Continuity equation in LlogL for the 2D Euler equations under the enstrophy measure

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Abstract

The 2D Euler equations with random initial condition has been investigates by S. Albeverio and A.-B. Cruzeiro in [1] and other authors. Here we prove existence of solutions for the associated continuity equation in Hilbert spaces, in a quite general class with LlogL densities with respect to the enstrophy measure.

1 Introduction

We consider the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, formulated in terms of the vorticity ω

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1}$$

where u is the velocity, divergence free vector field such that $\omega = \partial_2 u_1 - \partial_1 u_2$. We consider this equation in the following abstract Wiener space structure. We set $H = L^2(\mathbb{T}^2)$ with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\|\cdot\|_H$. Given $\delta > 0$, we consider the negative order Sobolev space $B := H^{-1-\delta}(\mathbb{T}^2)$, its dual $B^* = H^{1+\delta}(\mathbb{T}^2)$, and we write $\langle \cdot, \cdot \rangle$ for the dual pairing between elements of B and B^* . More generally, we shall use the notation $\langle \cdot, \cdot \rangle$ also for the dual pairing between elements of $C^{\infty}(\mathbb{T}^2)'$ and $C^{\infty}(\mathbb{T}^2)$; in all cases $\langle \cdot, \cdot \rangle$ reduces to $\langle \cdot, \cdot \rangle_H$ when both elements are in H. Let μ be the so called "enstrophy measure", the centered Gaussian measure on B (in fact it is supported on $H^{-1-}(\mathbb{T}^2) = \cap_{\delta>0} H^{-1-\delta}(\mathbb{T}^2)$; but not on $H^{-1}(\mathbb{T}^2)$) such that

$$\int_{B} \left\langle \omega, \phi \right\rangle \left\langle \omega, \psi \right\rangle \mu \left(d \omega \right) = \left\langle \phi, \psi \right\rangle_{H}$$

for all $\phi, \psi \in C^{\infty}(\mathbb{T}^2)$. Equation (1) has been investigated in this framework and it has been proved that, with a suitable interpretation of the nonlinear term of the equation, it

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has a (possibly non unique) solution for μ -almost every initial condition in B. Moreover, on a suitable probability space $(\Xi, \mathcal{F}, \mathbb{P})$, there exists a stationary process with continuous trajectories in B, with marginal law μ at every time t (in this sense we could say that μ is invariant for equation (1); see also the infinitesimal invariance [2]), whose trajectories are solutions of equation (1) in that suitable specified sense. These results have been proved first by Albeverio and Cruzeiro in [1] and proved with a different concept of solution (used below) in [12].

We want to study the *continuity equation*, associated to equation (1), for a density $\rho_t(\omega)$ with respect to μ . Let us introduce the notation

$$b\left(\omega\right) = -u\left(\omega\right) \cdot \nabla\omega$$

for the drift in equation (1), where we stress by writing $u(\omega)$ the fact that u depends on ω . The precise meaning of $b(\omega)$ is a nontrivial problem discussed below; for the time being, let us take it as an heuristic notation. Let $\mathcal{FC}_{b,T}^1$ be the set of all functionals $F: [0,T] \times C^{\infty} (\mathbb{T}^2)' \to \mathbb{R}$ of the form $F(t,\omega) = \sum_{i=1}^m \tilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$, with $\phi_1, ..., \phi_n \in C^{\infty} (\mathbb{T}^2), \ \tilde{f}_i \in C_b^1(\mathbb{R}^n), \ g_i \in C^1([0,T])$ with $g_i(T) = 0$. The weak form of the continuity equation is

$$\int_{0}^{T} \int_{B} \left(\partial_{t} F\left(t,\omega\right) + \left\langle b\left(\omega\right), DF\left(t,\omega\right) \right\rangle \right) \rho_{t}\left(\omega\right) \mu\left(d\omega\right) dt = -\int_{B} F\left(0,\omega\right) \rho_{0}\left(\omega\right) \mu\left(d\omega\right).$$
(2)

The most critical term, which requires a careful definition, is $\langle b(\omega), DF(t, \omega) \rangle$. Let us discuss this issue.

When $F(t,\omega) = \sum_{i=1}^{m} \tilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$ as above, given any element $\eta \in C^{\infty}(\mathbb{T}^2)'$ the limit

$$\lim_{\epsilon \to 0} \epsilon^{-1} \left(F\left(t, \omega + \epsilon \eta\right) - F\left(t, \omega\right) \right)$$

exists for every $(t, \omega) \in [0, T] \times C^{\infty} (\mathbb{T}^2)'$ and it is equal to

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \partial_{j} \widetilde{f}_{i} \left(\left\langle \omega_{t}, \phi_{1} \right\rangle, ..., \left\langle \omega_{t}, \phi_{n} \right\rangle \right) g_{i} \left(t \right) \left\langle \eta, \phi_{j} \right\rangle.$$

Assume we have defined $\langle b(\omega), \phi \rangle$ when ω is a typical element under μ and $\phi \in C^{\infty}(\mathbb{T}^2)$. Then we set

$$\langle b(\omega), DF(t,\omega) \rangle := \sum_{i=1}^{m} \sum_{j=1}^{n} \partial_{j} \widetilde{f}_{i} \left(\langle \omega_{t}, \phi_{1} \rangle, ..., \langle \omega_{t}, \phi_{n} \rangle \right) g_{i}(t) \left\langle b(\omega), \phi_{j} \right\rangle.$$
(3)

To complete the meaning of $\langle b(\omega), DF(t, \omega) \rangle$ we thus have to give a meaning to $\langle b(\omega), \phi \rangle$ for every $\phi \in C^{\infty}(\mathbb{T}^2)$. Formally

$$\left\langle b\left(\omega\right),\phi\right\rangle =-\left\langle u\left(\omega\right)\cdot\nabla\omega,\phi
ight
angle$$
 .

In Theorem 7 of Section 2 we shall define (for each $\phi \in C^{\infty}(\mathbb{T}^2)$) a random variable $\omega \mapsto \langle b(\omega), \phi \rangle$ on the space (B, \mathcal{B}, μ) (\mathcal{B} being the Borel σ -field on B). With this definition, identity (3) provides a rigorous definition of the measurable map $(\omega, t) \mapsto \langle b(\omega), DF(t, \omega) \rangle$, with certain integrability properties in ω coming from the results of Section 2.

Remark 1 To help the intuition, let us heuristically write equation (2) in the form

$$\partial_t \rho_t + \operatorname{div}_\mu \left(\rho_t b \right) = 0 \tag{4}$$

with initial condition $\rho_0(\omega)$, where $\operatorname{div}_{\mu}(v)$, when defined, for a vector field v on B, is (heuristically) defined by the identity

$$\int_{B} F(\omega) \operatorname{div}_{\mu}(v(\omega)) \mu(d\omega) = -\int_{B} \langle v(\omega), DF(\omega) \rangle \mu(d\omega)$$
(5)

for all $F \in \mathcal{FC}_b^1$, where \mathcal{FC}_b^1 is defined as $\mathcal{FC}_{b,T}^1$ but without the time-dependent components g_i .

In [12] it is proved that the random variable $\omega \mapsto \langle b(\omega), \phi \rangle$ on (B, \mathcal{B}, μ) has all finite moments; here we improve the result and show that it is exponentially integrable: given $\phi \in C^{\infty}(\mathbb{T}^2)$, it holds

$$\int_{B} e^{\epsilon |\langle b(\omega), \phi \rangle|} \mu(d\omega) < \infty$$
(6)

for some $\epsilon > 0$, which depends only on $\|\phi\|_{\infty}$; see Theorem 8 in Section 2 below.

This exponential integrability is a key ingredient to extend, to the 2D Euler equations, the result of the authors [7] for abstract equations in Hilbert spaces (in that work the measure μ is not necessarily Gaussian, but the nonlinearity is bounded). Indeed, we aim to prove existence in the class of densities $\rho_t(\omega)$ such that

$$\sup_{t \in [0,T]} \int_{B} \rho_t(\omega) \log \rho_t(\omega) \, \mu(d\omega) < \infty.$$
(7)

Since $ab \leq e^{\epsilon a} + \epsilon^{-1}b \left(\log \epsilon^{-1}b - 1 \right)$, if $\rho_t(\omega)$ satisfies (7) and property (6) is proved, then

$$\int_{B}\left\langle b\left(\omega\right),DF\left(t,\omega\right)\right\rangle \rho_{t}\left(\omega\right)\mu\left(d\omega\right)$$

is well defined. With these preliminaries we can give the following definition.

Definition 2 Given a measurable function $\rho_0 : B \to [0, \infty)$ such that $\int_B \rho_0(\omega) \log \rho_0(\omega) \mu(d\omega) < \infty$, we say that a measurable function $\rho : [0, T] \times B \to [0, \infty)$ is a solution of equation (4) of class LlogL if property (7) is satisfied and identity (2) holds for every $F \in \mathcal{FC}^1_{b,T}$.

Our main result, proved in Section 3, is:

Theorem 3 If

$$\int_{B} \rho_{0}(\omega) \log \rho_{0}(\omega) \mu(d\omega) < \infty$$

then there exists a solution of equation (4) of class LlogL.

2 Definition and properties of $\langle b(\omega), \phi \rangle$

We denote by $\{e_n\}$ the complete orthonormal system in $L^2(\mathbb{T}^2;\mathbb{C})$ given by $e_n(x) = e^{2\pi i n \cdot x}$, $n \in \mathbb{Z}^2$. As already said in the Introduction, given a distribution $\omega \in C^{\infty}(\mathbb{T}^2)'$ and a test function $\phi \in C^{\infty}(\mathbb{T}^2)$, we denoted by $\langle \omega, \phi \rangle$ the duality between ω and ϕ (namely $\omega(\phi)$), and we use the same symbol for the inner product of $L^2(\mathbb{T}^2)$. We set $\widehat{\omega}(n) = \langle \omega, e_n \rangle$, $n \in \mathbb{Z}^2$ and we define, for each $s \in \mathbb{R}$, the space $H^s(\mathbb{T}^2)$ as the space of all distributions $\omega \in C^{\infty}(\mathbb{T}^2)'$ such that

$$\|\omega\|_{H^s}^2 := \sum_{n \in \mathbb{Z}^2} \left(1 + |n|^2\right)^s |\widehat{\omega}(n)|^2 < \infty.$$

We use similar definitions and notations for the space $H^s(\mathbb{T}^2,\mathbb{C})$ of complex valued functions.

We want to define, for every $\phi \in C^{\infty}(\mathbb{T}^2)$, the random variable

$$\begin{aligned} \left\langle b\left(\omega\right),\phi\right\rangle &=-\left\langle u\left(\omega\right)\cdot\nabla\omega,\phi\right\rangle =-\int_{\mathbb{T}^{2}}u\left(\omega\right)\left(x\right)\cdot\nabla\omega\left(x\right)\phi\left(x\right)dx\\ &=\int_{\mathbb{T}^{2}}\omega\left(x\right)u\left(\omega\right)\left(x\right)\cdot\nabla\phi\left(x\right)dx\end{aligned}$$

where we have used integration by parts and the condition div u = 0 (the computation is heuristic, or it holds for smooth periodic functions; we are still looking for a meaningful definition). Recall that u is divergence free and associated to ω by $\omega = \partial_2 u_1 - \partial_1 u_2$. This relation can be inverted using the so called Biot-Savart law:

$$u(x) = \int_{\mathbb{T}^2} K(x-y) \,\omega(y) \,dy$$

where K(x, y) is the Biot-Savart kernel; in full space it is given by $K(x-y) = \frac{1}{2\pi} \frac{(x-y)^{\perp}}{|x-y|^2}$; on the torus its form is less simple but we still have K smooth for $x \neq y$, K(y-x) = -K(x-y),

$$|K(x-y)| \le \frac{C}{|x-y|}$$

for small values of |x - y|. See for instance [14] for details.

The difficulty in the definition of $\langle b(\omega), \phi \rangle$ is that ω is of class $H^{-1-\delta}(\mathbb{T}^2)$ and u of class $H^{-\delta}(\mathbb{T}^2)$, so we need to multiply distributions. The following remark recalls a trick used in several works on measure-valued solutions of 2D Euler equations, like [9], [10], [13], [14], [15].

Remark 4 If ω is sufficiently smooth and periodic, using Biot-Savart law we can write

$$\langle b(\omega), \phi \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \omega(x) \,\omega(y) \,K(x-y) \cdot \nabla \phi(x) \,dxdy$$

Since the double integral, when we rename x by y and y by x, is the same (the renaming doesn't affect the value), and K(y-x) = -K(x-y), we get

$$\left\langle b\left(\omega\right),\phi
ight
angle =\int_{\mathbb{T}^{2}}\int_{\mathbb{T}^{2}}\omega\left(x
ight)\omega\left(y
ight)H_{\phi}\left(x,y
ight)dxdy$$

where

$$H_{\phi}(x,y) := \frac{1}{2}K(x-y) \cdot \left(\nabla\phi(x) - \nabla\phi(y)\right).$$

The advantage of this symmetrization is that H_{ϕ} (opposite to $K(x-y) \cdot \nabla \phi(x)$) is a bounded function. It is smooth outside the diagonal x = y, discontinuous on the diagonal; more precisely, we can write

$$H_{\phi}(x,y) = \frac{1}{2\pi} \left\langle D^{2}\phi(x) \frac{x-y}{|x-y|}, \frac{(x-y)^{\perp}}{|x-y|} \right\rangle + R_{\phi}(x,y)$$
(8)

where $R_{\phi}(x, y)$ is Lipschitz continuous, with

$$\left|R_{\phi}\left(x,y\right)\right| \leq C\left|x-y\right|.$$

To summarize, when ω is sufficiently smooth and periodic, we have

$$\langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_{\phi} \rangle_{L^{2}(\mathbb{T}^{2} \times \mathbb{T}^{2})}$$

where $\omega \otimes \omega : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ is defined as $(\omega \otimes \omega)(x, y) = \omega(x)\omega(y)$.

Remark 5 The previous expression is meaningful when ω is a measure, since H_{ϕ} is Borel bounded. When ω is only a distribution, of class $H^{-1-\delta}(\mathbb{T}^2)$, one can define $\omega \otimes \omega$ as the unique element of $H^{-2-2\delta}(\mathbb{T}^2 \times \mathbb{T}^2)$ such that

$$\langle \omega \otimes \omega, f \rangle = \langle \omega, \varphi \rangle \langle \omega, \psi \rangle$$

for every smooth $f: \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ of the form $f(x, y) = \varphi(x) \psi(y)$, where the dual pairing $\langle \omega \otimes \omega, f \rangle$ is on $\mathbb{T}^2 \times \mathbb{T}^2$. But H_{ϕ} is not of class $H^{2+2\delta}(\mathbb{T}^2 \times \mathbb{T}^2)$, hence there is no simple deterministic meaning for $\langle \omega \otimes \omega, H_{\phi} \rangle$ when $\omega \in H^{-1-\delta}(\mathbb{T}^2)$. It is here that probability will play the essential role.

In [12] the following result has been proved. As remarked above, when $f \in H^{2+2\delta}(\mathbb{T}^2 \times \mathbb{T}^2)$, $\langle \omega \otimes \omega, f \rangle$ is well defined for all $\omega \in H^{-1-\delta}(\mathbb{T}^2)$, hence for a.e. ω with respect to the Entrophy measure μ .

Lemma 6 Assume $f \in H^{2+\epsilon} (\mathbb{T}^2 \times \mathbb{T}^2)$ for some $\epsilon > 0$. One has

$$\int_{B} |\langle \omega \otimes \omega, f \rangle|^{p} \, \mu \left(d\omega \right) \leq \frac{(2p)!}{2^{p} p!} \, \|f\|_{\infty}^{p}$$

for every positive integer $p \geq 2$,

$$\int_{B} \left\langle \omega \otimes \omega, f \right\rangle \mu \left(d\omega \right) = \int_{\mathbb{T}^{2}} f \left(x, x \right) dx$$

and, when f is also symmetric,

$$\int_{B} \left| \langle \omega \otimes \omega, f \rangle - \int_{\mathbb{T}^{2}} f(x, x) \, dx \right|^{2} \mu(d\omega) = 2 \int_{\mathbb{T}^{2}} \int_{\mathbb{T}^{2}} f(x, y)^{2} \, dx dy.$$

The consequence proved in [12] is:

Theorem 7 Let $\omega : \Xi \to C^{\infty} (\mathbb{T}^2)'$ be a white noise and $\phi \in C^{\infty} (\mathbb{T}^2)$ be given. Assume that $H^n_{\phi} \in H^{2+} (\mathbb{T}^2 \times \mathbb{T}^2)$ are symmetric and approximate H_{ϕ} in the following sense:

$$\lim_{n \to \infty} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(H^n_{\phi} - H_{\phi} \right)^2 (x, y) \, dx \, dy = 0$$
$$\lim_{n \to \infty} \int_{\mathbb{T}^2} H^n_{\phi} (x, x) \, dx = 0.$$

Then the sequence of r.v.'s $\langle \omega \otimes \omega, H_{\phi}^n \rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle b(\omega), \phi \rangle = \langle \omega \otimes \omega, H_{\phi} \rangle$$

its limit. Moreover, the limit is the same if H^n_{ϕ} is replaced by \widetilde{H}^n_{ϕ} with the same properties and such that $\lim_{n\to\infty} \int \int \left(H^n_{\phi} - \widetilde{H}^n_{\phi}\right)^2(x,y) \, dx \, dy = 0.$

A simple example of functions H_{ϕ}^{n} with these properties is given in [12]. In addition to these fact, here we prove exponential integrability, see (6).

Theorem 8 Given a bounded measurable f with $||f||_{\infty} \leq 1$, we have

$$\int_{B} e^{\epsilon |\langle \omega \otimes \omega, f \rangle|} \mu \left(d \omega \right) < \infty$$

for all $\epsilon < \frac{1}{2}$.

Proof.

$$\mathbb{E}\left[e^{\epsilon|\langle\omega\otimes\omega,f\rangle|}\right] = \sum_{p=0}^{\infty} \frac{\epsilon^p \mathbb{E}\left[|\langle\omega\otimes\omega,f\rangle|^p\right]}{p!} \le \sum_{p=0}^{\infty} \left(\frac{\epsilon}{2}\right)^p \frac{(2p)!}{p!p!}$$

This series converges for $\epsilon < \frac{1}{2}$ because (using ratio test)

$$\frac{\left(\frac{\epsilon}{2}\right)^{p+1}\frac{(2(p+1))!}{(p+1)!(p+1)!}}{\left(\frac{\epsilon}{2}\right)^p\frac{(2p)!}{p!p!}} = \frac{\epsilon}{2}\frac{(2p+2)(2p+1)}{(p+1)(p+1)} \to 2\epsilon.$$

3 Proof of Theorem 3

3.1 Approximate problem

Recall from the Introduction that $\delta > 0$ is fixed and we set $B = H^{-1-\delta}(\mathbb{T}^2)$, $H = L^2(\mathbb{T}^2)$; recall also from Section 2 that we write $e_n(x) = e^{2\pi i n \cdot x}$, $x \in \mathbb{T}^2$, $n \in \mathbb{Z}^2$, that is a complete orthonormal system in $H^{\mathbb{C}} := L^2(\mathbb{T}^2; \mathbb{C})$. Given $N \in \mathbb{N}$, let $H_N^{\mathbb{C}}$ be the span of e_n for $|n|_{\infty} \leq N$, $|n|_{\infty} := \max(|n_1|, |n_2|)$ for $n = (n_1, n_2)$; it is a subspace of $H^{\mathbb{C}}$. Let H_N be the subspace of $H_N^{\mathbb{C}}$ made of real-valued elements; it is a subspace of H and is characterized by the following property: $\omega = \sum_{|n|_{\infty} \leq N} \omega_n e_n$ is in H_N if and only if $\overline{\omega_n} = \omega_{-n}$, for all nsuch that $|n|_{\infty} \leq N$.

Let π_N be the orthogonal projection of H onto H_N . It is given by $\pi_N \omega = \sum_{|n|_{\infty} \leq N} \langle \omega, e_n \rangle_H e_n$, for all $\omega \in H$. We extend π_N to an operator on B by setting

$$\pi_N : B \to H_N$$
$$\pi_N \omega = \sum_{|n|_{\infty} \le N} \langle \omega, e_n \rangle e_r$$

where now $\langle \omega, e_n \rangle$ is the dual pairing. We may introduce the Dirichlet kernel

$$\theta_N(x_1, x_2) = \sum_{n_1 = -N}^N \sum_{n_2 = -N}^N e^{2\pi i (n_1 x_1 + n_2 x_2)} = \sum_{|n|_\infty \le N} e^{2\pi i n \cdot x}$$
(9)

for $x = (x_1, x_2) \in \mathbb{T}^2$, and check that

$$\pi_N \omega = \theta_N * \omega.$$

We define the operator

$$b_N: B \to H_N$$

as

$$b_N(\omega) = -\pi_N(u(\pi_N\omega) \cdot \nabla \pi_N\omega), \qquad \omega \in B$$

where $u(\pi_N \omega)$ denotes the result of Biot-Savart law applied to $\pi_N \omega$,

$$u(\pi_N\omega)(x) := \int_{\mathbb{T}^2} K(x-y)(\pi_N\omega)(y) \, dy.$$

The operator b_N has the following properties. We denote by div $b_N(\omega)$ the function

$$\operatorname{div} b_{N}(\omega) = \sum_{|n|_{\infty} \leq N} \partial_{n} \langle b_{N}(\omega), e_{n} \rangle_{H}$$

where, when defined, $\partial_n F(\omega) = \lim_{\epsilon \to 0} \epsilon^{-1} (F(\omega + \epsilon e_n) - F(\omega))$, for a function F defined on B. We say that div $b_N(\omega)$ exists if $\partial_n \langle b_N(\omega), e_n \rangle_H$ exists for all $|n|_{\infty} \leq N$. Moreover, we set

$$\operatorname{div}_{\mu} b_{N}(\omega) := \operatorname{div} b_{N}(\omega) - \langle \omega, b_{N}(\omega) \rangle$$

where $\langle \omega, b_N(\omega) \rangle$ is the dual pairing. It is easy to check that this definition is coherent with the general one (5) given in the Introduction.

Lemma 9 The divergence div $b_N(\omega)$ exists for all $\omega \in B$ and

$$\operatorname{div} b_{N}(\omega) = 0$$
$$\langle \omega, b_{N}(\omega) \rangle = 0$$

and thus

$$\operatorname{div}_{\mu} b_{N}(\omega) = 0$$

Proof. Step 1: A basic identity is

$$\left\langle \omega, b_N\left(\omega\right) \right\rangle = 0$$

for all $\omega \in B$, where as usual $\langle ., . \rangle$ denotes dual pairing. This identity holds because

$$\langle \omega, \pi_N \left(u \left(\pi_N \omega \right) \cdot \nabla \pi_N \omega \right) \rangle = \langle \pi_N \omega, u \left(\pi_N \omega \right) \cdot \nabla \pi_N \omega \rangle_H = 0$$

where the first equality can be checked by writing $\omega = \sum \langle \omega, e_n \rangle e_n$ (the series converges in *B*), and the second equality is true because

$$\langle v \cdot \nabla f, f \rangle = \frac{1}{2} \int_{\mathbb{T}^2} v(x) \cdot \nabla f^2(x) \, dx = -\frac{1}{2} \int_{\mathbb{T}^2} \operatorname{div} v(x) \, f^2(x) \, dx = 0$$

for all sufficiently smooth divergence free vector field v (we take $v = u(\pi_N \omega)$ that is a smooth divergence free vector field) and all sufficiently smooth functions f (we take $f = \pi_N \omega$).

Step 2: Recall that $u(e_n)(x)$ is periodic, divergence free, and such that $\nabla^{\perp} \cdot u(e_n) = e_n$ (it is also given by the Biot-Savart law $u(e_n)(x) := \int_{\mathbb{T}^2} K(x-y) e_n(y) dy$). Then we have

$$u(e_n)(x) \cdot \nabla e_n(x) = 0$$

for every $n \in \mathbb{Z}^2$. Indeed,

$$u(e_n)(x) \cdot \nabla e_n(x) = 2\pi i (u(e_n)(x) \cdot n) e_n(x)$$

and this is zero because $u(e_n)(x) \cdot n = 0$. To prove the latter property, it is necessary to understand the shape of $u(e_n)(x)$. Let us prove that

$$u(e_n)(x) = \frac{n^{\perp}}{|n|^2} e_n(x)$$

(which implies $u(e_n)(x) \cdot n = 0$ because $n^{\perp} \cdot n = 0$). The function $u(e_n)$ is uniquely defined by the conditions to be periodic, divengence free and $\nabla^{\perp} \cdot u(e_n) = e_n$, so we have to check these conditions for the function $\frac{n^{\perp}}{|n|^2}e_n(x)$. This is clearly periodic; it is divengence free because div $u(e_n)(x) = \frac{n^{\perp}}{|n|^2}e_n(x) \cdot n = 0$; and finally $\nabla^{\perp} \cdot \frac{n^{\perp}}{|n|^2}e_n(x) = \frac{n^{\perp}}{|n|^2}e_n(x) \cdot n^{\perp} = e_n(x)$. **Step 3**: Finally we can prove that div $b_N(\omega) = 0$. It is

$$\operatorname{div} b_{N}(\omega) = -\sum_{|n| \leq N} \partial_{n} \langle \pi_{N} \left(u \left(\pi_{N} \omega \right) \cdot \nabla \pi_{N} \omega \right), e_{n} \rangle_{H}.$$

We have

$$\partial_n \langle \pi_N \left(u \left(\pi_N \omega \right) \cdot \nabla \pi_N \omega \right), e_n \rangle_H \\= \partial_n \langle u \left(\pi_N \omega \right) \cdot \nabla \pi_N \omega, e_n \rangle_H \\= -\partial_n \langle \pi_N \omega, u \left(\pi_N \omega \right) \cdot \nabla e_n \rangle_H$$

(we have used integration by parts and div $u(\pi_N \omega) = 0$ in the last identity)

$$= - \langle \partial_n (\pi_N \omega), u (\pi_N \omega) \cdot \nabla e_n \rangle_H - \langle \pi_N \omega, \partial_n (u (\pi_N \omega) \cdot \nabla e_n) \rangle_H$$

= - \langle e_n, u (\pi_N \omega) \cdot \nabla e_n \rangle_H - \langle \pi_N \omega, u (\end{en}) \cdot \nabla e_n \rangle_H

because

$$\partial_n (\pi_N \omega) = \partial_n \left(\sum_{|n'| \le N} \langle \omega, e_{n'} \rangle e_{n'} \right) = \sum_{|n'| \le N} \partial_n (\langle \omega, e_{n'} \rangle) e_{n'} = \sum_{|n'| \le N} \delta_{nn'} e_{n'}$$
$$\partial_n (u (\pi_N \omega) \cdot \nabla e_n) = \partial_n \left(\sum_{|n''| \le N} \langle \omega, e_{n''} \rangle u (e_{n''}) \cdot \nabla e_n \right) = \sum_{|n''| \le N} \delta_{nn''} u (e_{n''}) \cdot \nabla e_n.$$

The first term, $\langle e_n, u(\pi_N \omega) \cdot \nabla e_n \rangle_H$, is zero by the same general rule recalled in Step 1. The second term is zero by Step 2. Therefore div $b_N(\omega) = 0$.

Consider the finite dimensional ordinary differential equation in the space H_N defined as

$$\frac{d\omega_t^N}{dt} = b_N\left(\omega_t^N\right), \qquad \omega_0^N \in H_N.$$
(10)

The function b_N , in H_N , is differentiable, bounded with bounded derivative on bounded sets. Hence, for every $\omega_0^N \in H_N$, there is a unique local solution ω_t^{N,ω_0^N} of equation (10) and the flow map $\omega_0^N \mapsto \omega_t^{N,\omega_0^N}$, where defined, is continuously differentiable, invertible with continuously differentiable inverse. The solution is global because of the energy estimate

$$\frac{d\left\|\boldsymbol{\omega}_{t}^{N}\right\|_{H}^{2}}{dt} = 2\left\langle b_{N}\left(\boldsymbol{\omega}_{t}^{N}\right), \boldsymbol{\omega}_{t}^{N}\right\rangle_{H} = 0$$

which implies $\sup_{t\in[0,\tau]} \|\omega_t^N\|_H^2 \leq \|\omega_0^N\|_H^2$ on any interval $[0,\tau]$ of local existence; the property $\langle b_N(\omega_t^N), \omega_t^N \rangle_H = 0$ holds by Lemma 9. We denote by $\Phi_t^N : H_N \to H_N$ the global flow defined as $\Phi_t^N(\omega_0^N) = \omega_t^{N,\omega_0^N}$.

Denote by $\mu^N(d\omega)$ the image measure, on H_N , of $\mu(d\omega)$ under the projection π_N . This measure is invariant under the flow Φ_t^N , because $\operatorname{div}_{\mu} b_N(\omega) = 0$: for every smooth $F: H_N \to [0, \infty)$, bounded with bounded derivatives,

$$\int_{H_N} \langle b_N(\omega), DF(\omega) \rangle_{H_N} \mu^N(d\omega) = \int_B \langle b_N(\omega), DF(\pi_N \omega) \rangle_H \mu(d\omega)$$
$$= -\int_B F(\pi_N \omega) \operatorname{div}_\mu b_N(\omega) \mu(d\omega) = 0.$$

3.2 Continuity equation for the approximate problem

Given a measurable function $\rho_0^N : H_N \to [0,\infty)$, with $\int_B \rho_0^N (\pi_N \omega) \, \mu (d\omega) < \infty$, consider the measure $\rho_0^N (\pi_N \omega) \, \mu^N (d\omega)$ and its push forward under the flow map Φ_t^N ; denote it by ν_t^N . By definition, for bounded measurable $F : H_N \to [0,\infty)$,

$$\int_{H_N} F(\omega) \nu_t^N(d\omega) = \int_{H_N} F(\Phi_t^N(\omega)) \rho_0^N(\omega) \mu^N(d\omega)$$

From the invariance of μ^N under the flow Φ_t^N , we have

$$\int_{H_N} F(\omega) \nu_t^N(d\omega) = \int_{H_N} F(\omega) \rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right) \mu^N(d\omega)$$

hence

$$\nu_t^N \left(d\omega \right) = \rho_t^N \left(\pi_N \omega \right) \mu^N \left(d\omega \right)$$

where

$$\rho_t^N(\omega) = \rho_0^N\left(\left(\Phi_t^N\right)^{-1}(\omega)\right), \qquad \omega \in H_N.$$
(11)

We have partially proved the following statement.

Lemma 10 Consider equation (10) in H_N , with the associated flow Φ_t^N . Given at time zero a measure of the form $\rho_0^N(\pi_N\omega)\mu^N(d\omega)$ with $\int_B \rho_0^N(\pi_N\omega)\mu(d\omega) < \infty$, its push forward at time t, under the flow map Φ_t^N , is a measure of the form $\rho_t^N(\pi_N\omega)\mu^N(d\omega)$, with $\int_B \rho_t^N(\pi_N\omega)\mu(d\omega) < \infty$. If in addition $\int_B \rho_0^N(\pi_N\omega)\log\rho_0^N(\pi_N\omega)\mu(d\omega) < \infty$, the same is true at time t and

$$\int_{B} \rho_{t}^{N}(\pi_{N}\omega) \log \rho_{t}^{N}(\pi_{N}\omega) \,\mu\left(d\omega\right) = \int_{B} \rho_{0}^{N}(\pi_{N}\omega) \log \rho_{0}^{N}(\pi_{N}\omega) \,\mu\left(d\omega\right).$$
(12)

If in addition ρ_0^N is bounded, then $\rho_t^N \leq \|\rho_0^N\|_{\infty}$. Finally. ρ_t^N satisfies the continuity equation

$$\int_{0}^{T} \int_{B} \left(\partial_{t} F\left(t,\omega\right) + \left\langle DF\left(t,\omega\right), b_{N}\left(\omega\right) \right\rangle_{H} \right) \rho_{t}^{N}\left(\pi_{N}\omega\right) \mu\left(d\omega\right) dt = -\int_{B} F\left(0,\omega\right) \rho_{0}^{N}\left(\pi_{N}\omega\right) \mu\left(d\omega\right) dt$$

$$(13)$$

$$(13)$$

$$(13)$$

Proof. The integrability of ρ_t^N comes from the invariance of μ^N under Φ_t^N , as well as the LlogL property; let us check this latter one. Using (11) we have

$$\begin{split} \int_{B} \rho_{t}^{N} \left(\pi_{N} \omega \right) \log \rho_{t}^{N} \left(\pi_{N} \omega \right) \mu \left(d \omega \right) &= \int_{H_{N}} \rho_{t}^{N} \left(\omega \right) \log \rho_{t}^{N} \left(\omega \right) \mu^{N} \left(d \omega \right) \\ &= \int_{H_{N}} \rho_{0}^{N} \left(\left(\Phi_{t}^{N} \right)^{-1} \left(\omega \right) \right) \log \rho_{0}^{N} \left(\left(\Phi_{t}^{N} \right)^{-1} \left(\omega \right) \right) \mu^{N} \left(d \omega \right) \\ &= \int_{H_{N}} \rho_{0}^{N} \left(\omega \right) \log \rho_{0}^{N} \left(\omega \right) \mu^{N} \left(d \omega \right) \\ &= \int_{B} \rho_{0}^{N} \left(\pi_{N} \omega \right) \log \rho_{0}^{N} \left(\pi_{N} \omega \right) \mu \left(d \omega \right). \end{split}$$

When ρ_0^N is bounded, we have

$$\rho_t^N\left(\omega\right) = \rho_0^N\left(\left(\Phi_t^N\right)^{-1}\left(\omega\right)\right) \le \left\|\rho_0^N\right\|_{\infty}$$

Finally, from the chain rule applied to $F(t, \Phi_t^N(\omega)), \omega \in H_N$, we get the weak form of the continuity equation.

Remark 11 We may construct ρ_t^N and prove (12) also by the following procedure, closer to [7]. We study the transport equation in H_N

$$\partial_t \rho_t^N + \left\langle b_N, D \rho_t^N \right\rangle_H = 0$$

with initial condition ρ_0^N , which has the solution (11) by the method of characteristics. Its weak form reduces to (13) because (for F like those of the Lemma)

$$\int_{H_N} F(t,\omega) \langle b_N(\omega), D\rho_t^N(\omega) \rangle_H \mu^N(d\omega)$$

= $\int_B F(t,\omega) \langle b_N(\omega), D\rho_t^N(\pi_N\omega) \rangle_H \mu(d\omega)$
= $-\int_B \langle DF(t,\omega), b_N(\omega) \rangle_H \rho_t^N(\pi_N\omega) \mu(d\omega)$

where we have used the property $\operatorname{div}_{\mu} b_N(\omega) = 0$. Finally, to prove (12) as in [7], we compute

$$\begin{split} \frac{d}{dt} \int_{H_N} \rho_t^N \left(\log \rho_t^N - 1 \right) d\mu^N \\ &= \int_{H_N} \log \rho_t^N \partial_t \rho_t^N d\mu^N = - \int_{H_N} \log \rho_t^N \left\langle b_N, D\rho_t^N \right\rangle d\mu^N \\ &= - \int_{H_N} \left\langle b_N, D\left[\rho_t^N \left(\log \rho_t^N - 1 \right) \right] \right\rangle d\mu^N \\ &= \int_{H_N} \left[\rho_t^N \left(\log \rho_t^N - 1 \right) \right] \operatorname{div}_{\mu} b_N d\mu^N = 0. \end{split}$$

3.3 Construction of a solution to the limit problem

3.3.1 First case: bounded continuous ρ_0

Consider first the case when ρ_0 is a bounded continuous function on B. Define the sequence of equibounded functions ρ_0^N on H_N by setting $\rho_0^N(\pi_N\omega) = \rho_0(\pi_N\omega)$. For each one of them, consider the associated function $\rho_t^N(\pi_N\omega)$ given by Lemma 10. There is a subsequence, still denoted for simplicity by $\rho_t^N(\pi_N\omega)$ which converges to some function ρ_t weak^{*} in $L^{\infty}([0,T] \times B)$; entropy is weakly lower semicontinuous in $L^1(B,\mu)$, hence

$$\int_{B} \rho_{t}(\omega) \log \rho_{t}(\omega) \mu(d\omega) \leq \liminf_{N \to \infty} \int_{B} \rho_{t}^{N}(\pi_{N}\omega) \log \rho_{t}^{N}(\pi_{N}\omega) \mu(d\omega).$$

By (12) we deduce

$$\int_{B} \rho_{t}(\omega) \log \rho_{t}(\omega) \mu(d\omega) \leq \liminf_{N \to \infty} \int_{B} \rho_{0}^{N}(\pi_{N}\omega) \log \rho_{0}^{N}(\pi_{N}\omega) \mu(d\omega)$$

But, by the definition above of ρ_0^N ,

$$\int_{B} \rho_{0}^{N}(\pi_{N}\omega) \log \rho_{0}^{N}(\pi_{N}\omega) \mu(d\omega) = \int_{B} \rho_{0}(\pi_{N}\omega) \log \rho_{0}(\pi_{N}\omega) \mu(d\omega).$$

Using Lebesgue dominated convergence theorem, this finally implies, by continuity of ρ_0 and of the function $x \log x$, and by boundedness of ρ_0 ,

$$\int_{B} \rho_{t}(\omega) \log \rho_{t}(\omega) \mu(d\omega) \leq \int_{B} \rho_{0}(\omega) \log \rho_{0}(\omega) \mu(d\omega)$$

Finally we have to prove that ρ_t satisfies the weak formulation. We have to pass to the limit in (13). The only problem is the term

$$\int_{0}^{T} \int_{B} \left\langle b_{N}(\omega), DF(t,\omega) \right\rangle_{H} \rho_{t}^{N}(\pi_{N}\omega) \,\mu\left(d\omega\right) dt.$$

We add and subtract the term

$$\int_{0}^{T} \int_{B} \left\langle b\left(\omega\right), DF\left(t,\omega\right) \right\rangle \rho_{t}^{N}\left(\pi_{N}\omega\right) \mu\left(d\omega\right) dt$$

and use integrability of $\langle b(\omega), D_H F(t, \omega) \rangle$ and weak* convergence of $\rho_t^N(\pi_N \omega)$ to $\rho_t(\omega)$ to pass to the limit in one addend. It remains to prove that

$$\lim_{N \to \infty} \int_0^T \int_B \left(\langle b_N(\omega), DF(t, \omega) \rangle_H - \langle b(\omega), DF(t, \omega) \rangle \right) \rho_t^N(\pi_N \omega) \, \mu(d\omega) \, dt = 0.$$

Keeping in mind again the weak^{*} convergence of $\rho_t^N(\pi_N\omega)$, it is sufficient to prove that $\int_B \langle b_N(\omega), D_H F(t, \omega) \rangle_H$ converges strongly to $\langle b(\omega), D_H F(t, \omega) \rangle$ in $L^1(0, T; L^1(B, \mu))$. Due to the form of F, it is sufficient to prove the following claim: given $\phi \in C^{\infty}(\mathbb{T}^2)$,

$$\lim_{N \to \infty} \int_{B} \left| \left\langle b_{N}(\omega), \phi \right\rangle_{H} - \left\langle b(\omega), \phi \right\rangle \right| \mu(d\omega) = 0.$$

The remainder of this subsection is devoted to the proof of this claim.

It is not restrictive to assume that $\phi \in H_{N_0}$ for some N_0 . Hence, for N large enough so that $\pi_N \phi = \phi$,

$$\langle b_N(\omega), \phi \rangle_H = - \langle \pi_N(u(\pi_N\omega) \cdot \nabla \pi_N\omega), \phi \rangle_H = - \langle u(\pi_N\omega) \cdot \nabla \pi_N\omega, \phi \rangle_H = \langle \pi_N\omega, u(\pi_N\omega) \cdot \nabla \phi \rangle_H = \langle (\pi_N\omega) \otimes (\pi_N\omega), H_\phi \rangle$$

where the last identity is proved as in Remark 4. We have

$$\langle (\pi_N \omega) \otimes (\pi_N \omega), H_\phi \rangle = \langle \omega \otimes \omega, (H_\phi)_N \rangle$$

where

$$(H_{\phi})_{N}(x,y) = \sum_{|n|_{\infty} \leq N} \sum_{|n'|_{\infty} \leq N} e_{n}(x) e_{n'}(y) \int_{\mathbb{T}^{2}} \int_{\mathbb{T}^{2}} e_{n'}(y') e_{n}(x') H_{\phi}(x',y') dx' dy'.$$

Therefore, our aim is to prove that, given $\phi \in C^{\infty}(\mathbb{T}^2)$,

$$\lim_{N \to \infty} \int_{B} \left| \left\langle \omega \otimes \omega, (H_{\phi})_{N} - H_{\phi} \right\rangle \right| \mu \left(d\omega \right) = 0$$

Thanks to Lemma 6 and Theorem 7, with a simple argument on Cauchy sequences one can see that it is sufficient to prove that $(H_{\phi})_N \to H_{\phi}$ in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$ and

$$\int_{\mathbb{T}^2} \left(H_\phi \right)_N (x, x) \, dx \to 0. \tag{14}$$

From the theory of Fourier series, $(H_{\phi})_N \to H_{\phi}$ in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$. The limit property (14) requires more work. The result is included in the next lemma, which completes the proof that ρ_t is a weak solution, in the case when ρ_0 is bounded.

Lemma 12 i) The Dirichlet kernel (9) has the two properties

$$\theta_N(x_1, x_2) = \theta_N(x_2, x_1)$$

$$\theta_N(-x_1, x_2) = \theta_N(x_1, x_2).$$

ii) If a kernel $\theta_N(x)$, $x \in T^2$, has these two properties, the kernel $W_N = \theta_N * \theta_N$ has the same properties.

iii) It follows that, for any symmetric matrix S,

$$\int_{\mathbb{T}^2} W_N(x) \left\langle S \frac{x}{|x|}, \frac{x^{\perp}}{|x|} \right\rangle dx = 0.$$

iv) It follows also that

$$\lim_{N \to \infty} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} W_N \left(x - y \right) H_\phi \left(x, y \right) dx dy = 0.$$

In the case when θ_N is the Dirichlet kernel, this property is the limit property (14).

Proof. Property (i) is obvious. The proof of (ii) is elementary, but we give the computations for completeness:

$$W_N(x_1, x_2) = \int_{\mathbb{T}^2} \theta_N(x_1 - y_1, x_2 - y_2) \,\theta_N(y_1, y_2) \,dy_1 dy_2$$

=
$$\int_{\mathbb{T}^2} \theta_N(x_2 - y_2, x_1 - y_1) \,\theta_N(y_2, y_1) \,dy_1 dy_2$$

=
$$W_N(x_2, x_1)$$

$$W_{N}(-x_{1}, x_{2}) = \int_{\mathbb{T}^{2}} \theta_{N} (-x_{1} - y_{1}, x_{2} - y_{2}) \theta_{N} (y_{1}, y_{2}) dy_{1} dy_{2}$$

$$= \int_{\mathbb{T}^{2}} \theta_{N} (x_{1} + y_{1}, x_{2} - y_{2}) \theta_{N} (y_{1}, y_{2}) dy_{1} dy_{2}$$

$$= \int_{\mathbb{T}^{2}} \theta_{N} (x_{1} - y_{1}, x_{2} - y_{2}) \theta_{N} (-y_{1}, y_{2}) dy_{1} dy_{2}$$

$$= \int_{\mathbb{T}^{2}} \theta_{N} (x_{1} - y_{1}, x_{2} - y_{2}) \theta_{N} (y_{1}, y_{2}) dy_{1} dy_{2}$$

$$= W_{N} (x_{1}, x_{2}).$$

Let us prove (iii). We can write

$$\left\langle S\frac{x}{|x|}, \frac{x^{\perp}}{|x|} \right\rangle = (S_{11} + S_{22})\frac{x_1x_2}{|x|^2} + S_{12}\frac{x_2^2 - x_1^2}{|x|^2}.$$

Let us show that the integrals corresponding to each one of the two terms vanish. We have

$$\int_{\mathbb{T}^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2$$

The integration in the second quadrant,

$$\int_{0}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{0} W_N(x) \frac{x_1 x_2}{|x|^2} dx_1 dx_2$$

cancels with the integration in the first quadrant,

$$\int_{0}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} W_{N}(x) \frac{x_{1}x_{2}}{|x|^{2}} dx_{1} dx_{2}$$

because of property $W_N(-x_1, x_2) = W_N(x_1, x_2)$ (point (ii)); similarly for the integrations in the other quadrants. So $\int_{\mathbb{T}^2} W_N(x) \frac{x_1 x_2}{|x|^2} dx = 0$. For the other integral, just by renaming the variables we have

$$\int_{\mathbb{T}^2} W_N(x_1, x_2) \frac{x_1^2}{|x|^2} dx_1 dx_2 = \int_{\mathbb{T}^2} W_N(x_2, x_1) \frac{x_2^2}{|x|^2} dx_2 dx_1$$

and then, using $W_N(x_1, x_2) = W_N(x_2, x_1)$ (point (ii))

$$= \int_{\mathbb{T}^2} W_N(x_1, x_2) \frac{x_2^2}{|x|^2} dx_1 dx_2$$

hence $\int_{\mathbb{T}^2} W_N(x) \frac{x_2^2 - x_1^2}{|x|^2} dx = 0$. We have proved (iii).

Finally, the limit in (iv) is a consequence of the decompositon (8). Indeed,

$$\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} W_N(x-y) \left\langle D^2 \phi(x) \frac{x-y}{|x-y|}, \frac{(x-y)^{\perp}}{|x-y|} \right\rangle dxdy$$
$$= \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}^2} W_N(z) \left\langle D^2 \phi(x) \frac{z}{|z|}, \frac{z^{\perp}}{|z|} \right\rangle dz \right) dx = 0$$

by (iii), and

$$\lim_{N \to \infty} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} W_N(x-y) R_{\phi}(x,y) \, dx \, dy = 0$$

because $R_{\phi}(x, y)$ is Lipschitz continuous with $|R_{\phi}(x, y)| \leq C |x - y|$. To complete the proof of the claims of part (iv), let us check that, when θ_N is the Dirichlet kernel, the property stated in (iv) coincides with the limit property (14). We have

$$\begin{split} \int_{\mathbb{T}^2} \left(H_{\phi} \right)_N (x, x) \, dx &= \sum_{|n'|_{\infty} \le N}^N \sum_{|n|_{\infty} \le N}^N \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} e^{2\pi i n' \cdot (x - x')} e^{2\pi i n \cdot (x - y')} H_{\phi} \left(x', y' \right) dy' dx' dx \\ &= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(\sum_{|n'|_{\infty} \le N}^N \sum_{|n|_{\infty} \le N}^N \int_{\mathbb{T}^2} e^{2\pi i n' \cdot (x' - x)} e^{2\pi i n \cdot (x - y')} dx \right) H_{\phi} \left(x', y' \right) dy' dx' \\ &= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} W_N \left(x' - y' \right) H_{\phi} \left(x', y' \right) dy' dx'. \end{split}$$

The proof is complete. \blacksquare

3.3.2 General case: ρ_0 of class LlogL

Assume now that ρ_0 satisfies only the assumptions of the main theorem. By Corollory C.3 in [7], there exists a sequence ρ_0^n of bounded continuous functions (in fact bounded smooth cylinder functions) that converges to ρ_0 in $L^1(B, \mu)$ and

$$C := \sup_{n \in \mathbb{N}} \int_{B} \rho_{0}^{n}(\omega) \log \rho_{0}^{n}(\omega) \mu(d\omega) < \infty.$$

For each n, apply the result of the first case and construct a weak solution ρ_t^n , which fulfills in particular

$$\int_{B} \rho_{t}^{n}\left(\omega\right) \log \rho_{t}^{n}\left(\omega\right) \mu\left(d\omega\right) \leq \int_{B} \rho_{0}^{n}\left(\omega\right) \log \rho_{0}^{n}\left(\omega\right) \mu\left(d\omega\right) \leq C.$$

From this inequality we deduce the existence of a subsequence, still denoted for simplicity by $\rho_t^n(\omega)$ which converges to some function ρ_t weak* in $L^1(0,T; L^1(B,\mu))$, which satisfies property (7), and moreover, from the duality of Orlicz spaces, such that

$$\int_{0}^{T} \int_{B} G(t,\omega) \rho_{t}^{n}(\pi_{N}\omega) \mu(d\omega) dt \to \int_{0}^{T} \int_{B} G(t,\omega) \rho_{t}(\omega) \mu(d\omega) dt$$

for all G such that, for some $\epsilon > 0$,

$$\sup_{t\in[0,T]} \int_{B} e^{\epsilon |G(t,\omega)|} \mu(d\omega) < \infty.$$
(15)

Due to these fact, in order to prove that ρ_t satisfies the weak formulation of the continuity equation, we have only to prove that

$$\int_{0}^{T} \int_{B} \left\langle b\left(\omega\right), DF\left(t,\omega\right) \right\rangle \rho_{t}^{n}\left(\omega\right) \mu\left(d\omega\right) dt \to \int_{0}^{T} \int_{B} \left\langle b\left(\omega\right), DF\left(t,\omega\right) \right\rangle \rho_{t}\left(\omega\right) \mu\left(d\omega\right) dt.$$

Since $G(t, \omega) := \langle b(\omega), DF(t, \omega) \rangle$ has property (15) by Theorem 8, this is true, and the proof is complete.

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