 such that for any $n \geq n_0$ (30) is satisfied. Then, we numerically verify that (13) holds up to n_0, and the lemma is proved. ■

Note that, by proving (13), the above Lemma proves also the following

Corollary *1:* If distribution π_X stochastically dominates distribution π_Y, and $g(n)$ is the function (12), then we have

$$\sum_{i=1}^{n} \varepsilon(i)\left(\pi_X(i) - \pi_Y(i)\right) \leq \sum_{i=1}^{n} g(i)\left(\pi_X(i) - \pi_Y(i)\right). \tag{31}$$

Corollary *2:* If distribution π_X stochastically dominates distribution π_Y, $g(n)$ is the function (12), and $l(n) = \nu\ln(n)$, ν as in (12), then we have

$$\sum_{i=1}^{n} \varepsilon(i)\left(\pi_X(i) - \pi_Y(i)\right) \leq \sum_{i=1}^{n} l(i)\left(\pi_X(i) - \pi_Y(i)\right), \tag{32}$$

where the thesis comes from Corollary 1 and Lemma 5, being $\Delta l(n) \geq \Delta g(n)$.

The next lemma provides a lower bound to $\Delta\varepsilon(n)$.

Lemma *2:* Given the function

$$\begin{aligned} g(n) &= \nu\ln(n) \qquad n \geq 3 \\ g(2) &= -0.5, \end{aligned}$$

for $\nu = 1$ the following inequality holds

$$\Delta\varepsilon(n) \geq \Delta g(n), \qquad n \geq 2. \tag{33}$$

Proof: The proof proceeds exactly as in Lemma 1, where now we set $g(n) = \nu\ln(n)$. We exploit again (15), carrying also the infinitesimal terms of expansion in Property 3c of Appendix B. This yields

$$\Delta\varepsilon(n+1) = -\frac{7}{24n^2} + \sum_{i=2}^{n} \varepsilon(i)\left(\pi'_{n+1}(i) - \pi_n(i)\right) + \mathcal{O}(n^{-3}) \tag{34}$$

for $n \geq 2$. We proceed by induction as in Lemma 1, where now (18) becomes $\overline{\varepsilon(i)} \geq \overline{g(i)}$. Here the value $g(2)$ has been chosen so as to satisfy $g(2) < \varepsilon(2) \approx -0.2817$. In order to lower bound $\overline{g(i)}$, relations (20) and (21) are respectively replaced by

$$g_1(x) = \sum_{i=1}^{4} (-1)^{i+1}\frac{(x-a)^i}{ia^i} \tag{35}$$

$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} \geq \sum_{i=0}^{3} (-1)^i \frac{(x-a)^i}{a^{i+1}} \tag{36}$$

for all $x \geq 1$ and $a \geq 1$. Using the same series expansions we get the corresponding of (23) as

$$\overline{\ln(i)} \geq n^{-1} + \frac{3-e}{2(e-1)}n^{-2} + \mathcal{O}(n^{-3}). \tag{37}$$

From (34), to prove the thesis $\Delta\varepsilon(n+1) \geq \Delta g(n+1)$, we must have

$$\begin{aligned} -\frac{7}{24}n^{-2} + \nu\,\overline{\ln(i)} + \mathcal{O}(n^{-3}) &> \Delta g(n+1) \\ &= \nu\left(n^{-1} - \frac{1}{2}n^{-2} + \frac{1}{3}n^{-3} + \ldots\right), \end{aligned} \tag{38}$$

and using (37) term n^{-1} simplifies, leading to condition

$$\frac{7}{24}n^{-2} \leq \nu\frac{1}{e-1}n^{-2} + \mathcal{O}(n^{-3}). \tag{39}$$

If we disregard term $\mathcal{O}(n^{-3})$, for any $\nu \geq 0.502$ the above inequality is always verified. This means that there is an $n_1(\nu)$ such that for $n \geq n_1(\nu)$ (39) is always verified, and the lemma is proved if the thesis is shown to hold numerically up to $n_1(\nu)$. This is the case, for example, with $\nu = 1$, and the thesis is proved. ■

Now we proceed to get bounds to error $\varepsilon(n)$ and to provide its asymptotic behavior.

Lemma *3:* The following inequality holds:

$$\varepsilon(n) > f(n) = \zeta\ln(n) + \frac{\lambda}{n}, \qquad n \geq 2, \tag{40}$$

where $\lambda = 1.2$ and

$$\zeta = -\frac{0.5}{\ln(1-e^{-1})} = 1.0900... \tag{41}$$

Proof: We start from relation (14) that, using Property 3b of Appendix B and $R_n/n = 1 - e^{-1} - \xi_n$, becomes

$$\varepsilon(n) = ne\xi_n + \sum_{2}^{n} \pi_n(i)\,\varepsilon(i),$$

and, solving for $\varepsilon(n)$, we get

$$\varepsilon(n) = \frac{ne\xi_n + \sum_{2}^{n-1} \pi_n(i)\,\varepsilon(i)}{1 - \pi_n(n)}. \tag{42}$$

In the following, we assume that inequality (40) is verified up to $n-1 \geq 2$, and show that it is also satisfied for n, proving the theorem by induction. This assumption, applied to (42), implies

$$\begin{aligned} \varepsilon(n) >\ & ne\xi_n + \sum_{2}^{n-1} \pi_n(i)\, f(i) \\ &= ne\xi_n + \sum_{2}^{n} \pi_n(i)\, f(i) - \pi_n(n)\, f(n). \end{aligned} \tag{43}$$

Similarly to preceding Lemmas we use inequalities

$$\ln(x) \geq \ln(a) + \frac{(x-a)}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \frac{(x-a)^4}{a^{3.5}} \tag{44}$$

$$\frac{1}{x} \geq \frac{1}{a} - \frac{(x-a)}{a^2} \tag{45}$$

for all $x \geq 1$ and $a \geq 1$, evaluated at $a = R_n$, which provide

$$\begin{aligned}\overline{f(i)} &= \sum_{2}^{n} \pi_n(i)\, f(i) \\ &\geq \zeta \left(\ln(R_n) - \frac{\mathsf{E}\left[(\mathcal{R}_n - R_n)^2\right]}{2R_n^2} + \frac{\mathsf{E}\left[(\mathcal{R}_n - R_n)^3\right]}{3R_n^3} \right. \\ &\quad \left. - \frac{\mathsf{E}\left[(\mathcal{R}_n - R_n)^4\right]}{R_n^{3.5}} \right) + \frac{\lambda}{R_n}. \end{aligned} \tag{46}$$

Using Property 3b of Appendix B, and substituting Taylor's expansions, (46) provides

$$\begin{aligned}\overline{f(i)} &\geq \zeta \left(\ln(1-e^{-1}) + \ln(n) - \frac{1}{(e-1)} n^{-1} \right) \\ &\quad + \frac{\lambda\, e}{(e-1)\, n + 1/2} + \mathcal{O}(n^{-2}). \end{aligned} \tag{47}$$

From inequality (43), and using the above, the thesis is true if the following holds

$$\begin{aligned}\varepsilon(n) >\ & ne\xi_n + \zeta\, \ln(1-e^{-1}) + \zeta\, \ln(n) - \frac{\zeta}{e-1} n^{-1} \\ & + \frac{\lambda\, e}{e-1} n^{-1} + \mathcal{O}(n^{-2}) \\ & \geq \zeta\, \ln(n) + \lambda\, n^{-1}. \end{aligned} \tag{48}$$

or

$$ne\xi_n + \zeta \ln(1-e^{-1}) - \frac{\zeta}{e-1} n^{-1} + \frac{\lambda}{e-1} n^{-1} + \mathcal{O}(n^{-2}) \geq 0. \tag{49}$$

Substituting expansion $ne\xi_n = \frac{1}{2} + \frac{7}{24n} + \mathcal{O}(n^{-2})$ and ζ, condition (49) becomes

$$\left(\frac{7}{24} - \frac{\zeta}{e-1} + \frac{\lambda}{e-1} \right) n^{-1} + \mathcal{O}(n^{-2}) \geq 0. \tag{50}$$

Disregarding term $\mathcal{O}(n^{-2})$, the above is true for $\lambda > \zeta - (7/24)(e-1) \approx 0.6$. This means that, in this case, there is some $n_0(\lambda)$ such that for all $n \geq n_0$ (50) is true. The thesis is then proved by showing that (40) numerically holds up to n_0. This happens for $\lambda = 1.2$. Since we are dealing with a lower bound, we are interested in taking λ as large as possible. However, we have found that as λ increases beyond 1.35 (40) does not hold from $n = 2$ up, and the lemma can not be proved. ∎

Lemma *4:* The following inequality holds:

$$\varepsilon(n) < g(n) = \zeta \ln(n) + K, \qquad n \geq 2, \tag{51}$$

where

$$\zeta = -\frac{0.5}{\ln(1-e^{-1})} = 1.0900.., \tag{52}$$

and $K = 0.7$.

Proof: The proof proceeds exactly as in Lemma 3, where term λ/n is replaced by the constant K. We assume that inequality (51) is verified up to $n-1 \geq 2$, and show that it is also satisfied for n, proving the theorem by induction. The corresponding of (43) is

$$\begin{aligned}\varepsilon(n) &< \frac{1}{1-\pi_n(n)} \left(ne\xi_n + \sum_{2}^{n} \pi_n(i)\, g(i) - \pi_n(n)\, g(n) \right) \\ &= ne\xi_n + \sum_{2}^{n} \pi_n(i)\, g(i) + \mathcal{O}(c^n) \\ &< ne\xi_n + \zeta\, \ln(R_n) + K + \mathcal{O}(c^n), \end{aligned} \tag{53}$$

where in the last step we applied Jensen's inequality. Again, using the expansion for $\ln(R_n)$ and Property 3b of Appendix B, that provides $ne\xi_n = \frac{1}{2} + \frac{7}{24} n^{-1} + \mathcal{O}(n^{-2})$, we have

$$\begin{aligned}\varepsilon(n) &< \frac{1}{2} + \frac{7}{24} n^{-1} + \zeta \ln(1-e^{-1}) + \zeta \ln(n) \\ &\quad - \zeta \frac{1}{2(e-1)} n^{-1} + K + \mathcal{O}(n^{-2}). \end{aligned} \tag{54}$$

The thesis holds if we show that $\varepsilon(n) < g(n)$, which, using the above and (52), gives the condition

$$\frac{7}{24} n^{-1} - \zeta \frac{1}{2(e-1)} n^{-1} + \mathcal{O}(n^{-2}) < 0. \tag{55}$$

Disregarding term $\mathcal{O}(n^{-2})$, the inequality above is always true. This means that there is some n_0 such that for all $n \geq n_0$ (55) is true. The value of constant K has no effect on the above inequality; in fact, it is taken as the practical smaller value that makes (51) true for $2 \leq n \leq n_0$. We have found that the thesis holds with $K = 0.7$. ∎

Using the preceding lemmas we may conclude:

Theorem *2:*

$$\frac{1.2}{n} + \zeta \ln(n) \leq \varepsilon(n) < 0.7 + \zeta \ln(n), \qquad n \geq 2, \tag{56}$$

$$\varepsilon(n) \sim \zeta \ln(n). \tag{57}$$

We now are in the position to prove the main theorem of this paper. Let $L(n,r)$ be the average IP when r is the length of the first frame, whereas for the remaining frames the Selected Strategy is adopted.

Theorem *3:* $L(n,r)$ is minimized by the strategy that at each frame sets the frame length r equal to the backlog size n.

Proof: We assume that the above strategy is used in all frames with backlog i, $2 \leq i \leq n-1$, and show that we have

$$\Delta L(n,k) = L(n,n) - L(n,n+k) < 0 \tag{58}$$

for $k \geq -(n-1)$ and $k \neq 0$, then the theorem is proved by induction starting from $n = 2$. From (5) and (6) we have

$$\begin{aligned}\Delta L(n,k) &= n + \sum_{i=2}^{n} \pi_{n,n}(i)\, L(i) \\ &\quad - \frac{n + k + \sum_{i=2}^{n-1} \pi_{n,n+k}(i)\, L(i)}{1 - \pi_{n,n+k}(n)}. \end{aligned} \tag{59}$$

We now proceed by proving (58) for the two cases, $k < 0$ and $k > 0$.

Part Ia. $k < 0$, or $h = -k > 0$. Since the range of h depends on n, we further set $h = \alpha n$, with $1/n \leq \alpha \leq (n-1)/n$. Equation (59) can be expressed as

$$\Delta L(n, -\alpha n) = n + \sum_{i=2}^{n} \pi_{n,n}(i)\, L(i) - \frac{n(1-\alpha) + \sum_{i=2}^{n} \pi_{n,n(1-\alpha)}(i)\, L(i) - \pi_{n,n(1-\alpha)}(n)\, L(n)}{1 - \pi_{n,n(1-\alpha)}(n)}. \tag{60}$$

By Theorem 2 we use inequalities

$$ie - \zeta \ln i - 0.7 \leq L(i) \leq ie - \zeta \ln(i),$$

to bound (60) as follows

$$\begin{aligned} \Delta L(n, -\alpha n) &\leq n + eR_{n,n} - \zeta \mathsf{E}\left[\ln \mathcal{R}_{n,n}\right] \\ &\quad - \frac{n(1-\alpha) + eR_{n,n(1-\alpha)} - \zeta\, \mathsf{E}\left[\ln \mathcal{R}_{n,n(1-\alpha)}\right]}{1 - \pi_{n,n(1-\alpha)}(n)} \\ &\quad + \frac{0.7 + \pi_{n,n(1-\alpha)}(n)\, (ne - \zeta \ln(n))}{1 - \pi_{n,n(1-\alpha)}(n)} \qquad (61) \\ &= -n(e-1) + e\, R_{n,n} + \zeta\left(\ln(n) - \mathsf{E}\left[\ln \mathcal{R}_{n,n}\right]\right) \\ &\quad - \frac{-n(e-1) + e\, R_{n,n(1-\alpha)}}{1 - \pi_{n,n(1-\alpha)}(n)} \\ &\quad - \frac{\zeta\left(\ln(n) - \mathsf{E}\left[\ln \mathcal{R}_{n,n(1-\alpha)}\right]\right) - n\alpha - 0.7}{1 - \pi_{n,n(1-\alpha)}(n)}. \qquad (62) \end{aligned}$$

In the last passage above we get term (61) that is of the order $\mathcal{O}(n^{-1})$. In fact, in Lemma 3 we have lower bounded function $\overline{f(i)}$ that includes term $\mathsf{E}\left[\ln \mathcal{R}_{n,n}\right]$. By result (47) we have

$$\ln(n) - \mathsf{E}\left[\ln \mathcal{R}_{n,n}\right] \leq -\ln(1-e^{-1}) + \frac{1}{(e-1)} n^{-1} + \mathcal{O}(n^{-2}). \tag{63}$$

Using Property 3b and Jensen's inequality $\mathsf{E}\left[\ln \mathcal{R}_{n,n(1-\alpha)}\right] \leq \ln(R_{n,n(1-\alpha)})$, term (61) becomes

$$-0.5 - \zeta \ln(1-e^{-1}) + \left(\frac{\zeta}{(e-1)} - \frac{7}{24}\right) n^{-1} + \mathcal{O}(n^{-2}) = \mathcal{O}(n^{-1}), \tag{64}$$

having exploited the relation $-0.5 - \zeta \ln(1-e^{-1}) = 0$. As for term (62), we use expansions

$$eR_{n,n(1-\alpha)} = n(e - e^{\frac{\alpha}{\alpha-1}}) + e^{\frac{\alpha}{\alpha-1}} \frac{(2\alpha-1)}{2(\alpha-1)^2} + \mathcal{O}(n^{-1})$$

$$\ln(R_{n,n(1-\alpha)}) = \ln(n) + \ln\left(1 - e^{\frac{1}{\alpha-1}}\right) - \frac{e^{\frac{1}{\alpha-1}}(2\alpha-1)}{2(e^{\frac{1}{\alpha-1}}-1)(\alpha-1)^2} n^{-1} + \mathcal{O}(n^{-2}) \tag{65}$$

and inequality (61)-(62) becomes

$$\Delta L(n, -\alpha n) \leq -\frac{s(n,\alpha)}{1 - \pi_{n,n(1-\alpha)}(n)} + \mathcal{O}(n^{-1}), \tag{66}$$

where

$$s(n,\alpha) = n(1 - \alpha - e^{\frac{\alpha}{\alpha-1}}) + e^{\frac{\alpha}{\alpha-1}} \frac{(2\alpha-1)}{2(\alpha-1)^2} - \zeta \ln\left(1 - e^{\frac{1}{\alpha-1}}\right) - 0.7. \tag{67}$$

Function $s(n,\alpha)$ is negative only in a small interval beyond $\alpha = 0$. It crosses the axis at α_0 that we find by expanding $s(n,\alpha)$ around $\alpha = 0$ up to the second power and for large n. We get

$$s(n,\alpha) = n\frac{\alpha^2}{2} - 0.7 + \mathcal{O}(n\alpha^3), \tag{68}$$

which shows that $s(n,\alpha)$ switches from negative to positive at about

$$\alpha_0 = \sqrt{1.4/n} + o(1/\sqrt{n}),$$

and then remains positive up to $\alpha = (n-1)/n$. Therefore, from (66) we see that an n_0 exists such that for $\alpha_0 < \alpha \leq 1$ and all $n > n_0$ we have $\Delta L(n, n(1-\alpha)) < 0$.

Part Ib. Here we consider the case $h/n = \alpha \leq \alpha_0$, i.e., $h/n \leq \alpha_0 = \sqrt{1.4/n} + o(1/\sqrt{n})$. This means that we have

$$h \leq \sqrt{1.4n} + o(1/\sqrt{n}). \tag{69}$$

The (59) can be bounded as follows

$$\begin{aligned} \Delta L(n, -h) &\leq n + \sum_{i=2}^{n} \pi_{n,n}(i)\, L(i) + \pi_{n,n-h}(n)\, L(n) \\ &\quad - \left(n - h + \sum_{i=2}^{n} \pi_{n,n-h}(i)\, L(i)\right) \\ &= h + \sum_{i=2}^{n} (\pi_{n,n}(i) - \pi_{n,n-h}(i))\, L(i) + \mathcal{O}(c^n), \end{aligned} \tag{70}$$

where we have made use of Property 2B.

By substituting the expression $L(i) = ie - \varepsilon(i)$ (Th. 1), condition (58) turns into

$$\begin{aligned} \overline{\varepsilon(i)} &= \sum_{i=2}^{n} \varepsilon(i)\, (\pi_{n,n-h}(i) - \pi_{n,n}(i)) \\ &< e\,(R_{n,n-h} - R_{n,n}) - h - \mathcal{O}(c^n). \end{aligned} \tag{71}$$

We use the expansion

$$R_{n,n-h} - R_{n,n} = he^{-1}\left(1 + \frac{h+1}{2} n^{-1} + \frac{4h^2 + 18h + 7}{24} n^{-2}\right) + \mathcal{O}(n^{-3}). \tag{72}$$

Furthermore, since $\pi_{n,n-h}$ stochastically dominates $\pi_{n,n}$ (Lemma 6), we use Corollary 2 to show that $\overline{\varepsilon(i)} \leq \overline{l(i)}$, where we have adopted the new distributions. Therefore, from

(71), the thesis is proved true by showing that

$$\begin{aligned}\overline{\varepsilon(i)} &\leq \overline{l(i)}\\ &= \sum_{i=2}^{n} l(i)\left(\pi_{n,n-h}(i) - \pi_{n,n}(i)\right)\\ &< \frac{(h+1)h}{2}n^{-1} + \frac{(4h^2+18h+7)h}{24}n^{-2} + \mathcal{O}(n^{-3}).\end{aligned} \tag{73}$$

We prove the above inequality by bounding $\overline{l(i)}$ exactly as we bounded $\overline{g(i)}$ in Lemma 1. Actually, this evaluation is simpler, as function $l(n)$ coincides with the first part of $g(n)$. We take the power series at $a = R_{n,n-h}$, and get the corresponding of (22), where now the moments are evaluated according to the distributions in (73). We then substitute the asymptotic expansions to get

$$\overline{l(i)} \leq \nu \frac{h}{e-1} n^{-1} + \mathcal{O}(n^{-2}), \tag{74}$$

then (73) is true if the following is true

$$\frac{\nu}{e-1} n^{-1} < \frac{h+1}{2} n^{-1} + \mathcal{O}(n^{-2}). \tag{75}$$

Disregarding term $\mathcal{O}(n^{-2})$, the above inequality is always verified as by Lemma 1 we have $\nu < e-1$. Under this hypothesis we can always find a finite n_1 such that for all $n > n_1$ inequality (75) holds.

Part II. The (59) becomes

$$\Delta L(n,k) = -k + \sum_{i=2}^{n} (\pi_{n,n}(i) - \pi_{n,n+k}(i)) L(i) + \mathcal{O}(c^n). \tag{76}$$

Substituting $L(i) = ie - \varepsilon(i)$, condition (58) transforms into

$$\sum_{i=2}^{n} \varepsilon(i)(\pi_{n,n}(i) - \pi_{n,n+k}(i)) \geq e(R_{n,n} - R_{n,n+k}) - k + \mathcal{O}(c^n). \tag{77}$$

If we disregard term $\mathcal{O}(c^n)$, the above inequality is always verified for any $k > 0$. In fact, by Property 4b in Appendix B, the right hand term is negative. On the other side, by Lemmas 2, 5, 6, and property (78), the left hand side cannot be negative (actually we can show it is positive). Therefore, we can find an n_2, independent of k, even when $k \to \infty$, such that the above inequality is satisfied for any $n > n_2$.

Then we numerically show that (58) holds up to $\max[n_0, n_1, n_2]$, and the whole theorem is proved. ∎

III. Conclusions

In this paper we have theoretically proved results about the Frame Aloha protocol that up to now were only numerically verified. In particular we have shown that the strategy that minimizes the time to the identification of a known number of tags is the one that sets at each frame the frame length r equal to the backlog n. Furthermore we have shown that the optimal asymptotic efficiency is e^{-1}, and derived tight upper and lower bounds to the identification time.

Appendix A

We make use of the concept of stochastic dominance of first order. Given two non negative RVs X and Y, the probability distribution π_X of X is said to stochastically dominate π_Y of Y if their cumulative distributions F_X and F_Y are such that

$$F_X(i) \leq F_Y(i), \qquad \forall i.$$

If the property above holds true, and $g(i)$ is a weakly increasing function, then the following property holds [15]:

$$\sum_i g(i)\left(\pi_X(i) - \pi_Y(i)\right) \geq 0. \tag{78}$$

Lemma *5:* If π_X stochastically dominates π_Y, and if $u(i) - u(i-1) \geq h(i) - h(i-1), \forall i$, we have

$$\sum_i u(i)\left(\pi_X(i) - \pi_Y(i)\right) \geq \sum_i h(i)\left(\pi_X(i) - \pi_Y(i)\right). \tag{79}$$

Proof: In fact, (79) holds if the following relation holds

$$\sum_i (u(i) - h(i))\left(\pi_X(i) - \pi_Y(i)\right) \geq 0. \tag{80}$$

The above is true if $u(i) - h(i)$ is weakly increasing, which holds by hypothesis. ∎

If $u(x)$ and $h(x)$ are defined over the real interval that comprises all the values of RVs X and Y, since

$$\frac{\mathrm{d}}{\mathrm{d}x}(u(x) - h(x)) \geq 0, \quad \forall x$$

is a sufficient condition for $u(i) - h(i) \geq 0, \forall i$, we have

Corollary *3:* If π_X stochastically dominates π_Y, and if $\mathrm{d}u(x)/\mathrm{d}x \geq \mathrm{d}h(x)/\mathrm{d}x, \forall x$, then inequality (80) holds.

With the notation used in the paper we have:

Lemma *6:*

1) Distribution $\pi_{n+1,r}$ stochastically dominates, in the first order, $\pi_{n,r}$;
2) $\pi_{n+1,r+1}$ and $\pi'_{n+1,r+1}$ stochastically dominate, in the first order, $\pi_{n,r}$;
3) $\pi_{n,r}$ stochastically dominates, in the first order, $\pi_{n,r+1}$.

Proof:

1) We must show that

$$\mathbb{P}(\mathcal{R}_{n+1,r} > i) > \mathbb{P}(\mathcal{R}_{n,r} > i), \qquad i = 0, \dots n.$$

The experiment that provides $\mathcal{R}_{n+1,r}$ can be composed of two subsequent experiments: the first is the experiment that provides $\mathcal{R}_{n,r}$, and the second experiment adds to the frame the $(n+1)$-th tag, which can be either collided or not. Let $\mathcal{R}_1$ denote the increase in the number of collided tags the second experiment causes, either 0, 1, or 2. Therefore we have $\mathcal{R}_{n+1,r} = \mathcal{R}_{n,r} + \mathcal{R}_1$, and also

$$\begin{aligned}\mathbb{P}(\mathcal{R}_{n+1,r} > i) &= \mathbb{P}(\mathcal{R}_{n,r} + \mathcal{R}_1 > i)\\ &= \mathbb{P}(\mathcal{R}_{n,r} > i) + \sum_{k\geq 0} \mathbb{P}(\mathcal{R}_{n,r} = i-k)\,\mathbb{P}(\mathcal{R}_1 > k)\\ &> \mathbb{P}(\mathcal{R}_{n,r} > i),\end{aligned}$$

which prove the first point.

2) We must show that

$$\mathbb{P}(\mathcal{R}_{n+1,r+1} > i) > \mathbb{P}(\mathcal{R}_{n,r} > i), \qquad i = 0, \dots n+1, \tag{81}$$

that is equivalent to show that

$$\mathbb{P}(\mathcal{S}_{n+1,r+1} < n-(i-1)) > \mathbb{P}(\mathcal{S}_{n,r} < n-i), \quad i = 0, \dots n+1,$$

and using the change of variable $i' = n - i$, the condition to check is

$$\mathbb{P}(\mathcal{S}_{n+1,r+1} < i' + 1) > \mathbb{P}(\mathcal{S}_{n,r} < i'), \qquad i' = -1, \dots n. \tag{82}$$

As suggested by (87), the ratio between $p_{n+1,r+1}(i+1)$ and $p_{n,r}(i)$ is a decreasing function of i:

$$\frac{p_{n+1,r+1}(i+1)}{p_{n,r}(i)} = \frac{n+1}{i+1}\left(\frac{r}{r+1}\right)^n, \quad 0 \le i \le \min\{n, r\}, \tag{83}$$

and this is a sufficient condition to ensure (82), i.e., that $\{p_{n,r}(i)\}$ stochastically dominates $\{p_{n+1,r+1}(i+1)\}$ [16]. This is equivalent to say that (81) holds, i.e., that $\pi_{n+1,r+1}$ stochastically dominates $\pi_{n,r}$.

Introducing the probabilities

$$p'_{n+1,r+1}(i+1) = \frac{p_{n+1,r+1}(i+1)}{1 - p_{n+1,r+1}(0)}, \qquad i = 0, \dots n,$$

that are scaled versions of $p_{n+1,r+1}(i+1)$, we note that their ratio with $p_{n,r}(i)$ is still a decreasing function of i, therefore $\pi'_{n+1,r+1}$ stochastically dominates $\pi_{n,r}$.

3) The thesis implies that distribution $p_{n,r+1}$ stochastically dominates $p_{n,r}$. Using the ratio $Y_{n,r}(i) = p_{n,r+1}(i)/p_{n,r}(i)$, owing to (87) we have the recursion

$$Y_{n,r}(i) = Y_{n-1,r-1}(i)\left(\frac{r^2}{r^2-1}\right)^{n-1}. \tag{84}$$

Let now assume that the ratio $Y_{n-1,r-1}(i)$ is a non-decreasing function of i and that

$$Y_{n-1,r-1}(i_0) \le 1, \qquad Y_{n-1,r-1}(i_0+1) \ge 1.$$

Then, owing to (84), $Y_{n,r}(i)$ is a non-decreasing function of i. Furthermore, condition $Y_{n,r}(i) > 1$ can not hold for all i, because this would imply $p_{n,r+1}(i) > p_{n,r}(i)$ for all i, clearly impossible. Therefore, for some i_1 it must hold

$$Y_{n,r}(i_1) \le 1, \qquad Y_{n,r}(i_1+1) \ge 1.$$

This proves the thesis by induction on the pair (n, r) starting from

$$\begin{aligned} &Y_{2,r}(0) = r/(r+1), \\ &Y_{2,r}(1) = 1, \\ &Y_{2,r}(2) = r^2/(r^2-1), \qquad r \ge 2, \end{aligned}$$

where $Y_{2,r}(1) = 1$ means that the two terms of the ratio are equal, though both equal to zero.

■

APPENDIX B

A recursive expression of $p_{n,r_n}(s, c)$, the probability of having s successes and c collided slots in the first frame, is given in [10]. A closed-form expression for $p_{n,r}(s)$ is given in the following

Property *1:* The distribution $p_{n,r}$ is given by

$$p_{n,r}(i) = \sum_{k=i}^{m} (-1)^{k+i}\binom{k}{i} X_{n,r}(k), \qquad 0 \le i \le m, \tag{85}$$

where $m = \min\{n, r\}$, and

$$X_{n,r}(k) = \binom{r}{k}\frac{n!}{(n-k)!}\left(\frac{1}{r}\right)^k\left(\frac{r-k}{r}\right)^{n-k}, \tag{86}$$

with $k \le m$.
Furthermore we have

$$p_{n+1,r+1}(i+1) = p_{n,r}(i)\frac{n+1}{i+1}\left(\frac{r}{r+1}\right)^n, \qquad 0 \le i \le m. \tag{87}$$

Proof: Let $A_1, A_2, \dots, A_r$ be r non-disjoint events. The probability that exactly t among these events jointly occur is given by [17]

$$\begin{aligned} P_t = X_t - \binom{t+1}{t}X_{t+1} + \binom{t+2}{t}X_{t+2} \\ - \dots + (-1)^{r-t}\binom{r}{t}X_r \end{aligned} \tag{88}$$

where

$$\begin{aligned} X_1 &= \sum \mathbb{P}(A_i) \\ X_2 &= \sum_{i \ne j} \mathbb{P}(A_i A_j) \\ X_3 &= \sum_{i \ne j \ne k} \mathbb{P}(A_i A_j A_k) \end{aligned} \tag{89}$$

and so on. Summations involve all possible combinations in such a way that each n-string appears just once, and the number of the terms X_k is $\binom{r}{k}$.

In our case the event A_i is defined as the occurrence of just one transmission, out of n, in slot i of a frame composed of r slots, and the probability of any of the k-string is given by

$$\mathbb{P}(A_{j_1} A_{j_2} \dots A_{j_k}) = \frac{n!}{(n-k)!}\left(\frac{1}{r}\right)^k\left(\frac{r-k}{r}\right)^{n-k},$$

$k \le m$, which by (88) and (89) proves the first part of the theorem.

The proof of the second part comes from (85) and (86), observing that

$$X_{n+1,r+1}(k+1) = \frac{n+1}{k+1}\left(\frac{r}{r+1}\right)^n X_{n,r}(k).$$

and rearranging terms. ■

Property *2:* The sequence $\{\pi_n(n)\}_n$ for $n > 16$ is bounded as:

$$\pi_n(n) < \bar{\pi}_n(n) = 3.47 \cdot 10^{-3} \cdot 0.9157^n + 59.79 \cdot 0.4157^n. \tag{90}$$

Proof: Once the number of tags that participates in a frame is fixed, adding a slot to the frame decreases the probability of having no successes, or, in other terms,

$$\pi_{n+1,n+1}(n+1) < \pi_{n+1,n}(n+1). \tag{91}$$

On the other side, considering the outcome of the $(n+1)$-th tag being added to the frame, we can write

$$\pi_{n+1,n}(n+1) \;=\; \pi_{n,n}(n)\,\mathbb{P}(X=1) + \pi_{n,n}(n-1)\,\mathbb{P}(X=2), \tag{92}$$

where X denotes the increase in the collided tags caused by the $(n+1)$-th tag. In the case represented by $\pi_{n,n}(n-1)$ there is only one success that the new added tag turns into two more collisions, and this happens with probability

$$\mathbb{P}(X=2) \;=\; 1/n.$$

In the case represented by $\pi_{n,n}(n)$ the added tag must select one of the collided slots, and this happens with probability

$$\mathbb{P}(X=1) = \frac{\mathsf{E}\left[\mathcal{C}_n \mid \mathcal{S}_n = 0\right]}{n} < \frac{n}{2}\cdot\frac{1}{n} = \frac{1}{2}.$$

From recursion (87) one has

$$\pi_n(n-1) = p_n(1) = p_{n-1}(0)\cdot S_n = \pi_{n-1}(n-1)\cdot S_n. \tag{93}$$

Using (91), (92) and (93) we finally get

$$\pi_{n+1}(n+1) < \frac{1}{2}\pi_n(n) + \frac{S_n}{n}\pi_{n-1}(n-1),$$

and, taking advantage of Property 3b, we may write

$$\pi_{n+1}(n+1) < \frac{1}{2}\pi_n(n) + \frac{S_{15}}{15}\pi_{n-1}(n-1),$$

for $n \geq 15$. This means that it is possible to build a sequence $\{\bar{\pi}_n(n)\}$, that upper bounds the actual sequence $\{\pi_n(n)\}$, through the recurrence

$$\bar{\pi}_{n+1}(n+1) = 0.5\;\bar{\pi}_n(n) + 0.381\;\bar{\pi}_{n-1}(n-1),$$

for $n \geq 16$, with initial conditions $\bar{\pi}_{14}(14) = \pi_{14}(14) \approx 1.285\cdot 10^{-3}$ and $\bar{\pi}_{15}(15) = \pi_{15}(15) \approx 8.106\cdot 10^{-4}$. The solution of the above difference equation is

$$\bar{\pi}_n(n) = 3.47\cdot 10^{-3}\cdot 0.9157^n + 59.79\cdot(-0.4157)^n, \tag{94}$$

for $n \geq 16$. From this, bound (90) is immediate. ∎

Property 2b:

$$\pi_{n,n/k}(n) = \mathcal{O}(c^n), \qquad c<1 \tag{95}$$

where $k>1$ is such that n/k is integer. This property can be proved exactly as the previous one by suitably choosing some coefficients.

In the analysis carried out in the paper we make use of properties of RVs $\mathcal{S}_n$ and $\mathcal{R}_n = n - \mathcal{S}_n$, listed below, that can be proved with standard tools.

Property *3:*

a) $\mathsf{E}\left[\mathcal{S}_{n,r}\right] = S_{n,r} = n\left(1-\frac{1}{r}\right)^{n-1}$;
b) R_n/n is an increasing function of n such that $\frac{R_n}{n} = 1 - e^{-1} - \frac{e^{-1}}{2n} - \frac{7e^{-1}}{24n^2} - \frac{3e^{-1}}{16n^3} + \mathcal{O}(n^{-4})$;
c) $R_{n+1} - R_n$ is a decreasing function of n with $(R_{n+1} - R_n) = 1 - e^{-1} + \frac{7e^{-1}}{24n^2} + \mathcal{O}(n^{-3})$.

Property *4:*

a) $R_{n,n+k-1} - R_{n,n+k} < e^{-1}$, for all $n \geq 1$ and $k \geq 1$;
b) $R_{n,n} - R_{n,n+k} < ke^{-1}$, for all $n \geq 1$ and $k \geq 1$.

Proof: For the first point it is

$$\begin{aligned} R_{n,n+k-1} - R_{n,n+k} &= S_{n,n+k} - S_{n,n+k-1} \\ &\leq \max_{k\in\{1,2,\ldots\}} \left(S_{n,n+k} - S_{n,n+k-1}\right) \\ &\leq \sup_{k\in[1,\infty)} \frac{\partial}{\partial k} S_{n,n+k} \end{aligned}$$

for all $n \geq 1$. The derivative with respect to k is

$$\frac{\partial}{\partial k} S_{n,n+k} = \frac{n}{n+k}\frac{S_{n-1,n+k}}{n+k}, \tag{96}$$

where $S_{n-1,n+k}/(n+k)$, the throughput per slot, is a decreasing function of k, for $k \geq 1$. This means that also (96) is a decreasing function of k, and therefore the maximum is achieved for $k=1$:

$$\begin{aligned} \frac{\partial}{\partial k} S_{n,n+k} &\leq \frac{n}{(n+1)^2} S_{n-1,n+1} \\ &= \frac{n(n-1)}{(n+1)^2}\left(1 - \frac{1}{n+1}\right)^{n-2} < e^{-1}. \end{aligned}$$

Point 4b comes straightforwardly from point 4a. ∎

Property *5:* Here we show how to derive moments of variable $\mathcal{S}_{n,r}$. Moments for variable $\mathcal{R}_{n,r}$ can be derived by the relation $\mathcal{R}_{n,r} = n - \mathcal{S}_{n,r}$. The first order moment is given above in Property 3a. For the evaluation of higher order moments we express $\mathcal{S}_{n,r}$ as the sum of binary variables $X_{n,r}(i)$, where $X_{n,r}(i)$ takes value 1 if in the corresponding i-th slot of the frame there is only one tag, i.e., a success. Hence

$$\mathsf{E}\left[\mathcal{S}^k_{n,r}\right] = \mathsf{E}\left[\left(\sum_{i=1}^{r} X_{n,r}(i)\right)^k\right].$$

We use the multinomial theorem, that gives

$$\mathsf{E}\left[\left(\sum_{i=1}^{r} X_{n,r}(i)\right)^k\right] = \sum_{k_1+k_2+\ldots+k_r=k} \frac{k!}{k_1!k_2!\ldots k_r!} Q_{n,r}$$

with

$$\begin{aligned} Q_{n,r} &= \mathsf{E}\left[\prod_{j=1}^{r} X^{k_j}_{n,r}(j)\right] \\ &= \sum \mathbb{P}\Big(X_{n,r}(1) = x_1; X_{n,r}(2) = x_2; \\ &\qquad \ldots X_{n,r}(r) = x_r\Big)\cdot \prod_{j\in\Omega} x_j^{k_j}, \end{aligned}$$

where the summation is extended to the whole space of outcomes, whereas the product is extended to indexes $j \in \Omega = \{j_1, j_2, \ldots, j_{|\Omega|}\}$ for which $k_j > 0$, and $1 \leq |\Omega| \leq \min\{r,k\}$.

We consider the case where $k \le r$. This allow us to write

$$Q_{n,r} = \mathbb{P}(X_{n,r}(j_1) = 1; X_{n,r}(j_2) = 1; \dots X_{n,r}(j_{|\Omega|}) = 1)$$
$$= \mathbb{P}(X_{n,r}(j_1) = 1) \cdot \prod_{t=2}^{|\Omega|} \mathbb{P}\Big(X_{n,r}(j_t) = 1 | X_{n,r}(j_{t-1}) = 1, X_{n,r}(j_{t-2}) = 1, \dots, X_{n,r}(j_1) = 1\Big)$$
$$= \prod_{t=1}^{|\Omega|} \mathbb{P}(X_{n-t+1,r-t+1}(j_t) = 1)$$
$$= \prod_{t=1}^{|\Omega|} \frac{S_{n-t+1,r-t+1}}{r-t+1},$$

where we have used the chain rule for probabilities and the fact that knowing the outcomes of some slots reduces the problem. Furthermore, sequences $k_1, k_2, \dots k_n$ that are permutations of the same sequence provide the same $Q_{n,r}$. As a consequence, in the case where $k \le r$, the k-th moment of RV $\mathcal{S}_{n,r}$ can be written as

$$\mathsf{E}\left[\mathcal{S}_{n,r}^k\right] = \sum_{i=1}^{k} a_i \prod_{t=1}^{i} \frac{S_{n-t+1,r-t+1}}{r-t+1}$$
$$= \frac{S_{n,r}}{r}\left(a_1 + \frac{S_{n-1,r-1}}{r-1} \cdot \left(a_2 + \dots \left(a_{k-1} + \frac{S_{n-k+1,r-k+1}}{r-k+1} a_k\right)\right)\right),$$

where a_i is the number of combinations where $|\Omega| = i$.

For example, for $k = 2 \le r$, we have r terms corresponding to $k_j = 2$, being all the others zero ($|\Omega| = 1$ and $a_1 = r$), and $r(r-1)$ terms of type $k_i = 1, k_j = 1$, $i \ne j$, being all the others zero ($|\Omega| = 2$ and $a_2 = r(r-1)$). This provides

$$\mathsf{E}\left[\mathcal{S}_{n,r}^2\right] = S_{n,r} + S_{n,r} S_{n-1,r-1}.$$

In a similar way we have found

$$\mathsf{E}\left[\mathcal{S}_{n,r}^3\right] = S_{n,r} + 3\, S_{n,r}\, S_{n-1,r-1} + S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2},$$
$$\mathsf{E}\left[\mathcal{S}_{n,r}^4\right] = S_{n,r} + 7\, S_{n,r}\, S_{n-1,r-1} + 6\, S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2} + S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2}\, S_{n-3,r-3},$$
$$\mathsf{E}\left[\mathcal{S}_{n,r}^5\right] = S_{n,r} + 15\, S_{n,r}\, S_{n-1,r-1} + 25\, S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2} + 10\, S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2}\, S_{n-3,r-3} + S_{n,r}\, S_{n-1,r-1}\, S_{n-2,r-2}\, S_{n-3,r-3}\, S_{n-4,r-4}.$$

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