FINITE SETTLING TIME STABILIZATION FOR LINEAR MULTIVARIABLE TIME-INVARIANT DISCRETE-TIME SYSTEMS:
An Algebraic Approach

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E. MILONIDIS, B.Sc.(Hons), M.Sc., M.Phil.

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CITY UNIVERSITY
CONTROL ENGINEERING RESEARCH CENTRE
DEPARTMENT OF ELECTRICAL ELECTRONIC & INFORMATION ENGINEERING
LONDON, EC1V 0HB

στη μνημη του πατερα μου

to the memory of my father

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A C K N O W L E D G E M E N T S

I would like to dedicate this work to the memory of my father whose support and encouragement made this work, and most of my studies possible. His sudden death is a great loss to all of us.

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D E C L A R A T I O N

ABSTRACT

The problem of ***Total Finite Settling Time Stabilization*** of linear time-invariant discrete-time systems is investigated in this thesis. This problem falls within the same area of the well known deadbeat (time-optimal) control and in particular, constitutes a generalization of this problem. That is, instead of seeking time-optimum performance, it is required that all internal and external variables (signals) of the closed-loop system settle to a new steady state after a finite time from the application of a step change to any of its inputs and for every initial condition. The state/output deadbeat control is a special case of the Total FSTS problem.

Using a mathematical and system theory framework based on sequences and the polynomial equation (algebraic) approach, we are able to tackle the FSTS problem in a unifying manner. The one-parameter (unity) feedback configuration is mainly used for the solution of the FSTS problem and FSTS related control strategies. The whole problem is reduced to the solution of a polynomial matrix Diophantine equation which guarantees not only internal stability but also internal FSTS and is further reduced to the solution of a linear algebra problem over $\mathbb{R}$. This approach enables the parametrization of the family of all FSTS controllers, as well as those which are causal, in a Youla-Bongiorno-Kucera type parametrization.

The minimal McMillan degree FSTS problem is completely solved for vector plants and a parametrization of the FSTS controllers according to their McMillan degree is obtained. In the MIMO case bounds of the minimum McMillan degree controllers are derived and families of FSTS controllers with given lower/upper McMillan degree bounds are provided in parametric form.

Having parametrized the family of all FSTS controllers, the state deadbeat regulation is treated as a special case of FSTS and complete parametrization of all the deadbeat

regulators is presented. In addition, further performance criteria, or design constraints are imposed such as, FSTS tracking and/or disturbance rejection, partial assignment of controller dynamics, ℓ^1-, ℓ^∞-optimization and robustness to plant parameter variations.

Finally, the Simultaneous-FSTS problem is formulated, and necessary as well as sufficient conditions for its solution are derived. Also, a two-parameter control scheme is introduced to alleviate some of the drawbacks of the one-parameter control. A parametrization of the family of FSTS controllers as well as the FSTS controllers for tracking and/or disturbance rejection is given as an illustration of the particular advantages of the two-parameter FSTS controllers.

A B B R E V I A T I O N S

det	determinant
diag	diagonal
ker	kernel
l.c.m.	least common multiple
max	maximum
min	minimum
sup	supremum
BIBO	Bounded Input Bounded Output
ED	Euclidean Domain
EDC	Extended Divisor Condition
E-FSTS	External FSTS
FIFO	Finite Input Finite Output
FIR	Finite Impulse Response
FST	Finite Settling Time
FSTS	FST Stabilization
FSTSP	FSTS Problem
I-FSTS	Internal FSTS
LP	Linear Program
MDP	Minimal Design Problem
MFD	Matrix Fraction Description
MIMO	Many Input Many Output
OP	Optimization Problem
PFM	Plant Family Matrix
PID	Principal Ideal Domain
PP	Parametrization Problem
RCMD (LCMD)	Right (Left) Common Matrix Divisor
RGCMD (LGCMD)	Right (Left) Greatest Common Matrix Divisor
RGMD (LGMD)	Right (Left) Greatest Matrix Divisor
RMD (LMD)	Right (Left) Matrix Divisor
R-PFM (L-PFM)	Right (Left) PFM

S-FSTS	Simultaneous FSTS
SISO	Single Input Single Output
SSC	Structure Space Condition
SSP	Simultaneous Stabilization Problem
TFSTS	Total FSTS
UFD	Unique Factorization Domain
YBK	Youla-Bongiorno-Kucera
ZOH	Zero-Order-Hold

NOTATION

A	the set of BIBO-stable operators
A_s	disc Algebra
$\mathcal{B}$	Boolean field
$\mathbb{C}$	complex numbers (plane)
$\mathbb{C}_e$	extended complex plane
$C_r(M)$	the rth compound of a matrix M
c_P (r_P)	column (row) complexity of the polynomial matrix P
$\mathcal{C}_k(A,B)$	kth-controllable subspace of (A,B)
$\mathcal{C}\Delta_k(A,C)$	kth-constructible subspace of (A,C)
d	indeterminate, delay operator
$\mathbb{D}[0,1)$	open unit disc
$E_{\mathcal{R}}$	$\mathcal{R}$-equivalent
$E^r_{\mathcal{R}}$ $(E^l_{\mathcal{R}})$	right (left) $\mathcal{R}$-equivalent
$\mathcal{F}$	field
$\mathcal{F}^{\mathbb{Z}}$, $\mathcal{F}<x>$	formal Laurent series in one indeterminate x over $\mathcal{F}$
$\mathcal{F}[[x]]$	formal power series in one indeterminate x over $\mathcal{F}$
$\mathcal{F}[x]$	formal polynomials in one indeterminate x over $\mathcal{F}$
$\mathcal{F}^{\mathcal{F}}$	the set of all functions from $\mathcal{F}$ to $\mathcal{F}$
$\mathcal{F}(x)$	formal rational series in one indeterminate x over $\mathcal{F}$
$F(\mathrm{LP})$	set of feasible solutions of the linear program LP
$\mathcal{F}(P)$	the family of causal FSTS controllers for the plant P
$\mathcal{F}^c_{c,min}(P)$ $(\mathcal{F}^c_{r,min}(P))$	family of FSTS controllers with minimum column (row) comlexity
$\mathcal{F}_{c,min}(P)$ $(\mathcal{F}_{r,min}(P))$	family of FSTS controllers with minimum column (row) scalar degree of their composite matrices

$\hat{\mathcal{F}}_k(P)$	family of FSTS controllers with scalar degree of their composite matrices less than or equal to μ - 1 or ν - 1
$g_p(G)$	gain of the ℓ^p-operator G
i_c	column vector with ones everywhere
I_l	$\ell \times \ell$ unity matrix
ℓ^p	space of sequences with bounded p-norm
$\underline{m}_i\wedge$	exterior (Grassmann) product of the vectors $\underline{m}_i$
$M(\mathcal{R})$	matrix with elements from the ring $\mathcal{R}$
M^t	transpose of M
M_r^+ (M_l^+)	right- (left-) annihilator of M
$M_r^\perp$ $(M_l^\perp)$	right- (left-) projector of M
$\mathbb{N}$	the set of natural numbers
$\mathcal{N}_r\{M\}$ $(\mathcal{N}_l\{M\})$	right (left) null space of M
$\mathcal{O}_k(A,C)$	kth-observable subspace of (A,C)
$p_P(d)$	the pole-polynomial of $P(d)$
$p_{\mathcal{S}}(d)$	the pole-polynomial of the system $\mathcal{S}$
Q_u	partitioned unimodular matrix
$\mathbb{R}$	the set of real numbers
$\mathbb{R}\{d\}$	the set of recurrent sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}(d)$	the set of rational sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}^0(d)$	the set of causal sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}^+(d)$	the set of stable sequences with one indeterminate d over $\mathbb{R}$
$\mathbb{R}[d]$	the set of polynomial sequences with one indeterminate d over $\mathbb{R}$
$\left.\begin{array}{l}\mathbb{R}(\mathrm{d})\\ \mathbb{R}^0(\mathrm{d})\\ \mathbb{R}^+(\mathrm{d})\end{array}\right\}$	corresponding functions over $\mathbb{R}$
$\left.\begin{array}{l}\mathbb{R}_{lm}(\mathrm{d})\\ \mathbb{R}^0_{lm}(\mathrm{d})\\ \mathbb{R}^+_{lm}(\mathrm{d})\end{array}\right\}$	corresponding $\ell \times m$ sequential matrices
$\mathcal{R}$	ring
$\mathcal{R}^{l\times m}$	$\ell \times m$ matrices with elements from $\mathcal{R}$

R_G (L_G)	right (left) composite matrix of G
$\mathcal{R}_k(A,B)$	kth-reachable subspace of (A,B)
$S_P^*(d)$	the essential part of the Smith-McMillan form of $P(d)$
$\mathcal{S}$	external description of a system
$\mathcal{T}$	the discrete time
$\mathcal{U}$	the input space
$U(\mathcal{R})$	unimodular matrices with elements from $\mathcal{R}$
V^T	the dual space of V
$\mathcal{X}$	the state space
$\mathcal{Y}$	the output space
z	the advance operator ($z = d^{-1}$)
$z_P(d)$	the zero-polynomial of $P(d)$
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of positive integers
$\delta_F(P(d))$	the Forney order of $P(d)$
$\delta_{\mathcal{M}}^{\Omega}(P(d))$	the McMillan degree of $P(d)$ over Ω
$\partial(f)$	the degree of the polynomial f
$\partial_s(P(d))$	the scalar degree of the polynomial matrix $P(d)$
μ	the maximum reachability index
μ_i	reachability indices
ν	the maximum observability index
ν_i	observability indices
Πf_k	product of the elements f_k
$\rho(P(d))$	normal rank of $P(d)$
$\rho_c(P(d))$	local rank of $P(d)$
Σ	internal description of a system
Σ_d	the dual of the system Σ
Σf_k	sum of the elements f_k
$\tau(f)$	the order of the sequence f
$\upsilon_\alpha(f(d))$	valuation of $f(d)$ at $d = \alpha$

SYMBOLS

$\forall$	for every
$\exists$	there exists
$\in$	belongs to
$\notin$	it does not belong to
$\subset$	proper subset
$\subseteq$	subset
$\cup$	union
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\therefore$	i.e.
$:=$	equal by definition
$\sim$	associate
$\cdot\sim$	extended associate
$\mid$	divides
$\oplus$	direct sum
$\otimes$	Kronecker product
$*$	convolutory multiplication
$\lVert\cdot\rVert_p$	p-norm
∞	infinity
$\square$	end of statement

Chapter 1

INTRODUCTION

Chapter 1

INTRODUCTION

The advent of the digital computers in the early fifties has brought an unprecedented revolution in our society. Industry in particular, could not escape from that. Numerous industrial control systems include now digital computers and operate using digital technology. Low-cost microprocessors and microcomputers are feasible solutions for even small-scale control systems.

This wide-spread use of digital computer technology in system analysis and design has arisen considerable interest among engineers and system theorists in the relatively new area of discrete system theory. Several methods, mainly as counterparts to the continuous-time methods, were proposed and applied; state-space techniques, difference equations as opposed to differential ones, and transformation techniques like z-Transform, representing the counterpart of the Laplace transform.

As in the case of continuous-time dynamical systems the algebraic approach has emerged as one of the most powerful methods for both analysis and design purposes. Recent work in this area is mainly based on what is termed as ***Fractional Representation Approach***, including the ***Polynomial*** one [Des., 2], [Kai., 1], [Kuc., 1], [Ros., 1], [Sae., 2], [Vid., 1], [You., 1], and corresponds to the use of matrix fractions with matrices having elements from special rings. This approach arises naturally from the need to describe algebraically the familiar problems of stability, realizability and performance of linear systems. As a consequence, most of the control system problems are reduced to the solution of (sets of) matrix equations and in many cases complete parametrization of the solutions is also possible [You., 1], [Kuc., 1].

The main feature in discrete-time systems is that they process signals that appear in discrete time instances, i.e. ***sequences***. The framework of studying discrete-time systems in terms of sequences and in particular ***formal power series***, has been used by Kalman [Kal., 1] and Kucera [Kuc., 1]. The most general mathematical representation of sequences is as ***formal Laurent series*** in one indeterminate over $\mathbb{R}$. A further classification of sequences as recurrent, rational, causal and stable, appeared first in the literature in the context of digital signal processing (see e.g. [Ope., 1] and references therein), and it was quickly adopted by control theorists. The aforementioned sets of sequences are either fields or rings, and sequential matrices can be described as matrix fractions with elements from these sets.

From the very early applications of discrete system theory a distinctive property of linear discrete-time systems was realized; namely, their ability to achieve steady state in finite time [Ber., 1]. This problem, and especially the time-optimal one, has intrigued control engineers for many years. In the continuous-time case it has resulted in nonlinear bang-bang control whereas the discrete-time case took the form of the so-called deadbeat control. Linear time-invariant discrete-time controllers, mostly in the form of constant state feedback, can be implemented in order to drive the states or the outputs of the discrete system to the origin in minimum time and for every initial condition. A large variety of versions of deadbeat control have appeared which differ due to the type of problem considered and the adopted approach. The approaches that are used for the solution of the deadbeat problem fall in either of the following two broad areas: the state-space approach and the algebraic (transfer function) approach the later having the advantages of complete parametrization of solutions.

The present work constitutes a generalization of the deadbeat problem. Instead of seeking time-optimum performance, it is required that all internal and external variables (signals) of the closed-loop system to settle to a new steady state after a finite time from the application of a step change to any of its inputs and for every initial condition. We call this problem ***Total Finite Settling Time Problem***. Deadbeat response is a special case of the FSTS problem and an algebraic formulation of this problem, that also guarantees internal FST behaviour, on one hand unifies existing results and on the other hand provides the solution, again in a unifying manner, of a variety of problems like the minimum design problem, tracking and/or disturbance rejection in FST sense, optimization, robustness and simultaneous FSTS.

In the following chapter we present a quick review of the basic tools of the algebraic approach within the context of discrete-time systems and an attempt is made for a unifying mathematical formalism. We refine Kucera's [Kuc., 1] treatment of discrete-time signals as sequences and attempt to unify his approach with Kalman's [Kal., 1] by providing all necessary tools and properties linking both approaches together. The important feature to this is the re-establishment of the isomorphism between certain classes of formal series in one indeterminate over $\mathbb{R}$ and series expansion of functions over $\mathbb{R}$. In addition some basic results of rational vector spaces are given and solvability conditions of matrix equations over rings are presented.

Chapter (3) is a summary of basic concepts and results of linear systems theory for discrete-time systems. The ring isomorphism between formal series and series expansions due to the infinite nature of $\mathbb{R}$, is exploited for a consistent treatment of discrete-time systems. To this end a unification between the *d*- and *z*-representations of discrete linear systems and the computation of the McMillan degree from either description is made possible. The internal and the external behaviour of the systems is described and the concepts of reachabililty/controllability, observability/

constructibility and stability are discussed in some detail. In the final section the properties of the unity feedback and its use as the main control scheme in this thesis are examined.

In the fourth chapter we place the deadbeat control problem within the context of the FSTS problem. We attempt a classification of the variety of deadbeat problems that have emerged over the past years and to survey the two main approaches, namely the state-space and the algebraic approach, for the solution of the deadbeat problem. As a consequence we extend some basic results. In particular, we extend the so-called main theorem of discrete-time systems given by Kalman [Kal., 2] to the MIMO case and in addition we obtain an explicit parametrization of the family of all deadbeat state regulators. It is pointed out that most of the work, if any, in FST has focused on deadbeat and thus examined special types of control problems. Therefore by taking the viewpoint that deadbeat response is a special case of the FST response, a considerable advance in the discrete control analysis and design can be achieved.

The core chapter of this thesis is chapter (5) where the problem of ***Total Finite Settling Time Stabilization*** for the case of SISO discrete-time systems is defined. The approach for its solution is purely algebraic based on the mathematical and system theory framework as it has been discussed in chapters (2) and (3). The whole problem is presented as a solution to a scalar polynomial Diophantine equation which can be reduced to the solution of a linear algebra problem over $\mathbb{R}$. This leads to the Youla-Bongiorno-Kucera type of parametrization of the family of FSTS controllers as well as their parametrization according to McMillan degree and the solution to the minimal McMillan degree problem. In the case of minimally realized plant and controller internal (state) FST, and not only internal stability, is guaranteed by the adopted approach. Further performance related problems are also considered in a unifying manner, such as strong FSTS, tracking and disturbance rejection of a class of signals in FST sense, FST controller design with partially assigned dynamics and FSTS for sampled-data systems exhibiting ripple-free FTS response.

Chapter (6) is the extension of the single variable case of the TFSTS problem to the case of multivariable time-invariant discrete-time systems. Using the same algebraic approach and a unity feedback scheme as in chapter (5), we tackle the FSTS problem as a solution of a polynomial matrix Diophantine equation. This enables the parametrization of the family $\mathcal{F}(P)$ of causal FSTS controllers in terms of a relatively simple generic condition and the computation of the family $\mathcal{F}(P)$ is reduced to the solution of a set of Toeplitz type linear equations over $\mathbb{R}$. All the problems discussed in chapter (5) are extended to the MIMO case in a straightforward manner except of the minimal design problem and the parametrization of the FSTS controllers according to McMillan degree.

The minimal design problem is examined by providing lower and upper bounds for the minimum McMillan degree of all FSTS controllers. In addition, a parametrization of all FSTS controllers according to column/row degrees or complexity is achieved allowing for the characterization of the family $\mathcal{F}(P)$ according to bounds on the McMillan degree. This parametrization leads to the complete characterization of all deadbeat regulators and the development of an algorithm for their computation. Finally, it is shown that in the case of vector plants a complete parametrization of FSTS controllers according to McMillan degree is possible which gives rise to the solution of the minimal design problem as well.

In chapter (7) we define the problem of ***Simultaneous-FSTS (S-FSTS)***, i.e. the problem of finding a discrete, linear, time-invariant controller that stabilizes in FST sense a family of distinct, linear, discrete-time plants. The motivation of this work comes from the so-called ***Simultaneous Stabilization Problem (SSP)*** and in the first section of this chapter we give a brief summary of some background results concerning the SSP. In the sequel, we consider the general case of S-FSTS of a family of MIMO plants. We associate to this family, a family plant matrix and its properties lead to a classification of the various types of families as well as general conditions for solvability of the S-FSTS problem. It is shown that for the left regular and coprime families a solution always exists, whereas for the right regular case of plant families what we call ***Space Structure Condition (SSC)***, i.e the existence of partitioned unimodular matrices in a given rational space, is the most significant condition for the solvability of the S-FSTS problem. Alternatively, with an approach similar to that in Vidyasagar and Viswanadham [Vid., 2], we derive necessary, sufficient conditions of the general S-FSTS problem. For the cases of families of vector plants SSC becomes the prevailing solvability condition and testable necessary and sufficient conditions are derived. These conditions may be tested using tools of the minimal basis theory of rational vector spaces, or equivalent standard linear algebra tests over $\mathbb{R}$. The derivation of computationally verifiable criteria for the SSC in the general case is still an open problem and under investigation.

The aim of the eighth chapter is twofold. On one hand we consider optimization, shaping and robustness problems in FSTS sense within the framework of the one-parameter feedback scheme used so far, on the other hand we introduce a two-parameter feedback scheme to alleviate some of the limitations of the one-parameter control design. The optimization, shaping and robustness problems reduce to the minimization of an ℓ^1-, or ℓ^∞-norm of certain vectors and they can be further reduced to the solution of corresponding finite linear programming problems where all the benefits of the linear programming optimization can be exploited.

In the final section of chapter (8) we replace the unity feedback scheme by a two-parameter control scheme. A complete parametrization of the family $\mathcal{F}_2(P)$ of all causal

FSTS controllers is derived, now in terms of two independent parameters T and $S \in M(\mathbb{R}[d])$ in an affine manner. The advantages of the two-parameter FSTS compensation over the one-parameter one, are demonstrated in the case of tracking and disturbance rejection in FST sense where it is clearly shown that the two performance requirements are independently affected by the two distinct parameters T and S.

Chapter 2

MATHEMATICAL PRELIMINARIES: A Unifying Mathematical Background for the Study of Discrete-Time Systems

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A Unifying Mathematical Background for the Study of Discrete-Time Systems

2.1 Introduction

In the study of discrete-time as well as continuous-time dynamical systems the algebraic approach has emerged as one of the most powerful methods for both analysis and synthesis purposes. The objective of this chapter is to present a quick review of the basic tools of this approach and to emphasize their use in the study of discrete-time control systems in particular. Furthermore, an attempt is made to provide a unifying mathematical formalism for the topic of discrete-time systems which is the subject of this thesis.

The notion of sequences - infinite as well as finite - over the field $\mathbb{R}$ of real numbers is introduced and a classification is given for control purposes. The framework for studying discrete-time systems in terms of sequences, or ***formal power series*** has been used by Kalman [Kal., 1] and Kucera [Kuc., 1]; in the present chapter we attempt to unify their approaches by providing all necessary tools and properties linking them together. In this context, polynomials are ***formal polynomials*** defined as finite causal sequences and are regarded as algebraic objects over $\mathbb{R}$ rather than ***polynomial functions*** on $\mathbb{R}$. It turns out that there is a relationship between the two polynomial notations which becomes a ring isomorphism when they are defined over an infinite field like $\mathbb{R}$ [Zar., 1].

In the sequel, fractions of sequences, sequential matrices and matrix fractions are introduced, the ***Smith form*** and ***Smith-McMillan form*** of a matrix are defined and some basic results on the algebraic structure of rational vector spaces are presented. Finally, necessary and sufficient conditions for the solution of matrix equations over rings are given and a parametrization of solutions, when possible, is presented.

2.2 Sequences - Formal Laurent and Power Series

Consider any field $\mathcal{F}$ and let $\mathbb{Z}$ be the set of integers (positive, negative or zero) and $\mathbb{N}$ be the set of natural numbers.

Definition 2.1: We shall denote by $\mathcal{F}^{\mathbb{Z}}$ the set of all infinite sequences

$$f = \{f_{-n}, f_{-n+1}, \ldots, f_{-1}; f_0, \ldots, f_k, \ldots\},\ f_k \in \mathcal{F} \text{ and } n \in \mathbb{N} \qquad (2.1a)$$

By convention we separate the elements of negative and non-negative indices by a semicolon. Alternatively, an $f \in \mathcal{F}^{\mathbb{Z}}$ may be represented by

$$f = \{f_n, f_{n+1}, \ldots, f_k, \ldots\} \quad f_k \in \mathcal{F} \quad \text{and } n \in \mathbb{Z} \qquad (2.1b)$$

and it describes a mapping $k \longmapsto f_k$ from the set of integers into the set $\mathcal{F}$.

□

We can introduce now the two binary operations $\mathcal{F}^{\mathbb{Z}} \times \mathcal{F}^{\mathbb{Z}} \longmapsto \mathcal{F}^{\mathbb{Z}}$ of addition and multiplication between elements of $\mathcal{F}^{\mathbb{Z}}$ and the binary operation $\mathcal{F} \times \mathcal{F}^{\mathbb{Z}} \longmapsto \mathcal{F}^{\mathbb{Z}}$ of multiplication between elements of $\mathcal{F}$ and elements of $\mathcal{F}^{\mathbb{Z}}$ as it is shown below:

a. <u>pointwise addition</u>: the pointwise sum of $f,\ g \in \mathcal{F}^{\mathbb{Z}}$ is an $h \in \mathcal{F}^{\mathbb{Z}}$ such that

$$h_n = (f + g)_n := f_n + g_n$$

b. <u>convolutory multiplication</u>: the convolution product of $f, g \in \mathcal{F}^{\mathbb{Z}}$ is an $h \in \mathcal{F}^{\mathbb{Z}}$ such that

$$h_n = (f*g)_n := \sum_{l+m=n} f_l g_m$$

c. <u>multiplication between $f \in \mathcal{F}$ and $g \in \mathcal{F}^{\mathbb{Z}}$</u>: the product of $f \in \mathcal{F}$ and $g \in \mathcal{F}^{\mathbb{Z}}$ is an $h \in \mathcal{F}^{\mathbb{Z}}$ such that

$$h_n = (fg)_n := fg_n$$

It is simple to establish that with the aforementioned two operations (a) and (b), $\mathcal{F}^{\mathbb{Z}}$ becomes a field.

Theorem 2.1 [Bur., 1]: Under the operations of pointwise addition and convolutory multiplication, $\mathcal{F}^{\mathbb{Z}}$ forms a field. The zero element of $\mathcal{F}^{\mathbb{Z}}$ is the sequence $\{0;0,0,...\}$ and the identity element is the sequence $\{0;1,0,0,...\}$.

□

The subset of all sequences of $\mathcal{F}^{\mathbb{Z}}$ of the type $w = \{0;a,0,...\}$ $a \in \mathcal{F}$ constitutes a subfield of $\mathcal{F}^{\mathbb{Z}}$ isomorphic to $\mathcal{F}$. Therefore, we shall no longer distinguish between $a \in \mathcal{F}$ and $\{0;a,0,...\} \in \mathcal{F}^{\mathbb{Z}}$. The elements of $\mathcal{F}$, regarded as sequences, will be called **constant sequences**, or just **constants**.

Definition 2.2: We call the sequence $\{0;0,1,0,0,...\}$ an **indeterminate** and we will denote it by x.

□

Clearly, $\{0;a,0,...\}*\{0;0,1,0,...\} = \{0;0,a,0,...\} = ax$. Also, for each $k \in \mathbb{Z}_+$ the (convolution) power $x^k := x*x*...*x$ with k factors, is the sequence $x^k : \mathbb{Z} \longmapsto \mathcal{F}$ with $(x^k)_i = \delta_{ik}$ $\forall\ i \in \mathbb{Z}$ and δ_{ik} the *Kronecker delta*, as it can be shown by induction on k. The inverse of x, denoted by x^{-1}, is the sequence $x^{-1} = \{1;0,...,0,...\}$ and $x^{-k} := x^{-1}*x^{-1}*...*x^{-1}$ with $k \in \mathbb{Z}_+$ factors, is the sequence with $(x^{-k})_i = \delta_{i(-k)}$ $\forall\ i \in \mathbb{Z}$. Hence, by defining by x^0 the identity sequence, x^k $k \in \mathbb{Z}$, is a sequence of zeros except for the element 1 at the kth position. Accordingly, the product

$$\{0;a,0,\ldots\}*x^k = \left\{ \begin{array}{ll} \{0;\underbrace{0,0,\ldots,0}_{k+1},a,0,\ldots\}, & k \geq 0 \\ \{\underbrace{a,0,\ldots,0}_{-k};0,\ldots\}, & k < 0 \end{array} \right\}$$

$$= ax^k \qquad (2.2)$$

is the sequence with zeros everywhere except at the kth position, where its value is $a \in \mathcal{F}$. Therefore, any sequence $f \in \mathcal{F}^{\mathbb{Z}}$ can be written in the form of a formal Laurent series

$$f = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \cdots, \quad n \in \mathbb{Z} \text{ fixed} \qquad (2.3)$$

Remark 2.1: It should be emphasized that x is simply a new symbol, or an **indeterminate** over $\mathcal{F}$, totally unrelated to the field $\mathcal{F}$ and in no sense represents an element of $\mathcal{F}$. In fact, x is an element of $\mathcal{F}^{\mathbb{Z}}$ and serves as a *'position-marker'* in the sequence. The series (2.3) is formal, it should not be interpreted as a function of x and there is no question of convergence whatsoever; it is just a convenient way of writing the sequence (2.1b).

□

The series notation (2.3) will be adopted to represent the elements of $\mathcal{F}^{\mathbb{Z}}$ and $\mathcal{F}^{\mathbb{Z}}$ itself will be denoted as $\mathcal{F}\langle x\rangle$ and shall be called the field of **formal Laurent series over** $\mathcal{F}$. To this extent, an element $f \in \mathcal{F}^{\mathbb{Z}}$ with indeterminate x will be denoted as $f(x)$. An important definition in connection with $\mathcal{F}\langle x\rangle$ is that of **order** given below.

Definition 2.3 [Kuc., 1]: If $f = \sum f_k x^k$ is a nonzero element of $\mathcal{F}\langle x\rangle$, then the smallest integer n such that x^n appears in the sequence is called the **order** of f and is denoted by $\tau(f)$. By convention $\tau(\{0\}) = \infty$.

□

The following result may be readily established.

Theorem 2.2 [Bur., 1]: If f, g are nonzero elements of $\mathcal{F}\langle x\rangle$, then

a. either $f + g = 0$, or $\tau(f+g) \geq \min\{\tau(f), \tau(g)\}$

b. $\tau(f*g) = \tau(f) + \tau(g)$

□

Using the notion of order we can define an important subring of $\mathcal{F}\langle x\rangle$, namely the ring of **formal power series**.

Definition 2.4: The set of all sequences in one indeterminate x over $\mathcal{F}$ with nonnegative order is called the set of **formal power series** over $\mathcal{F}$ and is denoted by $\mathcal{F}[[x]]$. Therefore, a sequence f of $\mathcal{F}[[x]]$ can be written in the form

$$f = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \cdots, \quad n \in \mathbb{N} \text{ fixed}$$

□

Under the operations of pointwise addition and convolutory multiplication the set $\mathcal{F}[[x]]$ forms a ring and the structure of it is given by the following theorem.

Theorem 2.3 [Bur., 1]: The set of formal power series $\mathcal{F}[[x]]$ is a *principal ideal domain*. The **units** of $\mathcal{F}[[x]]$ are the sequences of **zero** order, the **primes** of $\mathcal{F}[[x]]$ are the sequences of order **one** and the **nontrivial ideals** of $\mathcal{F}[[x]]$ are of the form (x^k), where $k \in \mathbb{Z}_+$; in fact $\mathcal{F}[[x]]$ is a local ring with (x) as its maximal ideal.

□

A simple relation between the ring of formal power series $\mathcal{F}[[x]]$ and the field of formal Laurent series $\mathcal{F}^{\mathbb{Z}}$ exists.

Theorem 2.4 [Bur., 1]: $\mathcal{F}^{\mathbb{Z}}$ is the **field of quotients**, or the **field of fractions** of the domain $\mathcal{F}[[x]]$.

□

Remark 2.2: Every $f \in \mathcal{F}^{\mathbb{Z}}$ can be expressed as a quotient

$$f = \frac{b(x)}{a(x)} = b(x)*a^{-1}(x) \qquad a(x),\ b(x) \in \mathcal{F}[[x]] \text{ and } a(x) \neq 0$$

where $a^{-1}(x)$ is the inverse in $\mathcal{F}^{\mathbb{Z}}$ of $a(x)$ and $\tau(f)=\tau(b)-\tau(a)$. The representation of f is not unique and $\mathcal{F}^{\mathbb{Z}}$

is the set of equivalence classes of these fractions. The representative of each equivalence class is a fraction with coprime numerator and denominator sequences.

□

Example 2.1: Let $a(x) = a_n x^n + a_{n+1} x^{n+1} + \cdots \in \mathcal{F}[[x]]$, and $b(x) = b_m x^m + b_{m+1} x^{m+1} + \cdots \in \mathcal{F}[[x]]$. Then, there is the inverse of $a(x)$ in $\mathcal{F}\langle x\rangle$ denoted by $g(x) = a^{-1}(x)$, i.e. $g*a = \{0;1,0,\ldots\}$. Therefore, $\tau(g) = -\tau(a) = -n$ and

$$a^{-1}(x) = g(x) = g_{-n} x^{-n} + g_{-n+1} x^{-n+1} + \cdots$$

where

$$\begin{aligned}
&a_n g_{-n} = 1 \therefore g_{-n} = 1/a_n \\
&a_n g_{-n+1} + a_{n+1} g_{-n} = 0 \therefore g_{-n+1} = -a_{n+1} g_{-n}/a_n \\
&a_n g_{-n+2} + a_{n+1} g_{-n+1} + a_{n+2} g_{-n} = 0 \therefore g_{-n+2} = \cdots
\end{aligned}$$

Hence,

$$b(x)*a^{-1}(x) = b_m g_{-n} x^{m-n} + (b_m g_{-n+1} + b_{m+1} g_{-n}) x^{m-n+1} + \cdots$$

is a Formal Laurent series with $\tau(b*a^{-1}) = m-n$.

□

Notation. From now on we will drop the symbol * for the convolutory multiplication. It will only be used when it is not clear from the context whether we refer to convolutory multiplication, or to pointwise one. In fact, under the formal series representation, convolutory multiplication becomes the usual multiplication.

Example 2.2: Let $f = \{f_n, f_{n+1}, \ldots\}$ and $g = \{g_k, g_{k+1}, \ldots\}$, f_i, $g_i \in \mathcal{F}$. The convolutory multiple of f, g is then

$$f*g = \{f_n g_k, f_n g_{k+1} + f_{n+1} g_k, \ldots\}$$

Consider now the power series representation of f and g, i.e.

$$f = f_n x^n + f_{n+1} x^{n+1} + \cdots, \text{ and } g = g_k x^k + g_{k+1} x^{k+1} + \cdots$$

The usual multiple of f and g, treating x as variable is then

$$fg = f_n g_k x^{n+k} + (f_n g_{k+1} + f_{n+1} g_k) x^{n+k+1} + \cdots$$

which is nothing more than the formal power series representation of $f*g$.

□

2.2.1 Polynomials and polynomial fractions over a field

An important subring of the domain of formal power series $\mathcal{F}[[x]]$ is the ring of **formal polynomials** defined next.

Definition 2.5: The set of all **finite** formal power series $\mathcal{F}[[x]]$ is called the set of **formal polynomials in one indeterminate** x **over the field** $\mathcal{F}$ and is denoted by $\mathcal{F}[x]$, i.e.:

$$\mathcal{F}[x] = \{ f_0 + f_1 x + \cdots + f_n x^n \mid f_k \in \mathcal{F} \}$$

□

According to the above definition, polynomials are sequences with nonnegative order and all but a finite number of elements zero. Therefore, they are regarded as algebraic objects with the indeterminate x over $\mathcal{F}$, rather than as functions of x. It is readily verified that

Theorem 2.5 [Har., 1]: The set of polynomials $\mathcal{F}[x]$ constitutes a subdomain of $\mathcal{F}[[x]]$. The **units** of $\mathcal{F}[x]$ are the constant polynomials $f(x) = f_0$ which are viewed as isomorphic with the nonzero elements of $\mathcal{F}$.

□

A concept similar to that of the order of a sequence is that of the degree of a polynomial defined below.

Definition 2.6 [Kuc., 1]: Given the nonzero polynomial $f = f_0 + \cdots + f_n x^n$ in $\mathcal{F}[x]$, we call f_n, $f_n \neq 0$ the **leading coefficient** of f and the integer n is referred to as the **degree** of the polynomial f and it is denoted by $\partial(f)$. By convention, the degree of the zero polynomial is defined as $-\infty$. Also, f is called **monic** if $f_n = 1$.

□

Thus ∂ is a function from $\mathcal{F}[x]$ to $\mathbb{N} \cup \{-\infty\}$ and we have the following properties.

Theorem 2.6 [Har., 1]: For every a, b in $\mathcal{F}[x]$

- **a.** $\partial(ab) = \partial(a) + \partial(b)$
- **b.** $\partial(a+b) \leq \max\{\partial(a), \partial(b)\}$
- **c.** $a|b \Rightarrow \partial(a) \leq \partial(b)$
- **d.** $\forall\ a, b \in \mathcal{F}[x]$ with $b \neq 0$, $\exists\ q, r \in \mathcal{F}[x]$ uniquely defined such that $a = bq + r$ and $\partial(r) < \partial(b)$.

□

Remark 2.3: The restriction of ∂ on $\mathcal{F}[x]-\{0\}$ is a function

$$\delta : \mathcal{F}[x]-\{0\} \longmapsto \mathbb{N}$$

which is a **Euclidean valuation** due to the properties (c) and (d) of theorem (2.6). Therefore, the integral domain $\mathcal{F}[x]$ is a *Euclidean Domain* and thus a *Principal Ideal Domain* as well as a *Unique Factorization Domain*. [Har., 1].

□

Definition 2.7 [God., 1]: Let $f = f_0 + f_1x +\cdots+ f_nx^n \in \mathcal{F}[x]$ and $\mathcal{G}$ be an overfield of $\mathcal{F}$. Then, for every $u \in \mathcal{G}$ we write $f(u)$ for the element $f_0 + f_1u + \cdots + f_nu^n \in \mathcal{G}$ and we call $f(u)$ the **value** of f at u. If $f(u) = 0$, then u is said to be a **zero** of f, or a **root** of f.

□

It can be easily shown that for a fixed $u \in \mathcal{G}$ the map $f \longmapsto f(u)$ is a ring homomorphism from $\mathcal{F}[x]$ into $\mathcal{G}$ [Har., 1]. In the sequel, we give some important properties of the ring of formal polynomials $\mathcal{F}[x]$. These properties help us to establish the relationship between formal polynomials and polynomial functions, which plays a key role in the present work; for this reason we give also their proofs as well. For an extensive treatment of the subject one could look in texts of abstract algebra e.g. [God., 1], [Har., 1], [Van., 1], [Zar., 1].

Lemma 2.1 [Har., 1]: Let $f \in \mathcal{F}[x]$ and $a \in \mathcal{F}$. Then, a is a root of f, if and only if $x - a$ divides f.

Proof. From theorem (2.6), $\exists\, q,\ r \in \mathcal{F}[x]\ :\ f = (x - a)q + r$ and $\partial(r) < \partial(x - a) = 1$. Therefore, r is a constant and applying the isomorphism $f \longmapsto f(a)$, i.e. substituting $x = a$ in the above equation, we have $f(a) = r = 0$ because a is a root of f. So, $f = (x - a)q$, i.e. $x - a$ divides f.

□

Theorem 2.7 [Har., 1]: A polynomial $f \in \mathcal{F}[x]$ of degree $n \geq 0$ has at most n distinct roots in $\mathcal{F}$.

Proof. We prove theorem (2.7) by induction on n. If $\partial(f)$ is zero, then f is a nonzero constant and therefore, has no zero, so that the theorem is true for $n = 0$. Suppose that it is also true for $n - 1$ and a is a root of f. Then according to lemma (2.1), f may be written as $f = (x - a)q$, where the degree of q is $n - 1$ (theor. 2.6). If b is another root of f in $\mathcal{F}$ we have $0 = (b - a)q(b)$ and so the zeros of f in $\mathcal{F}$, other than a are the zeros of q. Since q is of degree $n - 1$, it has at most $n - 1$ zeros in $\mathcal{F}$ by the inductive hypothesis and so f has at most n roots in $\mathcal{F}$.

□

We consider now the relationship between the **formal polynomials** already defined and the **polynomial functions.** Let $\mathcal{F}$ be a field and $\mathcal{F}^{\mathcal{F}}$ the set of all functions from $\mathcal{F}$ to $\mathcal{F}$. Then $\mathcal{F}^{\mathcal{F}}$ becomes a ring under the pointwise operations given by

$$(f + g)(u) := f(u) + g(u)$$
$$(-f)(u) := -f(u)$$
$$(fg)(u) := f(u)g(u)$$

for $f, g \in \mathcal{F}^{\mathcal{F}}$ and $u \in \mathcal{F}$.

Definition 2.8 : Let $f = f_0 + f_1 x + \cdots + f_n x^n \in \mathcal{F}[x]$. We can associate with f a function $\phi(f) : \mathcal{F} \longmapsto \mathcal{F}$ as follows

$$\phi(f)(u) = f_0 + f_1 u + \cdots + f_n u^n \qquad (u \in \mathcal{F}) \tag{2.4}$$

□

Thus ϕ is a map from $\mathcal{F}[x]$ to $\mathcal{F}^{\mathcal{F}}$; in fact, ϕ is a ring homomorphism as it can be shown using the pointwise operations in $\mathcal{F}^{\mathcal{F}}$ [Har., 1]. Then, $\mathrm{im}\phi$ is called the **ring of polynomial functions on** $\mathcal{F}$ and $\ker\phi$ consists of all elements of $\mathcal{F}[x]$ which vanish identically on $\mathcal{F}$. Therefore, two polynomials f, $g \in \mathcal{F}[x]$ determine the **same** function on $\mathcal{F}$ if and only if $f - g \in \ker\phi$ and this is precisely the reason why we cannot in general identify a **formal polynomial** in $\mathcal{F}$ with a **polynomial function** on $\mathcal{F}$. The criterion for this to be true, i.e. the map ϕ to be injective, is stated below.

Theorem 2.8 [Har., 1]: The map $\phi : \mathcal{F}[x] \longmapsto \mathcal{F}^{\mathcal{F}}$ is injective, if and only if $\mathcal{F}$ is an infinite field.

Proof. Let $\mathcal{F}$ be infinite and $f \in \ker\phi$. Then, $f(u) = 0$ for all $u \in \mathcal{F}$, i.e. every element of $\mathcal{F}$ is a root of f. Since any nonzero element of $\mathcal{F}$ has only finitely many roots then, f has to be zero. Therefore $\ker\phi = \{0\}$ and ϕ is injective. Assume now that $\mathcal{F}$ is finite and let $u_1,\dots,u_n$ be its elements. So, $(x-u_1)\cdots(x-u_n)$ is a nonzero element of $\mathcal{F}$ and has every element of $\mathcal{F}$ as its roots. Hence it belongs to $\ker\phi$ and so, $\ker\phi \neq \{0\}$ if $\mathcal{F}$ is finite.

□

Remark 2.4: According to theorem (2.8) the notions of **formal polynomials** and **polynomial functions** coincide when they are defined over an infinite field. Therefore,

$$f = f_0 + f_1x + \cdots + f_nx^n \qquad f_k \in \mathcal{F}$$

can be treated either as a finite sequence or as a polynomial function and x is an indeterminate i.e. a sequence by itself, or a variable in $\mathcal{F}$ correspondingly. We will use ***italic x*** to denote that x is an indeterminate and normal x if x is treated as variable. If no distinction is to be made between the two notions then ***italic x*** is used indiscriminately.

□

Example 2.3: Consider the ring $\mathcal{B}[x]$ of formal polynomials in one indeterminate x over the *Boolean field* $\mathcal{B} = \{0,1\}$. The formal polynomial

$$f = x^2 + x$$

is quite distinct from the formal zero polynomial in $\mathcal{B}[x]$, but $f(0) = f(1) = 0$ i.e. $f = 0\ \forall\ x \in \mathcal{B}$. So, $f \in \ker\phi$, where $\phi : \mathcal{B}[x] \longmapsto \mathcal{B}^{\mathcal{B}}$ and as a polynomial function cannot be distinguished from the zero polynomial.

□

Remark 2.5: Formal polynomials in $\mathcal{B}[x]$ are used in automata theory and in modeling of digital systems such as flip-flops, shift registers etc., where both the independent variable of time and the amplitude of the signals processed are quantized. It is an approximation, justified by the minimal effect of the amplitude quantization, that polynomials and sequences in general, are considered over the field $\mathbb{R}$ of real numbers when treating discrete control systems.

□

We close this section by a brief discussion on fractions of polynomials. As both **formal polynomials** and **polynomial functions** are integral domains we can construct their fields of fractions.

Definition 2.9: We call **rational fractions** or **rational sequences** in one indeterminate x over $\mathcal{F}$, the field of fractions of the **formal polynomials** and we denote it by $\mathcal{F}(x)$. Accordingly, the field of fractions of the **polynomial functions** is called the field of **rational functions** in $\mathcal{F}$ and is denoted by $\mathcal{F}(\mathrm{x})$.

□

Remark 2.6: Since $\mathcal{F}[x]$ is a subdomain of $\mathcal{F}[[x]]$, the field of **rational fractions** is isomorphic to a subfield of the field of formal Laurent series $\mathcal{F}\langle x\rangle$ (theorem 2.4). Therefore, the field of rational fractions is a set of sequences that can be written in the form (2.3) of formal Laurent series. We call this subfield of formal Laurent series the field of rational formal Laurent series and for this reason **rational fractions** are called **rational sequences** as well.

□

In the case where $\mathcal{F}$ is infinite, $\mathcal{F}(x)$ and $\mathcal{F}(\mathrm{x})$ are isomorphic as it can be shown with the help of the following lemma.

Lemma 2.2 [Zar., 1]**:** Let $\mathcal{R}$ and $\mathcal{R}'$ be two isomorphic integral domains, let T_0 be an isomorphism of $\mathcal{R}$ onto $\mathcal{R}'$, and let $\mathcal{S}$ and $\mathcal{S}'$ be respective fields of fractions. Then T_0 can be extended in a unique manner to an isomorphism T of $\mathcal{S}$ onto $\mathcal{S}'$.

□

Theorem 2.9: The field of rational fractions $\mathcal{F}(x)$ in one indeterminate x over $\mathcal{F}$ is isomorphic to the field of rational functions $\mathcal{F}(\mathrm{x})$ in $\mathcal{F}$, if and only if $\mathcal{F}$ is an infinite field.

Proof. According to theorem (2.8) the integral domain of formal polynomials $\mathcal{F}[x]$ is isomorphic to the integral domain of polynomial functions $\mathcal{F}[\mathrm{x}]$, if and only if $\mathcal{F}$ is an infinite field. Since $\mathcal{F}(x)$ and $\mathcal{F}(\mathrm{x})$ are the respective fields of fractions of $\mathcal{F}[x]$ and $\mathcal{F}[\mathrm{x}]$ they are isomorphic if and only if $\mathcal{F}$ is an infinite field (lemma 2.2).

□

Many times it is necessary to define the value of a rational fraction in one indeterminate x over $\mathcal{F}$ at points whose coordinates lie not in $\mathcal{F}$ but in an arbitrary extension field of $\mathcal{F}$. At the most elementary level it is clearly indispensable to be able to attribute a value to a rational fraction with **real** coefficients at a point with **complex** coordinates. We have then the following definition.

Definition 2.10 [God., 1]**:** Let $\mathcal{G}$ be a field, $\mathcal{F}$ a subfield of $\mathcal{G}$ and f a rational fraction in one indeterminate x over $\mathcal{F}$. We say that f **is defined at** $u \in \mathcal{G}$ or that u **is substitutable in** f, if there exist polynomials a, $b \in \mathcal{F}[x]$ such that

$$f = b/a \quad \text{and} \quad a(u) \neq 0$$

If $u \in \mathcal{G}$ is substitutable in f we define the **value of f in u** as the element $b(u)/a(u)$ of $\mathcal{G}$ and we write

$$f(u) = b(u)/a(u) \quad \text{and} \quad a(u) \neq 0$$

□

Example 2.4 [God., 1]: $u = 0$ is substitutable in the rational fraction

$$x^2/(x^2 + x)$$

because this fraction can also be written as

$$x/(x + 1)$$

in which form it is clear that the denominator does not vanish at $u = 0$.

□

Definition 2.11 [God., 1]: Let $f \in \mathcal{F}(x)$ and $\mathcal{G}$ be an overfield of $\mathcal{F}$. Then $u \in \mathcal{G}$ is a **pole** of f if f is not defined at u and f^{-1} is defined at u.

□

Remark 2.7: It is clear from definitions (2.10) and (2.11) that a rational fraction $f \in \mathcal{F}(x)$ is defined at $u \in \mathcal{G}$ if and only if there are uniquely defined coprime polynomials a, b and a is monic, such that $f(u) = b(u)/a(u) \in \mathcal{G}$ and $a(u) \neq 0$. In that case, u is a pole of f if and only if u is a root of the denominator $a \in \mathcal{F}[x]$.

□

To summarize, any rational Laurent series $f(x)$ over a field $\mathcal{F}$ of the form

$$f(x) = f_n x^n + f_{n+1} x^{n+1} + f_{n+2} x^{n+2} + \cdots, \quad n \in \mathbb{Z} \ \text{fixed}$$

can be regarded in two different ways. Either is formal, i.e. $f(x)$ is a **rational sequence**, x is an indeterminate over $\mathcal{F}$ and no question arises of convergence whatsoever, or it is a **rational function** in $\mathcal{F}$ which associates with at least one element $x \in \mathcal{F}$ another element $f(x) \in \mathcal{F}$. In the case where $\mathcal{F}$ is infinite no distinction need be made between the two points of view and $f(x)$ can be treated either as a **rational**

sequence, or as a **rational function**. In the most elementary case $\mathcal{F} = \mathbb{R}$, the set of real numbers, and as overfield of $\mathbb{R}$ is usually regarded $\mathbb{C}$, the set of complex numbers. Then

Remark 2.8: $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, i.e. the roots of polynomials with one indeterminate x over $\mathbb{R}$ lie in $\mathbb{C}$ and the isomorphism between formal polynomials over $\mathbb{R}$ and polynomial functions in $\mathbb{C}$ with coefficients in $\mathbb{R}$ is still valid [God., 1], [Zar., 1]. The irreducible polynomials in $\mathbb{R}[x]$ are of the form $\beta_0+\beta_1 x$, or $\gamma_0+\gamma_1 x+\gamma_2 x^2$ with $\gamma_1^2-\gamma_0\gamma_2<0$. We can also compactify the complex plane $\mathbb{C}$ by including the point at infinity and we can then talk about poles of rational fractions at infinity in the usual way.

□

Example 2.5: Consider the ring $\mathbb{R}(x)$ of rational sequences in one indeterminate x over the field of real numbers $\mathbb{R}$. Then, any $f(x) \in \mathbb{R}(x)$ can be written in the form of formal Laurent series as

$$f(x) = \frac{b(x)}{a(x)} = f_n x^n + f_{n+1} x^{n+1} + \cdots, \quad n \in \mathbb{Z} \tag{2.5}$$

$a(x)$, $b(x) \in \mathbb{R}[x]$ are coprime. Since $\mathbb{R}$ is an infinite field, $f(x)$ can be treated as a rational function as well with $x \in \mathbb{C}$ as variable (theor. 2.9, rem. 2.8). Therefore there is at least an $x = x_1 \in \mathbb{C}$ such that the series (2.5) converges. Due to this fact, it can be proved [Chu., 1], [Opp., 1] that series (2.5) converges uniformly $\forall\ x \in \mathbb{C} : |x| < |x_1|$, and $x \neq 0$, if $\tau(f) = n < 0$.

The greatest circle about the origin (the origin excluded if $n < 0$) such that the series converges, is called the **circle**, or **region of convergence** of the series (2.5). The series cannot converge at any point x_2 outside that circle, for in that case it would converge everywhere inside the circle centered at the origin and passing through x_2. The first circle could not then be the circle of convergence.

It can also be proved [Chu., 1] that series (2.5) represents a continuous function of x at each point interior to its circle of convergence, and since it can be represented by a rational function due to the isomorphism between rational fractions and rational functions, the region of convergence is bounded by the poles of $f(x) = b(x)/a(x)$. Therefore the roots of $a(x)$ cannot lie inside the region of convergence, and for this reason the region of convergence is called **forbiden region** as well.

Consider for example the formal series

$$f = \frac{b}{a} = \frac{\{0;1\}}{\{0;1,-1\}} = \{0;1,1,1,\ldots\} \quad \text{or}$$

$$f(x) = \frac{b(x)}{a(x)} = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Then the series $\sum_{n=0}^{\infty} x^n$ treated as a function, converges $\forall\ x \in \mathbb{C} : |x| < 1$, and therefore it can be represented by the rational function $1/(1-x)$ within the region of convergence. In other words, the series $\sum_{n=0}^{\infty} x^n$ represents the power series expansion of the function

$$f(x) = \frac{1}{1-x} \qquad |x| < 1$$

The function

$$f_1(x) = \frac{1}{1-x} \qquad x \neq 1$$

is defined and analytic everywhere except at the point $x = 1$ and is the analytic continuation of f into the domain of all points $x \in \mathbb{C}-\{1\}$ [Chu., 1]. It represents the series $\sum_{n=0}^{\infty} x^n$ only within the region of convergence $\mathcal{C}$. The region of convergence $\mathcal{C}$ is the open unit disc $\mathbb{D}[0,1)$ and is bounded by the pole of $f_1(x)$ at $x = 1$.

$\square$

2.2.2 Further classification and properties of the formal Laurent series

For the study of linear dynamical systems, discrete and continuous the field which is considered is the field of real numbers $\mathbb{R}$. In the case of discrete-time systems the indeterminate will be denoted by d and the field of formal

Laurent series over $\mathbb{R}$ by $\mathbb{R}\langle d\rangle$. An element f of $\mathbb{R}\langle d\rangle$ will be denoted by

$$f = f_n d^n + f_{n+1} d^{n+1} + f_{n+2} d^{n+2} + \cdot\cdot\cdot, \quad n \in \mathbb{Z} \quad \text{fixed} \tag{2.6}$$

Since $\mathbb{R}$ is an infinite field, series (2.6) can be regarded either as sequences over $\mathbb{R}$ or functions in $\mathbb{R}$, at least for the rational case.

If the series (2.6) is formal then f is a rational sequence and can be considered as the impulse response of a linear, lumped, discrete-time system though not necessarily causal as it will be shown in detail in chapter (3). In this case, $d = \{0;0,1,0,...\}$ is a sequence by itself, or an indeterminate, it serves as a position marker and the powers of d represent the discrete time.

Otherwise, the series (2.6) can be considered as a rational function in $\mathbb{R}$ with d being a variable. In fact, since $\mathbb{C}$ the field of complex numbers, is the algebraic closure of $\mathbb{R}$, series (2.6) can be regarded as a function from $\mathbb{C}$ to $\mathbb{C}$ with real coefficients (remark 2.8). In this case, the Laurent series can represent the transfer function of a linear, lumped, discrete-time system, and if we replace $d \in \mathbb{C}$ with z^{-1} the series (2.6) is no more than the z-Transform [Jur., 1] of the impulse response $f = \{f_n, f_{n+1}, ..., f_k, ...\}$.

Some basic definitions and results for the sequences in $\mathbb{R}\langle d\rangle$ which are related to system properties are given next.

Definition 2.12 [Kuc., 1]: A sequence $f \in \mathbb{R}\langle d\rangle$ is called **recurrent**, if there exist nonnegative integers r, s and reals $\lambda_1, ..., \lambda_r$ such that

$$f_{j+r} + \lambda_1 f_{j+r-1} + \cdots + \lambda_r f_j = 0, \quad j = n+s, n+s+1, ... \tag{2.7}$$

where n is the order of f. The set of recurrent sequences will be denoted by $\mathbb{R}\{d\}$.

□

The meaning of the recurrence condition (2.7) is that the sequence f can be generated by a linear system. The relation (2.7) is not unique in the sense that recurrence relations of different length can represent the same sequence. For this reason we will always consider that (2.7) is of a **minimum** recurrence length. The following theorem clarifies the foregoing statements.

Theorem 2.10 [Kuc. 1]: The set $\mathbb{R}\{d\}$ of recurrent sequences is isomorphic to the field $\mathbb{R}(d)$ of rational sequences.

Proof. According to definition (2.9) each rational sequence

$$f = f_n d^n + f_{n+1} d^{n+1} + f_{n+2} d^{n+2} + \cdots \qquad (2.8)$$

can be uniquely expressed as a polynomial fraction $b(d)/a(d)$ with a, b coprime in $\mathbb{R}(d)$ and $n=\tau(f)=\tau(b)-\tau(a)$. Therefore,

$$\frac{d^n(\mu_0+\mu_1 d+\cdots+\mu_m d^m)}{1+\lambda_1 d+\cdots+\lambda_r d^r} = f_n d^n + f_{n+1} d^{n+1} + f_{n+2} d^{n+2} + \cdots$$

By multiplying f by a and equating to the sequence b we have the recurrence relation (2.7) for the sequence f. Hence, f is a recurrent sequence. On the other hand, if f is recurrent there exist integers r, s and reals $\lambda_1,\ldots,\lambda_r$ such that

$$f_{j+r} + \lambda_1 f_{j+r-1} + \cdots + \lambda_r f_j = 0, \qquad j = n+s, n+s+1, \ldots$$

Setting $a(d) = d^k(1+\lambda_1 d+\cdots+\lambda_r d^r)$ with $k=|\min\{\tau(f),0\}|$, we deduce that $b(d)=f(d)a(d)$ is a polynomial. Moreover, a,b are coprime, because if they were not, a recurrence relation of smaller length would have existed.

$\square$

Remark 2.9: The significance of recurrent sequences is due to the fact that they can be expressed as polynomial fractions. Because $\mathbb{R}\{d\}$ is isomorphic to $\mathbb{R}(d)$, $\mathbb{R}\{d\}$ is also a field and we can refer to both fields of recurrent and rational sequences with the same notation $\mathbb{R}(d)$.

$\square$

Example 2.6: Consider the rational fraction

$$f(d) = \frac{b(d)}{a(d)} = \frac{1+d}{d^2+3d^3+2d^4}$$

$f(d)$ is a rational sequence which can be obtained by dividing $b(d)$ by $a(d)$ in ascending powers of d; it consists in finding for each integer $r \geq 0$, a polynomial f_p of degree $\leq r$ such that $b(d)-f_p(d)a(d)$ is a multiple of d^{r+1}. The polynomial f_p is obtained from the formal Laurent series $f(d)$ by deleting the terms of degree $> r$ [God., 1]. Here we apply a method in accordance to the results of section (2.2.1), i.e. $f(d)$ is a sequence of order $\tau(f) = \tau(b) - \tau(a) = 0 - 2 = -2$, and can be written as

$$f(d) = \frac{b(d)}{a(d)} = \frac{1+d}{d^2+3d^3+2d^4} = f_{-2}d^{-2}+f_{-1}d^{-1}+f_0+f_1d+f_2d^2+\cdots$$

Therefore $b(d) = a(d)f(d)$, or

$$\begin{aligned} 1+d = {} & f_{-2}+ f_{-1}d + f_0 d^2 + f_1 d^3+ f_2 d^4+\cdots \\ & +3f_{-2}d+3f_{-1}d^2+ 3f_0 d^3+3f_1 d^4+\cdots \\ & +2f_{-2}d^2+2f_{-1}d^3+2f_0 d^4+\cdots \end{aligned}$$

Equating corresponding terms of both sequences, we have

$$\left.\begin{aligned} & f_{-2} = 1 \\ & f_{-1} + 3f_{-2} = 1 \\ & f_{j+2} + 3f_{j+1} + 2f_j = 0 \qquad j = -2,-1,0,1,2,\ldots \end{aligned}\right\} \quad (2.9)$$

which shows that $f(d)$ is a recurrent sequence with $\lambda_1 = 3$ and $\lambda_2 = 2$.

Suppose now that $f(d)$ is a recurrent sequence satisfying the recurrence relations (2.9). Then $f(d)$ can be written as rational fraction. Define $a(d) \in \mathbb{R}[d]$ as $a(d) = d^k(1+\lambda_1 d+\lambda_2 d^2)$ where $\lambda_1 = 3$, $\lambda_2 = 2$ and $k = |\min\{\tau(f),0\}| = |\min\{-2,0\}| = 2$. Therefore, $a(d) = d^2(1+3d+2d^2)$ and $b(d) = a(d)f(d) = 1+d$ due to the recurrence relations (2.9). So

$$f(d) = \frac{b(d)}{a(d)} = \frac{1+d}{d^2+3d^3+2d^4}$$

If $a(d)$, $b(d)$ were not coprime in $\mathbb{R}[d]$, e.g. $a'(d)=(2+d)a(d)=d^2(2+7d+7d^2+2d^3)$ and $b'(d)=(2+d)b(d)=2+3d+d^2$, then the fraction $b'(d)/a'(d)$ gives rise to the same recurrent sequence $f(d)$ as above, but the recurrence relation is

$$2f_{j+3} + 7f_{j+2} + 7f_{j+1} + 2f_j = 0 \qquad j = -2,-1,0,1,...$$

which is longer than that of (2.9).

□

Definition 2.13 [Kuc., 1]: A recurrent sequence f is said to be **causal** and is denoted by $\mathbb{R}^0(d)$, if it has a nonnegative order. In addition, if f has a positive order it is called **strictly causal**.

□

Clearly, $\mathbb{R}^0(d)$ is a subset of formal power series $\mathbb{R}^{\mathbb{N}}$, or $\mathbb{R}[[d]]$. A **causal** sequence f may be represented by

$$f = f_0 + f_1 d + f_2 d^2 + \cdots \qquad (2.10)$$

The following properties of the set $\mathbb{R}^0(d)$ may be readily established and their proof is omitted.

Proposition 2.1 [Kuc., 1]: The set of causal sequences $\mathbb{R}^0(d)$ is a subdomain of $\mathbb{R}[[d]]$ with the following structure:

a. The **units** of $\mathbb{R}^0(d)$ are sequences of order 0, i.e.

$$f = f_0 + f_1 d + f_2 d^2 + \cdots, \quad f_0 \neq 0 \qquad (2.11)$$

b. The **primes** of $\mathbb{R}^0(d)$ are sequences of order 1, i.e.

$$f = f_1 d + f_2 d^2 + \cdots, \quad f_1 \neq 0 \qquad (2.12)$$

□

Definition 2.14 [Kuc., 1]: A causal sequence f is called **stable**, if it converges to zero, i.e.

$$\forall\ k_0 \in \mathbb{N}\ \exists\ \varepsilon(k_0) \in \mathbb{R} : |f_k| < \varepsilon\ \ \forall\ k \geq k_0$$

The set of stable sequences will be denoted by $\mathbb{R}^+(d)$.

□

Proposition 2.2 [Kuc., 1]: The set $\mathbb{R}^{+}(d)$ of stable sequences is an integral domain; moreover, $\mathbb{R}^{+}(d)$ is a subdomain of $\mathbb{R}^{0}(d)$.

□

The following proposition gives an alternative and very important, from the system theoretic point of view, characterization of the stable sequences.

Proposition 2.3 [Kol., 1]: Let $f=\{f_0, f_1, \ldots, f_k, \ldots\} \in \mathbb{R}^{0}(d)$ Then, $f \in \mathbb{R}^{+}(d)$ if and only if f is absolutely summable, i.e.

$$\sum_{k=0}^{\infty} |f_k| < \infty \tag{2.13}$$

□

The algebraic structure of $\mathbb{R}^{+}(d)$ is more complicated than that of $\mathbb{R}^{0}(d)$ and is left to be examined at the end of this section where it is shown that both domains can be expressed as polynomial fractions. It can be easily observed that the set $\mathbb{R}[d]$ of polynomials in one indeterminate d is a subdomain of the set of stable sequences $\mathbb{R}^{+}(d)$ and the following properties hold true.

Corollary 2.1: The rings of sequences $\mathbb{R}[d]$, $\mathbb{R}^{+}(d)$, $\mathbb{R}^{0}(d)$ and $\mathbb{R}(d)$ are at least integral domains and the following inclusion properties are valid

$$\mathbb{R}[d] \subset \mathbb{R}^{+}(d) \subset \mathbb{R}^{0}(d) \subset \mathbb{R}(d)$$

where by "$\subset$" we denote subring property.

□

Some important classes of polynomials are defined next which, with lemma (2.3) to follow, help to give a fractional characterization of all recurrent sequences.

Definition 2.15 [Kuc., 1]: Let $f \in \mathbb{R}[d]$. Then,

- **a.** f is called **causal** if it is a unit of $\mathbb{R}^{0}(d)$.
- **b.** f is called **stable** if it is a unit of $\mathbb{R}^{+}(d)$.

□

Recalling the properties of the ring $\mathbb{R}^0(d)$ we may characterize the family of causal polynomials.

Proposition 2.4: A polynomial $f = f_0 + f_1 d + \cdots + f_n d^n \in \mathbb{R}[d]$ is causal if and only if $f_0 \neq 0$, or $f(0) \neq 0$.

□

The characterization of stable polynomials is more complicated and will be examined later on. First we state the following important lemma.

Lemma 2.3 [Zar., 1]: Any ring between a Euclidean domain $\mathcal{R}$ and the quotient field of $\mathcal{R}$ is a quotient ring of $\mathcal{R}$ with respect to some suitable multiplicative system in $\mathcal{R}$.

□

Since $\mathbb{R}[d]$ is a Euclidean domain and the sets of causal and stable polynomials are such suitable multiplicative systems in $\mathbb{R}[d]$, we have the next theorem.

Theorem 2.11: Let $f = b/a \in \mathbb{R}(d)$ be a coprime polynomial fraction. Then

- **a.** The ring of causal sequences is a quotient ring of $\mathbb{R}[d]$ with a being a causal polynomial.
- **b.** The ring of stable sequences is a quotient ring of $\mathbb{R}[d]$ with a being a stable polynomial.

□

Theorem (2.11) allows for the characterization of the units of the various rational sequences.

Remark 2.10: The units of the causal and stable rational sequences are characterized as follows:

- **a.** The units of $\mathbb{R}^0(d)$ are rational fractions with both numerator and denominator causal polynomials.
- **b.** The units of $\mathbb{R}^+(d)$ are rational fractions with both numerator and denominator stable polynomials.

For this reason we call the units of $\mathbb{R}^0(d)$ **bicausal** sequences and the units of $\mathbb{R}^+(d)$ **bistable** sequences.

□

In the classification of the recurrent sequences so far we have used polynomials as formal polynomials; hence, causal and stable rational sequences and causal and stable polynomials have been treated as algebraic objects with $\boldsymbol{d}$ as indeterminate over $\mathbb{R}$, rather than as functions of $\boldsymbol{d}$. We define next the corresponding rational functions and we give the characterization of the stable formal polynomials.

Definition 2.16: Let $f = b/a \in \mathbb{R}(\mathrm{d})$ be the set of rational functions and a, b be polynomial functions over $\mathbb{R}$ with variable $\mathrm{d} \in \mathbb{C}$. Then,

- **a.** the set of rational functions with denominator a a polynomial function corresponding to a causal formal polynomial is called the set of causal rational functions and is denoted by $\mathbb{R}^0(\mathrm{d})$.
- **b.** the set of rational functions with denominator a a polynomial function corresponding to a stable formal polynomial is called the set of stable rational functions and is denoted by $\mathbb{R}^+(\mathrm{d})$.

□

Since $\mathbb{R}$ is an infinite field and the sets of formal polynomials $\mathbb{R}[\boldsymbol{d}]$ and rational sequences $\mathbb{R}(\boldsymbol{d})$ are isomorphic to the sets of polynomial functions $\mathbb{R}[\mathrm{d}]$ and rational functions $\mathbb{R}(\mathrm{d})$ respectively the following corollary may be readily established.

Corollary 2.2: The sets of sequences $\mathbb{R}[\boldsymbol{d}]$, $\mathbb{R}^+(\boldsymbol{d})$, $\mathbb{R}^0(\boldsymbol{d})$ and $\mathbb{R}(\boldsymbol{d})$ are isomorphic to the sets of rational functions $\mathbb{R}[\mathrm{d}]$, $\mathbb{R}^+(\mathrm{d})$, $\mathbb{R}^0(\mathrm{d})$ and $\mathbb{R}(\mathrm{d})$ respectively, they all are principal ideal domains and the following inclusion properties hold true

$$\mathbb{R}[\boldsymbol{d}] \subset \mathbb{R}^+(\boldsymbol{d}) \subset \mathbb{R}^0(\boldsymbol{d}) \subset \mathbb{R}(\boldsymbol{d})$$
$$\mathbb{R}[\mathrm{d}] \subset \mathbb{R}^+(\mathrm{d}) \subset \mathbb{R}^0(\mathrm{d}) \subset \mathbb{R}(\mathrm{d})$$

where by "$\subset$" we denote subring property.

□

We examine now the characterization of the stable polynomials.

Proposition 2.5: A polynomial sequence $a \in \mathbb{R}[d]$ is stable if and only if its roots lie outside the closed unit disc $\mathbb{D}$.

Proof. Since every stable polynomial sequence is a unit of the ring of stable rational sequences then $a=a_0+a_1d+\cdots+a_nd^n$ is such that

$$\frac{1}{a_0+a_1d+\cdots+a_nd^n} = f_0+f_1d+\cdots+f_kd^k+\cdots \in \mathbb{R}^+(d) \qquad (2.14)$$

But $f \in \mathbb{R}^+(d)$ *iff* is absolutely summable (proposition 2.3). By invoking the isomorphism between rational sequences and rational functions and performing the bilinear transformation $z^{-1} = d$, eqn. (2.14) becomes

$$\frac{z^n}{a_0z^n+a_1z^{n-1}+\cdots+a_n} = f_0+f_1z^{-1}+\cdots+f_kz^{-k}+\cdots, \quad z \in \mathbb{C} \qquad (2.15)$$

The left-hand side of eqn. (2.15) is the z-Transform in closed form of the sequence $\{f_k\}$, which is absolutely summable *iff* the roots of the denominator polynomial $\tilde{a}(z)$ lie inside the open unit disc [Jur., 1]. This is, due to the bilinear transformation, equivalent to the roots of the formal polynomial $a(d)$ lying outside the closed unit disc $\mathbb{D}$.

□

A test to check whether a nonzero polynomial $a \in \mathbb{R}[d]$ is stable is given by Kucera [Kuc., 1]. This test is similar to the table form based on the early work of Cohn [Coh., 1] and later on the work of Marden [Mar., 1] and Jury [Jur., 2].

We close this section by giving an alternative characterization of the rings of causal and stable rational sequences which constitutes the so called frequency domain criteria for causality and stability.

Remark 2.11: According to theorem (2.11) and propositions (2.4) and (2.5), the rings of causal and stable rational sequences can be described as follows.

Causal rational sequences $f(d)$ can be represented as coprime polynomial fractions $b(d)/a(d)$, where $a(d)$ has a nonzero constant term a_0 or $a(0) \neq 0$, i.e. $a(d)$ has no roots or $f(d)$ has no poles at zero.

Stable rational sequences $f(d)$ can be represented as coprime polynomial fractions $b(d)/a(d)$, where the roots of $a(d)$ or the poles of $f(d)$ lie outside the closed unit disc $\mathbb{D}[0,1]$.

□

Example 2.7: Let $f = \{f_0, f_1, f_2, \ldots\}$ be a causal recurrent sequence such that $f_0=1$, $f_1=0.1$, and $f_{j+2}+0.9f_{j+1}+0.2f_j=0$ for $j = 0,1,2,\ldots$ According to theorem (2.10) f can be written as a fraction of coprime polynomials $a(d)$, $b(d)$ where $a(d) = 1+0.9d+0.2d^2$ and $b(d) = 1+d$, i.e.

$$f(d) = \frac{b(d)}{a(d)} = \frac{1+d}{1+0.9d+0.2d^2} = f_0+f_1d+f_2d^2+\cdots$$

Then, $f(d)$ has an inverse in $\mathbb{R}^0(d)$, i.e. $f(d)$ is **bicausal**, because $b(0)=1\neq 0$. Indeed

$$f^{-1}(d) = g(d) = \frac{a(d)}{b(d)} = \frac{1+0.9d+0.2d^2}{1+d} \in \mathbb{R}^0(d)$$

Also $f(d)$ is **stable** since the roots of $a(d)$, $p_1=-2.5$ and $p_2=-2$, lie outside the unit disc $\mathbb{D}[0,1]$. In fact from the recurrence relation

$$f = \{0;1,0.1,-0.29,0.171,-0.096,0.052,-0.028,\ldots\}$$

i.e. f_k approaches zero as k tends to infinity. But f is **not bistable** since the root of $b(d)$ $z=1$, does not lie outside the closed unit disc $\mathbb{D}[0,1]$ and therefore the inverse $g(d)$ of $f(d)$ is not a stable sequence.

□

2.3 Matrices, Polynomial Matrices and Rational Vector Spaces

The objective of this section is to gather some of the fundamentals of the standard theory of polynomial and rational matrices [Gan., 1], [Kai., 1], [Ros., 1] which are essential for the further developments. The intention is not to give a proper review, but rather to summarize the basic results and introduce some notation.

Let $\mathcal{R}$ be a ring and $\mathcal{R}^{l\times m}$ denote the $\ell\times m$ matrices with elements from $\mathcal{R}$. Then, if we define addition and multiplication in the familiar way, the square matrices $\mathcal{R}^{m\times m}$ form a noncommutative ring. The units of this ring, i.e. the matrices in $\mathcal{R}^{m\times m}$ whose inverse belongs to $\mathcal{R}^{m\times m}$, are called **unimodular** matrices. Such matrices are products of elementary transformations over $\mathcal{R}$ -at least when $\mathcal{R}$ is a PID- and are characterized as follows.

Lemma 2.4: A matrix $U \in \mathcal{R}^{m\times m}$ is unimodular, if and only if $\det U$ is a unit in $\mathcal{R}$. We denote such a matrix as $U \in U(m,\mathcal{R})$ and we call it $\mathcal{R}$-**unimodular**.

□

Example 2.8 The $\mathbb{R}[d]$-unimodular matrices, i.e. the unimodular matrices in $\mathbb{R}^{m\times m}[d]$, are those polynomial matrices whose determinants are nonzero real constants.

□

Definition 2.17: Let $\mathcal{R}$ be a PID and A, $A' \in \mathcal{R}^{l\times m}$ such that

$$A = LA'R \tag{2.16}$$

where $L \in U(\ell,\mathcal{R})$ and $R \in U(m,\mathcal{R})$. Then A, A' are said to be $\mathcal{R}$-**equivalent** and this is denoted by $A\ E_{\mathcal{R}}\ A'$. If $\mathcal{R} = \mathbb{R}$, then they are called **strict equivalent** and if $L = I_l$, or $R = I_m$ they are called **right-**, **left-equivalent** and this is denoted by $A\ E^{r}_{\mathcal{R}}\ A'$, $A\ E^{l}_{\mathcal{R}}\ A'$ respectively. $E_{\mathcal{R}}$, $E^{r}_{\mathcal{R}}$, $E^{l}_{\mathcal{R}}$ are equivalence relations and the corresponding equivalence classes of A are denoted by $E_{\mathcal{R}}(A)$, $E^{r}_{\mathcal{R}}(A)$, $E^{l}_{\mathcal{R}}(A)$ respectively.

□

Next, we shall denote by $Q_{k,n}$ the strictly increasing, lexicographically ordered sequences of k integers chosen from

$\{1,2,...,n\}$ [Mar., 2]. Then we define the compound of a matrix as follows.

Definition 2.18 [Mar., 2]: Let $A \in \mathcal{R}^{\ell\times m}$ and $1 \le r \le \min\{\ell,m\}$. The **rth-compound** of A is the $\binom{\ell}{r}\times\binom{m}{r}$ matrix whose entries are the minors $|A[\alpha/\beta]|$, that is the determinants of the sub-matrices defined by the rows corresponding to $\alpha \in Q_{r,\ell}$ and columns corresponding to $\beta \in Q_{r,m}$. This matrix is denoted by $C_r(A)$.

□

Example 2.9: Suppose $A \in \mathcal{R}^{3\times3}$ and $r = 2$. Then

$$C_2(A) = \begin{bmatrix} A^{1,2}_{1,2} & A^{1,2}_{1,3} & A^{1,2}_{2,3} \\ A^{1,3}_{1,2} & A^{1,3}_{1,3} & A^{1,3}_{2,3} \\ A^{2,3}_{1,2} & A^{2,3}_{1,3} & A^{2,3}_{2,3} \end{bmatrix}$$

□

Definition 2.19: If $A \in \mathcal{R}^{\ell\times m}$, $\underline{a}_i$, $i \in \underline{m}$ denote the columns and $\underline{a}_j$, $j \in \ell$ the rows of A, then

- **a.** if $\ell \le m$, $C_\ell(A) := \underline{a}_1^t \wedge \cdots \wedge \underline{a}_\ell^t$ is an $1 \times \binom{m}{\ell}$-row vector and it is called the **exterior**, or **Grassmann product** of the rows of A.
- **b.** if $m \le \ell$, $C_m(A) := \underline{a}_1 \wedge \cdots \wedge \underline{a}_m$ is an $\binom{\ell}{m}$-column vector and it is called the **exterior**, or **Grassmann product** of the columns of A.

□

An important result relating the properties of the compound of the product of two matrices to the product of compounds is the *Binet-Cauchy theorem* stated below.

Theorem 2.12 [Mar., 2] *(Binet-Cauchy Theorem)*: Let $A \in \mathcal{R}^{\ell\times m}$, $B \in \mathcal{R}^{m\times k}$ and $1 \le p \le \min\{\ell,m,k\}$, then

$$C_p(AB) = C_p(A)C_p(B) \tag{2.17}$$

□

2.3.1 Polynomial and rational matrices

Polynomial and rational matrices appear throughout linear systems theory. The basic mathematical theory of polynomial matrices can be found in MacDuffee [Mac., 1], Gantmakher [Gan., 1] and Wedderburn [Wed., 1]. Some basic definitions, notations, and properties from the system theoretic point of view are summarized in this section (see, e.g, [Cal., 1], [Kai., 1], [Kar., 1], [Kuc., 1], [Ros., 1], [Var., 1], [Wol., 1] and the references therein).

Basic definitions and notation

Let $P(d) \in \mathbb{R}^{l\times m}(d)$ where $d \in \mathbb{C}_e := \mathbb{C} \cup \{\infty\}$. The rank of $P(d)$ over $\mathbb{R}(d)$ is denoted by $\rho(P(d)) = r \le \min\{\ell,m\}$ and is referred to as **normal rank,** whereas the rank $\rho_c(P(\lambda)) = r_\lambda$ of $P(d)$ over $\mathbb{C}_e$, for some $\lambda \in \mathbb{C}_e$, is called the **local rank** at $d = \lambda$. The tools for investigating rank properties are the Smith-McMillan forms. If $r = \min\{\ell,m\}$, then $P(d)$ is said to be **nondegenerate**, otherwise, i.e. if $r < \min\{\ell,m\}$, it will be called **degenerate**.

If $\underline{x}(d)=[x_1(d),\dots,x_l(d)]^t \in \mathbb{R}^l[d]$, then $\partial(\underline{x}(d))=\max\{\partial(x_i(d)), i \in \underline{\ell}\}$ is called the **degree** of $\underline{x}(d)$. If $P(d)=P_0+P_1d+\cdots+P_nd^n \in \mathbb{R}^{l\times m}[d]$, $P_i \in \mathbb{R}^{l\times m}$, then $P_n \neq 0$ is defined as the **leading coefficient matrix** of $P(d)$ and $n = \partial_s(P(d))$ as the **scalar degree** of $P(d)$. The polynomial matrix $P(d)$ is said to be **proper** or **regular** if its leading coefficient matrix P_n is nonsingular.

Let $P(d) = [\underline{p}_1(d),\dots,\underline{p}_m(d)] \in \mathbb{R}^{l\times m}[d]$ and $\rho(P(d)) = m$. Then the set of indices $I_P = \{\delta_i : \delta_i=\partial(\underline{p}_i(d)),\ i \in \underline{m}\}$ is defined as the **set of column degrees** and $c_P := \sum_{i=1}^m \delta_i$ as the **column complexity** of $P(d)$ (row degrees and row complexity are defined in a similar manner). The $\binom{\ell}{m}$ polynomial vector $C_m(P(d)) = \underline{p}_1(d) \wedge\cdots\wedge \underline{p}_m(d) := \underline{p}(d)\wedge$ is called the **Grassmann vector** of $P(d)$ [Kar., 2], and $\partial(P(d)) := \partial(\underline{p}(d)\wedge)$ is referred to as the **matrix degree**, or simply as the **degree** of $P(d)$. If $\rho(P(d)) = r \le \min\{\ell,m\}$, $\partial(P(d))$ is the maximum degree among the degrees of all maximal order minors ($r \times r$ minors) of $P(d)$ and if $P(d) = 0$, $\partial(P(d)) = -\infty$ [Ros., 2]. By using the Binet-Cauchy theorem we have the following important property.

Proposition 2.6: If $P(d) \in \mathbb{R}^{1xm}[d]$ and $Q(d) \in \mathbb{R}^{mxm}[d]$ with $\rho(P(d)) = \rho(Q(d)) = m$, then

$$\partial(P(d)Q(d)) = \partial(P(d)) + \partial(Q(d)) \tag{2.18}$$

□

If $\underline{p}_i(d) = \underline{p}_{i,h}d^{\delta i} + \cdots + \underline{p}_{i,0}$, then we may write

$$P(d) = [\underline{p}_{1,h}, \ldots, \underline{p}_{m,h}] \text{ diag } \{ d^{\delta_1}, \cdots, d^{\delta_m} \} + \hat{P}(d) \tag{2.19}$$

where the columns of $\hat{P}(d)$ have degrees less than δ_i. The matrix $P_h = [\underline{p}_{1,h}, \ldots, \underline{p}_{m,h}] := [P(d)]_h \in \mathbb{R}^{1xm}$ is referred to as the **high column coefficient matrix** of $P(d)$ and if $\rho(P_h)=m$ then $P(d)$ is called **column reduced** (high row coefficient matrix, and row reducedness are defined similarly). A very important characterization of the column reduced polynomial matrices is given by the next theorem.

Theorem 2.13 [For., 1], [Vek., 1] *(The Predictable-Degree Property)*: Let $P(d)$ be a polynomial matrix of full column rank, and for any polynomial vector $\underline{p}(d)$, let

$$\underline{q}(d) = P(d)\underline{p}(d) \tag{2.20}$$

Then $P(d)$ is column reduced, if and only if

$$\partial(\underline{q}(d)) = \max_{i:p_i(d)\neq 0} \{ \partial(p_i(d)) + \delta_i \} \tag{2.21}$$

where $\underline{p}_i(d)$ is the ith entry of $\underline{p}(d)$ and δ_i is the degree of the ith column of $P(d)$.

□

A very useful application of this result is the *invariance of column degrees of column reduced matrices* given by the following lemma.

Lemma 2.5 [Kai., 1]: Let $P(d)$ and $\overline{P}(d)$ be two column reduced matrices, with column degrees in ascending order, such that

$$P(d) = \overline{P}(d)U(d) \qquad U(d) \ \mathbb{R}[d]\text{-unimodular}$$

Then $P(d)$ and $\overline{P}$(d) have the same column degrees.

$\square$

Smith, Smith-McMillan forms

For a matrix $P(d) \in \mathbb{R}^{\ell\times m}(d)$ canonical forms may be defined under left, or right equivalence over $\mathcal{R}$ (where $\mathcal{R}$ is either $\mathbb{R}[d]$, or any other quotient ring of $\mathbb{R}[d]$ whose field of fractions is $\mathbb{R}(d)$). Such forms are referred to as **Hermite forms** if $P(d)$ is defined over $\mathcal{R}$, and **Hermite-McMillan forms** [Kar., 1] if $P(d)$ is rational in general. Under $\mathcal{R}$-equivalence we define respectively the **Smith forms,** if $P(d)$ is from $\mathcal{R}$, and **Smith-McMillan forms** if $P(d)$ is rational. The Smith, Smith-McMillan forms are central in the study of structure of linear systems and they are described next.

(a) <u>*Smith form over* $\mathbb{R}[d]$</u> [Gan., 1], [Kai., 1]

Let $P(d) \in \mathbb{R}^{\ell\times m}[d]$ and $\rho(P(d)) = r \le \min\{\ell,m\}$. Then there exist $U(d)$ and $V(d)$ $\mathbb{R}[d]$-unimodular matrices such that

$$U(d)P(d)V(d) = S_P(d) = \begin{array}{c} \left[\begin{array}{c|c} S_P^{\bullet}(d) & O \\ \hline \underbrace{O}_{r} & \underbrace{O}_{m-r} \end{array}\right] \begin{array}{l} \}\ r \\ \}\ \ell-r \end{array} \end{array} \tag{2.22}$$

$$S_P^{\bullet}(d) = \text{diag}\{f_1(d),\dots,f_r(d)\}, \quad f_i(d) \in \mathbb{R}[d]$$

$S_P(d)$ is called the **Smith form** and the monic polynomials $f_i(d)$, $i \in \underline{r}$ are called the **invariant polynomials** of $P(d)$ and satisfy the division property $f_i(d)|f_{i+1}(d) \ \forall \ i=1,\dots,r-1$. The set of $f_i(d)$ may be defined by the **Smith algorithm.** Let $d_0(d) = 1$ and $d_i(d)$ be the monic greatest common divisor of all $i\times i$ order minors, $i = 1,\dots,r$ of $P(d)$. Then $d_i(d)|d_{i+1}(d)$ $i = 0,\dots,r-1$ and

$$f_i(d) = d_i(d)/d_{i-1}(d), \quad i = 1,2,\ldots,r \tag{2.23}$$

The polynomial $z_P(d) = \prod_{i=1}^{r} f_i(d)$ is called the **zero polynomial** of $P(d)$. If $z_P(d)$ is factorized into irreducible factors over $\mathbb{C}$ as

$$z_P(d) = (d - z_1)^{\tau_1} \cdots (d - z_\mu)^{\tau_\mu}, \; z_i \in \mathbb{C}, \; z_i \neq z_j$$

then the set $\phi_P = \{ z_i, \; i \in \underline{\mu} \}$ is called the **root range**, z_i the **zero** of $P(d)$ and τ_i the **algebraic multiplicity** of z_i. The zeros are the finite values of d where $P(d)$ loses rank below its normal rank r and the number $v_i = r - \rho(P(z_i))$ is defined as the **geometric multiplicity** of z_i. Generally, $v_i \leq \tau_i$ and if equality holds, the zero is called **simple**. The matrix $P(d)$ is called **simple**, if all its zeros are simple, otherwise it is called **nonsimple**. By factoring each of the $f_j(d)$ into irreducible factors over $\mathbb{C}$ and collecting all terms corresponding to zero z_i, we define the set of **elementary divisors** for z_i,

$$D_{P,z_i} := \{ (d - z_i)^{q_{ik}}, \; k = 1,2,\ldots,v_i \}$$

where v_i is the geometric multiplicity and $\sum_{k=1}^{v_i} q_{ik} = \tau_i$.

(b) <u>*Smith-McMillan form over* $\mathbb{R}[d]$</u> [Kai., 1], [Vid., 1]

Let $P(d) \in \mathbb{R}^{l\times m}(d)$, $\rho(P(d)) = r \leq \min\{\ell,m\}$ and $d_p(d)$ be the least common multiple of the denominators of the elements of $P(d)$. Then $P(d) = d_p^{-1}(d)N(d)$, where $N(d) \in \mathbb{R}^{l\times m}[d]$. If

$$U(d)N(d)V(d) = S_N(d)$$

is the Smith reduction of $N(d)$, then the **Smith-McMillan form** of $P(d)$ is defined by

$$M_P(d) = \frac{1}{d_p(d)} S_N(d) \in \mathbb{R}^{l\times m}(d) \tag{2.24}$$

where all possible numerator-denominator cancellations in

$M_P(d)$ are assumed to have been carried out. Hence,

$$U(d)P(d)V(d) = M_P(d) = \left[\begin{array}{c|c} M_P^*(d) & 0 \\ \hline \underset{\longleftarrow r \longrightarrow}{0} & \underset{\longleftarrow m-r \longrightarrow}{0} \end{array}\right] \begin{array}{l} \}\ r \\ \}\ l-r \end{array} \tag{2.25}$$

$$M_P^*(d) = \mathrm{diag}\{\varepsilon_i(d)/\psi_i(d),\ i \in \underline{r}\}$$

The sets $\varepsilon_i(d)$, $\psi_i(d)$, $i \in \underline{r}$ are the **elementary zero- pole-polynomials** of $P(d)$ and satisfy the divisibility properties

$$\varepsilon_1(d)|\varepsilon_2(d)|\ldots|\varepsilon_r(d), \quad \psi_r(d)|\psi_{r-1}(d)|\ldots|\psi_1(d) \tag{2.26}$$

The polynomials $z_P(d) = \prod_{i=1}^{r}\varepsilon_i(d)$, $p_P(d) = \prod_{i=1}^{r}\psi_i(d)$ are defined as the **zero-, pole-polynomials** of $P(d)$ and $\partial(p_P(d)) := \delta_{\mathcal{M}}^{f}(P)$ is defined as the **finite McMillan degree** of $P(d)$.

Remark 2.12: The range of the poles and zeros of $P(d)$ is the range of the poles and zeros of the entries of $P(d)$. Their multiplicities are given by the Smith-McMillan form of $P(d)$.

□

Remark 2.13: Since for any $\mathbb{R}[d]$-unimodular matrix U, $\det U = c \in \mathbb{R} - \{0\}$ means that U has a **singular** structure at $d = \infty$, the Smith and Smith-McMillan forms give the zero-, pole-structure of $P(d)$ at $d \in \mathbb{C}$ and finite. The (highly non unique) unimodular matrices $U(d)$ and $V(d)$ in eqns. (2.22) and (2.25) may destroy the structure of $P(d)$ at infinity (see e.g. [Kai., 1]).

□

Remark 2.14 [Vid., 1]: As it was mentioned at the beginning of this section all the results of this section carry on, if $\mathbb{R}[d]$ is replaced by any PID $\mathcal{R}$ such that $\mathbb{R}(d)$ is its field of fractions. In particular, this is true for any quotient ring $\mathcal{R}$ of $\mathbb{R}[d]$. Then because the structure of $P(d)$ is defined up to units in $\mathcal{R}$, it is defined only over the so called forbidden region specified by $\mathcal{R}$.

□

Example 2.10: Consider the case where $\mathcal{R} = \mathbb{R}^{+}(d)$, the PID of stable rational functions in d, i.e. the functions whose poles lie outside the closed unit disc $\mathbb{D}[0,1]$. Therefore the forbidden region is the closed unit disc $\mathbb{D}[0,1]$. Then $\mathbb{R}(d)$ is the field of fractions of $\mathbb{R}^{+}(d)$ and in the Smith-McMillan form of $P(d) \in \mathbb{R}^{l\times m}(d)$ over $\mathbb{R}^{+}(d)$, $\varepsilon_i(d)$, $\psi_i(d)$ are stable rational functions instead of polynomials. Since they are defined up to units in $\mathbb{R}^{+}(d)$ (the bistable rational functions), they define the zeros and poles of $P(d)$ in the closed unit disc $\mathbb{D}[0,1]$ (see e.g. [Vid., 1]).

□

According to remark (2.14) the structure of the rational matrix $P(d)$ at $d = \infty$ can be determined by the Smith-McMillan form of $P(d)$ over $\mathbb{R}_{pr}(d)$, the ring of proper rational functions.

(c) *Smith-McMillan form at* $d = \infty$ [Var., 2]

Let $P(d) \in \mathbb{R}^{l\times m}(d)$ and $\rho(P(d)) = r$. There exist $\mathbb{R}_{pr}(d)$-unimodular matrices $U(d)$ and $V(d)$ such that

$$U(d)P(d)V(d) = M_P^{\infty}(d) = \begin{array}{c} \left[\begin{array}{c|c} M_P^{\infty *} & O \\ \hline \underset{\longleftarrow r \longrightarrow}{O} & \underset{\longleftarrow m-r \longrightarrow}{O} \end{array}\right] \begin{array}{l} \}\ r \\ \}\ l-r \end{array} \end{array} \tag{2.27}$$

$$M_P^{\infty *}(d) = \mathrm{diag}\{d^{q_1}, \ldots, d^{q_r}\},\ q_1 \geq \cdots \geq q_r$$

$M_P^{\infty}(d)$ is uniquely defined by $P(d)$ and is called the **Smith-McMillan form at** $d = \infty$ of $P(d)$. If p_∞ is the number of q_i's with $q_i > 0$, then we say that $P(d)$ has p_∞ **poles at infinity**, each one of order $q_i > 0$. If z_∞ is the number of q_i's with $q_i < 0$, then we say that $P(d)$ has z_∞ **zeros at infinity**, each one of order $|q_i|$. The number

$$\delta_{\mathcal{M}}^{\infty}(P) := \sum_{i=1}^{p_\infty} q_i, \quad q_i > 0$$

is defined as the **McMillan degree at infinity** of $P(d)$. If $P(d)$ is proper, it has no poles at infinity and $M_P^{\infty}(d) = S_P^{\infty}(d)$

is the **Smith form at** $d = \infty$ describing the infinite zero structure of $P(d)$.

$M_P^\infty(d)$ can be defined from the standard Smith-McMillan form of $P(1/w)$ [McM., 1], [Pug., 1] at $w = 0$. Alternatively, the q_i's may be computed by the **valuation algorithm** [Kun., 1] [Kai., 1], [Var., 2], [Ver. 1], [Ver., 2]. We recall that by **valuation at infinity** of a rational function $f(d)=b(d)/a(d)$ $a(d)$, $b(d)$ coprime polynomials, we define the map $\upsilon_\infty : \mathbb{R}(d) \longmapsto \mathbb{Z} \cup \{\infty\}$ such that

$$\upsilon_\infty(f(d)) := \begin{cases} \partial(a(d)) - \partial(b(d)), \text{ if } f(d) \neq 0 \\ \infty, \text{ if } f(d) = 0 \end{cases} \tag{2.28}$$

If $\upsilon_\infty^{(i)} \equiv$ the least valuation among the valuations of all $i \times i$ minors of $P(d)$, $i = 1,2,\ldots,r$ then

$$q_i = \upsilon_\infty^{(i-1)} - \upsilon_\infty^{(i)}, \quad i = 1,2,\ldots,r, \quad \upsilon_\infty^{(0)} = 0 \tag{2.29}$$

By adding up the left-hand side and the right-hand side of eqn. (2.29) we have the following relationship

$$\sum_{i=1}^{r} q_i = -\upsilon_\infty^{(r)} \tag{2.30}$$

or, that the number of infinite zeros minus the number of infinite poles of $P(d)$ equals $\upsilon_\infty^{(r)}$. $\upsilon_\infty^{(r)}$ plays an important role in systems theory and in the characterization of the Mc-Millan degree of a system as we will see in the next chapter. For this reason we give a formal definition and an important property of $\upsilon_\infty^{(r)}$.

Definition 2.20 [Kun., 1], [Var., 3], [Ver., 1]: Let $P(d) \in \mathbb{R}^{1\text{x}m}(d)$, $\rho(P(d))=r$ and define the map $\upsilon_\infty : \mathbb{R}(d) \longmapsto \mathbb{Z} \cup \{\infty\}$ such that

$$\upsilon_\infty(P(d)) := \begin{cases} \upsilon_\infty^{(r)}, \text{ if } P(d) \neq O \\ \infty, \text{ if } P(d) = O \end{cases} \tag{2.31}$$

where $\upsilon_\infty^{(r)}$ is the least valuation among the valuations of all $r \times r$ minors of $P(d)$. $\upsilon_{\infty}(P(d))$ is called the **valuation at $d = \infty$** of $P(d)$ and is denoted simply by $\upsilon_{\infty}(P)$.

□

Proposition 2.7 [Var., 3]: If $P(d) \in \mathbb{R}^{l \times m}(d)$ and $Q(d) \in \mathbb{R}^{m \times m}(d)$ with $\rho(P(d)) = \rho(Q(d)) = m$, then

$$\upsilon_{\infty}(P(d)Q(d)) = \upsilon_{\infty}(P(d)) + \upsilon_{\infty}(Q(d)) \tag{2.32}$$

□

According to the definition of the valuation at infinity of a rational matrix and eqn. (2.30), it follows

Remark 2.15: The valuation at infinity of a rational matrix $P(d)$ expresses the difference between the total number of infinite zeros and the total number of infinite poles of $P(d)$

$$\upsilon_{\infty}(P) = \{\# \text{ of infinite zeros}\} - \{\# \text{ of infinite poles}\} \tag{2.33}$$

□

Finally, from the Smith-Mcmillan forms over $\mathbb{R}[d]$ and at infinity we conclude the following for the McMillan degree of a rational matrix $P(d)$.

Remark 2.16: The **McMillan degree** $\delta_{\mathcal{M}}(P)$ of a rational matrix $P(d)$ is the number of finite as well as infinite poles of $P(d)$. Therefore

$$\delta_{\mathcal{M}}(P) := \delta_{\mathcal{M}}^{f}(P) + \delta_{\mathcal{M}}^{\infty}(P)$$

□

Matrix divisors and coprimeness of polynomial matrices

Let $P(d) \in \mathbb{R}^{l \times m}[d]$ and $\rho(P(d)) = m$. A matrix $R(d) \in \mathbb{R}^{m \times m}[d]$ such that

$$P(d) = P'(d)R(d)$$

is called a **right matrix divisor** (RMD) of $P(d)$. If $\overline{R}(d)$ is any other RMD and

$$R(d) = W(d)\overline{R}(d)$$

then $R(d)$ is called **right greatest matrix divisor** (RGMD) of $P(d)$. If $\rho(P(d)) = \ell$, the notions of **left matrix divisors** (LRD) and **left greatest matrix divisor** (LGMD) are defined similarly.

A matrix $P(d) \in \mathbb{R}^{l\times m}[d]$ with $\rho(P(d)) = m$ is called **right irreducible**, or **least degree** if all RMDs are $\mathbb{R}[d]$-unimodular. Nonunimodular RMDs contain a subset of the zeros of the original matrix. Therefore, a matrix $P(d)$ is right irreducible, if it has no finite zeros, i.e. $\rho(P(\lambda)) = m \;\forall\; \lambda \in \mathbb{C}$, or equivalently $S_P(d) = [I_m \;\; O]^t$. A **left irreducible** matrix is defined in a similar manner.

A matrix $P(d) \in \mathbb{R}^{l\times m}[d]$ with $\rho(P(d)) = m$ (or ℓ), is called a **minimal basis** [For., 1], if it is right (left) irreducible and column (row) reduced. Minimal bases have no finite and no infinite zeros.

If $P_r := \{P_i(d) \in \mathbb{R}^{l_i\times m}[d],\; i \in \underline{\nu}\}$ is a set of matrices, then the matrix

$$T_P^r(d) := \begin{bmatrix} P_1(d) \\ \vdots \\ P_\nu(d) \end{bmatrix} \in \mathbb{R}^{l\times m}[d], \quad \ell = \sum_{i=1}^{\nu} \ell_i$$

is called a **matrix representative** of P_r, and P_r is **right regular**, if $\rho(T_P^r(d)) = m$. If P_r is right regular, then a **right common matrix divisor** (RCMD) and a **right greatest common matrix divisor** (RGCMD) of P_r is defined as a RCMD and a RGCMD of $T_P^r(d)$ respectively. The set P_r is called **right coprime** (RC), if it is right regular and $T_P^r(d)$ is right irreducible. For a set of matrices with the same number of rows, the matrix representative, **left regularity** property, **left common matrix divisors** (LCMD), **left greatest common matrix divisor** (LGCMD) and **left coprimeness** (LC) are defined in a similar manner.

A very important property of the RGCMD of $P_i \in \mathcal{P}_r$, $i=1,\dots,\nu$ is given by the following lemma.

Lemma 2.6 [Ros., 1], [Vid., 1]: Let $P_i \in \mathcal{P}_r$, $i=1,\dots,\nu$ and R any RGCMD of $\mathcal{P}_r$. Then there exist polynomial matrices X_i, $i=1,\dots,\nu$ such that

$$R = \sum_{i=1}^{\nu} X_i P_i \tag{2.34}$$

□

A very useful generalization of lemma (2.6) when $i = 2$, is given next.

Lemma 2.7 [Kai., 1] ***(Generalized Bezout Identity)*:** Let $(N(d),D(d))$ be right coprime polynomial matrices, with $D(d)$ nonsingular. Then there exist polynomial matrices $X(d)$, $\tilde{X}(d)$ $Y(d)$, $\tilde{Y}(d)$ such that

$$\begin{bmatrix} -\tilde{X}(d) & \tilde{Y}(d) \\ \tilde{D}(d) & \tilde{N}(d) \end{bmatrix} \begin{bmatrix} -N(d) & Y(d) \\ D(d) & X(d) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{2.35}$$

$$\begin{bmatrix} -N(d) & Y(d) \\ D(d) & X(d) \end{bmatrix} \begin{bmatrix} -\tilde{X}(d) & \tilde{Y}(d) \\ \tilde{D}(d) & \tilde{N}(d) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{2.36}$$

We call (2.35) the **forward Bezout identity** and (2.36) the **reversed Bezout identity**.

□

We close this section by stating two important theorems about division of polynomial and general matrices.

Theorem 2.14 [Kai., 1]: Let $D(d) \in \mathbb{R}^{m\times m}[d]$ and also be non-singular. Then, for any $N(d) \in \mathbb{R}^{1\times m}[d]$, there exist unique polynomial matrices $Q(d)$, $R(d)$ such that $N(d) = Q(d)D(d)+R(d)$ and $R(d)D^{-1}(d)$ is strictly proper. If $D(d)$ is column reduced, uniqueness is ensured if the columns of $R(d)$ have degree strictly less than the corresponding column degrees of $D(d)$.

□

Theorem 2.15 [Vid., 1]: Suppose $\mathcal{R}$ is a Euclidean domain, $\boldsymbol{D},\boldsymbol{N}$ are right-coprime matrices with entries from $\mathcal{R}$ and D is square. Let $\boldsymbol{d}$ = det$D$ and n denote the smallest invariant factor of $\boldsymbol{N}$. Then the sets

$$\{d + rn : r \in \mathcal{R}\} \quad \textit{and} \quad \{|D + RN| : R \text{ matrix in } \mathcal{R}\}$$

are equal.

□

Matrix fraction descriptions

Suppose that $G(d) \in \mathbb{R}^{l\times m}(d)$ and $\rho(G(d)) = \min\{\ell,m\}$. Then it is well known that $G(d)$ can always be factored (in a non-unique way) as

$$G(d) = \tilde{D}^{-1}(d)\tilde{N}(d) = N(d)D^{-1}(d) \tag{2.37}$$

where $\tilde{N}(d)$, $N(d)$, $\tilde{D}(d)$, $D(d)$ are polynomial matrices with appropriate dimensions, and det$\tilde{D}(d)$, det$D(d) \neq 0$. The pair $(\tilde{D}(d), \tilde{N}(d))$ $(N(d), D(d))$ is called **left (right) matrix fraction description** (MFD) of the rational matrix $G(d)$. If $(\tilde{D}(d), \tilde{N}(d))$ $(N(d), D(d))$ are respectively left, right coprime, then the corresponding MFDs are referred to as **coprime**.

Coprime MFDs are not unique and the following theorem gives a characterization of their family.

Theorem 2.16 [Kai., 1], [Ros., 1]: Let $\{N_i(d)D_i^{-1}(d),\ i=1,2\}$ be two coprime MFDs. Then there exists an $\mathbb{R}[d]$-unimodular matrix $U(d)$ such that

$$D_1(d) = D_2(d)U(d), \quad N_1(d) = N_2(d)U(d) \tag{2.38}$$

□

Remark 2.17 [Kai., 1], [Ros., 1] : Let $(\tilde{D}(d), \tilde{N}(d))$, and $(N(d), D(d))$ be left, right coprime MFDs of the rational matrix $G(d)$. Then

a. The invariant polynomials of $\tilde{N}(d)$ and $N(d)$ and the elementary zero-polynomials of $G(d)$ are associates to each other.

b. The nonunit invariant polynomials of $\tilde{D}(d)$ and $D(d)$ and the elementary pole-polynomials of $G(d)$ are associates to each other. We call the matrices $D(d)$ and $\tilde{D}(d)$ **extended associates** and we denote that by $\cdot\sim$, i.e $D \cdot\sim \tilde{D}$.

□

Finally, we give a test for the properness of a rational matrix through its MFD and we state the notion of bicoprimeness and its important properties.

Lemma 2.8 [Kai., 1]: If $D(d)$ is column reduced, then $G(d) = N(d)D^{-1}(d)$ is strictly proper (proper), if and only if each column of $N(d)$ has degree less than (less than or equal to) the degree of the corresponding column of $D(d)$.

□

Definition 2.21 [Vid., 1]: Suppose $G(d)$ is a rational matrix. A quadruple $(N,D,\tilde{N},K)$ of polynomial matrices is a **bicoprime factorization** of $G(d)$ if

a. $\det D \neq 0$ and $G = ND^{-1}\tilde{N} + K$

b. (N, D) are right and $(D, \tilde{N})$ are left coprime.

□

Theorem 2.17 [Vid., 1]: Suppose $G(d) \in \mathbb{R}^{1\mathrm{x}m}(d)$ and $(\tilde{D},\tilde{N})$, (N,D), (C,A,B,K) are any left coprime, right coprime and bicoprime factorizations of $G(d)$ over $\mathbb{R}[d]$. Then $\det\tilde{D}$, $\det D$ and $\det A$ are associates, i.e

$$\det\tilde{D} \sim \det D \sim \det A$$

□

Due to the properties of coprime factorizations the following corollary is true.

Corollary 2.3 [Ros., 1], [Vid., 1]: Suppose $G(d) \in \mathbb{R}^{1\mathrm{x}m}(d)$ and $(\tilde{D},\tilde{N})$, (N,D), (C,A,B,K) are any left coprime, right coprime and bicoprime factorizations of $G(d)$ over $\mathbb{R}[d]$. Then

$G(d) \in \mathbb{R}^{l\mathrm{x}m}[d]$ if and ony if $\tilde{D}$, D, A are $\mathbb{R}[d]$-unimodular

□

Remark 2.18: Most of the results in section (2.3.1) were described for matrices with entries that are polynomials with real coefficients. Generally, these results hold true for entries drawn from any principal ideal domain and in particular from suitable quotient rings of $\mathbb{R}[d]$ whose field of fractions is the field of rational functions $\mathbb{R}(d)$. This is especially true for MFDs of such quotient rings [Vid., 1].

□

2.3.2 The algebraic structure of rational vector spaces

Let $G(d) \in \mathbb{R}^{l\mathrm{x}m}(d)$, $\ell \geq m$, and $\rho(G(d)) = m$. Let us also denote by $\mathcal{V}_G$ the set of all linear combinations of the columns of $G(d)$ with multipliers in $\mathbb{R}(d)$, i.e.

$$\begin{aligned} &\text{if } G(d) = [\underline{g}_1(d),\ldots,\underline{g}_m(d)], \text{ then} \\ &\mathcal{V}_G = \mathrm{span}_{\mathbb{R}(d)}\{\underline{g}_1(d),\ldots,\underline{g}_m(d)\} \end{aligned} \tag{2.39}$$

Then, $\mathcal{V}_G$ is a linear vector space over $\mathbb{R}(d)$, $\dim\mathcal{V}_G = m$, and it is called the **rational vector space generated by** $G(d)$.

From any rational basis $G(d)$ of $\mathcal{V}_G$ we can generate a polynomial basis of $\mathcal{V}_G$ by means of a right MFD of $G(d)$, i.e. if $G(d) = N(d)D^{-1}(d)$ with $N(d) \in \mathbb{R}^{l\mathrm{x}m}[d]$, $D(d) \in \mathbb{R}^{m\mathrm{x}m}[d]$, $\det D(d) \neq 0$, then clearly the columns of $N(d)$ define a polynomial basis of $\mathcal{V}_G$. More precisely, if $N(d)=[\underline{n}_1(d),\ldots,\underline{n}_m(d)]$ then

$$\begin{aligned} &\mathcal{V}_G = \mathrm{span}_{\mathbb{R}(d)}\{\underline{n}_1(d),\ldots,\underline{n}_m(d)\} \quad \text{and} \\ &\mathcal{M}_N = \mathrm{span}_{\mathbb{R}[d]}\{\underline{n}_1(d),\ldots,\underline{n}_m(d)\} \end{aligned} \tag{2.40}$$

where $\mathcal{M}_N$ denotes the set of all linear combinations of the columns of $N(d)$ with multipliers in $\mathbb{R}[d]$. The set $\mathcal{M}_N$ is a free $\mathbb{R}[d]$-module [Mac., 2] and it is called the **polynomial module generated by** $N(d)$.

Proposition 2.8 [Kou., 1]: Let $\mathcal{M}_{N1}$, $\mathcal{M}_{N2}$ be the polynomial modules generated by the $N_1(d)$, $N_2(d) \in \mathbb{R}^{1xm}[d]$, $\rho(N_1(d)) = \rho(N_2(d)) = m$. If $N_1(d) = N_2(d)Q(d)$, where $Q(d) \in \mathbb{R}^{mxm}[d]$, $\det Q(d) \neq 0$, then

$$\mathcal{M}_{N_1} \subseteq \mathcal{M}_{N_2}$$

□

Proposition 2.9 [Kou., 1]: Let $N_1(d)$, $N_2(d) \in \mathbb{R}^{1xm}[d]$ be two polynomial bases of the same polynomial module $\mathcal{M}_N$. Then, there exists a $\mathbb{R}[d]$-unimodular matrix $Q(d)$ such that

$$N_1(d) = N_2(d)Q(d)$$

Furthermore, for any $N_2(d)$ there is a $\mathbb{R}[d]$-unimodular $Q(d)$ such that $N_1(d)$ is column reduced.

□

Thus unimodular matrices represent coordinate transformations for a polynomial module.

Proposition 2.10 [Ros., 1]: Let $N(d)$ be a basis of the polynomial module $\mathcal{M}_N$. Then the matrix degree of $N(d)$ is an invariant of M_N, or in other words, if $N_1(d)$ is any other basis of $\mathcal{M}_N$ then $\partial(N(d)) = \partial(N_1(d))$.

□

From propositions (2.7), (2.8) and (2.9) we have the following important result.

Theorem 2.18: Let $N_1(d)$, $N_2(d) \in \mathbb{R}^{1xm}[d]$, $\ell \geq m$, $\rho(N_1(d)) = \rho(N_2(d)) = m$, and $d_1 = \partial(N_1(d))$, $d_2 = \partial(N_2(d))$. If

$$N_1(d) = N_2(d)Q(d), \quad Q(d) \in \mathbb{R}^{mxm}[d], \ \partial(Q(d)) = q \geq 1 \qquad (2.41)$$

then

a. $d_1 = d_2 + q$ (2.42)

b. $\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2}$ (2.43) where $\mathcal{M}_{N1}$, $\mathcal{M}_{N2}$ are the polynomial modules generated by the polynomial matrices $N_1(d)$, $N_2(d)$ respectively.

□

Clearly, eqn. (2.41) represents the extraction of a right divisor $Q(d)$ of the polynomial matrix $N_1(d)$. This observation leads us to the following conclusions.

Lemma 2.9: Let $N_1(d) \in \mathbb{R}^{l\times m}[d]$, $\ell \geq m$, $\rho(N_1(d)) = m$ be a polynomial matrix which can be written in terms of its columns as $N_1(d) = [\underline{n}_1^1(d), \ldots, \underline{n}_m^1(d)]$. Assume also that $N_1(d)$ is not irreducible and let

$$\mathcal{V} = \mathrm{span}_{\mathbb{R}(d)}\{\underline{n}_1^1(d), \ldots, \underline{n}_m^1(d)\}, \ \mathcal{M}_{N_1} = \mathrm{span}_{\mathbb{R}[d]}\{\underline{n}_1^1(d), \ldots, \underline{n}_m^1(d)\}$$

be the rational vector space $\mathcal{V}$ and the polynomial module $\mathcal{M}_{N1}$ spanned by its columns. Then if $Q_i(d)$, $i = 1,2,\ldots$ are right divisors of $N_1(d)$, i.e. $N_1(d) = N_{i+1}(d)Q_i(d)$, $i = 1,2,\ldots$ and $\partial(Q_i(d)) = q \geq 1$ are such that $q_1 \leq q_2 \leq q_3\cdots$, we have

$$\mathcal{M}_{N_1} \subset \mathcal{M}_{N_2} \subset \mathcal{M}_{N_3} \subset \cdots \tag{2.44}$$

and

$$\partial(N_1(d)) \geq \partial(N_2(d)) \geq \partial(N_3(d)) \geq \cdots \tag{2.45}$$

Moreover, if $Q_G(d)$ is a greatest right divisor of $N_1(d)$ so that $N_1(d) = N(d)Q_G(d)$, then

$$\mathcal{M}_{N_1} \subset \mathcal{M}_N \quad \text{and} \quad \partial(N_1(d)) \geq \partial(N(d)) \tag{2.46}$$

□

The polynomial module $\mathcal{M}_N$ is the **maximal submodule** of the rational vector space $\mathcal{V}$ and all its polynomial bases are least degree, or irreducible polynomial matrices. In other words, if we consider the set of all polynomial vectors in $\mathcal{V}$ then this set coincides with the module $\mathcal{M}_N$ defined above.

Clearly, although the ascending chain of modules which is defined by eqn. (2.44) is not unique (it depends on the choice of $Q_i(d)$), the maximal module $\mathcal{M}_N$ is defined uniquely. In the context of extraction of matrix divisors and modules, the previously defined concept of a minimal basis may also be defined as

Definition 2.22 [For., 1]: A matrix $N(d) \in \mathbb{R}^{l\times m}[d]$ with $\ell \geq m$ and $\rho(N(d)) = m$ is said to be a **minimal basis** of the rational vector space V spanned by the columns of a polynomial matrix $N_1(d) \in \mathbb{R}^{l\times m}[d]$, $\ell \geq m$, $\rho(N_1(d)) = m$, if and only if $N_1(d) = N(d)Q_G(d)$, where $Q_G(d)$ is a greatest right divisor of $N_1(d)$ and $N(d)$ has the following properties:

a. $N(d)$ is least degree, or irreducible.

b. $N(d)$ is column reduced.

□

Remark 2.19: Let $N_1(d) \in \mathbb{R}^{l\times m}[d]$, $\ell \geq m$, $\rho(N_1(d)) = m$. If $N(d)$, $N^*(d) \in \mathbb{R}^{l\times m}[d]$ are two minimal bases of the rational vector space V spanned by the columns of $N_1(d)$, then

$$N(d) = N^*(d)Q(d)$$

where $Q(d)$ is a $\mathbb{R}[d]$-unimodular matrix

□

Remark 2.20: Let $\underline{x}(d)$ be a polynomial vector of the rational vector space V spanned by the columns of $N_1(d) \in \mathbb{R}^{l\times m}[d]$, $\ell \geq m$ $\rho(N_1(d)) = m$, and let $N(d)$ be a minimal basis of V. Then $\underline{x}(d)$ can be expressed as a polynomial combination of the columns of $N(d)$.

□

Given in general, a $G(d) \in \mathbb{R}^{l\times m}(d)$, $\ell \geq m$, $\rho(G(d)) = m$, Forney [For., 1] describes a way of computing a minimal basis for the rational space V_G spanned by its columns. It is then shown that the column degrees $\delta_i = \partial(\underline{n}_i(d))$, $i = 1,\ldots,m$ of a minimal basis $N(d) = [\underline{n}_1(d),\ldots,\underline{n}_m(d)]$ are invariant for every minimal basis of V_G. Forney calls these degrees the **invariant dynamical indices** of V_G and

$$\delta_F := \sum_{i=1}^{m} \delta_i \tag{2.47}$$

is the **invariant dynamical order** of V_G.

Remark 2.21: Clearly, δ_F is the complexity of $N(d)$ and since $N(d)$ is a minimal basis, it is equal to the matrix degree of $N(d)$. We have to note that the invariant dynamical indices and the invariant order do not characterize V_G completely. A complete invariant for V_G is given by the ***echelon type minimal basis*** [For., 1], [Kai., 1].

□

Consider now the dual space $V_G^{\perp}$ to V_G, i.e. every $\underline{z} \in V_G^{\perp}$ is orthogonal to every $\underline{y} \in V_G$. Then we have the following theorem.

Theorem 2.19 [For., 1]: Let $V_G^{\perp}$ be the dual space to V_G. Then the invariant dynamical orders of $V_G^{\perp}$ and V_G are the same.

□

Finally, we define the minimal column and row indices and the Forney order of a polynomial MFD.

Definition 2.23: Let $(\tilde{D}(d), \tilde{N}(d))$ and $(N(d), D(d))$ be two left and right coprime polynomial MFDs of $G(d) \in \mathbb{R}^{1\mathrm{xm}}(d)$ with $[N^t(d)\ D^t(d)]^t$ column reduced and $[\tilde{N}(d)\ \tilde{D}(d)]$ row reduced. Consider then the spaces $\mathcal{X}_c := \text{col.sp.}_{\mathbb{R}(d)}\{[N^t(d)\ D^t(d)]^t\}$ and $\mathcal{X}_r := \text{rowsp.}_{\mathbb{R}(d)}\{[\tilde{N}(d)\ \tilde{D}(d)]\}$. We call the invariant dynamical indices μ_i $i = 1,...,m$ of $\mathcal{X}_c$, the **right**, or **column minimal indices** of $G(d)$, and the invariant dynamical indices ν_i $i = 1,...,\ell$ of $\mathcal{X}_r$, the **left**, or **row minimal indices** of $G(d)$. Also we call the invariant dynamical order of $\mathcal{X}_c$, or $\mathcal{X}_r = \mathcal{X}_c^{\perp}$, the **Forney dynamical order** of $G(d)$ and we denote it by $\delta_F(G)$, i.e.

$$\delta_F(G) := \sum_{i=1}^{m} \mu_i \qquad \text{or} \qquad \delta_F(G) := \sum_{i=1}^{l} \nu_i$$

□

2.4 Sequential Matrices

Matrices whose elements are sequences are called sequential matrices. In particular, we have the following notation.

Definition 2.24 [Kuc., 1]: The sets of $\ell \times m$ matrices with elements in $\mathbb{R}(d)$, $\mathbb{R}^0(d)$ and $\mathbb{R}^+(d)$ are denoted by $\mathbb{R}_{lm}(d)$, $\mathbb{R}^0_{lm}(d)$ and $\mathbb{R}^+_{lm}(d)$ and are called **rational**-sequence, **causal**-sequence and **stable**-sequence matrices respectively. Obviously, d is not a variable but an indeterminate over $\mathbb{R}$.

□

A matrix $P \in \mathbb{R}_{lm}(d)$ can be written as the matrix recurrent sequence

$$P = P_n d^n + P_{n+1} d^{n+1} + \cdots , \quad P_k \in \mathbb{R}^{l \times m} \tag{2.48}$$

If $P_n \neq 0$, then n is the **order** of P denoted by $\tau(P)$. Thus, if $\tau(P) \geq 0$ then P is a causal sequential matrix, and if $\tau(P) \geq 0$ and $P_k \longrightarrow 0$ as $k \longrightarrow \infty$ elementwise, i.e. $(p_{ij})_k \longrightarrow 0$ as $k \longrightarrow \infty$, then P is a stable sequential matrix.

Example 2.11: Let $P \in \mathbb{R}_{2x1}(d)$ be such that

$$P = \begin{bmatrix} 1 + ad + a^2d^2 + a^3d^3 + \cdots \\ + d - 2bd^2 + 2b^2d^3 - \cdots \end{bmatrix}$$

Then P can be written as

$$P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ 1 \end{bmatrix} d + \begin{bmatrix} a^2 \\ -2b \end{bmatrix} d^2 + \begin{bmatrix} a^3 \\ 2b^2 \end{bmatrix} d^3 + \cdots$$

Hence, $\tau(p) = 0$, so P is a causal matrix, and if $|a| < 1$ and $|b| < 1$, P is a stable matrix as well.

□

Polynomial-sequence matrices $P[d] \in \mathbb{R}^{l \times m}[d]$ are finite sequence matrices with order greater than, or equal to zero. Then the following inclusion property holds true.

$$\mathbb{R}_{lm}[d] \subset \mathbb{R}^+_{lm}(d) \subset \mathbb{R}^0_{lm}(d) \subset \mathbb{R}_{lm}(d) \tag{2.49}$$

Square sequential matrices form noncommutative rings and their units according to lemma (2.4) and remark (2.10) are characterized as follows.

Proposition 2.11: The units of rational (recurrent) sequence matrices ($\mathbb{R}(d)$-unimodular) are nonsingular matrices, the units of causal sequence matrices ($\mathbb{R}^0(d)$-unimodular) are those with determinant a bicausal sequence and the units of stable sequence matrices ($\mathbb{R}^+(d)$-unimodular) are those with determinant a bistable sequence.

□

We can define also **causal** and **stable polynomial** sequence matrices in a way similar to the scalar case, i.e.

Definition 2.25: A polynomial sequence matrix $P(d)$ is

a. causal if it is $\mathbb{R}^0(d)$-unimodular
b. stable if it is $\mathbb{R}^+(d)$-unimodular

□

According to propositions (2.4), (2.5) and (2.11) stable and causal polynomial matrices can be described as follows.

Proposition 2.12: A polynomial matrix $P(d)$ is

a. causal if and only if $\det P(0)\neq 0$, i.e. $\det P_0\neq 0$, where P_0 is the constant matrix term of $P(d)$
b. stable if and only if the roots of $\det P(d)$ lie outside the closed unit disc $\mathbb{D}[0,1]$.

□

Since $\mathbb{R}^+(d)$ and $\mathbb{R}^0(d)$ are quotient rings of $\mathbb{R}[d]$ and their field of fractions, including that of $\mathbb{R}[d]$, is the field of rational sequences $\mathbb{R}(d)$, formal matrix series (2.48) can be written in closed form and also most of the properties presented in section (2.3) carry on in this case. In particular, sequential matrices can be expressed as ***Polynomial Matrix Fractions***. Using the terminology introduced by definition (2.25), and according to remarks (2.12) and (2.17) we have the following important property.

Proposition 2.13: A coprime polynomial MFD $(\tilde{D}(d), \tilde{N}(d))$, $(N(d), D(d))$ is

a. **causal** if and only if $\tilde{D}(d)$, or $D(d)$ is **causal**
b. **stable** if and only if $\tilde{D}(d)$, or $D(d)$ is **stable**.

□

We conclude this chapter by a brief discussion of the basics of matrix equations over rings, which play an important role in our development.

2.5 Matrix Equations Over Rings

As it was mentioned in the introduction chapter, many control problems when using the algebraic approach, can be reduced to the solution of certain polynomial, or polynomial matrix linear equations. In this final section we give a quick review on the solution of matrix equations over rings. We will elaborate more if necessary, in subsequent chapters.

Suppose that $\mathcal{R}$ is a PID and $\mathcal{R}^{l\times m}$ is the set of $\ell\times m$ matrices in $\mathcal{R}$. We can distinguish mainly between two broad categories of matrix equations over $\mathcal{R}$, namely; ***linear equations*** of the form $AX = B$, and ***linear Diophantine equations*** (for more details see e.g. [Kuc., 1], [Kuc., 2], [Kar., 1] and references therein).

2.5.1 Linear matrix equations

Consider the matrix equation

$$AX = B \tag{2.50}$$

where $A \in \mathcal{R}^{l\times m}$, $B \in \mathcal{R}^{l\times k}$ are known matrices, and $X \in \mathcal{R}^{m\times k}$ is to be computed.

Theorem 2.20 [Kar., 1]: Eqn. (2.50) has a solution X over $\mathcal{R}$, if and only if either of the following conditions is satisfied:

a. A is a left divisor of B in $\mathcal{R}$
b. $[A\ B]\ E^{r}_{\mathcal{R}}\ [A\ O]$

If X_0 is a particular solution, then any solution is of the form

$$X = X_0 + WY \tag{2.51}$$

where W is a basis of the right null space of A and Y is an arbitrary $\mathcal{R}$-matrix with appropriate dimensions.

□

2.5.2 Linear Diophantine matrix equations

We distinguish two types of linear Diophantine matrix equations; either $AX + BY = C$ $(XA + YB = C)$, which is called **unilateral matrix equation**, or $AX + YB = C$ $(XA + BY = C)$, which is called **bilateral matrix equation**.

Unilateral Diophantine matrix equations

Consider the equation

$$AX + BY = C \tag{2.52}$$

$$XA + YB = C \tag{2.53}$$

where $A \in \mathcal{R}^{lxm}$, $B \in \mathcal{R}^{lxn}$, and $C \in \mathcal{R}^{lxk}$ (or any appropriate dimensions for eqn. (2.53)) are given matrices with elements in $\mathcal{R}$.

Theorem 2.21 [Kuc., 1]: Eqn. (2.52) (eqn. 2.53) has a solution pair X, Y in $\mathcal{R}$, if and only if the greatest common left (right) divisor G in $\mathcal{R}$ of matrices A and B is a left (right) divisor of C in $\mathcal{R}$.

□

It is possible to parametrize the solution to eqns. (2.52), (2.53) and the family of solutions is given by the next theorem.

Theorem 2.22 [Kuc., 1]: Let X_0, Y_0 be a particular solution of eqn. (2.52), $r = \rho_{\mathcal{R}}([A\ B])$ and $[-B_1^t\ A_1^t]^t$ be a basis of the right null space of $[A\ B]$. Then the general solution of eqn. (2.52) is given by

$$X = X_0 - B_1 T$$
$$Y = Y_0 + A_1 T \tag{2.54}$$

where $T \in \mathcal{R}^{(m+n-r)xk}$ is an arbitrary matrix in $\mathcal{R}$.

□

Remark 2.22: It can be shown [Kuc., 1] that one particular solution to eqn. (2.52) can be given by $(X_0, Y_0) = (P_1C_1, Q_1C_1)$ where P_1, Q_1 are a solution to the Diophantine equation

$$AP_1 + BQ_1 = G$$

with G a greatest common left divisor of A and B, and C_1 is such that $C = GC_1$. Then

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} P_1 & -B_1 \\ Q_1 & A_1 \end{bmatrix} \begin{bmatrix} C_1 \\ T \end{bmatrix} \tag{2.55}$$

Similar parametrization of solutions for the eqn. (2.53) can be obtained.

Bilateral Diophantine matrix equations

Consider the equation

$$AX + YB = C \tag{2.56}$$

where $A \in \mathcal{R}^{lxm}$, $B \in \mathcal{R}^{nxk}$ and $C \in \mathcal{R}^{lxk}$.

Theorem 2.23 [Kuc., 1]: Equation (2.56) has a solution if and only if the matrices

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ and } \begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

are $\mathcal{R}$-equivalent.

□

The parametrization of solutions of the eqn. (2.56) is not as simple as that of equation (2.52) and can be found in [Kuc., 1], [Emr., 1] and [Zak, 1]. Similar results can be obtained for the dual of equation (2.56).

2.6 Conclusions

A unifying mathematical background for the study of discrete-time systems has been presented in this chapter. The main objective has been to set up a framework in terms of sequences for the treatment of discrete-time systems.

In this context an attempt has been made to unify the approaches by Kalman [Kal., 1] and Kucera [Kuc., 1], use the distinguishing properties of the rings of formal power series and of the series expansion of functions and show under which conditions the two notions coincide, or the two rings are isomorphic. The main prerequisite for that is that formal power series are to be defined over an infinite field.

In the case of discrete-time systems, as opposed to digital systems, the field used is the field of real numbers $\mathbb{R}$, and the rings of recurrent formal Laurent series and rational functions are isomorphic. This important property is exploited in the next chapter where a formal definition and some basic system properties of discrete-time systems are presented. The whole approach is algebraic and provides a unifying mathematical formalism and a powerful method for both analysis and synthesis purposes.

Also, a brief review of polynomial and rational matrix theory has been given including Smith and Smith-McMillan forms, definition of poles and zeros over $\mathbb{C}_e := \mathbb{C} \cup \{\infty\}$, and coprime and bicoprime factorizations of rational matrices over $\mathbb{R}[d]$ with hints of how these results can be extended to suitable quotient rings of $\mathbb{R}[d]$ whose field of fractions is $\mathbb{R}(d)$.

Finally, we concluded this chapter by a quick discussion of the basics of matrix equations over rings with emphasis on Diophantine equations. Necessary and sufficient conditions for the solution of such equations were given including the parametrization of solutions when possible.

Chapter 3

DISCRETE LINEAR SYSTEMS: Unification of *d*- and *z*-Representations and Related Properties

Chapter 3

DISCRETE LINEAR SYSTEMS:
Unification of *d*- and *z*-Representations and Related Properties

3.1 Introduction

This chapter is a summary of the basic concepts and results of linear systems theory adapted to accommodate the needs of discrete-time signals and systems. The systems are studied by means of their mathematical models. According to Minsky [Min., 1], [Cel., 1], a model ($\mathcal{M}$) for a system ($\mathcal{S}$) and an experiment ($\mathcal{E}$) is anything to which $\mathcal{E}$ can be applied in order to answer questions about $\mathcal{S}$. The most basic requirements to determine such a model are to specify the time set on which the system behaviour is defined and the spaces of admissible input and output signals.

A *signal* can be considered as a function that conveys information, generally about the behaviour of a physical system. Mathematically it can be represented as a function of one or more independent variables drawn from a particular domain (or domains). The independent variable may be either continuous or discrete.

Continuous-time signals are signals that are defined at a continuum of times whereas *discrete-time* signals are defined at discrete times, i.e. they can be represented as sequences. If both time and amplitude are continuous, the signal is called *analog* and if both time and amplitude are discrete, the signal is called *digital*.

Systems that process discrete-time signals, i.e. their inputs and outputs are such signals, are called *discrete-time* *systems*. In sections (3.2) and (3.3) we deal with the two main system representations for linear time-invariant discrete-time systems, namely the input-output and the state-space one, and the relationships between the two representations. Also, the concepts of reachability/controllability, observability/constructibility and stability for this class of discrete-time systems are discussed in some detail. In section (3.4) we give the basic features and properties of the so-called ***unity***, or ***one-parameter feedback system*** which is the main control scheme we shall employ throughout this work.

3.2 Linear Discrete-Time Systems

Input-Output description

As it was mentioned in the introduction, discrete-time systems are systems that are stimulated at their inputs by sequences and produce sequences as responses at their outputs. We examine systems that process sequences in one indeterminate d over $\mathbb{R}$ and in particular those whose behaviour is governed by recurrent formal Laurent series.

We thoroughly exploit the fact that recurrent formal Laurent series and rational functions are isomorphic due to the infinite nature of $\mathbb{R}$, and use the algebraic framework presented in chapter (2) for a unifying treatment of discrete time systems. Many concepts, definitions and properties of this section can be found in a wide range of textbooks like Chen [Che., 1], Jury [Jur., 1], Kailath [Kai., 1], Kucera [Kuc., 1], Rosenbrock [Ros., 1], Sontag [Son., 1] Vardulakis [Var., 1], Vidyasagar [Vid., 1] and references therein. Our aim in this section is to unify the various approaches by using the dual nature of sequences as formal power series on one hand, and as power series expansions of functions on the other. We give now a formal definition of a discrete-time system.

Definition 3.1: A **discrete-time** system $\mathcal{S}$ with m inputs and ℓ outputs is a set $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},\mathcal{G})$ where $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$ is the **discrete time**, $\mathcal{U} \subseteq \mathbb{R}^{m}\langle d\rangle$ is the **input space**, $\mathcal{Y} \subseteq \mathbb{R}^{l}\langle d\rangle$ is the **output space**, and $\mathcal{G}$ is a **map** from $\mathcal{U}$ to $\mathcal{Y}$, i.e. $\mathcal{G} : \mathcal{U} \longmapsto \mathcal{Y}$.

$\square$

Therefore, mathematically a discrete-time system is a transformation $\mathcal{G}$ that maps uniquely a real input vector-sequence $\underline{u}$ to a real output vector-sequence $\underline{y}$, i.e.

$$\underline{y} = \mathcal{G}[\underline{u}] \tag{3.1}$$

as it is shown in fig. (3.1).

$$\underline{u} = \{\underline{u}_{p},\underline{u}_{p+1},\ldots\} \longrightarrow \boxed{\mathcal{G}[\ ]} \longrightarrow \underline{y} = \{\underline{y}_{q},\underline{y}_{q+1},\ldots\}$$

Figure (3.1): A discrete-time system

3.2.1 Linearity and time-invariance

Definition (3.1) covers a wide range of discrete-time systems. By imposing constraints on the map $\mathcal{G}$ we can have useful subclasses of discrete-time systems and such a class that will be dealt with, is that of **linear, time-invariant, discrete-time** systems. First, we consider the single-input/ single-output (SISO) case where $\mathcal{U},\mathcal{Y} \subseteq \mathbb{R}\langle d\rangle$, and then we generalize for the multivariable (MIMO) case. Before we give a formal characterization of this class of discrete-time systems we define a key sequence, namely the **impulse sequence** and the response to that, i.e. the **impulse response**.

Definition 3.2: The unit sequence $d^{0} = \{0;1,0,0,\ldots\}$ is called **impulse sequence**, or just **impulse** and is denoted by δ, i.e. $\delta := \{0;1,0,0,\ldots\}$. The response of the system $\mathcal{S}$ to the impulse δ is called **impulse response** and is denoted by g, i.e. $g := \mathcal{G}[\delta]$.

$\square$

Remark 3.1: It is clear from definition (3.2) that the impulse δ is a signal of unity amplitude applied at time zero and g is the response of the system to that signal. According to the definition of the indeterminate (def. 2.2) and to eqn. (2.2), $\delta d^k = d^k$ may be considered as an impulse applied at time k at the input of the discrete-time system. The response to d^k is denoted by $g^{(k)}$ and is the impulse response of the system $\mathcal{S}$ to an impulse applied at time k.

□

Now, a **linear system** is a system for which the principle of superposition is valid. Therefore

Definition 3.3: A discrete-time system $\mathcal{S}$ is called **linear** if for every u_1, $u_2 \in \mathcal{U}$, and a_1, $a_2 \in \mathbb{R}$, we have

$$\mathcal{S}[a_1 u_1 + a_2 u_2] = a_1 \mathcal{S}[u_1] + a_2 \mathcal{S}[u_2] \tag{3.2}$$

□

If the system $\mathcal{S}$ is *'smooth'* enough eqn. (3.2) can be extended for any number of inputs u_1, u_2, ... $\in \mathbb{R}\langle d\rangle$ and any constants a_1, a_2, ... $\in \mathbb{R}$ [Kai., 1]. By using the formal Laurent series representation for the input sequence, we have that

$$y = \mathcal{S}[u] = \mathcal{S}[\textstyle\sum_{k=p}^{\infty} u_k d^k] \tag{3.3}$$

and for a **linear** system eqn. (3.3) becomes

$$y = \textstyle\sum_{k=p}^{\infty} u_k \mathcal{S}[d^k] = \sum_{k=p}^{\infty} u_k g^{(k)} \tag{3.4}$$

Definition 3.4: A discrete-time system is called **time-invariant** if $\forall\ u \in \mathcal{U}$ $\mathcal{S}[d^k u] = d^k \mathcal{S}[u]$.

□

According to remark (3.1), definition (3.4), and eqn. (3.4), the output y of a linear, time-invariant (LTI), discrete-time system is given by

$$y = \textstyle\sum_{k=p}^{\infty} u_k \mathcal{S}[d^k] = \sum_{k=p}^{\infty} u_k \mathcal{S}[d^k \delta] = \sum_{k=p}^{\infty} u_k d^k \mathcal{S}[\delta] =$$

$$= \textstyle\sum_{k=p}^{\infty} u_k d^k g = \sum_{k=p}^{\infty} u_k d^k \sum_{l=r}^{\infty} g_l d^l \quad \text{i.e.}$$

$$y = g*u \quad \text{or} \quad y_j = \sum_{k+l=j} u_k g_l, \quad j \geq p+r \tag{3.5}$$

Therefore, for a LTI discrete-time system the system operator $\mathcal{G}$ can be replaced by its impulse response g and the convolutory multiplication. In the multivariable case, where there are m inputs and ℓ outputs, eqn. (3.4) becomes

$$y_i = \sum_{j=1}^{m} g_{ij}*u_j \ , \quad i = 1,\ldots,\ell$$

$$\text{or} \qquad \underline{y} := G*\underline{u} \ , \ G = (g_{ij}) \tag{3.6}$$

where g_{ij} is the impulse response at the ith output due to an impulse at the jth input. We have then the following definition.

Definition 3.5: A **linear, time-invariant, discrete-time** system $\mathcal{S}$ is a set $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$ where $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$ is the **discrete time**, $\mathcal{U} \subseteq \mathbb{R}^m\langle d\rangle$ is the **input space**, $\mathcal{Y} \subseteq \mathbb{R}^l\langle d\rangle$ is the **output space**, and $G \in \mathbb{R}_{lm}\langle d\rangle$ is a **linear map**, the **impulse response matrix,** from $\mathcal{U}$ to $\mathcal{Y}$ such that $\forall\ \underline{u} \in \mathcal{U}\ \exists\ \underline{y} \in \mathcal{Y} : \underline{y} = G*\underline{u}$.

□

Remark 3.2: It is clear from eqn. (3.5) that a linear time-invariant discrete-time system is a system with memory; that is the output at time k depends on the input applied before and/or after k. In fact, this is true for any discrete-time system described by the input-output relationship of the form of eqn. (3.1). Hence, an output $\underline{y}$ is uniquely determined for $k \geq k_0$ if the input is not only known for $k \geq k_0$ but also if its time history is known for $k < k_0$. This is the case with definitions (3.1) and (3.5) where the input space $\mathcal{U}$ is a subspace of the space of infinite sequences $\mathbb{R}^l\langle d\rangle$ and any $\underline{u} \in \mathcal{U}$ is well defined $\forall\ k \in \mathbb{Z} \cup \{\infty\}$ since $\underline{u}_k=0\ \forall\ k < \tau(\underline{u})$. Therefore, eqns. (3.1) and (3.5), or (3.6) uniquely define the output of the system by its input. If this is not the case, i.e. $\underline{u}_k$ is not known for $k < k_0$, then the system must be assumed **relaxed** or **at rest** at time k_0, and that the output is excited solely and uniquely by the input applied thereafter, or the so-called **initial conditions** have to be known, i.e. the effect of the input history on the output for $k \geq k_0$.

□

3.2.2 Causality and stability

We have seen in the previous section that the output at time k of a discrete-time system depends on the input applied before and/or after k. A more restricted class of linear time-invariant systems that arise naturally and are of practical importance is that of **causal** systems.

Definition 3.6: A discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},\mathcal{G})$ is **causal** if the output at any time k_0 depends on the input for $k \leq k_0$.

□

Due to eqns. (3.5) and (3.6), the causality conditions for a linear time-invariant discrete-time system are given by the following well known theorem.

Theorem 3.1: A linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$ is **causal**, if and only if $\tau(G) \geq 0$.

□

Another class of discrete-time systems of particular importance, is that of **stable** systems.

Definition 3.7: A discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},\mathcal{G})$ is **bounded input bounded output (BIBO) stable** or **externally stable** if for any bounded input the output is bounded.

□

The conditions for stability for a linear time-invariant discrete-time system are given by the next theorem [Opp., 1].

Theorem 3.2: A linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$, $\mathcal{U} \subseteq \mathbb{R}^m\langle d\rangle$, $\mathcal{Y} \subseteq \mathbb{R}^l\langle d\rangle$, is **BIBO stable**, if and only if

$$\sum_{k=\tau_{ij}}^{\infty} |(g_{ij})_k| < \infty \qquad i=1,\dots,\ell,\ j=1,\dots,m \tag{3.7}$$

where $G = (g_{ij})$, and $\tau_{ij} = \tau(g_{ij})$.

□

Consider now the discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},d,*)$, i.e. the system with impulse response $g = d$. Then, for any input

$$u = u_p d^p + u_{p+1} d^{p+1} + u_{p+2} d^{p+2} + \cdots$$

with order $\tau(u) = p$, the output $y = d*u$ is a sequence

$$y = u_p d^{p+1} + u_{p+1} d^{p+2} + u_{p+2} d^{p+3} + \cdots$$

with order $\tau(y) = p + 1$. Therefore, what applies as input at time k appears at the output at time $k + 1$ (fig. 3.2). For this reason the indeterminate d will be called alternatively the **delay sequence** or the **delay operator** and the sequence $z = d^{-1} = \{1;0,\dots\}$ will be denoted as the **advance sequence**, or the **advance operator**.

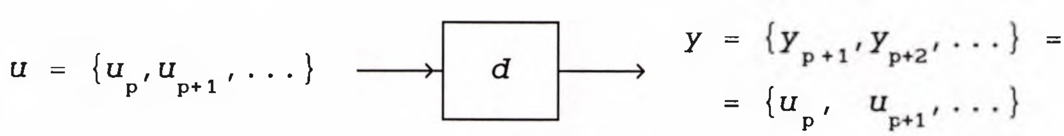

Figure (3.2): The delay operator

3.2.3 Lumped linear time-invariant discrete-time systems

As it was mentioned previously, LTI systems are those whose behaviour is characterized by the impulse response matrix G. For a multivariable system with m inputs and ℓ outputs, G can be any sequential matrix in one indeterminate d over $\mathbb{R}$, i.e. $G \in \mathbb{R}_{lm}\langle d\rangle$. However, the systems we study in this thesis are a subclass of linear time-invariant systems, namely the lumped systems, where the impulse response matrix is a recurrent, or rational sequential matrix.

Definition 3.8: A linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$ with m inputs and ℓ outputs is called **lumped**, if $G \in \mathbb{R}_{lm}(d)$.

$\square$

Causal and stable lumped LTI systems can be classified accordingly. Taking into account theorems (3.1) and (3.2), and the definition of causal and stable sequential matrices we have the following results.

Corollary 3.1: A lumped linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$ with $\mathcal{U} \subseteq \mathbb{R}^{m}\langle d\rangle$, $\mathcal{Y} \subseteq \mathbb{R}^{l}\langle d\rangle$, is **causal** if and only if G is a **causal** sequential matrix, i.e. $G \in \mathbb{R}^{0}_{lm}(d)$.

$\square$

Corollary 3.2: A causal lumped LTI system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,*)$ with $\mathcal{U} \subseteq \mathbb{R}^{m}\langle d\rangle$, $\mathcal{Y} \subseteq \mathbb{R}^{l}\langle d\rangle$, is **BIBO stable** if and only if G is a **stable** sequential matrix, i.e. $G \in \mathbb{R}^{+}_{lm}(d)$.

$\square$

Notation. From now on we will refer to **lumped linear** systems as **linear** systems, since we deal only with this class of systems in this thesis. Therefore by **linear** systems we mean systems having a **rational** sequential matrix as impulse response matrix. Also, we drop the convolutory multiplication symbol from the definition of system $\mathcal{S}$ which is not necessary anyway, when we use the formal Laurent series representation of the signals involved.

Rational sequential matrices can be expressed as polynomial matrix fractions and according to corollaries (3.1) and (3.2) and propositions (2.12) and (2.13), we have the following characterization of causal and stable linear systems.

Theorem 3.3: Consider a linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G)$, and let $(\tilde{D}(d), \tilde{N}(d))$, $(N(d), D(d))$ be left and right coprime polynomial MFDs of $G(d)$. Then $\mathcal{S}$ is

a. **causal** if and only if $\tilde{D}(d)$, or $D(d)$ is **causal**, i.e. $\det\tilde{D}(0) \neq 0$, or $\det D(0) \neq 0$

b. **stable** if and only if $\tilde{D}(d)$, or $D(d)$ is **stable**, i.e. the roots of $\det\tilde{D}(d)$, or $\det D(d)$ lie outside the closed unit disc $\mathbb{D}[0,1]$.

$\square$

We define now the transfer function matrix of a system.

Definition 3.9: Let $G = \{G_r, G_{r+1},\dots,G_k,\dots\}$, $G_k \in \mathbb{R}^{1\times m}$ be the impulse response matrix of a LTI system $\mathscr{S}$. We call the z-Transform of G the **transfer function matrix** $\tilde{G}(z)$ of $\mathscr{S}$, i.e.

$$\tilde{G}(z) = \sum_{k=r}^{\infty} G_k z^{-k}, \quad G_k \in \mathbb{R}^{1\times m}, \; z \in \mathbb{C} \tag{3.8}$$

$\square$

Remark 3.3: If we use the notation $z = d^{-1}$, then from the definition of the impulse response matrix $G(d)$ and the transfer function matrix $G(z)$ of a system $\mathscr{S}$, we have that

$$G(z^{-1}) = \tilde{G}(z)$$

Both $G(d)$ and $\tilde{G}(z)$ can be treated as sequential matrices or as rational functions. In the same way z can be either a complex variable and $z = d^{-1} \in \mathbb{C}$ represents a bilinear transformation, or a sequence $z = d^{-1} = \{1;0,\dots,0,\dots\}$ and can be an indeterminate like d but of different nature (advance operator, see fig. (3.2) and discussion there).

We use the notion **impulse response matrix** for $G(d)$ to stress out the **sequential** nature of $G(d)$, and **transfer function matrix** for $\tilde{G}(z)$ to distinguish the **functional** aspect of $\tilde{G}(z)$. This corresponds more to reality since the **delay operator** d is physically realizable, but of course $G(d)$ and $\tilde{G}(z)$ can be interpreted either way, i.e. as sequences or functions. In some instances when we want to stress the functional nature of both $G(d)$ and $\tilde{G}(z)$ we denote them as ***d*-transfer functions** and **z-transfer functions** respectively.

$\square$

Consider now the case of a SISO linear time-invariant system with impulse response the rational sequence $g(d)$. Then $g(d)$ can be expressed as a formal rational Laurent series over $\mathbb{R}$ and can be represented by a coprime polynomial fraction, i.e.

$$g(d) = g_l d^l + g_{l+1} d^{l+1} + \cdots = \frac{b_0 + b_1 d + \cdots + b_m d^m}{a_0 + a_1 d + \cdots + a_n d^n} = \frac{b(d)}{a(d)} \qquad (3.9)$$

where $a_n, b_m \neq 0$. By invoking the isomorphism between rational fractions and rational functions, $g(d)$ can be treated either as a formal series with one indeterminate d over $\mathbb{R}$, or as a function of the complex variable d. Series (3.9) can represent the behaviour of a SISO linear time-invariant system either way.

If series (3.9) is formal and causal, it represents a causal rational sequence $\{ g_l, g_{l+1}, \ldots, g_k, \ldots \}$ which can be regarded as the impulse response of a causal LTI discrete-time system. If series (3.9) is regarded as a causal rational function of d, then it can be considered as the transfer function of a linear causal discrete-time system. This becomes clear if we perform the bilinear transformation $z^{-1} = d$ in the series (3.9), i.e.

$$\tilde{g}(z) = g_l z^{-l} + g_{l+1} z^{-l-1} + \cdots = z^{n-m} \frac{b_0 z^m + \cdots + b_m}{a_0 z^n + \cdots + a_n} = g(z^{-1}) \qquad (3.10)$$

Then, $\tilde{g}(z)$ is the z-Transform of the causal impulse response $\{g_k\}$ and therefore a transfer function of a causal discrete-time system. In fact, because $a_0 = a(0) \neq 0$ and $a(d)$, $b(d)$ are coprime polynomials, it can be easily shown that $\tilde{g}(z)$ is a proper rational function in z.

Similarly, if (3.9) is formal and stable, it represents an absolutely summable sequence $\{g_k\}$ which can be regarded as the impulse response of a BIBO stable discrete-time system. On the other hand, if series (3.9) is a stable rational function of the variable d, it can be considered as a transfer function of a BIBO stable discrete-time system. Indeed, $\tilde{g}(z)$ represents the z-Transform of an absolutely summable impulse response $\{g_k\}$ and can be regarded as a transfer function of a BIBO stable system. In fact, since the roots of the denominator polynomial $a(d)$ in (3.9) lie outside the closed unit disc $\mathbb{D}[0,1]$, the poles of $\tilde{g}(z)$ lie inside the open unit disc $\mathbb{D}[0,1)$ which is an equivalent condition for $\tilde{g}(z)$ to be a transfer function of a causal BIBO stable discrete-time system.

Finally, taking into account the delay effect of the indeterminate d, it can be easily shown that the input-output behaviour of linear time-invariant discrete-time systems can be described by a difference equation between the input and output sequences [Opp., 1], [Jur., 1]. But before we pursue that further and generalize the above discussion to the MIMO case, we define the McMillan degree of a discrete linear system.

3.2.4 McMillan degree, relationship between $G(d)$ and $\tilde{G}(z)$

It is known (theorem 3.3), that causality and stability of a system can be characterized by the location of the poles of its impulse response matrix. In fact, the poles of the impulse response matrix do not account only for that; they are mainly responsible for the dynamic behaviour of the system. For this reason we identify them as a system property as follows.

Definition 3.10: Let $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G)$ be a LTI discrete-time system. We define the pole-polynomial of $G(d)$ over $\mathbb{R}[d]$ as the **pole-polynomial** of system $\mathcal{S}$ and we denote it by $p_{\mathcal{S}}(d)$, i.e.

$$p_{\mathcal{S}}(d) := p_G(d) \tag{3.11}$$

□

From the definition of the pole-polynomial of a rational matrix $G(d)$ and remark (2.17) we have the following corollary.

Corollary 3.3: Consider a linear time-invariant discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G)$, and let $(\tilde{D}(d), \tilde{N}(d))$, $(N(d), D(d))$ be left and right coprime polynomial MFDs of $G(d)$. Then

$$p_{\mathcal{S}}(d) = c_1 \det\tilde{D}(d) = c_2 \det D(d) \qquad c_1, c_2 \in \mathbb{R}-\{0\} \tag{3.12}$$

□

Definition 3.11: Let $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G)$ be a LTI discrete-time system. We define the McMillan degree of $G(d)$ as the **McMillan degree** of system $\mathcal{S}$ and we denote it by $\delta_{\mathcal{M}}(\mathcal{S})$, i.e.

$$\delta_{\mathcal{M}}(\mathcal{S}) := \delta_{\mathcal{M}}(G) \tag{3.13}$$

□

We recall (remark 2.16), that the McMillan degree $\delta_{\mathcal{M}}(G)$ of a rational matrix $G(d)$ expresses the total number of finite and infinite poles (multiplicities included) of $G(d)$ and can be found from the Smith-McMillan forms of $G(d)$ over $\mathbb{R}[d]$ and $\mathbb{R}_{pr}(d)$ respectively, i.e.

$$\delta_{\mathcal{M}}(G) := \delta_{\mathcal{M}}^{f}(G) + \delta_{\mathcal{M}}^{\infty}(G)$$

We derive now an alternative characterization of $\delta_{\mathcal{M}}(G)$ based on properties of $G(d)$ without resorting to the aforementioned decomposition. As far as we are aware, this approach, though not very formally proven, was first given by Kucera [Kuc., 3]. Later on Gevers [Gev., 1] and Janssen [Jan., 1] derived the same main result. Here we give a proof based on valuation theory. It is close to that given by Janssen but more elegant (see also [Kar., 3]). Before we present the main theorem we give some preliminary results.

The next proposition generalizes to the matrix case the definition of valuation at $d = \infty$ of a scalar function.

Proposition 3.1 [Var., 3]: Let $G(d) \in \mathbb{R}^{l\times m}[d]$, $\ell \geq m$ and $(N(d), D(d))$ a right polynomial MFD of $G(d)$ not necessarily coprime, i.e. $G(d) = N(d)D^{-1}(d)$. Then

$$\upsilon_{\infty}(G) = \partial(D) - \partial(N) \tag{3.14}$$

□

Consider now an $\ell \times m$ rational matrix $G(d)$. We define

$$T_G^r(d) := \begin{bmatrix} G(d) \\ I \end{bmatrix} \in \mathbb{R}^{(l+m)\times m}(d) \tag{3.15}$$

and refer to that as the **right composite matrix** of $G(d)$ (T_G^l is defined similarly). Then we have the following result which is a generalization of that given by Forney [For., 1].

Proposition 3.2: Let $\mathcal{R}$ be a quotient ring of $\mathbb{R}[d]$ defined over $\mathbb{C}_e - \Omega$, where Ω is a closed region in the complex plane, such that $\forall\ d \in \Omega \Rightarrow \bar{d} \in \Omega$ and not all the real axis belongs to Ω. Then for every $G(d) \in \mathbb{R}^{1\mathrm{xm}}(d)$, represented by an $\mathcal{R}$-coprime MFD $G(d) = N(d)D^{-1}(d)$, $N(d) \in \mathcal{R}^{1\mathrm{xm}}$ $D(d) \in \mathcal{R}^{\mathrm{mxm}}$, the composite matrix $T_G^{\mathrm{r}}(d)$ is given by the following $\mathcal{R}$-coprime MFD

$$T_G^{\mathrm{r}}(d) := \begin{bmatrix} G(d) \\ I \end{bmatrix} = \begin{bmatrix} N(d) \\ D(d) \end{bmatrix} D^{-1}(d) \tag{3.16}$$

Also, $T_G^{\mathrm{r}}(d)$ has no Ω-zeros and the same Ω-poles as $G(d)$, i.e.

$$\delta_{\mathcal{M}}^{\Omega}(T_G^{\mathrm{r}}) = \delta_{\mathcal{M}}^{\Omega}(G) \tag{3.17}$$

where $\delta_{\mathcal{M}}^{\Omega}(T_G^{\mathrm{r}})$ is the Ω-McMillan degree of $T_G^{\mathrm{r}}(d)$.

<u>*Proof*</u>. It is easy to prove (3.16) by substituting $G(d)$ by its $\mathcal{R}$-coprime MFD. Then, we must show that $([N^{\mathrm{t}}(d)\ \ D^{\mathrm{t}}(d)]^{\mathrm{t}}$, $D(d))$ are right $\mathcal{R}$-coprime. Indeed

$$\begin{bmatrix} N(d) \\ D(d) \\ D(d) \end{bmatrix} E_{\mathcal{R}}^{\mathrm{r}} \begin{bmatrix} N(d) \\ D(d) \\ O \end{bmatrix}$$

and since $(N(d),\ D(d))$ are $\mathcal{R}$-coprime, the first part of the above equivalence relationship is $\mathcal{R}$-coprime too. Also, since $[N^{\mathrm{t}}(d)\ \ D^{\mathrm{t}}(d)]^{\mathrm{t}}$ is the numerator of the $\mathcal{R}$-coprime MFD of T_G^{r} it follows (remarks 2.14 and 2.18) that $T_G^{\mathrm{r}}(d)$ has no Ω-zeros. Clearly, because $D(d)$ is common denominator to both $G(d)$ and $T_G^{\mathrm{r}}(d)$,

$$\delta_{\mathcal{M}}^{\Omega}(T_G^{\mathrm{r}}) = \delta_{\mathcal{M}}^{\Omega}(G)$$

□

Corollary 3.4: Let $G(d) \in \mathbb{R}^{1\mathrm{xm}}(d)$ and $T_G^{\mathrm{r}}(d)$ be the composite matrix of $G(d)$. Then, $T_G^{\mathrm{r}}(d)$ has no zeros in the extended complex plane $\mathbb{C}_e$ and the $\mathbb{C}_e$-poles of $T_G^{\mathrm{r}}(d)$ are the $\mathbb{C}_e$-poles of $G(d)$, i.e.

$$\delta_{\mathcal{M}}(T_G^{\mathrm{r}}) = \delta_{\mathcal{M}}(G) \tag{3.18}$$

Proof. From proposition (3.2) with $\mathcal{R} = \mathbb{R}[d]$, we have that $T^r_G(d)$ has no finite zeros and the same finite poles as $G(d)$. If $\mathcal{R} = \mathbb{R}_{pr}(d)$, then $T^r_G(d)$ has no infinite zeros and the same infinite poles as $G(d)$. Therefore, $T^r_G(d)$ has no $\mathbb{C}_e$-zeros and the same McMillan degree as $G(d)$.

□

We state now the main result of this subsection.

Theorem 3.4 [Jan., 1], [Kar., 3]: Let $G(d) \in \mathbb{R}^{1\mathrm{x}m}(d)$, $G(d) = N(d)D^{-1}(d)$ be any right coprime MFD over $\mathbb{R}[d]$ and $R_G(d) := [N^t(d)\ \ D^t(d)]^t$. Then

$$\delta_{\mathcal{M}}(G) := \delta^f_{\mathcal{M}}(G) + \delta^{\infty}_{\mathcal{M}}(G) = \partial(R_G(d)) \tag{3.19}$$

Proof. Consider the composite matrix $T^r_G(d)$ of $G(d)$. Then,

$$T^r_G(d) := \begin{bmatrix} G(d) \\ I \end{bmatrix} = \begin{bmatrix} N(d) \\ D(d) \end{bmatrix} D^{-1}(d) = R_G(d)D^{-1}(d) \tag{3.20}$$

Also (remark 2.15), $\upsilon_{\infty}(T^r_G) = \{\#\ \infty \text{ zeros}\} - \{\#\ \infty \text{ poles}\}$ and since $T^r_G(d)$ has no infinite zeros (corollary 3.4), we have

$$\upsilon_{\infty}(T^r_G) = -\{\#\ \infty \text{ poles of } T^r_G(d)\} := -\delta^{\infty}_{\mathcal{M}}(T^r_G)$$

which again by corollary (3.4) leads to

$$\upsilon_{\infty}(T^r_G) = -\delta^{\infty}_{\mathcal{M}}(G) \tag{3.21}$$

By proposition (3.1) we have that

$$\upsilon_{\infty}(T^r_G) = \partial(D) - \partial(R_G) \tag{3.22}$$

and given that $\partial(D) = \delta^f_{\mathcal{M}}(G)$ (corollary 3.3), eqns (3.21) and (3.22) lead to

$$-\delta^{\infty}_{\mathcal{M}}(G) = \delta^f_{\mathcal{M}}(G) - \partial(R_G(d)) \quad \text{or}$$

$$\delta_{\mathcal{M}}(G) := \delta^f_{\mathcal{M}}(G) + \delta^{\infty}_{\mathcal{M}}(G) = \partial(R_G(d))$$

□

Note that the degree of the matrix $R_G(d) := [N^t(d)\ \ D^t(d)]^t$ defined by a coprime MFD of $G(d)$ is invariant of the MFD and known as the Forney dynamical order of $G(d)$ (def. 2.22). Therefore we have the following corollary.

Corollary 3.5: Let $\delta_F(G)$ be the Forney dynamical order of $G(d)$. Then,

$$\delta_{\mathcal{M}}(\mathcal{S}) := \delta_{\mathcal{M}}(G) = \partial(R_G(d)) = \delta_F(G) \tag{3.23}$$

□

We consider now the relationship between the MFDs of the d- and z-transfer functions $G(d)$ and $\tilde{G}(z)$ of a system $\mathcal{S}$ and their corresponding McMillan degrees as well. First we examine the problem of constructing coprime MFDs for $\tilde{G}(z)$ from coprime MFDs of $G(d)$.

Definition 3.12: Consider the composite matrix $R_G(d) := [N^t(d) \quad D^t(d)]^t = [\underline{r}_{G1}, \ldots, \underline{r}_{Gm}]$ associated with a right coprime polynomial MFD of the $\ell \times m$ rational matrix $G(d)$ and let μ_i $i = 1, \ldots, m$ be the right minimal indices of $G(d)$. If $R_G(d)$ is column reduced and its columns are ordered in descending degree order, i.e.

$$\partial(\underline{r}_{G1}) := \mu_1 \geq \partial(\underline{r}_{G2}) := \mu_2 \geq \cdots \geq \partial(\underline{r}_{Gm}) := \mu_m$$

then the MFD is called **normal**.

□

If we apply the bilinear transformation $d = z^{-1}$ on $R_G(d)$, we have for each column of $R_G(d)$

$$\underline{r}_{Gi}(d) = \underline{r}_{Gi}(z^{-1}) = z^{-\mu_i}\tilde{\underline{r}}_{Gi}(z), \quad \tilde{\underline{r}}_{Gi}(z) \in \mathbb{R}^{l+m}[z] \tag{3.24}$$

Therefore

$$R_G(d) = R_G(z^{-1}) = [\tilde{\underline{r}}_{G1}(z), \ldots, \tilde{r}_{Gm}(z)]\mathrm{diag}\{z^{-\mu_1}, \ldots, z^{-\mu_m}\} \quad \therefore$$

$$\therefore \quad \begin{bmatrix} N(d) \\ D(d) \end{bmatrix} = \begin{bmatrix} \tilde{N}(z) \\ \tilde{D}(z) \end{bmatrix} S(z) \tag{3.25}$$

Definition 3.13: The pair $(\tilde{N}(z), \tilde{D}(z))$ constructed as in eqn. (3.25) will be called the **normal dual** of $(N(d), D(d))$.

□

The properties of the normal dual of a normal right coprime MFD of $G(d)$ have been studied by Kucera [Kuc., 3] and Wolovich and Elliott [Wol., 2]. Here, in the following proposition we relax some of their assumptions.

Proposition 3.3: Let $(N(d),\ D(d))$ be a normal right MFD of the rational matrix $G(d)$ and $(\tilde{N}(z),\ \tilde{D}(z))$ be the corresponding normal dual. Then,

a. $\tilde{G}(z) := G(z^{-1}) = \tilde{N}(z)\tilde{D}^{-1}(z)$

b. $(\tilde{N}(z),\ \tilde{D}(z))$ is a **normal right MFD** of $\tilde{G}(z)$ with the same right minimal indices as $G(d)$.

Proof.

a. From (3.25) we have $N(d) = \tilde{N}(z)S(z)$, $D(d) = \tilde{D}(z)S(z)$. So $\tilde{G}(z) := G(z^{-1}) = G(d) = N(d)D^{-1}(d) = \tilde{N}(z)S(z)S^{-1}(z)\tilde{D}^{-1}(z)$ $\therefore$

$$\tilde{G}(z) := G(z^{-1}) = \tilde{N}(z)\tilde{D}^{-1}(z) \tag{3.26}$$

b. From (3.26) it is clear that $(\tilde{N}(z),\ \tilde{D}(z))$ is a right MFD of $\tilde{G}(z)$; for the later MFD to be normal, it has to be shown that it is coprime, column reduced and ordered. Note, that $\forall\ z \in \mathbb{C} - \{0\}$, $S(z)$ has full rank (eqn. 3.25) and thus

$$\forall\ z \in \mathbb{C} - \{0\},\quad \rho([N^t(d)\ \ D^t(d)]^t) = \rho([\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t)$$

Therefore, $[\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t$ has no zeros in $\mathbb{C} - \{0\}$. Consider now $[\tilde{N}^t(0)\ \ \tilde{D}^t(0)]^t$. From the definition of the high coefficient matrix and eqn. (3.25) we have

$$R_{G,h} = \underset{d\longrightarrow\infty}{limit} \left\{ \begin{bmatrix} N(d) \\ D(d) \end{bmatrix} \mathrm{diag}\{d^{-\mu_1},\ldots,d^{-\mu_m}\} \right\} =$$

$$= \underset{z\longrightarrow 0}{limit} \left\{ \begin{bmatrix} \tilde{N}(z) \\ \tilde{D}(z) \end{bmatrix} \right\} = \begin{bmatrix} \tilde{N}(0) \\ \tilde{D}(0) \end{bmatrix}$$

and since $R_G(d)$ is column reduced, $R_{G,h}$ has full column rank, i.e. $R_{\tilde{G}} := [\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t$ has no zeros at $z = 0$. Similarly, $R_G(0) = \{[\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t\}_h$ and so $\{[\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t\}_h$ has full column rank. Hence

$$\rho(R_{\tilde{G}}(z)) = \rho([\tilde{N}^t(z)\ \ \tilde{D}^t(z)]^t) = m \quad \forall\ z \in \mathbb{C}_e$$

For $(\tilde{N}(z),\ \tilde{D}(z))$ to be normal, the column degrees of $R_{\tilde{G}}(z)$ have to be ordered in descending order. Indeed, since $R_{G}(0)$ has full column rank $\tau(\underline{r}_{Gi}(d)) = 0$, $i = 1,\ldots,m$ and from eqn. (3.24) we have

$$\partial(\tilde{\underline{r}}_{Gi}(z)) = \partial(\underline{r}_{Gi}(d)) := \mu_i, \quad i = 1,\ldots,m$$

□

According to theorem (3.4) and proposition (3.3) we can prove the following results which express no more than the obvious property of invariance of the McMillan degree under bilinear transformations.

Corollary 3.6 [Jan., 1], [Kar., 3]: Let $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G)$ be a LTI discrete-time system and $(N(d),\ D(d))$, $(\tilde{N}(z),\ \tilde{D}(z))$ be two right coprime MFDs of $G(d)$ and $\tilde{G}(z)$ respectively with corresponding composite matrices $R_G(d)$ and $R_{\tilde{G}}(z)$. Then,

$$\delta_{\mathcal{M}}(\mathcal{S}) := \delta_{\mathcal{M}}(G(d)) = \delta_{\mathcal{M}}(\tilde{G}(z)) \quad \text{or}$$
$$\delta_{\mathcal{M}}(\mathcal{S}) = \partial(R_G(d)) = \partial(R_{\tilde{G}}(z))$$

□

Remark 3.4: Causal rational matrices $G(d) \in \mathbb{R}^{0}_{1m}(d)$ {no poles at zero in the d-plane}, correspond to proper rational matrices $\tilde{G}(z) \in \mathbb{R}^{1\times m}_{pr}(z)$ {no poles at infinity in the z-plane} and stable rational matrices $G(d) \in \mathbb{R}^{+}_{1m}(d)$ {poles outside the closed unit disc $\mathbb{D}[0,1]$}, correspond to proper and stable rational matrices $\tilde{G}(z) \in \mathbb{R}^{1\times m}_{ps}(z)$ {poles inside the open unit disc $\mathbb{D}[0,1)$}.

The advantage of the d-plane description is that the forbidden region for stability and causality is rather simple, that is the closed unit disc of the d-plane. Testing for causality is similar in nature to testing for stability since for causality we have to test the existence of a pole at $d = 0$. Testing whether the pole-polynomial $p_{\mathcal{S}}(d)$ has no roots in the closed unit disc provides a criterion for both causality and BIBO stability.

□

3.2.5 Linear constant-coefficient difference equations

We close this section about the input-output description of discrete-time systems by referring to their representation as systems of linear constant-coefficient difference equations. This representation is a straightforward consequence of the impulse response description of discrete-time systems and it has not only theoretical significance but it may as well serve as a computational realization for them.

Consider first a causal linear time-invariant SISO discrete-time system $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},g(d))$ where $g(d) = b(d)/a(d)$ with $b(d)$, $a(d)$ coprime polynomials and $a(0) \neq 0$. Then $\forall\ u \in \mathcal{U}\ \ \exists\ y \in \mathcal{Y}$ such that

$$y(d) = \frac{b(d)}{a(d)}\, u(d) \quad \text{or} \quad a(d)y(d) = b(d)u(d)$$

and if $\partial(a(d)) = n$ and $\partial(b(d)) = m$, we have

$$(a_0+a_1d+\cdots+a_nd^n)\,y(d) = (b_0+b_1d+\cdots+b_md^m)\,u(d) \tag{3.27}$$

Taking into account the delay effect of the d-operator, eqn. (3.27) becomes

$$\begin{aligned} &a_0y_k+a_1y_{k-1}+\cdots+a_ny_{k-n} = \\ &\qquad = b_0u_k+b_1u_{k-1}+\cdots+b_mu_{k-m} \quad \forall\ k \in \mathcal{T} \end{aligned} \tag{3.28}$$

We then distinguish the following two cases.

a. $\mathcal{S}$ is at rest at $k = -\infty$ and a sequence $u \in \mathbb{R}\langle d\rangle$ applies at its input thereafter. So, u is completely known $\forall\ k \in \mathcal{T}$, and since $\mathcal{S}$ is causal y is described completely by the difference equation (3.28) $\forall\ k \in \mathcal{T}$.

b. $\mathcal{S}$ is at rest at $k = -\infty$ but the input sequence is not known for $k < k_0$, for example $k_0 = 0$. Then it can be easily shown that eqn. (3.28) describes y uniquely for $k \geq r := \max\{m,n\} = \delta_{\mathcal{M}}(\mathcal{S})$ when the vector of the ***initial conditions*** at $k = 0$, $\underline{y}_{in} = [y_0,\ldots,y_{r-1}]^t \in \mathbb{R}^r$ is known.

Example 3.1: Consider the causal LTI SISO system $\mathscr{S}(\mathcal{T},\mathcal{U},\mathcal{Y},g)$ where

$$g(d) = \frac{y(d)}{u(d)} = \frac{b_0+b_1d+b_2d^2}{a_0+a_1d}, \quad a_0 \neq 0$$

Then u, y satisfy the following difference equation.

$$a_0y_k+a_1y_{k-1} = b_0u_k+b_1u_{k-1}+b_2u_{k-2} \tag{3.29}$$

a. Suppose $\mathscr{S}$ is at rest at $k = -\infty$ and $u \in \mathbb{R}\langle d\rangle$ with order $\tau(u)$. Then $y \in \mathbb{R}\langle d\rangle$ with $\tau(y) = \tau(u)+\tau(b)-\tau(a)$ and if $b_0 \neq 0$ $\tau(b) = \tau(a) = 0$, i.e. $\tau(y) = \tau(u)$. Therefore,

$$y_k = \begin{cases} 0, \ \forall\ k \in \mathcal{T} : k < \tau(u) \\ a_0^{-1}(b_0u_k+b_1u_{k-1}+b_2u_{k-2}-a_1y_{k-1}), \ \forall\ k \in \mathcal{T} : k \geq \tau(u) \end{cases}$$

b. Suppose $\mathscr{S}$ is at rest at $k = -\infty$ and u is not known for $k < 0$, i.e. $u \in \mathbb{R}[[d]]$. Then $\delta_{\mathcal{M}}(\mathscr{S}) = \max\{\partial(a),\partial(b)\} = 2$, and given the initial conditions vector $\underline{y}_{in} = [y_0,y_1]^t$ y_k can be described by eqn. (3.29) for $k \geq 2$, i.e.

$$y_k = \begin{cases} \text{unknown} \ \forall\ k \in \mathcal{T} : k < 0 \\ y_0,\ y_1 \quad \text{for } k = 1,2 \\ a_0^{-1}(b_0u_k+b_1u_{k-1}+b_2u_{k-2}-a_1y_{k-1}), \ \forall\ k \in \mathcal{T} : k \geq 2 \end{cases}$$

□

Using polynomial MFDs for the impulse response matrix, we can easily generalize to the MIMO case [Kai., 1], [Ros., 1], [Var., 1]. To this extend, we have the following definition of a causal LTI discrete-time system.

Definition 3.14: A **lumped causal LTI discrete-time** system $\mathscr{S}$ with m inputs and ℓ outputs is a set $\mathscr{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,\underline{y}_{in})$, where $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$ is the **discrete time**, $\mathcal{U} \subseteq \mathbb{R}^m[[d]]$ is the **input space**, $\mathcal{Y} \subseteq \mathbb{R}^l[[d]]$ is the **output space**, $G \in \mathbb{R}^0_{lm}(d)$ is a **map** from $\mathcal{U}$ to $\mathcal{Y}$ and $\underline{y}_{in} \in \mathbb{R}^r$ is the **initial conditions vector** with $r = \delta_{\mathcal{M}}(\mathscr{S})$ the McMillan degree of $\mathscr{S}$.

□

Remark 3.5: Note that $\underline{y} \in \mathcal{Y}$ can be described uniquely from the initial conditions vector $\underline{u}_{in}$, which accounts for the past information about the system behaviour, and from the input $\underline{u} \in \mathcal{U}$ $\forall$ $k \in \mathcal{T} : k \geq 0$. $\underline{u}_{in}$ is no more than the ***initial state vector*** of the reachable and observable part of the physical system as we will see in the next section when we will discuss the ***state-space***, or the ***internal*** behaviour of the system.

□

3.3 Linear Discrete-Time Systems
State-Space description

In the previous section we have considered discrete-time systems from an input-output point of view describing them by their ***external*** behaviour through their input-output characteristics. A more comprehensive approach one could argue, is the one which takes into account the ***internal*** dynamics of the systems, the so-called **state-space** approach. As it has already been mentioned (remark 3.5), the ***state*** at time k_0 summarizes all the past information of the system so that together with the input after k_0 is all that is needed in order to determine its future behaviour.

Here we give a quick summary of the state-space approach for discrete linear systems and the important concepts of reachability/controllability, observability/constructibility and stability associated with this approach. On these grounds the relationship between the **external** and **internal** description of systems is presented as a final conclusion of this section. The mathematical description of discrete linear systems as well as the concepts of this part can be found in a large number of texts like Barnett [Bar., 1], Kailath [Kai., 1], Kalman Falb and Arbib [Kal., 1], Kucera [Kuc., 2], Padulo and Arbib [Pad., 1], Rosenbrock [Ros., 1], Zadeh and Desoer [Zad., 1] and references therein.

We give now the abstraction of the concept of a discrete-time system from the state-space perspective.

Definition 3.15 [Son., 1]: A **discrete-time** system Σ with m inputs and ℓ outputs is a set $\Sigma(\mathcal{T},\mathcal{X},\mathcal{U},\phi,\mathcal{Y},h)$ where $\mathcal{T}$ is the **discrete time**, i.e. $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$, $\mathcal{X}$ is the **state space**, $\mathcal{U} \subseteq \mathbb{R}^m\langle d\rangle$ is the **input space**, ϕ is a map from $\mathcal{D}_\phi$ to $\mathcal{X}$ called the **transition map** of Σ, which is defined on a subset $\mathcal{D}_\phi$ of

$$\{(s,r,\underline{x},\underline{u}) : r,s \in \mathcal{T},\ r \leq s,\ \underline{x} \in \mathcal{X},\ \underline{u} \in \mathcal{U}_{[r,s)}\}$$

such that the following properties hold:

nontriviality $\forall\ \underline{x} \in \mathcal{X}$, there is at least one pair $r < s$ in $\mathcal{T}$ and some $\underline{u} \in \mathcal{U}_{[r,s)}$ such that $\underline{u}$ **is admissible for** $\underline{x}$, that is, so that $(s,r,\underline{x},\underline{u}) \in \mathcal{D}_\phi$;

restriction If $\underline{u} \in \mathcal{U}_{[r,s)}$ is admissible for $\underline{x}$, then for each $t \in [r,s)$ the restriction $\underline{u}_1 := \underline{u}|_{[r,t)}$ of $\underline{u}$ to the subinterval $[r,t)$ is also admissible for $\underline{x}$ and the restriction $\underline{u}_2 := \underline{u}|_{[t,s)}$ is admissible for $\phi(t,r,\underline{x},\underline{u}_1)$;

semigroup If r, s, t are any three elements of $\mathcal{T}$ so that $r < s < t$, if $\underline{u}_1 \in \mathcal{U}_{[r,s)}$ and $\underline{u}_2 \in \mathcal{U}_{[s,t)}$ and if $\underline{x} \in \mathcal{X}$ so that

$$\phi(s,r,\underline{x},\underline{u}_1) = \underline{x}_1 \text{ and } \phi(t,s,\underline{x}_1,\underline{u}_2) = \underline{x}_2$$

then $\underline{u}=\{\underline{u}_1,\underline{u}_2\}$ is also admissible for $\underline{x}$ and $\phi(t,r,\underline{x},\underline{u}) = \underline{x}_2$;

identity For each $t \in \mathcal{T}$ and each $\underline{x} \in \mathcal{X}$, the empty sequence $\circ \in \mathcal{U}_{[t,t)}$ is admissible for $\underline{x}$ and $\phi(t,t,\underline{x},\circ) = \underline{x}$;

$\mathcal{Y} \subseteq \mathbb{R}^1[d]$ is the **output space** and h is a map from $\mathcal{T} \times \mathcal{X} \times \mathcal{U}$ to $\mathcal{Y}$ called the **readout** or **measurement map**.

□

Definition (3.15) captures the intuitive notion of a system that evolves in time according to the transition rules specified by ϕ. We call $\phi(s,r,\underline{x},\underline{u})$ *the state at time s resulting from starting at time r in state* $\underline{x}$ *and applying the input function* $\underline{u}$ [Son., 1].

3.3.1 Linearity time-invariance and causality

Taking into account definition (3.15) and following similar lines as in section (3.2), a lumped, linear, time-invariant, causal system is the one whose finite dimensional state at time k is a constant linear combination of the state and the input vectors at time $k - 1$, and its output at time k is a linear combination of the state and input vectors at time k. This gives rise to the following definition.

Definition 3.16: A **lumped, linear, time-invariant, causal, discrete-time** system Σ with n states, m inputs and ℓ outputs is a set $\Sigma(\mathcal{T},\mathcal{X},\mathcal{U},\mathcal{Y},A,B,C,D)$ where $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$ is the **discrete time**, $\mathcal{X} \subseteq \mathbb{R}^n\langle d\rangle$ is the **state space**, $\mathcal{U} \subseteq \mathbb{R}^m\langle d\rangle$ is the **input space**, $\mathcal{Y} \subseteq \mathbb{R}^l\langle d\rangle$ is the **output space** and A, B, C, D are real matrices with appropriate dimensions so that $\forall\ \underline{u} \in \mathcal{U}\ \ \exists\ \underline{x} \in \mathcal{X}$ and $\underline{y} \in \mathcal{Y}$ such that

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \quad \text{and} \quad \underline{y}_k = C\underline{x}_k + D\underline{u}_k \qquad \forall\ k \in \mathcal{T} \tag{3.30}$$

□

Note that the recursive equations (3.30) describe uniquely the state $\underline{x}$ and the output $\underline{y}$ of the system Σ assuming that it is at rest at $k = -\infty$. If this is not the case then the initial state $\underline{x}_0$ must be specified for time k_0. To this extend, as in the input-output approach, we may have the following definition.

Definition 3.17: A **lumped, linear, time-invariant, causal, discrete-time** system Σ with n states, m inputs and ℓ outputs is a set $\Sigma(\mathcal{T},\mathcal{X},\mathcal{U},\mathcal{Y},A,B,C,D,\underline{x}_0)$ where $\mathcal{T} = \mathbb{Z} \cup \{\infty\}$ is the **discrete time**, $\mathcal{X} \subseteq \mathbb{R}^n[[d]]$ is the **state space**, $\mathcal{U} \subseteq \mathbb{R}^m[[d]]$ is the **input space**, $\mathcal{Y} \subseteq \mathbb{R}^l[[d]]$ is the **output space**, A, B, C, D are real matrices with appropriate dimensions so that $\forall\ \underline{u} \in \mathcal{U}$, $\exists\ \underline{x} \in \mathcal{X}$ and $\underline{y} \in \mathcal{Y}$ such that

$$\left.\begin{aligned} \underline{x}_{k+1} &= A\underline{x}_k + B\underline{u}_k \qquad (3.31a) \\ \underline{y}_k &= C\underline{x}_k + D\underline{u}_k \qquad (3.31b) \end{aligned}\right\}\quad k \in \mathcal{T} : k \geq 0$$

$\underline{x}_0 \in \mathbb{R}^n$ is the **initial state** vector at $k = 0$ and n, the dimension of the $\mathcal{X}$ space is the **order** of the system.

□

It can be easily shown that the solution to the system equations (3.31) is

$$\underline{x}_k = A^k \underline{x}_0 + A^{k-1} B \underline{u}_0 + \cdots + B \underline{u}_{k-1} \tag{3.32a}$$

$$\underline{y}_k = C \underline{x}_k + D \underline{u}_k \tag{3.32b}$$

for $k = 0,1,\ldots$. Therefore the state and output sequences at time k are uniquely described by the initial state $\underline{x}_0$ and the input sequence up to time k.

The state response of the system (3.31) to the initial state $\underline{x}_0$ is described by the equation

$$\underline{x}_{k+1} = A \underline{x}_k, \qquad k \in \mathcal{T} : k \geq 0 \tag{3.33}$$

and represents the *free*, *unexcited*, or *natural dynamics* of the system. It is called the **zero input response** and is given by

$$\underline{x}_k = A^k \underline{x}_0, \qquad k \geq 0 \tag{3.34}$$

If λ is an *eigenvalue* of A, $\underline{v}_1$ is an *eigenvector* of A associated with λ and $\underline{v}_2,\ldots,\underline{v}_p$ is a chain of so-called *generalized eigenvectors*, then the response of (3.34) to the initial state $\underline{x}_0 = \underline{v}_p$, is

$$\underline{x}_k = \lambda^k \underline{v}_p + \binom{k}{1} \lambda^{k-1} \underline{v}_{p-1} + \cdots + \binom{k}{p-1} \lambda^{k-(p-1)} \underline{v}_1 \tag{3.35}$$

and is a composition of p motions along the generalized eigenvectors of A. These motions are called **modes** of A corresponding to eigenvalue λ. The modes corresponding to $\lambda = 0$, if any, are given by

$$\underline{x}_k = \begin{cases} \underline{v}_{p-1} & \text{for } k = 0,1,\ldots,p-1 \\ 0 & \text{for } k \geq p \end{cases}$$

they *all settle to zero in finite time*, and they are called **finite modes** [Kuc., 2].

The eigenvalues and the modes of A are invariant under $\mathbb{R}$-similarity transformations and they can be considered as system properties according to the following definition.

Definition 3.18: Consider the linear time-invariant causal discrete-time system $\Sigma(\mathcal{T},\mathcal{X},\mathcal{U},\mathcal{Y},A,B,C,D,\underline{x}_0)$. The ***eigenvalues*** of A are called the **eigenvalues** of Σ and the ***modes*** of A are called the **modes** of Σ.

□

Notation. From now on we drop the description of time, state, input and output spaces in the definition of systems. Therefore, $\mathcal{S}(\mathcal{T},\mathcal{U},\mathcal{Y},G,\underline{y}_{in})$ will be denoted by $\mathcal{S}(G,\underline{y}_{in})$ and $\Sigma(\mathcal{T},\mathcal{X},\mathcal{U},\mathcal{Y},A,B,C,D,\underline{x}_0)$ by $\Sigma(A,B,C,D,\underline{x}_0)$.

We investigate now separately the input-state and the state-output characteristics of $\Sigma(A,B,C,D,\underline{x}_0)$ by discussing the notions of reachability/controllability and observability/constructibility correspondingly. These notions were first formally introduced by Kalman [Kal., 2] and they are extensively treated in Kalman Falb and Arbib [Kal., 1] and Kailath [Kai., 1].

3.3.2 Reachability

As it has already been mentioned, reachability and controllability are properties referred to the underlying system $\Sigma(A,B,O,O)$, denoted by $\Sigma_{i-s}(A,B)$ and described by the input-state equation

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \quad k = 0,1,... \tag{3.36}$$

Definition 3.19: The system $\Sigma(A,B,C,D)$, or the pair (A,B) is **reachable from the origin**, or just **reachable**, if and only if for every $k > 0$ any state $\underline{x}_k$ can be reached from the origin in k steps by applying an input sequence $\underline{u}_0,...,\underline{u}_{k-1}$.

□

An important part in the investigation of reachability is played by the so-called kth-reachable spaces that we define next.

Definition 3.20: Consider the system $\Sigma_{i-s}(A,B)$ and define

$$\begin{aligned} \mathcal{R}_0(A,B) &= 0 \\ \mathcal{R}_k(A,B) &= \mathcal{I}m\ [B\ AB\ \cdots\ A^{k-1}B]\ , \quad k = 1,2,\ldots \end{aligned} \tag{3.37}$$

We call $\mathcal{R}_k(A,B)$, $k \geq 0$ the **kth-reachable subspace** of Σ.

□

Remark 3.6: According to (3.32a) with $\underline{x}_0 = 0$, we have

$$\underline{x}_k = A^{k-1}B\underline{u}_0 + \cdots + B\underline{u}_{k-1}$$

Therefore $\mathcal{R}_k(A,B) \subseteq \mathcal{X}$ and consists of the states $\underline{x}_k$ that can be reached from the origin in k steps by applying an input sequence $\underline{u}_0,\ldots,\underline{u}_{k-1}$. From eqn. (3.37) the following property is clear [Mul., 1]

$$\mathcal{R}_{k+1}(A,B) = A\mathcal{R}_k(A,B) + \mathcal{I}m\ B, \qquad k = 0,1,\ldots \tag{3.38}$$

□

We give now the necessary and sufficient conditions for a system Σ to be reachable.

Theorem 3.5 [Bar., 1], [Hau., 1], [Ros, 1]**:** The pair (A,B) is **reachable** if and only if any of the following equivalent conditions holds true.

a. There is a nonnegative integer r such that

$$\mathcal{R}_r(A,B) = \mathbb{R}^n$$

b. There is a nonnegative integer r such that

$$\rho([B\ AB\ \cdots\ A^{r-1}B]) = n.$$

c. $\underline{v}^tB = 0$ and $\underline{v}^tA = \lambda\underline{v}^t$ for some constant λ implies $\underline{v}^t = 0$.

d. The matrices $(zI_n - A,\ B)$ are relatively left prime over $\mathbb{R}[z]$.

□

Definition 3.21: The constant λ in theorem (3.5) is an eigenvalue of A. The modes associated with any such λ are called the **reachable modes** of the system Σ. Accordingly, the modes of Σ that are associated with any eigenvalue that does not satisfy condition (c) of theorem (3.5) are called **unreachable modes**.

□

Reachability Indices

Consider the pair (A,B) and define the integers

$$p_k = \dim \mathcal{R}_k(A,B) - \dim \mathcal{R}_{k-1}(A,B), \quad k = 1,2,... \tag{3.39}$$

Then we have the following important notion.

Definition 3.22: Consider the pair (A,B) and let p_k be as in (3.39) and $\mu_i^{(r)}$ be integers such that for $i = 1,2,...,m$

$$\mu_i^{(r)} := \text{number of } p_i\text{'s greater than or equal to } i$$

Then the integers $\mu_1^{(r)} \geq \mu_2^{(r)} \geq \cdots \geq \mu_m^{(r)}$ are called the **reachability indices** of (A,B).

□

Remark 3.7: It follows from (3.39) that

$$\sum_{i=1}^{m} \mu_i^{(r)} \leq n$$

with equality holding if and only if (A,B) is a reachable pair.

□

3.3.3 Controllability

Another important notion which is related to the input-state pair (A,B) is that of controllability. In this section we give the dynamic characterization of controllability, the criteria for a system to be controllable, and the relation between the two notions of reachability and controllablity.

Definition 3.23: The pair (A,B) is **controllable to the origin**, or just **controllable**, if and only if there is a finite $k > 0$, such that every initial state $\underline{x}_0$ can be steered to the origin in k steps by applying an input sequence $\underline{u}_0, \ldots, \underline{u}_{k-1}$.

□

Definition 3.24: Consider the system $\Sigma_{i-s}(A,B)$ and define

$$\begin{aligned} \mathcal{C}_0(A,B) &= 0 \\ \mathcal{C}_k(A,B) &= \{\underline{x} \in \mathcal{X} \subseteq \mathbb{R}^n : A^k\underline{x} \in \mathcal{R}_k(A,B)\}, \quad k = 1,2,\ldots \end{aligned} \tag{3.40}$$

We call $\mathcal{C}_k(A,B)$, $k \geq 0$ the **kth-controllable subspace** of Σ.

□

Remark 3.8: According to eqn. (3.32a) with $\underline{x}_k = 0$, we have

$$0 = A^k\underline{x}_0 + A^{k-1}B\underline{u}_0 + \cdots + B\underline{u}_{k-1}$$

Therefore $\mathcal{C}_k(A,B) \subseteq \mathcal{X}$ and consists of the initial states $\underline{x}_0$ that can be steered to the origin in k steps by applying an input sequence $\underline{u}_0, \ldots, \underline{u}_{k-1}$. From eqn. (3.40) the following property is clear [Mul., 1]

$$A\mathcal{C}_{k+1}(A,B) \subset \mathcal{C}_k(A,B) + \mathcal{I}m\ B, \qquad k = 0,1,\ldots \tag{3.41}$$

□

Theorem 3.6 [Bar., 1], [Hau., 1], [Ros, 1]: The pair (A,B) is **controllable** if and only if any of the following equivalent conditions holds true.

a. There is a nonnegative integer r such that

$$\mathcal{C}_r(A,B) = \mathbb{R}^n$$

b. There is a nonnegative integer r such that

$$\mathcal{I}m\ A^r \subset \mathcal{I}m\ [B\ AB \cdots A^{r-1}B].$$

c. $\underline{\upsilon}^tB = 0$ and $\underline{\upsilon}^tA = \lambda\underline{\upsilon}^t$ for some constant λ implies $\lambda = 0$ or $\underline{\upsilon}^t = 0$.

d. The matrices $(I_n - Az^{-1}, Bz^{-1})$ are relatively left prime over $\mathbb{R}[z^{-1}]$.

□

Remark 3.9 [Kuc., 2]: From theorems (3.5) and (3.6) it is evident that reachability implies controllability and that the two notions are identical when A is nonsingular. In fact, controllability is equivalent to reachability of the non-finite modes of Σ.

□

3.3.4 Observability

The notions of observability/constructibility are dual to the notions of reachability/controllability [Kal., 2]. They are related to the underlying system $\Sigma(A,O,C,O)$ denoted by $\Sigma_{s-o}(A,C)$ and described by the state-output equations

$$\left.\begin{aligned} \underline{x}_{k+1} &= A\underline{x}_k \\ y_k &= Cx_k \end{aligned}\right\} \quad k = 0,1,... \tag{3.42}$$

Definition 3.25: The system $\Sigma(A,B,C,D)$, or the pair (A,B) is called **observable**, if and only if every initial state $\underline{x}_0$ can be observed from a finite output sequence $\underline{y}_0,...,\underline{y}_{k-1}$.

□

Definition 3.26: Consider the system $\Sigma_{s-o}(A,C)$ and define

$$\begin{aligned} \mathcal{O}\ (A,C) &= 0 \\ \mathcal{O}_k(A,C) &= \mathcal{I}m^t \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix}, \qquad k = 1,2,... \end{aligned} \tag{3.43}$$

We call $\mathcal{O}_k(A,C)$, $k \geq 0$ the **kth-observable subspace** of Σ.

□

Remark 3.10 [Kuc., 1]: According to equations (3.32a) and (3.32b) with $\underline{u} = 0$, we have

$$\underline{y}_0 = C\underline{x}_0$$
$$\underline{y}_1 = CA\underline{x}_0$$
$$\vdots$$
$$\underline{y}_{k-1} = CA^{k-1}\underline{x}_0$$

Therefore $\mathcal{O}_k(A,C) \subseteq \mathbb{R}^{n^t}$ consists of the linear functionals through which the initial state $\underline{x}_0$ can be observed from the output sequence $\underline{y}_0,\ldots,\underline{y}_{k-1}$. The following property is evident from (3.43)

$$\mathcal{O}_{k+1}(A,C) = \mathcal{O}_k(A,C)A + \mathcal{I}m^t\, C, \qquad k = 0,1,\ldots \tag{3.44}$$

□

Theorem 3.7 [Bar., 1], [Hau., 1], [Ros, 1]: The pair (A,C) is **observable** if and only if any of the following equivalent conditions holds true.

a. There is a nonnegative integer r such that

$$\mathcal{O}_r(A,C) = \mathbb{R}^{n^t}$$

b. There is a nonnegative integer r such that

$$\rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}\right) = n.$$

c. $C\underline{v} = 0$ and $A\underline{v} = \lambda\underline{v}$ for some constant λ implies $\underline{v} = 0$.

d. The matrices $(zI_n - A,\ C)$ are relatively right prime over $\mathbb{R}[z]$.

□

Definition 3.27: The constant λ in theorem (3.7) is an eigenvalue of A. The modes associated with any such λ are called the **observable modes** of the system Σ. Accordingly, the modes of Σ that are associated with any eigenvalue that does not satisfy condition (c) of theorem (3.7) are called **unobservable modes**.

□

Observability Indices

Consider the pair (A,C) and define the integers

$$q_k = \dim \mathcal{O}_k(A,C) - \dim \mathcal{O}_{k-1}(A,C), \quad k = 1,2,... \tag{3.45}$$

Then we have the following important notion.

Definition 3.28: Consider the pair (A,C) and let q_k be as in (3.45) and $\nu_i^{(o)}$ be integers such that for $i = 1,2,...,\ell$

$$\nu_i^{(o)} := \text{number of } q_i\text{'s greater than or equal to } i$$

Then the integers $\nu_1^{(o)} \geq \nu_2^{(o)} \geq \cdots \geq \nu_l^{(o)}$ are called the **observability indices** of (A,C).

□

Remark 3.11: It follows from (3.45) that

$$\sum_{i=1}^{l} \nu_i^{(o)} \leq n$$

with equality holding if and only if (A,C) is an observable pair.

□

3.3.5 Constructibility

Another important notion which is related to the state-output pair (A,C) is that of constructibility.

Definition 3.29: The system $\Sigma(A,B,C,D)$, or the pair (A,C) is **constructible**, if and only if for every initial state $\underline{x}_0$, every state $\underline{x}_k$ can be constructed from the output sequence $y_0,...,y_{k-1}$.

□

Definition 3.30: Consider the system $\Sigma_{s-o}(A,C)$ and define

$$\begin{aligned} &\mathcal{C}\mathscr{s}_0(A,C) = 0 \\ &\mathcal{C}\mathscr{s}_k(A,C) = \{\underline{w}^t \in \mathbb{R}^{n^t} : \underline{w}^t A^k \in \mathcal{O}_k(A,C)\}, \quad k = 1,2,... \end{aligned} \tag{3.46}$$

We call $\mathcal{C}\mathscr{s}_k(A,C)$, $k \geq 0$ the ***k*th-constructible subspace** of Σ.

□

Remark 3.12 [Kuc., 1]: According to equations (3.32a) and (3.32b) with $\underline{u} = 0$, we have

$$\underline{x}_k = A^k \underline{x}_0$$

and

$$\begin{aligned} \underline{y}_0 &= C\underline{x}_0 \\ \underline{y}_1 &= CA\underline{x}_0 \\ &\vdots \\ \underline{y}_{k-1} &= CA^{k-1}\underline{x}_0 \end{aligned}$$

Therefore $\mathscr{C}\!\mathit{o}_k(A,C) \subseteq \mathbb{R}^{n^t}$ consists of the linear functionals through which the state $\underline{x}_k$ can be constructed from the output sequence $\underline{y}_0,\ldots,\underline{y}_{k-1}$. The following property is evident from eqns. (3.46)

$$\mathscr{C}\!\mathit{o}_{k+1}(A,C) \subset \mathscr{C}\!\mathit{o}_k(A,C)A + \mathscr{I}m^t\, C, \qquad k = 0,1,\ldots \tag{3.47}$$

□

Theorem 3.8 [Bar., 1], [Hau., 1], [Ros, 1]: The pair (A,C) is **constructible** if and only if any of the following equivalent conditions holds true.

a. There is a nonnegative integer r such that

$$\mathscr{C}\!\mathit{o}_r(A,C) = \mathbb{R}^{n^t}$$

b. There is a nonnegative integer r such that

$$\mathscr{I}m^t\, A^r \subset \mathscr{I}m^t \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}.$$

c. $C\underline{v} = 0$ and $A\underline{v} = \lambda\underline{v}$ for some constant λ implies $\lambda = 0$, or $\underline{v} = 0$.

d. The matrices $(I_n - Az^{-1},\ Cz^{-1})$ are relatively right prime over $\mathbb{R}[z^{-1}]$.

□

Remark 3.13 [Kuc., 2]: From theorems (3.7) and (3.8) it is evident that observability implies constructibility and that the two notions are identical when A is nonsingular. In fact, constructibility is equivalent to observability of the non-finite modes of Σ.

□

3.3.6 Duality

A comparison between (3.37) and (3.43) reveals that

$$\begin{aligned} \mathcal{R}_k(A^t,C^t) &= \mathcal{O}_k^t(A,C) \\ \mathcal{O}_k(A^t,B^t) &= \mathcal{R}_k^t(A,B) \end{aligned} \tag{3.48}$$

Similarly, from (3.40) and (3.46) we have

$$\begin{aligned} \mathcal{C}_k(A^t,C^t) &= \mathcal{C}\Delta_k^t(A,C) \\ \mathcal{C}\Delta_k(A^t,B^t) &= \mathcal{C}_k^t(A,B) \end{aligned} \tag{3.49}$$

It is clear that there is a symmetry between the properties of reachability/controllability and observability/constructibility and any result that involves one pair of properties can be translated into a corresponding result involving the other pair. This important property known as ***principle of duality***, was introduced by Kalman [Kal., 2] by defining the dual system $\Sigma_d(A^t,C^t,B^t,D^t)$, i.e.

$$\begin{aligned} \underline{w}_{k+1} &= A^t\underline{w}_k + C^t\underline{u}_k \\ \underline{y}_k &= B^t\underline{w}_k + D^t\underline{u}_k \end{aligned} \tag{3.50}$$

of the original system $\Sigma(A,B,C,D)$. Thus observability of Σ implies reachability of Σ_d and constructibility of Σ implies controllability of Σ_d and vice versa.

3.3.7 Invariants under state feedback and output injection

In this section we summarize the complete set of invariants under the group of static state feedback, static output injection, and state, input and output coordinate transformations.

Static state feedback relates to the underlying system $\Sigma_{i-s}(A,B)$ (eqns. 3.36) where

$$\underline{u}_k = -L\underline{x}_k + \underline{v}_k, \quad L \in \mathbb{R}^{m\times n} \tag{3.51}$$

and the closed-loop system is described by

$$\underline{x}_{k+1} = (A - BL)\underline{x}_k + B\underline{v}_k, \quad k = 0,1,... \qquad (3.52)$$

whereas static output injection is a concept dual to static state feedback and relates to the underlying system $\Sigma_{s-o}(A,C)$ (eqns. 3.42) where

$$\underline{x}_{k+1} = A\underline{x}_k - K\underline{y}_k, \quad K \in \mathbb{R}^{1xn} \qquad (3.53)$$

and the closed-loop system is described by

$$\left.\begin{aligned} \underline{x}_{k+1} &= (A - KC)\underline{x}_k \\ y_k &= Cx_k \end{aligned}\right\} \quad k = 0,1,... \qquad (3.54)$$

The invariants of $\Sigma(A,B,C,D)$ under the aforementioned group of transformations were studied extensively by Brunovský [Bru., 1], Kalman [Kal., 4], Karcanias [Kar., 6], Morse [Mor., 1], Popov [Pop., 1], Rosenbrock [Ros., 1] and Thorp [Tho., 1] and are summarized below.

Theorem 3.9: Let $\Sigma(A,B,C,D)$ be a LTI discrete-time system and L, K be any state feedback and output injection real matrices. Then for any $k = 0,1,...$

$$\mathcal{R}_k(A - BL,B) = \mathcal{R}_k(A,B) \qquad (3.55)$$

$$\mathcal{C}_k(A - BL,B) = \mathcal{C}_k(A,B) \qquad (3.56)$$

and

$$\mathcal{O}_k(A - KC,C) = \mathcal{O}_k(A,C) \qquad (3.57)$$

$$\mathcal{C}\Delta_k(A - KC,C) = \mathcal{C}\Delta_k(A,C) \qquad (3.58)$$

□

Remark 3.14: According to theorem (3.9) the reachability indices $\mu_1^{(r)}$, $\mu_2^{(r)}$, $\cdots$, $\mu_m^{(r)}$ of $\Sigma(A,B,C,D)$ form a ***complete set of invariants*** of the reachable pair $\Sigma_{i-s}(A,B)$ under the ***feedback group*** of transformations (state, input coordinate tranformations, and static state feedback)

$$\begin{aligned} (A,B) &\longrightarrow (T^{-1}AT,T^{-1}B), && T \text{ nonsingular} \\ (A,B) &\longrightarrow (A,BR), && R \text{ nonsingular} \\ (A,B) &\longrightarrow (A - BL,B), && L \text{ real} \end{aligned} \qquad (3.59)$$

Also, the observability indices $\nu_1^{(o)}, \nu_2^{(o)}, \cdots, \nu_l^{(o)}$ of $\Sigma(A,B,C,D)$ form a ***complete set of invariants*** of the observable pair $\Sigma_{s-o}(A,C)$ under the ***injection group*** of transformations (state, output coordinate transformations, and output injection)

$$\begin{aligned} (A,C) &\longrightarrow (T^{-1}AT, CT), && T \text{ nonsingular} \\ (A,C) &\longrightarrow (A, Q^{-1}C), && Q \text{ nonsingular} \\ (A,C) &\longrightarrow (A - KC, C), && K \text{ real} \end{aligned} \tag{3.60}$$

□

3.3.8 The Kalman decomposition

Using the concepts of reachability and observability we can always find an invertible state transformation $\underline{x}'_k = T\underline{x}_k$ to rewrite the state equations

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \qquad \underline{y}_k = C\underline{x}_k + D\underline{u}_k$$

in the form

$$\underline{x}'_{k+1} = \bar{A}\underline{x}'_k + \bar{B}\underline{u}_k, \qquad \underline{y}_k = \bar{C}\underline{x}'_k + \bar{D}\underline{u}_k \tag{3.61}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{ro} & O & \bar{A}_{13} & O \\ \bar{A}_{21} & \bar{A}_{r\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ O & O & \bar{A}_{\bar{r}o} & O \\ O & O & \bar{A}_{34} & \bar{A}_{\bar{r}\bar{o}} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} \bar{B}_{ro} \\ \bar{B}_{r\bar{o}} \\ O \\ O \end{bmatrix} \tag{3.62}$$

$$\bar{C} = [\, \bar{C}_{ro} \quad O \quad \bar{C}_{\bar{r}o} \quad O \,], \qquad \bar{D} = D$$

and

1. the subsystem

$$(\bar{A}_{ro}, \bar{B}_{ro}, \bar{C}_{ro}, \bar{D}_{ro})$$

is reachable and observable;

2. the subsystem

$$\left(\begin{bmatrix} \bar{A}_{ro} & O \\ \bar{A}_{21} & \bar{A}_{r\bar{o}} \end{bmatrix}, \begin{bmatrix} \bar{B}_{ro} \\ \bar{B}_{r\bar{o}} \end{bmatrix}, [\, \bar{C}_{ro} \quad O \,], \bar{D}_{r\bar{o}} \right)$$

is reachable and unobservable;

3. the subsystem

$$\left(\begin{bmatrix} \bar{A}_{ro} & \bar{A}_{13} \\ O & \bar{A}_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} \bar{B}_{ro} \\ O \end{bmatrix}, [\,\bar{C}_{ro} \quad \bar{C}_{\bar{r}o}\,], \bar{D}_{\bar{r}o}\right)$$

is unreachable and observable;

4. the subsystem

$$(\bar{A}_{\bar{r}\bar{o}}, O, O, \bar{D}_{\bar{r}\bar{o}})$$

is unreachable and unobservable.

This general decomposition was first enunciated by Gilbert [Gil., 1] and Kalman [Kal., 3] and is shown in figure (3.3), where *RO* stands for reachable and observable, $R\bar{O}$ for reachable but not observable, etc.

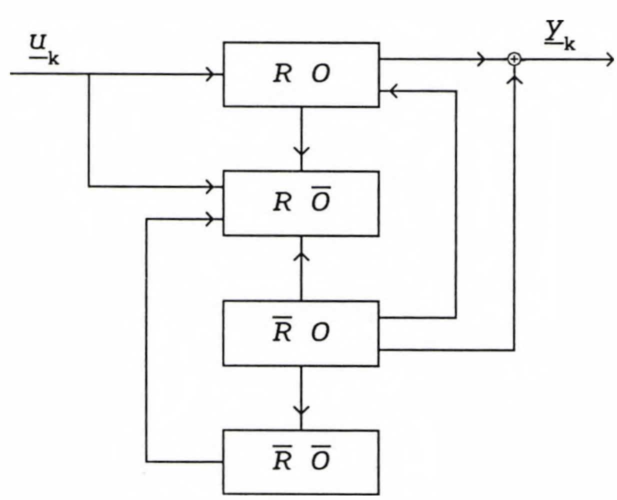

Figure (3.3): The Kalman decomposition

3.3.9 Stability stabilizability and detectability

The notion of stability of the system $\Sigma(A,B,C,D)$ refers to the dynamic behaviour of the unforced system

$$\underline{x}_{k+1} = A\underline{x}_k, \quad \underline{x}_0 \in \mathbb{R}^n \text{ given}, \quad k \in \mathcal{J} : k \geq 0$$

as *k* approaches infinity. We distinguish mainly between two types of stability, the ***bounded state stability*** and the ***asymptotic stability***.

Definition 3.31: The system $\Sigma(A,B,C,D)$ is **bounded state stable**, or **stable in the sense of Lyapunov**, if

$$\forall\ \underline{x}_0 \in \mathbb{R}^n \quad |x_k| < M < \infty \quad \forall\ k \in \mathcal{T} \text{ and } k \geq 0$$

□

Definition 3.32: The system $\Sigma(A,B,C,D)$ is **asymptotically stable**, if

$$\forall\ \underline{x}_0 \in \mathbb{R}^n \quad x_k \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty$$

□

The following well known theorems give the characterization of stable in the sense of Lyapunov and asymptotically stable systems.

Theorem 3.10 [Ros., 1]: The nth order system $\Sigma(A,B,C,D)$ is **stable in the sense of Lyapunov**, if and only if all the eigenvalues of A have moduli less than or equal to unity, i.e.

$$|\lambda_i(A)| \leq 1, \quad i = 1,2,\ldots,n$$

and the unity eigenvalue has a simple structure.

□

Theorem 3.11 [Ros., 1]: The system $\Sigma(A,B,C,D)$ is **asymptotically stable**, if and only if all the eigenvalues of A have moduli less than unity, i.e.

$$|\lambda_i(A)| < 1, \quad i = 1,2,\ldots,n$$

□

It is clear that non-repeated eigenvalues of A on the unit circle are consistent with bounded state stability but not with asymptotic stability. Asymtotic stability is a stricter notion to that of bounded state stability and it will be called alternatively internal stability. It is also known, as we shall see later in this section, that internal stabillity implies external (BIBO) one. For this reason we adopt here the notion of internal stability as opposed to the external one. We then introduce the following notation.

Definition 3.33: Consider the system $\Sigma(A,B,C,D)$. An eigenvalue λ of A is called **stable** if $|\lambda| < 1$ and the modes of A that are associated with a stable λ are called **stable modes.** If Σ is internally stable the matrix A is termed as **stable**.

□

We close this section by referring to the notions of stabilizability and detectability. They were first introduced by Wonham [Won., 1] in connection with the concepts of state feedback and output injection and they are the following.

Definition 3.34: The system $\Sigma(A,B,C,D)$ is said to be **stabilizable**, if its unreachable modes are stable.

□

Definition 3.35: The system $\Sigma(A,B,C,D)$ is said to be **detectable**, if its unobservable modes are stable.

□

Remark 3.15: Since controllability implies reachability of the non finite modes and the finite modes are stable, then $\Sigma(A,B,C,D)$ is stabilizable if its uncontrollable modes are stable. Accordingly, due to duality, $\Sigma(A,B,C,D)$ is detectable, if its unconstructible modes are stable.

□

3.3.10 Transfer functions and minimal realizations

We investigate now the relationship between the state-space and the input-output descriptions of discrete-time systems. Consider the system $\Sigma(A,B,C,D,\underline{x}_0)$ described by the equations

$$\left.\begin{aligned} \underline{x}_{k+1} &= A\underline{x}_k + C\underline{u}_k \\ \underline{y}_k &= B\underline{x}_k + D\underline{u}_k \end{aligned}\right\} \quad \underline{x}_0 \in \mathbb{R}^n \text{ given, } k \in \mathcal{T} : k \geq 0 \tag{3.63}$$

and apply the z-Transform with $\underline{x}_0 = 0$. Then

$$\underline{\tilde{y}}(z) = \tilde{G}(z)\underline{\tilde{u}}(z) \quad \text{or, if } d = z^{-1} \quad \underline{y}(d) = G(d)\underline{u}(d)$$

where

and

$$\tilde{G}(z) = C(zI_n - A)^{-1}B + D \tag{3.64a}$$

$$G(d) = dC(I_n - dA)^{-1}B + D \tag{3.64b}$$

$\tilde{G}(z)$ or $G(d)$ describe the input-output behaviour of the system Σ without necessarily taking fully into account its internal dynamics. Indeed, using the Kalman decomposition (3.62) $\tilde{G}(z)$ and $\tilde{G}(d)$ become

and

$$\tilde{G}(z) = \overline{C}_{ro}(zI_n - \overline{A}_{ro})^{-1}\overline{B}_{ro} + \overline{D}_{ro} \tag{3.65a}$$

$$G(d) = d\overline{C}_{ro}(I_n - d\overline{A}_{ro})^{-1}\overline{B}_{ro} + \overline{D}_{ro} \tag{3.65b}$$

i.e., as it can be seen from figure (3.3) also, the system $\Sigma(A,B,C,D)$ behaves as its reachable and observable part $\Sigma_{ro}(\overline{A}_{ro},\overline{B}_{ro},\overline{C}_{ro},\overline{D}_{ro})$. In fact, Σ_{ro} is the ***minimum order*** system, unique within state coordinate transformations, that has the same input-output behaviour as Σ. Obviously, $\tilde{G}(z)$ is the **transfer function matrix** and $G(d)$ the **impulse response matrix** of the physical system described by $\Sigma(A,B,C,D)$. In a similar way, given a system $\mathcal{S}(G,\underline{y}_{in})$ there is a family of systems $\Sigma(A,B,C,D)$ having the same external behaviour as $\mathcal{S}$.

Definition 3.36: A system $\Sigma(A,B,C,D)$ is a **realization** of $\mathcal{S}(G,\underline{y}_{in})$ if the quadruple (A,B,C,D) satisfies equations (3.65a), or (3.65b). If $\Sigma(A,B,C,D)$ is reachable and observable, it is called a **minimal realization** of $\mathcal{S}(G,\underline{y}_{in})$.

□

If $\Sigma(A,B,C,D)$ is a minimal realization of $\mathcal{S}(\tilde{G}(z),G(d))$ the following results are true.

Theorem 3.12: Let $\Sigma(A,B,C,D)$ be a minimal realization of $\mathcal{S}(\tilde{G}(z),G(d))$, $(\tilde{D}_l(z),\tilde{N}_l(z))$ and $(\tilde{N}_r(z),\tilde{D}_r(z))$ be any left, right coprime polynomial MFDs of $\tilde{G}(z)$ respectively, and $(D_l(d),N_l(d))$ and $(N_r(d),D_r(d))$ be any left, right coprime polynomial MFDs of $G(d)$ respectively. Then

$$\det(zI_n - A) \sim \det\tilde{D}_l(z) \sim \det\tilde{D}_r(z) \tag{3.66a}$$

$$\det(I_n - dA) \sim \det D_l(d) \sim \det D_r(d) \tag{3.66b}$$

Proof. Eqn. (3.66a) is a well known result (see e.g. Rosenbrock [Ros., 1]) but we give its proof here for reasons of completeness. According to theorem (2.17) it is enough to prove that $(C, zI_n - A, B, D)$ and $(dC, I_n - dA, B, D)$ are bicoprime factorizations of $\tilde{G}(z)$ and $G(d)$ respectively. Indeed, $(C, zI_n - A)$ are right coprime over $\mathbb{R}[z]$ due to the observability of Σ and $(zI_n - A, B)$ are left coprime over $\mathbb{R}[z]$ due to the reachability of Σ. Hence $(C, zI_n - A, B, D)$ is a bicoprime factorization of $\tilde{G}(z)$ over $\mathbb{R}[z]$ and (3.66a) is true.

Since observability implies constructibility (remark 3.13), $(dC, I_n - dA)$ are right coprime over $\mathbb{R}[d]$. So, what remains to be proved for (3.66b) to be true, is that $(I_n - dA, B)$ are left coprime over $\mathbb{R}[d]$. Indeed

$$\begin{bmatrix} I_n - dA & B \end{bmatrix} \begin{bmatrix} I_n & O \\ O & dI_m \end{bmatrix} = \begin{bmatrix} I_n - dA & dB \end{bmatrix} \tag{3.67}$$

and since reachability implies controllability (remark 3.9), $(I_n - dA, dB)$ are left coprime over $\mathbb{R}[d]$ and so $(I_n - dA, B)$ are left coprime over $\mathbb{R}[d] - \{0\}$ due to (3.67). Also, for $d = 0$, $\begin{bmatrix} I_n - dA & B \end{bmatrix}$ becomes $\begin{bmatrix} I_n & B \end{bmatrix}$ which is clearly coprime. Hence $(I_n - dA, B)$ are left coprime over $\mathbb{R}[d]$, $(dC, I_n - dA, B, D)$ is a bicoprime factorization of $G(d)$ over $\mathbb{R}[d]$ and so (3.66b) is true.

□

Remark 3.16 [Kai., 1], [Kuc., 1]**:** Obviously, the roots of $\det(zI_n - A)$ are the eigenvalues of A, whereas the roots of $\det(I_n - dA)$ are the inverses of those eigenvalues of A that correspond to the non-finite modes of $\Sigma(A, B, C, D)$. Then, according to eqns. (3.64) and (3.65) and to theorem (3.12), if Σ is a non-minimal realization of $\mathcal{S}(\tilde{G}(z), G(d))$

$$\det(zI_n - A) \sim f_{\bar{r}} f_{\bar{o}} \det\tilde{D}_l(z) \sim f_{\bar{r}} f_{\bar{o}} \det\tilde{D}_r(z) \tag{3.68a}$$

$$\det(I_n - dA) \sim f_{\bar{c}} f_{\overline{cs}} \det D_l(d) \sim f_{\bar{c}} f_{\overline{cs}} \det D_r(d) \tag{3.68b}$$

where $f_{\bar{r}}$, $f_{\bar{o}} \in \mathbb{R}[z]$ with roots associated to the unreachable and unobservable modes of Σ correspondingly, and $f_{\bar{c}}$, $f_{\overline{cs}} \in \mathbb{R}[d]$ with roots associated to the uncontrollable and unconstructible modes of Σ correspondingly. These modes that do not affect the input-output behaviour of Σ although are part of its internal dynamics are called **hidden modes**.

□

Theorem 3.13: Let $\Sigma(A,B,C,D)$ be a minimal realization of $\mathcal{S}(\tilde{G}(z),G(d))$. The reachability indices of Σ are the right minimal indices of $\tilde{G}(z)$, or $G(d)$ and the observability indices of Σ are the left minimal indices of $\tilde{G}(z)$, or $G(d)$.

Proof. It is known that this is true for $\tilde{G}(z)$ [Kal., 4], [Kuc., 2], [Ros., 1]. But according to proposition (3.3), $\tilde{G}(z)$ and $G(d)$ have the same minimal indices, and this completes the proof.

□

In accordance with the presentation of the input-output description in section (3.2) and due to the fact that d is physically realizable, we will mainly use the notion of the d-transfer function, or the impulse response matrix $G(d)$. Then the following definition comes naturally.

Definition 3.37: Let $\Sigma(A,B,C,D)$ be a discrete-time system. with order n. We call $\det(I_n - dA)$ the **characteristic polynomial** of Σ and we denote it by $f_\Sigma(d)$.

□

Remark 3.17: The polynomial $\det(I_n - dA)$ is the conjugate reciprocal of $\det(zI_n - A)$ which is usually referred to as characteristic, i.e.

$$f_\Sigma(d) := \det(I_n - dA) = d^n \det(zI_n - A)$$

Note that information about the eigenvalues at zero of A and consequently its order, is lost in $\det(I_n - dA)$. But if Σ is a minimal realization of $\mathcal{S}(G)$, then this information can be recovered since it can be easily shown that

$$n = \delta_{\mathcal{M}}(\mathcal{S}) := \delta_{\mathcal{M}}(G)$$

However, since the finite modes of $\Sigma(A,B,C,D)$ do not appear in $\det(I_n - dA)$ we cannot affect them by assigning, during the design procedure, the characteristic polynomial of Σ. This is not important in our case since our aim is to move all the eigenvalues of the system Σ to the origin as we shall see in chapter (5). Also, it is not important for stabilization purposes since the zero eigenvalues are stable.

□

Finally, we note that according to remark (3.16) internal stability of $\Sigma(A,B,C,D)$ implies external stability as well, i.e. the corresponding $\tilde{G}(z)$, or $G(d)$ are BIBO stable. The converse is true only if Σ is stabilizable and detectable. We formulate this result as the following corollary.

Corollary 3.7 [Kai., 1]: Suppose that $\Sigma(A,B,C,D)$ is stabilizable and detectable. Then $\Sigma(A,B,C,D)$ is asymptotically stable, if and only if the corresponding $\tilde{G}(z)$, or $G(d)$ are BIBO stable.

□

Remark 3.18: Under the stabilizability and detectability assumptions on a linear system $\Sigma(A,B,C,D)$, the notions of internal and external stability become equivalent.

□

Notation. We used up to now the script letter $\mathscr{S}$ to describe the external behaviour of a physical system and the greek letter Σ to denote the internal behaviour of it. From now on we will refer to a physical system by $\mathscr{S}$ and we will use italics to denote its impulse response or transfer function, and quadruples of the form (A,B,C,D) for its state-space description.

3.4 The Unity Feedback Configuration

Feedback systems are the systems most widely used for control system design. We will mainly use in this thesis the feedback system shown in figure (3.4), and we will refer to it as the **standard**, or **unity**, or **one-parameter** feedback configuration $\mathscr{S}_f$. Under this configuration, $\mathscr{S}_p$ represents a given physical LTI discrete-time system hereafter called the **plant**, and $\mathcal{S}_c$ is a LTI discrete-time system called the **controller**, or **compensator**, to be designed so that the feedback system has a desirable performance.

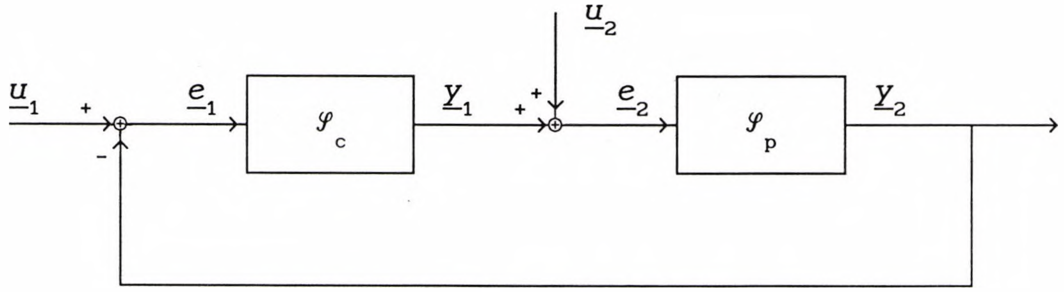

Figure (3.4): The unity feedback configuration

The principal aim of this section is the investigation of the problem of stability and stabilization of the unity feedback system of figure (3.4). First we give a set of various expressions for the pole-polynomial of the feedback system and the closed-loop impulse response matrices from any external signal to any signal along the directed path, with necessary and sufficient conditions for existence. Then we address the problem of internal and external stability and the problem of finding the family of all stabilizing controllers for a given plant known as *Youla-Bongiorno-Kucera parametrization.*

We refer to the work of Callier and Desoer [Cal., 1], Chen [Che., 1], Vidyasagar [Vid., 1], Youla, Bongiorno and Lu [You., 1] and Youla, Bongiorno and Jabr [You., 2] for the relevant properties and design issues of feedback systems. Here we closely follow the work of Karcanias [Kar., 4] and Kucera [Kuc., 1] in particular.

3.4.1 General aspects of the unity feedback configuration

Consider the unity feedback system of figure (3.4) and let $P \in \mathbb{R}^{l\times m}(d)$, $C \in \mathbb{R}^{m\times l}(d)$ represent the d-transfer function matrices, thereafter referred to as transfer function matrices, of the plant and controller respectively. $\underline{u}_1$, $\underline{u}_2$ denote the externally applied vector inputs, $\underline{e}_1$, $\underline{e}_2$ denote the vector inputs to controller, plant, and $\underline{y}_1$, $\underline{y}_2$ denote the vector outputs of the controller and plant respectively. All signals are vector sequences in d. Also, by $\mathcal{R}$ we will denote $\mathbb{R}[d]$, or any quotient ring of $\mathbb{R}[d]$ whose field of fractions is $\mathbb{R}(d)$, and $M(\mathcal{R})$ is the set of matrices with elements from $\mathcal{R}$ and appropriate dimensions.

Such a configuration, though not the ideal one for fairly complicated design cases, is quite versatile and may accomodate several control problems. For instance, in a problem of tracking, $\underline{u}_1$ would be a reference signal to be tracked by the output $\underline{y}_2$. In a problem of disturbance rejection, or desensitization to noise, $\underline{u}_1$ would be the disturbance/noise. Depending on whether $\underline{u}_1$, or $\underline{u}_2$ is the externally applied control signal (as opposed to noise etc.) the configuration can represent either feedback or cascade compensation.

Structure and external stability of the feedback system

The feedback system under consideration can be described by the transfer function

$$H_{e,y/u} \text{ from } \underline{u} \text{ to } \begin{bmatrix} \underline{e} \\ \underline{y} \end{bmatrix}$$

where

$$\underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}, \; \underline{e} = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix}, \; \underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix}$$

or, due to linearity, by the two transfer function matrices

$$H(P,C) \; : \; \underline{e} := H(P,C)\underline{u} \tag{3.69a}$$

$$W(P,C) \; : \; \underline{y} := W(P,C)\underline{u} \tag{3.69b}$$

The system equations can be written as

$$\underline{e} = (I + FG)^{-1}\underline{u} \quad \text{and} \quad \underline{y} = G(I + FG)^{-1}\underline{u} \tag{3.70}$$

where

$$F = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}, \; G = \begin{bmatrix} C & O \\ O & P \end{bmatrix} \text{ and } I + FG = \begin{bmatrix} I & P \\ -C & I \end{bmatrix} \tag{3.71}$$

We will refer to the unity feedback system of figure (3.4) as $\mathcal{S}_f$, or the pair (P,C). It can be readily verified (Schur formula), that

$$t(d) := \det(I + FG) = \det(I + PC) = \det(I + CP) \tag{3.72}$$

Definition 3.38: The pair (P,C) is **well-formed**, if $t(d)$ is a nonzero element of $\mathbb{R}(d)$, i.e. $t(d)$ is not identically zero for all $d \in \mathbb{C}$.

□

It is possible to obtain several equivalent expressions for $H(P,C)$ and $W(P,C)$ [Kar., 4], [Vid., 1] such that

$$\begin{aligned} H(P,C) &= \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} I-P(I+CP)^{-1}C & -P(I+CP)^{-1} \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ C(I+PC)^{-1} & I-C(I+PC)^{-1}P \end{bmatrix} \end{aligned} \tag{3.73}$$

$$\begin{aligned} W(P,C) &= \begin{bmatrix} C & O \\ O & P \end{bmatrix} \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \\ &= \begin{bmatrix} C-CP(I+CP)^{-1}C & -CP(I+CP)^{-1} \\ P(I+CP)^{-1}C & P(I+CP)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} C(I+PC)^{-1} & -C(I+PC)^{-1}P \\ PC(I+PC)^{-1} & P-PC(I+PC)^{-1}P \end{bmatrix} \end{aligned} \tag{3.74}$$

Remark 3.19: From (3.73) and (3.74) we see that the condition $t(d) \not\equiv 0$ (eqns 3.72) for well-formedness, is necessary and sufficient to ensure the existence of $H(P,C)$ and $W(P,C)$.

□

An important relationship between $H(P,C)$ and $W(P,C)$ is given next.

Lemma 3.1 [Vid., 1]: Suppose the feedback system $\mathcal{S}_f$ is well-formed. Then

$$W(P,C) = F(H(P,C) - I) \tag{3.75}$$

and so $W(P,C) \in M(\mathcal{R})$, if and only if $H(P,C) \in M(\mathcal{R})$.

□

Remark 3.20: According to lemma (3.1), the transfer function matrix $H(P,C)$ may be used in the investigation of causality and external stability of the unity feedback configuration of figure (3.4).

□

Theorem 3.14: Let the pair (P,C) be well-formed. Then the nonunit elementary pole-polynomials of $H_{e,y/u}$, $H(P,C)$ and $W(P,C)$ are associates.

Before we prove theorem (3.14) we give the proof of the following lemma.

Lemma 3.2: Let $G \in M(\mathbb{R}(d))$, (N, D) be a right coprime MFD of G over $\mathbb{R}[d]$ and $H \in M(\mathbb{R}[d])$. Then $((N+HD), D)$ is a right $\mathbb{R}[d]$-coprime MFD of $G + H$ and therefore the elementary pole-polynomials of G and $G + H$ are associates.

Proof. Since (N, D) is a polynomial right coprime MFD of G, then

$$G + H = ND^{-1} + H = ND^{-1} + HDD^{-1} = (N+HD)D^{-1}$$

Due to the coprimeness of (N, D) the pair $((N+HD), D)$ is right coprime.

□

Proof of theorem (3.14). We denote $H(P,C)$ and $W(P,C)$ by H and W in short. Then according to lemma (3.2) the elementary pole-polynomials of $W = FH - F$ and FH are associates. But $H \sim FH$ and therefore the elementary pole-polynomials of H, W are associates. Also

$$H_{e,y/u} = \begin{bmatrix} H \\ W \end{bmatrix} = \begin{bmatrix} H \\ FH-F \end{bmatrix} = \begin{bmatrix} H \\ FH \end{bmatrix} + \begin{bmatrix} O \\ -F \end{bmatrix}$$

and due to lemma (3.2), the elementary pole-polynomials of

$H_{e,y/u}$ and $[H^t \ H^tF^t]^t$ are associates. But

$$\begin{bmatrix} H \\ FH \end{bmatrix} \sim \begin{bmatrix} H \\ O \end{bmatrix}$$

and therefore the elementary pole-polynomials of $H_{e,y/u}$ are associates to the elementary pole-polynomials of H.

□

Remark 3.21: Theorem (3.14) is richer than lemma (3.1). According to theorem (3.14), the pole structure over $\mathbb{C}$ of the feedback system can be described by any of the transfer functions $H_{e,y/u}$, $H(P,C)$, or $W(P,C)$. It is common to use the error transfer function matrix $H(P,C)$ for this reason.

□

Definition 3.39: The pair (P,C), or the feedback system $\mathscr{S}_f$ is externally stable, if $H(P,C) \in M(\mathbb{R}^+(d))$.

□

Assume now that both plant and controller transfer function matrices are represented by MFDs over $\mathbb{R}[d]$, i.e.

$$P = \tilde{D}_p^{-1}\tilde{N}_p = N_p D_p^{-1} \tag{3.76}$$

$$C = \tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1} \tag{3.77}$$

Then by inserting (3.76), (3.77) into (3.73), (3.74), $H(P,C)$ and $W(P,C)$ become

$$H(P,C) = \begin{bmatrix} \tilde{D}_p & \tilde{N}_p \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix}^{-1} \begin{bmatrix} \tilde{D}_p & O \\ O & \tilde{D}_c \end{bmatrix} \tag{3.78a}$$

$$= \begin{bmatrix} D_c & O \\ O & D_p \end{bmatrix} \begin{bmatrix} D_c & N_p \\ -N_c & D_p \end{bmatrix}^{-1} \tag{3.78b}$$

$$= \begin{bmatrix} D_c \\ N_c \end{bmatrix} \tilde{\Delta}^{-1} \begin{bmatrix} \tilde{D}_p & -\tilde{N}_p \end{bmatrix} + \begin{bmatrix} O & O \\ O & I \end{bmatrix} \tag{3.78c}$$

$$= \begin{bmatrix} -N_p \\ D_p \end{bmatrix} \Delta^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} + \begin{bmatrix} I & O \\ O & O \end{bmatrix} \tag{3.78d}$$

$$W(P,C) = \begin{bmatrix} N_c \\ -D_c \end{bmatrix} \tilde{\Delta}^{-1} \begin{bmatrix} \tilde{D}_p & -\tilde{N}_p \end{bmatrix} + \begin{bmatrix} O & O \\ I & O \end{bmatrix} \tag{3.79a}$$

$$= \begin{bmatrix} D_p \\ N_p \end{bmatrix} \Delta^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} + \begin{bmatrix} O & -I \\ O & O \end{bmatrix} \tag{3.79b}$$

where

$$\Delta := \tilde{N}_c N_p + \tilde{D}_c D_p \quad \text{and} \quad \tilde{\Delta} := \tilde{N}_p N_c + \tilde{D}_p D_c \tag{3.80}$$

If

$$Q := \begin{bmatrix} D_c & N_p \\ -N_c & D_p \end{bmatrix} \quad \text{and} \quad \tilde{Q} := \begin{bmatrix} \tilde{D}_p & \tilde{N}_p \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix} \tag{3.81}$$

we have the following important theorem.

Theorem 3.15 [Kuc., 1]**:** Let the feedback system $\mathcal{S}_f$ be well-formed and P, C are represented by coprime polynomial MFDs. Then (3.78a) and (3.78b) are left, right coprime polynomial MFDs and (3.78c), (3.78d) are bicoprime polynomial MFDs of $H(P,C)$. In addition Δ, $\tilde{\Delta}$, Q and $\tilde{Q}$ are extended associates

$$\Delta \cdot\sim \tilde{\Delta} \cdot\sim Q \cdot\sim \tilde{Q}$$

i.e. they all share the same nonunit invariant polynomials. Also the elementary pole-polynomials of $H(P,C)$ are associates to the nonunit invariant polynomials of Δ, $\tilde{\Delta}$, Q and $\tilde{Q}$.

□

Corollary 3.8: According to theorem (3.15) the pole-polynomial of the feedback system $\mathcal{S}_f$ is given by

$$p_{\mathcal{S}_f}(d) := p_H(d) \sim \det\Delta \sim \det\tilde{\Delta} \sim \det Q \sim \det\tilde{Q} \tag{3.82}$$

□

Corollary 3.9: Consider the feedback system $\mathcal{S}_f$ with P, $C \in M(\mathbb{R}(d))$ the transfer function matrices of the plant and controller respectively, and let $(\tilde{D}_p, \tilde{N}_p)$, (N_p, D_p) be any left coprime MFD, right coprime MFD of P over $\mathbb{R}[d]$ and $(\tilde{D}_c, \tilde{N}_c)$, (N_c, D_c) be any left coprime MFD, right coprime MFD of C

over $\mathbb{R}[d]$. Then the following statements are equivalent.

a. (P,C) is externally stable

b. $\Delta := \tilde{N}_c N_p + \tilde{D}_c D_p$ is a stable polynomial matrix

c. $\tilde{\Delta} := \tilde{N}_p N_c + \tilde{D}_p D_c$ is a stable polynomial matrix

d. $Q := \begin{bmatrix} D_c & N_p \\ -N_c & D_p \end{bmatrix}$ is a stable polynomial matrix

e. $\tilde{Q} := \begin{bmatrix} \tilde{D}_p & \tilde{N}_p \\ -\tilde{N}_c & \tilde{D}_c \end{bmatrix}$ is a stable polynomial matrix.

□

Causality and well-posedness

It has been mentioned that the unity feedback system is well-formed if $\det(I + FG) = \det(I + PC) = \det(I + CP) \neq 0$ as a rational function. This allows the existence of the various closed-loop transfer functions. However, nothing has been said regarding the causality of them.

Definition 3.40 [Kar., 4]**:** A composite system is said to be **well-posed** if the transfer function of every subsystem is causal and the closed-loop transfer function from any point chosen as an input terminal to every other point along the directed path is well defined and causal.

□

In the design of feedback systems the well-posedness property is essential, if no signal is to be unduly amplified, or otherwise if the smoothness of signals throughout the system is to be preserved. Since causality is a special case of stability then, according to lemma (2.1), the causality of the feedback system $\mathcal{S}_f$ can be described by the causality of $H(P,C)$.

Theorem 3.16: Consider the unity feedback configuration of figure (3.4) and let P, $C \in M(\mathbb{R}^0(d))$ be the causal transfer functions of the plant and the controller. The feedback system is well-posed, if and only if

$$\det(I + P(0)C(0)) = \det(I + C(0)P(0)) \neq 0 \qquad (3.83)$$

Proof. It is enough to show that

$$H(P,C) = \begin{bmatrix} I & P \\ -C & I \end{bmatrix}^{-1} \in M(\mathbb{R}^0(d))$$

Since P, $C \in M(\mathbb{R}^0(d))$ then $\begin{bmatrix} I & P \\ -C & I \end{bmatrix}$ is causal and for $H(P,C)$ to be causal, $\begin{bmatrix} I & P \\ -C & I \end{bmatrix}$ must be bicausal. This is true if and only if (3.83) is valid.

□

According to theorem (3.15) the following corollary can be readily verified.

Corollary 3.10: Consider the feedback system $\mathcal{S}_f$ with P, $C \in M(\mathbb{R}^0(d))$ the causal transfer function matrices of the plant and controller respectively and let $(\tilde{D}_p, \tilde{N}_p)$, (N_p, D_p) be any left coprime MFD, right coprime MFD of P over $\mathbb{R}[d]$ and $(\tilde{D}_c, \tilde{N}_c)$, (N_c, D_c) be any left coprime MFD, right coprime MFD of C over $\mathbb{R}[d]$. Then the following statements are equivalent.

- **a.** (P,C) is well-posed
- **b.** $\det\Delta(0) \sim \det\tilde{\Delta}(0) \sim \det Q(0) \sim \det\tilde{Q}(0) \neq 0$.

□

Remark 3.22: If the plant is strictly causal, then $P(0) = 0$ and (3.83) is valid for any C. Therefore, the unity feedback system is well-posed for any causal controller C, if the plant P is strictly causal.

□

3.4.2 Internal description of the unity feedback system

The question of internal stability of the unity feedback configuration of figure (3.4) is examined briefly in this section. Assume that the plant and controller are characterized by the following sets of not necessarily minimal state-space equations.

$$\mathcal{S}_p: \quad \underline{x}_{p,k+1} = A_p\underline{x}_{p,k} + B_p\underline{e}_{2,k} \ , \quad \underline{y}_{2,k} = C_p\underline{x}_{p,k} + D_p\underline{e}_{2,k}$$

$$\mathcal{S}_c: \quad \underline{x}_{c,k+1} = A_c\underline{x}_{c,k} + B_c\underline{e}_{1,k} \ , \quad \underline{y}_{1,k} = C_c\underline{x}_{c,k} + D_c\underline{e}_{1,k}$$

If P, C are the transfer functions of the plant and the controller, then according to (3.64b) $D_p = P(0)$, $D_C = C(0)$ and the next corollary readily follows from theorem (3.16).

Corollary 3.11: The unity feedback configuration of figure (3.4) is well-posed, if and only if

$$\det(I + D_pD_c) = \det(I + D_cD_p) \neq 0 \tag{3.84}$$

□

We define now the closed-loop systems $\mathcal{S}_f$, $\mathcal{S}_h$ and $\mathcal{S}_w$ corresponding to transfer functions $H_{e,y/u}$, $H(P,C)$ and $W(P,C)$, i.e.

$$\mathcal{S}_f: \quad \underline{x}_f := [\underline{x}_p^t \ \underline{x}_c^t]^t, \ \underline{u}_f := \underline{u} \ , \ \underline{y}_f := [\underline{e}^t \ \underline{y}^t]^t \tag{3.85a}$$

$$\mathcal{S}_h: \quad \underline{x}_h := [\underline{x}_p^t \ \underline{x}_c^t]^t, \ \underline{u}_h := \underline{u} \ , \ \underline{y}_h := \underline{e} \tag{3.85b}$$

$$\mathcal{S}_w: \quad \underline{x}_w := [\underline{x}_p^t \ \underline{x}_c^t]^t, \ \underline{u}_w := \underline{u} \ , \ \underline{y}_w := \underline{y} \tag{3.85c}$$

We have then the following important property.

Lemma 3.3: Suppose that the feedback configuration of figure (3.4) is well-posed. Then, it may be described internally by the state-space representations of any of the systems $\mathcal{S}_f$, $\mathcal{S}_h$ or $\mathcal{S}_w$ given by the eqns. (3.85a) to (3.85c).

Proof. The three systems share the same state and input vectors, so they have common internal stability and reachability properties. From figure (3.4) we have that

$$\underline{y} = F\underline{e} - F\underline{u} \tag{3.86}$$

with F given by (3.71). If we ignore $\underline{u}$, F represents an output coordinate transformation for $\mathcal{S}_h$, $\mathcal{S}_w$ and $\mathcal{S}_h$ is observable if and only if $\mathcal{S}_w$ is observable. Also, since

$$\underline{y}_f := [\underline{e}^t \ \underline{y}^t]^t$$

$\mathcal{S}_f$ is observable, if and only if either of $\mathcal{S}_h$, $\mathcal{S}_w$ is observable (due to 3.86).

□

According to lemma (3.3), the closed-loop system

$$\mathcal{S}_h: \quad \underline{x}_{h,k+1} = A_h \underline{x}_{h,k} + B_h \underline{u}_k \ , \ \underline{e}_k = C_h \underline{x}_{h,k} + D_h \underline{u}_k \qquad (3.87)$$

with the corresponding transfer function $H(P,C)$, describes completely internally the feedback system $\mathcal{S}_f$. The following proposition is well known.

Proposition 3.4 [Vid., 1]: Consider the well-posed feedback system $\mathcal{S}_h$ with plant $\mathcal{S}_p$ and controller $\mathcal{S}_c$. Then

a. $\mathcal{S}_h$ is reachable, observable, if and only if both $\mathcal{S}_p$ and $\mathcal{S}_c$ are reachable, observable.

b. $\mathcal{S}_h$ is stabilizable, detectable, if and only if both $\mathcal{S}_p$ and $\mathcal{S}_c$ are stabilizable, detectable.

□

We may now state the main theorem of this section.

Theorem 3.17 [Vid., 1]: Consider the well-posed feedback system $\mathcal{S}_h$ with the plant $\mathcal{S}_p$ and the controller $\mathcal{S}_c$ both stabilizable and detectable. Under these assumptions, $\mathcal{S}_h$ is internally stable, if and only if $H(P,C)$ is BIBO stable.

□

By theorem (3.17) and proposition (3.4) it follows.

Corollary 3.12: Consider the well-posed feedback system $\mathcal{S}_h$ with the plant and controller systems $\mathcal{S}_p$, $\mathcal{S}_c$ minimal and the transfer functions P, C represented by the $\mathbb{R}[d]$ coprime MFDs

$$P = \tilde{D}_p^{-1}\tilde{N}_p = N_p D_p^{-1} \ , \quad C = \tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1}$$

Then the pole-polynomial of $\mathcal{S}_h$ and the characteristic polynomial of $\mathcal{S}_h$ are associates, i.e.

$$\det(I - dA_h) \sim \det(\tilde{N}_c N_p + \tilde{D}_c D_p) \sim \det(\tilde{N}_p N_c + \tilde{D}_p D_c) \qquad (3.88)$$

□

Remark 3.23 [Kuc., 1], [Kai., 1]: If the plant and controller are not free of hidden modes then

$$f_{\mathscr{S}_h}(d) := \det(I - dA_h) \sim f_p f_c \det\Delta \sim f_p f_c \det\tilde{\Delta}$$

where f_p, f_c are the hidden pole-polynomials of the plant, controller respectively.

□

3.4.3 Stabilization of the unity feedback system

The essence of feedback systems is to design a controller such that the overall system has a desired performance. The minimum required performance is clearly stability. According to theorem (3.15) and to corollary (3.9) one could pose the following two problems of stabilization.

(a) *Assignment of the elementary pole-polynomials*

Design a controller such that $H(P,C)$ has a desired set of elementary pole-polynomials $\{p_1, p_2, \ldots, p_r\}$, i.e. $\tilde{\Delta}$, Δ have $\{p_1, p_2, \ldots, p_r\}$ as nonunit invariant polynomials.

(b) *General stabilization*

Design a controller such that $H(P,C) \in M(\mathbb{R}^+(d))$, i.e. $\tilde{\Delta}$, Δ are stable polynomial matrices.

Definition 3.41: Consider the unity feedback system $\mathscr{S}_f$. Any controller that solves either problem (a) or problem (b) is called a **stabilizing controller**.

□

It turns out that both problems (a), (b) can always be solved and the solution is given by the following theorem.

Theorem 3.18 [Kuc., 1], [Vid., 1]: Consider the feedback configuration of fig. (3.4). Let $P \in \mathbb{R}^{l\times m}(d)$ be the transfer function of the plant, $(\tilde{D}_p, \tilde{N}_p)$, (N_p, D_p) be left, right coprime polynomial MFDs of P satisfying the Bezout identity

$$\begin{bmatrix} -\tilde{X}(d) & \tilde{Y}(d) \\ \tilde{D}_p(d) & \tilde{N}_p(d) \end{bmatrix} \begin{bmatrix} -N_p(d) & Y(d) \\ D_p(d) & X(d) \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{3.89}$$

and $\tilde{\Delta}$, Δ given stable polynomial matrices of appropriate dimensions. Then the family of controllers

$$C = \tilde{D}_c^{-1}\tilde{N}_c = N_c D_c$$

that assign the denominator matrix $\tilde{\Delta}$, Δ of the feedback system $\mathcal{S}_f$ satisfies either of the following Diophantine equations

$$\Delta := \tilde{N}_c N_p + \tilde{D}_c D_p \quad \text{or} \quad \tilde{\Delta} := \tilde{N}_p N_c + \tilde{D}_p D_c \tag{3.90}$$

and it is parametrized by the following manner.

$$\begin{bmatrix} N_c \\ D_c \end{bmatrix} = \begin{bmatrix} X & D_p \\ Y & -N_p \end{bmatrix} \begin{bmatrix} \tilde{\Delta} \\ R \end{bmatrix} \tag{3.91}$$

$$\begin{bmatrix} \tilde{D}_c & \tilde{N}_c \end{bmatrix} = \begin{bmatrix} \Delta & S \end{bmatrix} \begin{bmatrix} \tilde{Y} & \tilde{X} \\ -\tilde{N}_p & \tilde{D}_p \end{bmatrix} \tag{3.92}$$

where $\tilde{\Delta}$, $\Delta \in M(\mathbb{R}[d])$ have a set of given stable invariant polynomials $\{p_1, p_2, \ldots, p_r\}$, $r \le \min\{\ell, m\}$ (problem (a)), or $\tilde{\Delta}$, Δ are any stable polynomial matrices (problem (b)), and R, S are arbitrary polynomial matrices with the following properties.

a. (Δ, S) are left coprime, $(R, \tilde{\Delta})$ are right coprime to ensure coprimeness of $(\tilde{D}_c, \tilde{N}_c)$, (N_c, D_c) respectively.

b. $\det(\Delta(0)\tilde{Y}(0) - S(0)\tilde{N}_p(0)) \sim \det(Y(0)\tilde{\Delta}(0) - N_p(0)R(0)) \neq 0$ to ensure causality of the controller.

□

Remark 3.24: Theorem (3.18) gives the parametrization of all controllers that are stabilizing a given plant under the feedback configuration of fig. (3.4). This parametrization is affine in the free parameters R or S. The reason that the feedback system $\mathcal{S}_f$ is also called one-parameter feedback system, is that only one parameter (either R or S) is needed for the parametrization of the family of all stabilizing controllers. This parametrization is known as ***Youla-Bongiorno*** parametrization due to the work of Youla, Bongiorno and Lu [You., 1] and Youla, Bongiorno and Jabr [You., 2], but for the case of discrete-time systems it has appeared in an earlier work by Kucera [Kuc., 4], [Kuc., 5]. We will call the parametrization (3.91), (3.92) of the stabilizing controllers as ***Youla-Bongiorno-Kucera*** parametrization and we will denote it by YBK parametrization.

□

Remark 3.25: If the plant and the controller are minimal realizations of the MFDs given by theorem (3.18), then according to corollary (3.12) and remark (3.17), the design procedure of theorem (3.18) will assign the nonzero eigen-values of the closed-loop feedback matrix A_h to desired positions.

□

Proposition 3.5: Consider the pair (P,C). Then, any causal stabilizing controller C ensures well-posedness.

Proof. For any stabilizing controller C, Δ, $\tilde{\Delta}$ are stable and therefore causal polynomial matrices, i.e.

$$\det\Delta(0) \sim \det\tilde{\Delta}(0) \neq 0$$

Hence, since C is causal, the conditions of corollary (3.10) for well-posedness are satisfied.

□

Corollary 3.13: Consider the pair (P,C) with the plant strictly causal. Then any stabilizing controller C is causal and the pair (P,C) is well-posed for any C.

Proof. Let (N_p, D_p) be a right coprime polynomial MFD of P, i.e. $P = N_p D_p^{-1}$. Since P is strictly causal and N_p, D_p coprime we have $N_p(0) = 0$ and $D_p(0) \neq 0$. Then

$$\Delta(0) = \tilde{D}_c(0)D_p(0)$$

and $\det\Delta(0) \neq 0$ implies $\det\tilde{D}_c(0) \neq 0$, i.e causality of the controller and well-posedness of the feedback system.

□

3.5 Conclusions

The aim of this chapter was on one hand to summarize the basic concepts of system theory of linear discrete-time systems and on the other hand to describe the use and properties of the unity feedback control scheme. In doing so, we heavily relied on the algebraic framework introduced in chapter (2). We thoroughly exploited the fact that recurrent formal Laurent series in one indeterminate d over $\mathbb{R}$ and rational functions over $\mathbb{R}$ are isomorphic due to the infinite nature of $\mathbb{R}$. This resulted in particular in a unification between the d- and z-representations of linear discrete systems and the computation of their McMillan degree from either description. The internal and external descriptions of the systems with the system theoretic concepts specifically related to each of them were presented in some detail as well as the relationships between the two system descriptions.

In the final section the main features of the unity feedback configuration as a basic control scheme were described. A brief investigation of the problem of stability and stabilizability of the unity feedback scheme was given and was shown how the fraction representation approach leads to the well known Youla-Bongiorno-Kucera parametrization of the family of stabilizing controllers.

Chapter 4

FINITE SETTLING TIME CONTROL OF DISCRETE-TIME SYSTEMS:
A Survey and Some New Results

Chapter 4

FINITE SETTLING TIME CONTROL OF DISCRETE-TIME SYSTEMS: A Survey and Some New Results

4.1 Introduction

The problem of ***Finite Settling Time*** behaviour and especially the time-optimal one of a linear system has intrigued engineers for many years. Their efforts have resulted in nonlinear bang-bang control for continuous-time systems whereas in the case of discrete-time systems the FST problem, or the more commonly known deadbeat one, was solved through the use of linear time-invariant controllers. To this respect, deadbeat regulation, i.e. the forcing of the state, or the output vector of a system from any initial condition to the origin in minimum time, is unique in discrete-time systems. It was first introduced by Bergen and Ragazzini [Ber., 1] and then Kalman [Kal., 2] presented an elegant state-space solution of the discrete time-optimal linear regulation. In both cases the problem covered the single-variable case.

A large variety of versions of deadbeat have appeared which differ due to the type of problem considered and the adopted approach. Most of the work carried out until now has treated the case of state and output deadbeat regulation via state feedback, or in the case of inaccessible states via a combination of an observer and feedback of the state estimates. The main aim of the above problems is to shift all the eigenvalues (or almost all in the case of output deadbeat) of the closed-loop system to the origin. As the solution to the problem of eigenvalue, or pole placement is not uniquely determined in the multivariable case, specific types of deadbeat controllers have been proposed based on techniques varying from procedures on selecting independent vectors from a certain vector space to the solution of discrete Riccati equations [Ack., 1], [Kuc., 11], [Kuc., 12], [Led., 1], [Lew., 2], [O'R., 1], [O'R., 2].

On the other hand, Kucera has pioneered the use of polynomial algebra methods for the study of state and output deadbeat control problems of both single-variable and multivariable discrete-time systems [Kuc., 4] to [Kuc., 10]. Many other researchers followed this approach, e.g. [Eic., 1], [Wol., 3], with the most recent work of Zhao and Kimura, where the problem of robustness for multivariable deadbeat tracking is addressed [Zha., 1] to [Zha., 4].

In this chapter we attempt to survey the two main approaches, namely the state-space and the algebraic (transfer function) approach, for the solution of the deadbeat control problem and extend some basic results. In particular, we extend the main theorem of discrete-time systems given by Kalman [Kal., 2] to the MIMO case and by that we obtain an explicit parametrization of the family of all deadbeat state regulators.

4.2 Deadbeat Versus Finite Settling Time

Deadbeat and finite settling time (FST) are widely used in the literature many times interchangeably, meaning mainly the unique property of discrete-time systems to achieve specified response in finite time. Usually, deadbeat means in addition time-optimal response, and quite often the terminology is confusing.

In this brief introductory section we define formally the two terms; deadbeat and FST, and these definitions will be used consistently throughout the thesis.

Definition 4.1: A linear discrete-time system exhibits a **finite settling time** response, if it settles to **a steady state** in **finite time**.

□

Definition 4.2: A linear discrete-time system exhibits a **deadbeat** response, if it settles to a **specified steady state** in **minimum time**.

□

Remark 4.1: FST is a more relaxed concept than the deadbeat. It only assumes steady state behaviour in finite time whereas deadbeat apart from requiring time-optimal behaviour it also guarantees performance criteria, e.g. error deadbeat means zero steady state error in minimum time.

□

Most of the work carried out up to now is, or claims to be of the type of the deadbeat control rather that FST. For this reason we will devote the survey, if not entirely, at least mainly to the deadbeat control.

4.3 Deadbeat Control - A State-Space Approach

Consider an n-dimensional discrete-time system

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \quad \underline{y}_k = C\underline{x}_k + D\underline{u}_k, \quad k \geq 0 \qquad (4.1)$$

where $\underline{x}_k \in \mathbb{R}^n$, $\underline{u}_k \in \mathbb{R}^m$, $\underline{y}_k \in \mathbb{R}^l$, A not necessarily invertible and $\rho(B) = m$.

In this section we concentrate on the state-space solution of the deadbeat control and mainly on the state and output deadbeat regulation.

4.3.1 State deadbeat regulation by state feedback

Within the state-space framework two main approaches were developed. The one introduced by Kalman in its elegant work 'On The General Theory of Control Systems' [Kal., 2] is based on the concept of controllability as well as properties of kth controllable subspaces . We will refer to this approach as the **dynamic** approach. The other one is based on the result by Wonham on arbitrary eigenvalue assignment under state feedback [Won., 1], since the deadbeat regulation is equivalent to the assignment of the eigenvalues of the closed-loop system to zero. We call this approach, the **spectral** approach. We present here both approaches generalizing also for the multivariable case Kalman's result [Kal., 2] that state deadbeat regulation can be achieved, if and only if constant state feedback is used.

State deadbeat regulation : the dynamic approach

Consider the discrete-time system described by equations (4.1). The problem of state deadbeat regulation is to drive to zero in minimum time any initial state that can be driven to zero. According to section (3.3.3) the space of all initial states that can be driven to zero in at most k steps is the kth controllable subspace $\mathcal{C}_k(A,B)$, i.e.

$$\mathcal{C}_k(A,B) = \{\underline{x}_0 \; : \; A^k\underline{x}_0 = -A^{k-1}B\underline{u}_0 - \cdots -B\underline{u}_{k-1}\} \tag{4.2}$$

The following properties of $\mathcal{C}_k(A,B)$ are clear.

Lemma 4.1 [Aka., 1], [Kal., 2], [Mul., 1]: If $C_k(A,B)$ is given by (4.2) then

$$\begin{aligned} &\mathcal{C}_0 = \{0\} \\ &\mathcal{C}_{k+1} = A^{-1}\{\mathcal{C}_k + \text{range}(B)\} \\ &\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subseteq \mathbb{R}^n \end{aligned} \tag{4.3}$$

where A^{-1} is the functional inverse of A. Also, if

$$m_k = \dim\mathscr{C}_k - \dim\mathscr{C}_{k-1} \quad \text{and} \quad \mu = \min\{k : \mathscr{C}_{k+1} = \mathscr{C}_k\} \tag{4.4}$$

then

$$\mathscr{C}_k = \mathscr{C}_{k+1} \;\; \forall\; k \geq \mu \tag{4.5}$$

and μ is the maximum reachability index of the system.

□

We follow now an approach similar to that given by Mulis [Mul., 1], to prove for the multivariable case the main result of the state deadbeat regulation presented by Kalman [Kal., 2].

Lemma 4.2 [Mul., 1]: Let $Q_k = \{\underline{x} : A^k\underline{x} + B\underline{u} = 0,\ \underline{x} \notin \mathscr{C}_{k-1}\}$, $Q_0 = \{0\}$. Then

$$\mathscr{C}_k = \mathscr{C}_{k-1} \oplus Q_k = Q_0 \oplus \cdots \oplus Q_k, \quad \forall\; k \geq 0 \tag{4.6}$$

Proof. By induction. For $k = 1 \Rightarrow \mathscr{C}_1 = \{\underline{x}_0 : A\underline{x}_0 = -B\underline{u}_0\} = \{\underline{x}_0 : A\underline{x}_0 + B\underline{u}_0 = 0\} = \{0\}\oplus Q_1 = \mathscr{C}_0 \oplus Q_1 = Q_0 \oplus Q_1$. Suppose

$$\mathscr{C}_k = \mathscr{C}_{k-1} \oplus Q_k = Q_0 \oplus \cdots \oplus Q_k$$

and

$$Q_{k+1} = \{\underline{x} : A^{k+1}\underline{x} + B\underline{u} = 0,\ \underline{x} \notin \mathscr{C}_k\}$$

Consider also a space $V_{k+1} : \mathscr{C}_{k+1} = \mathscr{C}_k \oplus V_{k+1}$. Then $\forall\; \underline{x}_0 \in \mathscr{C}_{k+1}\ \exists\ \underline{w}_i \in Q_i,\ i = 1,\ldots,k \;\wedge\; \underline{v}_{k+1} \in V_{k+1}$ such that

$$\underline{x}_0 = \underline{w}_1 + \cdots + \underline{w}_k + \underline{v}_{k+1}$$

Therefore, $\exists\ \underline{u}_0,\ldots\underline{u}_{k-1} : A^i\underline{w}_i + B\underline{u}_{i-1} = 0,\ i = 1,\ldots,k$. So,

$$\underline{x}_1 = A\underline{x}_0 + B\underline{u}_0 = A\underline{w}_1 + A(\underline{w}_2 + \cdots + \underline{v}_{k+1}) + B\underline{u}_0 = A(\underline{w}_2 + \cdots + \underline{v}_{k+1})$$

$$\underline{x}_2 = A\underline{x}_1 + B\underline{u}_1 = A^2\underline{w}_2 + A^2(\underline{w}_3 + \cdots + \underline{v}_{k+1}) + B\underline{u}_1 = A^2(\underline{w}_3 + \cdots + \underline{v}_{k+1})$$

$$\vdots$$

$$\underline{x}_{k+1} = A^{k+1}\underline{v}_k + B\underline{u}_k = 0 \text{ and } \underline{v}_{k+1} \notin \mathscr{C}_k \quad \text{i.e. } \underline{v}_{k+1} \in Q_{k+1}$$

Hence,

$$\mathscr{C}_{k+1} \subseteq Q_0 \oplus \cdots \oplus Q_{k+1}$$

Clearly $Q_0 \oplus \cdots \oplus Q_{k+1} \subseteq \mathscr{C}_{k+1}$. Therefore $\mathscr{C}_{k+1} = Q_0 \oplus \cdots \oplus Q_{k+1}$

□

Remark 4.2: According to lemma (4.2), every $\underline{x} \in \mathscr{C}_k$ can be written uniquely as

$$\underline{x} = P_1\underline{\upsilon}_1 + \cdots + P_k\underline{\upsilon}_k \tag{4.7}$$

where $\underline{\upsilon}_i \in \mathbb{R}^{m_i}$ are arbitrary real vectors and P_i is a basis matrix of Q_i for $i = 1,\ldots,k$.

□

Lemma 4.3: For every $\underline{x} \in Q_k$ then $A^{k-1}\underline{x} \in Q_1$.

Proof. $\forall\ \underline{x} \in Q_k\ \exists\ \underline{u} \in \mathbb{R}^m : A^k\underline{x} + B\underline{u} = 0 \wedge \underline{x} \notin \mathscr{C}_{k-1}$. Then,

$$A^1(A^{k-1}\underline{x}) + B\underline{u} = 0 \tag{4.8}$$

i.e. $A^{k-1}\underline{x} \in \mathscr{C}_1$. For $A^{k-1}\underline{x}$ to belong to Q_1, $A^{k-1}\underline{x}$ must not belong to $\mathscr{C}_{1-1}$. Suppose $A^{k-1}\underline{x} \in \mathscr{C}_{1-1}$. Then,

$$A^{1-1}(A^{k-1}\underline{x}) = -A^{1-2}B\underline{u}_0 - \cdots -B\underline{u}_{1-2},\ \text{i.e.}$$

$$A^{k-1}\underline{x} = -A^{1-2}B\underline{u}_0 - \cdots -B\underline{u}_{1-2},\ \text{i.e.}$$

$$\underline{x} \in \mathscr{C}_{k-1}$$

which is not true because $\underline{x} \in Q_k$. Therefore

$$A^{k-1}\underline{x} \notin \mathscr{C}_{1-1} \tag{4.9}$$

So, from relationships (4.8), (4.9) we have that $A^{k-1}\underline{x} \in Q_1$.

□

Remark 4.3: According to lemma (4.2) and remark (4.2), every vector $\underline{x}_0$ that can be driven to the origin in at most k steps can be decomposed uniquely in k vectors $\underline{w}_1,\ldots,\underline{w}_k$ such that the ℓth component $\underline{w}_1$ is driven to zero by applying an input $\underline{u}_{1-1}$ such that $A^1\underline{w}_1 + B\underline{u}_{1-1} = 0$ while $\underline{w}_1$ is left free until time $\ell - 1$.

□

Remark 4.4: From lemma (4.1) and remark (4.2) we have that every initial state $\underline{x}_0$ of the system (A,B,C,D) that can be driven to the origin in any time k is uniquely expressed as

$$\underline{x}_0 = P_1\underline{v}_1 + \cdots + P_\mu\underline{v}_\mu \tag{4.10}$$

where μ is the maximum reachability index of the system.

□

Remark 4.5: $\mathscr{C}_\mu$ is the maximal space containing all the initial states $\underline{x}_0$ of the system (A,B,C,D) that can be driven to the origin. If $\mathscr{C}_\mu = \mathbb{R}^n$, then every $\underline{x}_0 \in \mathbb{R}^n$ can be driven to zero in at most $\mu \le n$ steps.

□

Theorem 4.1: Consider the discrete linear system (A,B,C,D) described by equations (4.1) with maximum reachability index μ. Every initial state $\underline{x}_0 \in \mathscr{C}_\mu(A,B)$ can be driven to zero in minimum time, if and only if $\underline{u}_k = -F\underline{x}_k$, where F is such that

$$F\cdot[P_1\ P_2\ \cdots\ P_\mu] = [B^+AP_1\ O\ \cdots\ O] \tag{4.11}$$

and B^+ is a, not uniqely defined, left inverse of B, i.e., $B^+B = I_m$.

<u>*Proof*</u>. $\forall\ \underline{x}_0 \in \mathscr{C}_k$, $k = 1,\ldots,\mu$, $\exists\ \underline{v}_i^{(0)} \in \mathbb{R}^{m_i}$, $\underline{w}_i^{(0)} \in Q_i$, $i = 1,\ldots,k$, such that

$$\underline{x}_0 = P_1\underline{v}_1^{(0)} + \cdots + P_k\underline{v}_k^{(0)} = \underline{w}_1^{(0)} + \cdots + \underline{w}_k^{(0)}$$

Then, $\exists\ \underline{u}_0 : \underline{x}_1 = A\underline{x}_0 + B\underline{u}_0 \in \mathscr{C}_{k-1}$ and $A\underline{w}_1^{(0)} + B\underline{u}_0 = 0$. So,

$$\begin{aligned}\underline{x}_1 &= A(\underline{w}_2^{(0)} + \cdots + \underline{w}_k^{(0)})\\ &= \underline{w}_1^{(1)} + \cdots + \underline{w}_{k-1}^{(1)},\ \underline{w}_i^{(1)} \in Q_i,\ i = 1,\ldots,k-1\\ \underline{u}_0 &= -B^+A\underline{w}_1^{(0)}\end{aligned}$$

Using the same procedure we can prove that

$$\underline{u}_k = -B^+A\underline{w}_1^{(k)},\quad \underline{w}_1^{(k)} \in Q_1,\ k = 0,1,\ldots,\mu-1 \tag{4.12}$$

Clearly relationships (4.12) constitute a control law of ***constant state feedback*** nature, where only part of the current state - namely the one that can be driven to zero in exactly one step - is used for the feedback. Therefore, if a feedback F is to be applied to the state $\underline{x}_k$ at each time k, it has to be orthogonal to $Q_2,\ldots,Q_\mu$ and $F|Q_1 = B^+A|Q_1$. Indeed, for every $\underline{x}_k$ that can be driven to zero

$$\begin{aligned}\underline{u}_k &= -F\underline{x}_k \quad \forall\ k = 0,1,\ldots,\mu-1\\ &= -F(P_1\underline{v}_1 + \cdots + P_\mu\underline{v}_\mu) \quad \forall\ \underline{v}_i \in \mathbb{R}^{m_i}\\ &= -B^+AP_1\underline{v}_1\end{aligned}$$

Hence

$$\begin{bmatrix} FP_1-B^+AP_1 & FP_2 & \cdots & FP_\mu \end{bmatrix}\begin{bmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_\mu \end{bmatrix} = O \quad \forall\ \underline{v}_i \in \mathbb{R}^{m_i}$$

This is true if and only if the left block matrix is zero, or

$$F\cdot\begin{bmatrix} P_1 & P_2 & \cdots & P_\mu \end{bmatrix} = \begin{bmatrix} B^+AP_1 & O & \cdots & O \end{bmatrix} \tag{4.13}$$

Equation (4.13) has always a solution since $P = [\ P_1\ \cdots\ P_\mu\ ]$ has full column rank and in the case of a controllable system $P \in \mathbb{R}^{nxn}$ and $\rho(P) = n$.

□

Remark 4.6: The requirement that $\underline{x}_0$ is driven to zero in minimum time **forces** us to use constant state feedback given by equation (4.11). This result was first proved by Kalman [Kal., 2] for the single-variable case and it was called *the fundamental theorem of linear control systems.* O'Reilly tried unsuccessfully to extend it to the multivariable case [O'R., 2]. Indeed, relationships (20) to (23) in his aforementioned work are not correct and do not readily come from his previous treatment of the problem as he claims. In fact, if we are to use a constant feedback a priori, then F must satisfy the equation

$$F\cdot[I\ \ A\ \cdots\ A^{\mu-1}] = [D_1G_1\ \cdots\ D_2G_2]$$

where D_i, G_i are constant matrices given in O'Reilly [O'R., 2], and not the equation

$$F\cdot[A^{-1}BD_1\ \cdots\ A^{-\mu}BD_\mu] = [-D_1\ \ O\ \cdots\ O]$$

as he claims.

□

Remark 4.7: According to theorem (4.1), what is needed for the computation of the constant state feedback matrix F is to derive bases $P_1, P_2, \ldots, P_\mu$ of the vector spaces $Q_1, Q_2, \ldots, Q_\mu$ and a left inverse B_0^+ of B. This can give us a parametrization of all the matrices F that achieve state deadbeat regulation as follows. If W is a basis of the left null space $\mathcal{N}_1\{B\}$ of B and $U_1, U_2, \ldots, U_\mu$ are nonsingular real matrices of appropriate dimensions, then the family of all matrices F is given by

$$F \cdot \left[\; P_1U_1 \;\; P_2U_2 \;\; \cdots \;\; P_\mu U_\mu \;\right] = \left[\; (B_0^+ + XW)AP_1U_1 \;\; O \;\; \cdots \;\; O \;\right]$$

where X is an arbitrary real matrix of appropriate dimensions.

□

From theorem (4.1) and remark (4.7) it is clear that the most important part for the computation of the constant state feedback matrix F for state deadbeat regulation is the derivation of the bases $P_1, P_2, \ldots, P_\mu$ of the vector spaces $Q_1, Q_2, \ldots, Q_\mu$. We demonstrate this by the following analysis, which also provides a new efficient algorithm for the computation of F.

Consider the discrete linear system described by the difference equations

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \tag{4.14}$$

To find the space $\mathcal{C}_k(A,B)$ we have to solve the following equations with respect to $\underline{x}_0, \ldots, \underline{x}_{k-1}, \underline{u}_0, \ldots, \underline{u}_{k-1}$.

$$\begin{aligned} \underline{x}_1 &= A\underline{x}_0 + B\underline{u}_0 \\ \underline{x}_2 &= A\underline{x}_1 + B\underline{u}_1 \\ &\vdots \\ \underline{x}_k &= A\underline{x}_{k-1} + B\underline{u}_{k-1} = 0 \end{aligned} \tag{4.15}$$

Equations (4.15) can be expressed in matrix form as

$$\begin{bmatrix} A\ B & & & & \\ -I\ O\ A\ B & & & 0 & \\ & -I\ O\ A\ B & & & \\ & & \ddots & & \\ 0 & & & & \\ & & & -I\ O\ A\ B \end{bmatrix} \begin{bmatrix} \underline{x}_{k-1} \\ \underline{u}_{k-1} \\ \vdots \\ \underline{x}_0 \\ \underline{u}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{4.16}$$

$$:= T_k(A,B)$$

Then the part of the right null space $\mathcal{N}_r\{T_k(A,B)\}$ that corresponds to $\underline{x}_0$ can give us the space $\mathcal{C}_k(A,B)$ and bases $P_1, P_2, \ldots, P_\mu$ of the vector spaces $Q_1, Q_2, \ldots, Q_\mu$. Note that if W_k is a basis of $\mathcal{N}_r\{T_k(A,B)\}$ then

$$W_k = \left[\begin{array}{c:c} O & \\ & \tilde{W}_k \\ W_{k-1} & \end{array} \right]$$

(see also proposition (6.2)). To illustrate the above we consider the following example.

Example 4.1: Consider the system

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.17}$$

Then

$$W_2 = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ \hdashline \boxed{0} & \boxed{1} & \boxed{1} & 1 \\ \boxed{1} & \boxed{0} & \boxed{0} & 0 \\ \boxed{-1} & \boxed{2} & \boxed{0} & 0 \\ \hdashline 0 & -1 & 0 & -1 \\ 0 & -2 & 0 & -1 \end{array} \right] \begin{array}{l} \\ \underline{x}_1 \\ \\ \underline{u}_1 \\ \\ \\ \underline{x}_0 \\ \\ \underline{u}_0 \\ \\ \end{array} \tag{4.18}$$

$$\qquad\qquad\qquad\qquad P_1 \qquad\qquad P_2$$

We note from (4.18), that a basis of the space $\mathscr{C}_2$ is $[P_1\ P_2]$. Therefore $\mathscr{C}_2 = \mathbb{R}^3$ which means that any initial state can be driven to the origin in at most two steps using constant state feedback. A left inverse B^+ of the matrix B of (4.17), is

$$B^+ = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the state feedback F with this particular P_1, P_2 and B^+ is a solution of the following equation

$$F \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

which gives

$$F = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & 1.0 & 1.0 \end{bmatrix}$$

Note that $(A - BF)^2 = O_3$. It is clear from the above procedure that the most important part for the computation of F is to bring the right null space of $T_k(A,B)$ in the form of W_k. This is always possible according to lemma (4.2) and provides an algorithm for the computation of the P_i's.

□

It is clear from the entire analysis of this section, that the most important aspect of the state deadbeat regulation is the characterization of the μth controllable space $\mathscr{C}_\mu(A,B)$. For an n-dimensional system (A,B,C,D), this reduces to the specification of an adequate selection procedure for determining n linearly independent columns of $\mathscr{C}_\mu$. Lemma (4.2) provides such a universal procedure by decomposing $\mathscr{C}_\mu$ to a direct sum of the uniquely defined spaces Q_i, $i=1,\dots,m$. This allows for the generalization of Kalman's ***fundamental theorem of linear control systems*** [Kal., 2].

This result has been rediscovered by many authors in a variety of different forms, not always correct and not always leading to time-optimal regulation as it is claimed by many of them, like Fahmy, Hanafy and Sakr [Fah., 1], Kucera [Kuc., 11], [Kuc., 12], Leden [Led., 2], Lin [Lin 1], Ludyk [Lud., 1] and Tou [Tou 1]. For a brief discussion of the most of the aforementioned work one could refer to O'Reilly's survey 'The Discrete Linear Time Invariant Time-Optimal Control Problem - An Overview' [O'R., 2].

State deadbeat regulation : the spectral approach

In the previous section we concentrated mainly on the structural properties of the kth controllable subspaces $\mathcal{C}_k(A,B)$ of the linear discrete-time system (A,B,C,D). Following theorem (4.1), we observe that the optimal closed-loop state matrix

$$K = A - BF.$$

has the following properties.

Property 4.1 [Cad., 1], [Far., 1], [Kuc., 11]: The matrix K is nilpotent, the maximum reachability index μ of the system being its nilpotency index, i.e. $(A - BF)^{\mu} = 0$.

□

Property 4.2 [Cad., 1], [Far., 1], [Kuc., 11]: The eigenvalues of $A - BF$ are all zero and its eigenvectors span the right null space $\mathcal{N}_r\{A - BF\}$ of $A - BF$.

□

Property 4.3 [Cad., 1], [Kha., 1], [Kuc., 11]: The Jordan canonical form J of the matrix K is composed of m nilpotent Jordan blocks J_{μ_j} of order μ_j, $j = 1,2,...,m$.

$$J = \begin{bmatrix} J_{\mu_1} & & & 0 \\ & J_{\mu_2} & & \\ & & \ddots & \\ 0 & & & J_{\mu_m} \end{bmatrix} \tag{4.19}$$

where $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ are the reachability indices of (A,B,C,D). Each Jordan block is a $\mu_j \times \mu_j$ matrix with ones on the first superdiagonal and zeros elsewhere, i.e.

$$J_{\mu_j} = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix} \tag{4.20}$$

□

From properties (4.1) to (4.3), it is clearly indicated that the state deadbeat regulation problem is a special eigenvalue assignment problem by constant state feedback; that is a problem where all the eigenvalues of the closed-loop system are assigned zero values and in addition its Jordan canonical form is of the form of (4.19). The structure of the Jordan canonical form (4.19), guarantees that every initial state of the system (A,B,C,D) that can be driven to the origin it is driven in the minimum possible time $\tau \leq \mu$. The minimum time as a whole is equal to the maximum reachability index μ and is attained by those states that can be steered to the origin in exactly and only μ steps. These minimality conditions define the state deadbeat regulation and force us to the use of constant state feedback.

We can relax the strict time-minimality conditions by allowing the states to move to the origin not in the absolute minimum time τ, but in time not greater than μ. We define this problem where the overall minimum deadbeat time is μ, as **relaxed state deadbeat regulation**. Taking into account the admissible Jordan forms for zero eigenvalue assignment [Bru., 1], the Jordan forms for the relaxed state deadbeat regulation must be of the form

$$J = \begin{bmatrix} J_{n_1} & & & 0 \\ & J_{n_2} & & \\ & & \ddots & \\ 0 & & & J_{n_m} \end{bmatrix} \tag{4.21}$$

with $n_1 \geq n_2 \geq \cdots \geq n_m$ and

$$\begin{aligned} n_1 &= \mu_1 = \mu \\ n_1 + n_2 &\geq \mu_1 + \mu_2 \\ &\vdots \\ n_1 + n_2 + \cdots + n_m &= \mu_1 + \mu_2 + \cdots + \mu_m = n \end{aligned} \tag{4.22}$$

where the last sum is equal to the system dimension n, if and only if the pair (A,B) is reachable. Obviously, if $n_1 > \mu$ the system response is not time-optimal any more but still it is achieved in at most finite time n_1 [Fah., 2].

In view of the above discussion the state deadbeat regulation problem can be reformulated as that of finding a control law of constant state feedback nature

$$\underline{u}_k = -F\underline{x}_k \tag{4.23}$$

such that the closed-loop system matrix $A - BF$ is similar to the Jordan form J of the form of (4.21). When strict inequalities are applied in (4.22), then F corresponds to the relaxed state deadbeat regulation problem.

The state deadbeat regulation problem may be readily achieved by transforming (A,B,C,D) into the controllable canonical form [Bru., 1], [Lue., 3]

$$\underline{\bar{x}}_{k+1} = \bar{A}\underline{\bar{x}}_k + \bar{B}\underline{u}_k \tag{4.24}$$

by means of the state similarity transformation

$$\bar{\underline{x}}_k = Q\underline{x}_k \tag{4.25}$$

If the non-trivial rows of $\bar{A}$ and $\bar{B}$ are given by the matrices $\bar{A}_m$ and $\bar{B}_m$ respectively, then we have the following result.

Theorem 4.2 [O'R., 2]**:** The state deadbeat regulation problem can be solved using constant state feedback

$$\underline{u}_k = -F\underline{x}_k$$

where

$$F = \bar{B}_m^{-1}\bar{A}_m Q \tag{4.26}$$

□

The deadbeat controller of theorem (4.2), like the one of theorem (4.1) has the properties (4.1) - (4.3). This result was originally introduced by Ackerman [Ack., 1] and was also derived independently by Prepelitá [O'R., 2]. Since then, it has been rediscovered in that form or in the form of relaxed, or non-time-optimal deadbeat, by many researchers like O'Reilly [O'R., 1], [O'R., 3], Pachter [Pac., 1], Porter [Por., 1], Fahmy et al. [Fah., 1].

Looking at the state deadbeat regulation as a special eigenvalue assignment problem, it is clear that there is no unique feedback matrix F that moves the eigenvalues of the closed-loop system to the origin. Since the transient response is determined by the eigenvectors as well, the deadbeat problem can be considered as an eigenstructure rather than an eigenvalue assignment problem and as such has been treated by Klein [Kle., 1], Porter and D'Azzo [Por., 2], Sebakhy and Abdel-Maneim [Seb., 1] and most recently by Elabdalla and Amin [Ela., 1] and Van Dooren [Van., 2] where the minimum Frobenius norm matrix F that assigns the eigenvalues to zero, is derived.

4.3.2 Parametrization of the state deadbeat regulators

It is clear from the analysis so far, that the multivariable state deadbeat regulators are not uniquely defined in general. Therefore, it is natural to require an explicit parametric solution if possible, to compensate for additional performance or robustness requirements. Theorems (4.1) and (4.2) provide the background for such a parametrization from two different points of view and in this section we investigate both viewpoints; the results here, as far as we are aware, are new. An investigation is needed to whether or not the proposed parametrizations are proper, i.e. whether only the genuine independent free parameters appear in the deadbeat controllers.

The following parametrization is based on theorem (4.1) and provides a complete and explicit parametrization of the family of all state deadbeat regulators.

Theorem 4.3: Consider the reachable n-dimensional system (A,B,C,D) with m inputs, ℓ outputs, $\rho(B) = m$ and μ the maximum reachability index. Let $Q_1,Q_2,...,Q_\mu$ be the spaces defined in lemma (4.2) and $P_1,P_2,...,P_\mu$ be any bases of $Q_1,Q_2,...,Q_\mu$ respectively. Then the family of all state feedback matrices that guarantee state deadbeat regulation is given by

$$\mathcal{F}(A,B) := \{ F : FP + XR = S \} \tag{4.27}$$

where

$$\begin{aligned} P &= [P_1 \; P_2 \; \cdots \; P_\mu] \\ R &= [-WAP_1 \; O \; \cdots \; O] \\ S &= [B_0^+AP_1 \; O \; \cdots \; O] \end{aligned} \tag{4.28}$$

B_0^+ is a left inverse of B and W is a basis of the left null space $\mathcal{N}_l\{B\}$ of B.

<u>*Proof*</u>. According to remark (4.7), the family of all matrices

F that achieve state deadbeat regulation is given by

$$F\cdot\begin{bmatrix} P_1U_1 & P_2U_2 & \cdots & P_\mu U_\mu \end{bmatrix} = \begin{bmatrix} (B_0^+ + XW)AP_1U_1 & O & \cdots & O \end{bmatrix} \tag{4.29}$$

where $U_1, U_2, \ldots, U_\mu$ are any nonsingular real matrices with appropriate dimensions. Equation (4.29) may be written as

$$F\cdot\begin{bmatrix} P_1 & P_2 & \cdots & P_\mu \end{bmatrix}U = \begin{bmatrix} (B_0^+ + XW)AP_1U_1 & O & \cdots & O \end{bmatrix} \tag{4.30}$$

where

$$U = \mathrm{diag}\{U_1, U_2, \ldots, U_\mu\}$$

Multiplying both sides of equation (4.30) from the right by the inverse of U, we have

$$F\begin{bmatrix} P_1 & P_2 & \cdots & P_\mu \end{bmatrix} + X\begin{bmatrix} -WAP_1 & O & \cdots & O \end{bmatrix} = \begin{bmatrix} B_0^+AP_1 & O & \cdots & O \end{bmatrix}$$

which completes the proof of theorem (4.3).

□

Theorem (4.3) gives an elegant parametrization of the family $\mathcal{F}(A,B)$ of all state deadbeat regulators as a solution of the matrix equation (4.27) over $\mathbb{R}$. An alternative parametrization is based on property (4.3) and on the treatment of the state deadbeat regulation as a special eigenstructure assignment problem.

According to property (4.4) the family of all state deadbeat regulators for the pair (A,B) is given by

$$\mathcal{F}(A,B) := \{ F \in \mathbb{R}^{m\mathrm{x}n} : A - BF \text{ is similar to } J \} \tag{4.31}$$

where J is the Jordan matrix of the form (4.19). Therefore

$$A - BF = XJX^{-1} \tag{4.32}$$

where $X \in \mathbb{R}^{n\mathrm{x}n}$ is any nonsingular matrix, or equivalently

$$AX - XJ - BFX = 0 \tag{4.33}$$

Hence, the whole problem of state deadbeat regulation is reduced to the solution of equation (4.33) with X a non-singular real matrix. The following theorem is based on the above analysis and allows for the parametrization of the family $\mathcal{F}(A,B)$.

Theorem 4.4: Consider the system (A,B,C,D) with reachability indices $\mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. If

$$J = \begin{bmatrix} J_{\mu_1} & & & 0 \\ & J_{\mu_2} & & \\ & & \ddots & \\ 0 & & & J_{\mu_m} \end{bmatrix}$$

with

$$J_{\mu_j} = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix}$$

then the family $\mathcal{F}(A,B)$ of the state deadbeat regulators is given by

$$\mathcal{F}(A,B) = \{ \, F : F = HX^{-1} \, \} \tag{4.34}$$

where H, X with X nonsingular is the solution pair of the equation

$$(I \otimes A - J^t \otimes I)\tilde{\underline{x}} - (I \otimes B)\tilde{\underline{h}} = 0 \tag{4.35}$$

with $\tilde{\underline{x}} = [\underline{x}_1^t \; \cdots \; \underline{x}_n^t]^t$ and $\tilde{\underline{h}} = [\underline{h}_1^t \; \cdots \; \underline{h}_n^t]^t$, $\underline{x}_1, \ldots, \underline{x}_n$ and $\underline{h}_1, \ldots, \underline{h}_n$ being the column vectors of X and H respectively.

Proof. If $H = FX$, then equation (4.33) is equivalent to

$$AX - XJ - BH = 0 \tag{4.36}$$

and $F = HX^{-1} \in \mathcal{F}(A,B)$ if X is nonsingular. It can be readily verified that equation (4.36) is equivalent to (4.35) (for more details see Bellman [Bel., 1]).

□

The problem of parametrizing the whole class of the state deadbeat regulators was introduced by Sebakhy and Abdel-Moneim [Seb., 2] by minimizing a quadratic performance index. Their approach has led to an overparametrized description of the family. The first attempt for a proper parametrization (minimum number of parameters) was given by Schlegel [Sch., 1] using a procedure similar to that of theorem (4.4) but his solution involves only systems with nonsingular transition matrix $\boldsymbol{A}$. Fahmy and O'Reilly [Fah., 2], under the same assumption on the invertibility of the transition matrix, treated the more general problem of parametrizing the non-time-optimal deadbeat regulators. Another approach, avoiding the invertibility assumption, was presented by Amin and Elabdalla [Ami., 1]. It provides an explicit parametrization of the state deadbeat regulators through the minimum parameters and is based on the theory of decoupling and the properties of square decouplable systems. Finally, Funahashi and Yamada [Fun., 1] gave a solution to the non-time-optimal (without restricting the dimensions of the Jordan blocks to the reachability indices) state deadbeat parametrization allowing for systems with singular transition matrices.

4.3.3 State deadbeat regulation with inaccessible states

Theorems (4.1) and (4.2) assume complete knowledge of the states of the system for the solution of the state deadbeat regulation problem. However, this is not usually the case and the state is available through the output equation

$$\underline{y}_k = C\underline{x}_k + D\underline{u}_k \tag{4.37}$$

where $\underline{y}_k \in \mathbb{R}^l$ and $\rho(C) = \ell$. The problem then can be solved in either of two ways. First, we can reconstruct the state, or a linear functional of the state in minimum time and then use it in a state feedback scheme for state deadbeat, or alternatively we can use constant output feedback instead of state feedback. In the sequel we describe briefly both approaches. For more details one could refer to [O'R., 2] and references therein.

State DB regulation using observers

One way to solve the inaccessible state deadbeat regulation problem is through the use of observers. A full order deadbeat observer can be used to reconstruct in minimum time the state $\underline{x}_k$ and then a feedback law provided by either theorem (4.1) or (4.2) can be applied on the reconstructed state $\hat{\underline{x}}_k$ [Por., 3], [Aka., 1]. The problem of deadbeat reconstruction is dual to deadbeat control, as it was first pointed out by Kalman [Kal., 2]. Therefore the state can be reconstructed in no more than ν steps, where ν is the maximum observability index of the system. Hence, the state can be driven to the origin in at most $\mu + \nu$ steps using a full order observer.

Taking into account that ℓ states can be computed from the inputs and outputs of the system using the output equation (4.37), we can use a reduced, or minimum order observer to reconstruct the remaining $n - \ell$ states in minimum time. Porter and Bradshaw [Por., 4] and Ichikawa [Ich., 1] have proposed such observers based on Luenberger's design method for reduced order observers [Lue., 4]. An alternative design has been proposed by O'Reilly [O'R., 1] Ichikawa [Ich., 1] and Inoue et al. [Ino., 1]. The advantage of the reduced order deadbeat observers is that they reconstruct the state of the system in at most ν - 1 steps rather than in ν.

Finally, instead of reconstructing the state of the system, we can reconstruct directly the deadbeat control law using a linear function observer. Therefore, the output $\hat{\underline{n}}_k$ of the observer estimates the linear functional $\underline{n}_k = -F\underline{x}_k$ in minimum time τ. It can be also required that the order of the dead-beat observer be minimum. The problem has been treated by Nagata et al. [Nag., 1], Kimura [Kim., 1] and Adachi et al.

[Ada., 1] where it is pointed out that the minimum settling time τ is less than or equal to the maximum observability index ν.

State DB using constant output feedback

Another way to solve the inaccessible state deadbeat regulation problem is by using constant output feedback of the form

$$\underline{u}_k = -L\underline{y}_k \tag{4.38}$$

The problem has been treated by some authors but the result is not time-optimal. Seraji [Ser., 1] finds L such that the closed-loop system matrix $A - BLC$ is nilpotent with index of nilpotency n rather than μ, whereas Chammas and Leondes [Cha., 3] propose a deadbeat output feedback controller that drives the system to the origin in at most $\mu \times \nu$ steps.

4.3.4 Bounded, minimum-cost state deadbeat regulation

It is often the case with the time-optimal control that the demand for minimum settling time produces unacceptably high control inputs, or states which may lead to saturation. For this reason and according to the design conditions of the particular problem, we may introduce amplitude constraints to the controls and/or the states. The time-optimal state regulation with constraints will be called **bounded** or **constrained state deadbeat regulation**. For an n-dimensional system

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k, \quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m} \tag{4.39}$$

the constrained state deadbeat regulation is defined as follows.

Find admissible controls and states such that

$$k_m = \min\{\ k : \underline{x}_k(\underline{x}_0;\underline{u}_0,\underline{u}_1,\ldots,\underline{u}_k),\ \underline{u}_i \in \Omega,\ \underline{x}_i \in \Gamma\ \} \tag{4.40}$$

where

Ω is a compact, convex polyhedral set defined by its vertices

$$\underline{\omega}_1,\ldots,\underline{\omega}_m;\ \text{containing the origin in its interior} \tag{4.41}$$

and

Γ is a compact, convex polyhedral set defined by its vertices

$$\underline{\gamma}_1,\ldots,\underline{\gamma}_m;\ \text{containing the origin in its interior} \tag{4.42}$$

It is clear that the minimum time attained for constrained deadbeat is greater than or equal to the time for unconstrained deadbeat.

The problem of constrained state deadbeat with bounded controls only and Ω being the unit cubic in the m-dimensional control space, was first introduced by Kalman [Kal., 5] and it was subsequently analyzed in detail by Wing and Desoer [Win., 1] in 1963. They introduced the geometrical concept of critical hypersurfaces in the state space and proposed a nonlinear state feedback control strategy. Their main result was a switching surface in $\mathbb{R}^n$ that separates the region of positive and negative control. The switching surface is easily developed for second-order systems, but the actual method of implementing the strategy for higher order systems was not considered because of its high complexity. A systematic algorithm has been developed by Brück [Brü., 2]. Based on the convexity of certain modified reachable sets a multi-input regulator was realized as a series connection of m single-input regulators with a total on-line computation effort considerably less than m times the single-input effort that was first introduced by Ludyk [Lud., 1]. A different procedure applicable to sampled-data systems and based on modal control was developed by Crossley and Porter [Cro., 1].

This led to a design algorithm which is computationally straightforward and does not involve the determination of switching surfaces.

Gutman and Cwikel [Gut 1], were the first to solve the constrained state deadbeat regulation problem (4.40) where state constraints are also included. However, their algorithm involves rather difficult computations and the determination of a minimum-time feedback law is not simple. Keerthi and Gilbert [Kee., 1] offered a new approach to the problem in which a facial description of the k-step admissible sets is used. The computations were more systematic and straightforward and the complete class of minimum-time feedback laws was characterized. Finally, we note that Kolev [Kol., 2] and Rasmy and Hamza [Ras., 1] have reduced constrained deadbeat regulation with bounded controls only, to a linear programing problem by minimizing an ℓ^1-type and ℓ^∞-type performance index of the controls correspondingly.

An alternative problem to the constrained state deadbeat regulation is the **minimum gain state deadbeat regulation**. We have already mentioned the work of Elabdalla and Amin [Ela., 1] and Van Dooren [Van., 2] which produces a minimum Frobenius norm deadbeat feedback matrix $\boldsymbol{F}$. Leden [Led., 1] arrives to a minimum feedback gain state deadbeat regulator by solving a Riccati equation involving a quadratic performance index which penalizes only the final state. However, the problem is restricted to systems with invertible transition matrix $\boldsymbol{A}$.

Taking into account the nonuniqueness of the state deadbeat regulators it is natural to introduce further performance criteria such as minimum-cost state deadbeat regulation. Emami-Naeini and Franklin [Ema., 1] have considered a quadratic cost function of the states and the controls and they discussed also the case of robust deadbeat tracking. Sebakhy and Abdel-Moneim [Seb., 2] have considered a similar quadratic cost function which has resulted in an optimal shaping of the transient response of the closed-loop system and to the parametrization of the class of state deadbeat regulator. However, as they note in their paper, their method leads to overparametrization.

4.3.5 Output deadbeat regulation

A relaxed version of the state deadbeat regulation is the **output deadbeat regulation**, where the output - and not the states - of the system $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$ is required to be driven to the origin in minimum time for any given initial state. Clearly, state deadbeat regulation implies output deadbeat regulation, but the opposite is not true in general.

The first state-space solution of the above problem was given by Kucera [Kuc., 12] for the case of SISO discrete-time systems. It is closely related to the theory of inverse systems and the output of the system is zeroed in a number of steps equal to the relative order of the system. Five years later, in 1977, Leden [Led., 3] tackled the multivariable output deadbeat regulation for sampled-data (strictly causal) systems, using a geometric approach. Leden's regulator involves linear state feedback and is essentially a cancellation controller since the invariant zeros of the plant are part of the poles of the closed-loop polynomial.

A special feedback law of the form

$$\underline{u}_k = -(\boldsymbol{CB})^{+}\boldsymbol{CA}\underline{x}_k \tag{4.43}$$

that achieves output regulation in one sampling instant was proposed by Corsetti and Houpis [Cor., 1]. Unlike Kucera's and Leden's output regulators, this controller does not require invertibility of the system transition matrix $\boldsymbol{A}$. The only requirement is the existence of the right inverse $(\boldsymbol{CB})^{+}$ of the matrix $\boldsymbol{CB}$. Again, if the set of invariant zeros of the plant is not empty, the aforementioned controller always assigns these zeros as closed-loop poles. This may lead to instability when the invariant zeros are located inside the unit disc in the d-plane.

To overcome the problem of instability, Leden proposed a suboptimal controller where only the stable invariant zeros of the open-loop system are allowed to appear in the closed-loop characteristic polynomial. The first to consider the stability requirements more systematically, were Akashi and Imai [Aka., 3]. In their approach, the output deadbeat regulator is required to have the additional property that the control input to the system converges to zero as time goes to infinity. Two configurations of such controllers together with the existence conditions were considered; one was a state feedback and the other was a dynamic controller using an observer. In 1981, Kimura [Kim., 2] treated the problem of stability in a systematic way by considering the problem of output deadbeat regulation with internal stability. He used two types of controllers; one was a constant state feedback regulator and the other was a regulator involving a minimum-time minimum-order function observer. In both cases the solvability conditions, the minimal settling time and the characterization of the minimal-time regulators were derived.

Akashi and Adachi [Aka., 2] used a geometric approach for the problem of output deadbeat regulation with incomplete state observation. The system considered was of the form

$$\begin{aligned} \underline{x}_{k+1} &= A\underline{x}_k + B\underline{u}_k \\ \underline{y}_k &= C\underline{x}_k \\ \underline{z}_k &= D\underline{x}_k \end{aligned} \tag{4.44}$$

where $\underline{x}_k \in \mathbb{R}^n$ is the state vector, $\underline{u}_k \in \mathbb{R}^m$ is the input vector, $\underline{y}_k \in \mathbb{R}^p$ is the output vector directly observed and $\underline{z}_k \in \mathbb{R}^l$ is the output vector to be regulated. Necessary and sufficient conditions for existence were given and a controller based on the separation principle was subsequently designed. A state estimator resembling a Kalman filter was proposed with a time-variable gain obtained as a solution to a singular Riccati equation. The feedback gain then is one of the minimum-time regulation gains in the case when complete state observation is available.

Two years later, Akashi and Imai [Aka., 4] considered the problem of output deadbeat under the presence of unmeasured disturbances. They defined this problem as the problem of disturbance localization and output deadbeat control. The model they used was similar to the one described by equations (4.44) by adding an additional input of the form $\boldsymbol{E}\underline{d}_i$ to the state equations, where $\underline{d}_i$ is the disturbance vector. Necessary and sufficient conditions were given for the existence of a controller using an observer that achieves simultaneous disturbance localization and output deadbeat regulation and an algorithm of designing such a controller was presented. But despite the output regulation performance under unmeasurable disturbances, the controller proposed by Akashi and Imai suffers a considerable drawback; it does not guarantee internal stability and so undesirable situations may occur.

Another way to solve the deadbeat output regulation problem is to formulate it, similarly to the state deadbeat regulation case, as an optimal control problem with no cost on the controls. Marrari, Emami-Naeini and Franklin [Mar., 3] have considered such a problem by minimizing the objective function

$$J = \frac{1}{2} \sum_{i=0}^{\infty} \underline{x}_i^t Q \underline{x}_i \qquad (4.45)$$

where $Q = \hat{H}^t\hat{H}$ such that the system $(\boldsymbol{A},\boldsymbol{B},\hat{H})$ has no finite transmission zeros except for all the stable transmission zeros of the plant. Due to the choice of Q, the controls are internally stable and drive the output of the system to zero in minimum time using state feedback. Two algorithms were provided for the solution. One is numerically stable and covers the most general case whilst the other, if the plant structure makes it feasible, allows some chosen subset of outputs to be driven to zero faster than the others using an output deadbeat controller. In 1991, an extension to the first algorithm, having particular advantages when dealing with the case of a plant with multiple stable transmission zeros, was presented by Spurgeon and Pugh [Spu., 1].

A different approach was used by F. Lewis for the solution of the output deadbeat regulation problem. By taking a geometrical approach, and by using a type of orthogonality in constructing bases for the kth controllable subspaces, he reduced the problem to the solution of a specialized Riccati equation whose solution sequence is used to compute the time-varying optimal state feedback gains.

4.3.6 Deadbeat tracking

Another aspect of the deadbeat control is that of deadbeat tracking where the states, or the outputs of the system are required either to settle to a specified value, or to follow a specified signal in minimum time. Yih-Shuh Jan [Jan., 2] presented a simple algorithm for the deadbeat control of a class of SISO linear time-invariant systems with time polynomial inputs. The system used was represented in phase-variable canonical form and it was required that any initial state be forced to a desired state $\underline{x}_s$ in minimum settling time whereas the output follows a reference polynomial signal. The restrictions of the method are that it can be used in a class of SISO systems and that the highest order of the polynomial inputs should be no larger than the system order.

Bradshaw and Porter developed a systematic design of linear MIMO discrete-time systems for tracking polynomial inputs. First, they considered the case of accessible states and later on they extended their results to the case of inaccessible states [Bra., 1], but their design does not necessarily lead to deadbeat control.

In 1982, De Vlieger et al. [Vli., 1] presented a time-optimal control algorithm allowing bounds on the control and state variables. Apart from the constraints the main control requirement was that the state $\underline{x}_k$ and the output $\underline{y}_k$ follow desired trajectories $\underline{x}_k^*$ and $\underline{y}_k^*$ after a minimum time k_s. The whole problem was reduced to a linear programming problem and for real-time applications feedback control was achieved by recalculating the control sequence each sampling period.

Finally Scott, see [Sco., 1] and references therein, proposed a unified approach to solving three optimal control problems under general constraints, using linear programming techniques. The main objective was to find a feasible input sequence which would drive the system from its initial state to the desired stable final state $\underline{x}_f$. The problems considered were: 1) deadbeat (time-optimal) control; 2) fuel-optimal control in fixed time by minimizing the ℓ^1-norm of the control sequence; 3) deadbeat control with control and state constraints.

4.4 Deadbeat Control - A Transfer Function Approach

In the previous section (4.3), we presented an extensive survey and some new results on deadbeat control based on a state-space approach. An alternative approach, which we will investigate in this section, is the transfer function approach. This was the approach used by Bergen and Ragazzini [Ber., 1], when they first introduced and solved the deadbeat tracking problem, and has been promoted ever since mainly by Kucera. Kucera's approach depends to a large extend on polynomial matrix techniques and the design involves the solution of linear matrix equations. For this reason it is called by Kucera the **polynomial equation**, or **algebraic approach** [Kuc., 1], [Kuc., 13], and it offers a straightforward and computationally efficient alternative to the state-space methods.

In the following sections we will briefly present the application of the algebraic methods to the deadbeat regulation and tracking problem. Most of this part of the survey is based on the work done by Kucera.

4.4.1 State deadbeat regulation by state feedback

Consider the n-dimensional discrete-time system

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \ , \quad k \geq 0 \tag{4.46}$$

where $\underline{x}_k \in \mathbb{R}^n$ and $\underline{u}_k \in \mathbb{R}^m$. If $d = z^{-1}$ is the delay operator (see chapter 2), the state and control sequences can be written as

$$\underline{x} = \underline{x}_0 + \underline{x}_1 d + \underline{x}_2 d^2 + \cdots$$
$$\underline{u} = \underline{u}_0 + \underline{u}_1 d + \underline{u}_2 d^2 + \cdots$$

Hence, an equivalent description of the system (4.46) is

$$\tilde{D}_p \underline{x} = \tilde{N}_p \underline{u} + \underline{x}_0 \tag{4.47}$$

where

$$\begin{aligned} \tilde{D}_p &= I_n - dA \in \mathbb{R}^{n\mathrm{x}n}[d] \\ \tilde{N}_p &= dB \quad \in \mathbb{R}^{n\mathrm{x}m}[d] \end{aligned} \tag{4.48}$$

As a generalization of the constant state feedback control law we are looking for all control laws of the form

$$\underline{u} = -N_c \underline{w} \ , \quad \underline{x} = D_c \underline{w} \tag{4.49}$$

where $\underline{w}$ is an internal variable and $D_c \in \mathbb{R}^{n\mathrm{x}n}[d]$, $N_c \in \mathbb{R}^{m\mathrm{x}n}[d]$. Substituting (4.49) into (4.47) we obtain

$$\begin{aligned} \underline{x} &= D_c(\tilde{D}_p D_c + \tilde{N}_p N_c)^{-1}\underline{x}_0 \\ \underline{u} &= -N_c(\tilde{D}_p D_c + \tilde{N}_p N_c)^{-1}\underline{x}_0 \end{aligned} \tag{4.50}$$

If $\underline{x}$ is to settle to zero in finite time, then $\underline{x}$ has to be a sequence of finite length, i.e. polynomial in d. Since this is to hold true for every initial state $\underline{x}_0$, D_c and N_c must satisfy the equation

$$\tilde{D}_p D_c + \tilde{N}_p N_c = I_n \tag{4.51}$$

Equation (4.51) has a solution if and only if $(\tilde{D}_p, \tilde{N}_p)$ are left coprime, and this is true if and only if the pair (A,B) is controllable. However, not all solutions of (4.51) qualify for deadbeat (minimum-time) regulation. For deadbeat regulation, $\underline{x}$ has to be not only polynomial but of **least degree** as well. Since

$$\underline{u} = -N_c \underline{x}_0 \ , \quad \underline{x} = D_c \underline{x}_0$$

we must take the solution that minimizes every column of D_c (and hence of N_c). We can state now the next theorem.

Theorem 4.5 [Kuc., 2], [Kuc., 6]: Consider the discrete-time system (A,B) defined by equation (4.46). Then, there exists a deadbeat regulator of state feedback form, if and only if the pair (A,B) is controllable. The deadbeat gain F is given by

$$F = -N_c D_c^{-1} \tag{4.52}$$

where N_c, D_c, with $\det D_c(0) \neq 0$, is a minimum column degree solution pair of the equation

$$\tilde{D}_p D_c + \tilde{N}_p N_c = I_n \tag{4.53}$$

and $\tilde{D}_p$, $\tilde{N}_p$ are given by the relationships (4.48).

□

It must be noted that the polynomial approach, in contrast to the recursive state-space methods of section (4.3.1), is straightforward and most important, it provides all the dynamic state feedback control laws in parametric form.

Remark 4.8 [Kuc., 6]: The state deadbeat regulator (4.52) is more general than the one using state-space techniques in that it includes dynamical control laws. Any solution of the Diophantine equation (4.53) assigns to the closed-loop system the characteristic polynomial z^n and therefore solves the relaxed state deadbeat regulation problem. For deadbeat regulation, the closed-loop characteristic polynomial must be split into invariant polynomials so that the minimal invariant polynomial must be z^{μ}, μ being the maximum reachability index of the system (A,B). This requirement is satisfied by the minimum column degree solutions of equation (4.53).

□

4.4.2 State deadbeat regulation by output feedback

It is often the case, as it has already been pointed out in section (4.3.3), that the state $\underline{x}$ of the system

$$\underline{x}_{k+1} = A\underline{x}_k + B\underline{u}_k \tag{4.54}$$

is not directly accessible. Instead, a linear combination of the state is available through the output equation

$$\underline{y}_k = C\underline{x}_k + D\underline{u}_k \tag{4.55}$$

Applying the d-Transform to equations (4.54) and (4.55) as in section (4.4.2), the system (A,B,C,D) can be described by the input-output equation

$$\tilde{D}_p\underline{y} = \tilde{N}_p\underline{u} + D_{p0} \tag{4.56}$$

where $\tilde{D}_p$, $\tilde{N}_p$ are polynomial matrices in d of appropriate dimensions and D_{p0} is a column vector accounting for the initial conditions.

Accordingly, the state deadbeat regulator is a dynamical system described by the equation

$$\tilde{D}_c\underline{u} = -\tilde{N}_c\underline{y} + D_{c0} \tag{4.57}$$

and the controller transfer function is given by

$$C(d) = \tilde{D}_c^{-1}\tilde{N}_c = N_cD_c^{-1} \tag{4.58}$$

Then, using similar arguments to those of section (4.4.1), we have the following theorem.

Theorem 4.6 [Kuc., 2], [Kuc., 10]: Consider the discrete-time system (A,B,C,D) and let $(\tilde{D}_p,\tilde{N}_p)$ be a left $\mathbb{R}[d]$-coprime fraction of the plant transfer function such that $[\tilde{D}_p \ \tilde{N}_p]$ is row reduced and row ordered. There exists a state deadbeat regulator with transfer function

$$C(d) = N_c D_c^{-1} \tag{4.59}$$

if and only if the system (A,B,C,D) is controllable and constructible. The pair (D_c,N_c) is the minimum column degree solution of the Diophantine equation

$$\tilde{D}_p D_c + \tilde{N}_p N_c = I_n \tag{4.60}$$

such that $D_c(0)$ is invertible.

□

Remark 4.9: The deadbeat controller (4.59) based on the Diophantine equation (4.60) provides an efficient alternative to the state-space methods described in section (4.3.3). The polynomial equation approach unifies the theory as well as the design of deadbeat controllers and includes as a subset all the observer-based deadbeat controllers of section (4.3.3). It is worth noting that the nonminimal solutions to equation (4.60), though not of deadbeat type, still drive any initial state of the system to zero.

□

4.4.3. Deadbeat tracking

We recall that by deadbeat tracking we mean the problem of designing a controller such that the closed-loop system settles to zero steady state error for a given reference sequence in minimum time independently of the initial conditions.

This is a unique feature of linear time-invariant discrete-time systems, as it was first mentioned by Bergen and Ragazzini [Ber., 1] for the case of SISO sampled-data systems as early as 1954. Their method was based on a unity feedback scheme incorporating a cancellation controller [Kuo., 1], [Ise., 1], whereby the absolute minimum settling time, allowing for realizability of the controller, was achieved by canceling out the undesired poles or zeros of the d-transfer function of the plant. This, they noted, could lead to instability. To avoid this drawback they proposed a design that did not allow pole-zero cancellations inside the unit disc. Such a controller was internally stable at the expense of minimum settling time.

In 1980, Jordan and Korn [Jor., 1] considered the case of deadbeat error control of multivariable processes with step reference signals. But again, their proposed design could neither guarantee stability of the closed-loop system, nor finite transient response to state disturbances. Thirty two years after the original work by Bergen and Ragazzini, a paper by Wang and Chen [Wan., 1] was published dealing with the problem of simultaneous deadbeat tracking for SISO discrete-time systems as they called it. Their approach, apart from the fact that it considered a prespecified class of inputs rather than a unique type of input for tracking, hardly distinguishes from the work of Bergen and Ragazzini.

All the aforementioned work was based on a transfer function approach where the transfer function of the system was expressed as a fraction of polynomial matrices. Kucera was the first to formulate the deadbeat tracking problem using a polynomial equation approach where equations of the form (4.56) were used for the dynamic description of the systems involved. In his work 'A Deadbeat Servo Problem' [Kuc., 7], a design is proposed for a deadbeat tracking of any reference signal from a prespecified class and for any initial condition of the single-variable plant. He pointed out that the control law based on a unity feedback scheme and operating on the tracking error only, is sub-optimal and in many cases the plant is required to be initially at rest. In this work Kucera argued that for time-optimal control the controller has to be placed in the feedback loop. A generalization of the polynomial equation approach to the multivariable deadbeat tracking case was given by Sebek [Seb., 3] and later on by Eichstaedt [Eic., 1] where necessary and sufficient conditions for finite settling time open-loop and closed-loop tracking were established. In Eichstaedt's treatment the initial conditions of the plant, the reference generator and controller were taken into account and an attempt was made to ensure asymptotic stability of the closed-loop system.

Later on, Kucera and Sebek [Kuc., 8] treated more systematically the multivariable deadbeat regulation and tracking problem by establishing necessary and sufficient conditions for the existence of various deadbeat controllers under the constraints of loop stability and causality based on a detailed analysis of the control sequence generated by any deadbeat controller. In a subsequent work Kucera and Hai [Kuc., 14] studied the effect of measurement dynamics on deadbeat performance in the case of single-variable discrete-time systems.

The problem of causality and stability of the inner loop under deadbeat tracking performance was also investigated by Wolovich [Wol., 3] using the polynomial matrix approach. The initial conditions of the plant and the controller were taken into account and necessary and sufficient conditions for ripple-free deadbeat tracking were established.

Chen et al. [Che., 2] investigated the problem of deadbeat unit step response with internal stability by using matrix fraction descriptions over the set of proper and stable z-rational functions. Their deadbeat controllers were confined to those achieving a sensitivity transfer function

$S(z)$ of particular structure. Two different control schemes were considered; one with unity feedback and another taking into account the sensor dynamics.

As in the state-space approach, the transfer function approach provides a family of deadbeat tracking controllers and in the later case in parametric form. There is therefore a possibility of imposing more performance requirements within the family of deadbeat controllers. Minimization of the squared errors, or minimum overshoot are the commonest of them [Pok., 1], [Pie., 1] [Lor., 1]. Another kind of deadbeat tracking was considered by Abdel-Nour and Mulholland [Abd., 1] where the single-variable plant is described by an autoregressive model with unknown parameters. An algorithm was presented combining both a deadbeat and a parameter identification task.

We have already mentioned that the main design feature of the deadbeat control is the placement of the poles of the closed-loop system at the origin of the z-plane. Clearly, this task is very sensitive to plant parameter variations and robust deadbeat design is needed. Work has been done on this area by Zhao and Kimura. In an earlier work [Zha., 1], [Zha., 2] they considered the problem of deadbeat robust tracking of step inputs using a unity feedback, or a one-parameter scheme. The characterization of all finite settling time tracking controllers was given based on the Youla-Bonjorno-Kucera parametrization of all stabilizing controllers. Robustness then was considered for multiplicative plant perturbations that were not violating the internal stability. The optimality robust design was formulated as the minimization of a robustness index that was essentially the L_2-norm of the sensitivity, and the optimally robust controller was computed for a specific finite settling time step. Later on, Zhao and Kimura considered the same robustness problem by allowing tracking of arbitrary prespecified input sequences and using a two-parameter control scheme [Zha., 3], [Zha., 4]. A comparison was given between the two schemes where the superiority of the two-parameter control scheme was proved.

4.5 Conclusions

Most of the work in deadbeat control area has been in the state-space set up and focused on specific type of problems largely dealing with minimum-time solutions. The transfer function approach has the advantages of parametrization, as it has been shown by Kucera's work based on polynomial equations, but again the work was focused on deadbeat and thus examined special type of control problems.

In this thesis, we have adopted the viewpoint that deadbeat response is a special case of the finite settling tine stabilization problem and that an algebraic formulation of this problem, that also guarantees internal finite settling time behaviour, can serve in a twofold way. On one hand it provides a unification of existing results, on the other hand the natural parametrization of solutions associated with polynomial matrix equations permits McMillan degree parametrizations, shows links with deadbeat and minimum McMillan degree solutions, provides a clear treatment of tracking and disturbance rejection in FST sense, and a useful set up for studying robustness in an FST sense. This unifying framework will be presented in the following chapters.

Chapter 5

TOTAL FINITE SETTLING TIME STABILIZATION:
The SISO Case

Chapter 5

TOTAL FINITE SETTLING TIME STABILIZATION: The SISO Case

5.1 Introduction

The aim of this chapter is to introduce the notion of *Total Finite Settling Time Stabilization (TFSTS)* for linear discrete-time systems and to consider in detail the SISO case. The ***Finite Settling Time Problem (FSTP)***, and more specifically the deadbeat performance - i.e. state or output regulation or tracking in minimum time - is unique in discrete-time systems [Ber., 1], [Kal., 2]. As it has already been described in chapter (4) (see references therein), most of the state or output deadbeat regulators are of the constant state feedback type, or in the case of inaccessible states a scheme involving observer and state feedback is used. The main aim of the above problems is to shift all the eigenvalues (or almost all, in the case of output deadbeat) of the closed-loop system to the origin. Kucera has pioneered the use of polynomial algebra methods for the study of state and output time-optimal control problems of both single-variable and multivariable discrete-time systems [Kuc., 4]-[Kuc., 8]. Subsequently, a number of researchers followed this approach, e.g. [Eic., 1], [Wol., 3] with the most recent work of Y. Zhao and H. Kimura [Zha., 1]-[Zha., 4] where the problem of robust deadbeat tracking is addressed.

The present work does not directly deal with the time-optimal (deadbeat) control, but with the rather more general problem of ***total finite settling time stabilization.*** That is, all internal and external variables (signals) of the closed-loop system are required to settle to a new steady state after a finite time from the application of a step change to any of its inputs and for every initial condition (see also [Lin., 1]). The state/output deadbeat regulation or tracking are then special cases of the TFSTS Problem abbreviated to FSTP ***(finite settling time problem).***

The unity feedback configuration of figure (3.4), shown again here in figure (5.1), is used for the solution of the FSTP and FST related control strategies as described in chapter (3) and will become more clear in this chapter. In this chapter we deal with the single-input/single-output FSTP where we address and solve a variety of control problems. The MIMO case is considered in the next chapter, where it is shown whether it is possible to extend the results of the SISO case to the MIMO one and what are the limitations of this process.

Using the mathematical and system theory framework as it has been presented in chapters (2) and (3) we are able to tackle the FSTP and a set of performance related problems in a unifying manner. The whole problem is reduced to the solution of a polynomial Diophantine equation which guarantees not only ***internal stability*** but also ***internal (state) FST.*** A Youla-Bongiorno-Kucera type parametrization of the class of all FST as well as causal FST controllers is derived and the necessary and sufficient conditions for the existence of stable FST controllers (strong FST) are obtained. The minimal design problem is considered together with the parametrization of the family of the FST controllers according to their McMillan degree.

Having solved in a parametric form the general problem of finite settling time stabilization we can impose more design constraints to the FST control problem such as FST tracking and disturbance rejection of a class of signals, FST controller design with partially given controller dynamics, and FST for sampled-data systems where a design for ripple-free FST is proposed. All of the aforementioned problems are considered in separate sections of this chapter.

5.2 Definition of the FSTP - Parametrization of the FST Controllers

Consider the closed-loop system of figure (5.1) where u_1, u_2 are the externally applied inputs and all the signals shown are scalar. Let also p, $c \in \mathbb{R}(d)$ be the impulse responses of the plant and controller respectively. Then, the problem of finite settling time is formulated as follows.

Definition 5.1: The unity feedback system of figure (5.1) exhibits a **finite settling time response**, if for a step change in any of the inputs u_1, u_2 and for any initial condition, all the signals e_1, e_2, or y_1, y_2 settle to a new steady state value in finite time. The values of the finite settling time and of the steady state are left free.

□

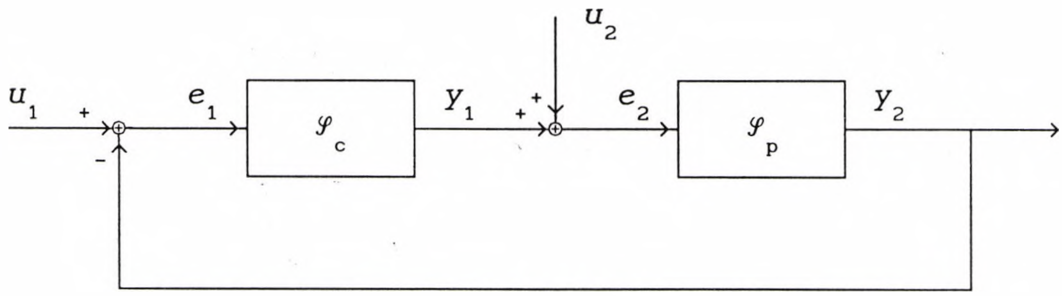

Figure (5.1): The SISO unity feedback configuration

We give next the characterization of the systems that exhibit FST response. For an alternative proof, not taking into account the initial conditions, see Isermann [Ise., 1].

Lemma 5.1: A causal discrete-time system characterized by an impulse response $g(d)$ exhibits an FST response, if and only if $g(d)$ is a polynomial in d, i.e. $g(d)$ is of finite duration.

Proof. According to definition (5.1), FST response is considered for a step input. To allow for the initial conditions, assume that the input u is a linear combination of two signals u^- and u^+, i.e.

$$u = u^- + u^+ \qquad (5.1)$$

where

$$u_k^- = \begin{cases} \text{any value for } -k_0 \le k \le -1 \\ 0, \text{ anywhere else} \end{cases} \qquad (5.2)$$

$$u_k^+ = \begin{cases} 1, & k \ge 0 \\ 0, & \text{anywhere else} \end{cases} \qquad (5.3)$$

"*if*". Suppose that $g(d) \in \mathbb{R}[d]$, i.e.

$$g = \{0;\ g_0, \ldots, g_m, 0, \ldots, 0, \ldots\} \qquad (5.4)$$

Then, the response of $\mathcal{S}(g(d))$ to u^- is

$$y_k^- = \sum_{n=0}^{\infty} g_n u_{k-n}^- = \sum_{n=0}^{m} g_n u_{k-n}^-, \quad k \ge 0 \qquad (5.5)$$

If $k > m - 1$, i.e. $k - m > -1$, (5.2) implies that $u_{k-n}^- = 0$, for $n = 0,\ldots,m$. Therefore

$$y_k^- = 0, \text{ for } k \ge m \qquad (5.6)$$

Also, the response of $\mathcal{S}(g(d))$ to u^+ is

$$y_k^+ = \sum_{n=0}^{\infty} g_n u_{k-n}^+ = \sum_{n=0}^{k} g_n u_{k-n}^+ = \sum_{n=0}^{k} g_n \qquad (5.7)$$

and for $k \ge m$, $g_k = 0$, so (5.7) gives

$$y_k^+ = \sum_{n=0}^{m} g_n, \quad k \ge m \qquad (5.8)$$

Hence, due to linearity, the response of the system to $u = u^- + u^+$ is $y = y^- + y^+$ and due to (5.6) and (5.8)

$$y_k = \sum_{n=0}^{m} g_n, \quad k \ge m \qquad (5.9)$$

i.e. y settles to the steady state value $\sum_{n=0}^{m} g_n$ after finite time $k_f = m$.

"*only if*". Suppose that y settles to a new steady state after finite time $k_f = m$ from the application of a step input and for any initial condition, i.e.

$$y_k = y_{k+1}, \ \forall \ k \geq m \tag{5.10}$$

This is true for any input u of the form (5.1), and therefore for $u = u^+$. In that case (5.7) implies that

$$y_{k+1} = \sum_{n=0}^{k+1} g_n = \sum_{n=0}^{k} g_n + g_{k+1} = y_k + g_{k+1} \tag{5.11}$$

and due to (5.10), $g_{k+1} = 0 \ \forall \ k \geq m$. Hence $\mathcal{S}(g(d))$ is a system with impulse response length $m + 1$ ($g_k \neq 0$ for $0 \leq k \leq m$).

□

Remark 5.1: Systems that are characterized by an impulse response of finite duration are known as **Finite Impulse Response (FIR)** systems in signal processing where they are used as filters. FIR systems exhibit a finite settling time response to almost any recurrent input and not just to step inputs. Indeed, consider any input $u(d)$ and express it in the form

$$u(d) = \frac{b(d)}{a(d)}, \quad a(d), \ b(d) \in \mathbb{R}[d] \text{ and coprime.}$$

Then, $y(d) = g(d)b(d)/a(d)$, $g(d) \in \mathbb{R}[d]$ and so $y(d)$ is of the same type as $u(d)$ if the zeros of $g(d)$ do not cancel any of the poles of $u(d)$ (zeros of $a(d)$). Needless to say that a FIR system exhibits an output of finite duration if it is subjected to any input of finite duration. Such systems are denoted by Kucera [Kuc., 9] as **Finite Input Finite Output (FIFO)** systems which is an equivalent way of describing their FIR property.

□

Consider now the unity feedback system of figure (5.1) and let n_p/d_p, n_c/d_c are coprime polynomial fractions in d of the transfer functions p, c of the plant and controller respectively. If we denote by $H(p,c)$ the transfer function from $\underline{u} = [u_1 \ u_2]^t$ to $\underline{e} = [e_1 \ e_2]^t$, and by $W(p,c)$ the transfer function from $\underline{u}$ to $\underline{y} = [y_1 \ y_2]^t$ as in section (3.4), then the denominators Δ, $\tilde{\Delta}$ (see 3.80), become scalar and equal to

$$\delta(p,c) := n_p n_c + d_p d_c \in \mathbb{R}[d] \tag{5.12}$$

and $H(p,c)$, $W(p,c)$ are expressed as follows.

$$H(p,c) = \begin{bmatrix} -n_p \\ d_p \end{bmatrix} \delta(p,c)^{-1} \begin{bmatrix} n_c & d_c \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.13a}$$

$$= \frac{1}{\delta(p,c)} \begin{bmatrix} d_p d_c & -n_p d_c \\ d_p n_c & d_p d_c \end{bmatrix} \tag{5.13b}$$

$$W(p,c) = \begin{bmatrix} d_p \\ n_p \end{bmatrix} \delta(p,c)^{-1} \begin{bmatrix} n_c & d_c \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \tag{5.14a}$$

$$= \frac{1}{\delta(p,c)} \begin{bmatrix} d_p n_c & -n_p n_c \\ n_p n_c & n_p d_c \end{bmatrix} \tag{5.14b}$$

According to lemma (3.1) and lemma (5.1) the following result is self-evident.

Lemma 5.2: The unity feedback system of figure (5.1) exhibits an FST response, if and only if $H(p,c) \in \mathbb{R}^{2x2}[d]$.

□

We are now able to give the solution to the finite settling time problem together with the parametrization of the FST controllers.

Theorem 5.1: Consider the closed-loop system of figure (5.1) and let $p = n_p/d_p$, $c = n_c/d_c$ be coprime polynomial fractions in d of the plant and controller transfer functions respectively. Then, the solution of the FSTP exists if and only if

$$\delta(p,c) := n_p n_c + d_p d_c \in \mathbb{R} - \{0\} \tag{5.15}$$

Moreover, the family of all causal FST controllers is given by

$$\mathcal{F}(p) = \{(n_c,d_c) : n_c = x + td_p,\ d_c = y - tn_p,$$
$$t \in \mathbb{R}[d] \text{ and } y(0) - t(0)n_p(0) \neq 0 \text{ if } n_p(0) \neq 0\} \tag{5.16}$$

where x, y is a particular solution pair of

$$n_p n_c + d_p d_c = 1 \tag{5.17}$$

Proof. According to lemma (5.2), $H(p,c)$ must be a polynomial matrix for FST response. Since (n_p,d_p), (n_c,d_c) are coprime, (5.13a) is a bicoprime fraction of $H(p,c)$, and according to corollary (2.3) $H(p,c) \in M(\mathbb{R}[d])$ if and only if $\delta(p,c)$ is a real constant, e.g. $\delta(p,c) = 1$ without loss of generality. The rest follows from theorem (3.18) and corollary (3.13).

□

It is clear from theorem (5.1) (see also proposition (3.5) and corollary (3.13)), that the following corollary holds true.

Corollary 5.1: The FST controller is causal

a. $\forall\ t(d) \in \mathbb{R}[d]$ if $n_p(0) = 0$

b. $\forall\ t(d) \in \mathbb{R}[d] : t(0) \neq \dfrac{y(0)}{n_p(0)}$ if $n_p(0) \neq 0$.

□

Corollary 5.2: Let n_p, d_p, n_c, d_c, be as in theorem (5.1). Then

$$H(p,c) = \begin{bmatrix} d_p d_c & -n_p d_c \\ d_p n_c & d_p d_c \end{bmatrix} \tag{5.18}$$

$$W(p,c) = \begin{bmatrix} d_p n_c & -n_p n_c \\ n_p n_c & n_p d_c \end{bmatrix} \tag{5.19}$$

□

Remark 5.2: According to theorem (5.1), FST control shifts the poles of $H(p,c)$ to infinity, i.e. outside the closed unit disc $\mathbb{D}[0,1]$ and therefore stabilizes externally the closed-loop system $\mathscr{S}_h$. In fact

a. If the plant and controller are minimal realizations of p, c, the feedback system is internally stable with all its eigenvalues shifted to zero, and so exhibits an internal (state) FST response as well.

b. If the plant and controller are stabilizable and detectable, then the feedback system is internally stable with at least the reachable and observable modes being finite.

The stabilizing nature of the FSTP suggests that the controllers defined by the FSTP may be referred to as **Total Finite Settling Time Stabilizing (TFSTS)** controllers, or simply **FSTS** controllers; the problem of defining them (FSTP), may also be referred to as **Total Finite Settling Time Stabilization Problem (TFSTSP)**, or simply **FSTSP**.

□

Example 5.1: Consider the feedback system of fig. (5.1) and let the transfer function of the plant be [Ise., 1]

$$p(d) = \frac{n_p(d)}{d_p(d)} = \frac{-0.0132d-0.0139d^2}{1-2.1889d+1.1618d^2}$$

According to theorem (5.1), the family of all controllers $c(d)$ that stabilize the plant in FST sense satisfy the Diophantine eqn. (5.17) and in this case the equation

$$(-0.0132d-0.0139d^2)n_c + (1-2.1889d+1.1618d^2)d_c = 1 \qquad (5.20)$$

It suffices to find one particular solution (x,y) of the equation (5.20). One way to obtain a particular solution is by reducing $[n_p \;\; d_p]$ to its Smith form. Since (n_p,d_p) are coprime, there exists a 2×2 $\mathbb{R}[d]$-unimodular matrix U such that [Kuc., 1], [Kai., 1]

$$[n_p \ \ d_p]U = [1 \ \ 0] \tag{5.21}$$

Such a U is

$$U = \begin{bmatrix} -105.38+66.685d & 105.38-230.67d+122.43d^2 \\ 1+0.7978d & 1.3911d+1.4648d^2 \end{bmatrix}$$

Therefore, according to (5.21), one particular solution (x,y) of (5.20) is

$$x = -105.38+66.685d$$
$$y = 1+0.7978d$$

Since the plant has a delay ($n_p(0) = 0$), the family of all causal FSTS controllers is

$$\mathcal{F}(p) = \{(n_c,d_c) : n_c = x + td_p,\ d_c = y - tn_p,\ \forall\ t \in \mathbb{R}[d]\}$$

Consider now the response at y_2 to step input applied at u_1. According to (5.19), the transfer function from u_1 to y_2 is $W_{21}(p,c) = n_p n_c$ and for $n_c = y$, $W_{21}(p,c)$ becomes

$$W_{21}(p,c) = 1.3911d + 0.5845d^2 - 0.9269d^3$$

For a step change at u_1, y_2 is the following (see also eqns. (5.7) and (5.9))

$$y_2 = 1.3911d + 1.9756d^2 + 1.0487(d^3 + d^4 + \cdots)$$

or

$$y_2 = \{0;0,\ 1.3911,\ 1.9756,\ 1.0487,\ 1.0487,\ 1.0487,...\}$$

We see that y_2 settles to the value of 1.0487 after two steps from zero but it does not track the input u_1 (the steady state error $e_{1,ss} = (u_1 - y_2)_{ss}$ is $e_{1,ss} = -0.0487$). Therefore, this particular controller $c = x/y$ is not good for tracking; it forces though all the states to zero in finite time. We will see in subsequent sections how we can select among the FSTS controllers those that guarantee tracking or other performance criteria.

□

5.3 Algebraic Computation of the Family $\mathcal{F}(p)$

It is clear from theorem (5.1) that the computation of the family $\mathcal{F}(p)$ requires only the computation of a particular solution (x,y) of the Diophantine equation (5.17). One way to do that is by reducing $[n_p \ d_p]$ to its Smith form [1 0], as it was illustrated in example (5.1). Another way is by using Toeplitz matrices [Che., 1] and it will be described in this section. This approach enables us to reduce the FSTP to a linear algebra problem in $\mathbb{R}$ and to obtain an immediate solution of the minimal design problem as well. Let

$$p = \frac{b_0+b_1d+\cdots+b_md^m}{1+a_1d+\cdots+a_nd^n} = \frac{n_p(d)}{d_p(d)} \tag{5.22}$$

be the transfer function of a causal plant and

$$c = \frac{c_0+c_1d+\cdots+c_\mu d^\mu}{f\ +f_1d+\cdots+f_\nu d^\nu} = \frac{n_c(d)}{d_c(d)} \tag{5.23}$$

be the transfer function of the controller in a unity feedback scheme, with (n_p,d_p) and (n_c,d_c) $\mathbb{R}[d]$-coprime respectively.

For FSTS, the pair n_c, d_c is a solution to the Diophantine eqn. (5.17) which implies that

$$\partial(n_pn_c) = \partial(d_pd_c) \therefore \partial(n_p) + \partial(n_c) = \partial(d_p) + \partial(d_c)$$

or

$$m + \mu = n + \nu \tag{5.24}$$

Using Toeplitz matrices for polynomial multiplication, eqn. (5.17) becomes

$$
\underbrace{\begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \vdots & & & b_0 \\ b_m & & & \vdots \\ 0 & b_m & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & b_m \end{bmatrix}}_{:=T_\mu(n_p)} \underbrace{\begin{bmatrix} c_0 \\ \vdots \\ c_\mu \end{bmatrix}}_{:=\underline{c}_\mu} + \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \vdots & & & 1 \\ a_n & & & \vdots \\ 0 & a_n & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{bmatrix}}_{:=T_\nu(d_p)} \underbrace{\begin{bmatrix} f_0 \\ \vdots \\ f_\nu \end{bmatrix}}_{:=\underline{f}_\nu} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{:=\underline{e}^1} \tag{5.25}
$$

or in compact form

$$
T_{\mu,\nu}(n_p, d_p) \begin{bmatrix} \underline{c}_\mu \\ \underline{f}_\nu \end{bmatrix} = \underline{e}^1 \tag{5.26}
$$

where

$$
T_{\mu,\nu}(n_p, d_p) = \begin{bmatrix} T_\mu(n_p) & T_\nu(d_p) \end{bmatrix}
$$

Therefore the solution of the FSTP reduces to the solution of equations (5.26).

Proposition 5.1: There always exists a unique FSTS controller $\hat{c} = \hat{n}_c/\hat{d}_c$ with $\hat{n}_c$, $\hat{d}_c$ having generic degrees $\partial(\hat{n}_c) = n$-1, $\partial(\hat{d}_c) = m$-1, and the parameter vectors $\underline{\hat{c}}_{n-1}$, $\underline{\hat{f}}_{m-1}$ of $\hat{n}_c$, $\hat{d}_C$ respectively, given by

$$
\begin{bmatrix} \underline{\hat{c}}_{n-1} \\ \underline{\hat{f}}_{m-1} \end{bmatrix} = \{T_{n-1,m-1}(n_p, d_p)\}^{-1} \underline{e}^1 \tag{5.27}
$$

The controller $\hat{c} = \hat{n}_c/\hat{d}_c$ will be referred to as the **prime** FSTS controller.

<u>*Proof*</u>. $T_{n-1,m-1}(n_p, d_p) \in \mathbb{R}^{(m+n)\mathrm{x}(m+n)}$ and it is sufficient to show that it has full rank. Indeed, the transpose of $T_{n-1,m-1}$ is the matrix

$$S(n_p,d_p) = \begin{bmatrix} b_0 & b_1 \cdots & \cdots b_m & 0 & \cdots & 0 \\ 0 & b_0 & & b_m & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ 0 \cdots\cdots & 0 & b_0 & \cdots & \cdots & b_m \\ 1 & a_1 & \cdots\cdots a_n & 0 & \cdots\cdots & 0 \\ 0 & 1 & & a_n & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ 0 \cdots\cdots & 0 & 1 & \cdots & \cdots & a_n \end{bmatrix}$$

which is known as the resultant, or Sylvester matrix of the coprime polynomials $n_p(d)$, $d_p(d)$ and therefore is full rank [Che., 1]. So, $T_{n-1,m-1}(n_p,d_p) = S^t(n_p,d_p)$ is a full rank matrix.

Now, it is possible that c_{n-1}, f_{m-1}, the last components of the solution parameter vectors $\hat{\underline{c}}_{n-1}$, $\hat{\underline{f}}_{m-1}$ of equation (5.27), to be zero (simultaneously, due to 5.24). This may be true for more pairs (c_{n-i},f_{m-i}) in the solution vector $[\hat{\underline{c}}^t_{n-1}\ \hat{\underline{f}}^t_{m-1}]^t$. The first nonzero pair c_{n-i}, f_{m-i} will constitute the leading coefficients of $\hat{n}_c(d)$, $\hat{d}_c(d)$ and therefore

$$\partial(\hat{n}_c) \le n - i \;, \quad \partial(\hat{d}_c) \le m - i \;, \quad i = 1,2,\ldots$$

Clearly, the generic values of the degrees are $\partial(\hat{n}_c) = n - 1$ and $\partial(\hat{d}_c) = m - 1$.

□

Remark 5.3: The prime FSTS controller may not be causal and so may not belong to the family $\mathcal{F}(p)$. Nevertheless, it can always be used for the parametrization of the causal FSTS controllers. If the plant possesses a delay all solutions to equation (5.17) are causal and so is the prime FSTS controller.

□

Example 5.2: Consider the unity feedback configuration of figure (5.1) with a plant as in example (5.1). Then, m=n=2 and the parameter vectors of the numerator, denominator polynomials of the prime FSTS controller are given by the following relationship

$$\begin{bmatrix} \underline{\hat{c}}_1 \\ \underline{\hat{f}}_1 \end{bmatrix} = \{T_{1,1}(n_p, d_p)\}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.28}$$

where

$$T_{1,1}(n_p, d_p) = \begin{bmatrix} 0.0 & 0.0 & 1.0 & 0.0 \\ -0.0132 & 0.0 & -2.1889 & 1.0 \\ -0.0139 & -0.0132 & 1.1618 & -2.1889 \\ 0.0 & -0.0139 & 0.0 & 1.1618 \end{bmatrix} \tag{5.29}$$

The solution of (5.28) with $T_{1,1}$ as in (5.29) is

$$\begin{bmatrix} \underline{\hat{c}}_1 \\ \underline{\hat{f}}_1 \end{bmatrix} = \begin{bmatrix} -105.3836 \\ 66.6854 \\ 1.0000 \\ 0.7978 \end{bmatrix}$$

and the **prime** FSTS controller is

$$\hat{c} = \frac{\hat{n}_c(d)}{\hat{d}_c(d)} = \frac{-105.3836 + 66.6854d}{1 + 0.7978d}$$

We see that $\partial(\hat{n}_c) = n - 1$, $\partial(\hat{d}_c) = m - 1$. Merely by chance, the prime FSTS controller coincides with the particular solution obtained by reduction to Smith form (example 5.1).

□

5.4 Parametrization of the FSTS Controllers Through their McMillan Degree

In this section we study the McMillan degree properties of the family of causal FSTS controllers $\mathcal{F}(p)$. We recall from section (3.2.4) that the McMillan degree $\delta_{\mathcal{M}}(g(d))$ of a rational function $g(d)$ denotes the total number of poles (finite and infinite) of $g(d)$. If $g(d)$ is causal, then $\delta_{\mathcal{M}}$ indicates the total number of poles (finite) of the proper transfer function $\tilde{g}(z) = g(z^{-1}) \in \mathbb{R}_{pr}(z)$. If $g(d)$ is not causal then $\tilde{g}(z) \in \mathbb{R}(z)$ and it is nonproper; in this case $\delta_{\mathcal{M}}$

denotes the total number of poles of $g(z)$, finite and infinite. Clearly $\delta_{\mathcal{M}}(g(d))$ is a measure of complexity of $g(d)$.

The importance of $\delta_{\mathcal{M}}$ in the study of properties of the $\mathcal{F}(p)$ family is demonstrated by the following self-evident statement.

Remark 5.4: Let $\mathcal{F}(p)$ be the family of all causal FSTS controllers. Then the following properties hold true.

a. The relationship $\mathcal{R}_{\mathcal{M}}$ on $\mathcal{F}(p)$ defined on every c_1, c_2 $\in \mathcal{F}(p)$ by

$$c_1 \mathcal{R}_{\mathcal{M}} c_2 \Longleftrightarrow \delta_{\mathcal{M}}(c_1) = \delta_{\mathcal{M}}(c_2)$$

is an equivalence relation.

b. Let $\mathcal{C}_{\mathcal{M}}(c)$ denote the equivalence class of $c \in \mathcal{F}(p)$ under $R_{\mathcal{M}}$. The family $\mathcal{C}^*_{\mathcal{M}}(p)$ of all equivalent classes $\mathcal{C}_{\mathcal{M}}(c)$ forms a partition of $\mathcal{F}(p)$.

□

Note that each equivalence class is parametrized by a distinct number namely, the McMillan degree of the equivalence class. A number of important problems that arise in this context are considered next. First define:

Definition 5.2: Let $\mathcal{F}(p)$ be the causal FSTS controller family, and $\mathcal{C}^*_{\mathcal{M}}(p)$ be the family of all equivalence classes $\mathcal{C}_{\mathcal{M}}(c_i)$, $c_i \in \mathcal{F}(p)$ with McMillan degree $\delta_{\mathcal{M}}(c_i) = i$.

a. Define $i = \delta_{\mathcal{M}}(c_i)$ as the **McMillan index** of $\mathcal{C}_{\mathcal{M}}(c_i)$ and $\mathcal{C}_{\mathcal{M}}(c_i)$ will be referred to as the i-subfamily.

b. The set of indices

$$I_{\mathcal{M}}(p) = \{i : i \text{ the McMillan index of } \mathcal{C}_{\mathcal{M}}(c_i),\ \forall\ \mathcal{C}_{\mathcal{M}}(c_i) \in \mathcal{C}^*_{\mathcal{M}}(p)\}$$

will be called the **McMillan index set** of $\mathcal{F}(p)$.

□

We may define now the following problems.

a. *Minimal Design Problem (MDP)*. Define the minimal McMillan degree $\delta_m(p)$ of all causal FSTS controllers.

b. *Parametrization Problem (I) (PP(I))*. Define the McMillan index set $I_{\mathcal{M}}(p)$ of $\mathcal{F}(p)$.

c. *Parametrization Problem (II) (PP(II))*. $\forall\ i \in I_{\mathcal{M}}(p)$ define a parametric expression of the $\mathcal{R}_{\mathcal{M}}$ equivalence class $\mathcal{C}_{\mathcal{M}}(c_i)$, where $\delta_{\mathcal{M}}(c_i) = i$.

In the sequel we give the solution to all three aforementioned problems.

Theorem 5.2: Consider the closed-loop system of figure (5.1) and let $p(d) = n_p(d)/d_p(d)$ be the plant transfer function with (n_p, d_p) coprime in $\mathbb{R}[d]$ and $\partial(n_p) = m$, $\partial(d_p) = n$. Then, the McMillan degree $\delta_{\mathcal{M}}(\hat{c})$ of the prime FSTS controller $\hat{c}(d) = \hat{n}_c(d)/\hat{d}_c(d)$ is the minimal McMillan degree of all FSTS controllers and is denoted by $\hat{\delta}_m(p)$, i.e.

$$\hat{\delta}_m(p) = \delta_{\mathcal{M}}(\hat{c}) \quad (= \max\{m,n\} - 1, \text{ generically})$$

Proof. Every FSTS controller $c = n_c(d)/d_c(d)$ can be parametrized as follows.

$$n_c = \hat{n}_c + t d_p \tag{5.30}$$

$$d_c = \hat{d}_c - t n_p \tag{5.31}$$

where $t \in \mathbb{R}[d]$ and $\hat{n}_c$, $\hat{d}_c$ is the prime FSTS controller. If $\partial(n_c) = \mu$ and $\partial(d_c) = \nu$, then according to theorem (3.4) $\delta_{\mathcal{M}}(c) = \max\{\mu,\nu\}$ and so

$$\hat{\delta}_m(p) := \min\{\delta_{\mathcal{M}}(c)\} = \min\{\max\{\mu,\nu\}\} \tag{5.32}$$

From equation (5.24) it is clear that μ, ν achieve simultaneously their minimum values and therefore it is enough to find the minimum value of either of them. We work out the minimum value of $\mu = \partial(n_c)$. According to proposition (5.1)

$$\partial(\hat{n}_c) \le n - 1 \tag{5.33}$$

Also $$\partial(td_p) = \partial(t) + \partial(d_p) \ge \partial(d_p) = n \tag{5.34}$$

Therefore, from (5.30), (5.33) and (5.34) we have that

$$\mu = \partial(n_c) \ge \partial(\hat{n}_c)$$

where the equality holds for $n_c = \hat{n}_c$ $(t = 0)$. Hence, the minimal McMillan degree FSTS controller is the prime FSTS controller. Since generically $\partial(\hat{n}_c) = n - 1$ and $\partial(\hat{d}_c) = m - 1$, we have that

$$\hat{\delta}_m(p) = \delta_{\mathcal{M}}(\hat{c}) \quad (= \max\{m,n\} - 1, \text{ generically}).$$

□

Definition 5.3: The minimal McMillan degree of the family $\mathcal{F}(p)$ will be denoted by $\delta_m(p)$, i.e.

$$\delta_m(p) := \min_{c \in \mathcal{F}(p)} (\delta_{\mathcal{M}}(c))$$

and will be referred to as the **McMillan characteristic** of the family $\mathcal{F}(p)$.

□

Using the prime FSTS controller for the parametrization of the family $\mathcal{F}(p)$ (eqns. 5.30 and 5.31), and the results of theorem (5.2), we may examine the minimal design problem and the parametrization problems (I) and (II).

Corollary 5.3 ***(MDP)*:** Let p, $\hat{c}$ be as in theorem (5.2). Then, the McMillan characteristic $\delta_m(p)$ of $\mathcal{F}(p)$ is given by

a. $\delta_m(p) = \delta_{\mathcal{M}}(\hat{c})$ $(= \max\{m,n\} - 1$, generically) if the prime FSTS controller $\hat{c}$ is causal.

b. $\delta_m(p) = \max\{m,n\}$ if the prime FSTS controller $\hat{c}$ is not causal.

Proof.

a. According to theorem (5.2), the prime FSTS controller $\hat{c}$ is the minimal McMillan degree controller among all FSTS controllers. Since $\hat{c}$ is causal, then it is the minimal McMillan degree controller within the family of causal FSTS controllers $\mathcal{F}(p)$ and therefore $\delta_m(p) = \delta_{\mathcal{M}}(\hat{c})$.

b. If the prime FSTS controller $\hat{c} = \hat{n}_c/\hat{d}_c$ is not causal, i.e. $\hat{c} \notin \mathcal{F}(p)$, then $\hat{d}_c(0) = 0$ and also (corollary 5.1), the plant p is not strictly causal ($n_p(0) \neq 0$). Hence,

$$\forall\ c = n_c/d_c \in \mathcal{F}(p)\ \exists\ t \in \mathbb{R}[d] - \{0\} :$$

$$n_c = \hat{n}_c + td_p, \quad d_c = \hat{d}_c - tn_p$$

and $t(0) \neq \hat{d}_c(0)/n_p(0) \neq 0$. Therefore, according to (5.33) and (5.34), $\partial(n_c) = \partial(t) + \partial(d_p)\ \forall\ c \in \mathcal{F}(p)$. So, $\min(\partial(n_c)) = \partial(d_p) = n$ is achieved for $t \in \mathbb{R} - \{0\}$, a real nonzero constant, and due to (5.24) $\min(\partial(d_c)) = \partial(n_p) = m$. Therefore $\delta_m(p) = \max\{m,n\}$.

□

The following two corollaries are given without proof. Their proofs are similar to that of corollary (5.3) and are omitted.

Corollary 5.4 *(PP(I))*: Let p be the plant transfer function as in theorem (5.2) and $I_{\mathcal{M}}(p)$ be the McMillan index set of $\mathcal{F}(p)$. Then

$$I_{\mathcal{M}}(p) = \{\delta_m(p);\ \max\{m,n\}+k,\ k = 0,1,2,\dots\}$$

□

Corollary 5.5 *(PP(II))*: Let the δ_j-subfamily of $\mathcal{F}(p)$ be as in definition (5.3). Then

a. The δ_m-subfamily of $\mathcal{F}(p)$ consists of

1. only the prime FSTS controller $\hat{c}$, if $\hat{c}$ is causal.

2. all the controllers $c = n_c/d_c$ that are parametrized by a real non zero constant, if $\hat{c}$ is not causal, i.e.

$$n_c = \hat{n}_c + td_p \text{ and } d_c = \hat{d}_c - tn_p \text{ with } t \in \mathbb{R} - \{0\}$$

b. The δ_j-subfamily, where $\delta_j = \max\{m,n\}+j$, $j = 0,1,\ldots$ is parametrized by

$$n_c = \hat{n}_c + td_p$$
$$d_c = \hat{d}_c - tn_p$$

where

1. $t \in \mathbb{R}[d]$, $\partial(t)=j$, if $n_p(0)=0$.

2. $t \in \mathbb{R}[d]$, $\partial(t)=j$, $t(0)\neq\hat{d}_c(0)/n_p(0)$, if $n_p(0)\neq 0$.

□

As it was mentioned in remark (5.1), all FSTS controllers guarantee finite input for finite output, i.e. finite-time regulation for any initial condition. Using the results of corollaries (5.3) to (5.5) it is possible to solve the deadbeat regulation problem, i.e. the time-optimal FSTS problem.

Theorem 5.3 *(Deadbeat Regulation)*: Let $p = n_p/d_p$ with $\partial(n_p) = m$, $\partial(d_p) = n$ be the transfer function of the plant and $c = n_c/d_c$ be the transfer function of the controller in the unity feedback control scheme of figure (5.1). If $\delta_m(p)$ is the McMillan characteristic of the family of FSTS controllers $\mathcal{F}(p)$, then every controller that belongs to the δ_m-subfamily of $\mathcal{F}(p)$ is a deadbeat regulator and forces the output y_2 to zero in at most

$$k = m + \delta_m(p)$$

steps, for any initial conditions.

Proof. To account for the initial conditions we assume that we apply before zero, finite inputs at u_1 and u_2. The transfer functions from u_1 to y_2 and u_2 to y_2 are $W_{21}(p,c) = n_p n_c$ and $W_{22}(p,c) = n_p d_c$ (see 5.19). Then, according to lemma (5.1) y_2 will be forced to zero in

$$k = \max\{\partial(n_p n_c),\partial(n_p d_c)\} = m + \max\{\partial(n_c),\partial(d_c)\} = m + \delta_{\mathcal{M}}(c)$$

steps. For deadbeat response k and therefore $\delta_{\mathcal{M}}(c)$ must be minimum, i.e. $\delta_{\mathcal{M}}(c) = \delta_m(p)$.

□

Example 5.3: Consider the closed-loop system of figure (5.1) with the plant

$$p(d) = \frac{n_p(d)}{d_p(d)} = \frac{-0.0132d-0.0139d^2}{1-2.1889d+1.1618d^2}$$

as in example (5.1). Then according to example (5.2), the prime FSTS controller is

$$\hat{c} = \frac{\hat{n}_c(d)}{\hat{d}_c(d)} = \frac{-105.3836 + 66.6854d}{1 + 0.7978d}$$

and it is causal as it was expected since the plant possesses a delay. The McMillan characteristic of the family $\mathcal{F}(p)$ is

$$\delta_m(p) = \max\{\partial(\hat{n}_c),\partial(\hat{d}_c)\} = 1$$

The prime FSTS controller is also the unique time-optimal (deadbeat) regulator and forces the output y_2 to zero in at most $k = \partial(n_p) + \delta_m(p) = 3$ steps.

□

5.5 Strong FSTS

Consider the unity feedback configuration of figure (5.1). The problem of strong FSTS is defined as the stabilization of the plant $\mathcal{S}_p$ in FST sense by a stable controller. Testable necessary and sufficient conditions for strong FSTS are derived in this section; it turns out that the plant must have the same ***parity interlacing property*** [Vid., 1], [You., 2] as in the case of usual strong stabilization where the domains of stability of the closed-loop system and the controller coincide.

We introduce now the notion of ***disc algebra*** A_s [Sim., 1], [Vid., 1] that will assist us to derive the solvability conditions for strong FSTS. A_s is the set of functions $f(d)$, $d \in \mathbb{C}$, over the real field $\mathbb{R}$ which are continuous in the closed unit disc $\mathbb{D}[0,1]$ and analytic in the interior of $\mathbb{D}$, i.e. the open unit disc $\mathbb{D}[0,1)$. If addition and multiplication between any two elements of A_s are defined pointwise and the norm of $f(d) \in A_s$ is

$$\|f\| = \sup_{d \in \mathbb{D}} |f(d)|$$

then, A_s is a commutative Banach algebra with identity over the real field. It is clear form the definition of the disc algebra A_s that every stable rational function belongs to A_s and the stable polynomials are the polynomial units of A_s.

The conditions for strong FSTS are given by the following theorem.

Theorem 5.4 *(Strong FSTS)*: A plant $p = n_p/d_p$, n_p, d_p coprime in $\mathbb{R}[d]$, is **strongly stabilizable** in FST sense, if and only if $d_p(d)$ has the same sign at all real zeros σ_i of $n_p(d)$ inside the unit disc $\mathbb{D}[0,1]$.

Proof.

"*only if*". Suppose that there is a stable FSTS controller $c = n_c/d_c$. Then

$$n_p n_c + d_p d_C = 1 \tag{5.35}$$

and d_c is a polynomial unit in the disc algebra A_s, i.e. d_c is a stable polynomial ($d_c(d) \neq 0 \ \forall \ d \in \mathbb{D}[0,1]$). Therefore, $d_c(\sigma_i)$ has the same sign $\forall \ \sigma_i \in [-1,1] \ : \ n_p(\sigma_i) = 0$. Also, equation (5.35) becomes for $d = \sigma_i$

$$d_p(\sigma_i) d_c(\sigma_i) = 1 \tag{5.36}$$

which means that $d_p(\sigma_i)$ has the same sign as $d_c(\sigma_i)$, i.e. $d_p(d)$ does not change sign at the real zeros of $n_p(d)$ inside the closed unit disc $\mathbb{D}[0,1]$.

"*if*". The proof of this part is constructive. Let $d_p(d)$ not change sign at the real zeros of $n_p(d)$ inside the closed unit disc $\mathbb{D}$ and $\bar{n}_c$, $\bar{d}_c$ be a particular solution of equation (5.35). Also

$$n_p = n_p^+ n_p^-$$

where

$$n_p^- = \prod_{i=1}^{\ell} (d - z_i)^{m_i} \in \mathbb{R}[d]$$

contains all the unstable zeros of n_p and $n_p^+ \in \mathbb{R}[d]$ contains all the stable zeros of n_p. Then, $d_p(\sigma_i)\bar{d}_c(\sigma_i) = 1$, $\forall\ \sigma_i \in [-1,1]$: $n_p(\sigma_i) = 0$ and therefore $\bar{d}_c(d)$ does not change sign at the real zeros σ_i of n_p inside the closed unit disc $\mathbb{D}$. Hence, there is a unit h in A_s (not necessarily polynomial) that either h or $-h$ possesses a logarithm (depending on whether $\bar{d}_c(\sigma_i) > 0$ or $\bar{d}_c(\sigma_i) < 0$ respectively) and which interpolates $\bar{d}_c(d)$ and its derivatives at the unstable poles of n_p [Vid., 1], i.e.

$$h^{(j)}(z_i) = \bar{d}_c^{(j)}(z_i),\ i = 1,\dots,\ell \quad j = 1,\dots,m_i - 1$$

Then, $n_p^+ | (h - \bar{d}_c)$ in A_s and let $p = (h - \bar{d}_c)/n_p^+ = c_1 n_p^-$, p, $c_1 \in A_s$ (because n_p^- is a polynomial unit in A_s), or

$$h - \bar{d}_c = c_1 n_p$$

Since polynomials are dense in A_s, there exists a polynomial c_2 such that

$$\|c_1 - c_2\| \le 1/\|h^{-1}\|\|n_p\|$$

define now $f = \bar{d}_c + c_2 n_p \in \mathbb{R}[d]$. Then,

$$\|h - f\| = \|(c_1 - c_2)n_p\| \le (1/\|h^{-1}\|\|n_p\|)\|n_p\|$$

i.e.

$$\|h - f\| \le 1/\|h^{-1}\|$$

Therefore, f is a polynomial unit in A_s [Vid., 1]. If $d_c = f$ and $t = -c_2$ then

$$n_c = \bar{n}_c + td_p, \quad d_c = \bar{d}_c - tn_p$$

define a stable FSTS controller.

□

Theorem (5.4) may be rephrased into the following corollary which expresses the ***parity interlacing property***.

Corollary 5.6 *(Parity Interlacing Property)*: There always exists a stable FSTS controller, if and only if the number of poles of $p(d)$ inside any interval of successive real zeros of $p(d)$ inside the closed unit disc, is even.

□

Example 5.4: Consider the plant of example (5.1). The real zeros of $n_p(d)$ inside $\mathbb{D}[0,1]$ are

$$\sigma_1 = -0.9496 \text{ and } \sigma_2 = 0$$

and the values of $d_p(d)$ at σ_1, σ_2 are

$$d_p(\sigma_1) = 4.1264 \text{ and } d_p(\sigma_2) = 1$$

Therefore, $d_p(d)$ does not change sign at σ_1, σ_2 and so it is strongly FST stabilizable. Indeed, one stable FSTS controller is the prime FSTS controller (example 5.2).

□

5.6 FST Tracking and Disturbance Rejection

An important problem in control system design is that of tracking where the output of a system has to follow a particular set of inputs. In the case of FSTS, it is required the output y_2 (fig. 5.1), tracks the input u_1 in ***finite*** time. The solution to this problem, as in the case of asymptotic tracking, is the well known ***internal model principle*** and is given by the next theorem.

Theorem 5.5 ***(FST Tracking)*:** Consider the feedback system of figure (5.1). Let $p = n_p/d_p$ be the transfer function of the plant and $c = n_c/d_c$ be the transfer function of any FSTS controller, with all fractions involved being coprime polynomial fractions. Suppose also that the input $u_1 = n_r/d_r$ belongs to a specified class of signals. Then, the output y_2 tracks the input u_1 in finite time, if and only if $d_r | d_p d_c$.

Proof. For any FSTS controller, the transfer function from u_1 to e_1 is $H_{11}(p,c) = d_p d_c$ (see 5.18). If $u_1 = n_r/d_r$, the error signal e_1 becomes

$$e_1(d) = H_{11}(p,c)u_1(d) = \frac{d_p d_c n_r}{d_r}$$

For finite tracking, the error sequence $e(d)$ has to vanish in finite time, i.e $e(d) \in \mathbb{R}[d]$. This is possible if and only if $d_r | d_p d_c$.

□

Another important problem usually encountered in control system design is that of disturbance rejection, i.e. every non desirable input signal has to vanish at the output of the system. In FSTS sense, if u_2 in figure (5.1) is the disturbance signal to be rejected at the output y_2, then y_2 has to reach a zero steady state after ***finite*** time. The conditions for disturbance rejection are summarized by the following theorem.

Theorem 5.6 *(FST Disturbance Rejection)*: Let p, c be as in theorem (5.5). The output y_2 rejects the input $u_2 = n_d/d_d$ in finite time, if and only if $d_d | n_p d_c$.

Proof. According to (5.19) the transfer function from u_2 to y_2 is $W_{22}(p,c) = n_p d_c$. The rest of the proof, as in theorem (5.5), follows from the fact that $y_2(d)$ must be polynomial for the particular set of inputs $u_2 = n_d/d_d$.

□

Remark 5.5: According to remark (5.1), every FSTS controller will reject at any output y_1, y_2 in finite time, any non persistent disturbance (disturbance of finite duration) applied at any input u_1, u_2.

□

From theorems (5.5) and (5.6) we can easily derive the solvability conditions for simultaneous FST tracking and disturbance rejection.

Corollary 5.7 *(FST Tracking and Disturbance Rejection)*: Consider the feedback system of figure (5.1). Let $p = n_p/d_p$ be the transfer function of the plant and $c = n_c/d_c$ be the transfer function of any FSTS controller, with all fractions involved being coprime polynomial fractions. Suppose also that the inputs $u_1 = n_r/d_r$, $u_2 = n_d/d_d$ belong to prespecified classes of signals. Then, the output y_2 tracks the input u_1 and rejects the output $\underline{u}_2$ in finite time, if and only if

$$\text{l.c.m.}\{d_r, d_d\} \mid d_c$$

where l.c.m. stands for the least common denominator.

□

Example 5.5: Consider the closed-loop system of figure (5.1) with the plant

$$p(d) = \frac{n_p(d)}{d_p(d)} = \frac{-0.0132d-0.0139d^2}{1-2.1889d+1.1618d^2}$$

as in example (5.1). The family of all FSTS controllers such that the output y_2 tracks parabolic signals from u_1 and rejects step changes applied at the input u_2, in finite time can be derived as follows.

u_1 should be of the form

$$\tilde{u}_1(z) = \frac{Az(z+1)}{(z-1)^3} = \frac{Ad(1+d)}{(1-d)^3} = \frac{n_r(d)}{d_r(d)}$$

and u_2

$$\tilde{u}_2(z) = \frac{Bz}{z-1} = \frac{B}{1-d} = \frac{n_d(d)}{d_d(d)}$$

According to theorem (5.5) d_r must divide $d_p d_c$, therefore d_r must divide d_c, and according to theorem (5.6) d_d must divide $n_p d_c$, so d_d must divide d_c. Hence, d_c must be of the form

$$d_c = (1-d)^3 d_c'$$

Therefore all FSTS controllers that achieve the required characteristics must satisfy the following Diophantine equation.

$$n_p n_c + d_p(1-d)^3 d_c' = 1$$

or

$$(-0.0132d-0.0139d^2)n_c+(1-2.1889d+1.1618d^2)(1-d)^3d_c' = 1 \qquad (5.37)$$

The least degree solution to equation (5.37) is

$$x(d) = -315.93+745.04d-793.96d^2+413.08d^3-85.137d^4$$

$$y'(d) = 1+1.0186d$$

which can be considered as the prime FSTS controller for the plant

$$p' = \frac{n_p(d)}{d_p(d)(1-d)^3}$$

Since $n_p(0) \neq 0$, all FSTS controllers are causal and the subfamily of the FSTS controllers that achieve the desired performance is given by

$$n_c = x + td_p(1-d)^3$$

$$d_c = y'(1-d)^3 - tn_p(1-d)^3$$

where $t \in \mathbb{R}[d]$. The minimal McMillan degree FSTS controller for tracking and rejection of the required signals is obtained for $t = 0$ and is the following.

$$c_m = \frac{n_{cm}(d)}{d_{cm}(d)} = \frac{-315.93+745.04d-793.96d^2+413.08d^3-85.137d^4}{1-1.9814d-0.0558d^2+2.0058d^3-1.0186d^4}$$

The responses of the unity feedback system using the controller c_m are shown in figure (5.2). We see that the system achieves its performance in six steps.

□

Remark 5.6: Example (5.5) provides a design procedure for the parametrization of the class of FST tracking and disturbance rejection controllers. If d_{ca} represents the common dynamics of the input signals $\underline{u}_1$ and $\underline{u}_2$, i.e.

$$d_{ca} = \text{l.c.m.}\{d_r, d_d\}$$

then, the FST tracking and disturbance rejection problem reduces to the nornal FST problem for the fictitious plant

$$p_f = n_{pf}/d_{pf} = n_p/d_p d_{ca}$$

The FST controllers for this plant are given as a solution to the Diophantine equation

$$n_p n_c + (d_p d_{ca}) d_c' = 1$$

and the family of all FSTS tracking and disturbance rejection controllers can be described by

$$c = n_c/d_c = n_c/d_{ca} d_c'$$

The following section is a generalization of the foregoing discussion.

□

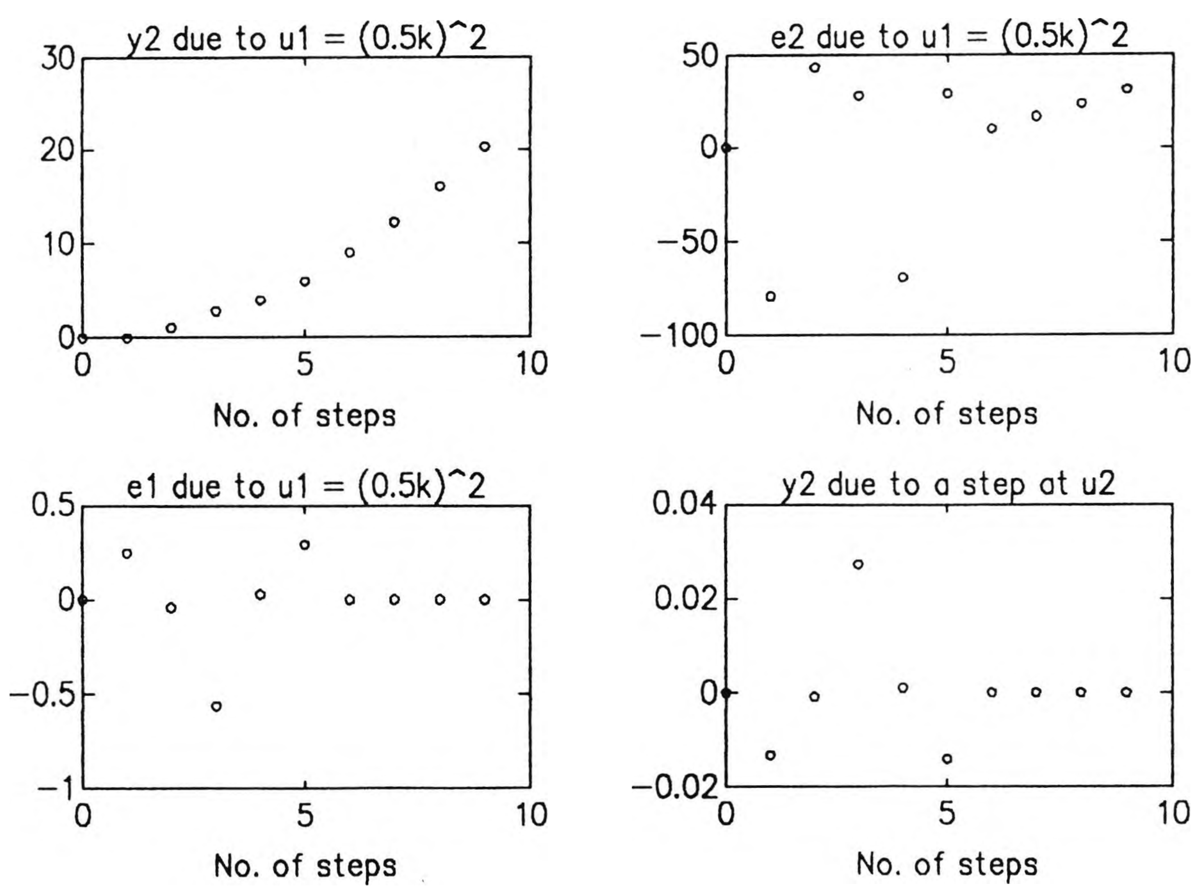

Figure (5.2): Responses for the example (5.5)

5.7 FSTS Controllers with Partially Assigned Dynamics

Sometimes it is desirable to assign a priori some of the dynamics of the controller; this is certainly the case in tracking and disturbance rejection as it has been illustrated in section (5.6). In this section we give the solution to this problem for the FSTS case.

Theorem 5.7: Let $p = n_p/d_p$ be the transfer function of a plant and $c = n_c/d_c$ be an FSTS controller, i.e. $c \in \mathcal{F}(p)$. The controller c may have partially assigned dynamics, i.e.

$$d_c = d_{ca}d_c' \;, \quad d_{ca} \in \mathbb{R}[d] \text{ given} \tag{5.38}$$

if and only if n_p, d_{ca} are $\mathbb{R}[d]$-coprime and $d_{ca}(0) \neq 0$, if $n_p(0) \neq 0$. Moreover, the family of all causal FSTS controllers with partially assigned dynamics will be denoted by $\mathcal{F}_{pad}(p)$ and is parametrized as follows.

$$\mathcal{F}_{pad}(p) = \{(n_c,d_c) \;:\; n_c = x + td_pd_{ca},\; d_c = (y - tn_p)d_{ca},$$
$$\forall\; t \in \mathbb{R}[d] \;:\; t(0) \neq y(0)/n_p(0) \text{ if } n_p(0) \neq 0\} \tag{5.39}$$

where (x,y) is a particular solution for (n_c,d_c') of the Diophantine equation

$$n_pn_c + d_pd_{ca}d_c' = 1$$

Proof. Every FSTS controller c satisfies the Diophantine equation

$$n_pn_c + d_pd_c = 1 \tag{5.40}$$

If d_c is as in equation (5.38), then (5.40) becomes

$$n_pn_c + d_pd_{ca}d_c' = 1 \tag{5.41}$$

Equation (5.41) has a solution if and only if n_p, d_{ca} are $\mathbb{R}[d]$-coprime. The family of solutions (n_c,d_c') of equation (5.41) is given by

$$\begin{aligned} n_c &= x + td_pd_{ca} \\ d_c' &= y - tn_p \end{aligned} \tag{5.42}$$

Then, according to (5.38), the family of all FSTS controllers with partial dynamics described by d_{ca} is the following.

$$n_c = x + td_p d_{ca}$$
$$d_c = (y - tn_p)d_{ca}$$

where t is an arbitrary polynomial in d. For causality $d_c(0)$ must be non zero. If $n_p(0) = 0$, then $y(0)$, $d_{ca}(0) \neq 0$ due to the coprimeness of (n_p, d_{ca}) hence, $d_c(0) = y(0)d_{ca}(0) \neq 0 \; \forall \; t \in \mathbb{R}[d]$. If $n_p(0) \neq 0$, then $t(0) \neq y(0)/n_p(0)$ and $d_{ca}(0) \neq 0$ for $d_c(0)$ to be non zero. And this proves (5.39).

□

5.8 FSTS for Sampled-Data Systems

In the previous sections we considered the ***Finite Settling Time Stabilization Problem*** for purely discrete-time systems. The discrete-time FSTS controllers guarantee finite settling time performance for the closed-loop system for any discrete-time instance. This is completely acceptable since the response of the system explicitly occurs in discrete-time only, but in many cases this may not be so. The plant may be a continuous-time process that is controlled by a discrete-time controller which processes information acquired at specific time instances. Such a system is known as a ***sampled-data*** system and its behaviour is of interest not only at the sampling points but for a continuum of time. Indeed, although the sampled-data system may exhibit an FST response at the sampling instants, the response may contain quite undesirable ***ripples*** in many cases.

The sampled-data system considered in this section is the closed-loop system of figure (5.3) where the digital to analog (D/A) converter is a Zero-Order-Hold (ZOH) device with transfer function

$$g_{ZOH} = \frac{1 - e^{-sT}}{s} = \frac{1 - d}{s} \tag{5.43}$$

where T is the sampling period. We consider first the notion of rippled response and give the reasons of its existence.

Definition 5.4 [Fra., 1]: We define as **ripple** in any sampled-data system, the error between sampling instants when the error at sampling instants is zero.

□

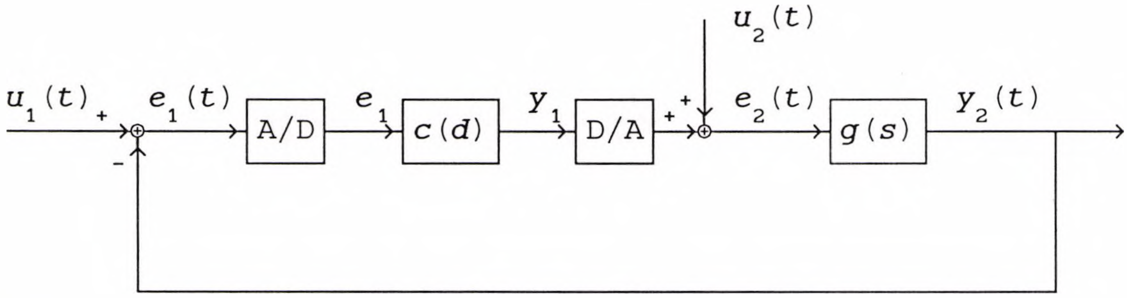

Figure (5.3): The unity feedback sampled-data system

The main reasons for the existence of ripples in a sampled-data system are the following.

a. Pole-zero cancellations between the controller and the discrete equivalent plant [Lew., 1], [Wil., 1].

b. Non constant discrete-time control signal y_1 that will result to a staircase type continuous signal $e_2(t)$ (see also fig. 5.4) [Rag., 1].

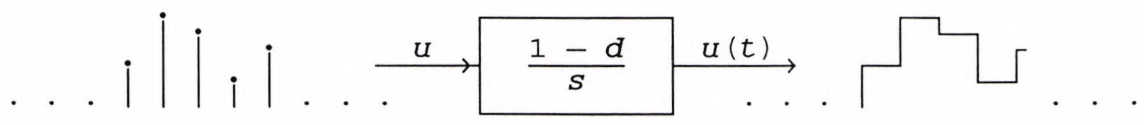

Figure (5.4): The Zero-Order-Hold device

In FSTS, pole-zero cancellations are avoided since all FSTS controllers satisfy the Diophantine equation

$$n_p n_c + d_p d_c = 1$$

which guarantees coprimeness of (n_c, d_p) and (d_c, n_p). Therefore, the only reason for ripples is a non constant control signal y_1. Consider e.g. the FST tracking case. If $u_1 = n_r / d_r$ then every FSTS controller for tracking would be of the form

$$c = \frac{n_c}{d_c} = \frac{n_c}{d_c' d_r}$$

and the control signal y_1 would be

$$y_1 = W_{11}(p, c) u_1 = d_p n_c \frac{n_r}{d_r}$$

i.e. y_1 is non constant and of infinite duration. If this signal is to be applied to the continuous-time process, it will create after the ZOH a staircase type signal $e_2(t)$ and although the error at sampling instants will be zero, by design, the plant in between sampling instants will respond to step changes. To avoid that, d_r has to divide d_p, or in other words a continuous-time rather than a discrete-time internal model has to be implemented. This will result to a finite duration discrete-time signal y_1 and to a ripple-free response of the continuous-time system. We summarize all the above discussion in the next theorem which is similar to the one presented in Franklin and Emami-Naeini [Fra., 1].

Theorem 5.8: Consider the sampled-data system of fig. (5.3) where the D/A converter is a Zero-Order-Hold device and $g(s)$ represents the transfer function of the continuous-time plant and any continuous-time controller. Then, the unity feedback system will be ripple-free, if and only if a continuous internal model of the input $u_1(t)$ that is observable from the output is implemented first and then a discrete FSTS controller is designed.

Proof. We assume that the input $u_1(t)$ possesses a rational Laplace transform and therefore can be written in the form

$$u_1(t) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} u_{il} \frac{t^l}{\ell !} e^{\lambda_1 t} \tag{5.44}$$

"*if*". Suppose that the system is designed to track at the sampling instants inputs of the form (5.44) with a continuous internal model of u_1 implemented. Then according to our previous discussion, the control signal y_1 goes to zero in finite time k_0 and so does $e_2(t)$. Hence the output $y_2(t)$ for $t \geq k_0$, is the open-loop response of the continuous-time system due to some initial conditions at k_0 and can be written in the form

$$y_2(t) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} \alpha_{il} \frac{t^l}{\ell !} e^{\lambda_1 t} + \sum_{j=1}^{q} \sum_{h=0}^{m_j - 1} \beta_{jh} \frac{t^h}{h!} e^{\lambda_h t} \tag{5.45}$$

The first set of terms is due to the continuous internal model and the second is due to the plant dynamics. At the sampling instants

$$y_2(kT) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} \alpha_{il} \frac{(kT)^l}{\ell !} e^{\lambda_1 kT} + \sum_{j=1}^{q} \sum_{h=0}^{m_j - 1} \beta_{jh} \frac{(kT)^h}{h!} e^{\lambda_h kT} \tag{5.46}$$

Also, due to the tracking property,

$$y_2(kT) = u_1(kT) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} u_{il} \frac{(kT)^l}{\ell !} e^{\lambda_1 kT} \tag{5.47}$$

The functions $\exp(\lambda_i t)$ are linearly independent and form a basis of the linear vector space of pointwise continuous functions. Therefore, since (5.46) and (5.47) represent the same signal at the points kT, $k \in \mathbb{Z}$, $k \geq k_0$, the corresponding coefficients must match, i.e.

$$\alpha_{il} = u_{il} \quad \text{and} \quad \beta_{jh} = 0 \quad \forall\, i,j,h \tag{5.48}$$

Hence, the steady state response of $y_2(t)$ is

$$y_2(t) = \sum_{i=1}^{p} \sum_{l=0}^{m_i - 1} u_{il} \frac{t^l}{\ell!} e^{\lambda_i t} \equiv u_1(t) \qquad (5.49)$$

and is ripple-free.

"*only if*". Suppose part of the internal model of the exogenous input is implemented in the discrete FSTS controller. Then, $e_2(t)$ will be of a staircase type (fig. 5.4) and although we will have tracking at sampling instants, in between the continuous-time system will respond to step inputs and this will create ripples.

□

Remark 5.7: In the case of tracking polynomial inputs one of the poles at $s = 0$ of the continuous internal model is provided by the ZOH. On the other hand, the zero at $d = 1$ of ZOH will cancel one of the poles at $d = 1$ of the discrete equivalent of the continuous internal model, plant. This means that this pole at $d = 1$ must be provided by the discrete FSTS controller. This design will create an unstable pole-zero cancellation between the FSTS controller and the ZOH but instability can be avoided if the continuous integrator is ***reset*** each cycle [Fra., 1]. A consequence of the above discussion is that there is no need of a continuous internal model for step inputs and the sampled-data system of figure (5.3) will be ripple-free using any FSTS controller for the discrete equivalent of the plant only, as it has been illustrated in previous sections.

□

Example 5.6: Consider the continuous-time plant with transfer function

$$g_p(s) = \frac{1}{s(s + 1)}$$

Using a ZOH with sampling period T = 1sec, we will design an FSTS controller such that the closed-loop system tracks parabolic inputs $u_1(t) = t^2$ and is also ripple-free. We use both designs (first discrete and then continuous internal model) to illustrate the existence and the absence of ripples.

a. The discrete equivalent of the plant $g_p(s)$ using a ZOH with sampling period $T = 1$ is

$$p(d) = \frac{0.3679d + 0.2642d^2}{1 - 1.3679d + 0.3679d^2}$$

We see that although the plant and ZOH have together a double pole at $s = 0$, $p(d)$ has only one pole at $d = 1$ (corresponding to $s = 0$) the second one being canceled by the zero at $d = 1$ of the ZOH. Therefore, since the discrete equivalent dynamics of $u_1(t)$ are given by $d_r = (1-d)^3$, the FSTS controller must have a partial dynamics factor $d_{ca} = (1-d)^2$. The least McMillan degree FSTS controller according to theorem (5.7) is

$$c(d) = \frac{n_c(d)}{d_c(d)} = \frac{7.2967 - 10.1378d + 5.3748d^2 - 0.9517d^3}{1 - 1.1364d - 0.3672d^2 + 0.6836d^3}$$

Using the above described $c(d)$ in the feedback system of fig. (5.3) with $g(s) = g_p(s)$ the error $e_1(t)$ for a parabolic input $u_1(t) = t^2$ is shown in figure (5.5). We notice the existence of ripples due to the absence of a continuous internal model of the exogenous input $u_1(t)$.

b. Suppose we implement now a continuous internal model of the input $u_1(t) = t^2$, i.e the ZOH-continuous internal model-plant must have a factor s^3 in their pole polynomial. Hence, we have to introduce a continuous internal model $g_i(s)$ with one pole at $s = 0$, i.e $g_i(s) = 1/s$. Therefore,

$$g(s) = g_i(s)g_p(s) = \frac{1}{s^2(s + 1)}$$

and the discrete equivalent of $g(s)$ is

$$p(d) = \frac{0.1321d + 0.4197d^2 - 0.0803d^3}{1 - 2.3679d + 1.7358d^2 - 0.3679d^3}$$

Again, $p(d)$ has a double pole at $d = 1$ and therefore the FSTS

controller must contain a factor $d_{ca} = 1 - d$. The minimal McMillan degree FSTS controller is

$$c(d) = \frac{n_c(d)}{d_c(d)} = \frac{10.4529 - 16.3777d + 9.1769d^2 - 1.6701d^3}{1 + 0.9868d - 1.6223d^2 - 0.3646d^3}$$

The error $e_1(t)$ due to a parabolic input $u_1(t)$ is shown in figure (5.6). We see that the closed-loop system settles in 6 secs and is ripple-free due to the presence of a continuous internal model of the exogenous input $u_1(t)$.

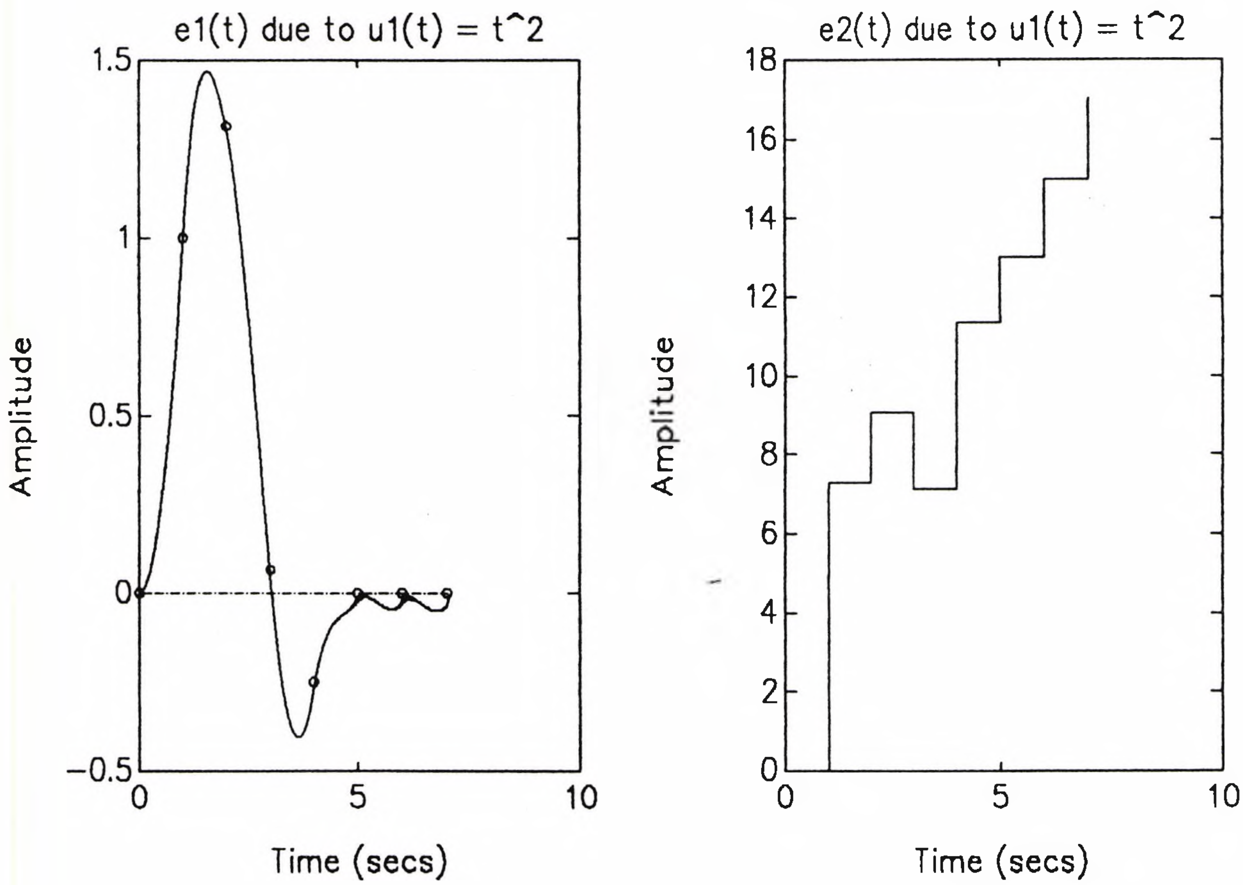

Figure (5.5): Behaviour with discrete internal model

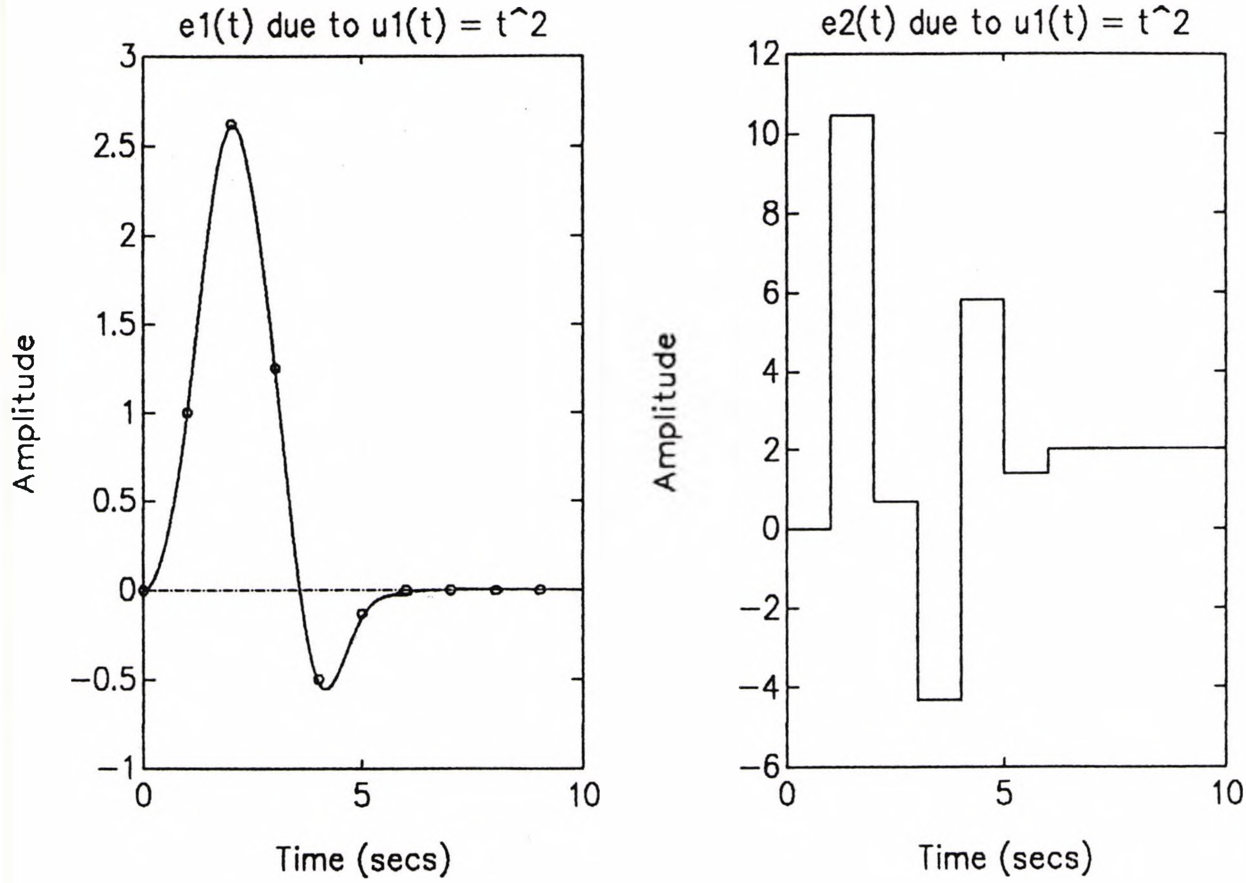

Figure (5.6): Behaviour with continuous internal model

□

Remark 5.8: It is possible that the sampled-data system of figure (5.3) to be ripple-free using a discrete rather than a continuous internal model. This may happen if the sampling period T is sufficiently small and so the elapsed time between samples is not sufficient for ripples to occur.

□

Example 5.7: The discrete plant $p(d)$ of example (5.5) has been obtained from the continuous plant

$$g_p(s) = \frac{1}{(1 + 5s)(1 - 2s)}$$

using a ZOH with sampling period T = 0.5 secs. The response of the closed-loop system to a ramp input $u_1(t) = t$, using the same FSTS controller as that in example (5.5), is ripple-free as can be seen from figure (5.7).

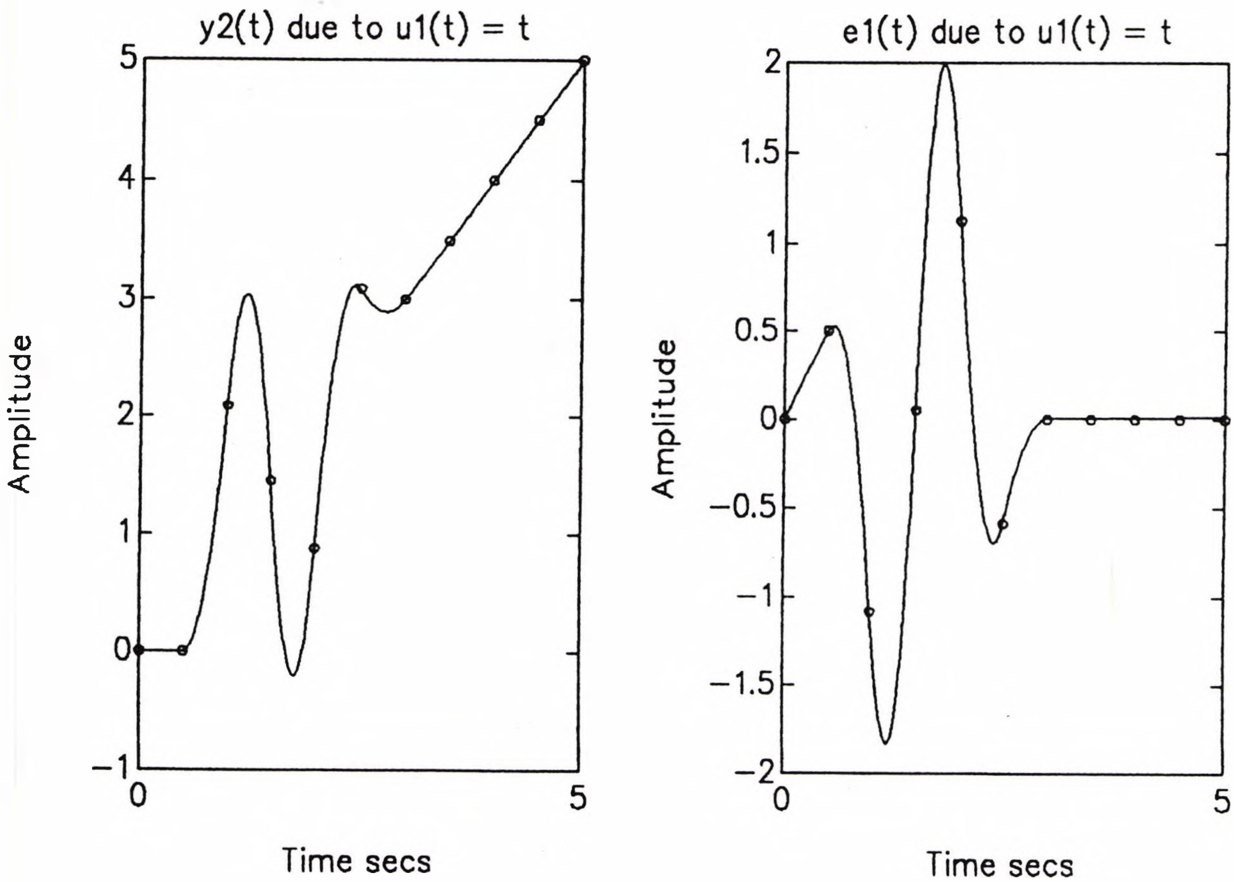

Figure (5.7): Behaviour with discrete internal model

□

5.9 Conclusions

The problem of *Total Finite Settling Time Stabilization* for single-input/single-output discrete-time systems has been addressed in this chapter. The approach used for its solution is purely algebraic and has led to a Youla-Bongiorno -Kucera type parametrization of all FSTS controllers with simple conditions for causality. All FSTS controllers are derived as a solution to a polynomial Diophantine equation which further leads to more simplification involving the solution of a Toeplitz type set of linear equations.

The solution to the linear algebra problem over $\mathbb{R}$ provides the minimal McMillan degree FSTS controller(s) and the nature of the FSTSP allows for the parametrization of the whole family of causal FSTS controllers $\mathcal{F}(p)$ according to McMillan degree. Other performance related problems such as time-optimal FSTS (deadbeat), tracking, disturbance rejection and partial assignment of the FSTS controller dynamics are also addressed. Finally, in the case of sampled-data systems, necessary and sufficient conditions for ripple-free FSTS are derived.

The FSTSP can be considered as a special case of stabilization where the ring of $\mathbb{R}[d]$-polynomials takes the place of $\mathbb{R}\mathbb{H}^{\infty}$ in Vidyasagar's approach [Vid., 1]. The forbidden region for stability (set of unstable points) becomes the entire complex plane whereas the stability region reduces to the point at infinity. Although the set of unstable points is not closed it does not seem to lead to convergence problems if the norm induced by the disc algebra $\boldsymbol{A}_s$ is used (section 5.5). This has enabled us to derive the conditions for strong FSTS and to prove that the well known ***parity interlacing property*** [Vid., 1], [You., 2] is valid in this case as well where the domains of stability of the controller (outside of the open unit disc) and of the feedback system (point at infinity) differ from each other.

Another problem that could be addressed within the FSTS framework is that of the transfer function and/or control signal shaping. We leave this problem for a subsequent chapter where a method is proposed for the design of ℓ^1 and ℓ^∞ FSTS controllers, i.e. FSTS controllers that minimize the ℓ^1- or ℓ^∞-norm of the error signal.

In the next chapter we will see how the FSTSP can be extended to the multivariable case. Using the same algebraic approach for its solution it will be shown which of the results of the SISO case it is possible to be extended to the MIMO one and if not which are the remaining best options.

Chapter 6

TOTAL FINITE SETTLING TIME STABILIZATION: The MIMO Case

Chapter 6

TOTAL FINITE SETTLING TIME STABILIZATION: The MIMO Case

6.1 Introduction

This chapter is an extension of the single variable case of the *Total Finite Settling Time Stabilization Problem* to the case of multivariable time-invariant discrete-time systems. Almost all the results of the SISO case carry on to the MIMO one with the only notable difference in the minimal McMillan degree problem and the parametrization of FSTS controllers according to McMillan degree.

In particular, using the same algebraic approach within the unity feedback system of figure (6.1), we tackle the multivariable FSTSP as a solution of a polynomial matrix Diophantine equation, which enables the parametrization of the family $\mathcal{F}(P)$ of causal FSTS controllers in terms of a relatively simple generic condition. The computation of the family $\mathcal{F}(P)$ is further reduced to the solution of a set of Toeplitz type linear equations over $\mathbb{R}$ which provides the family of all deadbeat (minimal-time) controllers.

The *parity interlacing property* is revisited again in the case of strong FSTS and necessary and sufficient conditions are derived for FST tracking, disturbance rejection and partial assignment of controller dynamics as in the SISO case.

As it was mentioned at the very beginning of the introduction, the only results that cannot be fully extended to the multivariable case are those concerning the minimal design problem and the parametrization of FSTS controllers according to McMillan degree. The best approximation to the exact solution is the specification of bounds for the minimal McMillan degree FSTS controller(s) and a semi parametrization according to McMillan degree where FSTS controllers of a given McMillan degree can be parametrized but not the entire subfamily.

Finally, the problem of FSTS for MIMO sampled-data systems is addressed and necessary and sufficient conditions for its solution are stated.

6.2 Definition of the FSTSP - Parametrization of the FSTS Controllers

Consider the one-parameter feedback configuration of figure (6.1) where $\underline{u}_1$, $\underline{u}_2$ are vector sequences in one indeterminate d over $\mathbb{R}$ representing the exogenous signals to the closed-loop system. Let $P \in \mathbb{R}^{l \times m}(d)$ and $C \in \mathbb{R}^{m \times l}(d)$ be the transfer functions of the plant $\mathcal{S}_p$ and controller $\mathcal{S}_c$ respectively and also let $M(\mathcal{R})$ denote the set of matrices and $U(\mathcal{R})$ the set of $\mathcal{R}$-unimodular matrices with elements from $\mathcal{R}$ and appropriate dimensions.

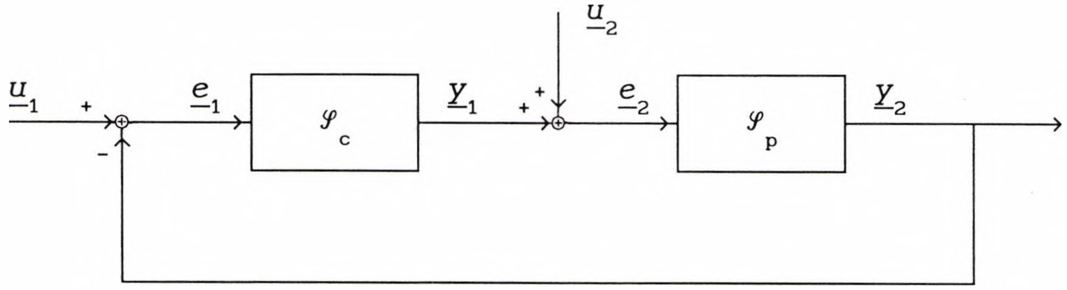

Figure (6.1): The MIMO unity feedback configuration

According to the presentation of section (5.2) we can have the following refined definition of the finite settling time response of a discrete linear system.

Definition 6.1: The discrete-time feedback system of figure (6.1) is said to exhibit

a. an **External Finite Settling Time (E-FST)** response, or to be **Externally FST-stable (E-FSTS)**, if for any step change in any of the components of the input vectors $\underline{u}_1$, $\underline{u}_2$ and for every initial conditions, all signals $\underline{y}_1$, $\underline{y}_2$ settle to a new steady state value in a finite number of steps

b. an **Internal Finite Settling Time (I-FST)** response, or to be **Internally FST-stable (I-FSTS)**, if for every initial state vector and any step change at the exogenous inputs $\underline{u}_1$, $\underline{u}_2$ all states settle to a new steady state in finite time.

□

Note that again the values of the finite settling time and of the steady state are left free. Also, the deadbeat response corresponds to the case of perfect tracking of step inputs in minimum number of steps and thus, it is a special case of the FST response. Due to linearity, we can readily extend lemma (5.2) to the following lemma.

Lemma 6.1: Consider the closed-loop system of figure (6.1) and let $H(P,C)$ denote the transfer function from $\underline{u} = [\underline{u}_1^t \ \underline{u}_2^t]^t$ to $\underline{e} = [\underline{e}_1^t \ \underline{e}_2^t]^t$. Then, the system exhibits an external FST response, if and only if $H(P,C) \in \mathbb{R}^{(l+m)x(l+m)}[d]$, i.e. the closed-loop system is a FIR system.

□

Remark 6.1: According to definition (6.1) and the properties of the unity feedback configuration (see section 3.4), the condition $H(P,C) \in M(\mathbb{R}[d])$ of lemma (6.1) implies that the closed-loop system of figure (6.1)

a. is *internally stable*, if and only if $\mathcal{S}_p$, $\mathcal{S}_c$ are stabilizable and detectable since all its controllable and constructible eigenvalues are shifted to zero; however, the later condition does not necessarily guarantee internal FST stability.

b. exhibits a ***total*** (***external*** as well as ***internal***) ***FST response***, if and only if $\mathcal{S}_p$, $\mathcal{S}_c$ are both controllable and constructible.

□

From lemma (6.1), remark (6.1) and the standard results from the analysis of the unity feedback configuration, as stated in section (3.4), we have the following theorem which provides the solution for the MIMO finite settling time problem.

Theorem 6.1: Let $P = N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$, $C = N_cD_c^{-1} = \tilde{N}_c\tilde{D}_c^{-1}$ be $\mathbb{R}[d]$ -coprime MFDs of the plant and controller transfer functions in the unity feedback configuration of figure (6.1). Then, the solution to the FSTSP exists, if and only if

$$\Delta := \tilde{N}_cN_p+\tilde{D}_cD_p \in U(\mathbb{R}[d]) \text{ or } \tilde{\Delta} := \tilde{N}_pN_c+\tilde{D}_pD_c \in U(\mathbb{R}[d]) \tag{6.1}$$

Moreover, the family of all causal FSTS controllers is given by

$$\mathcal{F}(P) = \{(N_c,D_c) : N_c = X + D_pR,\ D_c = Y - N_pR,$$
$$R \in M(\mathbb{R}[d]) \text{ and } |Y(0)-N_p(0)R(0)| \neq 0 \text{ if } N_p(0) \neq 0\} \tag{6.2a}$$

or

$$\mathcal{F}(P) = \{(\tilde{D}_c,\tilde{N}_c) : \tilde{N}_c = \tilde{X} + S\tilde{D}_p,\ \tilde{D}_c = \tilde{Y} - S\tilde{N}_p,$$
$$S \in M(\mathbb{R}[d]) \text{ and } |\tilde{Y}(0)-S(0)\tilde{N}_p(0)| \neq 0 \text{ if } \tilde{N}_p(0) \neq 0\} \tag{6.2b}$$

where R, S are arbitrary and X, Y, $\tilde{X}$, $\tilde{Y}$ are appropriate polynomial matrices satisfying the following Bezout identity

$$\begin{bmatrix} -\tilde{X} & \tilde{Y} \\ \tilde{D}_p & \tilde{N}_p \end{bmatrix}\begin{bmatrix} -N_p & Y \\ D_p & X \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{6.3}$$

Proof. According to lemma (6.1), $H(P,C)$ must be a polynomial matrix for FSTS. If we consider the expressions of $H(P,C)$ given by (3.78c) and (3.78d) we realize that they are bicoprime due to the coprimeness of the polynomial matrices involved. Then $H(P,C) \in M(\mathbb{R}[d])$, if and only if Δ, or $\tilde{\Delta}$ are unimodular polynomial matrices (corollary 2.3). The parametrization of the causal FSTS controllers readily follows from theorem (3.18) with Δ, $\tilde{\Delta}$ being identity matrices of appropriate dimensions.

$\square$

The family $\mathcal{F}(P)$ defines the solution to the External-FSTS problem if the plant and controller are both stabilizable and detectable. When both plant and controller are controllable and constructible the family $\mathcal{F}(P)$ defines the solution to the Total-FSTS problem. Also, from theorem (6.1), proposition (3.5) and corollary (3.13) it is clear that the following corollary holds true.

Corollary 6.1: Let (P,C) be FST-stable and R, S, Y, $\tilde{Y}$ be polynomial matrices as in theorem (6.1). The FSTS controller is causal

a. $\forall\ R \in M(\mathbb{R}[d])$, if $N_p(0)=0$, or $\forall\ S \in M(\mathbb{R}[d])$, if $\tilde{N}_p(0)=0$

b. $\forall\ R \in M(\mathbb{R}[d])\ :\ |Y(0)-N_p(0)R(0)|\neq 0$, if $N_p(0)\neq 0$, or $\forall\ S \in M(\mathbb{R}[d])\ :\ |\tilde{Y}(0)-S(0)\tilde{N}_p(0)|\neq 0$, if $\tilde{N}_p(0)\neq 0$.

$\square$

Theorem (6.1) characterizes all FSTS controllers that stabilize the plant P in FST sense in terms of a certain ***'free'*** parameter (R or S) such that there is a one-to-one correspondence between the parameter and the controller. By substituting the formula that generates all the FSTS controllers into the expressions of $H(P,C)$ and $W(P,C)$ in (3.78c) to (3.79b) we obtain the following parametrization of the closed-loop transfer functions (see also [Vid., 1]).

Corollary 6.2: Let (P,C) be FST-stable and N_p, D_p, $\tilde{N}_p$, $\tilde{D}_p$, be as in theorem (6.1). Then

$$H(P,C) = \begin{bmatrix} D_c\tilde{D}_p & -D_c\tilde{N}_p \\ N_c\tilde{D}_p & I-N_c\tilde{N}_p \end{bmatrix} \tag{6.4a}$$

$$= \begin{bmatrix} I-N_p\tilde{N}_c & -N_p\tilde{D}_c \\ D_p\tilde{N}_c & D_p\tilde{D}_c \end{bmatrix} \tag{6.4b}$$

$$W(P,C) = \begin{bmatrix} N_c\tilde{D}_p & -N_c\tilde{N}_p \\ I-D_c\tilde{D}_p & D_c\tilde{N}_p \end{bmatrix} \tag{6.5a}$$

$$= \begin{bmatrix} D_p\tilde{N}_c & D_p\tilde{D}_c-I \\ N_p\tilde{N}_c & N_p\tilde{D}_c \end{bmatrix} \tag{6.5b}$$

where (N_c,D_c), $(\tilde{D}_c,\tilde{N}_c)$ are given by (6.2a) and (6.2b) respectively.

□

Remark 6.2: In the characterization of the family $\mathcal{F}(P)$ the parameters R, or S are not entirely free. They have to satisfy condition (b) of corollary (6.1). This condition is not so strong and is valid for *'almost all'* R, or $S \in M(\mathbb{R}[d])$ for the following reasons. (Y,N_p) are left coprime due to (6.3). Then, according to theorem (2.15)

$$|Y - N_pR| = y - n_pr, \quad y,\ n_p,\ r \in \mathbb{R}[d]$$

where $y = \det Y$ and n_p is the least invariant polynomial of N_p. Then

$$|Y(0) - N_p(0)R(0)| \neq 0 \text{ if and only if } r(0) \neq y(0)/n_p(0)$$

This is true for *'almost all'* $r \in \mathbb{R}[d]$ and therefore for *'almost all'* $R \in M(\mathbb{R}[d])$. The analogous result can be proved for S.

□

6.3 Algebraic Computation of the Family $\mathcal{F}(P)$

According to theorem (6.1) the computation of the family $\mathcal{F}(P)$ of all causal FSTS controller requires only the computation of a particular solution of the Diophantine equation

$$\tilde{N}_c N_p + \tilde{D}_c D_p = I \quad \text{or} \quad \tilde{N}_p N_c + \tilde{D}_p D_c = I \tag{6.6}$$

One way to obtain such a particular solution is by reducing the problem to a standard linear algebra problem over $\mathbb{R}$ using Toeplitz matrices as it was illustrated in chapter (5) for the single variable case. The treatment here is similar to the one given by Chen [Che., 1] the only difference being that the right composite matrix of the plant transfer function is column reduced, rather than only the denominator matrix of its MFD. We give next the main result with its proof.

Theorem 6.2: Let $P = N_p D_p^{-1} \in \mathbb{R}^{l\times m}(d)$ with (N_p, D_p) right $\mathbb{R}[d]$ coprime and $[N^t \; D^t]^t$ column reduced and let μ_i $i = 1,\ldots,m$ be the right minimal indices of P, $\mu = \max\{\mu_i\}$, ν_j $j = 1,\ldots,\ell$ be the left minimal indices of P, $\nu = \max\{\nu_j\}$. Then, if $C = \tilde{D}_c^{-1}\tilde{N}_c$ and n is the maximum row degree of $[\tilde{N}_c \; \tilde{D}_c]$, i.e., $\partial_s([\tilde{N}_c \; \tilde{D}_c]) = n$, $\tilde{N}_c$, $\tilde{D}_c$ satisfy the condition

$$\tilde{N}_c N_p + \tilde{D}_c D_p = I \quad \text{for at least } n \geq \nu - 1 \tag{6.7}$$

Proof. N_p, D_p, $\tilde{N}_c$, $\tilde{D}_c$ can be written as

$$N_p = N_{p0} + N_{p1}d + \cdots + N_{p\mu}d^{\mu}$$

$$D_p = D_{p0} + D_{p1}d + \cdots + D_{p\mu}d^{\mu}$$

$$\tilde{N}_c = \tilde{N}_{c0} + \tilde{N}_{c1}d + \cdots + \tilde{N}_{cn}d^{n}$$

$$\tilde{D}_c = \tilde{D}_{c0} + \tilde{D}_{c1}d + \cdots + \tilde{D}_{cn}d^{n}$$

Then, equation (6.7) can be formed as

$$[\tilde{N}_{c0}\ \tilde{D}_{c0} \cdots \tilde{N}_{cn}\ \tilde{D}_{cn}] \begin{bmatrix} N_{p0} & \cdots & N_{p\mu} & & & & & 0 \\ D_{p0} & \cdots & D_{p\mu} & & & & & \\ & & N_{p0} & \cdots & N_{p\mu} & & & \\ & & D_{p0} & \cdots & D_{p\mu} & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & 0 & & & N_{p0} & \cdots & N_{p\mu} \\ & & & & & D_{p0} & \cdots & D_{p\mu} \end{bmatrix} =$$

$$:= T^{r}_{P,n+1}$$

$$= [I_m \quad O_m \quad \cdots \quad O_m \cdots O_m] \tag{6.8}$$

From the special structure of $T^{r}_{P,n+1}$ it can be derived relatively easily [Kun., 2], [Bit., 1] that

$$\rho(T^{r}_{P,n+1}) = m(n+1) + \sum_{i=1}^{l} \nu_i \quad \text{for } n \geq \nu - 1 \tag{6.9}$$

Also

$$T^{r}_{P,1} = \begin{bmatrix} N_{p0} & N_{p1} \cdots & N_{p\mu} \\ D_{p0} & D_{p1} \cdots & D_{p\mu} \end{bmatrix}$$

has at least $\ell_z = \sum_{i=1}^{m}(\mu - \mu_i)$ zero columns all of them occurring after the constant coefficient matrix $[N^{t}_{p0}\ D^{t}_{p0}]^{t}$. Every other $T^{r}_{P,k}$, $k > 2$ has the same minimum number of zero columns ℓ_z due to its structure [Che., 1]. Let $\tilde{T}^{r}_{P,n+1}$ be the matrix $T^{r}_{P,n+1}$ after deleting these zero columns. Then the number of columns of $\tilde{T}^{r}_{P,n+1}$ is

$$m(\mu + n + 1) - \ell_z = m(n+1) + \sum_{i=1}^{m} \mu_i \tag{6.10}$$

and by construction

$$\rho(\tilde{T}^{r}_{P,n+1}) = \rho(T^{r}_{P,n+1}) = m(n+1) + \sum_{i=1}^{l} \nu_i \tag{6.11}$$

But according to definition (2.23) and corollary (3.5)

$$\sum_{i=1}^{l} \nu_i = \sum_{i=1}^{m} \mu_i = \delta_{\mathcal{M}}(P)$$

and therefore, from (6.10) and (6.11), $\tilde{T}^{r}_{P,n+1}$ has full column rank. Also the positions of the zero columns of $[I_m\ O_m \ldots O_m]$ coincide with those of $T^{r}_{P,n+1}$ and so equation (6.8) has a solution for $n \geq \nu - 1$. It is possible that $[I_m\ O_m\ \ldots\ O_m]$ belongs to the row space of $T^{r}_{P,n+1}$ for $n < \nu - 1$ and this is the reason of the statement 'for at least $n \geq \nu - 1$' in the expression (6.7).

□

Remark 6.3: The family of solutions of equation (6.8) for $n = \nu - 1$, denoted by $\hat{\mathcal{F}}_{\nu-1}(P)$, may or may not be causal if the plant does not posses a delay. $\hat{\mathcal{F}}_{\nu-1}(P)$ consists of all FSTS controllers with the row degrees of the left composite matrix $[\tilde{D}_c\ \tilde{N}_c]$ less than or equal to $\nu - 1$. Any $C \in \hat{\mathcal{F}}_{\nu-1}(P)$, regardless of causality, can be used as a particular solution of the equation (6.7) and the family $\mathcal{F}(P)$ of all causal FSTS controllers can be parametrized according to theorem (6.1).

Similarly, if a row reduced composite matrix $[\tilde{N}_p\ \tilde{D}_p]$ of the plant transfer function is used, the dual equation of (6.8) will give us the family $\hat{\mathcal{F}}_{\mu-1}(P)$ of all FSTS controllers with column degrees of the right composite matrix $[N_c^t\ D_c^t]^t$ less than or equal to $\mu - 1$, where μ is the maximum right minimal index (reachability index for a minimal realization) of the plant transfer function $P(d)$.

□

6.4 McMillan Degree Bounds of the FSTS Controllers

In this section we study the McMillan degree properties of the family of causal FSTS controllers $\mathcal{F}(P)$. We recall from section (3.2.4) that the McMillan degree $\delta_{\mathcal{M}}(G)$ of a rational matrix $G(d)$ denotes the total number of poles (finite and infinite) of $G(d)$ and to this extend it is a measure of complexity of $G(d)$. If $G(d) = N(d)D^{-1}(d) = \tilde{D}^{-1}(d)\tilde{N}(d) \in \mathbb{R}^{1\times m}(d)$, where (N,D), $(\tilde{D},\tilde{N})$ are polynomial matrices not necessarily coprime, then according to theorem (3.4)

$$\delta_{\mathcal{M}}(G) = \partial([N^t\ D^t]^t) = \partial([\tilde{N}\ \tilde{D}]]) \qquad (6.12)$$

Let $R_G(d) = [N^t\ D^t]^t$ and $L_G(d) = [\tilde{N}\ \tilde{D}]$ be the right and left composite matrices of $G(d)$ and $\underline{r}_{Gi}$, $i = 1,2,\ldots,m$, $\underline{\ell}_{Gj}$, $j = 1,2,\ldots,\ell$ be the column and row vectors of $R_G(d)$ and $L_G(d)$ respectively. Then

$$\delta_{\mathcal{M}}(G) \le \sum_{i=1}^{m} \partial(\underline{r}_{Gi}), \quad \delta_{\mathcal{M}}(G) \le \sum_{j=1}^{l} \partial(\underline{\ell}_{Gj}) \tag{6.13}$$

with equality holding if and only if $R_G(d)$, $L_G(d)$ come from coprime MFDs and are column and row reduced respectively. In that case $\partial(\underline{r}_{Gi})$ become the right and $\partial(\underline{\ell}_{Gj})$ the left minimal indices of $G(d)$. In any case, inequalities (6.13) provide an upper bound for the McMillan degree of $G(d)$. We exploit mainly this property to establish bounds on the minimum McMillan degree of the FSTS controllers and to parametrize the family $\mathcal{F}(P)$ according to column, row degrees of $R_C(d)$ and $L_C(d)$ respectively.

Theorem 6.3 *(MDP)*: Let $R_P = [N_p^t\ D_p^t]^t$ and $L_P = [\tilde{N}_p\ \tilde{D}_p]$ be the right and left composite matrices of the plant $P \in \mathbb{R}^{l\times m}(d)$, where the MFDs involved are $\mathbb{R}[d]$-coprime. Let also $R_C = [N_c^t\ D_c^t]^t$, $L_C = [\tilde{N}_c\ \tilde{D}_c]$ be the right and left composite matrices of any FSTS controller and $R_{C\min}$ be a minimum column complexity and $L_{C\min}$ a minimum row complexity solution of the FSTS problem. If $C_l(R_C)$ and $C_m(L_C)$ are the Grassmann products of the column and row vectors of R_C and L_C respectively, then the minimum McMillan degree of the FSTS controllers lies within the following bounds

$$\begin{aligned}\max\{c_m^r(R_C), c_m^l(L_C)\} &\le \delta_{\mathcal{M}\min}(C) \\ &\le \min\{c_c(R_{C\min}), c_r(L_{C\min})\}\end{aligned} \tag{6.14}$$

where

$$c_m^r(R_C) = \max_i\{\min_{R_C}\{\partial(C_{l,i}(R_C))\}\}$$

$$c_m^l(L_C) = \max_i\{\min_{L_C}\{\partial(C_{m,i}(L_C))\}\}$$

and $c_c(R_{C\min})$, $c_r(L_{C\min})$ are the column and row complexities of $R_{C\min}$ and $L_{C\min}$ respectively.

Proof. Since the column and row complexities of $R_{C\min}$ and $L_{C\min}$ are the sum of the column and row degrees of $R_{C\min}$ and $L_{C\min}$ correspondingly (section 2.3.1), the right inequality of (6.14) comes straight from relations (6.13). For the first inequality we have from theorem (6.1) that for any FSTS controller

$$R_C = \begin{bmatrix} N_C \\ D_C \end{bmatrix} = \begin{bmatrix} X & D_p \\ Y & -N_p \end{bmatrix} \begin{bmatrix} I \\ R \end{bmatrix} \in \mathbb{R}^{(m+l)\times l}[d] \qquad (6.15)$$

Then

$$\delta_{\mathcal{M}}(C) = \partial(R_C(d)) = \partial(C_l(R_C))$$

where $C_l(R_C)$ is the Grassmann product of the columns of R_C; a $k = \binom{m+l}{l}$ column vector containing all the maximal order $\ell\times\ell$ minors of $R_C(d)$. Clearly

$$\delta_{\mathcal{M}\min}(C) = \min_{R_C}\{\partial(C_l(R_C))\}$$

From equation (6.15) we see that every minor of R_C, i.e. every entry $C_{l,i}(R_C)$ of $C_l(R_C)$ is given by

$$C_{l,i}(R_C) = \det(A_i + B_i R) \qquad (6.16)$$

where A_i is a corresponding square matrix from $[X^t \; Y^t]^t$ and B_i is the corresponding part of $[D_p^t \; -N_p^t]^t$ according to the row partitioning of

$$U = \begin{bmatrix} X & D_p \\ Y & -N_p \end{bmatrix}$$

According to theorem (6.1), U is an $\mathbb{R}[d]$-unimodular matrix and therefore A_i, B_i are left coprime. Hence, according to theorem (2.15), see also [Vid., 1]

$$\det(A_i + B_i R) = a_i + b_i r_i$$

where $a_i = \det A_i$, b_i is the least invariant polynomial of B_i and $r_i \in \mathbb{R}[d]$ depends on R. Since

$$\partial(C_l(R_C)) = \max_i\{\partial(C_{l,i}(R_C))\}$$

we proceed as follows for a lower bound on $\delta_{\mathcal{M}\min}(C)$. For

$i = 1,\ldots,k$, find r_i such that

$$\partial(a_i + b_i r_i) = minimum$$

by dividing a_i by b_i. Since the r_i's derived in this way do not necessarily correspond to the same matrix R, then

$$\max_i\{\partial(a_i + b_i r_i)\}$$

is a lower bound for $\partial(C_1(R_{\tilde{C}}))$ and is denoted by $c_m^r(R_{\tilde{C}})$. Another lower bound is $c_m^l(L_{\tilde{C}})$ which comes from the dual expression of the FSTS controllers using left coprime MFDs. So finally

$$\max\{c_m^r(R_{\tilde{C}}), c_m^l(L_{\tilde{C}})\} \le \delta_{\mathcal{M}\min}(C)$$

□

Theorem (6.3) is the multivariable analogue to the SISO minimal FST design problem. It is the best approximation to theorem (5.2) by establishing bounds for the minimum McMillan degree $\delta_{\mathcal{M}\min}(C)$ of the FSTS controllers instead of providing the exact solution. We consider next the analogues to corollaries (5.4) and (5.5), i.e. the parametrization of the family $\mathcal{F}(P)$ according to McMillan degree.

According to inequalities (6.13), the column and row complexities of any FSTS controller C constitute upper bounds for its McMillan degree. Therefore, fixed column or row degree solutions of the Diophantine equations (6.6) will lead to a partial parametrization of the family $\mathcal{F}(P)$ according to McMillan degree upper bounds. The following theorem gives the fixed column and row degree solutions of the Diophantine equations (6.6).

Theorem 6.4 *(Fixed Column/Row Degree Solutions)*: Consider the configuration of figure (6.1) where $P = \tilde{D}_p^{-1}\tilde{N}_p = N_p D_p^{-1} \in \mathbb{R}^{l\times m}(d)$ and $C = \tilde{D}_c^{-1}\tilde{N}_c = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ are the polynomial MFDs for the plant and any FSTS controller according to the parametrization of theorem (6.1). If $R_P = [N_p^t\ D_p^t]^t$ and $L_P = [\tilde{N}_p\ \tilde{D}_p]$ are normal right and left composite matrices of the plant, i.e. they are minimal bases ordered in column, row descending degree order (def. 3.12), and $R_{C\min} = [N_c^t \ D_c^t]^t_{\min}$, $L_{C\min} = [\tilde{N}_c \ \tilde{D}_c]_{\min}$ are any minimum column, row complexity solutions of the Diophantine equations (6.6), then the column and row degrees of any other solution are uniquely determined by the degrees of the columns and rows of the free parameters R, S respectively.

Proof. We give the proof for the fixed column degree solutions. The result for the fixed row degree solutions is derived in a similar manner. According to theorem (6.1) any right coprime MFD of any FSTS controller is given by

$$R_C = \begin{bmatrix} N_C \\ D_C \end{bmatrix} = \begin{bmatrix} N_C \\ D_C \end{bmatrix}_{\min} + \begin{bmatrix} D_p \\ -N_p \end{bmatrix} R \quad \in \mathbb{R}^{(m+l)\times l}[d]$$

$$\text{or} \quad R_C = R_{C\min} + Q \tag{6.17}$$

$$\text{where} \quad Q = \begin{bmatrix} D_p \\ -N_p \end{bmatrix} R$$

Let $\underline{r}_{Ci}$, $\underline{r}_{Ci,\min}$, $\underline{r}_i$, $\underline{q}_i$, $i = 1,\dots,\ell$ be the column vectors of R_C, $R_{C\min}$, R and Q respectively. Also, since $R_P = [N_p^t \ D_p^t]^t$ is normal, $[D_p^t \ -N_p^t]^t$ is coprime and column reduced with column degrees $\mu = \mu_1 \geq \cdots \geq \mu_m$. Then, according to the predictable degree property (theorem 2.13, [For., 1])

$$\partial(\underline{q}_j(d)) = \max_{i:r_{ij}(d)\neq 0} \{\partial(r_{ij}(d)) + \mu_i\}, \quad i = 1,\dots,m \quad j = 1,\dots,\ell$$

Hence, since the minimum column complexity is equivalent to minimum column degrees for each column we have that

$$\partial(\underline{r}_{Ci}(d)) = \begin{cases} \partial(\underline{r}_{Ci,\min}(d)), & \text{if } \partial(\underline{q}_i(d)) \leq \partial(\underline{r}_{Ci,\min}(d)) \\ \partial(\underline{q}_i(d)), & \text{if } \partial(\underline{q}_i(d)) > \partial(\underline{r}_{Ci,\min}(d)) \end{cases}$$

Therefore the column degrees of R_C are uniquely determined by the degrees of the entries of $R(d)$.

□

Remark 6.4: Theorem (6.4) provides a relatively simple parametrization of the family of all FSTS controllers according to column, or row degrees of the composite matrices of the controllers. This parametrization refers to the whole family of the FSTS controllers and not to the family $\mathcal{F}(P)$ of the causal FSTS controllers. For causality, the free parameters R, S should satisfy, in addition to the degree conditions of theorem (6.4), the causality conditions of corollary (6.1). These conditions are generic according to remark (6.2) and therefore the fixed degree solutions will lead to causal FSTS controllers for almost all R, S. In any case, fixed degree solutions provide upper bounds for the McMillan degrees and constitute the best approximations to the exact parametrization problems according to McMillan degree that have been successfully dealt with in the SISO case.

□

From the previous discussion it is clear that fixed degree solutions and in particular minimum column or row complexity solutions to the FSTSP play an important role in the parametrization issues of the FSTS controllers. In fact, not only provide upper bounds for the minimum McMillan degree problem and assist in the partial parametrization of the FSTS controllers according to McMillan degree, but the minimum column complexity solutions, may characterize the entire family of the deadbeat regulators as it will be seen later on in this section.

For this reason we give next the least column complexity solution to the Diophantine equation

$$\tilde{N}_p N_c + \tilde{D}_p D_c = I \tag{6.18}$$

Using left and right composite matrices for the plant and the controller respectively, equation (6.18) can be written as

$$L_P \cdot R_C = I \tag{6.19}$$

If $\underline{r}_{Ci}(d)$, $\underline{e}^i$, $i = 1,\ldots,\ell$ are the ith column vectors of $R_C(d)$ and I, equation (6.19) reduces to the following set of equations

$$L_P(d)\cdot\underline{r}_{Ci}(d) = \underline{e}^i, \quad i = 1,\ldots,\ell \tag{6.20}$$

and the next proposition is self-evident.

Proposition 6.1: A least column complexity solution $R_{C_{min}}$ of equation (6.19) corresponds to a set of least column degree solutions $\underline{r}_{Ci,min}(d)$, $i = 1,\ldots,\ell$ of equations (6.20) and vice versa.

□

Equations (6.20) can be written in a Toeplitz form in a dual way to theorem (6.2). If $k_i = \partial(\underline{r}_{Ci}(d))$, then

$$\underline{r}_{Ci}(d) = \underline{r}_{Ci,0} + \underline{r}_{Ci,1}d + \cdots + \underline{r}_{Ci,k_i}d^{k_i}, \quad \underline{r}_{Ci,j} \in \mathbb{R}^{(l+m)\times 1}$$

and equations (6.20) reduce to

$$T^l_{P,k_i+1}\begin{bmatrix} \underline{r}_{Ci,0} \\ \vdots \\ \underline{r}_{Ci,k_i} \end{bmatrix} = \begin{bmatrix} \underline{e}^i \\ \overline{0} \\ \vdots \\ 0 \end{bmatrix} \tag{6.21}$$

where T^l_{P,k_i+1} is the dual to T^r_{P,k_i+1} (theorem 6.2).

The following theorem gives the entire family of the least degree solutions of equations (6.20).

Theorem 6.5 *(Least Column Complexity Solutions)***:** Let L_P be a row reduced left composite matrix of the plant P and μ_i, $i = 1,\ldots,m$, ν_j, $j = 1,\ldots,\ell$ be the right and left minimal indices of P with $\mu = \max\{\mu_i\}$ and $\nu = \max\{\nu_j\}$. The least column complexity solution of the Diophantine equation (6.18) is given by the equations (6.21) where k_i is the first $k_i \le \mu - 1$ such that

$$\rho\left(\begin{bmatrix} T^1_{P,k_i+1} & \begin{bmatrix} \underline{e}^i \\ O \end{bmatrix} \end{bmatrix}\right) = (m+\ell)(k_i+1) - \sum_{j:\mu_j<(k_i+1)} (k_i+1-\mu_j) \qquad (6.22)$$

We denote this k_i by $k_{i,min}$ and $\max_i\{k_{i,min}\}$ by k_{min}.

<u>*Proof*</u>. Using duality and theorem (6.2), equations (6.21) have always a solution for every $k_i \le \mu - 1$ and therefore $\mu - 1$ is an upper bound for the least column degree solution. Using dual reasoning to the one described by Kung et all [Kun., 2] we have that

$$\rho(T^1_{P,k_i+1}) = (m+\ell)(k_i+1) - \sum_{j:\mu_j<(k_i+1)} (k_i+1-\mu_j) \qquad (6.23)$$

and therefore the first $k_i \le \mu - 1$ such that the relationship (6.22) holds allows for the least column degree solution of equations (6.21) and as a consequence for the least column complexity solution of equation (6.18).

□

From the discussion of this section, it is clear that we may distinguish between two types of solutions; the ones that provide for the least complexity and the ones that provide for the least degree of the composite matrices $R_C(d)$, or $L_C(d)$. We formalize that by the following definition.

Definition 6.2: Let $R_C = [N_c^t \;\; D_c^t]^t$ and $L_C = [\tilde{N}_c \;\; \tilde{D}_c]$ be the right and left composite matrices that satisfy the Diophantine equations (6.6) and let $\underline{r}_{Ci}$, $i = 1,\dots,m$, $\underline{\ell}_{Cj}$, $j = 1,\dots,\ell$ be the column and row vectors of R_C and L_C respectively. If $\underline{r}_{Ci,min}$ are the least column degree solutions of equations (6.20) and $\underline{\ell}_{Cj,min}$ are the least row degree solutions of the dual to equations (6.20) with

$$k_{i,min} = \partial(\underline{r}_{Ci,min}),\; k_{min} = \max\{k_{i,min}\},\; i = 1,\dots,m$$
$$n_{j,min} = \partial(\underline{\ell}_{Cj,min}),\; n_{min} = \max\{n_{j,min}\},\; j = 1,\dots,\ell$$

we define the family of all FSTS controllers such that $\partial(\underline{r}_{Ci})$

$= k_i$ ($\partial(\ell_{Cj}) = n_j$) as **least column (row) complexity** family and we denote it by $\mathcal{F}^c_{c,min}(P)$ ($\mathcal{F}^c_{r,min}(P)$). Also, we define the family of all FSTS controllers with minimum column (row) degree i.e., $\partial_s(R_C(d)) = k_{min}$ ($\partial_s(L_C(d)) = n_{min}$) as the **least column (row) degree** family and we denote it by $\mathcal{F}_{c,min}(P)$ ($\mathcal{F}_{r,min}(P)$).

□

According to the above definition and to theorems (6.2) and (6.5), the following corollary may be readily established.

Corollary 6.3: Let $\hat{\mathcal{F}}_{\mu-1}(P)$, $\hat{\mathcal{F}}_{\nu-1}(P)$ be the FSTS families as in remark (6.3) and $\mathcal{F}^c_{c,min}(P)$, $\mathcal{F}^c_{r,min}(P)$, $\mathcal{F}_{c,min}(P)$, $\mathcal{F}_{r,min}(P)$ be the FSTS families as in definition (6.2). Then, the following set inclusion properties hold true.

$$\mathcal{F}^c_{c,min}(P) \subseteq \mathcal{F}_{c,min}(P) \subseteq \hat{\mathcal{F}}_{\mu-1}(P)$$

$$\mathcal{F}^c_{r,min}(P) \subseteq \mathcal{F}_{r,min}(P) \subseteq \hat{\mathcal{F}}_{\nu-1}(P)$$

□

Remark 6.5: Among all the families of FSTS controllers depicted in corollary (6.3) the most important from the design point of view, are clearly the least complexity families. These families provide the best upper bounds for the minimum McMillan degree of the FSTS controllers and allow for the parametrization of all FSTS controllers according to column, or row degrees of the composite matrices R_C and L_C. Obviously, as we move from more to less restrictive families in the inclusion chains of corollary (6.3), the computational effort for their characterization decreases. Finally, it must be pointed out that only in the case of strictly causal plants all the FSTS controllers that belong to any of the families mentioned in corollary (6.3), are causal. Otherwise the causality conditions of corollary (6.1) must be satisfied.

□

6.4.1 FSTS controllers for deadbeat regulation

As it has already been mentioned in this section, the least column complexity FSTS controllers have another very important property. They may characterize completely the family of deadbeat regulators, i.e. the family of the controllers that guarantee minimum settling time to zero steady state of all the states of the unity feedback system of figure (6.1) for any initial conditions. We prove this result within the FSTS framework. For an alternative proof one could refer to Kucera [Kuc., 2], [Kuc., 10].

Theorem 6.6 *(Deadbeat Regulation)***:** Consider the feedback configuration of figure (6.1) where the plant $\mathcal{S}_p$ and the controller $\mathcal{S}_c$ are both controllable and constructible. Then, the family of all causal FSTS controllers $\boldsymbol{C} = \boldsymbol{N}_c\boldsymbol{D}_c^{-1}$ such that $\boldsymbol{L}_C = [\boldsymbol{N}_c^t \ \ \boldsymbol{D}_c^t]^t$ has the least possible column degrees is exactly the family of the deadbeat regulators.

__Proof__. It is known (chapter 4 and references therein) that for deadbeat regulation all the eigenvalues of the closed-loop system of figure (6.1) have to be moved to zero and in addition a minimum settling time has to be attained. Since the first task is accomplished by all FSTS controllers, the family of deadbeat regulators is a subfamily of the family $\mathcal{F}(P)$ of the causal FSTS controllers. What is needed more, is the minimum time requirement to be satisfied. To allow for the effect of the initial conditions we assume that finite sequences $\underline{u}_1 = \underline{u}_1^-$ and $\underline{u}_2 = \underline{u}_2^-$ with negative orders and only negative terms are applied at the inputs of the feedback system of fig. (6.1). If $\tau_1 = \tau(\underline{u}_1^-) < 0$ and $\tau_2 = \tau(\underline{u}_2^-) < 0$ are the orders of $\underline{u}_1^-$ and $\underline{u}_2^-$ respectively, then $\underline{u}^- = [\underline{u}_1^{-t} \ \ \underline{u}_2^{-t}]^t$ can be written as

$$\underline{u}^- = d^{\tau}\underline{u} \quad \text{where } \tau = \min\{\tau_1, \tau_2\},\ \underline{u} \in \mathbb{R}^{l+m}[d] \text{ and } \partial(\underline{u}) \le -\tau + 1$$

Then all the terms apart from the unity one of the transfer function matrix $\boldsymbol{W}(P,C)$ (eqn. 6.5a) contribute to the outputs $\underline{y}_1$, $\underline{y}_2$ for time $k \ge 0$, i.e.

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \end{bmatrix} = \begin{bmatrix} N_c\tilde{D}_p & -N_c\tilde{N}_p \\ -D_c\tilde{D}_p & D_c\tilde{N}_p \end{bmatrix} \cdot \underline{u}^-$$

$$= d^{\tau} \cdot \begin{bmatrix} N_c \\ -D_c \end{bmatrix} \begin{bmatrix} \tilde{D}_p & -\tilde{N}_p \end{bmatrix} \cdot \underline{u}$$

$$= d^{\tau} \cdot \begin{bmatrix} N_c \\ -D_c \end{bmatrix} \cdot \underline{p} = d^{\tau}\underline{q}$$

where $\underline{p}$, $\in \mathbb{R}^{m}[d]$ and $\underline{q} \in \mathbb{R}^{1+m}[d]$. Hence, $\underline{y}$ settles to zero in at most finite time $k_f = \tau + \partial(\underline{q}) + 1$. But

$$\partial(\underline{q}) \leq \max_{i:p_i(d)\neq 0} \{k_i + \partial(p_i)\}$$

where k_i are the column degrees of $[N_c^t \ \ -D_c^t]^t$. Then k_f is minimum for every $\underline{p}$, if and only if k_i becomes minimum and therefore any causal FSTS controller with the least possible column degrees is a deadbeat regulator.

□

Taking into account corollary (6.1) and the parametrization of FSTS controllers according to theorem (6.4), the following corollary may be readily established.

Corollary 6.4: Let (P,C) be an FSTS-stable pair in the feedback configuration of figure (6.1) and $\mathcal{F}^{c}_{c,min}(P)$ be the family of least column complexity controllers. Then

a. If P is strictly causal the family $\mathcal{F}^{c}_{c,min}$ coincides with the family of all deadbeat regulators.

b. If P is not strictly causal and $R_{C\min} = [N_c^t \ \ D_c^t]^t_{\min}$ is one least column complexity solution then

$$R_C = \begin{bmatrix} N_C \\ D_C \end{bmatrix} = \begin{bmatrix} N_C \\ D_C \end{bmatrix}_{\min} + \begin{bmatrix} D_p \\ -N_p \end{bmatrix} R \quad \in \mathbb{R}^{(m+1)\times l}[d]$$

defines the family of all deadbeat regulators, where R is any polynomial matrix such that R_C has the least possible column degrees and $\det D_c(0) \neq 0$.

□

It is clear from corollary (6.4) that the family $\mathcal{F}^{c}_{c,min}(P)$ of the least column complexity solutions plays an important role in the characterization of the entire family of deadbeat regulators. In the sequel we give a two-variant algorithm for the computation of the family $\mathcal{F}^{c}_{c,min}$. It is based mainly on theorems (6.4) and (6.5) and the interim results that we are considering next and it allows for an elegant parametrization of the entire family of the deadbeat regulators.

Proposition 6.2 *(Toeplitz Kernel Structure)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{1xm}[d]$ be a left coprime polynomial MFD of the plant transfer function and $L_P = [\tilde{N}_p \ \ \tilde{D}_p]$ be a left composite matrix of P with scalar degree $\partial_s(L_P) = \nu$. If $T^{1}_{P,k} \in \mathbb{R}^{(\nu+k)1x(m+1)k}$ is the dual to the Toeplitz matrix $T^{r}_{P,k}$ of theorem (6.2) and W_k is a basis of the right null space $\mathcal{N}_r\{T^{1}_{P,k}\}$ of $T^{1}_{P,k}$ then W_k may be expressed as

$$W_k = \left[\begin{array}{c:c} W_{k-1} & \tilde{W}_k \\ O & \end{array}\right] \tag{6.24}$$

$$\text{or} \quad W_k = \begin{bmatrix} W_1 & W_{12} & \cdots & W_{1k} \\ & W_{22} & & \vdots \\ & 0 & \ddots & \\ & & & W_{kk} \end{bmatrix} \tag{6.25}$$

<u>*Proof*</u>. L_P can be written as

$$L_P = L_{P0} + L_{P1}d + \cdots + L_{P\nu}d^{\nu}, \ L_{Pi} \in \mathbb{R}^{1x(m+1)}$$

Then

$$T^{1}_{P,k} = \left[\begin{array}{cc} \boxed{T^{1}_{P,k-1}} & \boxed{O_1} \\ \boxed{O_2} & \boxed{T^{1}_{P,1}} \end{array}\right]$$

where O_1, O_2 are zero matrices with dimensions $\ell\times(m+\ell)(k-1)$ and $\ell(k-1)\times(m+\ell)$ respectively and

$$T^1_{P,1} = \begin{bmatrix} L_{P0} \\ L_{P1} \\ \vdots \\ L_{P\nu} \end{bmatrix} \in \mathbb{R}^{(\nu+1)l\times(m+1)}$$

If W_{k-1} is a basis of the right null space $\mathcal{N}_r\{T^1_{P,k-1}\}$, then it is clear from the structure of $T^1_{P,k-1}$ that

$$\begin{bmatrix} W_{k-1} \\ O \end{bmatrix}$$

forms part of a basis of $\mathcal{N}_r\{T^1_{P,k}\}$ and therefore W_k is of the form of (6.24) and (6.25) follows recursively.

□

We are able now to give another parametrization, apart from the one given by theorem (6.4), for the minimum column complexity solutions of the Diophantine equation (6.18).

Corollary 6.5: Let $\bar{R}_{C\min} = [\underline{\bar{R}}_{C1,\min},\ldots,\underline{\bar{R}}_{Cl,\min}]$ be a real matrix of appropriate dimensions representing one particular least column complexity solution obtained by equations (6.21) according to theorem (6.5). If $k_{\min}$ is the maximum of the column degrees of the least complexity solution and $W_{k\min+1}$ is a basis of $\mathcal{N}_r\{T^1_{P,k\min+1}\}$ of the form of (6.25), then the family of all least column complexity solutions is given by

$$R_{C\min} = \bar{R}_{C\min} + W_{k_{\min}+1}\cdot\begin{bmatrix} \underline{r}_1 & \underline{r}_2 & \cdots & \underline{r}_l \\ \underline{O}_1 & \underline{O}_2 & \cdots & \underline{O}_l \end{bmatrix} \qquad (6.26)$$

where $\underline{r}_1,\ldots,\underline{r}_l$ are **real** and $\underline{O}_1,\ldots,\underline{O}_l$ are **zero** column vectors with appropriate dimensions.

Proof. Let, according to theorem (6.5), $k_{i,\min}$ be the column degree of the least column degree solution given by the ith equation (6.21) and

$$\overline{\underline{r}}_{Ci,min} = \begin{bmatrix} \overline{\underline{r}}_{Ci,0} \\ \vdots \\ \overline{\underline{r}}_{Ci,k_{i,min}} \end{bmatrix} \in \mathbb{R}^{(k_{i,min}+1)(l+m)\times 1}$$

be a particular solution corresponding to $T^{l}_{P,k_{i,min}+1}$, i.e.

$$\overline{\underline{r}}_{Ci,min}(d) = \overline{\underline{r}}_{Ci,0} + \cdots + \overline{\underline{r}}_{Ci,k_{i,min}} d^{k_{i,min}}$$

Then the family of the least degree solutions of the ith equation (6.21) is given by

$$\underline{r}_{Ci,min} = \overline{\underline{r}}_{Ci,min} + W_{k_{i,min}+1} \cdot \underline{r}_{i} \tag{6.27}$$

where

$W_{k_{i,min}+1}$ is a basis of $\mathcal{N}_{r}\{T^{l}_{P,k_{i,min}+1}\}$ and

$\underline{r}_{i} \in \mathbb{R}^{\rho^{r}_{i}\times 1}$ with $\rho^{r}_{i} = \rho(W_{k_{i,min}+1}) = \dim\mathcal{N}_{r}\{T^{l}_{P,k_{i,min}+1}\}$

Taking into account the structure of the right null space of $T^{l}_{P,k}$ as given by (6.25), relations (6.27) become

$$\underbrace{\begin{bmatrix} \underline{r}_{Ci,min} \\ \underline{O}_{i} \end{bmatrix}}_{:=\underline{R}_{Ci,min}} = \underbrace{\begin{bmatrix} \overline{\underline{r}}_{Ci,min} \\ \underline{O}_{i} \end{bmatrix}}_{:=\overline{\underline{R}}_{Ci,min}} + W_{k_{min}+1} \cdot \begin{bmatrix} \underline{r}_{i} \\ \underline{O}_{i} \end{bmatrix} \tag{6.28}$$

where $\underline{O}_{i}$ is a zero column vector of dimensions $(k_{min} - k_{i,min})(\ell + m)$ and therefore (6.26) holds true.

□

An algorithm for the computation of the least column complexity family $\mathcal{F}^{c}_{c,min}(P)$ based on theorem (6.4) and corollary (6.5) is presented next.

Algorithm for the computation of $\mathcal{F}^{c}_{c,\min}(P)$

Step 1: Given the plant transfer function $P \in \mathbb{R}^{l\times m}(d)$ find a left coprime polynomial MFD $(\tilde{D}_p, \tilde{N}_p)$ such that $L_P = [\tilde{N}_p \;\; \tilde{D}_p]$ is column reduced. Determine the left and right minimal indices ν_i, $i = 1,\dots,\ell$, $\nu = \max\{\nu_i\}$, μ_j, $j = 1,\dots,m$, $\mu = \max\{\mu_j\}$.

Step 2: For $i = 1,\dots,\ell$

a. Find the first $k_i \le \mu - 1$ such that

$$\rho\left(\begin{bmatrix} T^{1}_{P,k_i+1} & \begin{bmatrix} \underline{e}^{i} \\ \underline{O} \end{bmatrix} \end{bmatrix}\right) = (m+\ell)(k_i+1) - \sum_{j:\mu_j<(k_i+1)} (k_i+1-\mu_j)$$

Denote this k_i by $k_{i,\min}$ and also

$$\rho^{r}_{i} = (m+\ell)(k_{i,\min}+1) - \rho(T^{1}_{P,k_{i,\min}+1})$$

b. Find one particular solution $\bar{\underline{r}}_{Ci}$ of the equation

$$T^{1}_{P,k_{i,\min}+1}\cdot\begin{bmatrix} \underline{r}_{Ci,0} \\ \vdots \\ \underline{r}_{Ci,k_{i,\min}} \end{bmatrix} = \begin{bmatrix} \underline{e}^{i} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$:= \underline{r}_{Ci}$$

Denote by $k_{\min}$ the $\max_i\{k_{i,\min}\}$.

<u>*OPTION A*</u>

Step 3: Find a basis $W_{k_{\min}+1}$ of $\mathcal{N}_r\{T^{1}_{P,k_{\min}+1}\}$ of the form of (6.25). The family of least column complexity solutions is determined by the real matrix

$$R_{C\min} = \begin{bmatrix} \underline{R}_{C,0} \\ \vdots \\ \underline{R}_{C,k_{\min}} \end{bmatrix} = \bar{R}_{C\min} + W_{k_{\min}+1}\cdot\begin{bmatrix} \underline{r}_1 & \underline{r}_2 & \cdots & \underline{r}_l \\ \underline{O}_1 & \underline{O}_2 & \cdots & \underline{O}_l \end{bmatrix}$$

where

$$\bar{R}_{C\min} = \begin{bmatrix} \bar{\underline{r}}_{C1} & \bar{\underline{r}}_{C2} & \cdots & \bar{\underline{r}}_{Cl} \\ \underline{O}_1 & \underline{O}_2 & \cdots & \underline{O}_l \end{bmatrix}$$

and $\underline{r}_i \in \mathbb{R}^{\rho_i^r \times 1}$, $i = 1,\ldots,\ell$, with free entries.

Step 4: The family $\mathcal{F}^c_{c,min}(P)$ of all least column complexity FSTS controllers is given by

$$\mathcal{F}^c_{c,min}(P) = \{(N_c,D_c) \ : \ [N_c^t \ D_c^t]^t = R_{C,0} + \cdots + R_{C,k_{min}} d^{k_{min}}\}$$

Option B

Step 3: If $\bar{\underline{r}}_{Ci}(d) \in \mathbb{R}^{(l+m)\times 1}[d]$ is the vector

$$\bar{\underline{r}}_{Ci}(d) = \bar{\underline{r}}_{Ci,0} + \bar{\underline{r}}_{Ci,1}d + \cdots + \bar{\underline{r}}_{Ci,k_{i,min}} \cdot d^{k_{i,min}}$$

the family of all FSTS controllers is given by the matrix

$$R_C = \begin{bmatrix} N_c \\ D_c \end{bmatrix} = R_{Cmin} + \begin{bmatrix} D_p \\ -N_p \end{bmatrix} \cdot R \quad \in \mathbb{R}^{(m+l)\times l}[d]$$

$$\text{or} \quad R_C = R_{Cmin} + Q = R_{Cmin} + [\underline{q}_1 \ \cdots \ \underline{q}_l]$$

where $R_{Cmin} = [\bar{\underline{r}}_{C1}(d) \ \cdots \ \bar{\underline{r}}_{Cl}(d)]$.

Step 4: The family $\mathcal{F}^c_{c,min}(P)$ of all least column complexity FSTS controllers is given by

$$\mathcal{F}^c_{c,min}(P) = \{ \ (N_c,D_c) \ : \ \partial(\underline{q}_i(d)) \le \partial(\bar{\underline{r}}_{Ci}(d)) \ \}$$

Remark 6.6: Fast and stable algorithms for the solution of Toeplitz type equations that exploit the structure of the block Toeplitz matrices are available in the literature, see e.g. [Kun., 3], [Hei., 1] and references therein. Their existence makes the algorithm for the computation of the family $\mathcal{F}^c_{c,min}(P)$ computationally efficient and attractive. As it has already been mentioned this algorithm is a two-variant algorithm.

'Option A' allows for the parametrization of the family $\mathcal{F}^c_{c,min}(P)$ through a set of real vectors and in the case of

strictly causal plants provides the parametrization of the entire family of deadbeat regulators.

'Option B' provides the parametrization of the family of all FSTS controllers and allows for the parametrization of deadbeat regulators in the case of non strictly causal plants through the polynomial matrix *R*. Fixed column degree solutions can also be obtained by applying the conditions of theorem (6.4) on the degrees of the polynomial entries of the matrix *R*. Additional care, in this case, has to be taken for causality as well.

□

6.4.2 The case of vector plants

We consider in this section the case of vector plants and in particular those with many inputs and a single output (MISO) shown in the unity feedback configuration of figure (6.2).

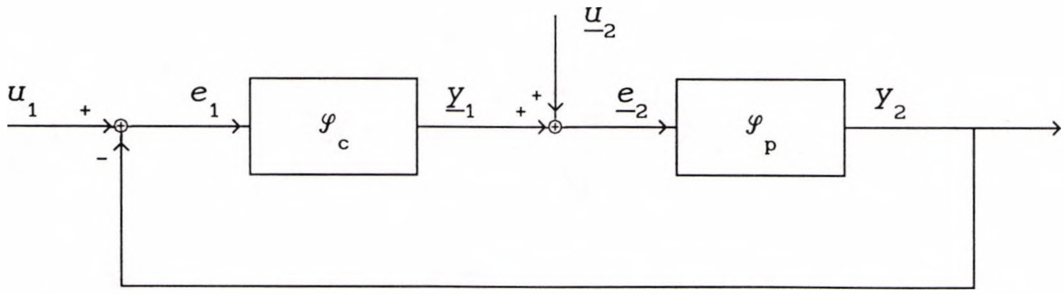

Figure (6.2): The MISO unity feedback configuration

If $\underline{p}^t$ and $\underline{c}$ are the plant and controller transfer functions, they can be written as follows using coprime polynomial MFDs.

$$\underline{p}^t = \tilde{d}_p^{-1}\underline{\tilde{n}}_p^t = \underline{n}_p^t D_p^{-1} \in \mathbb{R}^{1\mathrm{x}m}(d) \tag{6.29}$$

$$\underline{c} = \tilde{D}_c^{-1}\underline{\tilde{n}}_c = \underline{n}_c d_c^{-1} \in \mathbb{R}^{m\mathrm{x}1}(d) \tag{6.30}$$

where all the vectors are assumed column vectors (i.e. $\underline{p}^t$ is a row vector). Then the McMillan degrees of the plant and controller are

$$\delta_{\mathcal{M}}(\underline{p}^t) = \partial([\underline{\tilde{n}}_p^t \ \tilde{d}_p]) = \partial_s([\underline{\tilde{n}}_p^t \ \tilde{d}_p]) \tag{6.31}$$

$$\delta_{\mathcal{M}}(\underline{c}) = \partial([\underline{n}_c^t \ d_c]^t) = \partial_s([\underline{n}_c^t \ d_c]^t) \tag{6.32}$$

i.e., the McMillan degree reduces to the scalar column degree and therefore the results of the previous section can be used for the exact solution of the minimal FSTS design problem and the parametrization of the family $\mathcal{F}(\underline{p}^t)$ of all causal FSTS controllers according to their McMillan degree.

According to theorem (6.1), the family $\mathcal{F}(\underline{p}^t)$ is given by

$$\mathcal{F}(\underline{p}^t) = \{(\underline{n}_c, d_c) : \underline{n}_c = \underline{x} + D_p\underline{r},\ d_c = y - \underline{n}_p^t\underline{r},$$
$$\underline{r} \in \mathbb{R}^{m\mathrm{x}1}[d]) \text{ and } y(0)-\underline{n}_p^t(0)\underline{r}(0) \neq 0 \text{ if } \underline{n}_p^t(0) \neq 0\} \tag{6.33}$$

where $(\underline{x}, y)$ is a particular solution of the Diophantine equation

$$\underline{\tilde{n}}_p^t\underline{n}_c + \tilde{d}_p d_c = 1 \tag{6.34}$$

If $\partial_s([\underline{\tilde{n}}_p^t \ \tilde{d}_p]) = \nu$ and $\partial_s([\underline{n}_c^t \ d_c]^t) = k$, $\underline{\tilde{n}}_p^t$, $\tilde{d}_p$, $\underline{n}_c$, d_c can be expressed as

$$\underline{\tilde{n}}_p^t = \underline{\tilde{n}}_{p0}^t + \underline{\tilde{n}}_{p1}^t d + \cdots + \underline{\tilde{n}}_{p\nu}^t d^\nu$$

$$\tilde{d}_p = \tilde{d}_{p0} + \tilde{d}_{p1} d + \cdots + \tilde{d}_{p\nu} d^\nu$$

$$\underline{n}_c = \underline{n}_{c0} + \underline{n}_{c1} d + \cdots + \underline{n}_{ck} d^k$$

$$d_c = d_{c0} + d_{c1} d + \cdots + d_{ck} d^k$$

and equation (6.34) is equivalent to the following Toeplitz type system of equations over $\mathbb{R}$.

$$
T^{1}_{P,k+1} \cdot \begin{bmatrix} \underline{n}_{c0} \\ d_{c0} \\ \vdots \\ \underline{n}_{ck} \\ d_{ck} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{6.35}
$$

$$
:= \underline{r}_{ck}
$$

where

$$
T^{1}_{P,k+1} = \begin{bmatrix} \underline{\tilde{n}}^{t}_{p0} \ \tilde{d}_{p0} & & & 0 \\ \vdots & \underline{\tilde{n}}^{t}_{p0} \ \tilde{d}_{p0} & & \\ \underline{\tilde{n}}^{t}_{p\nu} \ \tilde{d}_{p\nu} & \vdots & \ddots & \underline{\tilde{n}}^{t}_{p0} \ \tilde{d}_{p0} \\ & \underline{\tilde{n}}^{t}_{p\nu} \ \tilde{d}_{p\nu} & & \vdots \\ & 0 & \ddots & \underline{\tilde{n}}^{t}_{p\nu} \ \tilde{d}_{p\nu} \end{bmatrix} \tag{6.36}
$$

$\longleftarrow$ k+1 blocks $\longrightarrow$

As in the SISO case, we consider now the following problems (see also remark (5.4) and definition (5.2)).

a. *Minimal Design Problem (MDP)*. Define the minimal McMillan degree $\delta_m(\underline{p}^t)$ of all causal FSTS controllers.

b. *Parametrization Problem (I) (PP(I))*. Define the McMillan index set $I_{\mathcal{M}}(\underline{p}^t)$ of $\mathcal{F}(\underline{p}^t)$.

c. *Parametrization Problem (II) (PP(II))*. $\forall\ i \in I_{\mathcal{M}}(\underline{p}^t)$ define a parametric expression of the $\mathcal{R}_{\mathcal{M}}$ equivalence class $\mathcal{C}_{\mathcal{M}}(\underline{c}_i)$, where $\delta_{\mathcal{M}}(\underline{c}_i) = i$.

Theorem 6.7 *(MDP)*: Let $(\underline{p}^t, \underline{c})$ be an FST-stable pair with the plant and controller transfer functions expressed as in (6.29) and (6.30) and let W_{k+1} be a basis of the right null space $\mathcal{N}_r\{T^1_{P,k+1}\}$ of $T^1_{P,k+1}$ and $\underline{\bar{r}}_{ck}$ a particular solution of equation (6.35). Then, the McMillan characteristic $\delta_m(\underline{p}^t)$ of the family $\mathcal{F}(\underline{p}^t)$, i.e. the minimum McMillan degree of the causal FSTS controllers can be derived as follows.

a. $\delta_m(\underline{p}^t)$ is the first k such that equations (6.35) have a solution, if the plant is strictly causal, otherwise

b. if $\bar{\underline{d}}_{c0}$ is the entry of $\bar{\underline{r}}_{c0}$ that corresponds to the constant term of the controller denominator and $\underline{w}_2^t$ is the second row of W_{k+1}, then $\delta_m(\underline{p}^t)$ is the first k such that either of the following conditions hold true

1. $\bar{\underline{d}}_{c0} \neq 0$, or

2. $\bar{\underline{d}}_{c0} = 0$ and $\underline{w}_2 \neq 0$.

Proof.

a. This part follows straight from theorem (6.5) and corollary (6.1).

b. If $\bar{\underline{r}}_{ck}$ is one particular solution of equation (6.35) for some k, then the family of solutions for this particular k is given by

$$\begin{bmatrix} \underline{n}_{c0} \\ d_{c0} \\ \vdots \\ \underline{n}_{ck} \\ d_{ck} \end{bmatrix} = \begin{bmatrix} \bar{\underline{n}}_{c0} \\ \bar{d}_{c0} \\ \vdots \\ \bar{\underline{n}}_{ck} \\ \bar{d}_{ck} \end{bmatrix} + W_{k+1} \cdot \underline{r} \qquad (6.37)$$

where $\underline{r}$ is a real column vector with free entries. For at least one causal solution to exist, either $\bar{\underline{d}}_{c0} \neq 0$, or if $\bar{\underline{d}}_{c0} = 0$, then $\underline{w}_2 \neq \underline{0}$ where $\underline{w}_2^t$ is the second row of W_{k+1}. Since the McMillan degree is the column degree k of the solution vector, the minimum McMillan degree is given by the first set of equations (6.35) that satisfy the aforementioned causality conditions.

□

Remark 6.7: Equation (6.37) for $k = \delta_m(\underline{p}^t)$, denoted for short by δ_m, and $\underline{r}$ such that $d_{c0} = \bar{d}_{c0} + \underline{w}_2^t \cdot \underline{r} \neq 0$, provides the parametrization of all least column degree causal FSTS controllers and therefore, according to theorem (6.6), the parametrization of the family of all deadbeat regulators.

Needless to say, that this family coincides with the family of minimum McMillan degree causal FSTS controllers. From equation (6.23) which describes the rank of the block Toeplitz matrices, we conclude that

$$\rho(W_{\delta_m+1}) = \dim\mathcal{N}^r\{T^1_{P,\delta_m+1}\} = \sum_{i:\mu_i<(\delta_m+1)} (\delta_m+1-\mu_i) \tag{6.38}$$

where μ_i $i = 1,\dots,m$ are the right minimal indices of $\underline{p}^t$. Hence, if

$$\delta_m \le \min_i\{\mu_i\} - 1 \tag{6.39}$$

the minimum McMillan degree FSTS controller is **unique**. This is clearly a general result and covers not only the MISO but the MIMO case as well.

□

Using theorem (6.7) we can define the McMillan index set $I_{\mathcal{M}}(\underline{p}^t)$ of the family $\mathcal{F}(\underline{p}^t)$, i.e. the set of McMillan degrees of the controllers that belong to $\mathcal{F}(\underline{p}^t)$.

Theorem 6.8 *(PP(I))*: Let δ_m be the McMillan characteristic of $\mathcal{F}(\underline{p}^t)$ and μ_{min} be the minimum of the right minimal indices of $\underline{p}^t$. Then

a. if $\delta_m \le \mu_{min} - 1$

$$I_{\mathcal{M}}(\underline{p}^t) = \{\delta_m;\ \mu_{min} + k,\ k = 0,1,\dots\}$$

b. if $\delta_m > \mu_{min} - 1$

$$I_{\mathcal{M}}(\underline{p}^t) = \{\delta_m + k,\ k = 0,1,\dots\}$$

Proof.

a. Let $\underline{p}^t = \underline{n}_p^t D_p^{-1} \in \mathbb{R}^{1\times m}(d)$ with $[\underline{n}_p\ \ D_p^t]^t$ $\mathbb{R}[d]$-coprime and column reduced and let $\underline{c}_m = \underline{n}_{cm} d_{cm}^{-1}$ be one FSTS controller with McMillan degree $\delta_{\mathcal{M}}(\underline{c}_m) = \delta_m$. The family $\mathcal{F}(\underline{p}^t)$ can be parametrized as follows

$$\begin{bmatrix} \underline{n}_c \\ d_c \end{bmatrix} = \begin{bmatrix} \underline{n}_{cm} \\ d_{cm} \end{bmatrix} + \begin{bmatrix} D_p \\ -\underline{n}_p^t \end{bmatrix}\cdot\underline{r} = \begin{bmatrix} \underline{n}_{cm} \\ d_{cm} \end{bmatrix} + \underline{q} \tag{6.40}$$

where $\underline{r} = [r_1 \ \ldots \ r_2]^t \in \mathbb{R}^{m\times 1}[d]$. Then, if $\mu_i \ i = 1,\ldots,m$ are the right minimal indices of $\underline{p}^t$, the degree of $\underline{q}(d)$ in (6.40) is given by

$$\partial(\underline{q}(d)) = \max_{i:r_i(d)\neq 0} \{\partial(r_i(d)) + \mu_i\} \geq \mu_{min} \quad \forall \ \underline{r} \in \mathbb{R}^{m\times 1}[d] - \underline{0}$$

Therefore, according to (6.40), if $\underline{c}$ is any FSTS controller with $\delta_{\mathcal{M}}(\underline{c}) \geq \delta_m$ and $\delta_m \leq \mu_{min} - 1$, $\delta_{\mathcal{M}}(\underline{c})$ takes any value from the set

$$\{\delta_m; \ \mu_{min} + k, \ k = 0,1,\ldots\}$$

In addition, $\underline{c}$ must belong to the family $\mathcal{F}(\underline{p}^t)$ of causal FSTS controllers, i.e.

$$d_c(0) = d_{cm}(0) - \underline{n}_p^t(0)\cdot\underline{r}(0) \neq 0 \tag{6.41}$$

This is always possible for the following reason. If the entries of $\underline{n}_p^t(0)$ that correspond to the nonzero entries of $\underline{r}(0)$ are zero, then $d_c(0) = d_{cm}(0) \neq 0$ because $\underline{c}_m$ is causal. Otherwise, the nonzero entries of $\underline{r}(0)$ are completely free and therefore there always exists an $\underline{r}(0)$ such that the causality conditions (6.41) are valid. This proves part (a).

b. If $\delta_m \geq \mu_m$, by the definition of δ_m, there does not exist a $\underline{c} \in \mathcal{F}(\underline{p}^t)$ such that $\delta_{\mathcal{M}}(\underline{c}) < \delta_m$ and therefore $\delta_{\mathcal{M}}(\underline{c})$ takes values from the set

$$\{\delta_m + k, \ k = 0,1,\ldots\}$$

Using similar arguments for causality as in part (a) we can prove that the aforementioned set coincides with $I_{\mathcal{M}}(\underline{p}^t)$ and this concludes the proof of theorem (6.8).

□

We conclude this section by giving the full parametrization of the family $\mathcal{F}(\underline{p}^t)$ according to the McMillan degree of the FSTS controllers. The proof is based on similar arguments used in theorems (6.7) and (6.8) and it is omitted.

Theorem 6.9 *(PP(II))*: Let $(\underline{p}^t,\underline{c})$ be an FST-stable pair and $\mathcal{C}_{\mathcal{M}}(\underline{c}_{\delta_j})$ be the δ_j-subfamily of $\mathcal{F}(\underline{p}^t)$ as in definition (5.2), i.e.

$$\forall\ \mathscr{C}_{\mathcal{M}}(\underline{c}_{\delta_j}) \Leftrightarrow \delta_{\mathcal{M}}(\underline{c}_{\delta_j}) = \delta_j$$

Then,

a. the δ_m-subfamily of $\mathscr{F}(\underline{p}^t)$ can be parametrized as follows.

$$\begin{bmatrix} \underline{n}_{c0} \\ d_{c0} \\ \vdots \\ \underline{n}_{c\delta_m} \\ d_{c\delta_m} \end{bmatrix} = \begin{bmatrix} \bar{\underline{n}}_{c0} \\ \bar{d}_{c0} \\ \vdots \\ \bar{\underline{n}}_{c\delta_m} \\ \bar{d}_{c\delta_m} \end{bmatrix} + W_{\delta_m+1}\cdot\underline{r}$$

$$:= \bar{\underline{r}}_{c\delta_m}$$

where $\bar{\underline{r}}_{c\delta_m}$ is a particular solution of equation (6.35) with $k = \delta_m$, W_{δ_m+1} is a basis of $\mathcal{N}_r\{T^1_{P,\delta_m+1}\}$ and $\underline{r}$ is any real column vector such that

$$d_{c0} = \bar{d}_{c0} + \underline{w}_2^t\cdot\underline{r} \neq 0$$

with $\underline{w}_2^t$ being the second row of W_{δ_m+1}.

b. the δ_j-subfamily, where

$$\delta_j = \delta_m + j \quad \text{or} \quad \delta_j = \mu_{min} - 1 + j, \quad j = 1,2,\ldots$$

consists of the following controllers.

$$\begin{bmatrix} \underline{n}_c \\ d_c \end{bmatrix} = \begin{bmatrix} \underline{n}_{cm} \\ d_{cm} \end{bmatrix} + \begin{bmatrix} D_p \\ -\underline{n}_p^t \end{bmatrix}\begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$$

where $(\underline{n}_{cm}, d_{cm})$ is a δ_m-controller, r_i, $i = 1,\ldots,m$ are any polynomials with the following degree properties

$$\partial(r_i(d)) \in -\infty \cup \{\ 0,\ldots,\delta_j-\mu_{\delta_j}\ \}$$

with at least one r_i taking its maximal degree value, and $\underline{r}(0)$ such that

$$d_c(0) = d_{cm}(0) - \underline{n}_p^t(0)\cdot\underline{r}(0) \neq 0$$

and $[D_p^t \ \underline{n}_p]^t$ is column reduced with column degrees

$$\mu_1, \mu_2, \ldots, \mu_m \text{ and } \mu_{min} = \min_i\{\mu_i\}$$

□

Remark 6.8: According to theorem (6.2) there is always a solution, not necessarily causal, of the Diophantine equation (6.34) with McMillan degree $k = \mu - 1$, where μ is the maximal of the right minimal indices of $\underline{p}^t$. Using this solution as a particular solution in the parametrization of the family $\mathcal{F}(\underline{p}^t)$, then any controller with McMillan degree greater than or equal to μ can be parametrized by a polynomial column vector $\underline{r}$ with all its entries comprising nonzero constant terms. This means that the causality conditions (6.41) can always be met for $k \geq \mu$, therefore

$$\delta_m \leq \mu$$

□

6.5 Strong FSTS

We recall that the problem of strong FSTS is defined as the stabilization of the plant P in FST sense by a stable controller, i.e. a controller with poles outside the closed unit disc $\mathbb{D}$. It turns out that the same ***parity interlacing property*** as in the case of usual stabilization must be satisfied by the plant. The proof is similar to the SISO strong FSTS case (see also [Vid., 1]). First we define the notion of blocking zeros of a transfer function that it will be used for the establishment of the conditions for strong FSTS.

Definition 6.3: The **blocking zeros** of a rational matrix $P(d)$ are all $z_i \in \mathbb{C}$ such that $P(z_i) = 0$.

□

Remark 6.9: If N_p, $\tilde{N}_p$ are the mumerator polynomial matrices of any right or left coprime polynomial MFD of a rational matrix P and n_p is the least invariant polynomial of either N_p or $\tilde{N}_p$, then the blocking zeros of P are the zeros of n_p.

□

The conditions for strong FSTS are given then by the following theorem.

Theorem 6.10 *(Strong FSTS)*: A plant $P = N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p$ is **strongly stabilizable** in FST sense, if and only if $\det D(d)$ or $\det\tilde{D}(d)$ has the same sign at all real blocking zeros σ_i of P inside the closed unit disc $\mathbb{D}$.

Proof. If X, Y satisfy the matrix polynomial Diophantine equation

$$\tilde{N}_pX + \tilde{D}_pY = I \tag{6.42}$$

then for every FSTS controller C, $D_c = Y - N_pR$. Therefore, according to remark (6.2)

$$|D_c(d)| = |Y(d) - N_p(d)R(d)| = y(d) - r(d)n_p(d) \tag{6.43}$$

where $y(d) = |Y(d)|$. For C to be stable, $|D_c(d)|$ has to be a polynomial unit in the disc algebra A_s. According to equation (6.43), $|D_c(d)|$ interpolates $y(d)$ and its derivatives at the zeros of $n_p(d)$ inside the closed unit disc $\mathbb{D}$, i.e. due to remark (6.9), the foregoing statement is true for the blocking zeros of P inside $\mathbb{D}[0,1]$. Therefore, according to theorem (5.4), $y(d)$ must have the same sign at the real blocking zeros σ_i of P inside the closed unit disc $\mathbb{D}$. Also, equation (6.42) becomes at the blocking zeros z_i of the plant P

$$\tilde{D}_p(z_i)Y(z_i) = I \quad \text{so,} \quad |\tilde{D}_p(z_i)|y(z_i) = 1$$

Therefore, $y(d)$ and $|\tilde{D}_p(d)|$ have the same sign at the blocking zeros of P, and this concludes the proof.

□

Theorem (6.10) can be restated as the following corollary which expresses the so called *parity interlacing property.*

Corollary 6.6 *(Parity Interlacing Property)*: There always exists a stable FSTS controller, if and only if the number of poles of P inside any interval of successive real blocking zeros of P inside the closed unit disc $\mathbb{D}$, is even

□

6.6 FST Tracking and Disturbance Rejection

One of the most fundamental performance requirements of a control system is that of tracking and/or rejection of a family of signals applied at its inputs. In the case of FST tracking and/or disturbance rejection, the required performance must be achieved in finite time. In this sense, most of the controllers for deadbeat tracking/disturbance rejection can be considered as time-optimum FST tracking/disturbance rejection controllers. The following two theorems give the conditions for FST tracking and disturbance rejection.

Theorem 6.11 *(FST Tracking)*: Let (P,C) be an FST-stable pair in the unity feedback system of figure (6.1) and $\underline{u}_1$ be expressed by a left $\mathbb{R}[d]$-coprime MFD as $\underline{u}_1 = \tilde{D}_r^{-1}\tilde{\underline{n}}_r$. Then $\underline{y}_2$ tracks the reference signal $\underline{u}_1$ in FST sense, if and only if either of the following two equivalent conditions are satisfied.

1. $\tilde{D}_r$ is a right divisor of $D_c\tilde{D}_p$, i.e.

$$\exists\ Q \in M(\mathbb{R}[d]) \ :\ D_c\tilde{D}_p = Q\tilde{D}_r$$

2. $\exists\ Q,\ R \in M(\mathbb{R}[d]) \ :\ Q\tilde{D}_r + N_pR\tilde{D}_p = Y\tilde{D}_p$

where all the matrices involved apart from Q are as in theorem (6.1).

Proof.

1. From equation (6.4a), $\underline{e}_1 = D_c\tilde{D}_p\underline{u}_1 = D_c\tilde{D}_p\tilde{D}_r^{-1}\tilde{\underline{n}}_r$. For FST tracking $\underline{e}_1$ must be polynomial, i.e. $D_c\tilde{D}_p\tilde{D}_r^{-1}\tilde{\underline{n}}_r \in \mathbb{R}^{m\times 1}[d]$ and because $\tilde{D}_r$, $\tilde{\underline{n}}_r$ are coprime, $D_c\tilde{D}_p\tilde{D}_r^{-1}$ must be polynomial [Vid., 1]. Therefore, $D_c\tilde{D}_p\tilde{D}_r^{-1} = Q \in M(\mathbb{R}[d])$, or

$$D_c\tilde{D}_p = Q\tilde{D}_r \tag{6.44}$$

2. From theorem (6.1), $D_c = Y - N_pR$ which if substituted in equation (6.44) gives

$$Q\tilde{D}_r + N_pR\tilde{D}_p = Y\tilde{D}_p \tag{6.45}$$

□

Theorem 6.12 *(FST Disturbance Rejection)*: Let (P,C) be an FST-stable pair in the unity feedback system of figure (6.1) and $\underline{u}_2$ be expressed by a left $\mathbb{R}[d]$-coprime MFD as $\underline{u}_2 = \tilde{D}_d^{-1}\tilde{\underline{n}}_d$. Then $\underline{u}_2$ is rejected at the output $\underline{y}_2$ in FST sense, if and only if any of the following conditions holds true.

1. $\tilde{D}_d$ is a right divisor of $N_p\tilde{D}_c$, i.e.

$$\exists\ Q \in M(\mathbb{R}[d])\ :\ N_p\tilde{D}_c = Q\tilde{D}_d$$

2. $\exists\ Q,\ S \in M(\mathbb{R}[d])\ :\ Q\tilde{D}_d + N_pS\tilde{N}_p = N_p\tilde{Y}$

3. $\tilde{D}_d$ is a right divisor of $D_c\tilde{N}_p$, i.e.

$$\exists\ Q \in M(\mathbb{R}[d])\ :\ D_c\tilde{N}_p = Q\tilde{D}_d$$

4. $\exists\ Q,\ R \in M(\mathbb{R}[d])\ :\ Q\tilde{D}_d + N_pR\tilde{N}_p = Y\tilde{N}_p$

where all the matrices involved apart from Q are as in theorem (6.1).

Proof. The proof of theorem (6.12) can be carried out in the same manner as that of theorem (6.11). We just note that condition (1) is derived using the expression of $W_{22}(P,C)$ in (6.5a) and (1) ⇔ (2). Also, condition (3) comes from the expression of $W_{22}(P,C)$ in (6.5b) and (3) ⇔ (4).

□

Remark 6.10: The solutions R from conditions (1) in theorem (6.11) and (4) in theorem (6.12) give the parametrization through their right MFDs of all the controllers that achieve

FST tracking and FST disturbance rejection respectively. The solutions Q from condition (2) in theorem (6.12) provide the parametrization through their left MFDs of all the controllers that guarantee disturbance rejection in FST sense. The criteria for FST tracking and disturbance rejection are testable since they are expressed in terms of equations that are linear with respect to the parameters R or S. We elaborate in the following corollary for the FST tracking case. The FST disturbance case can be treated in a similar manner.

□

Corollary 6.7: Let (P,C) be an FST-stable pair in the feedback scheme of figure (6.1), (N_p,D_p), $(\tilde{D}_p,\tilde{N}_p)$, (N_c,D_c), $(\tilde{D}_c,\tilde{N}_c)$ be any right, left coprime polynomial MFDs of P and C respectively and X, $Y \in M(\mathbb{R}[d])$ be any particular solution of the Diophantine equation $\tilde{N}_pX + \tilde{D}_pY = I$. Then the output $\underline{y}_2$ tracks the input $\underline{u}_1$ in FST sense, if and only if

$$\begin{bmatrix} N_p & O \\ O & D_1 \end{bmatrix} \sim \begin{bmatrix} N_p & S_0 \\ O & D_1 \end{bmatrix}$$

where S_0 is a particular solution of

$$Q\tilde{D}_r + S\tilde{D}_p = Y\tilde{D}_p$$

and $[-D_1 \ D_2]$ is a basis of the left null space of $[\tilde{D}_p^t \ \tilde{D}_r^t]^t$.

Proof. According to theorem (6.11) all FST controllers for tracking are parametrized through R by the equation (6.45). Equation (6.45) is equivalent to the following two equations.

$$Q\tilde{D}_r + S\tilde{D}_p = Y\tilde{D}_p \qquad (6.46a)$$

$$N_pR = S \qquad (6.46b)$$

Since $\tilde{D}_p$ appears in the right-hand side of equation (6.46a), equation (6.46a) has always a solution. If S_0 is one particular solution, then the family of solutions S is given according to theorem (2.22) by

$$S = S_0 - TD_1 \qquad (6.47)$$

where D_1 is such that $[-D_1 \;\; D_2]$ is a basis of $\mathcal{N}_1\{[\tilde{D}_p^t \;\; \tilde{D}_r^t]^t\}$ and T is any polynomial matrix of appropriate dimensions. Substituting (6.47) into (6.46b) we get the bilateral matrix Diophantine equation

$$N_pR + TD_1 = S_0$$

which, according to theorem (2.23), has a solution, if and only if

$$\begin{bmatrix} N_p & O \\ O & D_1 \end{bmatrix} \sim \begin{bmatrix} N_p & S_0 \\ O & D_1 \end{bmatrix}$$

□

The testable solvability conditions for FST disturbance rejection are similar to corollary (6.7). Finally we note the following concerning the problem of simultaneous FST tracking and disturbance rejection.

Remark 6.11 *(FST Tracking and Disturbance Rejection)*: If we want to design a controller such that $\underline{y}_2$ tracks in FST sense a family of inputs from $\underline{u}_1$ and rejects in FST sense a family of inputs from $\underline{u}_2$, then we must solve according to theorems (6.11) and (6.12), the following set of equations

$$\begin{aligned} Q_1\tilde{D}_r + N_pR\tilde{D}_p &= Y\tilde{D}_p \\ Q_2\tilde{D}_d + N_pR\tilde{N}_p &= Y\tilde{N}_p \end{aligned} \qquad (6.48)$$

for a common R. The FSTS controllers that guarantee both FST tracking and disturbance rejection will be $C = N_cD_c^{-1}$ such that $N_c = X + D_pR$, $D_c = Y - N_pR$ where X, Y are a particular solution of the equation $\tilde{N}_pX + \tilde{D}_pY = I$. Since equations (6.48) are linear with respect to Q_1, Q_2 and R, testable solvability conditions may be found but the whole solution procedure may become quite tedious. In addition, the performance of the controller depends only on one parameter R for both tracking and disturbance rejection which may result to a very restricted family of controllers. The answer to this problem is the use of a two-parameter feedback scheme. This design strategy is considered in chapter (8).

□

6.7 FSTS Controllers with Partially Assigned Dynamics

In the previous section we were able to parametrize the entire family of FSTS controllers that guarantee certain performance characteristics, namely FST tracking and/or disturbance rejection. Although the criteria for such performance reduce to the internal model principle in the SISO case as can be readily realized from condition (1) in theorem (6.11) and conditions (1) and (3) in theorem (6.12), the analogy is not as clear cut in the MIMO case in general.

The FSTS controllers provided by theorems (6.11) and (6.12) allow for a richer family. Indeed, condition (1) in theorem (6.11) could express the internal model principle only if

$$D_c = D'_c\tilde{D}_r \tag{6.49}$$

and $\tilde{D}_r$, $\tilde{D}_p$ commute, which may end up to be a very restrictive condition. The situation is considerably simpler in the FST disturbance rejection case where the relationship

$$\tilde{D}_c = \tilde{D}'_c\tilde{D}_d \tag{6.50}$$

satisfies condition (1) in theorem (6.12) and the main additional requirement is clearly

$$\tilde{N}_c N_p + \tilde{D}'_c\tilde{D}_d D_p = I$$

i.e. $(N_p, \tilde{D}_d D_p)$ must be right $\mathbb{R}[d]$-coprime. If the design emanating from either equation (6.49) or (6.50) is adopted, then the resulting controllers are only part of the family that can otherwise be obtained from theorems (6.11) or (6.12).

Equations (6.49) and (6.50) express no more than the ***internal model principle*** in the MIMO case and if they can be adopted, they result in an easier computation of FSTS controllers for tracking and disturbance rejection. Both equations can be regarded in more general terms as a partial assignment of controller dynamics; treating them as such, we have then the following result.

Theorem 6.13: Let (P,C) be an FST-stable pair in the feedback configuration of figure (6.1) and (N_p,D_p), $(\tilde{D}_p,\tilde{N}_p)$, (N_c,D_c), $(\tilde{D}_c,\tilde{N}_c)$ be any right, left coprime polynomial MFDs of P and C respectively. The controller C may have partially assigned dynamics, i.e.

$$\tilde{D}_c = \tilde{D}'_c\tilde{D}_{ca} \;, \quad \tilde{D}_{ca} \in M(\mathbb{R}[d]) \text{ given} \tag{6.51}$$

if and only if $(N_p,\tilde{D}_{ca}D_p)$ are right $\mathbb{R}[d]$-coprime and $|\tilde{D}_{ca}(0)| \neq 0$ if $n_p(0) \neq 0$, where n_p is the least invariant polynomial of N_p. Moreover, the family of all causal FSTS controllers with partially assigned dynamics will be denoted by $\mathcal{F}_{pad}(P)$ and is parametrized as follows.

$$\mathcal{F}_{pad}(P) = \{(\tilde{D}_c,\tilde{N}_c) \;:\; \tilde{D}_c = \tilde{X} + S\tilde{D}'_p,\; \tilde{D}_c = (\tilde{Y} - S\tilde{N}'_p)\tilde{D}_{ca},$$
$$\forall\; S \in M(\mathbb{R}[d]) \;:\; |\tilde{Y}(0) - S(0)\tilde{N}'_p(0)| \text{ if } n_p(0) \neq 0\} \tag{6.52}$$

where $(\tilde{X},\tilde{Y})$ is a particular solution for $(\tilde{N}_c,\tilde{D}'_c)$ of the Diophantine equation

$$\tilde{N}_cN_p + \tilde{D}'_c\tilde{D}_{ca}D_p = I \tag{6.53}$$

and $(\tilde{D}'_p,\tilde{N}'_p)$ is a left $\mathbb{R}[d]$-coprime pair such that

$$N_p(\tilde{D}_{ca}D_p)^{-1} = (\tilde{D}'_p)^{-1}\tilde{N}'_p \tag{6.54}$$

Proof. Clearly, every FSTS controller with denominator polynomial matrix $\tilde{D}_c$ described by (6.51) is a solution of the Diophantine equation (6.53). Equation (6.53) has a solution for $(\tilde{N}_c,\tilde{D}'_c)$ if and only if $(N_p,\tilde{D}_{ca}D_p)$ are right $\mathbb{R}[d]$-coprime. If $(\tilde{X},\tilde{Y})$ is a particular solution for $(\tilde{N}_c,\tilde{D}'_c)$

of (6.53) and $(\tilde{D}'_p, \tilde{N}'_p)$ is a left $\mathbb{R}[d]$-coprime pair satisfying (6.54) then

$$\tilde{N}_c = \tilde{X} + S\tilde{D}'_p, \quad \tilde{D}'_c = \tilde{Y} - S\tilde{N}'_p$$

and due to (6.51)

$$\tilde{N}_c = \tilde{X} + S\tilde{D}'_p, \quad \tilde{D}_c = (\tilde{Y} - S\tilde{N}'_p)\tilde{D}_{ca}$$

For causality, we must have $|\tilde{D}_c(0)| \neq 0$; however, according to remark (6.2), $|\tilde{D}_c(0)| = (\tilde{y}(0) - s(0)\tilde{n}'_p(0))|\tilde{D}_{ca}(0)|$, where $\tilde{y}(d) = |\tilde{Y}(d)|$ and $\tilde{n}'_p(d)$ is the least invariant polynomial of $\tilde{N}'_p(d)$. Due to the coprimeness of $(N_p, \tilde{D}_{ca}D_p)$ and $(\tilde{D}'_p, \tilde{N}'_p)$ and the relationship (6.54), $\tilde{n}'_p(d) = n_p(d)$ (up to a real constant) and $n_p(0)$, $|\tilde{D}_c(0)|$ are coprime. Hence,

$$|\tilde{D}_c(0)| = (\tilde{y}(0) - s(0)n'_p(0))|\tilde{D}_{ca}(0)|$$

and if $n_p(0) \neq 0$, $|\tilde{D}_c(0)| \neq 0$ if and only if $|\tilde{D}_{ca}(0)| \neq 0$.

□

6.8 FSTS for Sampled-Data Systems

We conclude this chapter by considering the case of FSTS for sampled-data systems. Assume that $G(s)$ in figure (6.3), is the transfer function of a continuous-time plant $\mathcal{S}_p$ and any continuous-time controller $\mathcal{S}_c$ which is discretized using Zero-Order-Hold (ZOH) devices as D/A converters in each input channel.

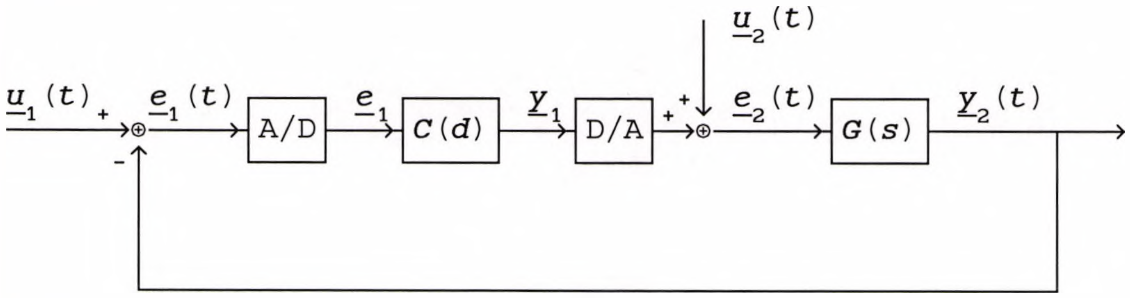

Figure (6.3): The unity feedback sampled-data system

For ripple-free FST response, we can easily extend theorem (5.8) to the MIMO case. Using the principle of superposition and remark (5.8) the following theorem may be readily established.

Theorem 6.14 *(Ripple-Free FSTS)*: Consider the sampled-data system of figure (6.3) where the D/A converters are Zero-Order-Hold devices and $G(s)$ represents the transfer function of the continuous-time plant and any continuous-time controller. Then, the unity feedback system will be ripple-free, if and only if a continuous internal model of the input $\underline{u}(t) = [\underline{u}_1^t(t) \;\; \underline{u}_2^t(t)]^t$ that is observable from the output is implemented first and then a discrete FSTS controller is designed.

□

Remark 6.12: If $F(s)$ represents the internal model transfer function and $G_p(s)$ is the transfer function of the plant, then for stability purposes the tandem connection of $F(s)$ and $G_p(s)$ need not be only observable but controllable as well. If $\underline{u}_1(s)$, $\underline{u}_2(s)$ are represented by coprime polynomial MFDs as

$$\underline{u}_1(s) = \tilde{D}_r^{-1}(s)\underline{\tilde{n}}_r(s) \text{ and } \underline{u}_2(s) = \tilde{D}_d^{-1}(s)\underline{\tilde{n}}_d(s) \tag{6.55}$$

then $F(s)$ may be of the form

$$\phi^{-1}(s)I_m \text{ or } \phi^{-1}(s)I_l \tag{6.56}$$

for an $\ell\times m$ plant, where $\phi(s)$ is the least common denominator of every denominator of $\tilde{D}_r^{-1}(s)$ and $\tilde{D}_d^{-1}(s)$. For the tandem connection of the plant $G_p(s)$ and $F(s)$ to be coprime, we have the following requirement [Che., 1].

The tandem connection of $\phi^{-1}(s)I_m$ followed by the $\ell\times m$ plant $G_p(s)$ ($G_p(s)$ followed by $\phi^{-1}(s)I_l$) is controllable and observable, if and only if

$$\rho\left(\begin{bmatrix} \lambda I - A_p & B_p \\ -C_p & D_p \end{bmatrix}\right) = n + m \quad (n + \ell) \quad \text{and } \lambda : \phi(\lambda)=0 \tag{6.57}$$

where (A_p, B_p, C_p, D_p) is any irreducible realization of $G_p(s)$ and n is the dimension of A_p.

□

According to theorem (6.14) and remark (6.12) we can develop the conditions for ripple-free FST tracking and disturbance rejection.

Theorem 6.15: Consider the sampled-data system of figure (6.3) with an $\ell \times m$ plant $G_p(s)$ and $G(s) = G_p(s)\phi^{-1}(s)I_m$ ($G(s) = \phi^{-1}(s)I_l G_p(s)$), where $\phi(s)$ is the least common denominator of every denominator of $\tilde{D}_r^{-1}(s)$, $\tilde{D}_d^{-1}(s)$ and $\tilde{D}_r^{-1}(s)$, $\tilde{D}_d^{-1}(s)$ are as in (6.55). Then the closed-loop sampled-data system exhibits a ripple-free FST tracking and disturbance rejection response, if and only if

$$\rho\left(\begin{bmatrix} \lambda I - A_p & B_p \\ -C_p & D_p \end{bmatrix}\right) = n + m \quad (n + \ell) \quad \text{and } \lambda : \phi(\lambda)=0$$

and an FST controller for tracking and disturbance rejection is designed for the discrete-equivalent system of $G(s)$.

□

Remark 6.13: Theorem (6.15) requires the use of the internal model $\phi^{-1}(s)$ in every input or output channel of the original plant and this may result to over design. Nevertheless, it provides by itself, a design procedure that guarantees ripple-free FST response.

□

6.9 Conclusions

The problem of ***Total Finite Settling Time Stabilization*** for MIMO discrete linear systems has been defined and solved algebraically using the one-parameter feedback scheme of figure (6.1). The family $\mathcal{F}(P)$ of all causal FSTS controllers has been derived as a solution to a polynomial matrix Diophantine equation and parametrized in an affine manner with respect to the free parameters R or S (theorem 6.1). Further, the computation of $\mathcal{F}(P)$ has been obtained by solving a set of Toeplitz type linear equations over $\mathbb{R}$.

The minimal design problem has been addressed by providing lower and upper bounds for the minimum McMillan degree of all FSTS controllers. Also, a parametrization of all FSTS controllers according to column/row degrees or complexity has been achieved allowing for the characterization of the family $\mathcal{F}(P)$ according to upper bounds on the McMillan degree. In addition, the aforementioned parametrization has led to the parametrization of the family of all deadbeat regulators and the development of a twofold algorithm for its characterization. In the case of vector plants, a complete parametrization of the FSTS controllers, according to McMillan degree, has been presented and the minimum McMillan degree FSTS controller(s) have been derived.

The *parity interlacing property* has been proved to be again a necessary and sufficient condition for strong FSTS and families of FSTS controllers that attain certain performance criteria as tracking, disturbance rejection and partial dynamics assignment have been obtained. Finally, necessary and sufficient conditions for ripple-free FSTS have been derived and as a result a design procedure for ripple-free FST tracking and disturbance rejection has been proposed.

Chapter 7

SIMULTANEOUS FST STABILIZATION

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SIMULTANEOUS FST STABILIZATION

7.1 Introduction, Background Results

The problem of finding a discrete, linear, time-invariant controller C that stabilizes in *Finite Settling Time (FST)* sense a family of $k + 1$ distinct, linear, discrete-time plants $\{P_i,\ i = 0,1,\ldots,k\}$ is referred to as *Simultaneous FST Stabilization Problem (S-FSTSP)* and is examined in this chapter. The motivation for this work comes from the so-called *Simultaneous Stabilization Problem (SSP)* introduced by Saeks and Murray [Sae., 1] and Vidyasagar and Viswanadham [Vid., 2].

The SSP is a type of robust stabilization problem and arises naturally in the synthesis of control systems with different modes of operation, due for instance to some structural changes; the SSP also naturally arises when $P_0,\ldots,P_k$ represent linearized models of a nonlinear plant around a number of operating points, and a common controller C is required to stabilize the whole family. Necessary and sufficient solvability conditions for the SSP have been given in [Sae., 1] and [Vid., 2], but these conditions are not computationally verifiable for the case of more than two plants. Vidyasagar and Viswanadham [Vid., 2] provided the generalization to the MIMO case of some of the SISO results of Saeks and Murray [Sae., 1]. To this end, they have shown that the problem of simultaneously stabilizing $k + 1$ plants is equivalent to the problem of simultaneously stabilizing k plants by a stable controller. Hence, in the case of two plants the problem reduces according to Youla Bongiorno and Lu [You., 2], to the satisfaction of the parity interlacing property by a single plant, but the problem of simultaneous stabilization of more than two plants is not just an interpolation but an interpolation/avoidance problem as it was introduced in Helton [Hel., 1] and pointed out subsequently by other authors e.g., Ghosh [Gho., 1], Vidyasagar [Vid., 3], Dorato et al. [Dor., 1] and Wei [Wei., 1].

It is also shown in Vidyasagar and Viswanadham [Vid., 2] that two $\ell \times m$ plants can be simultaneously stabilized generically if either ℓ or m is greater than one. This result has been generalized further by Ghosh and Byrnes [Gho., 2] and also by Ghosh [Gho., 3] - [Gho., 5] where it is shown that generic simultaneous stabilizability of r, $\ell \times m$ plants is guaranteed if $\max\{\ell, m\} \geq r$.

Nevertheless, despite many efforts the SSP has remained unsolved for $k \geq 2$ and is recognized as one of the hard open problems in linear systems theory [Blo., 1]. To this extend, many authors have provided necessary or sufficient conditions for the solution of the SSP. In Kwakernaak [Kwa., 1] it is shown that provided the high frequency behaviour of the plant transfer functions satisfies some restrictions, simultaneous stabilization is possible if all plants have the ***same number*** of transmission zeros which are located in the strict left half plane. Using a completely different proof, Barmish and Wei obtained a similar result first for a family of SISO plants [Bar., 2] and then for the MIMO case [Wei., 2] where a simultaneously stabilizing controller is constructed through a new iterative algorithm.

In view of the simultaneous stabilization conditions given by Vidyasagar and Viswanadham [Vid., 2], the issue of finding computationally verifiable tests for simultaneous stabilization was raised again by Alos [Alo., 1] and Emre [Emr., 2]. In the work of Alos simultaneous stabilization is confined to plants which result from a nominal one with possible loss of outputs or actuators whereas Emre considers the case of a family of $k + 1$ SISO plants where all the $k + 1$ closed-loop systems end up having the ***same*** characteristic monic polynomial.

The aim of this chapter is to examine the SSP in the context of discrete-time systems and for the special type of stabilization; namely the FST stabilization. To this extend, this part of the thesis can be considered as an extension of Emre's work [Emr., 2] where the characteristic polynomials of the $k + 1$ closed-loop MIMO systems are required to be nonzero real constants c_i.

The general case of S-FSTSP is considered first for plants of $\ell \times m$ common dimension and the results are then specialized to the case of $\ell \times 1$ or $1 \times m$ families of plants for which testable necessary and sufficient conditions are derived. With a family of $k + 1$ plants $\{P_i,\ i = 0,1,\ldots,k\}$, we may associate a family plant matrix and its properties lead to a classification of the various types of families as well as general conditions for solvability of the S-FSTSP. Also, in an approach very similar to that in Vidyasagar and Viswanadham [Vid., 2], necessary or sufficient solvability conditions of the S-FSTSP are derived. In the special case of vector plants ($m = 1$, or $\ell = 1$) testable necessary and sufficient conditions are given and when a solution exists, the family of S-FSTS controllers is derived. The necessary and sufficient conditions are expressed as properties of the plant family matrix and may be tested using tools of the minimal basis theory of rational vector spaces, or equivalent standard linear algebra tests over $\mathbb{R}$.

7.2 The Simultaneous FSTSP: Background Mathematics

In this section we consider the mathematical preliminaries needed for the solution of the simultaneous FSTS problem. In essence we deal with the matrix equation

$$AX = B \tag{7.1}$$

over the ring of polynomials $\mathbb{R}[d]$. One result that is readily available is theorem (2.20). Here we present another set of solvability conditions for the equation (7.1) which is more explicit than theorem (2.20).

The analysis that follows applies not only to polynomial matrices but to matrices with entries from any PID and describes in a more formal way the results given in Vidyasagar [Vid., 1]. A more extensive treatment of the subject can be found in Karcanias [Kar., 5]. First, we restate and extend some of the basic definitions of section (2.3.1).

Definition 7.1: A matrix $A \in \mathbb{R}^{pxq}[d]$ with $r = \rho(A) \leq \min\{p,q\}$ will be called:

- **a. Degenerate** if $r < \min\{p,q\}$ whereas if $r = \min\{p,q\}$ it will be called **nondegenerate**.
- **b. Coprime**, if it is nondegenerate and all its invariant polynomials are units in $\mathbb{R}[d]$.
- **c. Square** if $r = p = q$, **left regular** if $r = p \leq q$ and **right regular** if $r = q \leq p$.

□

Definition 7.2 [Kar., 5]: Let $A \in \mathbb{R}^{pxq}[d]$ and $r = \rho(A)$. Consider the Smith form decomposition of A defined by

$$UAV = \begin{bmatrix} S_A^* & O \\ O & O \end{bmatrix} = S_A \Leftrightarrow A = \hat{U}S_A\hat{V} \tag{7.2}$$

where S_A is the Smith form of A, $S_A^* = \mathrm{diag}\{a_1,\ldots,a_r\}$ with a_i the invariant polynomials of A and U, $\hat{U} = U^{-1}$, V, $\hat{V} = V^{-1}$ are $\mathbb{R}[d]$-unimodular matrices of appropriate dimensions. If U, V are partitioned according to the partitioning of S_A, i.e.

$$U = \begin{bmatrix} A_l^+ \\ \cdots \\ A_l^\perp \end{bmatrix}, \qquad V = [\, A_r^+ \,\vdots\, A_r^\perp \,] \tag{7.3}$$

then $A_l^+ \in \mathbb{R}^{r \times p}[d]$, $A_r^+ \in \mathbb{R}^{q \times r}[d]$ are called **left-, right-projectors** respectively and $A_l^\perp \in \mathbb{R}^{(p-r) \times p}[d]$, $A_l^+ \in \mathbb{R}^{q \times (q-r)}[d]$ are called **left-, right-annihilators** correspondingly of A.

□

According to definition (3.2), the following remark may be readily established.

Remark 7.1: Let $A \in \mathbb{R}^{p \times q}[d]$, $r = \rho(A)$ and let S_A^* be the essential part of the Smith form of A and $(A_l^+, A_l^\perp)$, $(A_r^+, A_r^\perp)$ be pairs of matrices as in definition (7.2). Then, due to the partitioning (7.3)

$$
\begin{aligned}
A_l^\perp A &= O \\
A A_r^\perp &= O \\
A_l^+ A A_r^+ &= S_A^*
\end{aligned}
\tag{7.4}
$$

and $A_l^\perp$, $A_r^\perp$ are minimal bases of the left, right null $\mathbb{R}(d)$-spaces $\mathcal{N}_l\{A\}$, $\mathcal{N}_r\{A\}$ of A respectively. Also, if A has full rank, then at least one of the annihilators does not exist. In particular

$$
\begin{aligned}
A_l^\perp &= O, \ A_l^+ = I, \quad \text{if and only if} \quad r = p \\
A_r^\perp &= O, \ A_r^+ = I, \quad \text{if and only if} \quad r = q
\end{aligned}
$$

□

Using the aforementioned concepts we may express the solvability of the matrix equation (7.1) in the following way.

Lemma 7.1 [Kar., 5]: Let $A \in \mathbb{R}^{p \times q}[d]$, $B \in \mathbb{R}^{p \times s}[d]$, $r = \rho(A)$ and consider the matrix equation

$$
AX = B, \quad X \in \mathbb{R}^{q \times s}[d] \tag{7.5}
$$

over $\mathbb{R}[d]$.

a. For any pair $(A_l^+, A_l^\perp)$, there exists a pair $(A_r^+, A_r^\perp)$ such that, if

$$X = [\, A_r^{+} \; \vdots \; A_r^{\perp} \,] \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = A_r^{+} X_1 + A_r^{\perp} X_2 \tag{7.6}$$

the equation (7.5) is equivalent to (7.6) together with

$$A_1^{\perp} B = O \tag{7.7}$$

$$S_A^{*} X_1 = A_1^{+} B \tag{7.8}$$

b. Conditions (7.7), (7.8) are necessary and sufficient for the solvability of (7.5). If these conditions are satisfied, then for any polynomial X_1 solving (7.8), there exists a family of X matrices defined by (7.6), where X_2 is an arbitrary polynomial matrix of appropriate dimensions.

□

According to remark (7.1), lemma (7.1) reduces to the following corollary, if the matrix A has full rank.

Corollary 7.1: Let all the matrices concerned be as in definition (7.1) and lemma (7.1). Then

a. If A is left regular (full row rank), i.e. $r = p$, equation (7.5) is equivalent to

$$S_A^{*} X_1 = B \tag{7.9}$$

$$X = A_r^{+} X_1 + A_r^{\perp} X_2 \tag{7.10}$$

b. If A is right regular (full column rank), i.e. $r = q$, equation (7.5) is equivalent to

$$A_1^{\perp} B = O \tag{7.11}$$

$$S_A^{*} X = A_1^{+} B \tag{7.12}$$

□

From lemma (7.1) we can recover the result given in Vidyasagar [Vid., 1] regarding the solvability conditions of equation (7.5).

Lemma 7.2 [Vid., 1]: Let $A \in \mathbb{R}^{p \times q}[d]$, $r = \rho(A)$ and let U, V be $\mathbb{R}[d]$-unimodular matrices such that $UAV = S_A$, where S_A is the Smith form of A with a_i, $i = 1,\ldots,r$ denoting the invariant polynomials of A. The equation $AX = B$, $B \in \mathbb{R}^{p \times s}[d]$ has a polynomial solution X if and only if

a. a_i divides every element of the ith row of UB

b. all the other rows of UB are zero.

□

We are now ready to give algebraic solvability conditions for the simultaneous finite settling time stabilization problem.

7.3 The Simultaneous FSTSP: Statement of the Problem and General Results

In this section we deal with the derivation of the solvability conditions for the simultaneous stabilization in FST sense of a family of discrete, linear systems by a discrete, linear, time-invariant controller. Both, the generic as well as the general MIMO cases are treated and algebraic testable conditions are provided when possible.

Definition 7.3: Let $\Sigma_k = \{P_i : P_i \in \mathbb{R}^{1 \times m}(d),\ i = 0,1,\ldots,k\text{-}1\}$ be a family of k discrete-time plants represented by their transfer function matrices P_i or by their $\mathbb{R}[d]$-coprime left or right MFDs $(\tilde{D}_i, \tilde{N}_i)$, (N_i, D_i) respectively. The problem of finding the conditions under which there exists a causal controller C that stabilizes in FST sense all the plants of the Σ_k family is referred to as **Simultaneous Finite Settling Time Stabilization Problem (S-FSTSP)** and the controller that solves the S-FSTSP will be called **S-FSTS controller**.

□

Definition 7.4: The set of all families Σ_k of k, $\ell \times m$ plants will be denoted by $\mathcal{S}^k_{1,m}$. If $\Sigma_k \in \mathcal{S}^k_{1,m}$ and the S-FSTSP is solvable, then Σ_k will be called **S-FSTS family**.

□

If $C = N_c D_c^{-1} = \tilde{D}_c^{-1}\tilde{N}_c \in \mathbb{R}^{m\times l}(d)$, then according to theorem (6.1), C is an S-FSTS controller, if and only if

$$\tilde{N}_i N_c + \tilde{D}_i D_c = U_i \in U(\mathbb{R}[d]), \quad i = 0,1,\ldots,k\text{-}1 \tag{7.13}$$

or equivalently

$$\tilde{N}_c N_i + \tilde{D}_c D_i = U_i \in U(\mathbb{R}[d]), \quad i = 0,1,\ldots,k\text{-}1 \tag{7.14}$$

The above conditions may be expressed as

$$\tilde{T}_k R_C = Q_u \tag{7.15}$$

where

$$\tilde{T}_k := \begin{bmatrix} \tilde{N}_0 & \tilde{D}_0 \\ \vdots & \vdots \\ \tilde{N}_{k-1} & \tilde{D}_{k-1} \end{bmatrix}, \quad R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix}, \quad Q_u := \begin{bmatrix} U_0 \\ \vdots \\ U_{k-1} \end{bmatrix} \tag{7.16}$$

or

$$L_C T_k = \tilde{Q}_u \tag{7.17}$$

where

$$T_k := \begin{bmatrix} N_0 & \cdots & N_{k-1} \\ D_0 & \cdots & D_{k-1} \end{bmatrix}, \quad L_C := [\ \tilde{N}_c\ \tilde{D}_c\ ] \tag{7.18}$$

$$\tilde{Q}_u := [\ U_0\ \cdots\ U_{k-1}]$$

Definition 7.5: The matrices $\tilde{T}_k$, T_k are referred to as **left-, right-plant family matrix (L-PFM, R-PFM)** respectively. The matrices $Q_u \in \mathbb{R}^{kl\times l}[d]$, $\tilde{Q}_u \in \mathbb{R}^{m\times km}[d]$ are called **partitioned unimodular** and the corresponding sets will be denoted by $U_{k,l}[d]$, $\tilde{U}_{k,m}[d]$.

□

In the study of the S-FSTSP either (7.15), or (7.17) may be used. We shall refer to (7.15), (7.17) as the **right**, **left formulation** of the S-FSTSP respectively. In the following we will work with the right formulation of the S-FSTSP and all definitions and results can be translated to the left formulation in the obvious manner. According to (7.15) the solvability of the S-FSTSP may be summarized as follows.

Remark 7.2: The S-FSTSP is solvable, if and only if

$$\tilde{T}_k R_C = Q_u \tag{7.19}$$

has an $\mathbb{R}[d]$-coprime, causal ($|D_c(0)| \neq 0$) solution R_C.

□

Some preliminary properties of $\tilde{T}_k$ matrices are given next.

Remark 7.3: For any $\Sigma_k \in \mathcal{S}^k_{1,m}$ the L-PFM $\tilde{T}_k$ is uniquely defined modulo permutations of the k row blocks and premultiplication by $\mathrm{diag}\{U_0,\dots,U_{k-1}\}$, where $U_i \in U(\mathbb{R}[d])$, $i = 0,1,\dots,k\text{-}1$.

□

Proposition 7.1: Let $\Sigma_k \in \mathcal{S}^k_{1,m}$, $\tilde{T}_k \in \mathbb{R}^{k1\times(1+m)}[d]$ be a L-PFM of Σ_k, $\ell \geq m$ and $r = \rho(\tilde{T}_k)$. Then the following hold true.

a. if $\{\tilde{t}_i(d),\ i = 1,\dots,r\}$ is the set of invariant polynomials of $\tilde{T}_k$, then $\tilde{t}_1(d) = \cdots = \tilde{t}_1(d) = 1$.

b. $\ell \leq r \leq \ell + m$

<u>*Proof*</u>.

a. Since $(\tilde{D}_i, \tilde{N}_i)$ are left $\mathbb{R}[d]$-coprime for any i, then the matrix $[\tilde{N}_i \ \ \tilde{D}_i]$ is a rank ℓ coprime matrix and part (a) follows as well as that $\ell \leq r$.

b. $\tilde{T}_k$ is a $k\ell \times (\ell + m)$ polynomial matrix and therefore $\rho(\tilde{T}_k) \leq \min\{k\ell, \ell{+}m\}$. But $\ell \geq m$, i.e. $k\ell \geq 2\ell \geq m + \ell$, $\forall\ k{\geq}2$. Hence, $\min\{k\ell,\ell{+}m\} = \ell + m$ and this proves part (b).

□

Remark 7.4: The requirement that $\ell \geq m$ in proposition (7.1) is not really restrictive. If this is not the case, i.e, if $\ell < m$, then analogous results can be shown for the R-PFM T_k. In this chapter, the treatment of the S-FSTSP is based on the assuption that $\ell \geq m$.

□

Remark 7.5: For any $\Sigma_k \in \mathcal{S}^k_{1,m}$ the Smith form of any L-PFM $\tilde{T}_k$ with $r = \rho(\tilde{T}_k)$, is of the form

$$\tilde{S}_k := S_{\tilde{T}_k} = \begin{bmatrix} I_l & & O \\ & \tilde{S}^* & \\ O & & O \end{bmatrix} = \begin{bmatrix} \tilde{S}^*_k & O \\ O & O \end{bmatrix} \tag{7.20}$$

$$\tilde{S}^* = \mathrm{diag}\{\tilde{t}_{l+1}, \dots, \tilde{t}_r\}$$

□

According to proposition (7.1), the rank of $\tilde{T}_k$ has well defined bounds. We consider first the case of the minimum rank $\tilde{T}_k$.

Proposition 7.2: Let $\Sigma_k \in \mathcal{S}^k_{l,m}$ and $\tilde{T}_k$ be a L-PFM of Σ_k with $\rho(\tilde{T}_k) = \ell$. Then

a. For every pair of systems described by the coprime MFDs $(\tilde{D}_i, \tilde{N}_i)$, $(\tilde{D}_j, \tilde{N}_j)$ there exists a $U_{ij} \in \mathcal{U}(\mathbb{R}[d])$ such that

$$[\tilde{N}_i \;\; \tilde{D}_i] = U_{ij} [\tilde{N}_j \;\; \tilde{D}_j] \;\; \forall \; i, j \in \underline{k} = \{0, 1, \dots, k\text{-}1\}$$

b. There exists a family of S-FSTS controllers, which is the family $\mathcal{F}(P_i)$, i.e. the family that stabilizes in FST sense any pair $(\tilde{D}_i, \tilde{N}_i) \in \Sigma_k$.

Proof.

a. If $\rho(\tilde{T}_k) = \ell$, then any $[\tilde{N}_i \;\; \tilde{D}_i]$ matrix, which is by definition left coprime, defines a least degree basis for the $\mathbb{R}(d)$-row space $\mathcal{X}_r$ or for the maximal row module $\mathcal{M}_r$ of $\tilde{T}_k$. Clearly, any two bases of $\mathcal{M}_r$ are related by $\mathbb{R}[d]$-unimodular matrices. Part (b) readily follows from part (a).

□

From remark (7.2) it is clear that the S-FSTSP can be reduced to the solution of the matrix equation (7.5) over $\mathbb{R}[d]$. Therefore, lemma (7.1) can provide the necessary tools for the study of the S-FSTSP. First we consider the generic cases with the help of corollary (7.1).

Theorem 7.1: Let $\Sigma_k \in \mathcal{S}^k_{l,m}$ and assume that any L-PFM $\tilde{T}_k$ is both left regular and coprime. Then the S-FSTSP is always solvable on the Σ_k family. Furthermore, there exist a pair of a right projector and right annihilator $(\tilde{T}_r^+, \tilde{T}_r^\perp)$ such that the family of solutions of the S-FSTSP is given by

$$R_C = \tilde{T}_r^+ Q_u + \tilde{T}_r^\perp X \tag{7.21}$$

where $Q_u \in U_{k,l}[d]$ is a partitioned unimodular and X is an appropriate dimension polynomial matrix such that

$$\det(\hat{T}_l(0) \begin{bmatrix} Q_u(0) \\ X(0) \end{bmatrix}) \neq 0 \tag{7.22}$$

where $\hat{T}_l$ is the last ℓ-row block of the matrix $[\tilde{T}_r^+ \ \tilde{T}_r^\perp]$.

Proof. According to remark (7.2), the S-FSTSP is solvable, if and only if the equation

$$\tilde{T}_k R_C = Q_u \tag{7.23}$$

is solvable for $R_C = [N_c^t \ D_c^t]^t$ and $|D_c(0)| \neq 0$. Eqn. (7.23) is of the form $AX = B$, $A = \tilde{T}_k$, $X = R_C$ and $B = Q_u \in U_{k,l}[d]$. Since $\tilde{T}_k$ is left regular, equation (7.23) can be solved according to part (a) of corollary (7.1) and because $\tilde{T}_k$ is coprime, $S_A^* = \tilde{S}_k^* = I$. Therefore, if $(\tilde{T}_r^+, \tilde{T}_r^\perp)$ is a pair of right projector and right annihilator of $\tilde{T}_k$ respectively, then

$$R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix} = [\ \tilde{T}_r^+ \ \tilde{T}_r^\perp\ ] \begin{bmatrix} Q_u \\ X \end{bmatrix} \tag{7.24}$$

where $\tilde{T}_r^+$, $\tilde{T}_r^\perp$ correspond to A_r^+, $A_r^\perp$, Q_u to X_1 and X to X_2 of equation (7.10). Equation (7.24) is true for any partitioned unimodular Q_u and any polynomial matrix X. If $\hat{T}_l$ is the last ℓ-row block of $[\tilde{T}_r^+ \ \tilde{T}_r^\perp]$, then

$$D_c(0) = \hat{T}_l(0) \begin{bmatrix} Q_u(0) \\ X(0) \end{bmatrix}$$

and therefore the relationship (7.22) must be satisfied for causality to hold true.

$\square$

Theorem 7.2: Let $\Sigma_k \in \mathcal{S}^k_{l,m}$ and assume that any L-PFM $\tilde{T}_k$ is both right regular and coprime. If $(\tilde{T}^+_l, \tilde{T}^\perp_l)$ is a pair of left projector and left annihilator of $\tilde{T}_k$ respectively and $\hat{T}^+_l$ is the last ℓ-row block of $\tilde{T}^+_l$, then necessary and sufficient condition for the causal S-FSTSP to be solvable is that there exists a $Q_u \in U_{k,l}[d]$ such that

$$\tilde{T}^\perp_l Q_u = 0 \tag{7.25}$$

$$|\hat{T}^+_l(0) Q_u(0)| \neq 0 \tag{7.26}$$

If the above conditions are satisfied, then the solution is given by

$$R_C = \tilde{T}^+_l Q_u \tag{7.27}$$

Proof. Using similar arguments to those of theorem (7.1) we can easily prove that equations (7.25) and (7.27) are a straightforward consequence of part (b) of corollary (7.1) with $S^*_A = \tilde{S}^*_k = I$. Also, from (7.27)

$$D_c = \hat{T}^+_l Q_u$$

and therefore (7.26) must be satisfied for causality to hold true.

□

Remark 7.6: Theorems (7.1), (7.2) cover the generic cases where $k\ell < m + \ell$, or where $k\ell > m + \ell$, since nondegeneracy and coprimeness are generic properties.

□

Using lemma (3.1), we may state now the conditions characterizing the solvability for the general case of the S-FSTSP.

Theorem 3.3: Let $\Sigma_k \in \mathcal{S}^k_{l,m}$, $\tilde{T}_k$ be an L-PFM of Σ_k and let $\rho(\tilde{T}_k) = r$. For any pair $(\tilde{T}^+_l, \tilde{T}^\perp_l)$, there exists a pair $(\tilde{T}^+_r, \tilde{T}^\perp_r)$ such that, if

$$R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix} = [\ \tilde{T}_r^+ \ \tilde{T}_r^\perp\ ] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \tilde{T}_r^+ X_1 + \tilde{T}_r^\perp X_2 \tag{7.28}$$

then the S-FSTSP is solvable if there exist $Q_u \in U_{k,l}[d]$ such that

$$\tilde{T}_l^\perp Q_u = 0 \tag{7.29}$$

$$\tilde{S}_k^* X_1 = \tilde{T}_l^+ Q_u \tag{7.30}$$

where $\tilde{S}_k^*$ is the essential part of the Smith form $\tilde{S}_k$ of $\tilde{T}_k$. If equations (7.29), (7.30) are satisfied for some Q_u, then the the family of solutions is given by equation (7.28) where X_2 is an arbitrary polynomial matrix of appropriate dimensions and such that together with Q_u, $|D_c(0)| \neq 0$.

□

Remark 7.7: If $\mathcal{N}_l\{\tilde{T}_k\} \neq 0$, condition (7.29) is present and expresses the fact that for the solvability of the S-FSTSP, it is necessary that the $\mathbb{R}(d)$-column space $\mathcal{X}_c$ of $\tilde{T}_k$ contains vectors which form a partitioned unimodular matrix. This alternative formulation of (7.29) is also valid when $\mathcal{N}_l\{\tilde{T}_k\} = \{0\}$. We shall refer to this condition as the **Space Structure Condition (SSC)** of the S-FSTSP.

□

Remark 7.8: Since $\tilde{S}_k^*$, X_1 in (7.30) have dimensions $r \times r$ and $r \times \ell$ respectively and $r \geq \ell$, the solvability of equation (7.30) is not a trivial divisor condition, unless $r = \ell$. Note that if $r = \ell$, it follows by proposition (7.1) that $\tilde{S}_k^*$ is a unity matrix ($\tilde{S}_k^* = I_l$) and therefore (7.30) is solvable for any Q_u. Condition (7.30) is thus an essential condition when $r > \ell$ and we shall refer to it as the **Extended-Divisor Condition (EDC)** of the S-FSTSP.

□

Conditions (7.29), (7.30) may be combined to give the following alternative formulation of theorem (3.3) and hence of the S-FSTSP.

Corollary 7.2: Let $\Sigma_k \in \mathscr{S}^k_{l,m}$, $\tilde{T}_k$ be an L-PFM of Σ_k and let $r = \rho(\tilde{T}_k)$. The S-FSTSP is solvable if for any pairs $(\tilde{T}^+_l, \tilde{T}^\perp_l)$, $(\tilde{T}^+_r, \tilde{T}^\perp_r)$ and the associated $\tilde{S}^*_k$ (the essential part of the Smith form $\tilde{S}_k$ of $\tilde{T}_k$), the following conditions are satisfied

a. There exists a solution $X \in \mathbb{R}^{(kl+r)\times l}[d]$ of the equation

$$\begin{bmatrix} \tilde{S}^*_k & -\tilde{T}^+_l \\ O & \tilde{T}^\perp_l \end{bmatrix} \cdot X = 0, \quad X = \begin{bmatrix} X_1 \\ Q_u \end{bmatrix}, \quad Q_u \in U_{k,l}[d] \tag{7.31}$$

b. For any X_1, Q_u solution of (7.31), there exists X_2 of appropriate dimensions such that

$$R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix} = [\, \tilde{T}^+_r \ \tilde{T}^\perp_r \,] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \tilde{T}^+_r X_1 + \tilde{T}^\perp_r X_2 \tag{7.32}$$

is causal, i.e. $|D_c(0)| \neq 0$.

□

Remark 7.9: Equation (7.31) reduces the overall S-FSTSP to an investigation of the existence of a matrix which is partitioned unimodular and has its columns from a given rational vector space. It is worth pointing out again that if $k\ell \neq m + \ell$, then generically the families $\mathscr{S}^k_{l,m}$ are non-degenerate and coprime. The space structure condition thus becomes the most significant. For special families of systems, this condition takes a rather simple form that allows the derivation of testable solvability conditions.

□

7.4 Necessary, Sufficient Conditions for the S-FSTSP

The analysis so far has shown that if $k\ell < m + \ell$, then for a generic family Σ_k the S-FSTSP is solvable (theorem 3.1), whereas if $k\ell > m + \ell$ and the family Σ_k is once more generic, then the solvability of the S-FSTSP is reduced to a testing of the space structure condition (SSC) (theorem 3.2). The general problem associated with the SSC, that is finding the conditions for the existence of partitioned unimodular matrices in a given rational vector space is still open. However, we can provide either necessary or sufficient testable conditions for the solution of the S-FSTSP.

Let $\Sigma_{k+1} = \{P_0,\ldots,P_k\}$ be a family of $k+1$, $\ell \times m$ plants represented by their left $\mathbb{R}[d]$-coprime polynomial MFDs $(\tilde{D}_i, \tilde{N}_i)$, $i = 0,1,\ldots,k$. Any S-FSTS controller C of the family Σ_{k+1} must belong to any family $\mathcal{F}(P_i)$ and thus to $\mathcal{F}(P_0)$. According to theorem (6.1) $\mathcal{F}(P_0)$ consists of all the FSTS controllers C with the following right composite matrices R_C

$$R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix} = \begin{bmatrix} X & D_0 \\ Y & -N_0 \end{bmatrix} \begin{bmatrix} I \\ R \end{bmatrix} \tag{7.33}$$

where X, Y satisfy the polynomial Diophantine equation

$$\tilde{N}_0 X + \tilde{D}_0 Y = I \tag{7.34}$$

and R is an arbitrary polynomial matrix of appropriate dimensions such that $|D_c(0)| \neq 0$. Then a solution to the S-FSTSP exists, if and only if there is an R satisfying (7.33) and its causality requirements such that

$$[\, \tilde{N}_i \;\; \tilde{D}_i] \begin{bmatrix} X & D_0 \\ Y & -N_0 \end{bmatrix} \begin{bmatrix} I \\ R \end{bmatrix} = U_i \in U(\mathbb{R}[d]), \; i = 1,\ldots,k \tag{7.35}$$

or

$$[\, \tilde{A}_i \;\; \tilde{B}_i] \begin{bmatrix} I \\ R \end{bmatrix} = U_i \in U(\mathbb{R}[d]), \quad i = 1,\ldots,k \tag{7.36}$$

where

$$[\, \tilde{A}_i \;\; \tilde{B}_i] = [\, \tilde{N}_i \;\; \tilde{D}_i] \begin{bmatrix} X & D_0 \\ Y & -N_0 \end{bmatrix} \tag{7.37}$$

or

$$\tilde{A}_i = \tilde{N}_i X + \tilde{D}_i Y \in \mathbb{R}^{l\times l}[d], \quad \tilde{B}_i = \tilde{N}_i D_0 - \tilde{D}_i N_0 \in \mathbb{R}^{l\times m}[d] \tag{7.38}$$

In view of equations (7.33) and (7.36), the S-FSTSP can be formulated as follows.

Remark 7.10: Let $\Sigma_{k+1} = \{P_0, \ldots, P_k\}$ be a $\mathcal{S}^{k+1}_{1,m}$ family, X, Y be a polynomial solution of the Diophantine equation (7.34) and $\tilde{A}_i$, $\tilde{B}_i$ be polynomial matrices defined by equations (7.38). The S-FSTSP is equivalent to that of finding a polynomial matrix R such that

$$\det(\tilde{A}_i + \tilde{B}_i R) = c_i \in \mathbb{R} - \{0\}, \quad i = 1, \ldots, k \tag{7.39}$$

and $|D_c(0)| \neq 0$, where D_c is a denominator polynomial matrix of the FSTS controller $C \in \mathcal{F}(P_0)$ parametrized by R according to (7.33), i.e.

$$R_C := \begin{bmatrix} N_c \\ D_c \end{bmatrix} = \begin{bmatrix} X & D_0 \\ Y & -N_0 \end{bmatrix} \begin{bmatrix} I \\ R \end{bmatrix}$$

□

The following theorem gives necessary conditions for the S-FSTSP.

Theorem 7.4: Let $\Sigma_{k+1} = \{P_0, \ldots, P_k\}$ be a $\mathcal{S}^{k+1}_{1,m}$ family and $\tilde{A}_i$, $\tilde{B}_i$ be polynomial matrices defined by equations (7.38). If $\tilde{a}_i = \det\tilde{A}_i$ and $\tilde{b}_i$, $i = 1, \ldots, k$ is the least invariant polynomial of $\tilde{B}_i$, then a necessary condition for the solvability of the S-FSTSP is that

> the remainder of the division of $\tilde{a}_i$ by $\tilde{b}_i$ must be a real nonzero constant for $i = 1, \ldots, k$.

<u>*Proof*</u>. Since $\tilde{D}_i$, $\tilde{N}_i$ are left $\mathbb{R}[d]$-coprime and the matrix

$$\begin{bmatrix} X & D_0 \\ Y & -N_0 \end{bmatrix}$$

is unimodular (theorem 6.1), then according to (7.37) $\tilde{A}_i$, $\tilde{B}_i$ are left $\mathbb{R}[d]$-coprime. Then due to theorem (2.15)

$$\det(\tilde{A}_i + \tilde{B}_i R) = \tilde{a}_i + \tilde{b}_i r_i \tag{7.40}$$

Hence, for the S-FSTSP to be solvable, it is necessary that

$$\tilde{a}_i + \tilde{b}_i r_i = c_i \in \mathbb{R} - \{0\} \tag{7.41}$$

i.e. $\partial(\tilde{b}_i) \leq \partial(\tilde{a}_i)$ and the condition of theorem (7.4) follows.

□

Remark 7.11: Equations (7.36) express in the FSTS case the well known result given by Vidyasagar [Vid., 1], [Vid., 2], namely that the simultaneous stabilization of $k + 1$ plants can be reduced to the simultaneous stabilization of k plants by a stable controller. Indeed, $\tilde{A}_i$, $\tilde{B}_i$ can be regarded as the left coprime MFDs of the fictitious plants $P_{fi} = \tilde{A}_i^{-1}\tilde{B}_i$ (remark 7.10), and equations (7.36) simply require that the family $\Sigma_{fk} = \{P_{f1},...,P_{fk}\}$ is simultaneously stabilized by the polynomial (stable in FSTS sense) controller R.

□

Remark 7.12: From theorem (7.4) a necessary solvability condition for the S-FSTSP is the existence of $r_i \in \mathbb{R}[d]$ and $c_i \in \mathbb{R} - \{0\}$, such that

$$\tilde{a}_i + \tilde{b}_i r_i = c_i$$

This is always possible when $\tilde{b}_i$ is a unit in $\mathbb{R}[d]$ ($\tilde{b}_i = 1$ without loss of generality). In this case, r_i is uniquely determined apart from its constant term.

If $\tilde{b}_i \neq 1$, then c_i is the remainder of the division of $\tilde{a}_i$ by $\tilde{b}_i$ and therefore r_i, c_i are uniquely determined polynomials. This is certainly the case of sampled-data systems due to the structure of $\tilde{B}_i$ (equations 7.38), and the necessary condition of theorem (7.4) becomes very important.

□

According to remark (7.10), the S-FSTSP is reduced to the solution of the system of equations (7.39) for a common R. This problem is a multilinear problem and its solution over $\mathbb{R}[d]$ remains open. By allowing only one column of R in (7.39) to be free, we can linearize the solution to (7.39) and therefore obtain sufficient solvability conditions for the S-FSTSP.

Theorem 7.5: Consider the S-FSTSP described by the set of equations (7.39) in remark (7.10), i.e.

$$\det(\tilde{A}_i + \tilde{B}_i R) = c_i \in \mathbb{R} - \{0\}, \quad i = 1,\ldots,k$$

and let $R \in \mathbb{R}^{m\times l}[d]$ be a polynomial matrix with all its entries fixed polynomials apart from the entries of one column, e.g. the first column $\underline{r}_1$, i.e.

$$R = \underline{r}_1 + \overline{R}\ , \quad \overline{R} \in \mathbb{R}^{m\times(l-1)}[d] \quad \text{fixed}$$

The S-FSTSP is reduced then to the solution of the system of equations

$$g_{i0} + g_{i1}r_{11} + \cdots + g_{im}r_{m1} = c_i, \quad i = 1,\ldots,k \tag{7.42}$$

where g_{ij}, $i = 1,\ldots,k$, $j = 0,1,\ldots,k$ are known polynomials depending on the entries of $\tilde{A}_i$, $\tilde{B}_i$ and $\overline{R}$, and c_i, $i = 1,\ldots,k$ are real nonzero constants.

Proof. The $\mathbb{R}[d]$-unimodular matrix $U_i = \tilde{A}_i + \tilde{B}_i R$ can be written as

$$U_i = [\ \tilde{A}_i\ \tilde{B}_i] \begin{bmatrix} I \\ R \end{bmatrix} = [\ \tilde{A}_i\ \tilde{B}_i] \begin{bmatrix} & I & \\ \underline{r}_1 & & \overline{R} \end{bmatrix}$$

where $\overline{R}$ is an $m \times (\ell\text{-}1)$ fixed polynomial matrix. If we take the determinant of U_i by considering the grassmann products of

$$[\ \tilde{A}_i\ \tilde{B}_i] \text{ and } \begin{bmatrix} & I & \\ \underline{r}_1 & & \overline{R} \end{bmatrix}$$

we will have that

$$g_{i0} + g_{i1}r_{11} + \cdots + g_{im}r_{m1} = c_i \in \mathbb{R} - \{0\}, \quad i = 1,\ldots,k$$

where g_{ij} are polynomials depending on the entries of $\tilde{A}_i$, $\tilde{B}_i$ and $\overline{R}$.

□

Theorem (7.5) may be restated as the following corollary.

Corollary 7.3: Let R be the polynomial matrix in (7.39) with all its entries apart from its first column $\underline{r}_{1}$ fixed polynomials. The S-FSTSP is reduced to the solution of the system

$$G\cdot\underline{r}_{1} = \underline{g} \tag{7.43}$$

where

$$G := \begin{bmatrix} g_{11} \cdots g_{1m} \\ \vdots \quad\quad \vdots \\ g_{k1} \cdots g_{km} \end{bmatrix}, \quad \underline{g} := \begin{bmatrix} g_{10} - c_1 \\ \vdots \quad\quad \vdots \\ g_{k0} - c_k \end{bmatrix}, \quad c_i \in \mathbb{R} - \{0\} \tag{7.44}$$

□

Corollary (7.3) can provide sufficient solutions for the S-FSTSP using the analysis of section (7.3). If $\tilde{b}_i$ are not real constants, the c_i are completely determined and equation (7.43) can be solved straightforward using lemma (7.1).

7.5 The S-FSTSP on Families of Vector Plants

As it was pointed out in section (7.3) the most significant condition for the solvability of the S-FSTSP is the *space structure condition (SSC)*, i.e. the existence of partitioned unimodular matrices in a given rational vector space. The general problem associated with the SSC is still open; however, this problem takes a rather simple form in certain special cases and in particular in the case of vector plants which we examine in this section. From the formulation of the S-FSTSP we note the following.

Remark 7.13: For families $\mathscr{F}^{k}_{1,m}$ and $\mathscr{F}^{k}_{1,1}$ the solvability of the S-FSTSP is reduced to the study of the following equations.

a. $\mathscr{F}^{k}_{1,m}$ families. From equations (7.15), (7.16) we have

$$\tilde{T}_k\underline{r}_C = \underline{q}_u, \quad \tilde{T}_k \in \mathbb{R}^{k\times(m+1)}[d], \quad \underline{r}_C \in \mathbb{R}^{m+1}[d], \quad \underline{q}_u \in \mathbb{R}^{k}_{0} \tag{7.45}$$

where

$$\tilde{T}_k := \begin{bmatrix} \tilde{\underline{n}}_0^t & \tilde{d}_0 \\ \vdots & \vdots \\ \tilde{\underline{n}}_{k-1}^t & \tilde{d}_{k-1} \end{bmatrix}, \quad \underline{r}_c := \begin{bmatrix} \underline{n}_c \\ d_c \end{bmatrix}, \quad \underline{q}_u := \begin{bmatrix} c_0 \\ \vdots \\ c_{k-1} \end{bmatrix}, \quad c_i \neq 0 \tag{7.46}$$

b. $\mathcal{S}^k_{1,1}$ families. From equations (7.17), (7.18) we have

$$\underline{\ell}_c^t T_k = \tilde{\underline{q}}_u^t, \quad T_k \in \mathbb{R}^{(1+1)xk}[d], \quad \underline{\ell}_c \in \mathbb{R}^{1+1}[d], \quad \tilde{\underline{q}}_u \in \mathbb{R}^k_0 \tag{7.47}$$

where

$$T_k := \begin{bmatrix} \underline{n}_0^t & \cdots & \underline{n}_{k-1}^t \\ d_0 & \cdots & d_{k-1} \end{bmatrix}, \quad \underline{\ell}_c^t := [\ \tilde{\underline{n}}_c \ \ \tilde{d}_c\ ]$$
$$\tilde{\underline{q}}_u^t := [\ c_0 \ \cdots \ c_{k-1}], \quad c_i \neq 0 \tag{7.48}$$

and $\mathbb{R}^k_0$ denote all vectors of $\mathbb{R}^k$ with all coordinates nonzero.

□

The families $\mathcal{S}^k_{1,m}$, $\mathcal{S}^k_{1,1}$ contain systems with either one output or one input and thus they have vector transfer functions. We shall refer to such families as ***families of vector plants***. It is clear, that the study of the S-FSTSP on such families is simpler, since the partitioned unimodular matrices become constant vectors with all components nonzero. In the following, the case of many-input single-output (MISO) families is considered, whereas the results for the single-input many-output (SIMO) case are similar. The case of the left regular coprime families has already been discussed (theorem 7.1). Since we want to explore the SSC we shall assume throughout this section that the families are right regular and coprime. The following result is a consequence of theorem (7.2).

Theorem 7.6: Let $\Sigma_k \in \mathcal{S}^k_{1,m}$ be a right regular coprime family. If $\tilde{T}_k$ is a L-PFM of Σ_k, $(\tilde{T}^+_k, \tilde{T}^\perp_l)$ a pair of left projector, left annihilator of $\tilde{T}_k$ and $\underline{t}^+_l$ is the last row of $\tilde{T}^+_l$, then the causal S-FSTSP is solvable on Σ_k, if and only if there exists a $\underline{q}_u \in \mathbb{R}^k_0$ such that

$$\tilde{T}_1^{\perp}\underline{q}_u = 0 \tag{7.49}$$

$$\hat{\underline{t}}_1^{+}(0)\underline{q}_u \neq 0 \tag{7.50}$$

If the above conditions are satisfied, then the solution is given by

$$\underline{r}_C := [\underline{n}_c^t \ d_c]^t = \tilde{T}_1^{+}\underline{q}_u \tag{7.51}$$

□

The significance of condition (7.49) is emphasized by the following result.

Corollary 7.4: Let $\Sigma_k \in \mathcal{S}^k_{1,m}$, $\tilde{T}_k$ be a L-PFM and assume that $\mathcal{N}_1\{\tilde{T}_k\} \neq 0$. Necessary conditions for the S-FSTSP to be solvable on Σ_k is that either of the following equivalent conditions hold true.

a. If $\mathcal{M}_c$ is the column module of $\tilde{T}_k$, then $\mathcal{M}_c$ has at least a zero dynamical index; furthermore, if $\mathcal{M}_c^0$ is the submodule characterized by the zero dynamical indices, then $\mathcal{M}_c^0 \cap \mathbb{R}_0^k \neq 0$.

b. If $\tilde{T}_1^{\perp}[d] = T_0 + \cdots + d^n T_n$ and $\hat{T} = [T_0^t \ \cdots \ T_n^t]^t$, then $\mathcal{N}_r\{\hat{T}\} \cap \mathbb{R}_0^k \neq 0$.

Proof. Note that necessary condition for the solvability of the S-FSTSP is that $\tilde{T}_1^{\perp}(d)\underline{q}_u = 0$, $\underline{q}_u \in \mathbb{R}_0^k$. But by definition of $\tilde{T}_1^{\perp}(d)$, $\tilde{T}_1^{\perp}(d)\tilde{T}_k(d) = 0$ and part (a) follows. Clearly, part (b) expresses the condition $\tilde{T}_1^{\perp}(d)\underline{q}_u = 0$.

□

Remark 7.14: Corollary (7.4) provides tools for the computation of the vectors $\underline{q}_u \in \mathbb{R}_0^k$ which satisfy the space structure condition (7.49). In particular, from part (b) of corollary (7.4), we have the following testable condition. If W is a basis matrix for $\mathcal{N}_r\{\hat{T}\}$, then the space structure condition is satisfied, if and only if the matrix W has no zero rows.

□

Remark 7.15: The space structure condition is necessary for the general case of the S-FSTSP for vector plant families where the L-PFM $\tilde{T}_k$ is neither right regular or coprime. In that case the equation

$$\tilde{S}^*_k \underline{x}_1 = \tilde{T}^+_1 \underline{q}_u$$

has to be solved for $\underline{x}_1$, for any $\underline{q}_u$ that satisfies the SSC. According to theorem (7.4) and remark (7.12), in the case of sampled-data systems, $\underline{q}_u$ is uniquely defined up to multiplication by a nonzero real constant and therefore, the SSC can be tested straight away. Also, the causality conditions are always valid since all FSTS controllers are causal in this case.

□

7.6 Conclusions

The *Simultaneous Finite Settling Time Stabilization Problem* has been addressed in this chapter and necessary and sufficient conditions for its solvability have been given. It has been shown that for the left regular and coprime families a solution always exists, whereas for the right regular case of plant families the *space structure condition* is the key one. For the case of families of vector plants the latter condition is readily testable using standard linear algebra tools.

The derivation of computationally verifiable criteria for the SSC in the general case is still open and under investigation. Alternatively, necessary or sufficient testable conditions for the S-FSTSP have been provided in section (7.4).

As it was mentioned in the introduction, the simultaneous stabilization problem in general, is a particular case of robustness problem, where the plants to be simultaneously stabilized are distinct and well defined. The next extension to that is to assume that the family of plants is an infinite one defined by all the plants whose transfer functions entries are rational functions having interval polynomials as numerators and denominators (see e.g. [Bar., 2], [Wei., 2]). Using our notion of FSTS and a two-parameter control scheme, Junhua Chang [Cha., 1] has been able to solve this problem for the FSTS case of SISO plants by reducing it to the one described in section (7.5) and testing mainly for the SSC of a finite family of plants. The general MIMO case remains still unsolved.

Chapter 8

FURTHER DESIGN ASPECTS FOR FSTS:
Optimization, Shaping , Robustness and Two-Parameter FSTS

Chapter 8

FURTHER DESIGN ASPECTS FOR FSTS: Optimization, Shaping, Robustness and Two-Parameter FSTS

8.1 Introduction

In this last chapter we consider some further performance requirements apart from tracking and disturbance rejection imposed on the family $\mathcal{F}(P)$ of FSTS controllers of a plant P. In particular, using the parametrization of $\mathcal{F}(P)$ obtained in chapters (5) and (6), it is shown how optimal, or robust FSTS controllers can be constructed and finally we try to alleviate some of the restrictions of the one-parameter feedback scheme by adopting the most general linear control scheme of the two-parameter, or two-degrees-of-freedom compensation.

Thus, the problems we deal with in this chapter are twofold. On one hand, we consider optimization, shaping and robustness FSTS problems within the framework of one-parameter unity feedback compensation. On the other hand, we use a two-parameter feedback scheme to solve problems of similar nature to those encountered in chapter (6).

Specifically, in the next three sections we examine:

a. *ℓ^1-, ℓ^∞-optimization* where among the FSTS controllers we select those with a minimum ℓ^1- or ℓ^∞-norm of a certain error vector.

b. *shaping*, where error, control, or transfer function shaping in addition to FST performance is obtained, and

c. *robustness*, where robust FSTS controllers are designed for plants with multiplicative uncertainty.

The common denominator for the solution of the aforementioned problems is that they are mainly linear, time-domain problems and they can all be reduced to the solution of corresponding finite, linear programming problems where the full advantage of the linear programming optimization can be exploited.

In the final section the FSTSP is defined for a two-parameter control scheme where a full parametrization of all the FSTS controllers is derived depending now on two rather than on one parameter and the benefit of this dependence is exploited for tracking and disturbance rejection purposes.

8.2 ℓ^1- and ℓ^∞-Optimal FSTS

With the emergence of the YBK parametrization and the introduction of the H^∞-control theory, the optimization problem in the form of min-max optimization has attracted a considerable amount of attention [Cha., 2], [Fra., 1], [Fra., 2], [Saf., 1], [Vid., 1]. Subsequently, the ℓ^1-approach, introduced by Vidyasagar [Vid., 4] and followed by Dahleh and Pearson [Dah., 1] to [Dah., 4], has complemented the H^∞-optimization by handling problems that cannot be treated by the H^∞-theory.

The objective in both methods is to minimize the maximum amplitude of the system error when the system inputs are signals with bounded norm but otherwise arbitrary. In general the error transfer function can be written as

$$\hat{\Phi} = \hat{H} - \hat{U}\hat{Q}\hat{V}$$

where $\hat{H}$, $\hat{U}$, $\hat{V}$ are given stable rational matrices and $\hat{Q}$ is an arbitrary stable rational matrix. The objective of the minimization problem is to minimize a suitable norm of $\hat{\Phi}$ with respect to $\hat{Q}$. In particular, the H^∞-theory handles signals with finite energy, i.e. with bounded ℓ^2-norm and minimizes the H^∞-norm $\|\hat{\Phi}\|_\infty$ of $\hat{\Phi}$, and the ℓ^1-theory handles signals with bounded magnitude, i.e. with finite ℓ^∞-norm and minimizes the

ℓ^1-norm $\|\hat{\Phi}\|_1$ of $\hat{\Phi}$ [Dah., 4]. Therefore, the H^∞-optimization is effectively a frequency-domain approach where signal energy is most adequately represented, whereas the ℓ^1-optimization is a time-domain approach that treats persistent signals, i.e. signals that act more or less continuously with time and do not satisfy the bounded energy condition of the H^∞-theory [Dah., 4].

In many cases of optimal control, the interest might be directed to the minimization of the error amplitude for specified signals such as steps, ramps, sinusoids etc., and the ℓ^1-theory as such does not provide a solution to that problem. Dahleh and Pearson [Dah., 5] considered such a problem by minimizing the amplitude of the regulated output due to a specific bounded input for the SISO discrete-time case.

In the FSTS case, the system output is of the same type as the system input after finite time though the system error is not zero. Therefore, by minimizing the ℓ^1- or the ℓ^∞-norm of the system error due to a specified input for a specific settling time we can have an ***'approximate tracking'*** FSTS controller, in the sense of attaining minimum error in finite time. In this case the optimization problem is reduced to a finite linear programming problem, whereas in the general ℓ^1-optimization a semi-infinite linear programming problem must be solved.

In the sequel, we give a brief review of some suitable norms for discrete-time systems and also a brief review of the linear programming method before we deal with the problem of FSTS optimization.

8.2.1 Norms for discrete-time systems

In this section we introduce various norms that define the problem of FST optimization and that of FST robustness that we treat in section (8.4). For a detailed account of the norm properties given subsequently, we refer to [Des., 1] and [Vid., 5]. Here we follow the approach and terminology given in Dahleh and Pearson [Dah., 4].

Consider the set of formal power series $\mathbb{R}[[d]]$ with one indeterminate d over the field of real numbers $\mathbb{R}$. Then,

$$\forall \; f = \{f_i\} = \sum_{i=0}^{\infty} f_i d^i \in \mathbb{R}[[d]] \tag{8.1}$$

the expressions

$$\|f\|_p := \left\{ \sum_{i=0}^{\infty} |f_i|^p \right\}^{1/p} \quad , \; 1 \le p < \infty \tag{8.2}$$

$$\|f\|_\infty := \sup_i |f_i| \tag{8.3}$$

define a norm which is denoted as the p-norm of f. The space of all sequences f such that $\|f\|_p$ is defined, i.e. $\|f\|_p < \infty$, is denoted by ℓ^p.

Any sequence $f \in \mathbb{R}[[d]]$ can represent the impulse response of a linear time-invariant system. It is a well-known fact (theorem 3.2), that the system is BIBO-stable, if and only if f is an ℓ^1 sequence. We recall that the series (8.1) are formal and d is an indeterminate and not a variable. If f is an ℓ^1 sequence, the series (8.1) are summable for some $d \in \mathbb{C}$ and f may also represent a function of the complex variable d. To distinguish the two expressions we represent the function corresponding to the sequence f by $\hat{f}(d)$, or just by $\hat{f}$. In system theory terms, $f = \{f_i\}$ is the impulse response of a linear time-invariant system whereas $\hat{f}(d)$, which is no more than the z-Transform of $\{f_i\}$ with $d = z^{-1}$, is the transfer function of the system .

Let A denote the space of all matrices with elements BIBO-stable functions. Therefore, for every $\hat{G} \in A$, $G = (g_{ij})$ $i = 1,\ldots,m$, $j = 1,\ldots,n$ is its impulse response matrix with $g_{ij} \in \ell^1$, or $G \in \ell^1_{mn}$. If ℓ^∞_n is the space of all bounded vector sequences $\underline{f}$ with a norm

$$\|\underline{f}\|_\infty := \max_j \|f^j\|_\infty, \quad \underline{f} = (f^1, \ldots, f^n)^t \tag{8.4}$$

we can regard $\mathcal{A}$ as the space of bounded linear time-invariant operators on ℓ^∞_n, i.e. $\forall\ \hat{G} \in \mathcal{A}$, $\underline{f} \in \ell^\infty_n$, then

$$\hat{G} : \ell^\infty_n \longrightarrow \ell^\infty_m$$
$$\hat{G}\underline{f} = G{*}\underline{f}$$

The induced norm on $\mathcal{A}$ is given by

$$\|\hat{G}\|_{\mathcal{A}} := \sup_{\|\underline{f}\|_\infty \neq 0} \frac{\|\hat{G}\underline{f}\|_\infty}{\|\underline{f}\|_\infty} = \max_i \sum_{j=1}^{n} \|g_{ij}\|_1 := \|G\|_1 \tag{8.5}$$

Therefore, $\mathcal{A}$ and ℓ^∞_{mn} are identified with each other with the above norm. We can generalize the previous concepts as follows.

Definition 8.1 [Des., 1]: An operator $\hat{G}$ is said to be **ℓ^p-stable**, $1 \le p \le \infty$, if and only if $\hat{G}$ is a map from ℓ^p to ℓ^p and the **gain** of the operator defined as

$$g_p(\hat{G}) := \sup_{\|\underline{f}\|_p \neq 0} \frac{\|\hat{G}\underline{f}\|_p}{\|\underline{f}\|_p} < \infty \tag{8.6}$$

□

Remark 8.1: For linear time-invariant systems, the gain of $\hat{G}$ is the induced norm of $\hat{G}$ as a bounded linear operator on ℓ^p. In the case of lumped LTI systems, $\mathcal{A}$ is the space of rational matrices with poles outside the closed unit disc and ℓ^1_{mn} is the set $\mathbb{R}^+_{mn}(d)$ of sequential matrices with entries stable sequences. In this case, according to the analysis in chapter (2), these two spaces are isomorphic. Thus, $\hat{G}(d)$ and G represent the same algebraic entities and may not be distinguished from each other. From the definition of the gain of the operator we have

$$g_\infty(\hat{G}) = \|\hat{G}\|_{\mathcal{A}} \tag{8.7}$$

$$g_2(\hat{G}) = \|\hat{G}\|_\infty \tag{8.8}$$

□

We conclude this section with the following theorem.

Theorem 8.1 [Vid., 5]: Given a lumped linear time-invariant system described by the transfer function $\hat{G}$, then the following statements are equivalent.

1. $\hat{G}$ is ℓ^∞-stable.
2. $\hat{G}$ is ℓ^1-stable.
3. $\hat{G}$ is ℓ^p-stable for all $p \in [1\ \infty]$.
4. $\hat{G} \in A$.

Furthermore, the ℓ^∞-induced norm on $\hat{G}$ bounds from above all other ℓ^p-induced norms, or equivalently

$$g_p(\hat{G}) \leq g_\infty(\hat{G}) = \|\hat{G}\|_{\bar{A}} \tag{8.9}$$

□

8.2.2 A brief review of the linear programming method

As it has already been mentioned in the introduction of this chapter, both optimal and robust FSTS reduce to the solution of a finite linear program. In this section we give the fundamentals of the finite linear programming optimization.

A **linear program (LP)**, is an optimization problem with linear objective function and linear constraints. If the number of variables and the number of constraints is finite, then the linear program is called a **finite linear program**. In this work we deal only with finite linear programs and so we will subsequently refer to them as linear programs. In 1947, D.G. Dantzing [Dan., 1] discovered the simplex algorithm for the solution of such problems. Its elegance and completeness make the LP the most appealing of all optimization techniques. The main reasons for the popularity of the simplex method seem to be its extreme efficiency and its ability to provide a complete solution to the optimization problem. As well as the optimal solution, the simplex algorithm contains useful information about the sensitivity of the solution to data variations, crucial when these data are known only imprecisely.

In the past 25 years, there has been a movement towards an abstract approach to optimization which has resulted to a better understanding of the optimization theory. A landmark in this area is the work of D.G. Luenberger [Lue., 1] and the most recent work is that of E.J. Anderson and P. Nash [And., 1] both encompassing finite and infinite dimensional problems. We give now the definition of a linear program.

Let A be a linear map from the linear vector space X to the linear vector space Z, b an element of Z and let c^* be a linear functional on X. The linear program is

$$\begin{aligned} \text{LP:} \quad & \text{minimize} && \langle x, c^* \rangle \\ & \text{subject to} && Ax = b, \\ & && x \in X,\ x \geq 0. \end{aligned} \tag{8.10}$$

In the case of real linear vector spaces the LP (8.10) becomes

$$\begin{aligned} \text{LP:} \quad & \text{minimize} && \underline{c}^t \underline{x} \\ & \text{subject to} && A\underline{x} = \underline{b}, \\ & && \underline{x} \in \mathbb{R}^n,\ \underline{x} \geq 0. \end{aligned} \tag{8.11}$$

where $\underline{c} \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and the $m \times n$ matrix A are given. The positivity constraint $\underline{x} \geq 0$ means that $x_i \geq 0$, $i = 1,\ldots,n$. The linear program described by equations (8.11) will be referred to as **standard form LP**.

Remark 8.2: LP (8.11) is an equality-constrained program. In the case of an inequality-constrained program we can transform the inequality constraints to equality form by adding **surplus** and/or **slack variables**. Thus the constraint

$\underline{Ax} \geq \underline{b}$ becomes $\underline{Ax} - \underline{z} = \underline{b}$, $\underline{z} \geq 0$, $\underline{z}$ being the surplus variables, and the constraint $\underline{Ax} \leq \underline{b}$ becomes $\underline{Ax} + \underline{z} = \underline{b}$, $\underline{z} \geq 0$, $\underline{z}$ being the slack variables.

□

Definition 8.2: For any linear program LP, we call $\underline{x}$ **feasible**, if $\underline{x}$ satisfies the constraints of LP, including any positivity constraints. The set of all feasible solutions of LP is denoted by F(LP), or just F, i.e.

$$F(\text{LP}) = \{\ \underline{x}: \underline{Ax} = \underline{b},\ \underline{x} \geq 0\ \}$$

A program for which at least one feasible solution exists is called **consistent** and a feasible solution that minimizes the objective function is called **optimal**.

□

Definition 8.3: A solution $\underline{x}$ to the constraints $\underline{Ax} = \underline{b}$ of the LP is called **basic**, if the number of non-zero components of $\underline{x}$ is no greater than the rank of A. $\underline{x}$ is called **non-degenerate**, if this number is equal to the rank of A, and **degenerate** otherwise.

□

Definition 8.4: A basic solution to the LP, which is also feasible is called a **basic feasible solution**.

□

We can state now the fundamental theorem of linear programming.

Theorem 8.2 [Lue., 2] ***(The Fundamental Theorem of LP)*:** Given a linear program LP in standard form, then whenever LP has an optimal solution one can be found among the basic feasible solutions of it.

□

We conclude this section by considering the case of free variables and the treatment of absolute values within the LP framework.

Remark 8.3: If one or more of the unknowns in the standard LP form is unconstrained in sign, the problem can be transformed to standard form as follows. Suppose that the restriction $x_1 \geq 0$ is not present. We can then write

$$x_1 = x_1^+ - x_1^- \ , \quad x_1^+, \ x_1^- \geq 0 \tag{8.12}$$

If we substitute $x_1^+ - x_1^-$ for x_1 everywhere in (8.11), the linearity of the problem is preserved and all the variables satisfy the nonnegative constrains. The problem is expressed in terms of the $n + 1$ variables x_1^+, x_1^-, x_2, ..., x_n. There is obviously a certain degree of redundancy introduced by (8.12). However, this does not hinder the simplex method of solution. Indeed, according to definition of basic variables and theorem (8.2), we can always keep either x_1^+ or x_1^- out of the basic solution when the other one appears in it. Hence, at least one of x_1^+, x_1^- can be always zero and we can write the absolute value of x_1

$$|x_1| = x_1^+ + x_1^- \tag{8.13}$$

We will use these useful comments extensively in the following sections.

□

8.2.3 Definition and solution of the optimal FSTS

The whole FSTS approach encompasses the design of a unity feedback system that guarantees finite settling time behaviour for step inputs but by no means tracks them or is time-optimal. In chapter (6) we treated the case of tracking and/or disturbance rejection of a family of signals and also we proposed time-optimal controllers within the family $\mathcal{F}(P)$ of the FSTS controllers. Here, we follow a different approach. Instead of seeking perfect tracking, we select the FSTS controller that minimizes the ℓ^1-norm of the steady state error, or the ℓ^∞-norm of a certain error vector; this may result of course in perfect tracking. We consider the case of step inputs but any other fixed bounded input can be treated similarly.

Consider the discrete linear unity feedback system of figure (8.1), where $P = N_pD_p^{-1} = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{1\times m}(d)$, $C = N_cD_c^{-1} = \tilde{D}_c^{-1}\tilde{N}_c \in \mathbb{R}^{m\times 1}(d)$ and (N_p, D_p), $(\tilde{D}_p, \tilde{N}_p)$, (N_c, D_c), $(\tilde{D}_c, \tilde{N}_c)$ are pairs of coprime polynomial matrices.

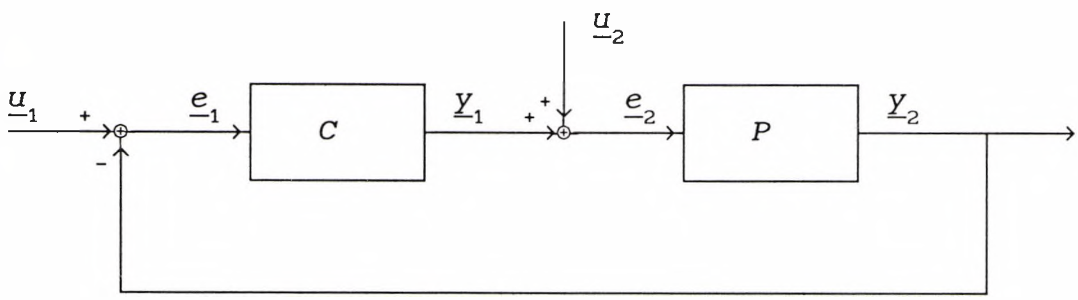

Figure (8.1): The MIMO unity feedback configuration

If the system is FST stabilized, then the error transfer function from $\underline{u}_1$ to $\underline{e}_1$ is

$$H_{11}(P,C) = D_c\tilde{D}_p \tag{8.14}$$

Suppose the scalar degrees of the polynomial matrices are $\partial_s(\tilde{D}_p) = \nu$ and $\partial_s(D_c) = n$. Then $\tilde{D}_p$, D_c may be written as

$$\tilde{D}_p = \tilde{D}_{p0} + \tilde{D}_{p1}d + \cdots + \tilde{D}_{p\nu}d^{\nu} \tag{8.15}$$

$$D_c = D_{c0} + D_{c1}d + \cdots + D_{cn}d^{n} \tag{8.16}$$

The error $\underline{e}_1$ due to a step at the input $\underline{u}_1$ of the form

$$\underline{u}_1 = \{0; \underline{u}_c, \underline{u}_c, \ldots\} = \sum_{i=0}^{\infty} \underline{u}_c d^i, \quad \underline{u}_c \in \mathbb{R}^1 \tag{8.17}$$

may be derived as follows. First we calculate the vector

$$\underline{u}_p = \tilde{D}_p\underline{u}_1 \tag{8.18}$$

This gives

$$\underline{u}_p = \underline{u}_{p0} + \underline{u}_{p1}d + \cdots + \underline{u}_{p\nu}(d^{\nu} + d^{\nu+1} + \cdots) \tag{8.19}$$

where

$$\underline{u}_{pk} = \sum_{i=0}^{k} \tilde{D}_{pi}\underline{u}_c, \quad k = 0,\ldots,\nu \tag{8.20}$$

Then according to (8.14) and (8.18)

$$\underline{e}_1 = D_c\underline{u}_p \quad \text{or} \quad \underline{e}_{1k} = \sum_{i+j=k} D_{ci}\underline{u}_{pj} \tag{8.21}$$

Therefore,

$$\begin{aligned} \underline{e}_{10} &= D_{c0}\underline{u}_{p0} \\ \underline{e}_{11} &= D_{c0}\underline{u}_{p1} + D_{c1}\underline{u}_{p0} \\ &\vdots \\ \underline{e}_{1k} &= (D_{c0} + D_{c1} + \cdots + D_{cn})\underline{u}_{p\nu}, \quad k \geq n + \nu \end{aligned} \tag{8.22}$$

Therefore, $\underline{e}_1$ reaches its steady state value in at most $n + \nu$ steps. For the rest of the analysis we define the following error vectors.

$$\underline{E}_1 := [\ \underline{e}_{10}^t \quad \underline{e}_{11}^t \quad \cdots \quad \underline{e}_{1(n+\nu)}^t\ ]^t \tag{8.23}$$

and

$$\underline{e}_{1,ss} := \underline{e}_{1(n+\nu)} = (D_{c0} + D_{c1} + \cdots + D_{cn})\underline{u}_{p\nu} \tag{8.24}$$

We can now consider the following optimization problems.

a. *Optimization Problem (I), ℓ^1-minimization*. Find an FSTS controller that minimizes the ℓ^1-norm $\|\underline{e}_{1,ss}\|_1$ of the steady state error for a given settling time.

b. *Optimization Problem (II), ℓ^∞-minimization*. Find an FSTS controller that minimizes the ℓ^∞-norm $\|\underline{E}_1\|_\infty$ of the error for a given settling time.

Let $L_P = [\tilde{N}_p \ \tilde{D}_p]$ be a left composite matrix of the plant, $R_C = [N_c^t \ D_c^t]^t$ be a right composite matrix of any FSTS controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$ and $k \geq k_{min}$ where k_{min} is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$. Then, for every $k = \partial_s(R_C) \geq k_{min}$, the solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$ is given by

$$T^l_{P,k+1} \begin{bmatrix} N_{co} \\ D_{c0} \\ \vdots \\ N_{ck} \\ D_{ck} \end{bmatrix} = \begin{bmatrix} I_l \\ O \\ \vdots \\ O \\ O \end{bmatrix} \tag{8.25}$$

where $T^l_{P,k+1}$ is the left Toeplitz plant matrix (chapter 6). If $\underline{e}^i$ is the ith column of the unity matrix I_l and $\underline{n}_{cji}$, $\underline{d}_{cji}$ is the ith column of N_{cj} and D_{cj} respectively, equation (8.25) becomes

$$\underbrace{\mathrm{diag}\{T^l_{P,k+1}, \ldots, T^l_{P,k+1}\}}_{\longleftarrow \ \ell \ \longrightarrow} \begin{bmatrix} \underline{n}_{c01} \\ \underline{d}_{c01} \\ \vdots \\ \underline{n}_{ckl} \\ \underline{d}_{ckl} \end{bmatrix} = \begin{bmatrix} \underline{e}^1 \\ \underline{O} \\ \vdots \\ \underline{e}^l \\ \underline{O} \end{bmatrix} \tag{8.26}$$

Rearranging the order of the solution vector in (8.26), i.e. reordering the columns of the block diagonal matrix in (8.26), we have

$$\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{y} \end{bmatrix} = \underline{b}_1 \tag{8.27}$$

where

$$\underline{y} = \begin{bmatrix} \underline{n}_{c01} \\ \underline{n}_{c11} \\ \vdots \\ \underline{n}_{c(k-1)l} \\ \underline{n}_{ckl} \end{bmatrix}, \quad \underline{z} = \begin{bmatrix} \underline{d}_{c01} \\ \underline{d}_{c11} \\ \vdots \\ \underline{d}_{c(k-1)l} \\ \underline{d}_{ckl} \end{bmatrix}, \quad \underline{b}_1 = \begin{bmatrix} \underline{e}^1 \\ \underline{0} \\ \vdots \\ \underline{e}^1 \\ \underline{0} \end{bmatrix} \tag{8.28}$$

The steady state error $\underline{e}_{1,ss}$ is given by (8.24) where $\underline{u}_{p\nu}$ is a known vector according to (8.20). If $\underline{r}^t_{cji}$ is the *i*th row of D_{cj}, equation (8.24) becomes

$$\begin{bmatrix} e_{1,ss1} \\ \vdots \\ e_{1,ssl} \end{bmatrix} = \begin{bmatrix} \underline{u}^t_{p\nu} & \cdots & \underline{u}^t_{p\nu} & & & 0 \\ & 0 & & \ddots & & \\ & & & \underbrace{\underline{u}^t_{p\nu} \quad \cdots \quad \underline{u}^t_{p\nu}}_{\leftarrow\ k+1\ \rightarrow} & & \end{bmatrix} \begin{bmatrix} \underline{r}_{c01} \\ \underline{r}_{c11} \\ \vdots \\ \underline{r}_{ck1} \\ \cdots \\ \vdots \\ \cdots \\ \underline{r}_{c0l} \\ \underline{r}_{c1l} \\ \vdots \\ \underline{r}_{ckl} \end{bmatrix} \tag{8.29}$$

By rearranging the order of the entries of the solution vector in (8.29), we have

$$A_{21}\underline{z} - \underline{e}_{1,ss} = \underline{0} \tag{8.30}$$

Also, according to definition of $\underline{E}_1$ in (8.23) and to equations (8.22) $\underline{E}_1$, like $\underline{e}_{1,ss}$, can be written as

$$\underline{E}_1 = \hat{A}_{21}\underline{z} \tag{8.31}$$

where $\hat{A}_{21}$ is a known real matrix and $\underline{z}$ is given by (8.28).

We can give now the solution to the two optimization problems *OP(I)* and *OP(II)*.

Theorem 8.3 *(ℓ^1-Optimal FSTS)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{l\times m}(d)$ be a left $\mathbb{R}[d]$-coprime MFD of the plant transfer function and $C = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ be a right $\mathbb{R}[d]$-coprime MFD of any FSTS controller. Let also $L_P = [\tilde{N}_p \ \tilde{D}_p]$ be the left composite matrix of the plant and $R_C = [N_c^t \ D_c^t]^t$ be the right composite matrix of the controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$. Then, there always exists an FSTS controller such that $\|\underline{e}_{1,ss}\|_1 = minimum$, if $k \geq k_{min}$ where k_{min} is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$.

Proof. For every $k \geq k_{min}$, equations (8.27) give the solution to the FSTS problem. Combining equations (8.27) and (8.30) together with the minimization of the ℓ^1-norm of $\underline{e}_{1,ss}$ we end up to the following problem.

$$\text{minimize} \qquad \|\underline{e}_{1,ss}\|_1$$

$$\text{subject to} \qquad \underbrace{\begin{bmatrix} A_{11} & A_{12} & O \\ A_{21} & O & -I_l \end{bmatrix}}_{:=A} \underbrace{\begin{bmatrix} \underline{z} \\ \underline{y} \\ \underline{e}_{1,ss} \end{bmatrix}}_{:=\underline{x}} = \underbrace{\begin{bmatrix} \underline{b}_1 \\ \underline{O} \end{bmatrix}}_{:=\underline{b}} \tag{8.32}$$

Since $\underline{x}$ is not necessarily positive, (8.32) reduces to the following linear program (see remark 8.3).

$$\begin{aligned} &\text{minimize} \qquad \sum_{i=1}^{l} (e^+_{1,ssi} + e^-_{1,ssi}) \\ &\text{subject to} \qquad [\,A \quad -A\,] \begin{bmatrix} \underline{x}^+ \\ \underline{x}^- \end{bmatrix} = \underline{b} \end{aligned} \tag{8.33}$$

□

Theorem 8.4 *(ℓ^∞-Optimal FSTS)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{l\times m}(d)$ be a left $\mathbb{R}[d]$-coprime MFD of the plant transfer function and $C = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ be a right $\mathbb{R}[d]$-coprime MFD of any FSTS controller. Let also $L_P = [\tilde{N}_p \ \tilde{D}_p]$ be the left composite matrix of the plant and $R_C = [N_c^t \ D_c^t]^t$ be the right composite matrix of the controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$. Then, there always exists an FSTS controller such that $\|\underline{E}_1\|_\infty$ $=$ *minimum*, if $k \geq k_{\min}$ where $k_{\min}$ is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_1$.

Proof. For every $k \geq k_{\min}$ the solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_1$ can be expressed by the system of equations (8.27). Since $\|\underline{E}_1\|_\infty = \max|E_{1i}|$, $i = 1,\ldots,\ell(k+\nu)$ the problem in consideration can be described as

$$\text{minimize} \qquad \max_i |E_{1i}|$$

$$\text{subject to} \qquad \begin{bmatrix} A_{11} & A_{12} & O \\ \hat{A}_{21} & O & -I_1 \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{y} \\ \underline{E}_1 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{O} \end{bmatrix} \tag{8.34}$$

where A_{11}, A_{12}, $\underline{z}$, $\underline{y}$, $\underline{b}_1$ are as in theorem (8.3) and $\hat{A}_{21}$ is given by equation (8.31). The above problem can be reduced to a standard linear program as follows (see also Späth [Spä., 1]). Let

$$\max_i |E_{1i}| = r \in \mathbb{R}_{\geq 0} \tag{8.35}$$

Then $-r \leq E_{1i} \leq r$, $i = 1,\ldots,\ell(k+\nu)$, and if the ith row of $\hat{A}_{21}$ denoted by $\hat{\underline{a}}^t_{21i}$, we have from (8.33)

$$\hat{\underline{a}}^t_{21i}\underline{z} + r \geq 0 \quad \text{and} \quad -\hat{\underline{a}}^t_{21i}\underline{z} + r \geq 0 \tag{8.36}$$

Hence, by introducing surplus variables $\underline{u}_1 \geq 0$, $\underline{u}_2 \geq 0$, and denoting by i_C the column vector with ones everywhere, problem (8.34) becomes

$$\text{minimize} \qquad r$$

$$\text{subject to} \qquad \begin{bmatrix} A_{11} & A_{12} & O & O & O \\ \hat{A}_{21} & O & i_C & -I_1 & O \\ -\hat{A}_{21} & O & i_C & O & -I_1 \end{bmatrix} \begin{bmatrix} \underline{z} \\ \underline{y} \\ r \\ \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{O} \end{bmatrix} \tag{8.37}$$

Since $\underline{z}$, $\underline{y}$ are not necessarily positive, we have finally the following linear program.

minimize r

subject to

$$\begin{bmatrix} A_{11} & -A_{11} & A_{12} & -A_{12} & 0 & 0 & 0 \\ \hat{A}_{21} & -\hat{A}_{21} & 0 & 0 & i_C & -I_1 & 0 \\ -\hat{A}_{21} & \hat{A}_{21} & 0 & 0 & i_C & 0 & -I_1 \end{bmatrix} \begin{bmatrix} \underline{z}^+ \\ \underline{z}^- \\ \underline{y}^+ \\ \underline{y}^- \\ r \\ \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} = \begin{bmatrix} \underline{b}_1 \\ \underline{0} \end{bmatrix} \tag{8.38}$$

□

Remark 8.4: The absolute minimum of the ℓ^1-norm $\|\underline{e}_{1,ss}\|_1$ of the steady state error is clearly zero, which corresponds to perfect tracking. Therefore, if for $k = \partial_s(R_C)$ there exists a tracking FSTS controller, this will be obtained as solution to the optimization problem (8.33) of theorem (8.3). The solution to this problem may not be unique and this may create convergence problems to the optimization algorithm. One way to aleviate these problems is to introduce more constraints to the LP problem as it will be demonstrated in the following section.

□

8.3 Error, Control and Transfer Function Shaping

In the previous section we rigorously demonstrated how two optimization problems, namely the minimization of the ℓ^1-norm of the steady state error and the minimization of the maximum error amplitude, can be transformed to normal finite linear programs. The essence of this transformation is the linearity of the signals involved with respect to controller parameters. We proved for instance, that the error signal

$$\underline{e}_1 = D_c \tilde{D}_p \underline{u}_1 \tag{8.39}$$

can be written as

$$\underline{E}_1 = \hat{A}_{21}\underline{z} \tag{8.40}$$

where $\underline{z}$ is the vector of parameters of D_c (see 8.28), $\hat{A}_{21}$ is a known real matrix and $\underline{u}_1$ is a step input of the form of (8.17). Similarly, the control signal

$$\underline{e}_2 = N_c\tilde{D}_p\underline{u}_1 \tag{8.41}$$

and the transfer function

$$W_{21}(P,C) = I - D_c\tilde{D}_p \tag{8.42}$$

from $\underline{u}_1$ to $\underline{y}_2$, can be expressed in the form of (8.40), where $\underline{z}$ represents the appropriate parameter vector. It is then natural , within the LP framework, to impose further shaping constraints on the error and/or the control signals, or the closed-loop transfer function (see also the recent work by McDonald and Pearson [McD., 1]).

If for a matrix A we denote by $|A|$ the matrix with elements the absolute values of the elements of A, and for any two matrices A, B of equal dimensions $A \le B$ means $a_{ij} \le b_{ij}$, then we can define the following problems.

a. *__Error Shaping Problem__*. Find an FSTS controller that minimizes $\|\underline{e}_{1,ss}\|_1$, or $\|\underline{E}_1\|_\infty$ and shapes the error signal $\underline{e}_1$ due to a step input $\underline{u}_1$, for a given settling time.

b. *__Control Shaping Problem__*. Find an FSTS controller that minimizes $\|\underline{e}_{1,ss}\|_1$, or $\|\underline{E}_1\|_\infty$ and shapes the control signal $\underline{u}_2$ due to a step input $\underline{u}_1$, for a given settling time.

c. *__Transfer Function Shaping Problem__*. Find an FSTS controller that minimizes $\|\underline{e}_{1,ss}\|_1$, or $\|\underline{E}_1\|_\infty$ and shapes the transfer function $W_{21}(P,C)$ for given settling time.

The following theorems give the answers to the aforementioned problems. The proofs are omitted since they follow in an identical manner from theorems (8.3) and (8.4).

Theorem 8.5 *(Error Shaping)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{l\times m}(d)$ be a left $\mathbb{R}[d]$-coprime MFD of the plant transfer function and $C = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ be a right $\mathbb{R}[d]$-coprime MFD of any FSTS controller. Let also $L_P = [\tilde{N}_p \;\; \tilde{D}_p]$ be the left composite matrix of the plant and $R_C = [N_c^t \;\; D_c^t]^t$ be the right composite matrix of the controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$. Then, the problem of minimizing the ℓ^1-norm of $\underline{e}_{1,ss}$, or the ℓ^∞-norm of $\underline{E}_1$ and shaping the error $\underline{e}_1$ as

$$|\underline{e}_{1i}| \le \underline{\varepsilon}_i \;, \quad \underline{\varepsilon}_i \text{ given}$$

can be reduced to a linear program if $k \ge k_{min}$, where k_{min} is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$.

□

Theorem 8.6 *(Control Shaping)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{l\times m}(d)$ be a left $\mathbb{R}[d]$-coprime MFD of the plant transfer function and $C = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ be a right $\mathbb{R}[d]$-coprime MFD of any FSTS controller. Let also $L_P = [\tilde{N}_p \;\; \tilde{D}_p]$ be the left composite matrix of the plant and $R_C = [N_c^t \;\; D_c^t]^t$ be the right composite matrix of the controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$. Then, the problem of minimizing the ℓ^1-norm of $\underline{e}_{1,ss}$, or the ℓ^∞-norm of $\underline{E}_1$ and shaping the control signal $\underline{e}_2$ as

$$|\underline{e}_{2i}| \le \underline{\varphi}_i \;, \quad \underline{\varphi}_i \text{ given}$$

can be reduced to a linear program if $k \ge k_{min}$, where k_{min} is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$.

□

Theorem 8.7 *(Transfer Function Shaping)*: Let $P = \tilde{D}_p^{-1}\tilde{N}_p \in \mathbb{R}^{l\times m}(d)$ be a left $\mathbb{R}[d]$-coprime MFD of the plant transfer function and $C = N_c D_c^{-1} \in \mathbb{R}^{m\times l}(d)$ be a right $\mathbb{R}[d]$-coprime MFD of any FSTS controller. Let also $L_P = [\tilde{N}_p \;\; \tilde{D}_p]$ be the left composite matrix of the plant and $R_C = [N_c^t \;\; D_c^t]^t$ be the right composite matrix of the controller with $\partial_s(L_P) = \nu$ and $\partial_s(R_C) = k$. Then, the problem of minimizing the ℓ^1-norm of $\underline{e}_{1,ss}$, or the ℓ^∞-norm of $\underline{E}_1$ and shaping the closed-loop transfer function $W_{21}(P,C)$ as

$$|W_{21}(P,C)| \le \Phi \ , \quad \Phi \text{ given}$$

can be reduced to a linear program if $k \ge k_{min}$, where k_{min} is the degree of any minimum column degree solution of the Diophantine equation $\tilde{N}_p X + \tilde{D}_p Y = I_l$.

□

Remark 8.5: Theorems (8.5) to (8.7) state that the shaping problem can be reduced to a linear program. The solvability of the shaping problem subject to the given constraints, comes from the solvability of the corresponding linear program. To this extent, the completeness of the solution of the simplex method, including the sensitivity analysis due to variation in the shaping constraints, can be fully exploited [And., 1], [Dan., 1].

□

8.4 Robust FSTS

It is clear from the analysis in chapters (5) and (6), that the main feature of FST stabilization is the placement of the poles of the closed-loop d-transfer function at infinity, or the eigenvalues of the closed-loop system at zero. This makes the FSTS design very sensitive to plant uncertainty. A naïve selection of R in the case of tracking for example (see theorem 6.11), may result as a system with poor performance against plant parameter variations and model inaccuracy. Since R is the free parameter that specifies the FSTS controller, and for the case of tracking R is specified not uniquely by the solution of the equation

$$Q\tilde{D}_r + N_p R\tilde{D}_p = Y\tilde{D}_p \tag{8.43}$$

we can select R for robust performance.

Zhao and Kimura [Zha., 1] to [Zha., 4] have considered the problem of robust output deadbeat control, as opposed to FST control. In this section, making full use of the norm properties of section (8.2.1), we introduce a different and more efficient robustness index to that used by Zhao and Kimura. This enables us to exploit the linearity of the formula (8.43) and to reduce the whole robustness problem to a linear program. In the sequel, we assume that the nominal plant perfectly tracks in FST sense, inputs with known dynamics and bounded ℓ^∞-norm, and that it is subjected to multiplicative perturbations.

Consider the unity feedback scheme of figure (8.1) and let P_0 denote the transfer function of the nominal plant and P the dynamics of the actual plant. Then, under multiplicative perturbation we may have

$$P - P_0 = \Delta P \cdot P, \quad \Delta P \ \ell^\infty\text{-stable and } g_\infty(\Delta P) = p \tag{8.44}$$

The nominal closed-loop transfer function from $\underline{u}_1$ to $\underline{y}_2$ is

$$G_0 := W_{21}(P_0, C) = P_0 C(I + P_0 C)^{-1} \tag{8.45}$$

and the perturbed closed-loop transfer function is

$$G := W_{21}(P, C) = PC(I + PC)^{-1} \tag{8.46}$$

After straightforward manipulation we have

$$G - G_0 = \Delta G \cdot G \tag{8.47}$$

where

$$\Delta G = (I - G_0)\Delta P \tag{8.48}$$

Therefore

$$g_p(\Delta G) \le g_p(I - G_0) g_p(\Delta P) \tag{8.49}$$

and since the induced ℓ^∞-norm bounds from above all the other induced norms we may choose as robustness index the ℓ^∞-norm of $I - G_0$, i.e.

$$\rho := g_\infty(I - G_0) \tag{8.50}$$

In the case of FST stabilization, $G_0 = I - D_c\tilde{D}_p$, and (8.50) becomes

$$\rho = g_\infty(D_c\tilde{D}_p) \tag{8.51}$$

The following theorem is the solution to robust FSTS.

Theorem 8.8 *(Robust FSTS)*: Consider the feedback configuration of figure (8.1) and let $P_0 = \tilde{D}_{p0}^{-1}\tilde{N}_{p0} = N_{p0}D_{p0}^{-1} \in \mathbb{R}^{1\times m}(d)$ $C = \tilde{D}_c^{-1}\tilde{N}_c = N_cD_p^{-1} \in \mathbb{R}^{m\times 1}(d)$ be left and right $\mathbb{R}[d]$-coprime MFDs of the nominal plant and controller respectively. If $\underline{u}_1 = \tilde{D}_r^{-1}\underline{\tilde{n}}_r$, $\|\underline{u}_1\|_\infty < \infty$ is a left MFD of the input to be tracked, then the robust FSTS to multiplicative perturbations can be described by the following linear program

$$\begin{aligned} &\text{minimize} \quad g_\infty((Y - N_pR)\tilde{D}_p) = \|(Y - N_pR)\tilde{D}_p\|_1 \\ &\text{subject to} \quad Q\tilde{D}_r + N_pR\tilde{D}_p = Y\tilde{D}_p \end{aligned} \tag{8.52}$$

for some particular $k = \partial_s(R)$ for which the equation

$$Q\tilde{D}_r + N_pR\tilde{D}_p = Y\tilde{D}_p \tag{8.53}$$

has a solution, and Y is a particular solution of the equation $\tilde{N}_pX + \tilde{D}_pY = I_l$.

<u>*Proof*</u>. If (X,Y) is a particular solution of the equation

$$\tilde{N}_pX + \tilde{D}_pY = I_l$$

then the family of FSTS controllers that track the input $\underline{u}_1 = \tilde{D}_r^{-1}\underline{\tilde{n}}_r$, is given by

$$N_c = X + D_pR, \quad D_c = Y - N_pR \tag{8.54}$$

where R satisfies eqn. (8.53). For robustness, if $k = \partial_s(R)$

$$\rho_k := g_\infty(D_c\tilde{D}_p) = g_\infty((Y - N_pR)\tilde{D}_p) = \|(Y - N_pR)\tilde{D}_p\|_1$$

must be minimum. This results to the optimization problem (8.52) which is a linear program according to the analysis of section (8.2.3).

□

Remark 8.6: Due to the nature of the linear programming, the optimal solution ρ_k^* for a particular $k = \partial_s(R)$ is a suboptimal solution to the optimization problem with $\ell = \partial_s(R) > k$. Therefore, ρ_k^* is a monotonically decreasing function of k, i.e.

$$\rho_l^* \leq \rho_k^*, \quad \ell \geq k$$

Hence, we can improve the robustness performance of the closed-loop system by increasing the settling time of its response. This is also true for the optimal FSTS and the shaping problems of sections (8.2) and (8.3).

□

Remark 8.7: The controller of theorem (8.8) does not necessarily guarantee stability of the perturbed closed-loop transfer function. Indeed, from equation (8.47) we have that

$$G = (I - \Delta G)^{-1} G_0$$

For stability $(I - \Delta G)^{-1}$ must be stable, and this is possible ([Des., 1]), if

$$g_\infty(\Delta G) = g_\infty((I - G_0)\Delta P) < 1$$

Therefore, a $k = \partial_s(R)$ must be chosen such that

$$g_\infty(I - G_0) = \textit{minimum} \quad \text{and} \quad g_\infty((I - G_0)\Delta P) < 1$$

□

Remark 8.8: If the plant is not strictly causal, the solution to the optimal, or robust FSTS does not guarantee a causal FSTS controller. Nevertheless, according to remark (6.2) a suboptimal causal FSTS controller can be found such that the objective function is as close to optimal as desired.

□

8.5 Two-Parameter FSTS

The FSTS analysis and design up to now, was based on the unity, or one-parameter feedback scheme of figure (8.1). Under this scheme, the FSTS controllers can be tuned with respect to one free parameter either R or S (theorem 6.1). This gives a considerable degree of freedom for accomplishing further performance requirements like tracking, disturbance rejection, optimization, or robustness but puts some limits in the case of multitask control which may result to be severe and to poor performance. For example, in the case of both tracking and disturbance rejection, a *'single'* parameter $R \in M(\mathbb{R}[d])$ determines both performance requirements.

In this section the unity feedback system is replaced by a more general scheme which employs a controller which is usually referred to as *two-degrees-of-freedom*, or *two-parameter* controller. This controller performs the most general linear time-invariant scheme, that is

$$\underline{e} = C_1\underline{u} - C_2\underline{y}\ , \quad C_1,\ C_2 \in M(\mathbb{R}^0(d)) \tag{8.55}$$

and is shown in figure (8.2).

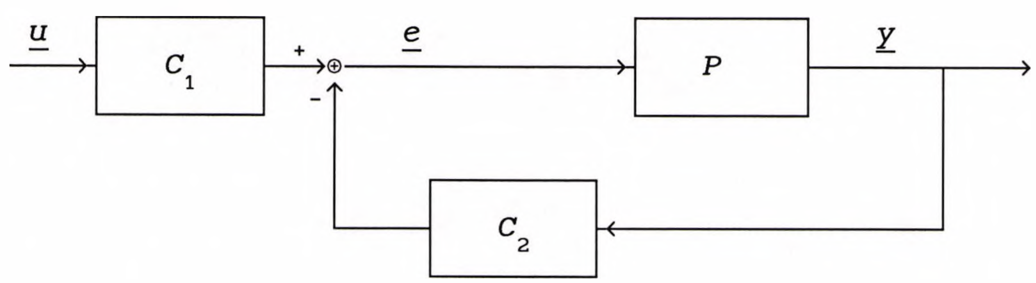

Figure (8.2): Infeasible implementation of a two-parameter controller

The implementation of figure (8.2) does not make sense for reasons of internal stability unless C_1 is stable. A feasible implementation of the two-parameter controller is shown in figure (8.3) where we assume that $C = [C_1\ \ C_2]$ and $(\tilde{D}_c, [\tilde{N}_{c1}\ \ \tilde{N}_{c2}])$ is a left $\mathbb{R}[d]$-coprime MFD of C. For a more extensive discussion of the two-parameter controller one is referred to Vidyasagar [Vid., 1] and references therein.

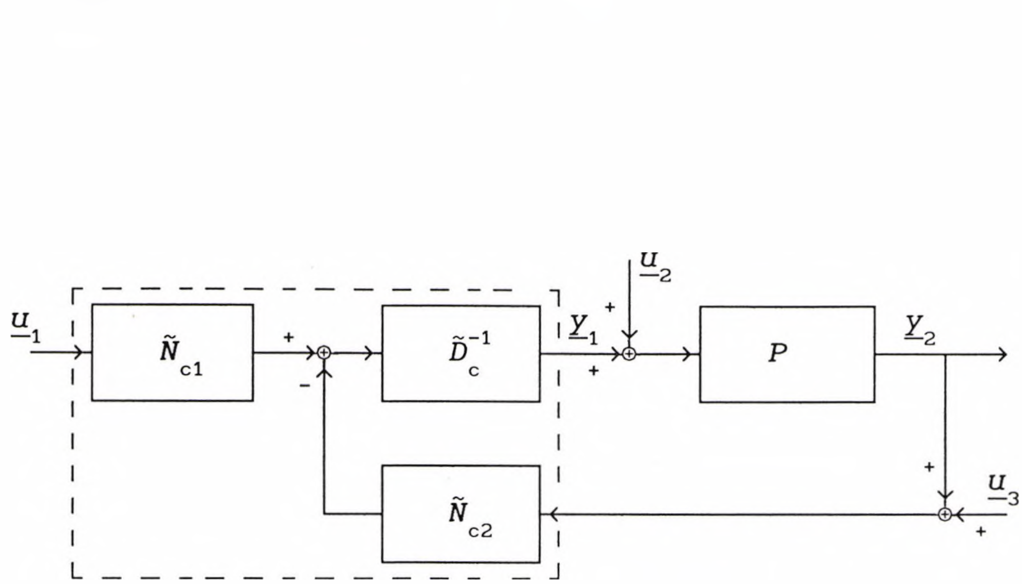

Figure (8.3): Feasible implementation of a two-parameter controller

8.5.1 Definition and solution of the two-parameter FSTSP

In this section we define the FSTS problem under the two-parameter control scheme. The results are generalizations of those of chapter (6), or may be derived in a manner similar to that given by Vidyasagar [Vid., 1].

Definition 8.5: Consider the feedback scheme of figure (8.3). The closed-loop system exhibits an FST response, if for a step change to any of the inputs $\underline{u}_1$, $\underline{u}_2$, or $\underline{u}_3$ all outputs $\underline{y}_1$, $\underline{y}_2$ settle to a new steady state in finite time.

□

Lemma 8.1: The closed-loop system of figure (8.3) exhibits an FST response if and only if the transfer function $W(P,C)$ from $\underline{u} = [\underline{u}_1^t\ \underline{u}_2^t\ \underline{u}_3^t]^t$ to $\underline{y} = [\underline{y}_1^t\ \underline{y}_2^t]^t$, is a polynomial matrix in d.

<u>*Proof*</u>. According to definition (8.5), $\underline{y}_j$, $j = 1,2$, must exhibit an FST response to a step input at $\underline{u}_i$, $i = 1,2,3$. This holds true, if and only if the transfer function from $\underline{u}_i$ to $\underline{y}_j$, $i = 1,2,3$, $j = 1,2$, is polynomial in d (lemma 5.1), i.e., if $W(P,C)$ is a polynomial matrix in d.

□

Theorem 8.9: Consider the two-parameter feedback scheme of figure (8.3) and let $P \in \mathbb{R}^{1\times m}(d)$ be a given plant transfer function. Suppose (N_p, D_p), $(\tilde{D}_p, \tilde{N}_p)$ are any right, left $\mathbb{R}[d]$-coprime MFDs of P, $(\tilde{D}_c, [\tilde{N}_{c1} \ \ \tilde{N}_{c2}])$ is a left $\mathbb{R}[d]$-coprime MFD of $C = [C_1 \ \ C_2]$, and that $\tilde{X}$, $\tilde{Y}$ satisfy the Diophantine equation $\tilde{X}N_p + \tilde{Y}D_p = I$. Then the closed-loop system of figure (8.3) is FST-stable, if and only if

$$\Delta := \tilde{N}_{c2}N_p + \tilde{D}_cD_p \in U(\mathbb{R}[d]) \tag{8.56}$$

Moreover, the set $\mathcal{F}_2(P)$ of all causal two-parameter FSTS controllers is given by

$$\mathcal{F}_2(P) = \{ (\tilde{Y}-S\tilde{N}_p)^{-1}[T \ \ \tilde{X}+S\tilde{D}_p], \ S, \ T \in M(R[d]) \text{ and } |\tilde{Y}(0)-S(0)\tilde{N}_p(0)| \neq 0 \text{ if } \tilde{N}_p(0) \neq 0\} \tag{8.57}$$

The set of all transfer matrices $W(P,C)$ is of the form

$$W(P,C) = \begin{bmatrix} D_pT & D_p(\tilde{Y}-S\tilde{N}_p)-I & -D_p(\tilde{X}-S\tilde{D}_p) \\ N_pT & N_p(\tilde{Y}-S\tilde{N}_p) & -N_p(\tilde{X}-S\tilde{D}_p) \end{bmatrix} \tag{8.58}$$

Proof. The tranfer function $W(P,C)$ can be written as

$$W(P,C) = N_rD^{-1}\tilde{N}_1$$

where

$$N_r = \begin{bmatrix} I & O \\ O & N_p \end{bmatrix}, \quad D = \begin{bmatrix} I & -D_p \\ \tilde{D}_c & \tilde{N}_{c2}N_p \end{bmatrix}, \quad \tilde{N}_1 = \begin{bmatrix} O & -I & O \\ \tilde{N}_{c1} & O & \tilde{N}_{c2} \end{bmatrix}$$

with (N_r, D) right $\mathbb{R}[d]$-coprime and $(D, \tilde{N}_1)$ left $\mathbb{R}[d]$-coprime [Vid., 1]. For $W(P,C)$ to be polynomial, D must be unimodular in $\mathbb{R}[d]$. This is true, if and only if

$$\Delta := \tilde{N}_{c2}N_p + \tilde{D}_cD_p \in U(\mathbb{R}[d]), \quad \text{[Vid., 1]}$$

The parametrization of the family of FSTS controllers is derived in a straightforward manner as a solution of the Diophantine equation (8.56).

□

Remark 8.9: Equation (8.58) shows that $W(P,C)$ depends on two free parameters S and T and this is the reason that the FSTS controller of figure (8.3) is called a two-parameter controller as opposed to the one-parameter controller of figure (8.1). Thus the two-parameter scheme offers greater flexibility in that the transfer matrix from $\underline{u}_1$ to the outputs can be adjusted independently from that between $\underline{u}_2$ and the outputs. This is not so, in the case of the unity feedback configuration. This flexibility is illustrated in the next section where we design FSTS controllers for tracking and disturbance rejection.

□

8.5.2 Two-parameter FSTS for tracking and disturbance rejection

In this section we show how we can exploit the flexibility of the two-parameter FSTS controllers in order to achieve tracking and disturbance rejection of a family of signals in FST sense, i.e. the performance requirements are accomplished in finite time. The proofs of the results are similar to the one-parameter case (theorems (6.11) and (6.12)) and are omitted.

Theorem 8.10 *(Two-Parameter FST Tracking)*: Let (P,C) be an FST-stable pair in the feedback system of figure *(8.3)* and $\underline{u}_1$ be expressed by a left $\mathbb{R}[d]$-coprime MFD as $\underline{u}_1 = \tilde{D}_r^{-1}\underline{\tilde{n}}_r$. Then $\underline{y}_2$ tracks the reference signal $\underline{u}_1$ in FST sense, if and only if there are T, $W \in M(\mathbb{R}[d])$ such that

$$N_pT + W\tilde{D}_r = I \tag{8.59}$$

where all the matrices involved apart from W are as in theorem (8.9).

□

Theorem 8.11 *(Two-Parameter FST Disturbance Rejection)*: Let (P,C) be an FST-stable pair in the feedback system of figure (8.3) and $\underline{u}_2$ be expressed by a left $\mathbb{R}[d]$-coprime MFD as $\underline{u}_2$ =

$\tilde{D}_d^{-1}\tilde{n}_d$. Then $\underline{u}_2$ is rejected at the output $\underline{y}_2$ in FST sense, if and only if there are S, $W \in M(\mathbb{R}[d])$ such that

$$W\tilde{D}_d + N_p S\tilde{N}_p = N_p\tilde{Y} \tag{8.60}$$

where all the matrices involved apart from W are as in theorem (8.9).

□

It is clear from theorems (8.10) and (8.11), that FST tracking and disturbance rejection can be acheived independently of each other by tuning the two free parameters T and $S \in M(\mathbb{R}[d])$ of the FSTS controller. This is the main advantage of the two-parameter-controllers over the one-parameter controllers.

8.6 Conclusions

In this final chapter we considered some advanced design problems for FST stabilization. On one hand we dealt with the problems of optimization, shaping and robustness within the framework of the unity feedback configuration used throughout this thesis. On the other hand we introduced more flexibility in the FST stabilization problem by using a two-parameter control scheme.

In particular, we demonstrated how within the unity feedback scheme the problems of

minimization of the ℓ^1-norm of the steady state error

minimization of the ℓ^∞-norm of the error

shaping of the error, control or transfer function

robustness to multiplicative plant perturbations

can be transformed to finite linear programs where all the benefits (effectiveness, efficiency and ability to provide complete solution to the optimization problem) of the simplex method can be exploited.

In the final section we have replaced the unity feedback scheme by a two-parameter control scheme for FSTS purposes. Under this more flexible configuration, the FSTS problem was redefined and the parametrization of the family $\mathcal{F}_2(P)$ of all causal two-parameter FSTS controllers was derived. The flexibility of the two-degrees-of freedom FSTS compensation was further demonstrated in the case of FST tracking and disturbance rejection where tracking and disturbance rejection can be affected independently by two distinct parameters T and $S \in M(\mathbb{R}[d])$.

Further problems like strong stabilization, simultaneous stabilization, optimization and robustness can be treated in an analogous way to the one-parameter case. There are indications that the minimal design problem can be solved completely in the case of two-parameter compensation, and this is a subject of further research.

Chapter 9

CONCLUSIONS

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CONCLUSIONS

The motivation for this thesis was to provide a generalization and a unification of the deadbeat control problem. Deadbeat response is a fascinating and unique feature of linear discrete-time systems and has intrigued many engineers over the last forty years. Most of the work in deadbeat control has been in the state-space set up and focused on a variety of versions of deadbeat, using constant state feedback or state estimation and then constant feedback of the estimated states. By reassessing Mullis's work [Mul., 1] on discrete time-optimal control, we were able to generalize to the MIMO case Kalman's theorem [Kal., 2] about the inevitability of the use of constant state feedback and to give a complete parametrization of the family of state deadbeat regulators. On the other hand, the transfer function approach has by its nature, the advantages of parametrization, as it has been shown by Kucera's work based on polynomial equations, but again his work was focused on deadbeat and thus examined special type of control problems.

In this thesis, we have adopted the viewpoint that deadbeat response is a special case of the finite settling time stabilization problem. An algebraic formulation of this problem, which also guarantees internal finite settling time behaviour provides a unification of existing results. Moreover, the natural parametrization of solutions associated with polynomial matrix equations permits McMillan degree parametrizations, shows links with deadbeat and minimum McMillan degree solutions and provides a clear treatment for a variety of performance related FSTS problems.

Indeed, the algebraic (polynomial equation) approach has been extremely beneficial and powerful for a unifying treatment of the FSTS problem. The review and refinement of the concept of sequences (formal series) in one indeterminate d over $\mathbb{R}$, and the isomorphism between them and the corresponding series expansions of functions, has provided a better understanding of the nature of the input-output description of discrete linear systems by enabling us to treat the impulse responses and transfer functions as identical algebraic objects - a feature which has no counterpart in the continuous-time case.

This mathematical framework within the one-parameter (unity) feedback compensation and the requirement that both the error and control sequences settle to their steady state values in finite time for a step change to any of the system inputs, leads to the solution of a polynomial matrix Diophantine equation of the form $AX + BY = I$, which guarantees not only ***internal stability*** but ***internal FSTS*** - both features missing in many of the previouly tried approaches. In addition, the reduction of the FSTS problem to the solution of a unilateral Diophantine equation, naturally results in a Youla-Bongiorno-Kucera parametrization of the family of all FSTS controllers in an affine manner with respect to a ***'free'*** polynomial matrix.

It is worth noting, that the FSTS problem can be considered as a special case of stabilization where the ring of polynomials in d replaces $\mathbb{R}\mathbb{H}^{\infty}$ in Vidyasagar's approach [Vid., 1]. The forbidden region for stability becomes the entire complex plane and consequently the stability region reduces to the point at infinity. Although the set of unstable points is not closed (essential in Vidyasagar's treatment), it does not lead to convergence problems if the norm induced by the disc algebra A_s is used. This enabled us to derive the conditions for ***strong FSTS*** and to prove that the well known ***parity interlacing property*** [Vid., 1], [You., 2] is valid in the FSTS case as well, where the domains of stability of the controller (exterior of $\mathbb{D}[0,1)$) and of the feedback system (point at infinity) differ from each other.

A further reduction of the solution of the polynomial matrix Diophantine equation to a linear algebra problem over $\mathbb{R}$ together with the YBK parametrization of FSTS controllers leads to the characterization of the solutions according to column/row degrees. This enables us to obtain upper and lower bounds of the minimum McMillan degree FSTS controllers and to characterize the family of FSTS controllers according to upper and lower bounds of their McMillan degrees. Moreover, the state deadbeat regulation problem comes naturally as a special case of the FSTS problem (minimum column complexity solutions), and a complete parametrization of the family of all deadbeat regulators follows readily. In addition, within the same FSTS framework, further performance criteria and design constraints may be imposed such as, tracking and/or disturbance rejection, partial assignment of controller dynamics, ripple-free response, ℓ^1-, ℓ^∞-optimization, shaping, and robustness to parameter variations.

In the cases of optimization, shaping, and robustness, the problems are reduced to finite linear programming problems with the finite settling time serving as a design parameter and with the optimal/robust controller being the outcome of the solution. This approach differs distinctively from the approaches adopted so far, in the following manner.

In the case of the state-space set up (section 4.3.4), the linear programming solutions provide the optimum control sequence (and not the controller parameters), and therefore they constitute an open-loop control strategy. In order to obtain a closed-loop control law the linear programming optimization has to be performed on-line for each sampling instance using the results of the previous run as initial conditions. This clearly leads to a considerably costly control strategy and to the need for a search of time-efficient linear programming methods.

In the case of the algebraic approach [Dah., 1] to [Dal., 4], the YBK parametrization over the ring $\mathbb{R}^+(d)$ of stable sequences leads to a semi-infinite linear program. This semi-infinite linear programming problem is further reduced to a finite linear program by truncation, but the truncating parameter is not directly related to a performance criterion like the finite settling time.

Finally, within the unity feedback FSTS framework, the ***Simultaneous-FSTS*** problem has been formulated and computationally verifiable necessary and sufficient conditions for the case of families of vector plants have been given. It has been shown that the so-called ***Space Structure Condition (SSC)*** is the most important condition to be met for the solution of the MIMO S-FSTS problem. The existence of testable criteria for the SSC is still an open issue. Alternatively, necessary or sufficient testable conditions for the solution of the general S-FSTS problem have been provided.

This work presented a coherent approach for the solution of the Total FSTS problem and a unifying framework for the solution of a complete variety of FSTS related problems. However, there are two areas, one of theoretical and the other of numerical nature, that are in need of further investigation.

In particular, the MIMO minimal FSTS design problem, the parametrization of MIMO FSTS controllers according to McMillan degree, and the derivation of computationally verifiable criteria for the SSC in the general S-FSTS case remain still open and under investigation. In addition, the numerical aspects of the solution of the FSTS problem are of extreme importance. The reduction of the solution of the Diophantine equation to a Toeplitz type linear algebra problem over $\mathbb{R}$ is only a partial answer to this problem. Some work has been done concerning the scalar case [Kuc., 15]. However, very little work, of experimental nature, has taken place regarding the solution of the general matrix polynomial equation [Kra., 1] and at least a theoretical comparison of the existing algorithms is essential.

Finally, as an alternative design procedure, a two-parameter feedback scheme has been introduced and the FSTS problem has been defined and solved within this framework. The family of all two-parameter FSTS controllers has been completely parametrized in an affine manner with respect to two *'free'* polynomial matrices, and the controllers that guarantee tracking and disturbance rejection in FST sense have been provided. The superiority of the two-parameter feedback scheme has been demonstrated in this case, where the FST tracking and disturbance rejection controllers can be tuned with respect to two independent parameters. Further performance requirements can be tackled in a similar manner to the one-parameter FST case.

REFERENCES

[Abd 1] Abdel-Nour G.M. and Mulholland R.J.,
Algorithm for Deadbeat Tracking.
IEEE Trans. Auto. Control, AC-34, No. 7, July 1989, pp. 773-775.

[Ada 1] Adachi M. and Akashi H.,
Minimal Order of Minimal-Time Deadbeat Function Observers.
Int. J. Control, Vol. 35, No. 2, 1982, pp. 355-365.

[Ack 1] Ackerman J.,
Zeitoptimale Mehrafach-Abtastregelsysteme.
Preprint IFAC Symp. Multiv. Control Systems Band 1, Düsseldorf, 1968.

[Aka 1] Akashi H., Imai H. & Adachi M.,
On Time Optimal Control of Linear Discrete-Time Systems by Geometric Approach.
Int. J. Control, Vol. 28, No. 6, 1978, pp. 853-867.

[Aka 2] Akashi H. and Adachi M.,
Minimum Time Output Regulation Problem of Linar Discrete-Time Systems.
IEEE Trans. Auto. Control, AC-22, No. 6, Dec. 1977, pp. 939-942.

[Aka 3] Akashi H. and Imai H.,
Output Deadbeat Controllers with Stability for Discrete-Time Multivariable Linear Systems.
IEEE Trans. Auto. Control, AC-23, No. 6, Dec. 1978, pp. 1036-1043.

[Aka 4] Akashi H. and Imai H.,
Disturbunce Localization and Output Deadbeat Control Through an Observer in Discrete-Time Linear Multivariable Systems.
IEEE Trans. Auto. Control, AC-24, No. 4, Aug. 1979, pp. 621-627.

[Alo 1] Alos A.,
Stabilization of a Class of Plants with Possible Loss of Outputs or Actuator Failures.
IEEE Trans. Auto. Control, AC-28, No. 2, Feb. 1982, pp. 231-233.

[Ami 1] Amin M.H. and Elabdalla A.M.,
Parametrization of a Class of Deadbeat Controllers Via the Theory of Decoupling.
IEEE Trans. Auto. Control, AC-33, No. 12, Dec. 1988, pp. 1185-1188.

[And 1] Anderson E.J. and Nash P.,
Linear Programming in Infinite-Dimensional Spaces
John Wiley & Sons Ltd., 1987.

[Bar 1] Barnett S.,
Polynomials and Linear Control Systems
Dekker, New York, 1983.

[Bar 2] Barmish B.R. and Wei K.H.,
Simultaneous Stabilizability of Single Input Single Output Systems.
Modeling, Identification and Robust Control, The Netherlands: North-Holland, Amsterdam, 1986.

[Bel 1] Bellman R.,
Introduction to Matrix Analysis
McGraw-Hill, New York, 1960.

[Ber 1] Bergen A.R. and Ragazzini J.R.,
Sampled-Data Techniques for Feedback Control Systems.
AIEE Trans., Vol. 73(II), 1954, pp. 236-247.

[Bit 1] Bitmead R.R., Kung S.-Y., Anderson B.D.O. & Kailath T.,
Greatest Common Divisors via Generalized Sylvester and Bezout Matrices.
IEEE Trans. Auto. Control, AC-23, No. 6, Dec. 1978, pp. 1043-1047.

[Blo 1] Blondel V., Gevers M., Mortini R. & Rupp R.,
Simultaneous Stabilization of Three or More Plants: Conditions on the Positive Real Axis do not Suffice.
SIAM J. Control and Optimization, Vol. 32, No. 2, March 1994, pp. 572-590.

[Bra 1] Bradshaw A. and Porter B.,
Design of Linear Multivariable Discrete-Time Tracking Systems for Plants with Inaccessible States.
Int. J. Control, Vol. 24, No. 2, 1976, pp. 275-281.

[Bru 1] Brunovský P.,
A Classification of Linear Controllable Systems.
Kybernetika, Vol. 6, 1970, pp. 173-187.

[Brü 2] A New Result in Time-Optimal Control of Discrete Systems.
IEEE Trans. Auto. Control, Oct.1974, pp. 597-598.

[Bur 1] Burton D.M.,
A First Course in Rings and Ideals
Addison Wesley, Reading, Massachusetts, 1970.

[Cad 1] Cadzow J.A.,
Nilpotency Property of the Discrete Regulator.
IEEE Trans. Auto. Control, AC-13, Dec. 1968, pp. 734-735.

[Cal 1] Callier F.M. and Desoer C.A.,
Multivariable Feedback Systems
Springer-Verlag, New York, 1982.

[Cal 2] Callier F.M. and Desoer C.A.,
Linear System Theory
Springer-Verlag, New York, 1991.

[Cel 1] Cellier F.E.,
Continuous System Modeling
Springer-Verlag, New York, 1991.

[Cha 1] Chang J.,
Robust Finite Settling Time Stabilization of Discrete Time Systems with Structured Uncertainty.
Systems Contr. Lett., Vol. 16, 1991, pp. 49-58.

[Cha 2] Chang B.C. and Pearson J.B.,
Optimal Disturbance Rejection in Linear Multivariable Systems.
IEEE Trans. Auto. Control, AC-29, Oct. 1984, pp. 231-233.

[Cha 3] Chammas A.B. and Leondes C.T.,
On the Finite Time Control of Linear Systems by Piecewise Constant Output Feedback.
Int. J. Control, Vol. 30, No. 2, 1979, pp. 227-234.

[Che 1] Chen C.T.,
Linear System Theory and Design
Holt-Saunders International Editions, Japan, 1984.

[Che 2] Chen B.-S., Chiang C.-C. & Hsiao F.-H.,
Deadbeat Controller Synthesis: Multivariable Case.
Int. J. Control, Vol. 40, No. 6, 1984, pp. 1207-1213.

[Chu 1] Churchill R.V., Brown J.W. & Verhey R.F.,
Complex Variables and Applications
McGraw-Hill Inc., 3rd ed., 1974.

[Coh 1] Cohn A.,
Über die Anzahl der Wurzeln einer Algebraischen Gleichung in einem Kreise.
Math. 2, Vol. 14, August 1922, pp. 110-148.

[Cor 1] Corsetti C.D., and Houpis C.H.,
An Algebraic Approach to the Time-Optimal Output Regulator.
Int. J. Control, Vol. 41, No. 3, 1985, pp. 641-652.

[Cro 1] Crossley T.R. and Porter B.,
Dead-Beat Control of Sampled-Data Systems With Bounded Input.
Int. J. Control, Vol., 19, No. 5, 1974, pp. 869-876

[Dah 1] Dahleh M.A. and Pearson J.B.,
ℓ^1 Optimal Feedback Controllers for Discrete-Time Systems.
Proc. ACC, Seattle WA, June 1986, pp. 1964-1968.

[Dah 2] Dahleh M.A. and Pearson J.B.,
ℓ^1 Optimal Feedback Controllers for MIMO Discrete-Time Systems.
IEEE Trans. Auto. Control, AC-32, Apr. 1987, pp. 314-322.

[Dah 3] Dahleh M.A. and Pearson J.B.,
L^1 Optimal Compensators for Continuous-Time Systems.
IEEE Trans. Auto. Control, AC-32, Oct. 1987, pp. 889-895.

[Dah 4] Dahleh M.A. and Pearson J.B.,
Optimal Rejection of Persistent Disturbances, Robust Stability, and Mixed Sensitivity Minimization.
IEEE Trans. Auto. Control, AC-33, No. 8, Aug. 1988, pp. 722-731.

[Dah 5] Dahleh M.A. and Pearson J.B.,
Minimization of a Regulated Response to a Fixed Input.
IEEE Trans. Auto. Control, AC-33, No. 10, Oct. 1988, pp. 924-930.

[Dan 1] Dantzing G.,
Linear Programming and Extensions
Princeton University Press, Princeton N.J., 1963.

[Des 1] Desoer C.A. and Vidyasagar M.,
Feedback Systems: Input-Output Properties
New York Academic, 1975.

[Des 2] Desoer C.A., Liu R.W., Murray J.W. & Saeks R.,
Feedback Systems Design: The Fractional Representation Approach to Analysis and Synthesis.
IEEE Trans. Auto. Control, AC-25, No. 3, 1980, pp. 399-412.

[Dor 1] Doratto P., Park H. & Li Y.,
An Algorithm for Interpolation with Units in H^∞, with Applications to Feedback Stabilization.
Automatica, Vol. 25, No. 3, 1989, pp. 427-430.

[Eic 1] Eichstaedt B.,
Multivariable Closed-Loop Deadbeat Control: a Polynomial-Matrix Approach.
Automatica, Vol. 18, 1982, pp. 589-593.

[Ela 1] Elabdalla A.M. and Amin M.H.,
Minimum-Gain Minimum-Time Deadbeat Controllers.
Systems Contr. Lett., Vol. 11, 1988, pp. 213-219.

[Ema 1] Emami-Naeini A. and Franklin G.F.,
Deadbeat Control and Tracking of Discrete-Time Systems.
IEEE Trans. Auto. Control, AC-27, No. 1, Feb. 1982, pp. 176-181.

[Emr 1] Emre E. and Silverman L.M.,
The Equation $XR + QY = \Phi$: A Characterization of solutions.
SIAM J. Control and Optimization, Vol. 19, No. 1, Jan. 1981, pp. 33-38.

[Emr 2] Emre E.,
Simultaneous Stabilization with Fixed Closed-Loop Characteristic Polynomial.
IEEE Trans. Auto. Control, AC-28, No. 1, Jan. 1983, pp. 103-104.

[Fah 1] Fahmy M.M., Hanafy A.A.R. & Sakr M.F.,
On the Discrete Time-Optimal Regulator Control Problem.
Information and Control, Vol. 44, 1980, pp. 223-235.

[Fah 2] Fahmy M.M. and O'Reilly J.,
Dead-Beat Control of Linear Discrete-Time Systems.
Int. J. Control, Vol. 37, No. 4, 1983, pp. 685-705.

[Far 1] Farison J.B. and Fu F.C.,
The Matrix Properties of Minimum-Time Discrete Linear Regulator Control.
IEEE Trans. Auto. Control, AC-15, June 1970, pp. 390-391.

[For 1] Forney G.D. Jr.,
Minimal Bases of Rational Vector Spaces, with Applications to Multivariable Linear Systems.
SIAM J. Control, Vol. 13, 1975, pp. 493-520.

[Fra 1] Francis B.A., Helton J.W. & Zames G.,
H^{∞}-Optimal Feedback Controllers for Linear Multivariable Systems.
IEEE Trans. Auto. Control, AC-29, No. 10, Oct. 1984, pp. 888-900.

[Fra 2] Francis B.A.,
A Course in H^{∞} Control Theory
Springer-Verlang, New York, 1987.

[Fun 1] Funahashi Y. and Yamada M.,
Explicit Parametrization of State Deadbeat Controllers.
IEEE Trans. Auto. Control, AC-37, No. 10, Oct. 1982, pp. 1584-1588.

[Gan 1] Gantmakher F.R.,
Theory of Matrices Vols. I and II
Chelsea Publicing Co., New York, 1977.

[Gev 1] Gevers M.R.,
ARMA Models, their Kronecker Indices and their McMillan Degree.
Int. J. Control, Vol. 43, No. 6, 1986, pp. 1745-1761.

[Gho 1] Ghosh B.K.,
Transcendental and Interpolation Methods in Simultaneous Stabilization and Simultaneous Partial Pole Placement Problems.
SIAM J. Control and Optimization, Vol. 24, 1986, pp. 1091-1109.

[Gho 2] Ghosh B.K. and Byrnes C.I.,
Simultaneous Stabilization and Pole-Placement by Nonswitching Dynamic Compensation.
IEEE Trans. Auto. Control, AC-28, No. 6, June 1983, pp. 735-741.

[Gho 3] Ghosh B.K.,
Some New Results on the Simultaneous Stabilizability of a Family of Single Input Single Output Systems.
Systems Contr. Lett., Vol. 6, 1985, pp. 39-45.

[Gho 4] Ghosh B.K.,
An Approach to Simultaneous System Design, Part I: Semialgebraic Geometric Methods.
SIAM J. Control and Optimization, Vol. 24, 1986, pp. 480-496.

[Gho 5] Ghosh B.K.,
An Approach to Simultaneous System Design, Part II: NonSwitching Gain and Dynamic Feedback Compensation by Algebraic Geometric Methods.
SIAM J. Control and Optimization, Vol. 26, 1988, pp. 919-963.

[Gil 1] Gilbert E.G.,
Controllability and Observability in Multivariable Control Systems.
SIAM J. Control, Vol. 1, 1963, pp. 128-151.

[God 1] Godement R.,
Algebra
Hermann, Paris, 1968.

[Gut 1] Gutman P.-O. and Cwikel M.,
An Algorithm to Find Maximal State Constraint Sets for Discrete-Time Linear Dynamical Systems with Bounded Controls and States.
IEEE Trans. Auto. Control, AC-32, No. 3, March 1987 pp. 251-254.

[Har 1] Hartley B. and Hawkes T.O.,
Rings, Modules and Linear Algebra
Chapman and Hall Ltd., 1987.

[Hau 1] Hautus M.L.J.,
Controllability and Observability conditions of Linear Autonomous Systems.
Nederl. Akad. Wetensch. Proc., Ser. A, Vol. 72, 1969, pp. 443-448.

[Hei 1] Heinig G. and Rost K.,
Algebraic Methods for Toeplitz-like Martices and Operators
Birkhäuser Verlag, 1984.

[Hel 1] Helton J.W.,
Worst Case Analysis in the Frequency Domain: The H^{∞} Approach to Control.
IEEE Trans. Auto. Control, AC-30, No. 12, Dec. 1985, pp. 1154-1170.

[Ich 1] Ichikawa K.,
Design of Discrete-Time Deadbeat Reduced-Order State Observer.
Int. J. Control, Vol., 29, 1979, pp. 93-101.

[Ino 1] Inoue A., Fisher D.G., Shah S.L. & Seborg D.E., Minimal-Time, Minimal-Order Observers for Discrete Systems.
Int. J. Control, Vol. 30, No. 4, 1979, pp. 597-615.

[Ise 1] Isermann R.,
Digital Control Systems, Vol. I
Springer-Verlag, Berlin Heidelberg, 1989.

[Jan 1] Janssen P.H.M.,
General Results on the McMillan Degree and the Kronecker indices of ARMA and MFD Models.
Int. J. Control, Vol. 48, No. 2, 1988, pp. 591-608.

[Jan 2] Jan Y.-S.,
Deadbeat Response to Time Polynomial Inputs.
Proc. IEEE, Vol. 65, No. 12, Dec. 1977, pp. 1728-1730.

[Jor 1] Jordan D. and Korn J.,
Deadbeat Algorithms for Multivariable Process Control.
IEEE Trans. Auto. Control, AC-25, No. 3, June 1980, pp. 486-491.

[Jur 1] Jury E.I.,
Theory and Application of the z-Transform Method
Robert E. Krieger Publishing Company Inc., Florida, 1982.

[Jur 2] Jury E.I. and Blanchard J.,
A Stability Test for Linear Discrete Systems in Table Form.
Proc. IRE, Vol. 44, Dec. 1961, pp. 1947-1948.

[Kai 1] Kailath T.,
Linear Systems
Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1980.

[Kal 1] Kalman R.E., Falb P.L. & Arbib M.A.,
Topics in Mathematical System Theory
McGraw-Hill Inc., New York, 1969.

[Kal 2] Kalman R.E.,
On The General Theory of Control Systems.
Proc. 1st IFAC Congress, Moscow, 1969, pp. 481-491.

[Kal 3] Kalman R.E.,
Mathematical Description of Linear Systems.
SIAM J. Control, Vol. 1, 1963, pp. 152-192.

[Kal 4] Kalman R.E.,
Kronecker Invariants and Feedback.
Ordinary Differential Equations, Academic Press, New York, 1972, pp. 459-471.

[Kal 5] Kalman R.E.,
Optimal Nonlinear Control of Saturating Systems By Intermittent Action.
IRE Wescon Convention, pt. 4, 1957, pp. 130-135.

[Kar 1] Karcanias N.,
The Basic Theory of Polynomial and Rational Matrices for Linear Multivariable Control
Research Notes, Control Engineering Centre, City University, London, 1988.

[Kar 2] Karcanias N.,
Notes on Exterior Algebra and Systems
Research Notes, Control Engineering Centre, City University, London, 1985.

[Kar 3] Karcanias N.,
The Extended McMillan Degree of a Rational Matrix
Research Notes, Control Engineering Centre, City University, London, 1988.

[Kar 4] Karcanias N.,
On Internal and External Stability of Feedback Systems
Research Notes, Control Engineering Centre, City University, London, 1988.

[Kar 5] Karcanias N.,
Matrix Equations Over Principal Ideal Domains
Research Notes, Control Engineering Centre, City University, London, 1987.

[Kar 6] Karcanias N.,
Linear Systems: Kronecker Canonical Forms and Invariants.
Systems & Control Encyclopedia, Editor-in-Chief Sing M.G., Pergamon Press, 1988, pp. 2866-2871.

[Kee 1] Keerthi S.S. and Gilbert E.G.,
Computation of Minimum-Time Feedback Control Laws for Discrete-Time Systems with State-Control Constraints.
IEEE Trans. Auto. Control, AC-32, No. 5, May 1987, pp. 432-435.

[Kha 1] Khadr A.S.,
Rosenbrock's Theorem and the Minimum Time State Regulator Problem.
IEEE Trans. Auto. Control, AC-26, No. 2, April 1981, pp. 537-538.

[Kim 1] Kimura K.,
Deadbeat Function Observers for Discrete-Time Linear Systems.
SIAM J. Control, Vol. 16, 1978, pp. 880-894.

[Kim 2] Kimura H. and Tanaka Y.,
Minimal-Time Minimal-Order Deadbeat Regulator with Internal Stability.
IEEE Trans. Auto. Control, AC-26, No. 6, Dec., 1981 pp. 1276-1282.

[Kle 1] Klein G.,
On the Relationship Between Controllability Indexes, Eigenvector Assignment, and Deadbeat Control.
IEEE Trans. Auto. Control, AC-29, No. 1, Jan. 1984, pp. 77-79.

[Kol 1] Kolmogorov A.N. and Fomin S.V.,
Introductory Real Analysis
Dover Publications Inc., New York, 1975.

[Kol 2] Kolev L.V.,
Minimum-Fuel Minimum-Time Control of Linear Discrete Systems.
Int. J. Control, Vol. 27, No. 1, 1978, pp. 21-29.

[Kou 1] Kousiouris T.,
Ph.D. thesis, UMIST, Manchester, UK, 1977.

[Kra 1] Kraffer F., Pejchova S. & Sebek M.,
Polynomial Operations: Numerical Performance in Matrix Diophantine Equations.
To appear in Kybernetika.

[Kuc 1] Kucera V.,
Discrete Linear Control: The polynomial equation approach
John Wiley & Sons Inc., London, 1979.

[Kuc 2] Kucera V.,
Analysis and Design of Discrete Linear Control
Prentice Hall International (UK) Ltd., 1991.

[Kuc 3] Kucera V.,
Dynamical Indices and Order of Delay-Operator Models.
IEEE Trans. Auto. Control, AC-25, No. 2, Apr. 1980, pp. 269-270.

[Kuc 4] Kucera V.,
Algebraic Theory of Discrete Optimal Control for Single-Variable Systems, Parts I, II and III.
Kybernetika, Vol. 9, 1973, pp. 94-107, 206-221 and 291-312.

[Kuc 5] Kucera V.,
Algebraic Theory of Discrete Optimal Control for Multivariable Systems.
Kybernetika, published in instalments, Vols. 10-12, 1974, pp. 1-240.

[Kuc 6] Kucera V.,
Polynomial Design of Deatbeat Control Laws.
Kybernetika, Vol. 16, 1980, pp. 431-441.

[Kuc 7] Kucera V.,
A Dead-Beat Servo Problem.
Int. J. Control, Vol. 32, No. 1, 1980, pp. 107-113.

[Kuc 8] Kucera V. and Šebek M.,
On Deadbeat Controllers.
IEEE Trans. Auto. Control, AC-29, 1984, pp. 719-722.

[Kuc 9] Kucera V.,
The Algebraic Approach to Control System Design.
Polynomial Methods in Optimal Control and Filtering
Edited by Hunt K.J., Peter Peregrinus Ltd., London, UK., 1993.

[Kuc 10] Kucera V.,
Observer-Based Deadbeat Controllers: A Polynomial Design.
Kybernetika, Vol. 16, No. 5, 1980, pp. 431-441.

[Kuc 11] Kucera V.,
The Structure and Properties of Time-Optimal Discrete Linear Control.
IEEE Trans. Auto. Control, AC-16, Aug. 1971, pp. 375-377.

[Kuc 12] Kucera V.,
State Space Approach to Discrete Linear Control.
Kybernetika, Vol. 8, No. 3, 1972, pp. 233-251.

[Kuc 13] Kucera V.,
Algebraic Approach to Discrete Linear Control.
IEEE Trans. Auto. Control, Feb. 1975, pp. 116-120.

[Kuc 14] Kucera V. and Hai H.M.,
Deadbeat Performance Under Measurement Dynamics.
Kybernetica, Vol. 22, No. 3, 1986, pp. 218-225.

[Kuc 15] Kucera V., Jezek J. & Krupicka M.,
Numerical Analysis of Diophantine Equations.
Advanced Methods in Adaptive Control for Industrial Applications
Edited by Warwick K., Karny M. & Halouskova A., Springer Verlang, Berlin, 1991, pp. 128-136.

[Kun 1] Kung S.-Y. and Kailath T.,
Some Notes on Valuation Theory in Linear Systems.
Proc. IEEE Conf. on Decision and Conrol, San Diego, California, 1978, pp. 515-517.

[Kun 2] Kung S.-Y., Kailath T. & Morf M.,
A Generalized Resultant Matrix for Polynomial Matrices.
Proc. IEEE Conf. on Decision and Conrol, Florida, Dec. 1976, pp. 892-895.

[Kuo 1] Kuo B.C.,
Digital Control Systems
Saunders HBJ, 2nd Edition, 1992.

[Kwa 1] Kwakernaak H.,
A Condition for Robust Stabilizability.
Systems Contr. Lett., Vol. 2, No. 1, 1982, pp. 1-5.

[Led 1] Leden B.,
Dead-Beat Control and the Riccati Equation.
IEEE Trans. Auto. Control, AC-21, Oct. 1976, pp. 791-792.

[Led 2] Leden B.,
Multivariable Dead-Beat Control.
Automatica, Vol. 13, 1977, pp. 185-188.

[Led 3] Leden B.,
Output Dead-beat Control — A Geometric Approach.
Int. J. Control, Vol. 26, No. 4, 1977, pp. 493-507.

[Lew 1] Lewis F.L.,
Applied Optimal Control and Estimation
Prentice Hall, Englewood Clifs, New Jersey, 1992.

[Lew 2] Lewis F.L.,
A General Riccati Equation Solution to the Deadbeat Control Problem.
IEEE Trans. Auto. Control, AC-27, No. 1, Feb. 1972, pp. 186-188.

[Lew 3] Lewis F.L.,
Riccati Equation Solution to the Minimum Time Output Control Problem.
IEEE Trans. Auto. Control, AC-26, No. 3, June 1981, pp. 763-766.

[Lin 1] Lin W.-S.,
An Algorithm for Computing the Minimum-Time Regulator of Multivariable Discrete Linear Systems.
Int. J. Control, Vol. 44, No. 2, 1986, pp. 543-548.

[Lor 1] Lorchirachoonkul V. and Pierre D.A.,
Deadbeat Response of Time-Delayed systems with Minimal Overshoot Compromise.
IEEE Trans. Auto. Control, Aug. 1967, pp. 468-469.

[Lud 1] Ludyk G.,
Zeitoptimale Diskrete Systeme
Habilitationschrift, 1970.

[Lue 1] Luenberger D.G.,
Optimization by Vector Space Methods
John Wiley & Sons Inc., 1969.

[Lue 2] Luenberger D.G.,
Linear and Nonlinear Programming
Addison-Wesley Inc., 1989.

[Lue 3] Luenberger D.G.,
Canonical Forms for Linear Multivariable Systems.
IEEE Trans. Auto. Control, AC-12, 1967, pp. 290-293.

[Lue 4] Luenberger D.G.,
Observers for Multivariable Systems.
IEEE Trans. Auto. Control, AC-11, 1966, pp. 190-197.

[Mac 1] MacDuffee C.C.,
The Theory of Matrices
Springer, Berlin, 1930; reprinted by Chelsea Publicing Co., New York, 1950.

[Mac 2] MacLane S. and Birkhoff G.,
Algebra
Collier Macmillan, New York, 2nd ed., 1979.

[Mar 1] Marden M.,
The Geometry of the Zeros of a Polynomial in a Complex Variable.
Amer. Math. Soc., New York, 1949, pp. 152-157.

[Mar 2] Marcus M. and Minc H.,
A Survey of Matrix Theory and Matrix Inequalities
Prindle Weber & Schmidt, Boston, 1964; reprinted by Dover Publications Inc., New York, 1992.

[Mar 3] Marrari M.R., Emami-Naeini & Franklin G.F.,
Output Deadbeat Control of Discrete-Time Multivariable Systems.
IEEE Trans. Auto. Control, AC-34, No. 6, June 1989, pp. 644-648.

[McD 1] McDonald J.S. and Pearson J.B.,
ℓ^1-Optimal Control of Multivariable Systems with Output Norm Constraints.
Automatica, Vol. 27, No. 2, 1991, pp. 317-329.

[McM 1] McMillan B.,
Introduction to Formal Realization Theory.
Bell Syst. Tech. J., Vol. 31, 1952, pp. 217-279, 541-600.

[Min 1] Minsky M.,
Models, Minds, Machines.
Proceedings IFIP Congress, 1965, pp. 45-49.

[Mor 1] Morse A.S.,
Structural Invariants of Linear Multivariable Systems.
SIAM J. Control, Vol. 11, 1973, pp. 446-465.

[Mul 1] Mullis C.T.,
Time Optimal Discrete Regulator Gains.
IEEE Trans. Auto. Control, AC-17, 1972, pp. 265-266.

[Nag 1] Nagata A., Nishimura T. & Ikeda M.,
Linear Function Observer for Linear Discrete-Time Systems.
IEEE Trans. Auto. Control, AC-20, 1975, pp. 401-407.

[Opp 1] Oppenheim A.V. and Schafer R.W.,
Digital Signal Processing
Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1975.

[O'R 1] O'Reilly J.,
The Deadbeat Control of Linear Multivariable Systems with Inaccessible State.
Int J. Control, Vol. 31, No. 4, 1980, pp. 645-654.

[O'R 2] O'Reilley J.,
The Discrete Linear Time Invariant Time-Optimal Control — An Overview.
Automatica, Vol. 17, No. 2, 1981, pp. 363-370.

[O'R 3] O'Reilly J.,
Comments on an Explicit Pole-Assigning Feedback Formula with Application to Deadbeat Feedback Construction in Discrete Linear Systems.
IEEE Trans. Auto. Control, AC-25, 1980, pp. 1012-1013.

[Pac 1] Pachter M.,
An Explicit Pole-Assigning Feedback Formula with Application to Deadbeat Feedback Construction in Discrete Linear Systems.
IEEE Trans. Auto. Control, AC-22, April 1977, pp. 263-265.

[Pad 1] Padulo L. and Arbib M.A.,
System Theory: A Unified State Space Approach and Discrete Systems
Hemisphere Publishing, Washington D.C., 1974.

[Pie 1] Pierre D.A. and Lorchirachoonkul V.,
A Performance Limit for a Class of Linear Sampled-Data Control Systems.
IEEE Trans. Auto. Control, Feb. 1967, pp. 112-113.

[Pok 1] Pokoski J.L. and Pierre D.A.,
Deadbeat Response to Parabolic Inputs with Minimum-Squared Error Restrictions on Ramp and Step Responses.
IEEE Trans. Auto. Control, April 1969, pp. 199-200.

[Pop 1] Popov V.M.,
Invariant Description of Linear, Time-Invariant Controllable Systems.
SIAM J. Control, Vol 10, 1972, pp. 252-264.

[Por 1] Porter B.,
Deadbeat Control of Linear Multivariable Discrete Time Systems.
Electronics Letters, Vol. 9, No. 4, 1973, pp. 83-84.

[Por 2] Porter B. and D'Azzo J.J.,
Closed-Loop Eigenstructure Assignment By State Feedback in Multivariable Linear Systems.
Int. J. Control, Vol. 27, No. 3, 1978, pp. 487-492.

[Por 3] Porter B. and Bradshaw A.,
Design of Dead-Beat Controllers and Full-Order Observers for Linear Multivariable Discrete-Time Plants.
Int. J. Control, Vol. 22, No. 2, 1975, pp. 149-155.

[Por 4] Porter B. and Bradshaw A.,
Design of Dead-Beat Controllers and Reduced- Order Observers for Linear Multivariable Discrete- Time Plants.
Trans. ASME J. Dynamics Syst., Meas. and Control, Vol. 98, 1976, pp. 152-155.

[Pug 1] Pugh A.C. and Ratcliffe P.A.,
On the Zeros and Poles of a Rational Matrix.
Int. J. Control, Vol. 30, 1979, pp. 213-226.

[Rag 1] Ragazzini J. and Franklin G.F.,
Sampled Data Control Systems
McGraw-Hill, New York, 1959.

[Ras 1] Rasmy M.E. and Hamza M.H.,
Minimum-Effort Time-Optimal Control of Linear Discrete Systems.
Int. J. Control, Vol. 21, No. 2, 1975, pp. 293-304.

[Ros 1] Rosenbrock H.H.,
State-Space and Multivariabe Theory
John Wiley & Sons Inc., New York, 1970.

[Ros 2] Rosenbrock H.H.,
Order, Degree and Complexity
Int. J. Control, Vol. 19, No. 2, 1979, pp. 323-331.

[Sae 1] Saeks R. and Murray J.,
Fractional Representations, Algebraic Geometry and the Simultaneous Stabilization Problem.
IEEE Trans. Auto. Control, AC-27, No. 4, Aug. 1982, pp. 895-903.

[Sae 2] Saeks R. and Murray J.,
Feedback Systems Design: The Tracking and Disturbance Rejection Problems.
IEEE Trans. Auto. Control, AC-26, No. 1, 1981, pp. 203-217.

[Saf 1] Safonov M.G. and Verma M.S.
Multivariable L^{∞}-Sensitivity Optimization and Hankel Approximation.
Proc. ACC, June 1983, pp. 1280-1283.

[Sch 1] Schlegel M.,
Parametrization of the Class of Deadbeat Controllers.
IEEE Trans. Auto. Control, AC-27, No. 3, June 1982, pp. 231-233.

[Sco 1] Scott M.,
Time/Fuel Optimal Control of Constrained Linear Discrete Systems.
Automatica, Vol. 22, No. 6, 1986, pp. 711-715.

[Seb 1] Sebakhy O.A. and Abdel-Moneim T.M.,
State Regulation in Linear Discrete-Time Systems in Minimum Time.
IEEE Trans. Auto. Control, Ac-24, 1979, pp. 84-88.

[Seb 2] Sebakhy O.A. and Abdel-Moneim T.M.,
Design of Optimal Dead-Beat Controllers.
IEEE Trans. Auto. Control, Ac-25, No. 3, June 1980, pp. 84-88.

[Seb 3] Sebek M.,
Multivariable Deadbeat Servo Problem.
Kybernetica, Vol. 16, 1980, pp. 442-450.

[Ser 1] Seraji H.,
Deadbeat Control of Discrete-Time Systems Using Output Feedback.
Int. J. Control, Vol. 21, No. 2, 1975, pp. 213-223.

[Sim 1] Simmons G.,
Introduction to Topology and Modern Analysis
McGraw-Hill, New York, 1966.

[Son 1] Sontag E.D.,
Mathematical Control Theory: Deterministic finite dimensional systems
Springer-Verlag, New York, 1990.

[Spä 1] Späth H.,
Mathematical Algorithms for Linear Regression
Academic Press Inc., 1987

[Spu 1] Spurgeon S.K. and Pugh A.C.,
On Output Deadbeat Control of Discrete-Time Mutivariable Systems.
IEEE Trans, Auto. Control, AC-36, No. 7, July 1991, pp. 894-896.

[Tho 1] Thorp J.P.,
The Singular Pencil of Linear Dynamical Systems.
Int. J. Control, Vol. 18, 1973, pp. 577-596.

[Tou 1] Tou J.T.,
Modern Control Theory
McGraw-Hill, 1964.

[Van 1] Van der Waerden B.L.,
Modern Algebra Vols. I and II
Springer-Verlang, Berlin, 1930.

[Van 2] Van Dooren P.,
Deadbeat Control : A Special Inverse Eigenvalue Problem.
BIT, Vol. 24, 1984, pp. 681-699.

[Var 1] Vardulakis A.I.G.,
Linear Multivariable Control: Algebraic analysis and synthesis methods
John Wiley & Sons Ltd., Chichester (UK), 1991.

[Var 2] Vardulakis A.I.G., Limebeer D.N.J. & Karcanias N.,
Structure and Smith-McMillan form of a Rational Matrix at Infinity.
Int. J. Control, Vol. 35, No. 4, 1982, pp. 701-725.

[Var 3] Vardulakis A.I.G. and Karcanias N.,
Classification of Proper Bases of Rational Vector Spaces: Minimal McMillan Degree Bases.
Int. J. Control, Vol. 38, 1983 , pp. 779-809.

[Vek 1] Vekua N.P.,
Systems of Singular Integral Equations
Noordoff, The Netherlands, 1967. (in Russian 1950)

[Ver 1] Verghese G.,
Infinite-Frequency Behavior in Generalized Dynamical Systems
Ph.D. thesis, Dept. of Electrical Engineering, Stanford University, Stanford, Calif., Dec. 1978.

[Ver 2] Verghese G. and Kailath T.,
Rational Matrix Structure
IEEE Trans. Auto. Control, AC-26, No. 2, Apr. 1981, pp. 434-439.

[Vid 1] Vidyasagar M.,
Control System Synthesis: A factorization approach
The MIT Press, Cambridge, Massachusetts, 1985.

[Vid 2] Vidyasagar M. and Viswanadham N.,
Algebraic Design Techniques for Reliable Stabilization.
IEEE Trans. Auto. Control, AC-27, No. 5, Oct. 1982, pp. 1085-1095.

[Vid 3] Vidyasagar M.,
Some Results on Simultaneous Stabilization with Multiple Domains of Stability.
Automatica, Vol. 23, No. 4, 1987, pp. 535-540.

[Vid 4] Vidyasagar M.,
Optimal Rejection to Persistent Bounded Disturbances.
IEEE Trans. Auto. Control, AC-31, No. 6, June 1986, pp. 527-534.

[Vid 5] Vidyasagar M.,
Nonlinear System Analysis.
Englewood Cliffs, Prentice-Hall, New Jersey, 1978.

[Vli 1] De Vlieger J.H., Verbruggen H.B. & Bruijn P.M., A Time-Optimal Control Algorithm for Digital Computer Control. Automatica, Vol. 18, No. 2, 1982, pp. 239-244.

[Wan 1] Wang S.-S. and Chen B.-S., Simultaneous Deadbeat Tracking Controller Synthesis. Int. J. Control, Vol. 44, No. 6, 1986, pp. 1579-1586.

[Wed 1] Wedderburn J.H.M., Lectures on Matrices American Mathematical Society, New York, 1934.

[Wei 1] Wei K., The Solution of a Transcendental Problem and its Applications in Simultaneous Stabilization Problems. IEEE Trans. Auto. Control, AC-37, No. 9, Sep. 1992, pp. 1305-1315.

[Wei 2] Wei K. and Barmish B.R., An Iterative Design Procedure for Simultaneous Stabilization of MIMO Systems. Automatica, Vol. 24, No. 5, 1988, pp. 643-652.

[Wil 1] Williamson D., Digital Control and Implementation Prentice Hall of Australia Pty Ltd., 1991.

[Win 1 Wing J and Desoer C.A., The Multiple-Input Minimal Time Regulator Problem (General Theory). IEEE Trans. Auto. Control, AC-8, April 1963, pp. 125- 136.

[Wol 1] Wolovich W.A.,
Linear Multivariable Systems
Springer-Verlag, New York, 1974.

[Wol 2] Wolovich W.A. and Elliott H.,
Discrete Models for Linear Multivariable Systems.
Int. J. Control, Vol. 38, No. 2, 1983, pp. 337-357.

[Wol 3] Wolovich W.A.,
Deadbeat Error Control of Discrete Multivariable Systems.
Int. J. Control, Vol. 37, No. 3, 1983, pp. 567-582.

[Won 1] Wonham W.M.,
Linear Multivariable Control, A Geometric Approach
Springer-Verlag, 3r edn., New York, 1986.

[You 1] Youla D.C., Bongiorno J.J. & Jabr H.A.,
Modern Wiener-Hopf Design of Optimal Controllers, Parts I and II.
IEEE Trans. Auto. Control, AC-21, 1976, pp. 3-13 and 319-338.

[You 2] Youla D.C., Bongiorno J.J. & Lu C.N.,
Single-Loop Feedback Stabilization for Linear Multivariable Dynamical Plants.
Automatica, Vol. 10, 1974, pp. 159-173.

[Zad 1] Zadeh L.A. and Desoer C.A.,
Linear Systems Theory, The State Space Approach
McGraw-Hill, New York, 1963.

[Zak 1] Zak S.H.,
On the Polynomial Matrix Equation $\boldsymbol{AX} + \boldsymbol{YB} = \boldsymbol{C}$.
IEEE Trans. Auto. Control, AC-30, No. 12, Dec. 1985, pp. 1240-1242.

[Zar 1] Zariski O. and Samuel P., <u>Commutative Algebra</u>, Vols. I and II Springer-Verlang, New York, 1986.

[Zha 1] Zhao Y. and Kimura H., Dead-Beat Control with Robustness. <u>Int. J. Control</u>, Vol. 43, No. 5, 1986, pp. 1427-1440.

[Zha 2] Zhao Y. and Kimura H., Multivariable Dead-Beat Control with Robustness. <u>Int. J. Control</u>, Vol. 47, No. 1, 1988, pp. 229-255.

[Zha 3] Zhao Y. and Kimura H., Two-Degrees-of-Freedom Dead-Beat Control System with Robustness. <u>Int. J. Control</u>, Vol. 48, No. 1, 1988, pp. 303-315.

[Zha 4] Zhao Y. and Kimura H., Two-Degrees-of-Freedom Dead-Beat Control System with Robustness: Multivariable Case. <u>Int. J. Control</u>, Vol. 49, No. 2, 1989, pp. 667-679.