Classe di SCIENZE
Corso di perfezionamento in MATEMATICA
XXXIV ciclo
Tesi in cotutela con l'Università di Strasbourg
Anno Accademicao 2021-2022

A study of the spherical Hecke category via derived algebraic geometry

Tesi di perfezionamento in Matematica
MAT/03

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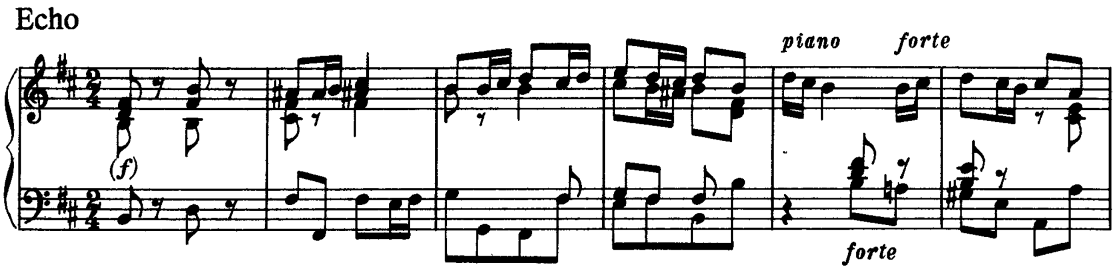
> […] *a child at play, playing draughts;*
> *a child's kingship.*
>
> Heraclitus



J.S. Bach, *Ouvertüre nach Französischer Art*, Echo

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Acknowledgements

I started to learn derived geometry and geometric Langlands for the first time at the beginning of my PhD, mostly during the pandemic. That has not been always easy, and I wish to deeply thank first of all my parents, with whom I lived during most of the time, for their constant and warm support and affection, and also for their help with my English writing.[1]

My advisors, Mauro Porta and Gabriele Vezzosi, were the ones who suggested the topic of this thesis to me in the first place and encouraged me to embrace the language of derived algebraic geometry. Geographical distance and the pandemic have often prevented me from having a frequent interaction with them, and our longest time together remains Spring 2019 in Berkeley, from which I will remember Mauro's elven verses from the foundational Noldor poem *Higher Topos Theory*, and Gabriele's unmistakable smile behind G. Karapet's portrait of a derived Artin stack.

Emanuele Pavia experienced with me "all the sorrows, but also the joy and marvel, of this journey": I owe him empathy, mathematics, many curses (especially in the Ghost type variant), and his company during two chocolate-based months of lockdown in Strasbourg.

I wish to send a special thank to Morena Porzio, for an infinite list of shared visions, feelings and laughs, midnight proofs, understanding support, perseverance, and good omens.

Thanks to Angelo Vistoli for sharing with me his knowledge and irony, Carlo Gasbarri for accepting to be my thesis director in Strasbourg, Pierre Baumann and Francesco Sala for their help through my quest for postdocs. I am deeply indebted to Roman Bezrukavnikov for hosting me at MIT in April and May 2022. Thanks also go to Dario Beraldo, Ivan Di Liberti, Jeremy Hahn, Andrea Gagna, Dennis Gaitsgory, Owen Gwilliam, Jacob Lurie, Valerio Melani and Sam Raskin for the fruitful oral and/or written conversations I carried on with them.

Thanks to my collaborators and friends: Peter Haine, Mark Macerato, Michele Pernice, Marco Volpe: I was lucky to carry on mathematical investigations with all of them, for their mixture of rigour and sense

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[1] About this, I must say that I often wished I knew better the English language, to whose inclination to a certain somberness I owe the tone of this page; I hope that my next things will be, if not more interesting, better written.

of wonder. A special thank to Peter, for sharing their knowledge, enthusiasm, and many thoughts about mathematical life, to Michele for willing to undertake such a thorough yet satisfying journey through derived geometry, and to Marco for the equally fruitful and fun period we spent in Regensburg earlier this year.

Finally, thanks, for their friendship, to Leonardo Abate, Gioacchino Antonelli, Marco Barbato, Lorenzo Benedini (who was so kind to share his house in Milan with me for five weeks), Andrea Cappelletti, Julian Demeio (to whom I owe one, and he knows why), Fabio Ferri, Alberto Morelli, Daniele Palombi, and especially to Davide Del Gusto for the many late-evening exchanges of thoughts and *boutades*. Thanks to Fabio Bentivoglio and Massimo Piccolomini, for their affection during the years.

And to my grandfather, source of joy and affection, goes my greatest hug and wishes to recover soon.

Chapter 1

Introduction: derived geometry in the Geometric Langlands Program

Contents

1.1 Background: Geometric Satake

1.1.1 Motivation

A classical problem in representation theory is the study of a reductive group G (e.g. GL_n, SL_n, $\mathbb{P}\mathrm{GL}_n$) and its Langlands dual $\check{G}$ (e.g. $\check{\mathrm{GL}}_n = \mathrm{GL}_n$, $\check{\mathrm{SL}}_n = \mathbb{P}\mathrm{GL}_n$). A celebrated result in the study of Langlands duality is the Satake theorem, which establishes an isomorphism between the $\mathbb{C}$-algebra of compactly supported $G(\mathbb{Z}_p)$-biinvariant functions on $G(\mathbb{Q}_p)$, called the *Hecke algebra* of G, and the (complexified) Grothendieck ring of finite dimensional representations of $\check{G}$. Ginzburg [Gin95] and later Mirkovic and Vilonen [MV07] provided a "sheaf theoretic" analogue (actually a categorification) of this theorem, called the **Geometric Satake Equivalence**: here G is a complex reductive group, and the statement has the form of an equivalence of tensor categories between the derived category of *equivariant perverse sheaves* $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$ and the category of finite dimensional representations of $\check{G}$ (see Section 1.1.2 below). The key new object here is the **affine Grassmannian** Gr_G, an infinite dimensional algebro-geometric object with the property that $\mathrm{Gr}_G(\mathbb{C}) = G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$.

The importance of results such as Geometric Satake, Derived Satake and their variants is related to the more general (and partly conjectural) **Geometric Langlands Duality**, which was introduced by Beilinson and Drinfeld in analogy to the celebrated arithmetic Langlands conjecture (see e.g. [BD05]). One can say that if the Geometric Langlands Duality deals with algebraic and geometric data related to a reductive group and a smooth complex curve X, the Geometric Satake Equivalence is a "specialization" which looks at the same data near a chosen closed point in X; the affine Grassmannian Gr_G itself, for example, is related to the local geometry of smooth curves, see Proposition 1.1.5 below. The use of techniques from homotopy theory and derived algebraic geometry in this field has provided many powerful results, and the current and most convincing formulations of the Geometric Langlands Duality are themselves derived in nature. One of the currently most accepted statements for a Geometric Langlands Conjecture is in [AG15].

Intuitively, the statement should satisfy the following requirements.

Conjecture 1.1.1. *Let G be a reductive complex group, and X a smooth projective complex curve. Then the Geometric Langlands Duality should at least:*

- *establish an equivalence between some category of sheaves over the stack* $\mathrm{Bun}_G(X)$ *of G-torsors over X and a category of sheaves over the stack of $\check{G}$-local systems on X.*
- *agree with the Geometric Satake Equivalence (see below) when specialized "at" any closed point of X.*

In order to explain Conjecture 1.1.1 we now present a short overview of the Geometric Satake Equivalence. The expert reader can skip directly to Section 1.3. We refer the reader seeking for a detailed explanation on this matter to the excellent notes [Zhu16], covering all of the next subsection and much more, including background, motivations and further developments.

1.1.2 Statement of the Geometric Satake Theorem

Theorem 1.1.2 (Geometric Satake Equivalence)**.** *Fix a reductive algebraic group G over $\mathbb{C}$, and a commutative ring k for which the left-hand-side of the following formula is defined: for example, k could be $\mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{Z}/\ell^n\mathbb{Z}, \overline{\mathbb{F}_\ell}$. There exists a symmetric monoidal structure $\star$ on $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G)$, called* convolution*, and an equivalence of symmetric monoidal abelian categories*

$$(\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k), \star) \simeq (\mathrm{Rep}^{\mathrm{fin}}(\check{G}, k), \otimes).$$

Let us explain the meaning of this statement. Here $\check{G}$ is the Langlands dual of G, obtained by dualising the root datum of the original group G, and the right-hand-side is the abelian category of finite-dimensional R-representations of $\check{G}$, equipped with a tensor (i.e. symmetric monoidal) structure given by the tensor product of representations.

In order to define the left hand side, we need to introduce some further definitions: by $G_\mathcal{O}$ we mean the representable functor $\mathbb{C}$-algebras $\to$ $\mathcal{S}$et, $R \mapsto \mathrm{Hom}(R[\![t]\!], G)$ (also denoted by $G(\mathbb{C}[\![t]\!])$), by $G_\mathcal{K}$ we mean the ind-representable functor $\mathbb{C}$-algebras $\to$ $\mathcal{S}$et, $R \mapsto \mathrm{Hom}(R(\!(t)\!), G)$ (also denoted by $G(\mathbb{C}(\!(t)\!))$), and by Gr_G we mean the **affine Grassmannian**, that is the stack quotient $\mathrm{Gr}_G = G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[\![t]\!])$. Ind-representability of $G_\mathcal{K}$ (and of Gr by consequence) comes from the fact that there is a natural filtration in finite-dimensional projective schemes $\mathrm{Gr}_{\leq N}$, $N \geq 0$, induced by [Zhu16, Theorem 1.1.3]. We will call this filtration the **lattice filtration**.

Remark 1.1.3. There is a natural action of $G_\mathcal{O}$ on Gr_G by left multiplication, whose orbits define an algebraic stratification of Gr_G over the poset $\mathbb{X}_\bullet(T)^+$. When viewed from the point of view of the complex-analytic topology, this stratification satisfies the so-called **Whitney conditions** (for a proof, see [Mat]). This allows to define the category $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_G, k)$, namely the abelian category of $G_\mathcal{O}$-equivariant perverse sheaves on Gr_G with values in k-modules. This category is defined as

$$\mathrm{colim}_N\, \mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr}_{\geq N}, k)$$

in the sense of [Zhu16, 5.1 and A.1.4].

The ind-scheme Gr_G is related to the theory of curves in the following sense: if X is a smooth projective complex curve, the formal neighborhood $\widehat{X}_x$ at a given closed point x of X is given by the map

$\phi_x : \operatorname{Spec} \mathbb{C}[\![t]\!] \to X$. The inclusion $\mathbb{C}[\![t]\!] \subset \mathbb{C}(\!(t)\!)$ induces a map $\operatorname{Spec} \mathbb{C}(\!(t)\!) \to \operatorname{Spec} \mathbb{C}[\![t]\!] \xrightarrow{\phi_x} X$ which is a model for the punctured formal neighbourhood $\mathring{X}_x$ of x.

Definition 1.1.4. Let $\mathbf{Bun}_G$ be the moduli stack of principal G-bundles. If a scheme Z over $\mathbb{C}$ is given, we define the relative version

$$\mathbf{Bun}_G^Z : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpd}$$

$$R \mapsto \{\text{principal } G\text{-bundles over } X \times \operatorname{Spec} R, \text{ flat over } \operatorname{Spec} R\}.$$

In the language of mapping stacks, we can write

$$\mathbf{Bun}_G^Z \simeq \mathbf{Map}_{\mathrm{Stacks}}(Z, \mathbf{Bun}_G).$$

Proposition 1.1.5. *For any closed point x of a smooth projective complex curve X, the functor Gr_G is equivalent to the following:*

$$\mathrm{Gr}_G^{\mathrm{loc}} : R \to \{\mathcal{F} \in \mathrm{Vect}_n(\hat{X}_x \times \operatorname{Spec} R), \alpha : \mathcal{F}|_{\mathring{X}_x \times \operatorname{Spec} R} \xrightarrow{\sim} \mathcal{T}_{GR}|_{\mathring{X}_x \times \operatorname{Spec} R}\} \tag{1.1.1}$$

where $\mathcal{T}_{GR}$ is the trivial G-torsor on $\hat{X}_x \times \operatorname{Spec} R$. In other words, Gr_G is equivalent to the fiber at the trivial bundle of the functor $\mathbf{Bun}_G^{\hat{X}_x} \to \mathbf{Bun}_G^{\mathring{X}_x}$.

Proof. The proof is explained for instance in [Zhu16, Proposition 1.3.6]. ☐

We will need the following version of the affine Grassmannian as well.

Construction 1.1.6. Define $\mathrm{Gr}_G^{\mathrm{glob}}$ as the fiber of the restriction map $\mathbf{Bun}_G^X \to \mathbf{Bun}_G^{X \setminus \{x\}}$, i.e. as the stack

$$R \to \{\mathcal{F}, \alpha : \mathcal{F}|_{(X\setminus\{x\}) \times \operatorname{Spec} R} \xrightarrow{\sim} \mathcal{T}_{GR}|_{(X\setminus\{x\}) \times \operatorname{Spec} R}\}.$$

Indeed, in the diagram

$$\begin{tikzcd}
\mathrm{Gr}_G^{\mathrm{glob}} \arrow[r] \arrow[d] & \mathbf{Bun}_G^X \arrow[r] \arrow[d] & \mathbf{Bun}_G^{\widehat{X_x}} \arrow[d] \\
\{\mathcal{T}_{G X\setminus\{x\}}\} \arrow[r] & \mathbf{Bun}_G^{X\setminus\{x\}} \arrow[r] & \mathbf{Bun}_G^{\widehat{\mathring{X}_x}}
\end{tikzcd}$$

the right-hand square is cartesian by the so-called Formal Gluing Theorem ([HPV16]), extending the theorem of Beauville and Laszlo [BL95]. Since the left-hand square is cartesian by definition, the outer square is cartesian. Therefore, $\mathrm{Gr}_G^{\mathrm{glob}}$ is isomorphic to the fiber of the restriction map $\mathbf{Bun}_G^{\widehat{X_x}} \to \mathbf{Bun}_G^{\widehat{\mathring{X}_x}}$, which is exactly $\mathrm{Gr}_G^{\mathrm{loc}}$. For more details, see [Zhu16, Theorem 1.4.2].

1.1.3 Convolution product of equivariant perverse sheaves

Now we explain the tensor structure on both sides. The category $\mathrm{Rep}(\check{G}, k)$ is equipped with the standard tensor product of representations; we define now the tensor structure given by **convolution product** on $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$. A more detailed account is given in [Zhu16, Section 1, Section 5.1, 5.4]. Consider the diagram

$$\begin{tikzcd}
& G_{\mathcal{K}} \times \mathrm{Gr}_G \arrow[r, "q"] \arrow[dl, "p"'] & G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G \arrow[dr, "m"] & \\
\mathrm{Gr}_G \times \mathrm{Gr}_G & & & \mathrm{Gr}_G
\end{tikzcd} \tag{1.1.2}$$

where $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$ is the stack quotient of the product $G_{\mathcal{K}} \times \mathrm{Gr}_G$ with respect to the "anti-diagonal" left action of $G_{\mathcal{O}}$ defined by $\gamma \cdot (g, [h]) = (g\gamma^{-1}, [\gamma h])$. The map p is the projection to the quotient on the first factor and the identity on the second one, the map q is the projection to the quotient by the "anti-diagonal" action of $G_{\mathcal{O}}$, and the map m is the multiplication map $(g, [h]) \mapsto [gh]$. It is important to remark that this construction, like everything else in this section, does not depend on the chosen $x \in X(\mathbb{C})$, since the formal neighbourhoods of closed points in a smooth projective complex curve are all (noncanonically) isomorphic.

Note also that the left multiplication action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}}$ and on Gr_G induces a left action of $G_{\mathcal{O}} \times G_{\mathcal{O}}$ on $\mathrm{Gr}_G \times \mathrm{Gr}_G$. It also induces an action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times \mathrm{Gr}_G$ given by (left multiplication, id) which canonically projects to an action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$. Note that p, q and m are equivariant with respect to these actions.

Now if $\mathcal{A}_1, \mathcal{A}_2$ are two $G_{\mathcal{O}}$-equivariant perverse sheaves on Gr_G, one can define a convolution product

$$\mathcal{A}_1 \star \mathcal{A}_2 = m_* \tilde{\mathcal{A}} \tag{1.1.3}$$

where m_* is the derived direct image functor, and $\tilde{\mathcal{A}}$ is any perverse sheaf on $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}_G$ which is equivariant with respect to the left action of $G_{\mathcal{O}}$ and such that $q^*\tilde{\mathcal{A}} = p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2)$. (Of course, the tensor product must be understood as a derived tensor product in the derived category.) Note that such an $\tilde{\mathcal{A}}$ exists because q is the projection to the quotient and $\mathcal{A}_2$ is $G_{\mathcal{O}}$-equivariant.

This is the tensor structure that we are considering on $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$.

Remark 1.1.7. Note that m_* carries perverse sheaves to perverse sheaves: indeed, it can be proven that m is ind-proper, i.e. it can be represented by a filtered colimit of proper maps of schemes compatibly with the lattice filtration. By [KW01, Lemma III.7.5], and the definition of $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k)$ as a direct limit, this ensures that m_* carries perverse sheaves to perverse sheaves.

Remark 1.1.8. Note that the convolution product can be described as follows. Consider the diagram of stacks

$$\begin{tikzcd}
 & (G_{\mathcal{O}} \times G_{\mathcal{O}})\backslash(G_{\mathcal{K}} \times \mathrm{Gr}) \arrow[r, "\sim"] \arrow[dl, "p"'] & G_{\mathcal{O}}\backslash(G_{\mathcal{K}}\tilde{\times}\mathrm{Gr}) \arrow[dr, "m"] & \\
G_{\mathcal{O}}\backslash\mathrm{Gr} \times G_{\mathcal{O}}\backslash\mathrm{Gr} & & & G_{\mathcal{O}}\backslash\mathrm{Gr}
\end{tikzcd}$$

where all the actions are induced by the *left* multiplication action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}}$. Then:

- the horizontal map is an equivalence;
- a $G_{\mathcal{O}}$-equivariant perverse sheaf on Gr is the same thing as a perverse sheaf on $G_{\mathcal{O}}\backslash\mathrm{Gr}$;
- the convolution product is equivalently described (up to shifts and perverse truncations) by

$$\mathcal{A}_1 \star \mathcal{A}_2 = \bar{m}_*(\bar{p}^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2)).$$

Observations similar to Construction 1.1.6 prove the following:

Proposition 1.1.9. *We have the following equivalences of schemes or ind-schemes:*

- $G_{\mathcal{O}} \simeq \mathrm{Aut}_{\widehat{X}_x}(\mathcal{T}_G)$
- $G_{\mathcal{K}}(R) \simeq \{\mathcal{F} \in \mathrm{Bun}_G(X \times \operatorname{Spec} R), \alpha : \mathcal{F}|_{(X\backslash\{x\})\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{(X\backslash\{x\})\times\operatorname{Spec} R}, \mu : \mathcal{F}|_{\widehat{X}_x\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{\widehat{X}_x} \times \operatorname{Spec} R\}$
- $(G_{\mathcal{K}} \times \mathrm{Gr}_G)(R) \simeq \{\mathcal{F}, \alpha : \mathcal{F}|_{(X\backslash\{x\})\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{(X\backslash\{x\})\times\operatorname{Spec} R}, \mu : \mathcal{F}|_{\widehat{X}_x\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{\widehat{X}_x\times\operatorname{Spec} R}, \mathcal{G}, \beta : \mathcal{F}|_{(X\backslash\{x\})\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{(X\backslash\{x\})\times\operatorname{Spec} R}\}$
- $(G_{\mathcal{K}}\times^{G_{\mathcal{O}}}\mathrm{Gr}_G)(R) \simeq \{\mathcal{F}, \alpha : \mathcal{F}|_{(X\backslash\{x\})\times\operatorname{Spec} R} \simeq \mathcal{T}_G|_{(X\backslash\{x\})\times\operatorname{Spec} R}, \mathcal{G}, \eta : \mathcal{F}|_{(X\backslash\{x\})\times\operatorname{Spec} R} \simeq \mathcal{G}_{(X\backslash\{x\})\times\operatorname{Spec} R}\}$.

Let us finally explain the meaning of Conjecture 1.1.1. Thanks to Proposition 1.1.5 and Construction 1.1.6, there is a way to interpret $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ as the "specialization at any point of x" of the "$\mathrm{Bun}_G(X)$" side of the Geometric Langlands Conjecture, and $\mathrm{Rep}(\check{G})$ as the specialization of the "local systems" side. In this sense, one wants the Geometric Langlands Conjecture to agree with the Geometric Satake Equivalence.

1.2 The spherical Hecke category

1.2.1 Equivariant constructible sheaves

We now review the notion of equivariant constructible sheaves on the affine Grassmannian. Recall that the affine Grassmannian admits a stratification in Schubert cells. Let k be a finite ring. We can consider the ∞-category of sheaves with coefficients in k which are constructible with respect to that stratification. We denote this category by $\mathcal{C}\mathrm{ons}(\mathrm{Gr}, \mathscr{S}; k)$. Here we are not assuming any finite-dimensionality constraint on the stalks nor on the cohomology of our sheaves. We denote by $\mathcal{C}\mathrm{ons}^{\mathrm{fd}}(\mathrm{Gr}, \mathscr{S}; k)$ the small subcategory of $\mathcal{C}\mathrm{ons}(\mathrm{Gr}, \mathscr{S}; k)$ spanned by constructible sheaves with finite stalks. The ∞-category $\mathcal{C}\mathrm{ons}(\mathrm{Gr}, \mathscr{S}; k)$ admits a t-structure whose heart is the category of perverse sheaves which are constructible with respect to $\mathscr{S}$. We also consider the category $\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; k)$ of $G_{\mathcal{O}}$-equivariant constructible sheaves with respect to $\mathscr{S}$, defined as

$$\lim \Big(\dots \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \mathcal{C}\mathrm{ons}(G_{\mathcal{O}} \times \mathrm{Gr}, \mathscr{S}; k) \substack{\longleftarrow \\ \longleftarrow} \mathcal{C}\mathrm{ons}(\mathrm{Gr}, \mathscr{S}; k) \Big), \tag{1.2.1}$$

where the stratification on $G_{\mathcal{O}} \times \dots \times \dots G_{\mathcal{O}} \times \mathrm{Gr}$ is trivial on the first factors and $\mathscr{S}$ on the last one. Note that there exists a notion of category of equivariant constructible sheaves with respect to *some* stratification (instead of a fixed one, like $\mathscr{S}$ in our case). In full generality, let us fix an algebraic group H acting on a scheme X defined over a field K, and let us denote by $\mathscr{S}$ the orbit stratification on X. We have a pullback square of triangulated (or dg, or stable ∞-[1]) categories (full faithfulness of the vertical arrows comes from the fact that the transition maps in the colimits are fully faithful).

$$\begin{tikzcd}
\mathcal{C}\mathrm{ons}_H(X, \mathscr{S}; k) \arrow[r] \arrow[d, hook] & \mathcal{C}\mathrm{ons}(X, \mathscr{S}; k) \arrow[d, hook] \\
\mathcal{D}_{\mathrm{c},H}(X; k) \arrow[r] & \mathcal{D}_{\mathrm{c}}(X; k).
\end{tikzcd} \tag{1.2.2}$$

where

$$\mathcal{D}_{\mathrm{c}}(X; k) = \mathrm{colim}_{\mathscr{S} \text{ stratification of } X} \, \mathcal{C}\mathrm{ons}(X, \mathscr{S}; k)$$

and

$$\mathcal{D}_{\mathrm{c},H}(X; k) = \lim \Big(\dots \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \mathcal{D}_{\mathrm{c}}(H \times X; k) \substack{\longleftarrow \\ \longleftarrow} \mathcal{D}_{\mathrm{c}}(X; k) \Big).$$

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[1] In this thesis, we adopt the perspective of ∞-categories, which is systematically exposed in [Lur09]. Both dg-categories and ∞-categories are ways to encode the idea of "categories with a notion of homotopy and homotopy equivalence" in a way that is particularly useful to deal with derived categories and homotopy theory. One of the simplest formulations of this concept is the notion of categories enriched in topological spaces or simplicial sets.

Now, the horizontal arrows in (1.2.2) are not equivalences, although they are while restricted to the abelian subcategories of perverse sheaves. Indeed, the forgetful functor

$$\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; k) \to \mathcal{P}\mathrm{erv}(\mathrm{Gr}, \mathscr{S}; k)$$

is an equivalence (see for example [BR18, Section 4.4]), but

$$\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}, \mathscr{S}; k) \to \mathcal{C}\mathrm{ons}(\mathrm{Gr}, \mathscr{S}; k)$$

is not: its essential image only generates the target as a triangulated category ([Ric]).

On the contrary, the left vertical arrow of (1.2.2) is an equivalence. Indeed, the functor is fully faithful because the transition maps in the diagram of which we take the colimit are, and essentially surjective by the following argument. First of all, we reduce to the finite-dimensional terms of the filtration $\{\mathrm{Gr}_N\}_{N\in\mathbb{N}}$ (which can be done by the very definition of category of constructible sheaves on an infinite-dimensional variety). Then we apply the following lemma:

Lemma 1.2.1. *Let H be an group scheme acting on a finite-dimensional scheme Y, and suppose that the orbits form a stratification $\mathscr{S}$ of Y. Then the two categories $\mathcal{D}_{\mathrm{c},H}(Y)$ and $\mathcal{C}\mathrm{ons}_H(Y, \mathscr{S})$ are equivalent.*

Proof. Let us consider an equivariant constructible sheaf $\mathcal{F}$ (with respect to some stratification) and the maximal open subset U of Y where the sheaf is locally constant: this is nonempty since we know that $\mathcal{F}$ is constructible with respect to some stratification. Then U is H-stable by equivariancy of $\mathcal{F}$ and maximality of U, and thus its complementary is, and we can apply Noetherian induction. ☐

Remark 1.2.2. Let $\mathrm{Gr}_G^{\mathrm{an}}$ (resp. $G_{\mathcal{O}}^{\mathrm{an}}$) be the analytic ind-variety (resp. analytic group) corresponding to Gr_G (resp $G_{\mathcal{O}}$). There is an equivalence between the (triangulated, or dg/∞-) category of *algebraic* $G_{\mathcal{O}}$-equivariant constructible sheaves over the affine Grassmannian, and that of *analytic* $G_{\mathcal{O}}^{\mathrm{an}}$-equivariant constructible sheaves over $\mathrm{Gr}_G^{\mathrm{an}}$. Indeed, by [BGH20, Proposition 12.6.4] (in turn building on [Art72, Théorème XVI. 4.1]), there is an equivalence of categories

$$\mathcal{D}_{\mathrm{c}}^{\mathrm{alg}}(\mathrm{Gr}_G) \simeq \mathcal{D}_{\mathrm{c}}^{\mathrm{an}}(\mathrm{Gr}_G^{\mathrm{an}}),$$

and the construction which adds the equivariant structure is the same on both sides (the analytification functor commutes with colimits and finite limits).

1.2.2 Derived Satake Theorem and the $\mathbb{E}_3$-monoidal structure

The Geometric Langlands Conjecture is currently formulated as a "derived" statement (see [AG15]. Bezrukavnikov and Finkelberg [BF07] have proven the so-called **Derived Satake Theorem**. There, the abelian category $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G; k)$ is replaced by $\mathrm{Sph}^{\mathrm{loc.c.}}(G)$ (the small *spherical Hecke category*), which is a higher category admitting the following presentations:

- as the dg- or ∞-category $\mathrm{Cons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}_G; k)$ of $G_{\mathcal{O}}$-equivariant constructible sheaves on Gr_G with finite-dimensional stalks;
- as the dg- or ∞-category $\text{D-mod}_{G_{\mathcal{O}}}(\mathrm{Gr}_G; k)$ $G_{\mathcal{O}}$-equivariant D-modules on Gr_G with finite-dimensional stalks.

Theorem 1.2.3 (Derived Satake Theorem,[BF07, Theorem 5])**.** *There is an equivalence*

$$\mathcal{C}\mathrm{ons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}_G) \simeq \mathrm{Coh}(\mathrm{Spec}\,\mathrm{Sym}(\check{\mathfrak{g}}^*[1]))^{\check{G}}.$$

The category $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ is the heart of a t-structure on $\mathrm{Sph}^{\mathrm{loc.c}}(G)$, and the Geometric Satake Theorem is indeed recovered from the Derived Satake Theorem by passing to the heart. The same diagram as in (1.1.2) provides the formula for the convolution product of constructible sheaves, but the commutativity of the product is lost. However, in [Noc20] we recovered techniques similar to the ones that provide the commutativity of $\star$ in the perverse case, in order to prove a subtler result.

Theorem 1.2.4 ([Noc20])**.** *The ∞-category* $\mathrm{Sph}^{\mathrm{loc.c}}(G)$ *admits an $\mathbb{E}_3$-monoidal structure in $\mathcal{C}\mathrm{at}^{\times}_{\infty}$, extending the symmetric monoidal convolution product of perverse sheaves.*

Theorem 1.2.5. *The ∞-category* $\mathrm{Sph}^{\mathrm{loc.c}}(G)$ *is equivalent, as an $\mathbb{E}_3$-∞-category, to the $\mathbb{E}_2$-center of the derived ∞-category of finite-dimensional representations*

$$\mathrm{DRep}^{\mathrm{fd}}(\check{G}, k).$$

Both theorems were originally stated by Gaitsgory and Lurie in unpublished work. The second one follows essentially from the Derived Satake Theorem and work of Ben-Zvi, Francis, Nadler and Preygel [BFN10],[BNP17], and it implies the first one. In Chapter 2, we prove Theorem 1.2.4 independently, building the sought $\mathbb{E}_3$-monoidal structure in an intrinsic way. This is in the same spirit of the Tannakian reconstruction explained above for the case of perverse sheaves, where the existence of a symmetric monoidal structure on the category is a part of the initial datum, and only a posteriori it is interpreted as the natural tensor product in a category of representations. To be precise, we do not work exactly with the usual small spherical category, but with a big version which is presentable Theorem 2.3.6, and then we deduce the sought result as a corollary Corollary 2.3.7.

Remark 1.2.6. It is worth noticing that the heart of an $\mathbb{E}_3$ stable ∞-category $\mathcal{C}^{\otimes}$ with a compatible t-structure is a symmetric monoidal category, whereas for $\mathcal{C}$ $\mathbb{E}_1$-monoidal one only recovers a monoidal category. In other words, an $\mathbb{E}_3$-monoidal structure for $\mathrm{Sph}^{\mathrm{loc.c}}(G)$ is the "least level" of commutativity allowing to recover the full symmetric monoidal structure on $\mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr})$ in a purely formal way.

Remark 1.2.7. In recent yet unpublished work [Cam22], Campbell and Raskin proved that, up to a certain renormalization of both sides, one can put a natural factorizable structure on the RHS of the Derived Satake Equivalence, and that the equivalence can be promoted to a factorizable/$\mathbb{E}_3$-monoidal equivalence.

Remark 1.2.8. It is worth stressing that the complex topology takes on a prominent role in our proofs. When taking constructible sheaves, we always look at the underlying complex-analytic topological space $\mathrm{Gr}_G^{\mathrm{an}}$ of Gr_G, with its complex stratification in Schubert cells. With this topology, $\mathrm{Gr}_G^{\mathrm{an}}$ is actually homotopy equivalent to $\Omega^2\mathrm{B}(G^{\mathrm{an}})$ (although we do not use this in our paper). This is an equivalent way to derive the $\mathbb{E}_2$-algebra structure on the spherical category. However, by Remark 1.2.2, the category $\mathrm{Sph}^{\mathrm{loc.c}}$ is the same both from the algebraic or the complex-analytic point of view and therefore our result works in both settings.

1.3 An overview

The main guiding principle of the present dissertation (excluding Chapter 4) is the application of the homotopy theory of stratified spaces (as introduced in [Lur17, Appendix A]) to the study of the affine Grassmannian and the spherical Hecke category. This idea was not part of the project at the beginning, but arose in response to some issues in the description of the tensor structure on the spherical Hecke category; later, it took several other forms. We will now highlight which ones and how they arose.

Before doing this, let us just sketch the structure of the present dissertation. This thesis covers my first three works (two of them with coauthors). Chapter 2 is dedicated to the preprint [Noc20], which has been the start of many questions leading to the other works. Chapter 3 contains my work with Marco Volpe [NV21], which deals with problems of more topological nature and exploits phenomena related to the formalism of conically smooth spaces. Chapter 4 contains my work with Michele Pernice on Derived Azumaya Algebras [NP22]. I will explain the connection of this work to my PhD project in Section 1.3.3.

1.3.1 The affine Grassmannian as a Whitney stratified space

As mentioned in Remark 1.1.3, the stratification in Schubert cells of the affine Grassmannian satisfies the Whitney conditions. Now, a stratification can be seen as a continuous map s from the topological space Y

to a poset P endowed with the Alexandrov topology (Section 2.A.1). A very important consequence of this property is recorded in the following:

Proposition 1.3.1. *Let* (Y, P, s) *be a stratified space satisfying the Whitney conditions. Then the stratification is conical in the sense of [Lur17, Definition A.5.5].*

Proof. This is a consequence of [Mat70, Proposition 6.2], as proven in Section 3.2.1. ☐

Theorem 1.3.2 ([Lur17, Theorem A.3.9])**.** *Let* (Y, P, s) *be a stratified space which is locally of singular shape (see [Lur17, Definition A.4.15]) and conically stratified, and suppose that* P *satisfies the ascending chain condition. Then there exists an* ∞*-category* $\mathrm{Exit}(Y, P, s)$ *and an equivalence of the form*

$$\mathcal{C}\mathrm{ons}(Y, P, s; \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{Exit}(Y, P, s), \mathcal{S})$$

where $\mathcal{S}$ *is the* ∞*-category of spaces.*

Two easy but important consequences of this theorem are the following:

Corollary 1.3.3. *Let* (Y, P, s) *be a stratified space which is locally of singular shape and conically stratified. Then its* ∞*-category of constructible sheaves is presentable.*

Corollary 1.3.4. *Suppose that* $(Y, P, s) \to (Z, P, t)$ *is a stratified homotopy equivalence of stratified spaces over the same poset* P *(i.e. a stratified map inducing equivalences at the level of exit-paths-categories). Then it induces an equivalence of* ∞*-categories*

$$\mathcal{C}\mathrm{ons}(Y, P, s; \mathcal{S}) \simeq \mathcal{C}\mathrm{ons}(Z, P, t; \mathcal{S}).$$

Another important consequence regards the functoriality and the symmetric monoidality properties of the association $(Y, P, s) \mapsto \mathcal{C}\mathrm{ons}(Y, P, s; \mathcal{S})$. The exact same arguments work if we replace $\mathcal{S}$ by the derived ∞-category Mod_k, with k a ring. We record all these consequences (proving those which have not been proven before) in Section 2.A.

These facts allow to study the affine Grassmannian and the spherical Hecke category $\mathrm{Sph}(G)$ from the viewpoint of **stratified homotopy theory**. Thanks to this, we may deduce properties about $\mathrm{Sph}(G)$ from homotopy-theoretic considerations about the affine Grassmannian. This is the main content of Chapter 2, and leads to prove Theorem 1.2.4 by arguments which are purely intrinsic to the topology of Gr_G and its variants, and do not involve anything regarding the spectral side of Geometric Langlands.

1.3.2 The affine Grassmannian as a conically smooth stratified space

A much subtler version of Proposition 1.3.1 can be proven. Indeed, the notion of "conical chart" around a point can be promoted to something more rigid, which represents the analogue in the stratified setting of a smooth differential structure on a topological space. This notion is made rigorous in [AFT17, Section 3], where the Authors define what is called a **conically smooth structure** on a conically stratified space[2]. Roughly speaking, the idea is the same as that of a differential structure: a conically smooth structure is an equivalence class of atlases, and an atlas is a system of conical charts with a "smooth change of charts" property. A stratified space (Y, P, s) together with a conically smooth structure $\mathcal{A}$ is called a *conically smooth space*. The Authors conjecture that every stratified space satisfying the Whitney conditions (which is conical by Proposition 1.3.1) admits such a conically smooth structure. In Chapter 3 (joint work with Marco Volpe) we prove this conjecture:

Theorem 1.3.5 (Theorem 3.2.7)**.** *Let (Y, P, s) be a stratified space satisfying the Whitney conditions. Then it admits a canonical conically smooth structure.*

This result is somehow useful *per se*, in that it adds a very natural class of examples to the newly introduce theory of conically smooth spaces. However, our middle-term plan together with Marco Volpe is to use this fact to prove additional properties of the affine Grassmannian. Indeed, by Theorem 1.3.5 Gr_G admits a canonical conically smooth structure, and so does its Ran version $\mathrm{Gr}_{\mathrm{Ran}}$ (see Definition 2.1.3,Proposition 2.2.5). Now, [AFT17] introduce the notion of **constructible bundle**, which is a certain class of maps between conically smooth spaces representing the stratified analogue of the notion of smooth fiber bundle.

Our conjecture together with Marco Volpe is the following:

Conjecture 1.3.6. *The stratified map $\mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}}) \to \mathrm{Ran}(X^{\mathrm{an}})$ (see Section 2.2) is a constructible bundle, and it satisfies suitably strong surjectivity properties.*

Also, in analogy to what happens with fiber bundles, we conjecture the following:

Conjecture 1.3.7. *Every constructible bundle $(Y, P, s, \mathcal{A}) \to (Z, Q, t, \mathcal{B})$ satisfying the aforementioned surjectivity properties satisfies the exit homotopy lifting property. That is, any diagram of stratified maps the form*

$$\begin{tikzcd}
H \times \{0\} \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
H \times I \arrow[r, "h"] \arrow[ur, dotted] & Z
\end{tikzcd}$$

---

[2] To be precise, their notion of "conically stratified space" is slightly different from that of [Lur17], and is defined by induction.

where (H, R, u) *is a compact stratified space and* I *is the unit interval stratified with a closed stratum at* 0 *and an open stratum at* $(0, 1]$*, admits a filling.*

As a corollary, we would obtain that the map $\mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}}) \to \mathrm{Ran}(X^{\mathrm{an}})$ (see Construction 2.2.3) satisfies the exit homotopy lifting property. This is a much used "folklore" fact, whose proof we have not been able to locate anywhere. We use this fact in Section 2.B.4, and it is a crucial step in the proof of Theorem 1.2.4.

1.3.3 Towards an affine Grassmannian for surfaces

A long-term goal in the Geometric Langlands Program is to provide a statement of the Geometric Langlands Conjecture regarding a moduli space of sheaves (or stacks) over a surface S, which should replace $\mathbf{Bun}_G(X)$ where X is a curve. Such a statement is expected to have connections to representation-theoretic objects such as double Hecke algebras or similar constructions.

A part of this program is to provide statements analogous to the Geometric or Derived Satake Theorem, again following the idea that they should represent the "specialization at a point" of the "global" statement. Therefore, one is naturally led to seek for an analogue of the affine Grassmannian Gr_G at the level of surfaces.

For instance, one could fix a complex smooth projective surface S and a point $s \in S(\mathbb{C})$, and define

$$\mathrm{Gr}_G(S, s)(R) = \{\mathcal{F} \in \mathrm{Bun}_G(S_R), \alpha : \mathcal{F}|_{(S \setminus \{x\}) \times \mathrm{Spec}\, R} \xrightarrow{\sim} \mathcal{T}_{G,(S\setminus\{x\})\times \mathrm{Spec}\, R}\}.$$

However, one can prove by means of the Hartogs theorem that this functor is "trivial", because every trivialization away from a closed subset of codimension at least 2 extends to S_R, and therefore our moduli space is equivalent to G (seen as the automorphism group of $\mathcal{T}_{G,S}$).

Alternatively, one could fix an algebraic curve C in S and define

$$\mathrm{Gr}_G(S, C)(R) = \{\mathcal{F} \in \mathrm{Bun}_G(S_R), \alpha : \mathcal{F}|_{(S \setminus C) \times \mathrm{Spec}\, R} \xrightarrow{\sim} \mathcal{T}_{G,(S\setminus C)\times \mathrm{Spec}\, R}\}.$$

This last definition presents an important difference with respect to the setting of curves when it comes to the convolution product. Indeed, one can figure out suitable versions of the convolution diagram, but the analogue of the map m in (1.1.2) is not ind-proper ([Kapoo, Proposition 2.2.2]), and therefore we are not granted that the pushforward (or the proper pushforward) along that map takes perverse sheaves to perverse sheaves. However, $m_!$ should preserve constructible sheaves, provided that it carries a sufficiently strong stratified structure. Therefore, although a convolution product of perverse sheaves over $\mathrm{Gr}_G(S, x)$ or $\mathrm{Gr}_G(S, C)$ is probably not well-defined, there is a good chance that it is well-defined at the level of some ∞-category of (equivariant) constructible sheaves $\mathcal{C}$.

Another possible version of an affine Grassmannian for surfaces is the following (the additional "g" stays for "gerbes"). Let S be a smooth complex surface and $s \in S(\mathbb{C})$. We define

$$\mathrm{Grg}_{G,S,s}(R) = \{\mathcal{G} \text{ a } G\text{-gerbe over } S_R, \alpha : \mathcal{G}|_{(S\setminus\{s\})\times \mathrm{Spec}\, R} \xrightarrow{\sim} \mathrm{BG} \times (S \setminus \{s\}) \times \mathrm{Spec}\, R)\}.$$

This definition circulated in the mathematical community years ago, and I am currently studying the properties of such an object in an ongoing project. This motivated by interest in the theory of gerbes. In the case $G = \mathrm{GL}_1 = \mathbb{G}_\mathrm{m}$, the following result of Toën holds:

Theorem 1.3.8 ([Toë10], see also Section 1.3.4)**.** *If S is a quasicompact quasiseparated scheme, then there is an isomorphism of abelian groups between* $\mathrm{H}^2(S, \mathbb{G}_\mathrm{m}) \times \mathrm{H}^1(S, \mathbb{Z})$ *and the group* $\mathrm{dBr}_{\mathrm{Az}}$ *of* ***derived Azumaya algebras*** *over S up to Morita equivalence. In particular, if S is normal, there is an isomorphism*

$$\mathrm{H}^2(S, \mathbb{G}_\mathrm{m}) \simeq \mathrm{dBr}_{\mathrm{Az}}.$$

This correspondence is useful in proving properties of $\mathrm{Grg}_{G,S,s}$, although $\mathrm{Grg}_{\mathbb{G}_\mathrm{m},S,s}$ is itself a fairly trivial object. Indeed, Jacob Lurie [Lur] suggested to us a way to prove that a "Beauville-Laszlo theorem for G-gerbes" is true (and thus, that $\mathrm{Grg}_{G,S,s}$ is independent of S and s) that reduce to the case when G is abelian and then uses a reformulation of Toën's result in terms of prestable $\mathcal{O}_S$-linear prestable presentable ∞-categories (we will talk about this refurmulation in Section 1.3.4).

Motivated by such applications, Michele Pernice and I studied some further properties of the correspondence established in Theorem 1.3.8. This is the content of Chapter 4, which we summarize in Section 1.3.4.

A final remark: we suspect that techniques analogous to the ones used for the construction of the $\mathbb{E}_3$-structure on $\mathrm{Sph}(G)$ can be used to build an $\mathbb{E}_5$-structure[3] on a suitable category of equivariant constructible sheaves/D-modules on Grg_G. A naïve topological motivation comes from the fact that we can define a "topological affine Grassmannian of G-gerbes on a surface" as

$$\mathrm{Map}(\mathrm{S}^3, \mathrm{AutB}(G^{\mathrm{an}})/\,\mathrm{Map}(\mathrm{D}^4, \mathrm{AutB}(G^{\mathrm{an}}))$$

where S^3 is the real 3-sphere, D^4 is the real 4-disk, G^{an} is the complex analytic topological group associated to G, and $\mathrm{AutB}(G^{\mathrm{an}})$ is the 2-group of automorphisms of the trivial topological G^{an}-gerbe. Now this space is homotopy equivalent to

$$\Omega^3\mathrm{AutB}(G^{\mathrm{an}}) \sim \Omega^4\mathrm{BAutB}(G^{\mathrm{an}}),$$

---

[3] Here $\mathbb{E}_1$ would stand for an "associativity" coming from some convolution diagram, and $\mathbb{E}_4$ would come from the real dimension of S, just like $\mathbb{E}_2$ comes from the real dimension of the curve X in our construction.

where $\mathrm{BAutB}(G^{\mathrm{an}})$ is the higher classifying space of $\mathrm{AutB}(G^{\mathrm{an}})$, and on the latter space we have an evident $\mathbb{E}_4$-structure which is the exact analogue of the $\mathbb{E}_2$-structure on $\Omega^2\mathrm{B}(G^{\mathrm{an}}) \sim \mathrm{Gr}_G^{\mathrm{an}}$. Not that the relationship between $\mathrm{Grg}_G^{\mathrm{an}}$ and this topological counterpart is at the moment unclear to us (we do not know an immediate argument to deduce that they are equivalent; the analogous statement for Gr_G is already nontrivial), and therefore the existence of the $\mathbb{E}_4$-structure on $\Omega^4\mathrm{BAut}(G^{\mathrm{an}})$ remains just a heuristic motivation.

1.3.4 The derived Brauer map via twisted sheaves

Let us fix a quasicompact quasiseparated scheme X over some field k of arbitrary characteristic.

In 1966, Grothendieck [Gro66] introduced the notion of *Azumaya algebra* over X: this is an étale sheaf of algebras which is locally of the form $\mathcal{E}\mathrm{nd}(\mathcal{E})$, the sheaf of endomorphisms of a vector bundle $\mathcal{E}$ over X. This is indeed a notion of "local triviality" in the sense of *Morita theory*: two sheaves of algebras A, A' are said to be *Morita equivalent* if the categories

$$\mathcal{LM}\mathrm{od}_A = \{\mathcal{F} \text{ quasicoherent sheaf over } X \text{ together with a left action of } A\}$$

and its counterpart $\mathcal{LM}\mathrm{od}_{A'}$ are (abstractly) equivalent; one can prove that $\mathcal{LM}\mathrm{od}_{\mathcal{E}\mathrm{nd}(\mathcal{E})}$ is equivalent to $\mathrm{QCoh}(X)$ via the functor $M \mapsto \mathcal{E}^\vee \otimes_{\mathcal{E}\mathrm{nd}(\mathcal{E})} M$.

The classical *Brauer group* $\mathrm{Br}_{\mathrm{Az}}(X)$ of X is the set of Azumaya algebras up to Morita equivalence, with the operation of tensor product of sheaves of algebras. Grothendieck showed that this group injects into $\mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}})$ by using cohomological arguments: essentially, he used the fact that a vector bundle corresponds to a GL_n-torsor for some n, and that there exists a short exact sequence of groups

$$1 \to \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

The image of $\mathrm{Br}_{\mathrm{Az}}(X)$ inside $\mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}})$ is contained in the torsion subgroup, which is often called the *cohomological Brauer group* of X.[4]

One of the developments of Grothendieck's approach to the study of the Brauer group is due to Bertrand Toën and its use of derived algebraic geometry in [Toë10]. There, he introduced the notion of *derived Azumaya algebra* as a natural generalization of the usual notion of Azumaya algebra. Derived Azumaya algebras over X form a dg-category $\mathcal{D}\mathrm{eraz}_X$. There is a functor $\mathcal{D}\mathrm{eraz}_X \to \mathbb{D}g^c(X)$, this latter being (in Toën's notation) the dg-category of presentable stable $\mathcal{O}_X$-linear dg-categories[5] which are compactly generated, with, as morphisms, functors preserving all colimits. The functor is defined by

$$A \mapsto \mathcal{LM}\mathrm{od}_A = \{\text{quasicoherent sheaves on } X \text{ with a left action of } A\}$$

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[4] Other authors, however, use this name for the whole $\mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}})$.

[5] See Section 4.1.2 for a definition of $\mathcal{O}_X$-linear category in the ∞-categorical setting.

where all terms have now to be understood in a derived sense. One can prove that this functor sends the tensor product of sheaves of algebras to the tensor product of presentable $\mathcal{O}_X$-linear dg-categories, whose unit is $\mathrm{QCoh}(X)$. Building on classical Morita theory, Toën defined two derived Azumaya algebras to be Morita equivalent if the dg-categories of left modules are (abstractly) equivalent. This agrees with the fact mentioned above that $\mathcal{L}\mathcal{M}\mathrm{od}_{\mathcal{E}\mathrm{nd}(\mathcal{E})} \simeq \mathrm{QCoh}(X)$ for any $\mathcal{E} \in \mathrm{Vect}_n(X)$.

In [Toë10, Proposition 1.5], Toën characterized the objects in the essential image of the functor $A \mapsto \mathcal{L}\mathcal{M}\mathrm{od}_A$ as the compactly generated presentable $\mathcal{O}_X$-linear dg-categories which are *invertible* with respect to the tensor product, i.e. those M for which there exists another presentable compactly generated $\mathcal{O}_X$-linear dg-category $M^\vee$ and equivalences $\mathbf{1} \xrightarrow{\sim} M \otimes M^\vee$ and $M^\vee \otimes M \xrightarrow{\sim} \mathbf{1}$. The proof of this characterization goes roughly as follows: given a compactly generated invertible dg-category M, one can always suppose that M is generated by some single compact generator $\mathcal{E}_M$. Now, compactly generated presentable $\mathcal{O}_X$-linear dg-categories satisfy an important descent property [Toë10, Theorem 3.7]; from this and from [Toë10, Proposition 3.6] one can deduce that the $\mathrm{End}_M(\mathcal{E}_M)$ has a natural structure of a quasicoherent sheaf of $\mathcal{O}_X$-algebras A, and one can prove that $M \simeq \mathcal{L}\mathcal{M}\mathrm{od}_A$.

Both Antieau-Gepner [AG14] and Lurie [Lur18, Chapter 11] resumed Toën's work, using the language of ∞-categories in replacement of that of dg-categories. Lurie also generalized the notion of Azumaya algebra and Brauer group to spectral algebraic spaces, see [Lur18, Section 11.5.3]. He considers the ∞-groupoid of compactly generated presentable $\mathcal{O}_X$-linear ∞-categories which are invertible with respect to the Lurie tensor product $\otimes$, and calls it the *extended Brauer space* $\mathcal{B}\mathrm{r}^\dagger_X$. This terminology is motivated by the fact that the set $\pi_0 \mathcal{B}\mathrm{r}^\dagger_X$ has a natural abelian group structure, and by a result of Toën is isomorphic to $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m}) \times \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z})$: in particular, it contains the usual cohomological Brauer group of X. At the categorical level, Lurie proves that there is an equivalence of ∞-groupoids between $\mathcal{B}\mathrm{r}^\dagger(X)$ and $\mathrm{Map}_{\mathrm{St}_k}(X, \mathrm{B}^2\mathbb{G}_\mathrm{m} \times \mathrm{B}\mathbb{Z})$ (St_k is the ∞-category of stacks over the base field k). In particular, $\mathcal{B}\mathrm{r}^\dagger(X)$ is categorically equivalent to a 2-groupoid.

We can summarize the situation in the following diagram:

$$\mathcal{D}\mathrm{eraz}_X[\mathrm{Morita}^{-1}]^{\simeq} \xrightarrow{\sim} \mathcal{B}\mathrm{r}^\dagger(X) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{St}_k}(X, \mathrm{B}^2\mathbb{G}_\mathrm{m} \times \mathrm{B}\mathbb{Z}) \qquad (1.3.1)$$

where the left term is the maximal ∞-groupoid in the localization of the ∞-category of derived Azumaya algebras to Morita equivalences. At the level of dg-categories, this chain of equivalences is proven in [Toë10, Corollary 3.8]. At the level of ∞-categories, this is the combination of [Lur18, Proposition 11.5.3.10] and [Lur18, p. 11.5.5.4]

Note that, while in the classical case we had an injection $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$, in the derived setting one has a surjection $\mathrm{dBr}^\dagger_{\mathrm{Az}}(X) := \pi_0(\mathcal{D}\mathrm{eraz}_X[\mathrm{Morita}^{-1}])^{\simeq} \to \mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$. If $\mathrm{H}^1(X, \mathbb{Z}) = 0$ (e.g. when X is a normal scheme), then the surjection becomes an isomorphism of abelian groups.

While the first equivalence in (1.3.1) is completely explicit in the works of Toën and Lurie, the second one leaves a couple of questions open:

- since the space $\mathrm{Map}(X, \mathrm{B}^2\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{Z})$ is the space of pairs (G, P), where G is a $\mathbb{G}_{\mathrm{m}}$-gerbe over X and P is a $\mathbb{Z}$-torsor over X, it is natural to ask what are the gerbe and the torsor naturally associated to an element of $\mathcal{B}\mathrm{r}^{\dagger}(X)$ according to the above equivalence. This is not explicit in the proofs of Toën and Lurie, which never mention the words "gerbe" and "torsor", but rather computes the homotopy group sheaves of a sheaf of spaces $\underline{\mathcal{B}\mathrm{r}}^{\dagger}_X$ over X whose global sections are $\mathcal{B}\mathrm{r}^{\dagger}(X)$.
- conversely: given a pair (G, P), what is the ∞-category associated to it along the above equivalence?

The goal of Chapter 4 is to give a partial answer to the two questions above. The reason for the word "partial" is that we will neglect the part of the discussion regarding torsors, postponing it to a forthcoming work, and focus only on the relationship between linear ∞-categories/derived Azumaya algebras and $\mathbb{G}_{\mathrm{m}}$-gerbes. Our precise result is stated in Theorem 4.1.19.

Chapter 2

$\mathbb{E}_3$-monoidal structure on the spherical Hecke category

Contents

The content of this chapter is the paper [Noc20]. The aim is to provide an extension of the convolution product of equivariant perverse sheaves on the affine Grassmannian, whose definition will be recalled in the first subsection of this Introduction, to the ∞-category of $G_{\mathcal{O}}$-equivariant constructible sheaves on the affine Grassmannian[1]. We will endow this extension with an $\mathbb{E}_3$-algebra structure in ∞-categories, which is the avatar of Mirkovic and Vilonen's commutativity constraint [MV07, Section 5]. The final result is Theorem 2.3.6.

Theorem 2.0.1 (Theorem 2.3.6)**.** *If G is a reductive complex group and k is a finite ring of coefficients, there is an object $\mathcal{A} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathrm{Cat}^{\times}_{\infty,k})$ describing an associative and braided product law on the k-linear ∞-category $\mathcal{C}\mathrm{ons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr}_G, k)$ of $G_{\mathcal{O}}$-equivariant constructible sheaves over the affine Grassmannian (see Section 1.2.1). The restriction of this product law to the abelian category of equivariant perverse sheaves coincides up to shifts with the classical (commutative) convolution product of perverse sheaves [MV07].*

From now on, we will fix the reductive group G and denote the affine Grassmannian associated to G simply by Gr.

The theorem will be proven by steps.

- in Section 2.1, we encode the convolution diagram Remark 1.1.8 in a semisimplicial 2-Segal stack $\mathrm{Gr}_{x,\bullet}$, thus providing an associative algebra object in $\mathrm{Corr}(\mathrm{StrStk}_{\mathbb{C}})$ which, after quotienting by the analogue of the $G_{\mathcal{O}}$-action, describes the span given in Remark 1.1.8. This is done in order to express the convolution product, formally, as push-pull of constructible sheaves along this diagram.

- Let $\mathrm{Ran}(X)$ be the algebraic Ran space of X, i.e. the prestack parametrising finite sets of points in X (see Definition 2.1.1). The affine Grassmannian Gr admits a variant $\mathrm{Gr}_{\mathrm{Ran}}$, called the Ran

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[1] Our notion of constructible sheaf does not require finitely dimensional stalks. However, the full subcategory obtained by imposing this condition (which is the one usually considered) is closed under the product law that we describe.

Grassmannian (see Definition 2.1.3), living over $\mathrm{Ran}(X)$, and parametrizing G-torsors on X together with a trivialization away from a finite system of points. This object is a reformulation of the classical Beilinson-Drinfeld Grassmannian, and just like that it allows to define the so-called fusion product of perverse sheaves over the affine Grassmannian (see [MV07, Section 5] for the definition of the fusion product via Beilinson-Drinfeld Grassmannian). This fusion product is built on the fact that the Beilinson-Drinfeld Grassmannian satisfies the so-called factorization property (see [Zhu16, Section 3.1]). This property is formulated in a very convenient way for $\mathrm{Gr}_{\mathrm{Ran}}$, taking advantage of the features of the Ran space.

Again in Section 2.1, we give a version of $\mathrm{Gr}_\bullet$ living over $\mathrm{Ran}(X)$. Formally, what we obtain is a semisimplicial 2-Segal (stratified) prestack $\mathrm{Gr}_{\mathrm{Ran},\bullet}$ over $\mathrm{Ran}(X)$. This is done in order to take into account the factorization structure of $\mathrm{Gr}_{\mathrm{Ran}}$ and give the setup of the extension of the fusion product to the setting of constructible sheaves. The same is done for $G_{\mathcal{O}}$, thus defining a "global" object $G_{\mathcal{O},\mathrm{Ran}}$; all the preceding constructions are $G_{\mathcal{O},\mathrm{Ran}}$-equivariant.

- Given a finite-type (stratified) scheme Y over the complex numbers, one can consider its analytification Y^{an} (see [Rey71]). This is a complex analytic (stratified) space with an underlying (stratified) topological space which we denote by $\mathfrak{strtop}(Y)$. This procedure can be extended to a functor $\mathfrak{strtop} : \mathrm{PSh}(\mathrm{StrSch}_{\mathbb{C}}) \to \mathrm{Sh}(\mathrm{StrTop}, \mathcal{S})$, where $\mathcal{S}$ is the ∞-category of spaces. In Section 2.2, we consider the analytification of all the constructions performed in Section 2.1. The goal is to exploit the topological and homotopy-theoretic properties of the analytic version of $\mathrm{Ran}(X)$, which are reflected all the way up to $\mathrm{Gr}_{\mathrm{Ran}}$ and to the definition of the fusion product. In particular, these properties allow to apply [Lur17, Theorem 5.5.4.10] and realize the fusion product as a consequence of the very existence of the map $\mathrm{Gr}^{\mathrm{an}}_{\mathrm{Ran},k} \to \mathrm{Ran}(X)^{\mathrm{an}}$ for each k. Combining this with the simplicial structure of $\mathrm{Gr}_{\mathrm{Ran},\bullet}$, we obtain that $\mathfrak{strtop}(G_{\mathcal{O}}\backslash\mathrm{Gr})$ carries a natural $\mathbb{E}_3$-algebra structure in $\mathrm{Corr}(\mathrm{Sh}(\mathrm{StrTop}, \mathcal{S}))$ (by taking quotients in Theorem 2.2.26).
- In Section 2.A we show, building on [Lur17, Appendix A] and many other contributions, that there is a natural "equivariant k-valued constructible sheaves" functor $\mathfrak{Cons}$ from a certain category of stratified topological spaces with action of a group towards the ∞-category of presentable k-linear ∞-categories, which is also symmetric monoidal (Corollary 2.A.11). A very important feature of this functor is that it sends stratified homotopy equivalences to equivalences of presentable ∞-categories. Also, it satiesfies the Beck-Chevalley properties, that allow to take push-pull along correspondences of stratified spaces in a functorial way. In our case, the needed formal properties are contained essentially in the fact that the affine Grassmannian is a stratified ind-scheme and its strata are exactly the orbits with respect to the action of $G_{\mathcal{O}}$. Thanks to all these properties, in Section 2.3

> we formally obtain that the category $\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr})$ admits a structure of $\mathbb{E}_3$-algebra in presentable k-linear ∞-categories (with respect to the Lurie tensor product) inherited from the structure built on $\mathfrak{strtop}(G_{\mathcal{O}}\backslash\mathrm{Gr})$ in the previous point. As a corollary, $\mathcal{C}\mathrm{ons}^{\mathrm{fd}}_{G_{\mathcal{O}}}(\mathrm{Gr})$ has an induced $\mathbb{E}_3$-algebra structure in small k-linear categories (with respect to the usual Cartesian product).

Some of the techniques used in the present chapter are already "folklore" in the mathematical community; for example, application of Lurie's [Lur17, Theorem 5.5.4.10] to the affine Grassmannian appears also in [HY19], though in that paper the Authors are interested in the (filtered) topological structure of the affine Grassmannian and do not take constructible sheaves. Up to our knowledge, the formalism of constructible sheaves via exit paths and exodromy has never been applied to the study of the affine Grassmannian and the spherical Hecke category. Here we use it in order to take into account the homotopy invariance of the constructible sheaves functor, which is strictly necessary for the application of Lurie's [Lur17, Theorem 5.5.4.10].

2.1 Convolution over the Ran space

The aim of this section is to expand the construction of the Ran Grassmannian defined for instance in [Zhu16, Definition 3.3.2] in a way that allows us to define a convolution product of constructible sheaves in the Ran setting.

2.1.1 The presheaves $\mathbf{Gr}_{\mathbf{Ran},k}$

The Ran Grassmannian

Let us recall the definition of the basic objects that come into play. Let $\mathrm{Alg}_{\mathbb{C}}$ be the category of (discrete) complex algebras and X a smooth projective curve over $\mathbb{C}$.

Definition 2.1.1. The **algebraic Ran space** of X is the presheaf

$$\mathrm{Ran}(X) : \mathrm{Alg}_{\mathbb{C}} \to \mathcal{S}\mathrm{et}$$

$$R \mapsto \{\text{finite subsets of } X(R)\}.$$

Remark 2.1.2. Let $\mathcal{F}\mathrm{in}_{\mathrm{surj}}$ be the category of finite sets with surjective maps between them. Then we have an equivalence of presheaves

$$\mathrm{Ran}(X) \simeq \mathrm{colim}^{\mathrm{PSh}(X)}_{I\in\mathcal{F}\mathrm{in}_{\mathrm{surj}}} X^I.$$

Definition 2.1.3. The **Ran Grassmannian** of X is the functor $\mathrm{Gr}_{\mathrm{Ran}} : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpd}$

$$R \mapsto \{S \in \mathrm{Ran}(X)(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus \Gamma_S}\},$$

where $\mathcal{T}_G$ is the trivial G-bundle on X_R, Γ_S is the union of the graphs of the s_i inside X_R, $s_i \in S$, and α is a trivialization, i.e. an isomorphism of principal G-bundles with the trivial G-bundle. This admits a natural forgetting map towards $\mathrm{Ran}(X)$.

Remark 2.1.4. Consider the twisted tensor product defined in [Zhu16, (3.1.10)]

$$\mathrm{Gr}_X = \hat{X} \times^{\mathrm{Aut}(\mathrm{Spec}\,\mathbb{C}[\![t]\!])} \mathrm{Gr}_G,$$

where $\hat{X}$ is the space of formal parameters defined in *loc. cit.*. This parametrizes

$$\mathrm{Gr}_X(R) = \{x \in X(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_x} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus \Gamma_x}\}.$$

For each finite set I, there is a multiple version

$$\mathrm{Gr}_{X^I}(R) = \{(x_1, \dots, x_{|I|}) \in X(R)^I, \mathcal{F} \in \mathrm{Bun}_G(X_R),$$
$$\alpha : \mathcal{F}|_{X_R \setminus \Gamma_{x_1} \cup \dots \cup \Gamma_{x_{|I|}}} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus \Gamma_{x_1} \cup \dots \cup \Gamma_{x_{|I|}}}\}.$$

Then we have

$$\mathrm{Gr}_{\mathrm{Ran}} = \mathrm{colim}_{I \in \mathcal{F}\mathrm{in}_{\mathrm{surj}}} \mathrm{Gr}_{X^I}.$$

Definition 2.1.5. We can define $G_{\mathcal{K},X^I}, G_{\mathcal{O},X^I}$ in the same way. Let also $G_{\mathcal{K},\mathrm{Ran}}$ be the functor

$$R \mapsto \{S \in \mathrm{Ran}(X)(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus \Gamma_S},$$
$$\mu : \mathcal{F}|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\sim} \mathcal{T}_G|_{\widehat{(X_R)}_{\Gamma_S}}\} \simeq \mathrm{colim}_I\, G_{\mathcal{K};X^I}.$$

Let $G_{\mathcal{O},\mathrm{Ran}}$ be the functor

$$R \mapsto \{S \in \mathrm{Ran}(X)(R), g \in \mathrm{Aut}_{\widehat{(X_R)}_{\Gamma_S}}(\mathcal{T}_G)\} \simeq \mathrm{colim}_I\, G_{\mathcal{O};X^I}.$$

By means of [Zhu16, Proposition 3.1.9] one has that $\mathrm{Gr}_{\mathrm{Ran}} \simeq G_{\mathcal{K},\mathrm{Ran}}/G_{\mathcal{O},\mathrm{Ran}}$ in the sense of a quotient in $\mathrm{PSh}(\mathrm{Stk}_{\mathbb{C}})$, i.e.

$$\mathrm{Gr}_{\mathrm{Ran}} \simeq \mathrm{colim}_I\, G_{\mathcal{K},X^I}/G_{\mathcal{O},X^I}$$

where each term is the étale stack quotient in the category of stacks (recall that $G_{\mathcal{K}}$ is an ind-scheme and $G_{\mathcal{O}}$ is a group scheme) and is equivalent to Gr_{X^I}.

Definition 2.1.6. We define a $\mathrm{Gr}_{\mathrm{Ran},k}$ to be the functor $\mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpd}$

$$\overbrace{G_{\mathcal{K},\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} G_{\mathcal{K},\mathrm{Ran}}}^{k-1 \text{ times}} \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}},$$

that is

$$R \mapsto \{S \in \mathrm{Ran}(X)(R), \mathcal{F}_i \in \mathrm{Bun}_G(X_R),$$

$$\alpha_i \text{ trivialization of } \mathcal{F}_i \text{ outside } \Gamma_S, i = 1, \dots, k,$$

$$\mu_i \text{ trivialization of } \mathcal{F}_i \text{ on the formal neighborhood of } \Gamma_S, i = 1, \dots, k-1\}.$$

We call $r_k : \mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X)$ the natural forgetting map. Note that $\mathrm{Gr}_{\mathrm{Ran},0} = \mathrm{Ran}(X)$, $\mathrm{Gr}_{\mathrm{Ran},1} = \mathrm{Gr}_{\mathrm{Ran}}$.

A priori, $\mathrm{Gr}_{\mathrm{Ran},k}$ is groupoid-valued, because if R is fixed the $\mathcal{F}_i$'s may admit nontrivial automorphisms that preserve the datum of the α_i's and the μ_j's. Actually, this is not the case, just like for the classical affine Grassmannian which is ind-representable:

Proposition 2.1.7. *For any $k \geq 0$ the functor*

$$\mathrm{Gr}_{\mathrm{Ran},k} : \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Grpd}$$

factorises through the inclusion $\mathcal{S}\mathrm{et} \to \mathrm{Grpd}$.

Proof. See Section 2.B.1. □

It is worthwhile to remark that the map $\mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X)$ is ind-representable, although $\mathrm{Gr}_{\mathrm{Ran},k}$ itself is not.

Definition 2.1.8. Let $x \in X(\mathbb{C})$ be a closed point of X. There is a natural map $\{x\} \to X \to \mathrm{Ran}(X)$, represented by the constant functor $R \mapsto \{x\} \in \mathcal{S}\mathrm{et}$. Let us denote $\mathrm{Gr}_{x,k} = \mathrm{Gr}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(X)} \{x\}$.

Proposition 2.1.9. $\mathrm{Gr}_{k,x}$ *is independent from the choice of X and x, and*

$$\mathrm{Gr}_{1,x} \simeq \mathrm{Gr}_G$$

$$\mathrm{Gr}_{2,x} \simeq G_{\mathcal{K}} \times \mathrm{Gr}_G.$$

Proof. Note first that

$$\mathrm{Gr}_{k,x}(R) = \{(\mathcal{F}_i \in \mathrm{Bun}_G(X_R), \alpha_i : \mathcal{F}_i|_{X_R \setminus (\{x\} \times \mathrm{Spec}\, R)} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus (\{x\} \times \mathrm{Spec}\, R)},$$

$$\mu_i : \mathcal{F}_i|_{\widehat{(X_R)}_{(\{x\}\times \operatorname{Spec} R)}} \xrightarrow{\sim} \mathcal{T}_G|_{\widehat{(X_R)}_{(\{x\}\times \operatorname{Spec} R)}})_{i=1,\dots,k}\}.$$

By the Formal Gluing Theorem [HPV16] this can be rewritten as

$$\operatorname{Gr}_{k,x}(R) = \{(\mathcal{F}_i \in \operatorname{Bun}_G(\widehat{(X_R)}_{\{x\}\times \operatorname{Spec} R}),$$

$$\alpha_i : \mathcal{F}|_{(\mathring{X}_R)_{\{x\}\times \operatorname{Spec} R}} \xrightarrow{\sim} \mathcal{T}_G|_{(\mathring{X}_R)_{\{x\}\times \operatorname{Spec} R}},$$

$$\mu_i : \mathcal{F}|_{\widehat{(X_R)}_{(\{x\}\times \operatorname{Spec} R)}} \xrightarrow{\sim} \mathcal{T}_G|_{\widehat{(X_R)}_{(\{x\}\times \operatorname{Spec} R)}})_{i=1,\dots,k}\} \simeq$$

$$\simeq \{(\mathcal{F}_i \in \operatorname{Bun}_G(\widehat{X}_{\{x\}} \times \operatorname{Spec} R), \alpha_i : \mathcal{F}|_{\mathring{X}_{\{x\}}\times \operatorname{Spec} R} \xrightarrow{\sim} \mathcal{T}_G|_{\mathring{X}_{\{x\}}\times \operatorname{Spec} R},$$

$$\mu_i : \mathcal{F}|_{\widehat{X}_{\{x\}}\times \operatorname{Spec} R} \xrightarrow{\sim} \mathcal{T}_G|_{\widehat{X}_{\{x\}}\times \operatorname{Spec} R})_{i=1,\dots,k}\},$$

but $\widehat{X}_{\{x\}}$ is independent from the choice of the point x, being (noncanonically) isomorphic to Spec $\mathbb{C}[\![t]\!]$ (and the same for $\mathring{X}_x$). The rest of the statement is clear from the definitions. □

2.1.2 The 2-Segal structure

Face maps

We now establish a semisimplicial structure on the collection of the $\operatorname{Gr}_{\operatorname{Ran},k}$.

Construction 2.1.10. Let ∂_i be the face map from $[k-1]$ to $[k]$ omitting i. We define the corresponding map $\delta_i : \operatorname{Gr}_{\operatorname{Ran},k} \to \operatorname{Gr}_{\operatorname{Ran},k-1}$ as follows. A tuple $(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_k, \alpha_k)$ is sent to a tuple

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_{i-1}, \alpha_{i-1}, \mu_{i-1},$$

$$\operatorname{Fgl}(\mathcal{F}_i, \mathcal{F}_{i+1}, \alpha_i, \alpha_{i+1}, \mu_i), \mu_i', \dots, \mathcal{F}_k, \alpha_k),$$

where:

- $\operatorname{Fgl}(\mathcal{F}_i, \mathcal{F}_{i+1}, \alpha_i, \alpha_{i+1}, \mu_i)$ is the pair $(\mathcal{F}_i', \alpha_i')$ formed as follows: the Formal Gluing Theorem ([HPV16]) allows us to glue the sheaves $\mathcal{F}_i|_{X_R \setminus \Gamma_S}$ and $\mathcal{F}_{i+1}|_{\widehat{(X_R)}_{\Gamma_S}}$ along the isomorphism $\mu_i^{-1}|_{(\mathring{X}_R)_{\Gamma_S}} \circ \alpha_{i+1}|_{(\mathring{X}_R)_{\Gamma_S}}$. This is our $\mathcal{F}_i'$. Also, $\mathcal{F}_i'$ inherits a trivialization over $(\mathring{X}_R)_{\Gamma_S}$ described by

  $$\alpha_i|_{(\mathring{X}_R)_{\Gamma_S}} \mu_i^{-1}|_{(\mathring{X}_R)_{\Gamma_S}} \alpha_{i+1}|_{(\mathring{X}_R)_{\Gamma_S}},$$

  which is the second datum.

- μ_i' coincides with μ_{i+1} via the canonical isomorphism between the glued sheaf and $\mathcal{F}_{i+1}$ over the formal neighbourhood of Γ_S.

Proposition 2.1.11. *This construction defines a semisimplicial object* $\mathrm{Gr}_{\mathrm{Ran},\bullet} : \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Alg}_{\mathbb{C}}, \mathcal{S}\mathrm{et})$ *because the given maps satisfy the simplicial identities.*

Proof. Let k be fixed. We check the face identities $\delta_i \delta_j = \delta_{j-1}\delta_i$ for $i < j$.
The essential nontrivial case is when $k = 3, i = 0, j = 1$ or $i = 1, j = 2$ or $i = 2, j = 3$. Otherwise the verifications are trivial since, if $i < j - 1$, then the two gluing processes do not interfere with one another. The cases $i = 0, j = 1$ and $i = 2, j = 3$ are very simple, because there is only one gluing and one forgetting ($\mathcal{F}_1$ or $\mathcal{F}_3$ respectively). In the remaining case, we must compare $\mathcal{F}_{1,23} = \mathrm{Fgl}(\mathcal{F}_1, \mathrm{Fgl}(\mathcal{F}_2, \mathcal{F}_3))$ with $\mathcal{F}_{12,3} = \mathrm{Fgl}(\mathrm{Fgl}(\mathcal{F}_1, \mathcal{F}_2), \mathcal{F}_3)$. (we omit the S, α_i, μ_i from the notation for short). We have:

- $\mathcal{F}_{1,23}|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_1|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_{12}|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_{12,3}|_{X_R \setminus \Gamma_S}$
- $\mathcal{F}_{1,23}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{23}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_2|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{12}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{12,3}|_{\widehat{(X_R)}_{\Gamma_S}}$.

This tells us that the two sheaves are the same, and from this it is easy to deduce that the same property holds for the trivializations. ☐

We thus have a semisimplicial structure on $\mathrm{Gr}_{\mathrm{Ran},\bullet}$, together with maps $r_k : \mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X)$ which commute with the face maps by construction.

Verification of the 2-Segal property

The crucial property of the semisimplicial presheaf $\mathrm{Gr}_{\mathrm{Ran},\bullet}$ is the following:

Proposition 2.1.12. *For any* $R \in \mathrm{Alg}_{\mathbb{C}}$*, the semisimplicial set* $\mathrm{Gr}_{\mathrm{Ran},\bullet}(R)$ *enjoys the 2-Segal property, that is the equivalent conditions of [DK19, Proposition 2.3.2].*

Proof. *Case* "$0, l \leq k$". With the natural notations appearing in [DK19, Proposition 2.3.2], there is a map $\mathrm{Gr}_{\mathrm{Ran},\{0,1,\dots,l\}} \times_{\mathrm{Gr}_{\mathrm{Ran},\{0,l\}}} \mathrm{Gr}_{\mathrm{Ran},\{0,l,l+1,\dots,k\}} \to \mathrm{Gr}_{\mathrm{Ran},\{0,\dots,k\}} = \mathrm{Gr}_{\mathrm{Ran},k}$ inverse to the natural projection. The map sends

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_l, \alpha_l, \mathcal{F}'_l, \alpha'_l, \mu'_l,$$

$$\xi : (\mathcal{F}'_l, \alpha'_l) \xrightarrow{\sim} \mathrm{Fgl}(\{\mathcal{F}_i, \alpha_i, \mu_j\}_{i=1,\dots,l, j=1,\dots,l-1}), \mathcal{F}'_{l+1}, \alpha'_{l+1}, \mu'_{l+1}, \dots, \mathcal{F}'_k \alpha'_k)$$

$$\mapsto (\mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_l, \alpha_l, \mu''_l, \mathcal{F}'_{l+1}, \alpha'_{l+1}, \mu'_{l+1}, \dots, \mathcal{F}'_k, \alpha'_k)$$

where $\mu''_{l,l+1}$ is the trivialization of $\mathcal{F}_l$ on the formal neighbourhood of x defined as $\mathcal{F}_l|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\sim}$ $\mathrm{Fgl}(\{\mathcal{F}_i, \alpha_i, \mu_j\}_{i=1,\dots,l, j=1,\dots,l-1})|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\xi^{-1}} \mathcal{F}'_l|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\mu'_l} \mathcal{T}_{G_{\widehat{(X_R)}_{\Gamma_S}}}$.
This map is indeed inverse to the natural map arising from the universal property of the fibered product, thus establishing the 2-Segal property in the case "$0, l$". The case "l, k" can be tackled in a similar way. ☐

Notation 2.1.13. Given a category or an ∞-category $\mathcal{C}$, we denote by $2\text{-Seg}^{\text{ss}}(\mathcal{C})$ the (∞-)category of 2-Segal semisimplicial objects in $\mathcal{C}$.

2.1.3 Action of the arc group in the Ran setting

We now introduce analogues of the "arc group" $G_{\mathcal{O}}$ to our global context, namely group functors over $\text{Ran}(X)$ denoted by $\text{Arc}_{\text{Ran},k}$, each one acting on $\text{Gr}_{\text{Ran},k}$ over $\text{Ran}(X)$.

The Ran version of the arc group

Construction 2.1.14. Consider the functor

$$G_{\mathcal{O},\text{Ran}} : \text{Alg}_{\mathbb{C}} \to \mathcal{S}\text{et}$$

from Definition 2.1.5. It is immediate to see that this functor takes values in $\mathcal{S}\text{et}$ just like $\text{Gr}_{\text{Ran},k}$, and admits a map towards $\text{Ran}(X)$.
This functor is a group functor over $\text{Ran}(X)$ under the law

$$(S, g) \cdot (S, h) = (S, g \cdot h).$$

There is a semisimplicial group object of $(\text{PSh}_{\mathbb{C}})_{/\text{Ran}(X)}$ assigning

$$[k] \mapsto \overbrace{G_{\mathcal{O},\text{Ran}} \times_{\text{Ran}(X)} \cdots \times_{\text{Ran}(X)} G_{\mathcal{O},\text{Ran}}}^{k\ times}.$$

The face maps are described by

$$\delta_i : (S, g_1, \dots, g_k) \mapsto (S, g_1, \dots, \overbrace{g_i g_{i+1}}^{i}, \dots, g_k).$$

Definition 2.1.15. We denote this semisimplicial group functor over $\text{Ran}(X)$ by $\text{Arc}_{\text{Ran},\bullet}$.[2]

Note that $\text{Arc}_{\text{Ran},0} = \text{Ran}(X)$, and $\text{Arc}_{\text{Ran},1} = G_{\mathcal{O},\text{Ran}}$. It is also useful to define the version of $\text{Arc}_{\text{Ran},k}$ over X^I:

$$\text{Arc}_{X^I,k} := \overbrace{G_{\mathcal{O},\text{Ran}} \times_{\text{Ran}(X)} \cdots \times_{\text{Ran}(X)} G_{\mathcal{O},\text{Ran}}}^{k\ times}.$$

As usual, the "Ran" version is the colimit over $I \in \mathcal{F}\text{in}_{\text{surj}}$ of the "X^I" versions.

Remark 2.1.16. The simplicial object $\text{Arc}_\bullet$ enjoys the 2-Segal property. The verification is straightforward thanks to the multiplication structure.

---

[2] We replace the notation "$G_{\mathcal{O}}$" with "Arc" for typographical reasons.

The action on $\mathrm{Gr}_{\mathrm{Ran},\bullet}$

The first observation now is that $\mathrm{Arc}_{\mathrm{Ran},k-1}$ acts on $\mathrm{Gr}_{\mathrm{Ran},k}$ on the left over $\mathrm{Ran}(X)$ in the following way:

$$(S, g_2, \ldots, g_k).(S, \mathcal{F}_1, \alpha_1, \mu_1, \ldots, \mathcal{F}_k, \alpha_k) =$$

$$= (S, \mathcal{F}_1, \alpha_1, \mu_1 g_2^{-1}, \mathcal{F}_2, g_2\alpha_2, \mu_2 g_3^{-1}, \ldots, \mathcal{F}_k, g_k\alpha_k)$$

where $g_i\alpha_i$ is the modification of α_i by g_i on $(\overset{\circ}{X}_R)_{\Gamma_S}$ which, by the usual "local-global" reformulation, induces a new trivialization of $\mathcal{F}_i$ outside S.
We denote by:

- $\Phi_{k-1,k}$ this action of $\mathrm{Arc}_{\mathrm{Ran},k-1}$ on $\mathrm{Gr}_{\mathrm{Ran},k}$.
- $\Xi_{1,k}$ the left action of $\mathrm{Arc}_{\mathrm{Ran},1}$ on $\mathrm{Gr}_{\mathrm{Ran},k}$ altering the first trivialization α_1.
- $\Phi_{k,k}$ the left action of $\mathrm{Arc}_{\mathrm{Ran},k}$ on $\mathrm{Gr}_{\mathrm{Ran},k}$ obtained as combination of $\Xi_{1,k}$ and $\Phi_{k-1,k}$.

Definition 2.1.17. Define $\mathrm{Conv}_{\mathrm{Ran},k}$, the **Ran version of the convolution Grassmannian**, as the quotient of $\mathrm{Gr}_{\mathrm{Ran},k}$ by the left action $\Phi_{k-1,k}$ described above. That is, $\mathrm{Conv}_{\mathrm{Ran},k} = \mathrm{colim}_I\, \mathrm{Arc}_{X^I,k-1}\backslash \mathrm{Gr}_{X^I,k}$, where the terms of the colimits are quotient stacks with respect to the étale topology and $\mathrm{Arc}_{X^I,k-1}$ acts through the pullback to X^I of the action $\Phi_{k-1,k}$.

Remark 2.1.18. ConvGr_k is a presheaf which can alternatively be described as follows:

$$R \mapsto \{S \subset X(R), \mathcal{F}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k \in \mathrm{Bun}_G(X_R), \alpha_1 : \mathcal{F}_1|_{X_R\setminus\Gamma_S} \xrightarrow{\sim} \mathcal{T}_G|_{X_R\setminus\Gamma_S},$$

$$\eta_2 : \mathcal{G}_2|_{X_R\setminus\Gamma_S} \xrightarrow{\sim} \mathcal{F}_1|_{X_R\setminus\Gamma_S}, \ldots, \eta_k : \mathcal{G}_k|_{X_R\setminus\Gamma_S} \xrightarrow{\sim} \mathcal{G}_{k-1}|_{X_R\setminus\Gamma_S}\}.$$

This is proven in [Rei12, Proposition III.1.10, (1)], because if we take $m = k$ then $\widetilde{\mathrm{Gr}}_p|_\Delta$ in *loc. cit.* is the pullback along $X^n \to \mathrm{Ran}(X)$ of the functor described in Remark 2.1.18, and Conv_n^m is the pullback along $X^n \to \mathrm{Ran}(X)$ of our $\mathrm{Conv}_{\mathrm{Ran},m}$.

2.1.4 The convolution product over Ran(X)

Connection with the Mirkovic-Vilonen convolution product

Consider the action of $\mathrm{Arc}_{\mathrm{Ran},k}$ on ConvGr_k induced by $\Xi_{1,k}$, which we still call $\Xi_{1,k}$ by abuse of notation, and consider also $\Phi_{1,1}$ as an action of $\mathrm{Arc}_{\mathrm{Ran},1}$ on $\mathrm{Gr}_{\mathrm{Ran}}$. Note that $\Phi_{1,1}^{\times k}$ is an action of $\mathrm{Arc}_{\mathrm{Ran},1}^{\times_{\mathrm{Ran}(X)} k}$ on

$\mathrm{Gr}_{\mathrm{Ran}}^{\times_{\mathrm{Ran}(X)} k}$. The actions $\Phi_{1,1}^{\times k}$ on $\mathrm{Gr}_{\mathrm{Ran}}^{\times_{\mathrm{Ran}(X)} k}$, $\Phi_{k,k}$ on $\mathrm{Gr}_{\mathrm{Ran},k}$, $\Xi_{1,k}$ on ConvGr_k and $\Phi_{1,1}$ on $\mathrm{Gr}_{\mathrm{Ran}}$ are compatible with the *k-associative convolution diagram*

$$\begin{tikzcd}
 & \mathrm{Gr}_{\mathrm{Ran},k} \arrow[dl, "p"'] \arrow[r, "q"] & \mathrm{ConvGr}_k \arrow[dr, "m"] & \\
\mathrm{Gr}_{\mathrm{Ran}}^{\times_{\mathrm{Ran}(X)} k} & & & \mathrm{Gr}_{\mathrm{Ran}}
\end{tikzcd}$$

(2.1.1)

where:

- p is the map that forgets all the trivializations μ_i.

- q is the projection to the quotient with respect to $\Phi_{k-1,k}$, alternatively described as follows: we keep $\mathcal{F}_1$ and α_1 intact, and define $\mathcal{G}_h$ by induction as the formal gluing of $\mathcal{G}_{h-1}$ (or $\mathcal{F}_1$ if $h = 1$) and $\mathcal{F}_h$ along μ_{h-1} and α_h: indeed, μ_{h-1} is a trivialization of $\mathcal{G}_{h-1}$ over the formal neighbourhood of Γ_S via the canonical isomorphism between $\mathcal{G}_{h-1}$ and $\mathcal{F}_{h-1}$ on that formal neighbourhood. The isomorphism
$$\eta_h : \mathcal{G}_h|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{G}_{h-1}|_{X_R \setminus \Gamma_S}$$
is provided canonically by the formal gluing procedure.

- m is the map sending
$$(S, \mathcal{F}_1, \alpha_1, \mathcal{G}_2, \eta_2, \ldots, \mathcal{G}_k, \eta_k) \mapsto (S, \mathcal{G}_k, \alpha_1 \circ \eta_2 \circ \cdots \circ \eta_k).$$

Remark 2.1.19. Consider the special case $k = 2$. Note first of all that, since this diagram lives over $\mathrm{Ran}(X)$, we can take its fiber at $\{x\} \in \mathrm{Ran}(X)$. Under the identifications of Proposition 1.1.9, we obtain the diagram (1.1.2).

We can consider the diagram

$$\begin{tikzcd}
 & \Phi_{k,k} \backslash \mathrm{Gr}_{\mathrm{Ran},k} \arrow[dl] \arrow[r, "\sim"] & \Xi_{1,k} \backslash \mathrm{ConvGr}_k \arrow[dr] & \\
\Phi_{1,1}^{\times k} \backslash \mathrm{Gr}_{\mathrm{Ran},1}^{\times_{\mathrm{Ran}(X)} k} & & & \Phi_{1,1} \backslash \mathrm{Gr}_{\mathrm{Ran}}.
\end{tikzcd}$$

(2.1.2)

The horizontal map is an equivalence since it exhibits its target as the quotient $\Xi_{1,k} \Phi_{k-1,k} \backslash \mathrm{Gr}_{\mathrm{Ran},k} \simeq \Phi_{k,k} \backslash \mathrm{Gr}_{\mathrm{Ran},k}$. For $k = 2$ one obtains:

$$\begin{tikzcd}
 & \Phi_{2,2} \backslash \mathrm{Gr}_{\mathrm{Ran},2} \arrow[dl] \arrow[r, "\sim"] & \Xi_{1,2} \backslash \mathrm{ConvGr}_2 \arrow[dr] & \\
\Phi_{1,1}^{\times 2} \backslash \mathrm{Gr}_{\mathrm{Ran}}^{\times_{\mathrm{Ran}(X)} 2} & & & \Phi_{1,1} \backslash \mathrm{Gr}_{\mathrm{Ran}}.
\end{tikzcd}$$

Remark 2.1.20. Take again the fiber of this diagram at the point $\{x\} \in \operatorname{Ran}(X)$. This results in a diagram of the form

$$\begin{tikzcd}
& \Phi_{x,2,2}\backslash \mathrm{Gr}_{x,2} \arrow[r, "\sim"] \arrow[dl] & \Xi_{x,1,2}\backslash \mathrm{ConvGr}_{x,2} \arrow[dr] & \\
\Phi_{1,1,x}^{\times 2}\backslash \mathrm{Gr}_{x,1}^{\times 2} & & & \Phi_{1,1,x}\backslash \mathrm{Gr}_{x,1}.
\end{tikzcd}$$

Recall now the identifications of Proposition 1.1.9.

Here, the action $\Phi_{x,1,1}$ is the usual left-multiplication action of $G_{\mathcal{O}}$ over Gr. The action $\Phi_{x,2,2}$ is the action of $G_{\mathcal{O}} \times G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times \mathrm{Gr}$ given by $(g_1, g_2).(h, \gamma) = (g_1 h g_2^{-1}, g_2\gamma)$. Finally, the action $\Xi_{x,1,2}$ on $\mathrm{ConvGr}_{x,2}$ is the action of $G_{\mathcal{O}}$ on $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \mathrm{Gr}$ given by $g[h, \gamma] = [gh, \gamma]$.

Consider now two perverse sheaves $\mathcal{F}$ and $\mathcal{G}$ on the quotient $\Phi_{x,1,1}\backslash \mathrm{Gr}_{x,1}$. This is equivalent to the datum of two $G_{\mathcal{O}}$-equivariant perverse sheaves over Gr. We can perform the external product $\mathcal{F} \boxtimes \mathcal{G}$ living over $\Phi_{x,1,1}^{\times 2}\backslash \mathrm{Gr}^{\times 2}$, and then pull it back to $\Phi_{x,2,2}\backslash \mathrm{Gr}_{x,2}$. Under the equivalence displayed above, this can be interpreted as a $\Xi_{1,2}$-equivariant perverse sheaf over ConvGr_2. By construction, this is exactly what [MV07] call $\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ (up to shifts). Its pushforward along m is therefore the sheaf $\mathcal{F} \star \mathcal{G} \in \mathcal{P}\mathrm{erv}_{G_{\mathcal{O}}}(\mathrm{Gr})$.

The same construction can be done with constructible sheaves instead of perverse sheaves. In the rest of the chapter we will describe an algebra structure on the category $\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ of equivariant constructible sheaves over the affine Grassmannian. This ∞-category is equivalent to $\mathcal{C}\mathrm{ons}(G_{\mathcal{O}}\backslash \mathrm{Gr}_G)$, and from this point of view the product law is exactly the one described here.

2.1.5 Stratifications

Stratification of Gr, $\mathbf{Gr}_{\mathbf{Ran}}$ and $\mathbf{Gr}_{\mathbf{Ran},k}$

Construction 2.1.21. Recall from Remark 1.1.3 that the affine Grassmannian has a stratification in Schubert cells. We have explained in Section 1.2.1 that we are interested in considering constructible sheaves on the affine Grassmannian which are equivariant and constructible with respect to this stratification. While in principle the equivariant structure on the sheaf/D-module is sufficient in that implies the constructibility condition by what observed in Section 1.2.1, it will be very important for us to consider "stratified homotopy equivalences" when in the topological setting (see Definition 2.A.21). Therefore, it is essential to keep track of the stratifications. We will now extend the stratification from Gr to $\mathrm{Gr}_{\mathrm{Ran},k}$. We give/recall the definition of stratified schemes and presheaves in Definition 2.A.2 and Construction 2.A.3.

In that formalism, the stratification in Schubert cells can be seen as a continuous map of ind-topological spaces $\mathrm{zar}(\mathrm{Gr}_G) \to \mathbb{X}_\bullet(T)^+$ where $\mathrm{zar}(\mathrm{Gr}_G)$ is the Zariski ind-topological space associated to Gr_G, and $\mathbb{X}_\bullet(T)^+$ is the poset of dominant coweights of any maximal torus $T \subset G$. Therefore, the datum $(\mathrm{Gr}_G, \mathscr{S})$ may be interpreted as an object of $\mathrm{StrPSh}_{\mathbb{C}}$. The global version Gr_X admits a stratification described in [Zhu16, eq. 3.1.11], which detects the monodromy of the pair (*bundle*, *trivialization*) at the the chosen point. By filtering Gr_X by the lattice filtration (see discussion after Theorem 1.1.2) at every point of X, we can exhibit Gr_X as a stratified ind-scheme, or more generally a stratified presheaf, whose indexing poset is again $\mathbb{X}_\bullet(T)^+$.[3]

Notation 2.1.22. From now on, an arrow of the form $\mathcal{X} \to P$, where $\mathcal{X}$ is a complex presheaf and P is a poset, will denote a geometric morphism $\mathrm{zar}(\mathcal{X}) \to P$, $\mathrm{zar}(\mathcal{X})$ being $\mathrm{colim}_{X \to \mathcal{X} \text{ scheme}} \mathrm{zar}(X)$.

We will now construct maps $\mathrm{Gr}_{X^I} \to (\mathbb{X}_\bullet(T)^+)^I$ for any I. Recall from [Zhu16] the so-called factorising property of the Beilinson-Drinfeld Grassmannian. For $|I| = 2$, it says the following:

Proposition 2.1.23 ([Zhu16, Proposition 3.1.13]). *There are canonical isomorphisms* $\mathrm{Gr}_X \simeq \mathrm{Gr}_{X^2} \times_{X^2,\Delta} X, c : \mathrm{Gr}_{X^2}|_{X^2 \setminus \Delta} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_{X^2 \setminus \Delta}$.

For an arbitrary I, the property is stated in [Zhu16, Theorem 3.2.1].
This property allows us to define a stratification on Gr_{X^I} over $(\mathbb{X}_\bullet(T)^+)^I$ endowed with the lexicographical order:

Definition 2.1.24. For $|I| = 2$, $(\mathrm{Gr}_{X^I})_{\leq(\mu,\lambda)} \subset \mathrm{Gr}_{X^2}$ is defined to be the closure of $\mathrm{Gr}_{X,\leq\mu} \times \mathrm{Gr}_{X,\leq k} \subset (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_{X^2\setminus\Delta} \xrightarrow{\sim} (\mathrm{Gr}_{X^2})|_{X^2\setminus\Delta}$ inside Gr_{X^2}.

For an arbitrary I, the definition uses the small diagonals. This stratification coincides with the partition in Arc_{X^I}-orbits of Gr_{X^I}. We now consider the map $s_I : \mathrm{Gr}_{X^I} \to \mathbb{X}_\bullet(T)^+$ given by

$$\mathrm{Gr}_{X^I} \to (\mathbb{X}_\bullet(T)^+)^I \to \mathbb{X}_\bullet(T)^+$$

where the first map is the one described above, and the second one is the map

$$(\mu_1, \ldots, \mu_{|I|}) \mapsto \Sigma_{i=1}^n \mu_i.$$

The map s_I is a stratification. Also, by applying [Zhu16, Proposition 3.1.14], one proves that the diagonal map $X^J \to X^I$ induced by a surjective map of finite sets $I \to J$ is stratified with respect to s_I, s_J. Now, $\mathrm{Gr}_{\mathrm{Ran}}$ is the colimit $\mathrm{colim}_{I \in \mathcal{F}\mathrm{in}_{\mathrm{surj}}} \mathrm{Gr}_{X^I}$, and therefore $\mathrm{Gr}_{\mathrm{Ran}}$ inherits a map towards $\mathbb{X}_\bullet(T)^+$. This stratification coincides with the stratification in $\mathrm{Arc}_{\mathrm{Ran}}$-orbits of $\mathrm{Gr}_{\mathrm{Ran}}$.

Finally, $\mathrm{Gr}_{\mathrm{Ran},k}$ admits a map towards $\mathrm{Ran}(\mathbb{X}_\bullet(T)^+)^k$, inherited from the bundle map

$$\mathrm{Gr}_{\mathrm{Ran},k} \to \overbrace{\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}}^{k}.$$

Definition 2.1.25. We denote the induced stratification $\mathrm{Gr}_{\mathrm{Ran},k} \to (\mathbb{X}_\bullet(T)^+)^k$ by σ_k.

---

[3] One can see that this is again a stratification by the translational invariance property, i.e. by the fact that $\mathrm{Gr}_{\mathbb{A}^1} \simeq \mathrm{Gr} \times \mathbb{A}^1$.

Interaction with the semisimplicial structure and with the action of $\mathrm{Arc}_{\mathrm{Ran},\bullet}$

Now we want to study the interaction between the stratifications and the semisimplicial structure. In particular, we want to prove that the simplicial maps agree with the stratifications σ_k (and consequently τ_k), thus concluding that $\mathrm{Gr}_{\mathrm{Ran},\bullet}$ upgrades to a semisimplicial object in $(\mathrm{StrPSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)}$.

Definition 2.1.26. Consider the semisimplicial group

$$\mathrm{Cw}_{\bullet}^{(I)}$$

(coweights) defined by

$$\mathrm{Cw}_k = (\mathbb{X}_{\bullet}(T)^+)^k$$

$$\delta_j : (\mu_1, \dots, \mu_k) \mapsto (\mu_1, \dots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \dots, \mu_k).$$

Note that the formal gluing procedure

$$\mathrm{Gr}_{\mathrm{Ran},2} \to \mathrm{Gr}_{\mathrm{Ran}}$$

sends the stratum $((\mu, \lambda), n)$ to the stratum $(\mu + \lambda, n)$. Indeed, unwinding the definitions and restricting to the case of points of cardinality 1 in $\mathrm{Ran}(X)$, formal gluing at a point amounts to multiplying two matrices, one with coweight μ and the other with coweight λ, hence resulting in a matrix of coweight $\mu\lambda$. Thus if $S : \mathbb{X}_{\bullet}(T)^+ \times \mathbb{X}_{\bullet}(T)^+ \to \mathbb{X}_{\bullet}(T)^+$ is the sum map, the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{\mathrm{Ran},2} \arrow[r, "\partial_1"] \arrow[d, "\tau_2"] & \mathrm{Gr}_{\mathrm{Ran},1} \arrow[d, "\tau_1"] \\
\mathrm{Cw}_2 = \mathbb{X}_{\bullet}(T)^+ \times \mathbb{X}_{\bullet}(T)^+ \arrow[r, "S"] & \mathrm{Cw}_1 = \mathbb{X}_{\bullet}(T)^+
\end{tikzcd}$$

commutes.

The case of an arbitrary I uses similar arguments. We can therefore say that for any face map $\phi : [h] \to [k]$ the induced square

$$\begin{tikzcd}
\mathrm{Gr}_{\mathrm{Ran},k} \arrow[r, "\mathrm{Gr}_{\mathrm{Ran}}(\phi)"] \arrow[d, "\tau_k"] & \mathrm{Gr}_{\mathrm{Ran},h} \arrow[d, "\tau_h"] \\
\mathrm{Cw}_k \arrow[r, "\mathrm{Cw}(\phi)"] & \mathrm{Cw}_h
\end{tikzcd}$$

commutes. Note that the top row is a map of presheaves, so the correct interpretation of this diagram is: the diagrams

$$\begin{tikzcd}
X^I \times_{\mathrm{Ran}(X)} (\mathrm{Gr}_{\mathrm{Ran},k})_{\leq N} \arrow[r] \arrow[d] & X^I \times_{\mathrm{Ran}(X)} (\mathrm{Gr}_{\mathrm{Ran},h})_{\leq N} \arrow[d] \\
\mathrm{Cw}_k \arrow[r] & \mathrm{Cw}_h
\end{tikzcd}$$

commute for every I and N, where N refers to the lattice filtration.

Lemma 2.1.27. *The strata of the stratification* σ_k *are the orbits of the action of* $\mathrm{Arc}_{\mathrm{Ran},k}$ *on* $\mathrm{Gr}_{\mathrm{Ran},k}$ *over* $\mathrm{Ran}(X)$.

Proof. We know that the orbits of the Schubert stratification of the affine Grassmannian are the orbits of the action of $G_{\mathcal{O}}$. The same is true at the Ran level. □

Remark 2.1.28. Both $(\mathrm{Gr}_{\mathrm{Ran},\bullet}, \sigma_\bullet)$ and $(\mathrm{Arc}_{\mathrm{Ran},\bullet})$ (unstratified) enjoy the 2-Segal property.

Proof. We want to use the unstratified version of the same result, proved in Section 2.1.2. In order to do this, it suffices to prove that the functor $\mathrm{StrPSh}_{\mathbb{C}} \to \mathrm{PSh}_{\mathbb{C}}$ preserves and reflects finite limits. We can reduce this statement to the one that $\mathrm{StrSch}_{\mathbb{C}} \to \mathrm{Sch}_{\mathbb{C}}$ does. By definition, this follows from Lemma 2.A.4. □

We can thus summarise the content of this whole section as follows, again in the notations of Section 2.A.3. Recall that $\mathrm{Ran}(X)$ is seen as a trivially stratified presheaf over $\mathrm{Sch}_{\mathbb{C}}$.

Theorem 2.1.29. *There exists a functor*

$$\mathbb{A}\mathrm{ctGr}_\bullet : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathbb{A}\mathrm{ct}((\mathrm{StrPSh}_{\mathbb{C}})_{/(\mathrm{Ran}(X))})$$

$$[k] \mapsto ((\mathrm{Gr}_{\mathrm{Ran},k}, \sigma_k) \to (\mathrm{Ran}(X)), (\mathrm{Arc}_{\mathrm{Ran},k} \to \mathrm{Ran}(X),$$

$$\Phi_{k,k} \text{ stratified action of } \mathrm{Arc}_{\mathrm{Ran},k} \text{ on } \mathrm{Gr}_{\mathrm{Ran},k} \text{ over } \mathrm{Ran}(X)),$$

which enjoys the 2-Segal property, and such that:

- $\mathbb{A}\mathrm{ctGr}_1 = (\mathrm{Gr}_{\mathrm{Ran}} \to \mathrm{Ran}(X), G_{\mathcal{O},\mathrm{Ran}} \to \mathrm{Ran}(X), \Phi_{1,1} : G_{\mathcal{O},\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}} \to \mathrm{Gr}_{\mathrm{Ran}})$
- *the values of* $\mathbb{A}\mathrm{ctGr}_k$ *object for higher k's describe the Mirkovic-Vilonen convolution diagram and its associativity in the sense of Section 2.1.4.*

2.2 Fusion over the Ran space

2.2.1 Analytification

Topological versions of $\mathbf{Gr}_{\mathbf{Ran},k}$ and $\mathbf{Arc}_{\mathbf{Ran},k}$

In order to take into account the topological properties of the affine Grassmannian and of its global variants, we will now analytify the construction performed in the previous section. This will allow us to consider the complex topology naturally induced on the analytic analogue of the prestacks $\mathrm{Gr}_{\mathrm{Ran},k}$ by

the fact that X is a complex curve, as well as a naturally induced stratification on the resulting complex analytic spaces.
In 2.A.1 we describe the stratified analytification functor, which in turn induces a functor

$$\mathfrak{strtop} : \mathrm{PSh}(\mathrm{StrSch}_{\mathbb{C}}) \to \mathrm{PSh}(\mathrm{StrTop})$$

(see Construction 2.A.3). Recall that this functor preserves finite limits.

Hence, if we precompose $\mathfrak{strtop}$ with $\mathrm{Gr}_{\mathrm{Ran},\bullet} : \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{StrPSh}_{\mathbb{C}}$ we obtain a 2-Segal semisimplicial object in stratified spaces. Also, since $\mathfrak{strtop}$ sends stratified étale coverings to stratified coverings in the topology of local homeomorphisms, it extends to a functor between the categories of sheaves.

Notation 2.2.1. For simplicity, we set $\mathcal{G}\mathrm{r}_{\mathcal{R}\mathrm{an}} = \mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}})$, $\mathcal{G}\mathrm{r}_{\mathrm{Ran},k} = \mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran},k})$.

This construction admits a relative version over $\mathrm{Ran}(X)$ which is not exactly the natural one, because of a change of topology we are going to perform on $\mathfrak{top}(\mathrm{Ran}(X))$.

Remark 2.2.2. Let $M = \mathfrak{top}(X)$, which is a real topological manifold of dimension 2. In [Lur17, Definitions 5.5.1.1, 5.5.1.2] J. Lurie defines the Ran space $\mathrm{Ran}(M)$ of a topological manifold. By definition, there is a map of topological spaces $\mathfrak{top}(\mathrm{Ran}(X)) \to \mathrm{Ran}(M)$. Indeed,

$$\mathfrak{top}(\mathrm{colim}_I\, X^I) \simeq \mathrm{colim}_I\, \mathfrak{top}(X^I) \simeq \mathrm{colim}_I (\mathfrak{top} X)^I = \mathrm{colim}_I\, M^I,$$

because $\mathfrak{top}$ is a left Kan extension. Now each term of the colimit is the space of I-indexed collections of points in $X(\mathbb{C})$, and hence it admits a map of sets towards $\mathrm{Ran}(M)$. This is a continuous map: indeed, let $f : I \to X(\mathbb{C})$ be a function such that $f(I) \in \mathrm{Ran}(\{U_i\})$ for some disjoint open sets U_i. Then there is an open set V in $\mathrm{Map}_{\mathrm{Top}}(I, \mathfrak{top}(X))$ containing f and such that $\forall g \in V$, $g(I) \in \mathrm{Ran}(\{U_i\})$: for instance,

$$V = \bigcap_i \{g : I \to X \mid g(f^{-1}(f(I) \cap U_i)) \subset U_i\}$$

suffices.
This induces a continuous map from colim M^I to $\mathrm{Ran}(M)$ by the universal property of the colimit topology, and therefore a continuous map $\mathfrak{top}(\mathrm{Ran}(X)^{\mathrm{an}}) \to \mathrm{Ran}(M)$ which is the identity set-theoretically (and thus it is compatible with the stratifications).

Construction 2.2.3. Composition with the map that we have just described yields a functor which we call again

$$\mathfrak{strtop} : (\mathrm{StrPSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)} \to \mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)}. \tag{2.2.1}$$

In particular, we obtain a map $\rho_\bullet : \mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran},\bullet}, \sigma_\bullet) \to \mathrm{Ran}(M)$, which we consider as an object of 2-Seg$^{\mathrm{ss}}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)})$ (we abuse of notation by denoting the cardinality stratification again by κ).

Analogously, the functor $\mathrm{ActGr}_{\mathrm{Ran},\bullet} : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathbb{A}\mathrm{ct}((\mathrm{StrPSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)})$ induces a functor $\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}_\bullet : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathbb{A}\mathrm{ct}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)})$.

Remark 2.2.4. An important remark: with the notations of Section 2.A.3, the functor $\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}_\bullet$ takes values in $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)})$.

This is the combination of Lemma 2.1.27 and the following result:

Proposition 2.2.5. *Every $\mathcal{G}\mathrm{r}_k$ is a colimit of objects belonging to* $\mathrm{StrTop}_{\mathrm{con}} \subset \mathrm{StrTop}$.

Proof. See Section 2.B.3. ☐

Let us stress that we are consider $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)})$ and not $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop})$ because the whole affine Grassmannian is stratified by a poset which does not satisfy the ascending chain condition, whereas the stratifying posets of the truncations at the level N do.

Preimage functors

For every open $U \subset \mathrm{Ran}(M)$, there exists a "preimage space" $\mathcal{G}\mathrm{r}_{U,k} \in \mathrm{StrTop}$, whose underlying set can be described as

$$\{\text{tuples in } \mathrm{Gr}_{\mathrm{Ran},k}(\mathbb{C}) \text{ such that } S \text{ lies in } U\}.$$

Formally:

Definition 2.2.6. We define functors $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k : \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M)) \to \mathrm{StrTop}_{/\mathrm{Ran}(M)}$ as

$$\mathcal{O}\mathrm{pen}(\mathrm{Ran}(M)) \subset \mathrm{Top}_{/\mathrm{Ran}(M)} \xrightarrow{\rho_k^{-1}} \mathrm{StrTop}_{/\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k} \to \mathrm{StrTop}_{/\mathrm{Ran}(M)}$$

sending U to $(U, \kappa|_U)$ and finally to $(\rho_k^{-1}(U) \to \mathrm{Ran}(M), \sigma_k|_{\rho_k^{-1}(U)})$.

This operation is compatible with the semisimplicial structure, and therefore we obtain a functor:

$$\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_\bullet : \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M)) \to \text{2-Seg}^{\mathrm{ss}}(\mathrm{StrTop}_{/\mathrm{Ran}(M)}).$$

We can perform the same restriction construction as above for $\mathcal{A}\mathrm{rc}_{\mathrm{Ran},k}$ and obtain stratified topological groups

$$\mathcal{A}\mathrm{rc}_{U,k}$$

acting on $\mathcal{G}r_{U,k} \in \mathrm{StrTop}_{/\mathrm{Ran}(M)}$, functorially in

$$U \in \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M))$$

and k. We denote the functor $U \mapsto (\mathcal{A}\mathrm{rc}_{U,k} \to \mathrm{Ran}(M))$ by

$$\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k : \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M)) \to \mathrm{Grp}(\mathrm{StrTop}_{/\mathrm{Ran}(M)}).$$

Again, this construction is functorial in k.

Remark 2.2.7. The two functors $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k$ and $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k$ are hypercomplete cosheaves with values in $\mathrm{PSh}(\mathrm{StrTop}_{/\mathrm{Ran}(M)})$.

2.2.2 Fusion

Definition 2.2.8. Let

$$\mathrm{StrTop}^{\odot}_{/\mathrm{Ran}(M)}$$

be the following symmetric monoidal structure on $\mathrm{StrTSpc}_{/(\mathrm{Ran}(M),\kappa)}$: if $\xi : \mathcal{X} \to \mathrm{Ran}(M)$, $\upsilon : \mathcal{Y} \to \mathrm{Ran}(M)$ are continuous maps, we define $\xi \odot \upsilon$ to be the **disjoint product**

$$(\mathcal{X} \times \mathcal{Y})_{\mathrm{disj}} = \{x \in \mathcal{X}, y \in \mathcal{Y} \mid \xi(x) \cap \upsilon(y) = \varnothing\}$$

together with the map towards $\mathrm{Ran}(M)$ induced by the map

$$\mathrm{union} : (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} \to \mathrm{Ran}(M)$$

$$(S, T) \mapsto S \sqcup T.$$

Recall the definition of the operad $\mathrm{Fact}(M)^{\otimes}$ from [Lur17, Definition 5.5.4.9]. The aim of this subsection is to extend the $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k$'s and the $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k$'s to maps of operads respectively $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{StrTSpc}^{\odot}_{/\mathrm{Ran}(M)}$ and $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{Grp}(\mathrm{StrTSpc}_{/\mathrm{Ran}(M)})^{\odot}$: the idea is that the first one should encode the gluing of sheaves trivialised away from disjoints systems of points, and the second one should behave accordingly.

The gluing map

We turn back for a moment to the algebraic side.

Definition 2.2.9. Let $(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}$ be the subfunctor of $\mathrm{Ran}(X) \times \mathrm{Ran}(X)$ parametrising those $S, T \subset X(R)$ for which $\Gamma_S \cap \Gamma_T = \varnothing$.
Let also $(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}}$ be the preimage of $(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}$ with respect to the map $r_k \times r_k : \mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X) \times \mathrm{Ran}(X)$.

Proposition 2.2.10. *There is a map of stratified presheaves* $\chi_k : (\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}} \to \mathrm{Gr}_{\mathrm{Ran},k}$ *encoding the gluing of sheaves with trivializations outside disjoint systems of points.*

Proof. The map χ_k is defined as follows: we start with an object

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_k, \alpha_k), (T, \mathcal{G}_1, \beta_1, \nu_1, \dots, \mathcal{G}_k, \beta_k),$$

where $S \cap T = \varnothing$. We want to obtain a sequence $(P, \mathcal{H}_1, \gamma_1, \zeta_1, \dots, \mathcal{H}_k, \gamma_k)$. Since the graphs of S and T are disjoint, $X_R \setminus \Gamma_S$ and $X_R \setminus \Gamma_T$ form a Zariski open cover of X_R. Therefore, by the descent property of the stack $\mathbf{Bun}_G$, every couple $\mathcal{F}_i$, $\mathcal{G}_i$ can be glued by means of α_i and β_i.

Each of these glued sheaves, that we call $\mathcal{H}_i$, inherits a trivialization γ_i outside $\Gamma_S \cup \Gamma_T$, which is well-defined up to isomorphism (it can be seen both as $\alpha_i|_{X_R \setminus (\Gamma_S \cup \Gamma_T)}$ or as $\beta_i|_{X_R \setminus (\Gamma_S \cup \Gamma_T)}$). Now set $P = S \cup T$ (in the usual sense of joining the two collections of points).

It remains to define the glued formal trivializations. However, to define a trivialization ζ_i of $\mathcal{H}_i$ over the formal neighbourhood of Γ_P amounts to look for a trivialization of $\mathcal{H}_i$ on the formal neighbourhood of $\Gamma_S \sqcup \Gamma_T$. But the first part of this union is contained in $X_R \setminus \Gamma_T$, where $\mathcal{H}_i$ is canonically isomorphic to $\mathcal{F}_i$ by construction; likewise, the first part of the union is contained in $X_R \setminus \Gamma_S$, where $\mathcal{H}_i$ is canonically isomorphic to $\mathcal{G}_i$ by construction. Hence, the originary trivializations μ_i and ν_i canonically provide the desired datum ζ_i, and the construction of the map is complete.
Moreover, this map is stratified. Indeed, we have the torsor $\mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}$, and the stratification on $\mathrm{Gr}_{\mathrm{Ran},k}$ is the pullback of the one on $\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}$. Now for any I, J finite sets, the map $(\mathrm{Gr}_{X^I} \times \mathrm{Gr}_{X^J})_{\mathrm{disj}} \to \mathrm{Gr}_{X^{I \sqcup J}}$ is stratified by definition (cfr. Definition 2.1.24). Therefore,

$$((\overbrace{\mathrm{Gr}_{X^I} \times_{X^I} \cdots \times_{X^I} \mathrm{Gr}_{X^I}}^{k}) \times (\overbrace{\mathrm{Gr}_{X^J} \times_{X^J} \cdots \times_{X^J} \mathrm{Gr}_{X^J}}^{k}))_{\mathrm{disj}} \to$$

$$\to \overbrace{\mathrm{Gr}_{X^{I \sqcup J}} \times_{X^{I \sqcup J}} \cdots \times_{X^{I \sqcup J}} \mathrm{Gr}_{X^{I \sqcup J}}}^{k}$$

is stratified, and taking the colimit for $I \in \mathcal{F}\mathrm{in}_{\mathrm{surj}}$, we obtain that

$$((\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}) \times (\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}))_{\mathrm{disj}}$$

is stratified. Finally, since the stratification on $\mathrm{Gr}_{\mathrm{Ran},k}$ is induced by the one on $\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}$ via the torsor map $\mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}$, we can conclude. $\square$

Construction of $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot}$

Remark 2.2.11. Consider two independent open subsets U and V of $\mathrm{Ran}(M)$. We have the following diagram

$$\begin{tikzcd}
\mathcal{G}\mathrm{r}_{U,k} \times \mathcal{G}\mathrm{r}_{V,k} \arrow[r] \arrow[d, "\pi"] & \mathfrak{strtop}((\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}}) \arrow[r] \arrow[d] & \mathcal{G}\mathrm{r}_{\mathrm{Ran},k} \arrow[d] \\
U \times V \arrow[r, "\subset"] & (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} \arrow[r, "\mathrm{union}"] & \mathrm{Ran}(M),
\end{tikzcd} \tag{2.2.2}$$

where the left hand square is a pullback of topological spaces, and the right top horizontal map is induced by Proposition 2.2.10 by applying $\mathfrak{strtop}$. Here we use that the underlying complex-analytical topological space of $\mathrm{Ran}(X)$ is – set-theoretically – the space of points of M, and therefore the map $\mathfrak{strtop}(\mathrm{Ran}(X) \times \mathrm{Ran}(X)) \to \mathrm{Ran}(M) \times \mathrm{Ran}(M)$ restricts to a well-defined map $\mathfrak{strtop}((\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}) \to (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}}$.

Note also that the bottom composition coincides with $U \times V \to U \star V \hookrightarrow \mathrm{Ran}(M)$, the first map being the one taking unions of systems of points; hence, by the universal property of the fibered product of topological spaces, $(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(U) \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(V)$ admits a map towards $(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(U \star V) = \mathcal{G}\mathrm{r}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(M)} (U \star V)$, which we call $p_{U,V,k}$. Of course the triangle

$$\begin{tikzcd}
\mathcal{A}\mathrm{rc}_{U,k} \times \mathcal{A}\mathrm{rc}_{V,k} \arrow[rr, "p_{U,V,k}"] \arrow[dr, "\mathrm{union} \circ \pi"] & & \mathcal{A}\mathrm{rc}_{U \star V,k} \arrow[dl] \\
& \mathrm{Ran}(M) &
\end{tikzcd} ,$$

commutes.

Proposition 2.2.12. *Remark 2.2.11 induces well-defined maps of operads* $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{StrTop}^{\odot}_{/\mathrm{Ran}(M)}$ *encoding the gluing of sheaves trivialised outside disjoint systems of points. That is, we have*

- $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot}(U)$ *is the map* $(\mathcal{A}\mathrm{rc}_{U,k}) \to U \hookrightarrow \mathrm{Ran}(M)$ *for every* U *open subset of* $\mathrm{Ran}(M)$.
- *the image of the morphism* $(U, V) \to (U \star V)$ *for any independent* $U, V \in \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M))$ *is the commuting triangle*

$$\begin{tikzcd}
(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(U) \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(V) \arrow[rr] \arrow[dr] & & (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)(U \star V) \arrow[dl] \\
& \mathrm{Ran}(M) &
\end{tikzcd} ,$$

where the top map is the gluing of sheaves trivialised outside disjoints systems of points, and the left map is the map that remembers the two disjoint systems of points and takes their union.

Proof. See Section 2.B.2. ☐

Remark 2.2.13. The constructions performed in the proof of Proposition 2.2.12 are compatible with the face maps of $\mathcal{G}\mathrm{r}_{\mathrm{Ran},\bullet}$. Indeed, for any two independent open subsets $U, V \subset \mathrm{Ran}(M)$, the squares

$$\begin{tikzcd}
\mathcal{G}\mathrm{r}_{U,k} \times \mathcal{G}\mathrm{r}_{V,k} \arrow[r] \arrow[d] & \mathcal{G}\mathrm{r}_{U\star V,k} \arrow[d] \\
\mathcal{G}\mathrm{r}_{U,k-1} \times \mathcal{G}\mathrm{r}_{V,k-1} \arrow[r] & \mathcal{G}\mathrm{r}_{U\star V,k-1}
\end{tikzcd}$$

are commutative because the original diagrams at the algebraic level commute. That is,

$$\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{Ran},k} \arrow[d] \\
(\mathrm{Gr}_{\mathrm{Ran},k-1} \times \mathrm{Gr}_{\mathrm{Ran},k-1})_{\mathrm{disj}} \arrow[r] & \mathrm{Gr}_{\mathrm{Ran},k-1}
\end{tikzcd}$$

commutes, since the construction involved in the horizontal maps, as we have seen, does not change the formal trivializations, and, by the independence hypothesis, the non-formal trivializations do not change in the punctured formal neighbourhoods involved in the formal gluing procedure.

Proposition 2.2.14. *The maps of operads* $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{StrTop}^{\odot}_{/\mathrm{Ran}(M)}$ *assemble to a map of operads* $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{\bullet}^{\odot} : \mathrm{Fact}(M)^{\otimes} \to (2\text{-}\mathrm{Seg}^{\mathrm{ss}}(\mathrm{StrTop}))^{\odot}_{/\mathrm{Ran}(M)}$.

Proof. Since we have already noticed that the functor $\mathfrak{strtop}$ preserve finite limits, the condition that $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{\bullet}$ is 2-Segal can be recovered from the algebraic setting. Now the map

$$\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(U) \to \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{\{0,\dots,l\}}(U) \times_{\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{\{0,l\}}(U)} \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{\{l,\dots,k\}}(U)$$

is the pullback of

$$\mathcal{G}\mathrm{r}_{\mathrm{Ran},k} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran},\{0,\dots,l\}} \times_{\mathcal{G}\mathrm{r}_{\mathrm{Ran},\{0,l\}}} \mathcal{G}\mathrm{r}_{\mathrm{Ran},\{0,l,\dots,k\}}$$

along $U \to \mathrm{Ran}(M)$, hence it is a homeomorphism (and the same holds for the $\{l, k\}$ case). ☐

Construction of $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k^{\odot}$

Construction 2.2.15. We can perform a similar construction for

$$\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k : \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M)) \to \mathrm{Grp}(\mathrm{StrTop}_{/\mathrm{Ran}(M)})$$

as well. Indeed, we can define

$$(\mathrm{Arc}_{\mathrm{Ran},k} \times \mathrm{Arc}_{\mathrm{Ran},k})_{\mathrm{disj}}(R) = \{(S, g_1, \dots, g_k) \in \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(R),$$

$$(T, h_1, \dots, h_k) \in \mathrm{Arc}_{\mathrm{Ran},k}(R) \mid \Gamma_S \cap \Gamma_T = \varnothing\}$$

and maps

$$(\mathrm{Arc}_{\mathrm{Ran},k} \times \mathrm{Arc}_{\mathrm{Ran},k})_{\mathrm{disj}}(R) \to \mathrm{Arc}_{\mathrm{Ran},k}(R)$$

$$((S, g_1, \dots, g_k), (T, h_1, \dots, h_k)) \mapsto (S \cup T, \widetilde{g_1 h_1}, \dots, \widetilde{g_k h_k}),$$

where $\widetilde{g_i h_i}$ is the automorphism of $\mathcal{T}_{G_{\widehat{(X_R)}_{\Gamma_S \cup \Gamma_T}}}$ defined separatedly as g_i and h_i on the two components, which are disjoint by hypothesis. The rest of the construction is analogous, and provides maps of operads

$$\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{Grp}(\mathrm{StrTop}_{/\mathrm{Ran}(M)})^{\odot}$$

which are, as usual, natural and 2-Segal in $k \in \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$.

2.2.3 The factorising property

Our aim now is to verify the so-called factorization property (see [Lur17, Theorem 5.5.4.10]) for the functors

$$\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{StrTop}^{\odot}_{/\mathrm{Ran}(M)}$$

and

$$\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k^{\odot} : \mathrm{Fact}(M)^{\otimes} \to \mathrm{Grp}(\mathrm{StrTop}_{/\mathrm{Ran}(M)})^{\odot}.$$

This will immediately imply the property.

Proposition 2.2.16 (Generalised factorization property)**.** *If U, V are independent, then the maps $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(U) \times \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(V) \to \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(U \star V)$, resp. $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(U) \times \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(V) \to \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(U \star V)$, are stratified homeomorphisms over* $\mathrm{Ran}(M)$*, resp. homeomorphisms of topological groups over* $\mathrm{Ran}(M)$.

Proof. Note that the right-hand square in Diagram (2.2.2) is Cartesian. Indeed, let us now prove that its algebraic counterpart

$$\begin{tikzcd}
(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Gr}_{\mathrm{Ran},k} \arrow[d] \\
(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \arrow[r] & \mathrm{Ran}(X)
\end{tikzcd}$$

is cartesian in $\mathrm{PSh}_{\mathbb{C}}$.

The pullback of the cospan computed in $\mathrm{PSh}_{\mathbb{C}}$ is, abstractly, the functor parametrising tuples of the form $(S, T), (P, \mathcal{H}_i, \gamma_i, \zeta_i)$, where $(S, T) \in (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}(R)$, $P = S \cup T$, $(P, \mathcal{H}_i, \gamma_i, \zeta_i) \in \mathrm{Gr}_{\mathrm{Ran},k}(R)$. From this we can uniquely reconstruct a sequence $(S, \mathcal{F}_i, \alpha_i, \mu_i, T, \mathcal{G}_i, \beta_i, \nu_i)$ in $(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}}(R)$. To do so, define $\mathcal{F}_i \in \mathrm{Bun}_G(X_R)$ as the gluing of $\mathcal{H}_i$ with the trivial G-bundle around

T, which comes with a trivialization $\alpha_i : \mathcal{F}_i|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{T}_G|_{X_R \setminus \Gamma_S}$. We also define $\mathcal{G}_i$ as the gluing of $\mathcal{H}_i$ with the trivial G-bundle around S, coming with a trivialization β_i outside T. As for the formal part of the datum, the ζ_i's automatically restrict to the desired formal neighbourhoods.

This construction is inverse to the natural map $(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}} \to \mathrm{Gr}_{\mathrm{Ran},k} \times_{\mathrm{Ran}(X)} (\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}$.

But now, the diagram

$$\begin{tikzcd}
\mathfrak{strtop}(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathfrak{strtop}(\mathrm{Ran}(X)) \arrow[d] \\
(\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} \arrow[r] & \mathrm{Ran}(M)
\end{tikzcd}$$

is again Cartesian, since, set-theoretically, the vertical maps are the identity, and we are just performing a change of topology on the bottom map. Hence the right-hand square in Diagram (2.2.2) is Cartesian, because the functor $\mathfrak{strtop}$ preserves finite limits.

This concludes the proof since the outer square in (2.2.2) is Cartesian, and therefore the natural map $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(U) \times \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(V) \to \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(U \star V)$ is a homeomorphism of topological spaces.

Now we turn to $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k$. It suffices to prove that the square

$$\begin{tikzcd}
(\mathrm{Arc}_{\mathrm{Ran},k} \times \mathrm{Arc}_{\mathrm{Ran},k})_{\mathrm{disj}} \arrow[r] \arrow[d] & \mathrm{Arc}_{\mathrm{Ran},k} \arrow[d] \\
(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \arrow[r] & \mathrm{Ran}(X)
\end{tikzcd}$$

is Cartesian in $\mathrm{PSh}_{\mathbb{C}}$. But this is clear once one considers the map

$$(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} \times_{\mathrm{Ran}(X)} \mathrm{Arc}_{\mathrm{Ran},k} \to (\mathrm{Arc}_{\mathrm{Ran},k} \times \mathrm{Arc}_{\mathrm{Ran},k})_{\mathrm{disj}}$$

given by

$$(S, T, \{g_i \in \mathrm{Aut}_{\widehat{(X_R)}_{\Gamma_S \cup \Gamma_T}}(\mathcal{T}_G)\}) \mapsto ((S, \{g_i|_{\widehat{(X_R)}_{\Gamma_S}}\}), (S, \{g_i|_{\widehat{(X_R)}_{\Gamma_T}}\})).$$

Since the graphs are disjoint, this map is an equivalence. This concludes the proof. □

2.2.4 Local constancy

The aim of this subsection is to prove that the functors $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k^{\odot}$ and $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k^{\odot}$ satisfy a "local constancy" property in a homotopical sense, which will be used in the following to apply [Lur17, Theorem 5.5.4.10]. First of all, we need a lemma.

Lemma 2.2.17. *Let G be a (stratified) group scheme. If $f : S \to T$ is a morphism of (stratified) schemes which is a (stratified) G-torsor with respect to the étale topology, then $(\mathfrak{str})\mathfrak{top}(f) : (\mathfrak{str})\mathfrak{top}(S) \to (\mathfrak{str})\mathfrak{top}(T)$ is a principal (stratified) G-bundle.*

Proof. First we perform the proof in the unstratified setting. First of all, to be locally trivial with respect to the étale topology at the algebraic level implies to be locally trivial with respect to the analytic topology at the analytic level (see [Rey71, Section 5] and [Bha, Section 5]). Now the analytic topology on an analytic manifold is the topology whose coverings are jointly surjective families of local homeomorphisms. We want to prove that a trivialising covering for $\mathfrak{top}(f)$ with respect to this topology is the same of a classical trivialising open covering. Any open embedding is a local homeomorphism. Conversely, let us suppose that we have a trivialising local homeomorphism, that is a local homeomorphism of topological spaces $w : W \to \mathfrak{top}(T)$ such that $\mathfrak{top}(S) \times_{\mathfrak{top}(T)} W$ is isomorphic to $\mathfrak{top}(G) \times W$. Now, by definition of local homeomorphism, for every x in the image of w we have that w restricts to an open embedding on some open set $U \subset W$ whose image contains x. Moreover, $U \to W \to \mathfrak{top}(T)$ is trivialising *a fortiori*, that is $U \times_{\mathfrak{top}(T)} \mathfrak{top}(S) \simeq U \times \mathfrak{top}(G)$. In conclusion, if we have a jointly surjective family of trivialising local homeomorphisms, the above procedure yields a covering family of trivialising open sets.
In the stratified setting, the proof is the same, since the relevant maps are étale-stratified on the algebraic side, and therefore become analytic-stratified on the topological side. ☐

Proposition 2.2.18. *For every nonempty finite collection of disjoint disks $D_1, \dots, D_n \subseteq M$ containing open subdisks $E_1 \subseteq D_1, \dots, E_n \subseteq D_n$, the maps*

$$\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(\{E_i\})) \to \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(\{D_i\}))$$

and

$$\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(\{E_i\})) \to \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(\{D_i\}))$$

are stratified homotopy equivalences over $(\mathbb{X}_\bullet(T)^+)^k$ in the sense of Definition 2.A.21.

Proof. The factorising property tells us that these maps assume the form $\prod_{i=1}^n \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(E_i)) \to \prod_{i=1}^n \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(D_i))$ and $\prod_{i=1}^n \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(E_i)) \to \prod_{i=1}^n \mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(D_i))$ respectively, and hence it suffices to perform the checks term by term, i.e., to assume $n = 1$.
We deal first with the case $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(\{E_i\})) \to \mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k(\mathrm{Ran}(\{D_i\}))$. Now we observe that we can reduce to the case $k = 1$: indeed, $\mathrm{Gr}_{\mathrm{Ran},k}$ is a stratified $\mathrm{Arc}_{\mathrm{Ran},k-1}$-torsor over the k-fold product $\mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \cdots \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{\mathrm{Ran}}$, and therefore $\mathcal{G}\mathrm{r}_{\mathrm{Ran},k} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}} \times_{\mathrm{Ran}(M)} \cdots \times_{\mathrm{Ran}(M)} \mathcal{G}\mathrm{r}_{\mathrm{Ran}}$ is a stratified $\mathcal{A}\mathrm{rc}_{\mathrm{Ran},k-1}$-principal topological bundle, in particular it is a Serre fibration. Thus, if we prove that $(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(E)}) \to (\mathcal{G}\mathrm{r}_{\mathrm{Ran}}(D))$ is a stratified homotopy equivalence, then the same will be true for

$\mathcal{G}\mathrm{r}_{\mathrm{Ran}(E)} \times_{\mathrm{Ran}(E)} \cdots \times_{\mathrm{Ran}(E)} \mathcal{G}\mathrm{r}_{\mathrm{Ran}(E)} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)} \times_{\mathrm{Ran}(D)} \cdots \times_{\mathrm{Ran}(D)} \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)}$ and therefore at the upper level for $\mathcal{G}\mathrm{r}_{\mathrm{Ran}(E),k} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D),k}$.
Thus we only need to prove the statement for $\mathcal{G}\mathrm{r}_{\mathrm{Ran}}$. This is done in [HY19, Proposition 3.17]. More precisely, [HY19, Proposition 3.17] states that

(*) The diagram

$$\begin{tikzcd}
\mathcal{G}\mathrm{r}_x \arrow[r] \arrow[d] & \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)} \arrow[d] \\
\{x\} \arrow[r] & \mathrm{Ran}(D)
\end{tikzcd}$$

is a pullback in StrTop and the map $\mathcal{G}\mathrm{r}_x \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)}$ is a stratified homotopy equivalence over $\mathbb{X}_\bullet(T)^+$.

where $\mathrm{Gr}_x = \mathrm{Gr}_{\mathrm{Ran}} \times_{\mathrm{Ran}(X)} \{x\}$ was defined in Definition 2.1.8 (this is the case $k = 1$). We thank Jeremy Hahn and Allen Yuan for explaining the details of their proof to us. We report their explanation in Section 2.B.4.

An analogous (and simpler) argument works for $\mathcal{A}\mathrm{rc}_{\mathrm{Ran},k}$, because

$$G_{\mathcal{O},\mathrm{Ran}}|_{\mathrm{Ran}_{card=k}(M)} \simeq (X^{\times k})_{\mathrm{disj}} \times G_{\mathcal{O}}^{\times k}$$

(see Section 2.B.4). ☐

Note that (*) implies the following:

Proposition 2.2.19. *If D is a disk around x in M, there is a natural map $\mathcal{G}\mathrm{r}_{x,k} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D),k}$ which is a stratified homotopy equivalence over $(\mathbb{X}_\bullet(T)^+)^k$. In a similar way, the natural map $\mathcal{A}\mathrm{rc}_{x,k} \to \mathcal{A}\mathrm{rc}_{\mathrm{Ran}(D),k}$ is a homotopy equivalence of topological groups, and the above map $\mathcal{G}\mathrm{r}_{x,k} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran}(D),k}$ is equivariant with respect to this latter map.*

Notation 2.2.20. From now on, we will denote the relation "being stratified homotopy equivalent" by $\overset{\mathfrak{she}}{\sim}$ and "being homotopy equivalent as topological groups" by $\overset{\mathfrak{ghe}}{\sim}$.

We can express this property in a more suitable way by considering the ∞-categorical localization $\mathrm{PSh}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop}_{/\mathrm{Ran}(M)})[\mathfrak{esh}^{-1}])^\times$ see Definition 2.A.21. Since $\mathfrak{she}$ is closed under the symmetric monoidal structure of $\mathrm{PSh}(\mathrm{StrTop})^\times$, the localization functor extends to a map of operads

$$(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{PSh}(\mathrm{StrTop}))^\times \to \mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{PSh}(\mathrm{StrTop})[\mathfrak{esh}^{-1}]^\times$$

and therefore to

$$\mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{PSh}(\mathrm{StrTop}))^{\times} \to \mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{PSh}(\mathrm{StrTop})[\mathfrak{esh}^{-1}])^{\times}.$$

This provides a functor

$$\mathcal{A}\mathrm{ct}\mathfrak{G}\mathrm{r}^{\times} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)}))[\mathfrak{esh}^{1}])^{\times,\odot} \tag{2.2.3}$$

which is a map of operads in both variables with respect to the respective symmetric monoidal structures on the target, has the factorization property in the first variable and sends the usual inclusions of Ran spaces of systems of disks to equivalences in the target category.

2.2.5 Interaction of convolution and fusion over Ran(M)

Construction 2.2.21. Let thus $\mathcal{A}\mathrm{ct}\mathfrak{G}\mathrm{r}_{\bullet}(-)^{\cdot} : \mathrm{Fact}(M)^{\otimes} \times \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{PSh}(\mathbb{A}\mathrm{ct}((\mathrm{StrTop})_{/\mathrm{Ran}(M)}))^{\odot}$ denote the functor defined by $(U,k) \mapsto (\mathfrak{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k(U), \mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_k(U), \Phi_{k,k}(U))$. Recall that this functor satisfies the 2-Segal property in k. If we apply (the semisimplicial variant of) [DK19, Theorem 11.1.6] to $\mathcal{A}\mathrm{ct}\mathfrak{G}\mathrm{r}_{\bullet}(-)^{\odot}$, we obtain a functor

$$\mathcal{A}\mathrm{ct}\mathfrak{G}\mathrm{r}(-,-)^{\odot,\times} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}(\mathrm{Top}_{/\mathrm{Ran}(M)})))^{\odot,\times}$$

This functor is lax monoidal in the variable $\mathrm{Fact}(M)^{\otimes}$ with respect to the structure $\odot$ on the target, and in the variable $\mathbb{E}_1^{\mathrm{nu}}$ with respect to the structure $\times$ on the target.

Remark 2.2.22. Now we make the two algebra structures interact. Consider $\mathfrak{G}\mathrm{r}_{\mathrm{Ran},1} = \mathfrak{G}\mathrm{r}_{\mathrm{Ran}}$: we have two ways of defining an "operation" on it:

- restrict to $\mathfrak{G}\mathrm{r}_{\mathrm{Ran},1} \times_{\mathrm{Ran}(M)} \mathfrak{G}\mathrm{r}_{\mathrm{Ran},1}$ and consider the correspondence

$$\begin{tikzcd}
 & \mathfrak{G}\mathrm{r}_{\mathrm{Ran},2} \arrow[dl] \arrow[dr] & \\
\mathfrak{G}\mathrm{r}_{\mathrm{Ran},1} \times_{\mathrm{Ran}(M)} \mathfrak{G}\mathrm{r}_{\mathrm{Ran},1} & & \mathfrak{G}\mathrm{r}_{\mathrm{Ran},1}
\end{tikzcd}$$

- restrict to $(\mathfrak{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathfrak{G}\mathrm{r}_{\mathrm{Ran},1})_{\mathrm{disj}}$, or more precisely to $\mathfrak{G}\mathrm{r}_{U,1} \times \mathfrak{G}\mathrm{r}_{V,1}$ for independent open sets $U, V \subset \mathrm{Ran}(M)$, and consider the map $\mathfrak{G}\mathrm{r}_{U,1} \times \mathfrak{G}\mathrm{r}_{V,1} \to \mathfrak{G}\mathrm{r}_{U\star V,1}$ induced by Remark 2.2.11.

Formally, these "restrictions" are obtained by forgetting both structures to StrTop. Indeed, the forgetful functor $\mathrm{StrTop}_{/\mathrm{Ran}(M)} \to \mathrm{StrTop}$ induces a functor

$$\mathrm{Corr}(\mathrm{PSh}(\mathrm{StrTop})_{/\mathrm{Ran}(M)}) \to \mathrm{Corr}(\mathrm{PSh}(\mathrm{StrTop}))$$

which is lax monoidal with respect to both $\times_{\mathrm{Ran}(X)}$ and $\odot$. Indeed, there are maps

$$\mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times_{\mathrm{Ran}(M)} \mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathcal{G}\mathrm{r}_{\mathrm{Ran},1}$$

and

$$(\mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathcal{G}\mathrm{r}_{\mathrm{Ran},1})_{\mathrm{disj}} \to \mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathcal{G}\mathrm{r}_{\mathrm{Ran},1}$$

which can be encoded as correpondences from $\mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathcal{G}\mathrm{r}_{\mathrm{Ran},1}$ to $\mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times_{\mathrm{Ran}(M)} \mathcal{G}\mathrm{r}_{\mathrm{Ran},1}$ and $(\mathcal{G}\mathrm{r}_{\mathrm{Ran},1} \times \mathcal{G}\mathrm{r}_{\mathrm{Ran},1})_{\mathrm{disj}}$ respectively. Note that the context of correspondences here is very useful to encode this "restriction" procedure.

By this argument, the functor obtained from

$$\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}(-,-)^{\odot,\times} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}_{/\mathrm{Ran}(M)}))^{\odot,\times}$$

by composition with

$$\mathrm{Corr}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}_{/\mathrm{Ran}(M)}))^{\odot,\times} \to \mathrm{Corr}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}))^{\times}$$

is a lax monoidal functor in both variables

$$\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}(-,-)^{\times} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}))^{\times}.$$

2.2.6 The stalk of the factorising cosheaf

We can apply [Lur17, Theorem 5.5.4.10] to the map of operads (2.2.3), since we have proven in the previous subsections that the hypotheses of the theorem are satisfied. We denote the operads $\mathbb{E}_n^{\otimes}, \mathbb{E}_M^{\otimes}$ by $\mathbb{E}_n, \mathbb{E}_M$.

Corollary 2.2.23. *The functor* $\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}^{\times}$ *induces a nonunital* $\mathbb{E}_M$*-algebra object*

$$\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}_M^{\times} \in \mathrm{Alg}_{\mathbb{E}_M}^{\mathrm{nu}}(\mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}(\mathrm{Top})[\mathfrak{csh}^{-1}])^{\times})).$$

Fix a point $x \in M$. The main consequence of the above result is that the stalk of $\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}^{\times}$ at the point $\{x\} \in \mathrm{Ran}(M)$ inherits an $\mathbb{E}_2^{\mathrm{nu}}$-algebra structure in 2-$\mathrm{Seg}^{\mathrm{ss}}(\mathrm{PSh}(\mathbb{A}\mathrm{ct}(\mathrm{Top})[\mathfrak{csh}^{-1}])^{\times}))$. We will now explain how, running through [Lur17, Chapter 5] again.

Remark 2.2.24. By [Lur17, Example 5.4.5.3], a nonunital $\mathbb{E}_M$-algebra object $A^{\otimes}$ in a symmetric monoidal ∞-category $\mathcal{C}$ induces a nonunital $\mathbb{E}_n$-algebra object in $\mathcal{C}$, where n is the real dimension of the topological manifold M, by taking the stalk at a point $x \in M$. More precisely, there is an object in $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ whose underlying object is $\lim_{\{x\}\in U \in \mathcal{O}\mathrm{pen}(\mathrm{Ran}(M))} A(U)$, which coincides with $\lim_{x \in D \in \mathrm{Disk}(M)} A(\mathrm{Ran}(D))$, since the family of Ran spaces of disks around x is final in the family of open neighbourhoods of $\{x\}$ inside Ran(M). Now each $A^{\otimes}|_{\mathrm{Fact}(D)^{\otimes}}$ induces a nonunital $\mathbb{E}_D$-algebra by Lurie's theorem [Lur17, p. 5.5.4.10]. But [Lur17, Example 5.4.5.3] tells us that $\mathbb{E}_D$-algebras are equivalent to $\mathbb{E}_n$ algebras. Also, by local constancy (i.e. constructibility), the functor $D \mapsto A(D)$ is constant over the family $x \in D \in \mathrm{Disk}(M)$, and therefore the stalk A_x coincides with any of those $\mathbb{E}_n$-algebras. This also implies that all stalks at points of M are (noncanonically) isomorphic.[4]

Also, the content of [Lur17, Subsection 5.5.4] tells us how the $\mathbb{E}_n$-multiplication structure works concretely. Choose a disk D containing x. We interpret this as the only object in the $\langle 1\rangle$-fiber of $\mathbb{E}_D$. Recall that a morphism in $\mathbb{E}_D$ lying over the map

$$\langle 2\rangle \to \langle 1\rangle$$

$$1, 2 \mapsto 1$$

is the choice of an embedding $D \amalg D \hookrightarrow D$. Call n_D the unique object lying over $\langle n\rangle$ in $\mathbb{E}_D$.

Consider the canonical map $\mathbb{E}_D \to \mathbb{E}_M$. If A_M is the $\mathbb{E}_M^{\mathrm{nu}}$-algebra object appearing in the conclusion of Lurie's theorem, call A_D its restriction to $\mathbb{E}_D$. Recall from the proof of Lurie's theorem that A_M is obtained by operadic left Kan extension of the restriction $A|_{\mathrm{Disk}(M)}$ along the functor $\mathrm{Disk}(M)^{\otimes} \to \mathbb{E}_M$. Then we have that

Lemma 2.2.25.

$$A_D(1_D) = A(\mathrm{Ran}(D)),$$

and

$$A_D(2_D) = \mathrm{colim}\{A(\mathrm{Ran}(E)) \otimes A(\mathrm{Ran}(F)) \mid E, F \in \mathrm{Disk}(M),$$

$$E \cap F = \varnothing, D \amalg D \xrightarrow{\sim} E \amalg F \hookrightarrow D\} \simeq$$

$$\simeq A(\mathrm{Ran}(E_0)) \otimes A(\mathrm{Ran}(F_0))$$

for any choice of an embedding $D \amalg D \xrightarrow{\sim} E_0 \amalg F_0 \hookrightarrow D$.

Proof. We need to prove that the colimit degenerates. Take indeed two couples of disks as in the statement, say E_1, F_1 and E_2, F_2. By local constancy, we can suppose that all four disks are pairwise disjoint. Now we can embed both E_1 and E_2 into some E, and F_1, F_2 into some F, in such a way that $E \cap F = \varnothing$. Then we have canonical equivalences

$$A(\mathrm{Ran}(E_1)) \otimes A(\mathrm{Ran}(F_1)) \xrightarrow{\sim} A(\mathrm{Ran}(E)) \otimes A(\mathrm{Ran}(F))$$

---

[4] Note that this is true only for points of Ran(M) coming from single points of M. If we allow the cardinality of the system of points to vary, stalks may take different values. In fact, the factorization property tells us that a system of cardinality m will give the m-ary tensor product in $\mathcal{C}$ of the stalk at the single point.

and

$$A(\mathrm{Ran}(E_2)) \otimes A(\mathrm{Ran}(F_2)) \xrightarrow{A} (\mathrm{Ran}(E)) \otimes A(\mathrm{Ran}(F)).$$

$\square$

This discussion implies that the operation μ on A_x encoded by Lurie's theorem has the form

$$\begin{aligned} A_x \otimes A_x \simeq A(\mathrm{Ran}(D)) \otimes A(\mathrm{Ran}(D)) &\xrightarrow{\sim} A(\mathrm{Ran}(E_0)) \otimes A(\mathrm{Ran}(F_0)) \\ &\xrightarrow{\sim} A(\mathrm{Ran}(E_0) \star \mathrm{Ran}(F_0)) \to A(\mathrm{Ran}(D)) \simeq A_x, \end{aligned} \tag{2.2.4}$$

where:

- the first and last equivalences come from local constancy;
- the second equivalence is induced by the chosen embedding $D \amalg D \xrightarrow{\sim} E_0 \amalg F_0 \hookrightarrow D$;
- the third equivalence is the factorization property.

The discussion about the stalk leads to the main theorem of this section:

Theorem 2.2.26. *The stalk at $x \in M$ of the $\mathbb{E}_M$-algebra object $\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}_M$ from Corollary 2.2.23 can be naturally viewed as an object of*

$$\mathcal{A}\mathrm{ct}\mathcal{G}\mathrm{r}_x^{\times} \in \mathrm{Alg}_{\mathbb{E}_2}^{\mathrm{nu}}(\mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{nu}}(\mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}[\mathfrak{esh}^{-1}])^{\times}))$$

encoding simultaneously the convolution and fusion procedures on $G_{\mathcal{O}} \backslash \mathrm{Gr}_G$.

2.3 Product of constructible sheaves

2.3.1 Taking constructible sheaves

Definition 2.3.1. We denote by $\mathcal{A}^{\otimes,\mathrm{nu}}$ the functor

$$\mathcal{C}\mathrm{ons}_{\mathrm{corr}}^{\otimes} \circ \mathrm{act}^{\times} : \mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}.$$

By construction, one has that $\mathcal{A}(1,1) = \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr})$ and more in general

$$(\langle m \rangle, \langle k \rangle) \mapsto \overbrace{\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}^{\times k}}(G_{\mathcal{K}}^{\times k-1} \times \mathrm{Gr}) \otimes \cdots \otimes \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}^{\times k}}(G_{\mathcal{K}}^{\times k-1} \times \mathrm{Gr})}^{m}.$$

2.3.2 Units and the main theorem

Remark 2.3.2. Let us inspect the behaviour of the $\mathbb{E}_2$ product. We had a map

$$\mu : \mathcal{H}\mathrm{ck}_x \times \mathcal{H}\mathrm{ck}_x \to \mathcal{H}\mathrm{ck}_x$$

from (2.2.4). By construction, when we apply the functor $\mathcal{C}\mathrm{ons}$, this is sent forward with lower shriek functoriality. Therefore, for $k = 1$, we end up with a map

$$\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}) \times \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}) \xrightarrow{\mu_!} \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}.$$

Recovering the original structure of the map μ, $\mu_!$ decomposes as

$$\begin{gathered}
\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}) \times \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}) \\
\xrightarrow{\sim} \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(D)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)}) \times \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(D)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)}) \\
\xrightarrow{\sim} \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(E)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(E)}) \times \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(F)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(F)}) \\
\xrightarrow{\sim} \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(E)\star\mathrm{Ran}(F)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(E)\star\mathrm{Ran}(F)}) \\
\to \mathcal{C}\mathrm{ons}_{\mathcal{A}\mathrm{rc}_{\mathrm{Ran}(D)}}(\mathcal{G}\mathrm{r}_{\mathrm{Ran}(D)}) \xrightarrow[j_x^*]{\sim} \mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}),
\end{gathered}$$

where $j_x : \{x\} \to \mathrm{Ran}(M)$ is the inclusion.

Proposition 2.3.3. *Let $x \in M$ be a point. Consider the functor $\mathcal{A}_x^{\otimes} : \mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$. Then this can be upgraded to a map of operads $\mathbb{E}_2 \times \mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$.*

Proof. We can apply [Lur17, Theorem 5.4.4.5], whose hypothesis is satisfied since for any ∞-category $\mathcal{C}$ the functor $\mathcal{C}^{\times} \to \mathcal{F}\mathrm{in}_*$ is a coCartesian fibration of ∞-operads ([Lur17, Proposition 2.4.1.5]): therefore, it suffices to exhibit a quasi-unit for any $\mathcal{A}^{\otimes}(-, \langle k\rangle)$, functorial in $\langle k\rangle \in \mathbb{E}_1^{\mathrm{nu}}$. We can consider the map (natural in k) $u_k : \mathrm{Spec}\,\mathbb{C} \to \mathrm{Gr}_{x,k}$ represented by the sequence $(\mathcal{T}_G, \mathrm{id}|_{X\setminus\{x\}}, \mathrm{id}|_{\hat{X}_x}, \dots, \mathcal{T}_G, \mathrm{id}_{X\setminus\{x\}}) \in \mathrm{Gr}_{x,k}$. Note now that this induces a map of spaces

$$* \to \mathcal{G}\mathrm{r}_{x,k}.$$

The formal gluing property evidently commutes with this map at the various levels, so this construction is natural in k. As usual, let us denote by $\mathcal{H}\mathrm{ck}_{x,k}$ be the evaluation of $\mathcal{H}\mathrm{ck}_x^{\times}$ at $\langle 1\rangle \in \mathbb{E}_2^{\mathrm{nu}}$, $\langle m\rangle \in \mathbb{E}_1^{\mathrm{nu}}$. We have an induced map $* \to \mathcal{H}\mathrm{ck}_{x,k}$ for every k. Now if $\mu_k : \mathcal{H}\mathrm{ck}_{x,k} \times \mathcal{H}\mathrm{ck}_{x,k} \to \mathcal{H}\mathrm{ck}_{x,k}$ is the multiplication in $\mathrm{Sh}(\mathrm{StrTop})$, we can consider the composition

$$\mathcal{H}\mathrm{ck}_{x,k} \overset{\mathfrak{she}}{\sim} \mathcal{H}\mathrm{ck}_{x,k} \times * \xrightarrow{\mathrm{id}, u_k} \mathcal{H}\mathrm{ck}_{x,k} \times \mathcal{H}\mathrm{ck}_{x,k} \xrightarrow{\mu_k} \mathcal{H}\mathrm{ck}_{x,k}$$

and we find that this composition is the identity. Therefore, u_k is a right quasi-unit, functorially in k. The condition that it is a left-quasi unit is verified analogously. □

Remark 2.3.4. Let us now inspect the behaviour of the $\mathbb{E}_1$ product. Let us fix the $\mathbb{E}_2$ entry equal to $\langle 1\rangle$ for simpplicity. Then the product law is described by the map

$$\mathcal{A}_x(\langle 1\rangle, \langle 2\rangle) = \mathcal{C}\mathrm{ons}_{G_\mathcal{O}}(\mathrm{Gr}) \otimes \mathcal{C}\mathrm{ons}_{G_\mathcal{O}}(\mathrm{Gr}) \xrightarrow{\boxtimes} \mathcal{C}\mathrm{ons}_{G_\mathcal{O}\times G_\mathcal{O}}(\mathrm{Gr}\times \mathrm{Gr})$$

$$\xrightarrow{\mathcal{H}\mathrm{ck}_x(\partial_2\times\partial_0)^*=p^*} \mathcal{C}\mathrm{ons}_{G_\mathcal{O}\times G_\mathcal{O}}(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_{2,x}) = \mathcal{C}\mathrm{ons}(\mathcal{H}\mathrm{ck}_x) \cong$$

$$\mathcal{C}\mathrm{ons}(G_\mathcal{O}\backslash(G_\mathcal{K}\times^{G_\mathcal{O}}\mathrm{Gr})) \xrightarrow{\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}(\delta_1)_*=m_*} \mathcal{C}\mathrm{ons}_{G_\mathcal{O}}(\mathrm{Gr}) = \mathcal{A}_x(\langle 1\rangle, \langle 1\rangle).$$

The “pullback” and “pushforward” steps come from the construction of the functor out of the category of correspondences, which by construction takes a correspondence to the “pullback-pushforward” transform between the categories of constructible sheaves over the bottom vertexes of the correspondence. Note that the most subtle step is the equivalence in the penultimate step. If one is to compute explicitly a product of two constructible sheaves $F, G \in \mathcal{C}\mathrm{ons}_{G_\mathcal{O}}(\mathrm{Gr})$, one must reconstruct the correct equivariant sheaf over $\mathrm{ConvGr}_{x,2}$ whose pullback to $\mathrm{Gr}_{x,2}$ is $p^*(F\boxtimes G)$, and then push it forward along m (in the derived sense of course). We stress again that this, when restricted to $\mathcal{P}\mathrm{erv}_{G_\mathcal{O}}(\mathrm{Gr})$, is exactly the definition of the convolution product of perverse sheaves from [MV07] (up to shift and t-structure).

Proposition 2.3.5. *The map of operads* $\mathcal{A}_x^{\otimes} : \mathbb{E}_2\times\mathbb{E}_1^{\mathrm{nu}} \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$ *can be upgraded to a map of operads* $\mathcal{A}_x^{\otimes} : \mathbb{E}_2\times\mathbb{E}_1 \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$.

Proof. Again, it suffices to exhibit a quasi-unit. In this case, this is represented by the element $* \xrightarrow{\mathbf{1}} \mathcal{C}\mathrm{ons}_{k,G_\mathcal{O}}(\mathrm{Gr})$. Here $\mathbf{1}$ is the pushforward along the trivial section $t : * \to \mathrm{Gr}$, $t(*) = (\mathcal{T}_G, \mathrm{id}|_{X\backslash x})$, of the constant sheaf with value k.

The proof is given in [Rei12, Proposition IV.3.5]. We denote by $\star$ the $\mathbb{E}_1$-product of equivariant constructible sheaves on Gr described by $\mathcal{A}_x(-)$. By Remark 2.3.4 for any $F \in \mathcal{C}\mathrm{ons}_{k,G_\mathcal{O}}(\mathrm{Gr})$ we can compute the product via the convolution diagram

$$\begin{tikzcd}
 & G_\mathcal{K}\times\mathrm{Gr} = \mathrm{Gr}_{x,2} \arrow[r, "q"] \arrow[dl, "p"'] & G_\mathcal{K}\times^{G_\mathcal{O}}\mathrm{Gr} = \mathrm{ConvGr}_{2,x} \arrow[dr, "m"] & \\
\mathrm{Gr}\times\mathrm{Gr} & & & \mathrm{Gr}.
\end{tikzcd}$$

In our specific case, we are given a diagram

$$\begin{tikzcd}
 & & G_\mathcal{K}\times\mathrm{Gr} \arrow[r, "q"] \arrow[dl, "p"'] & G_\mathcal{K}\times^{G_\mathcal{O}}\mathrm{Gr} \arrow[dr, "m"] & \\
*\times\mathrm{Gr} \arrow[r, "t\times\mathrm{id}"] \arrow[urrr, bend left, "j"] & \mathrm{Gr}\times\mathrm{Gr} & & & \mathrm{Gr},
\end{tikzcd}$$

where j is the closed embedding $(\mathcal{F}, \alpha) \mapsto (\mathcal{T}_G, \mathrm{id}|_{X\setminus x}, \mathcal{F}, \alpha)$ whose image is canonically identified with Gr. Let $F \in \mathcal{C}\mathrm{ons}_{k,G_{\mathcal{O}}}(\mathrm{Gr})$. We want to prove that $\mathbf{1}\tilde{\boxtimes}F \simeq j_*(k \boxtimes F)$, i.e. that

$$q^* j_*(k \boxtimes F) \simeq p^*(t \times \mathrm{id})_*(k \boxtimes F).$$

Note that because of the consideration about the image of j the support of both sides lies in $G_{\mathcal{O}} \times \mathrm{Gr} \subset G_{\mathcal{K}} \times \mathrm{Gr}$, and this yields a restricted diagram

$$\begin{tikzcd}
 & G_{\mathcal{O}} \times \mathrm{Gr} \arrow[r, "\tilde{q}"] \arrow[dl, "\tilde{p}"'] & \mathrm{Gr} \arrow[dr, "\tilde{m}"] & \\
\mathrm{Gr} \arrow[urr, "\sim", "j"'] & & & \mathrm{Gr}.
\end{tikzcd}$$

This proves the claim. By applying m_* we obtain

$$\mathbf{1} \star F \simeq m_*(j_*(k \boxtimes F)) = k \boxtimes F = F$$

since $mj = \mathrm{id}$. □

Thanks to these results, our functor $\mathcal{A}_x^{\otimes}$ is promoted to a map of operads $\mathbb{E}_2 \times \mathbb{E}_1 \to \mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$. By the Additivity Theorem ([Lur17, Theorem 5.1.2.2]), we obtain an $\mathbb{E}_3$-algebra object in $\mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes}$. Summing up:

Theorem 2.3.6 (Main theorem)**.** *Let G be a complex reductive group and k be a finite ring of coefficients. There is an object $\mathcal{A}_x^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_3}(\mathcal{P}\mathrm{r}_k^{\mathrm{L},\otimes})$ describing an associative and braided product law on the ∞-category*

$$\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$$

of $G_{\mathcal{O}}$-equivariant constructible sheaves over the affine Grassmannian. The restriction of this product law to the abelian category of equivariant perverse sheaves coincides, up to shifts and perverse truncations, with the classical (commutative) convolution product of perverse sheaves [MV07].

Corollary 2.3.7. *There is an induced $\mathbb{E}_3$-monoidal structure in $\mathcal{C}\mathrm{at}_\infty^{\times}$ on $\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}^{\mathrm{fd}}(\mathrm{Gr})$.*

Proof. The inclusion $\mathcal{P}\mathrm{r}_k^{\mathrm{L}} \to \mathrm{Cat}_{\infty,k}$ is lax monoidal with respect to the $\otimes$-structure on the source and the $\times$-structure on the target. Therefore, $\mathcal{C}\mathrm{ons}_{G_{\mathcal{O}}}(\mathrm{Gr})$ has an induced $\mathbb{E}_3$-algebra structure in $\mathrm{Cat}_{\infty,k}^{\times}$. One can easily check that the convolution product restricts to the small (not presentable) subcategory of finite-dimensional sheaves, and this concludes the proof. □

2.A Constructible sheaves on stratified spaces: theoretical complements

2.A.1 Stratified schemes and stratified analytic spaces

Definitions

The following definitions are particular cases of [BGH20, p. 8.2.1] and ff. .

Definition 2.A.1. Let Top be the 1-category of topological spaces. The category of stratified topological spaces is defined as

$$\mathrm{StrTop}_{\mathbb{C}} = \mathrm{Fun}(\Delta^1, \mathrm{Top}) \times_{\mathrm{Top}} \mathrm{Poset},$$

where the map $\mathrm{Fun}(\Delta^1, \mathrm{Top}) \to \mathrm{Top}$ is the evaluation at 1, and $\mathrm{Alex} : \mathrm{Poset} \to \mathrm{Top}$ assigns to each poset P its underlying set with the so-called Alexandrov topology (see [BGH20, Definition 1.1.1]).

Definition 2.A.2. Let $\mathrm{StrSch} = \mathrm{Sch} \times_{\mathrm{Top}} \mathrm{StrTop}$, where the map $\mathrm{Sch} \to \mathrm{Top}$ sends a scheme X to its underlying Zariski topological space, and the other map is the evaluation at $[0]$, be the category of stratified schemes, and StrAff its full subcategory of stratified affine schemes.

Analogously, one can define stratified complex schemes $\mathrm{StrSch}_{\mathbb{C}}$ and stratified complex affine schemes $\mathrm{StrAff}_{\mathbb{C}}$. The key point now is that there is an analytification functor $\mathrm{an} : \mathrm{Aff}_{\mathbb{C}} \to \mathrm{Stn}_{\mathbb{C}}$, the category of Stein analytic spaces. This is defined in [Rey71, Théorème et définition 1.1] (and for earlier notions used there, see also [Gro57, p. 6]). In this way we obtain a Stein space, which is a particular kind of complex manifold with a sheaf of holomorphic functions. We can forget the sheaf and the complex structure and recover an underlying Hausdorff topological space (which corresponds to the operation denoted by $| - |$ in [Rey71]) thus finally obtaining a functor

$$\mathfrak{top} : \mathrm{Sch}_{\mathbb{C}} \to \mathrm{Top}.$$

A reference for a thorough treatment of analytification (also at a derived level) is [HP18].

Let now $\mathrm{StrStn}_{\mathbb{C}} = \mathrm{Stn}_{\mathbb{C}} \times_{\mathrm{Top},\mathrm{ev}_0} \mathrm{StrTop}$. There is a natural stratified version of the functor $\mathfrak{top}$, namely the one that assigns to a stratified affine complex scheme $(S, s : \mathrm{zar}(S) \to P)$ the underlying topological space of the associated complex analytic space, with the stratification induced by the map of ringed spaces $u : S^{\mathrm{an}} \to S$:

$$\mathrm{StrAff}_{\mathbb{C}} \to \mathrm{StrStn}_{\mathbb{C}} \to \mathrm{StrTop}$$

$$(S, s) \mapsto (S^{\mathrm{an}}, s \circ u) \mapsto (|S^{\mathrm{an}}|, s \circ u).$$

Construction 2.A.3. We can define the category $\mathrm{StrPSh}_{\mathbb{C}}$ as $\mathrm{PSh}(\mathrm{StrAff}_{\mathbb{C}})$. Note that StrTop is cocomplete, because Top, $\mathrm{Fun}(\Delta^1, \mathrm{Top})$ and Poset are. By left Kan extension we have a functor

$$\mathfrak{strtop} : \mathrm{StrPSh}_{\mathbb{C}} \to \mathrm{StrTop}. \tag{2.A.1}$$

By construction, this functor preserves small colimits and finite limits (since both $-^{\mathrm{an}} : \mathrm{Aff}_{\mathbb{C}} \to \mathrm{Stn}_{\mathbb{C}}$ and $| - | : \mathrm{Stn}_{\mathbb{C}} \to \mathrm{Top}$ preserve finite limits, see [Rey71]).

Pullbacks of stratified spaces

Lemma 2.A.4. *The forgetful functor* StrTop $\to$ Top *preserves and reflects finite limits.*

Proof. Since $\mathrm{StrTop} = \mathrm{Fun}(\Delta^1, \mathrm{Top}) \times_{\mathrm{ev}_1, \mathrm{Top}, \mathrm{Alex}} \mathrm{Poset}$ (see Section 2.A.1), it suffices to show that:

- the functor $\mathrm{ev}_0 : \mathrm{Fun}(\Delta_1, \mathrm{Top}) \to \mathrm{Top}$ preserves and reflects pullbacks.
- the functor $\mathrm{Alex} : \mathrm{Poset} \to \mathrm{Top}$ preserves and reflects pullbacks;

Now, the first point follows from the fact that limits in categories of functors are computed componentwise. The second point can be verified directly, by means of the following facts:

- the functor preserves binary products. Indeed, given two posets P, Q, then the underlying sets of $P \times Q$ and $\mathrm{Alex}(P) \times \mathrm{Alex}(Q)$ coincide. Now, the product topology on $\mathrm{Alex}(P) \times \mathrm{Alex}(Q)$ is coarser than the Alexandrov topology $\mathrm{Alex}(P \times Q)$. Moreover, there is a simple base for the Alexandrov topology of a poset P, namely the one given by "half-lines" $P_{p_0} = \{p \in P \mid p \geq p_0\}$. Now, if we choose a point $(p_0, q_0) \in \mathrm{Alex}(P \times Q)$, the set $(P \times Q)_{(p_0, q_0)}$ is a base open set for the topology of $\mathrm{Alex}(P \times Q)$, but it coincides precisely with $P_{p_0} \times Q_{q_0}$. Therefore the Alexandrov topology on the product is coarser that the product topology, and we conclude. Note that this latter part would not be true in the case of an infinite product.
- equalizers are preserved by a simple set-theoretic argument. Therefore, we can conclude that finite limits are preserved.
- Alex is a full functor (by direct verification). Since we have proved that it preserves finite limits, then it reflects them as well.

□

Topologies

In our setting, there are two specially relevant Grothendieck topologies to consider: the étale topology on the algebraic side and the topology of local homeomorphisms on the topological side (which has however the same sheaves as the topology of open embeddings). We have thus sites $\mathrm{Aff}_{\mathbb{C}}, \text{ét}$ and $\mathrm{Top}, \mathrm{loc}$. We can therefore consider the following topoi:

- $\mathrm{Sh}_{\text{ét}}(\mathrm{Sch}_{\mathbb{C}})$;
- $\mathrm{PSh}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sch}_{\mathbb{C}}))$, which we interpret as "étale sheaves over the category of complex presheaves";
- $\mathrm{Sh}_{\mathrm{loc}}(\mathrm{Top})$. This last topos is indeed equivalent to the usual topos of sheaves over the category of topological spaces and open covers.

Now, there are analogs of both topologies in the stratified setting. Namely, we can define strét as the topology whose coverings are stratification-preserving étale coverings, and strloc as the topology whose coverings are jointly surjective famiilies of stratification-preserving local homemorphisms. Therefore, we have well-defined stratified analogues:

- $\mathrm{Sh}_{\text{strét}}(\mathrm{StrSch}_{\mathbb{C}})$
- $\mathrm{Sh}_{\text{strét}}(\mathrm{StrPSh}_{\mathbb{C}}) := \mathrm{PSh}(\mathrm{Sh}_{\text{strét}}(\mathrm{StrSch}_{\mathbb{C}}))$
- $\mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTop})$.

The stratification functor $\mathfrak{strtop} : \mathrm{StrSch}_{\mathbb{C}} \to \mathrm{StrTop}$ sends stratified étale coverings to stratified coverings in the topology of local homeomorphism, and thus induces a functor

$$\mathrm{Sh}_{\text{strét}}(\mathrm{StrPSh}_{\mathbb{C}}) \to \mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTop}).$$

2.A.2 Symmetric monoidal structures on the constructible sheaves functor

Constructible sheaves on conically stratified spaces locally of singular shape

Fix a finite ring of coefficients k (this can be extended to the ℓ-adic setting and to more general rings of coefficients, but we will not do this in the present work. For a reference, see [LZ17]).

Remark 2.A.5. Let (X, s) be a stratified topological space, and k be a torsion ring. Suppose that (X, s) is *conically stratified*, that X is locally of singular shape and that P, the stratifying poset, satisfies the ascending chain condition (see [Lur17, Definition A.5.5 and A.4.15 resp.]). By [Lur17, Theorem A.9.3] the

∞-category of constructible sheaves on X with respect to s with coefficients in k, denoted by $\mathcal{C}\mathrm{ons}_k(X, s)$, is equivalent to the ∞-category

$$\mathrm{Fun}(\mathrm{Exit}(X, s), \mathrm{Mod}_k).$$

Here $\mathrm{Exit}(X, s)$ is the ∞-category of exit paths on (X, s) (see [Lur17, Definition A.6.2], where it is denoted by $\mathrm{Sing}^A(X)$, A being the poset associated to the stratification). We will often write $\mathcal{C}\mathrm{ons}$ instead of $\mathcal{C}\mathrm{ons}_k$.

We review the definition of conical stratifications in more detail in Definition 3.1.11.

Remark 2.A.6. In recent work by Porta and Teyssier [PT22], the hypothesis of "being locally of singular shape" has been removed. For simplicity, we will often work in this higher degree of generality.

Let $\mathrm{StrTop}_{\mathrm{con}}$ denote the 1-category of stratified topological spaces (X, P, s) such that the stratification is conical, X is locally of singular shape, and P satisfies the ascending chain condition. This category admits finite products because the product of two cones is the cone of the join space. Therefore, there is a well-defined symmetric monoidal Cartesian structure $\mathrm{StrTop}_{\mathrm{con}}^{\times}$.

Corollary 2.A.7. *Let $(X, P, s) \in \mathrm{StrTop}_{\mathrm{con}}$, and k be a ring. Then $\mathcal{C}\mathrm{ons}_k(X, P, s)$ is a presentable stable k-linear category.*

Therefore, the ∞-category $\mathcal{P}\mathrm{r}_k^{\mathrm{L}}$ of presentable stable k-linear ∞-categories will be our usual environment from now on.

Symmetric monoidal structure

Lemma 2.A.8. *The functor*

$$\mathrm{Exit} : \mathrm{StrTop}_{\mathrm{con}} \to \mathcal{C}\mathrm{at}_{\infty}$$

$$(X, s : X \to P) \mapsto \mathrm{Exit}(X, s) = \mathrm{Sing}^P(X)$$

carries a symmetric monoidal structure when we endow both source and target with the Cartesian symmetric monoidal structure. That is, it extends to a symmetric monoidal functor

$$\mathrm{StrTop}_{\mathrm{con}}^{\times} \to \mathcal{C}\mathrm{at}_{\infty}^{\times}.$$

Proof. Given two stratified topological spaces $X, s : X \to P, Y, t : Y \to Q$, in the notations of [Lur17, A.6], consider the commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Sing}^{P\times Q}(X\times Y) \arrow[r] \arrow[d] & \mathrm{Sing}(X\times Y) \arrow[r, "\sim"] \arrow[d] & \mathrm{Sing}(X)\times\mathrm{Sing}(Y) \arrow[dd] \\
N(P\times Q) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Sing}(P\times Q) \arrow[dr, "\sim"] & \\
N(P)\times N(Q) \arrow[rr] & & \mathrm{Sing}(P)\times\mathrm{Sing}(Q).
\end{tikzcd}$$

The inner diagram is Cartesian by definition. Therefore the outer diagram is Cartesian, and we conclude that $\mathrm{Sing}^{P\times Q}(X\times Y)$ is canonically equivalent to $\mathrm{Sing}^{P}(X)\times\mathrm{Sing}^{Q}(Y)$. Since $\mathrm{Sing}^{P}(X)$ models the ∞-category of exit paths of X with respect to s, and similarly for the other spaces, we conclude. ☐

Remark 2.A.9. There exists a functor $\mathcal{P}^{(*)} : \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}$ sending an ∞-category $\mathcal{C}$ to the presheaf ∞-category $\mathcal{P}(\mathcal{C})$, a functor $F : \mathcal{C} \to \mathcal{D}$ to the functor $F^{*} : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ given by given by restriction under F.

We would like to set a symmetric monoidal structure on this functor. However, this is slightly complicated. Up to our knowledge, the symmetric monoidal structure is well-studied on its "covariant version", in the following sense.

Lemma 2.A.10 ([Lur17, Remark 4.8.1.8 and Proposition 4.8.1.15])**.** *There exists a symmetric monoidal functor $\mathcal{P}_{(!)} : \mathcal{C}\mathrm{at}_{\infty}^{\times} \to \mathcal{P}\mathrm{r}^{L,\otimes}$ sending an ∞-category $\mathcal{C}$ to the ∞-category of $\mathcal{S}$-valued presheaves $\mathcal{P}(\mathcal{C})$, a functor $F : \mathcal{C} \to \mathcal{D}$ to the functor $\mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$ given by $F_{!} = \mathrm{Lan}_{F}(-)$.*

Proof. The existence of an oplax-monoidal structure follows from [Lur17, p. 4.8.1]. As for symmetric monoidality, apparently, a detail in the proof of [Lur17, Proposition 4.8.1.15] needs to be fixed: for any pair of ∞-categories $\mathcal{C}$, $\mathcal{D}$, the equivalence $\mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{D})$ follows from the universal property of the tensor product of presentable categories, and not from [Lur17, Corollary 4.8.1.12]. Indeed, for any cocomplete ∞-category $\mathcal{E}$ one has:

$$\mathrm{Cocont}(\mathcal{P}(\mathcal{C}\times\mathcal{D}),\mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}\times\mathcal{D},\mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C},\mathrm{Fun}(\mathcal{D},\mathcal{E})) \simeq$$

$$\simeq \mathrm{Fun}(\mathcal{C}_0,\mathrm{Cocont}(\mathcal{P}(\mathcal{D}),\mathcal{E})) \simeq \mathrm{Cocont}(\mathcal{P}(\mathcal{C}),\mathrm{Cocont}(\mathcal{P}(\mathcal{D}),\mathcal{E})) \simeq$$

$$\simeq \mathrm{Bicocont}(\mathcal{P}(\mathcal{C})\times\mathcal{P}(\mathcal{D}),\mathcal{E}).$$

☐

Corollary 2.A.11. *There is a well-defined symmetric monoidal functor*

$$\mathcal{C}\mathrm{ons}^{\otimes}_{(!)} : \mathrm{StrTop}^{\times}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{k}$$

$$(X, s) \mapsto \mathcal{C}\mathrm{ons}_k(X, s)$$

$$f \mapsto f^{\mathrm{formal}}_{!} = \mathrm{Lan}_{\mathrm{Exit}(f)}.$$

Proof. The previous constructions provide us with a symmetric monoidal functor

$$\mathrm{StrTop}^{\times} \xrightarrow{\mathrm{Exit}(-)} \mathcal{C}\mathrm{at}^{\times}_{\infty} \xrightarrow{\mathrm{op}} \mathcal{C}\mathrm{at}^{\times}_{\infty} \xrightarrow{\mathcal{P}(-)} \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}$$

sending

$$(X, s) \mapsto \mathrm{Fun}(\mathrm{Exit}(X, s), \mathcal{S}),$$

$$f \mapsto \mathrm{Lan}_{\mathrm{Exit}(f)}.$$

But now, with the notations of [Lur17, Subsection 1.4.2], for any ∞-category $\mathcal{C}$ we have

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}) = \mathrm{Sp}(\mathrm{Fun}(\mathcal{C}, \mathcal{S})).$$

Then we can apply [Rob14, Remark 4.2.16] and finally [Rob14, Theorem 4.2.5], which establish a symmetric monoidal structure for the functor $\mathrm{Sp}(-) : \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{stable}}$. The upgrade from Sp to Mod_k is straightforward and produces a last functor $\mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{\mathrm{stable}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{k}$ (the ∞-category of presentable stable k-linear categories). ☐

From now on, we will often omit the "linear" part of the matter and prove statements about the functor $\mathcal{C}\mathrm{ons} : \mathrm{StrTop}^{\times}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}$ and its variations, because tha passage to the stable k-linear setting is symmetric monoidal.

Remark 2.A.12. We denote the functor $\mathrm{Lan}_{\mathrm{Exit}(f)}$ by $f^{\mathrm{formal}}_{!}$ because, in general, it does not coincide with the proper pushforward of sheaves. We will see in the next subsection that it does under some hypothesis on f.

We will also see that, as a corollary of Corollary 2.A.11, there exists a symmetric monoidal structure on the usual contravariant version

$$\mathcal{C}\mathrm{ons}^{(*)} : \mathrm{StrTop}^{\mathrm{op}}_{\mathrm{con}} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}$$

$$(X, s) \mapsto \mathcal{C}\mathrm{ons}(X, s)$$

$$f \mapsto f^{*} = - \circ \mathrm{Exit}(f)$$

as well.

2.A.3 Constructible sheaves and correspondences

First of all, we need to recall some properties of the category of constructible sheaves with respect to an unspecified stratification.

Definition 2.A.13. Let X be a topological space. Then there is a well-defined ∞-category of constructible sheaves with respect to a non-fixed stratification

$$\mathcal{D}_c(X) = \operatorname{colim}_{s:X\to P \text{ stratification}} \mathcal{C}\mathrm{ons}(X,s).$$

where the colimit is taken over the category $\mathrm{StrTop} \times_{\mathrm{Top}} \{X\}$ of stratifications of X and refinements between them.

Our aim is to prove the following theorem (we will indeed prove a more powerful version, see Theorem 2.A.22).

Theorem 2.A.14. *There is an ∞-functor*

$$\mathcal{D}_{\mathrm{c}}^{\mathrm{corr}} : \mathrm{Corr}(\mathrm{Top}) \to \mathcal{C}\mathrm{at}_\infty$$

that coincides with $\mathcal{D}_{\mathrm{c}}$ when restricted to

$$\mathrm{Top}^{\mathrm{op}} \hookrightarrow \mathrm{Corr}(\mathrm{Top}).$$

It sends morphisms in $\mathfrak{horiz}$ to pullback functors along those morphisms, and morphisms in $\mathfrak{vert}$ to proper pushforward functors along those morphisms.

Unlike in the stratified case (i.e. the case when the stratification is fixed at the beginnig, treated in the previous subsections), for $\mathcal{D}_{\mathrm{c}}(-)$ there is a well-defined six functor formalism. In particular

Lemma 2.A.15. *For any continuous map $f : X \to Y$, there are well defined functors $f^*, f^! : \mathcal{D}_{\mathrm{c}}(Y) \to \mathcal{D}_{\mathrm{c}}(X)$. Moreover, f^* has a right adjoint Rf_* and $f^!$ has a left adjoint $Rf_!$.*

From now on, we will write f_* for Rf_* and $f_!$ for $Rf_!$. With these notations, the Proper Base Change Theorem (stated e.g. in [Kim15, Theorem 6] at the level of abelian categories, and in [Vol21] at the level of derived ∞-categories) holds:

Theorem 2.A.16 (Base Change theorem for constructible sheaves)**.** *For any Cartesian diagram of unstratified topological spaces*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{2.A.2}$$

there is a canonical transformation of functors $\mathcal{D}_c(X) \to \mathcal{D}_c(Y')$

$$f'_! g'^* \to g^* f_!$$

which is an equivalence of functors. The same holds for $f^! g_* \to g'_* f^!$ *as functors* $\mathcal{D}_c(Y) \to \mathcal{D}_c(X)$.

We are now ready to prove Theorem 2.A.14. We follow the approach used by D. Gaitsgory and N. Rozenblyum in [GR17, Chapter 5] to construct IndCoh as a functor out of the category of correspondences. As in these works, we need a theory of $(\infty, 2)$-categories. In particular, we need to extend $\mathcal{C}\mathrm{at}_\infty$ to an $(\infty, 2)$-category, which we shall denote by $\mathcal{C}\mathrm{at}_\infty^{\text{2-cat}}$: this is done in [GR17], informally by allowing natural transformations of functors which are not natural equivalences. We proceed by steps.

Step 1: (f^*, f_*) adjunction and proper base change.

The previous discussion tells us that $\mathcal{D}_c^{\mathrm{op}} : \mathrm{Top} \to \mathcal{C}\mathrm{at}_\infty^{\text{2-cat,op}}$ satisfies the *left Beck-Chevalley condition* [GR17, Chapter 7, 3.1.5] with respect to $\mathfrak{adm} = \mathfrak{vert} = \mathfrak{all}, \mathfrak{horiz} = \mathfrak{proper}$, taking $\Phi = \mathcal{D}_c : \mathrm{StrTop}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$. Indeed, for any $f \in \mathfrak{horiz}$ we set $\Phi^!(f) = f_*$, which is right adjoint to $f^* = \Phi(f)$. Finally, the base change property for the diagram (2.A.2) completes the proof of the left Beck-Chevalley property.

Therefore by [GR17, Chapter 7, Theorem 3.2.2.(a)] the functor $\mathcal{C}\mathrm{ons}^{\mathrm{op}} : \mathrm{StrTop} \to \mathcal{C}\mathrm{at}_\infty^{2-\mathrm{cat,op}}$ extends to a functor

$$(\mathcal{C}\mathrm{ons}^{\mathrm{op}})_{\mathfrak{all;proper}} : \mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{proper}}_{\mathfrak{all;proper}} \to \mathcal{C}\mathrm{at}_\infty^{\text{2-cat,op}},$$

or equivalently a functor

$$\mathcal{C}\mathrm{ons}_{\mathfrak{all;proper}} : (\mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{proper},}_{\mathfrak{all;proper}})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty^{2-\mathrm{cat}}.$$

Now we restrict this functor to the $(\infty, 1)$-category

$$(\mathrm{Corr}(\mathrm{StrTop}^{\mathrm{op}})^{\mathfrak{isom}}_{\mathfrak{all;proper}})^{\mathrm{op}},$$

which is equivalent to the more familiar $\mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{isom}}_{\mathfrak{proper;all}}$ (horizontal and vertical maps are interchanged while considering the opposite of the correspondence category). We have an $(\infty, 1)$-functor

$$\mathcal{C}\mathrm{ons}_{\mathfrak{proper,all}} : \mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{isom}}_{\mathfrak{proper,all}} \to \mathcal{C}\mathrm{at}_\infty. \tag{2.A.3}$$

Step 2: Nagata compactification and proper pushforward

Consider the classes of morphisms in StrTop given by $\mathfrak{horiz} = \mathfrak{all}, \mathfrak{vert} = \mathfrak{all}, \mathfrak{adm} = \mathfrak{open} \subset \mathfrak{horiz}, \mathfrak{co\text{-}adm} = \mathfrak{proper}\cap \subset \mathfrak{vert}$ (note that we need $\mathfrak{adm} \subset \mathfrak{horiz}$ and $\mathfrak{co\text{-}adm} \subset \mathfrak{vert}$ instead of the converse, because we are performing a kind of construction "dual" to that used for IndCoh in [GR17]). We want to apply [GR17, Chapter 7, Theorem 5.2.4] and extend our functor to $\mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{open}}_{\mathfrak{all},\mathfrak{all}} \supset \mathrm{Corr}(\mathrm{StrTop})^{\mathfrak{isom}}_{\mathfrak{proper},\mathfrak{all}}$. We perform the verifications necessary to the application of that theorem:

- The pullback and 2-out-of-3 properties for $\mathfrak{horiz} = \mathfrak{all}, \mathfrak{co\text{-}adm} = \mathfrak{proper}, \mathfrak{isom}$, which are immediately verified.

- Every map in $\mathfrak{adm} \cap \mathfrak{co\text{-}adm} = \mathfrak{open} \cap \mathfrak{proper}$ should be a monomorphism. But the class $\mathfrak{open} \cap \mathfrak{proper}$ consists of embeddings of unions of connected components.

- For any $\alpha : X \to Y$ in $\mathfrak{vert} = \mathfrak{all}$, consider the ordinary category $\mathrm{Factor}(\alpha)$, whose objects are

  $$X \xrightarrow{\varepsilon} \overline{X} \xrightarrow{\gamma} Y,$$

  where $\varepsilon \in \mathfrak{adm} = \mathfrak{open}$ and $\gamma \in \mathfrak{co\text{-}adm} = \mathfrak{proper}$, and whose morphisms are commutative diagrams

  $$\begin{tikzcd} & \overline{X}' \arrow[dd] \arrow[dr] & \\ X \arrow[ur] \arrow[dr] & & Y \\ & \overline{X}'' \arrow[ur] & \end{tikzcd}.$$

  Consider the $(\infty, 1)$-category $\mathrm{N}(\mathrm{Factor}(\alpha))$. We require $\mathrm{N}(\mathrm{Factor}(\alpha))$ to be contractible for any $\alpha \in \mathfrak{vert}$. But this is exactly Nagata's compactification theorem, see [GR17, Chapter 5, Proposition 2.1.6].

- $\mathcal{D}_{\mathrm{c}}|_{\mathrm{Top}}$ should satisfy the *right* Beck-Chevalley condition with respect to $\mathfrak{adm} = \mathfrak{open} \subset \mathfrak{horiz} = \mathfrak{all}$. This is true, because for every $f \in \mathfrak{open}$ we have that $f^* = f^!$: now this admits a left adjoint $f_!$, and the Base Change Theorem holds.

- Given a Cartesian diagram

  $$\begin{tikzcd} X \arrow[r, "\varepsilon_0"] \arrow[d, "\gamma_1"'] & Y \arrow[d, "\gamma_0"] \\ Z \arrow[r, "\varepsilon_1"] & W \end{tikzcd}$$

  with $\varepsilon_i \in \mathfrak{open}$ and $\gamma_i \in \mathfrak{proper}$, the Beck-Chevalley condition is satisfied (every functor from correspondences must satisfy it; this is the "easy" part of the extension theorems). Hence there is an equivalence

  $$(\gamma_1)_* \varepsilon_0^* \simeq \varepsilon_1^* (\gamma_0)_*.$$

Now we use the *other* Beck-Chevalley condition, the one introduced and checked in the previous point, using the fact that $\varepsilon_1 \in \mathfrak{open}$. This new set of adjunctions gives us a morphism

$$(\varepsilon_1)_!(\gamma_1)_* \to (\gamma_0)_*(\varepsilon_0)_!.$$

We require that this morphism is an equivalence. But since $\gamma_i \in \mathfrak{proper}$, we have that $(\gamma_i)_* \simeq (\gamma_i)_!$ and we conclude by commutativity of the diagram and functoriality of the proper pushforward.

Note that we have used exactly once that, respectively, for an open embedding $f^* \simeq f^!$ and for a proper morphism $f_* = f_!$.

This completes the proof of Theorem 2.A.14: the application of [GR17, Chapter 7, Theorem 5.2.4] provides us with an $(\infty, 2)$-functor from

$$\mathrm{Corr}(\mathrm{Top})^{\mathfrak{open}}_{\mathfrak{all},\mathfrak{all}}$$

to $\mathfrak{Cat}^{2\text{-cat}}_\infty$ which we restrict to an $(\infty, 1)$-functor from

$$\mathrm{Corr}(\mathrm{Top})^{\mathfrak{isom}}_{\mathfrak{all},\mathfrak{all}}$$

to $\mathfrak{Cat}_\infty$.

Equivariant constructible sheaves

Definition 2.A.17. Let $\mathbb{A}\mathrm{ctStrSch}_\mathbb{C}$ be the category with the following objects

$$\{H \text{ group scheme over } \mathbb{C}, (Y, P, s) \text{ stratified varieties of finite type over } \mathbb{C},$$

$$\Phi \text{ action of } H \text{ over } Y, \text{ such that } P \text{ is finite and the strata of } (Y, P, s) \text{ coincide with the orbits of } \Phi\}$$

and the following morphisms

$$\{f : H \to H' \text{ morphism of group schemes}, g : Y \to Y' \ f\text{-equivariant morphism of schemes }\}.$$

We take the definition of "conically stratified space"

Definition 2.A.18. Let $\mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}$ be the category with the following objects

$$\{H \text{ topological group}, (Y, P, s) \text{ locally compact conically stratified topological space locally of singular shape},$$

$$\Phi \text{ action of } H \text{ on } Y \text{ such that the strata of}$$

$$(Y, P, s) \text{ coincide with the orbits of } \Phi, \text{ and } P \text{ satisfies the ascending chain condition}\}$$

and morphisms analogue to the previous definition.

Remark 2.A.19. All algebraic actions with finitely many orbits induce a Whitney stratification by orbits, and hence their analytic counterpart lies in $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top})$. Also, by [use], the analytification of algebraic varieties is locally of singular shape. Formally, this means that the functor $\mathfrak{strtop}$ induces a functor $\mathbb{A}\mathrm{ctStrSch}_{\mathbb{C}} \to \mathbb{A}\mathrm{ct}_{\mathrm{con}}\mathrm{StrTop}$.

Remark 2.A.20. There exist functors

- $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop}) \to \mathrm{StrTop}_{\mathrm{con}}$ (which remembers only the stratification in H-orbits)
- $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop}) \to \mathrm{Sh}_{\mathrm{loc}}(\mathrm{Top})$ sending (Y, H, Φ) to the quotient Y/H defined as the colimit of the usual diagram in the category of sheaves. Note that here the stratification is forgotten.

All the preceding constructions can be extended to presheaves, by replacing $\mathrm{StrSch}_{\mathbb{C}}$ by $\mathrm{PSh}(\mathrm{StrSch}_{\mathbb{C}})$ and StrTop by PSh(StrTop).

Definition 2.A.21. We define $\mathfrak{she}$ to be the class of maps in $\mathrm{StrTop}_{\mathrm{con}}$ which admit a homotopy inverse which is itself stratified. We define $\mathfrak{esh}$ to be the class of maps $f : H \to H', g : Y \to Y'$ in $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop})$ which admit a homotopy inverse $\bar{f} : H' \to H, \bar{g} : H' \to H$ also living in $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop})$ (i.e. $\bar{f}, \bar{g}$ stratified and $\bar{g}$ $\bar{f}$-equivariant). These classes easily extend to the the setting of presheaves.

Theorem 2.A.22. *There is a well-defined "equivariant constructible sheaves" functor*

$$\mathcal{C}\mathrm{ons}^{\otimes}_{\mathrm{act,corr}} : \mathrm{Corr}(\mathcal{P}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{PSh}(\mathrm{StrTop}))[\mathfrak{esh}^{-1}]))^{\times} \to \mathcal{P}\mathrm{r}^{\mathrm{L},\otimes}_{k}.$$

Proof. We argue as in [GR17, Chapter 5, 3.4]. First of all, we have defined above a (symmetric monoidal) functor $q : \mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop})^{\times} \to \mathrm{Sh}(\mathrm{StrTop})^{\times}$, in turn inducing a symmetric monoidal functor q_{corr} between the categories of correspondences. We now show that there exists a symmetric monoidal functor

$$\mathrm{Corr}(\mathrm{Sh}(\mathrm{Top}))^{\times} \to \mathrm{Pr}^{\mathrm{L},\otimes}.$$

By right Kan extension of $\mathcal{D}_{\mathrm{c}}$ along the Yoneda embedding we obtain a functor $\mathcal{P}(\mathrm{Top})^{\mathrm{op}} \to \mathrm{Cat}_{\infty,k}$. By using the same arguments as in the proof of [GR17, Chapter 5, Theorem 3.4.3], Theorem 2.A.14 provides an extension to

$$\mathrm{Corr}(\mathcal{P}(\mathrm{Top})).$$

To replace $\mathcal{P}(\mathrm{Top})$ with the category of sheaves we use the descent properties of the functor $\mathcal{D}_{\mathrm{c}}(-)$. We call $\mathcal{D}_{\mathrm{c,corr}}$ the extension $\mathrm{Corr}(\mathrm{Sh}(\mathrm{Top})) \to \mathcal{C}\mathrm{at}_{\infty}$. Let $\mathcal{C}\mathrm{ons}_{\mathrm{act,corr}} = \mathcal{D}_{\mathrm{c,corr}} \circ q_{\mathrm{corr}} : \mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{StrTop})) \to \mathrm{Cat}_{\infty,k}$. Note that if we restrict the functor we just obtained to $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top}) \hookrightarrow \mathrm{Corr}(\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top}))_{\mathfrak{vert}}$, then coincides with the composition $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top}) \to \mathrm{StrTop}_{\mathrm{con}} \xrightarrow{\mathcal{C}\mathrm{ons}_{(!)}} \mathrm{Pr}^{\mathrm{L}}$. Now:

- this proves that $\mathfrak{Cons}_{\mathrm{act,corr}}$ takes values in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_k$, since it does on objects and on vertical morphism, and for what concerns horizontal morphisms we have that any f^* is a left adjoint;

- the functor $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top}) \to \mathrm{StrTop}_{\mathrm{con}}$ sends the class $\mathfrak{esh}$ to stratified homotopy equivalences, and the by the Exodromy Theorem the functor $\mathrm{StrTop}_{\mathrm{con}} \xrightarrow{\mathfrak{Cons}_{(!)}} \mathcal{P}\mathrm{r}^{\mathrm{L}}_k$ sends stratified homotopy equivalences to equivalences of ∞-categories. Therefore, $\mathcal{D}_{\mathrm{c,corr}}$ sends equivariant stratified homotopy equivalences to equivalences as well, and therefore it factors through the localization;

- the functor $\mathbb{A}\mathrm{ct}_{\mathrm{con}}(\mathrm{Top}) \to \mathrm{StrTop}_{\mathrm{con}}$ is Cartesian symmetric monoidal, and by Corollary 2.A.11 the functor $\mathrm{StrTop}_{\mathrm{con}} \xrightarrow{\mathfrak{Cons}_{(!)}} \mathrm{Pr}^{\mathrm{L}}$ is symmetric monoidal with respect to the Cartesian structure on the source and to the Lurie tensor product on the target. Therefore, to conclude we can apply arguments similar to the ones used in [GR17, Chapter 5, 4.1.5], that is essentially [GR17, Chapter 9, Proposition 3.2.4], whose hypotheses are trivially verified since $\mathfrak{horiz} = \mathfrak{all}$.

□

In particular, the functor

$$\mathfrak{Cons} : \mathrm{StrTop}^{\mathrm{op}}_{\mathrm{con}} \to \mathfrak{Cat}_\infty$$

$$(X, s) \mapsto \mathfrak{Cons}(X, s)$$

$$f \mapsto f^* = - \circ \mathrm{Exit}(f)$$

is symmetric monoidal, as we had announced in Remark 2.A.12.

2.B Omitted proofs and details

2.B.1 Proof of Proposition 2.1.7

Proof. Fix a discrete complex algebra R, and let $\xi = (S, \mathcal{F}_i, \alpha_i, \mu_i)_i$ be a vertex of the groupoid $\mathrm{Gr}_{\mathrm{Ran},k}(R)$. We must prove that $\pi_1(\mathrm{Gr}_{\mathrm{Ran},k}(R), \xi) = 0$. We know that $\mathrm{Ran}(X)$ is a presheaf of sets over complex algebras. Therefore it suffices to prove that for every $S \in \mathrm{Ran}(X)(R)$, the fiber of $\mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X)$ at S is discrete.

Consider then an automorphism of a point $(S, \mathcal{F}_i, \alpha_i, \mu_i)$: this is a sequence of automorphisms ϕ_i for each bundle $\mathcal{F}_i$, such that the diagrams

$$\begin{tikzcd}
\mathcal{F}_i|_{X_R\setminus\Gamma_S} \arrow[r, "\alpha_i"] \arrow[d, "\phi_i|_{X_R\setminus\Gamma_S}"'] & \mathcal{T}_G|_{X_R\setminus\Gamma_S} \\
\mathcal{F}_i|_{X_R\setminus\Gamma_S} \arrow[ur, "\alpha_i"'] &
\end{tikzcd}$$

(for $i = 1, \dots, k$) and

$$\begin{tikzcd}
\mathcal{F}_i|_{\widehat{(X_R)}_{\Gamma_S}} \arrow[r, "\mu_i"] \arrow[d, "\phi_i|_{\widehat{(X_R)}_{\Gamma_S}}"'] & \mathcal{T}_G|_{\widehat{(X_R)}_{\Gamma_S}} \\
\mathcal{F}_i|_{\widehat{(X_R)}_{\Gamma_S}} \arrow[ur, "\mu_i"'] &
\end{tikzcd}$$

(for $i = 1, \dots, k-1$) commute. (Actually, only the commutation of the former set of diagrams is relevant to the proof.)

The first diagram implies that ϕ_i is the identity over $X_R \setminus \Gamma_S$. We want to show that ϕ_i is the identity. In order to show this, we consider the relative spectrum $\underline{\mathrm{Spec}}_{X_R}(\mathrm{Sym}(\mathcal{F}_i))$ of $\mathcal{F}_i$, which comes with a map $\pi : Y = \underline{\mathrm{Spec}}_{X_R}(\mathrm{Sym}(\mathcal{F}_i)) \to X_R$. An automorphism of $\mathcal{F}_i$ corresponds to an automorphism f_i of Y over X_R, which in our case is the identity over the preimage of $X_R \setminus \Gamma_S$ inside Y. Let U be the locus $\{f_i = \mathrm{id}\}$. This set is topologically dense, because it contains the preimage of the dense open set $X_R \setminus \Gamma_S$. We must see that it is schematically dense, that is the restriction map $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$ is injective. If we do so, then $\phi_i = \mathrm{id}$ globally.
The remaining part of the proof was suggested to us by Angelo Vistoli.

We may suppose that R (and therefore Y and X_R) are Noetherian. Indeed, we can reduce to the affine case and suppose $X_R = \mathrm{Spec}\, P$ and Y of the form Spec $P[t_1, \dots, t_n]$ from the Noether Lemma (observe that Y is finitely presented over X_R). Any global section f of Spec $P[t_1, \dots, t_n]$ lives in a smaller Noetherian subalgebra $P'[t_1, \dots, t_n]$, because it has a finite number of coefficients in P. Analogously, we can suppose U to be a principal open set of Spec $P[t_1, \dots, t_n]$ and thus $f|_U$ can be seen as a section in some noetherian subalgebra of $P[t_1, \dots, t_n]_g$, with g a polynomial in $P[t_1, \dots, t_n]$. Therefore we conclude that the proof that f is zero can be carried out over a Noetherian scheme.

Let us recall the following facts.

- ([Mat89, page 181]) If $A \to B$ is a flat local homomorphism of local noetherian rings, then

  $$\mathrm{depth}\, B = \mathrm{depth}\, A + \mathrm{depth}\, B/\mathfrak{m}B,$$

  where $\mathfrak{m}$ is the maximal ideal of A.

- If $f : S \to T$ is a flat morphism of noetherian schemes, $p \in S$, then p is associate in S if and only if p is associate in the fiber of $f(p)$ and $f(p)$ is associate in T.

Let now $S = Y$ and $T = X_R$. First of all, if we consider the composition $Y^{\text{red}} \to X_R$ we have that $U^{\text{red}} = Y^{\text{red}}$, because two morphisms between separated and reduced schemes coinciding on an open dense set coincide everywhere.

Now we note that U contains the generic points of every fiber. Indeed, every $f^1(x) \subset Y$ factors through $Y^{\text{red}} \to Y$ because the fibers are integral, and hence through $U^{\text{red}} = Y^{\text{red}} \to Y$.

Now if y is an associate point in Y then it is associate in $f^{-1}(f(y))$. Therefore it is a generic point of $f^{-1}(f(y))$, because every fiber of a principal G-bundle is isomorphic to G, which is integral. But U contains all generic points of the fibers, which are their associated points because the fibers are integral.

This implies that U is schematically dense. □

2.B.2 Proof of Proposition 2.2.12

Proof. Since the inclusions $U \hookrightarrow V$ of open sets in $\mathrm{Ran}(X)$ induce inclusions $(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_U \hookrightarrow (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_V$ and do not alter the datum of $(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_U$, it suffices to prove that the maps $p_{U,V,k}$ make the diagram

$$\begin{tikzcd}
& (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_{U\star V} \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_W \arrow[dr, "p_{U\star V,W,k}"] & \\
(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_U \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_V \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_W \arrow[ur, "p_{U,V,k}\times \mathrm{id}"] \arrow[dr, "\mathrm{id}\times p_{V,W,k}"'] & & (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_{U\star V\star W} \\
& (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_U \times (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_{U\star V} \arrow[ur, "p_{U,V\star W,k}"'] &
\end{tikzcd}$$

commute in StrTop. Now this is true because of the following. Define $(\mathrm{Ran}(X) \times \mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}}$ as the subfunctor of $\mathrm{Ran}(X) \times \mathrm{Ran}(X) \times \mathrm{Ran}(X)$ parametrising those $S, T, P \subset X(R)$ whose graphs are pairwise disjoint in X_R. Let $(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathrm{disj}}$ be its preimage under $r_k \times r_k \times r_k$:

$\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k} \to \mathrm{Ran}(X) \times \mathrm{Ran}(X) \times \mathrm{Ran}(X)$. Then the diagram

$$\begin{tikzcd}
 & (\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathsf{disj}} \arrow[dr, "\chi_k"] & \\
(\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathsf{disj}} \arrow[ur, "\chi_k \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times \chi_k"'] & & \mathrm{Gr}_{\mathrm{Ran},k} \\
 & (\mathrm{Gr}_{\mathrm{Ran},k} \times \mathrm{Gr}_{\mathrm{Ran},k})_{\mathsf{disj}} \arrow[ur, "\chi_k"'] &
\end{tikzcd}$$

commutes because the operation of gluing is associative, as it is easily checked by means of the defining property of the gluing of sheaves.

Note also that everything commutes over $\mathrm{Ran}(M)$.

Finally, to prove that the functor defined in Remark 2.2.11 is a map of operads, we use the characterization of inert morphisms in a Cartesian structure provided by [Lur17, Proposition 2.4.1.5]. Note that:

- An inert morphism in $\mathrm{Fact}(M)^{\otimes}$ is a morphism of the form

$$(U_1, \ldots, U_m) \to (U_{\phi^{-1}(1)}, \ldots U_{\phi^{-1}(n)})$$

  covering some inert arrow $\phi : \langle m \rangle \to \langle n \rangle$ where every $i \in \langle n \rangle^{\circ}$ has exactly one preimage $\phi^{-1}(i)$.

- An inert morphism in $\mathrm{StrTop}^{\times}$ is a morphism of functors $\bar{\alpha}$ between $f : \mathscr{P}(\langle m \rangle^{\circ})^{\mathrm{op}} \to \mathrm{Top}$ and $g : \mathscr{P}(\langle n \rangle^{\circ})^{\mathrm{op}} \to \mathrm{Top}$, covering some $\alpha : \langle m \rangle \to \langle n \rangle$, and such that, for any $S \subset \langle n \rangle$, the map induced by $\bar{\alpha}$ from $f(\alpha^{-1}S) \to g(S)$ is an equivalence in StrTop.

By definition, $\mathcal{C}\mathcal{G}\mathrm{r}_k^{\times}((U_1, \ldots, U_m))$ is the functor f assigning

$$T \subset \langle m \rangle^{\circ} \mapsto \prod_{j \in T} (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_{U_j},$$

and analogously $\mathcal{C}\mathcal{G}\mathrm{r}_k^{\times}((U_{\phi^{-1}(1)}, \ldots, U_{\phi^{-1}(m)}))$ is the functor g assigning

$$S \subset \langle n \rangle^{\circ} \mapsto \prod_{i \in S} (\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_{U_{\phi^{-1}(i)}}.$$

But now, if $\alpha = \phi$ and $T = \phi^{-1}(S)$, we have the desired equivalence. □

2.B.3 Proof of Proposition 2.2.5

Proof. It is sufficient to prove that each $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k$ is conically stratified and locally of singular shape (although as remarked in Remark 2.A.6 this condition is not strictly necessary). Indeed, each $(\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k)_U$, being an open set of $\mathcal{F}\mathrm{act}\mathcal{G}\mathrm{r}_k$ with the induced stratification, will be conically stratified and locally of singular shape as well.

Moreover, it suffices to show that $\mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}})$ is conically stratified and locally of singular shape. Indeed, this will imply the same property for the k-fold-product of copies of $\mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}})$ over $\mathrm{Ran}(M)$, and $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k$ is a principal bundle over this space, with unstratified fiber. This consideration implies the property for $\mathcal{F}\mathrm{act}\mathcal{A}\mathrm{rc}_k$.

Let us then prove the property for $\mathfrak{strtop}(\mathrm{Gr}_{\mathrm{Ran}})$. First of all, the Ran Grassmannian is locally of singular shape because of the following argument.

Proposition 2.B.1. *Let* $\mathcal{U} : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{O}\mathrm{pen}(X)$ *be a hypercovering. Then* $\mathrm{Sing}(X) \simeq \mathrm{colim}\, \mathrm{Sing}_{n\in\mathbf{\Delta}^{\mathrm{op}}}(\mathcal{U}_n)$.

Proof. We use [Lur17, Theorem A.3.1]. Condition (*) in *loc.cit.* is satisfied for the following modification of $\mathcal{U}$. Since $\mathcal{U}$ is a hypercovering, one can choose for any $[n]$ a covering $(U_n^i)_i$ of $\mathcal{U}_n$, functorially in n. We can thus define a category $\mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}$ as the unstraightening of the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

$$[n] \mapsto \{(U_n^i)_i\}.$$

Then there is a functor $\tilde{\mathcal{U}} : \mathcal{C} \to \mathcal{O}\mathrm{pen}(X)$, $([n], U_n^i) \mapsto U$. This functor satisfies (*) in [Lur17, Theorem A.3.1], and therefore $\mathrm{colim}_{([n],U)\in\mathcal{C}}\,\mathrm{Sing}(U) \simeq \mathrm{Sing}(X)$. Now note that $\mathcal{U}$ is the left Kan extension of $\tilde{\mathcal{U}}$ along $\mathcal{C} \to \mathbf{\Delta}^{\mathrm{op}}$. Therefore,

$$\mathrm{Sing}(X) \simeq \mathrm{colim}_{\mathcal{C}}\,\mathrm{Sing}(U) \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}}\,\mathrm{Sing}(\mathcal{U}_n).$$

□

This allows us to apply the proof of [Lur17, Theorem A.4.14] to any hypercovering of $\mathrm{Ran}(M)$. Now, the space $\mathrm{Ran}(M)$ is of singular shape because is contractible and homotopy equivalences are shape equivalences. Therefore, so are elements of the usual prebase of its topology, namely open sets of the form $\prod_i \mathrm{Ran}(D_i)$. One can construct a hypercovering of $\mathrm{Ran}(M)$ by means of such open subsets, and therefore we can conclude by applying the modified version of [Lur17, Theorem A.4.14] that we have just proved.

It remains to prove that the Ran Grassmannian is conically stratified. Indeed, it is Whitney stratified. The proof of such property has been suggested to us by David Nadler [Nad], and uses very essential properties of the Ran Grassmannian. Consider two strata X and Y of $\mathrm{Gr}_{\mathrm{Ran}}$. We want to prove that they satisfy Whitney's conditions A and B; that is:

- for any sequence $(x_i) \subset X$ converging to y, such that $T_{x_i}X$ tends in the Grassmanian bundle to a subspace τ_y of $\mathbb{R}^m$, $T_yY \subset \tau_y$ (Whitney's Condition A for $X, Y, (x_i), y$);[5]
- when sequences $(x_i) \subset X$ and $(y_i) \subset Y$ tend to y, the secant lines x_iy_i tend to a line v, and $T_{x_i}X$ tends to some τ_y as above, then $v \in \tau_y$ (Whitney's Condition B for $X, Y, (x_i), (y_i), y$).[6]

The only case of interest is when $\overline{X} \cap Y$ is nonempty. Observe that, when the limit point $y \in Y$ appearing in Whitney's conditions is fixed, conditions A and B are local in Y, i.e. we can restrict our stratum Y to an (étale) neighbourdhood U of the projection of y in $\mathrm{Ran}(M)$. Also, both Y and the y_i in Condition B live over some common stratum $\mathrm{Ran}_n(M)$ of $\mathrm{Ran}(M)$. Using the factorization property, which splits components and tangent spaces, we can suppose that $n = 1$. Therefore, X projects onto the "cardinality 1" component of $\mathrm{Ran}(M)$, that is M itself. By the locality of conditions A and B explained before, we can suppose that y and the (y_i) involved in Whitney's conditions live over $\mathbf{A}^1$. There, the total space is $\mathrm{Gr}_{\mathbb{A}^1}$ and this is simply the product $\mathrm{Gr} \times \mathbb{A}^1$ because on the affine line the identification is canonical. From this translational invariance it follows that we can suppose our stratum Y concentrated over a fixed point $0 \in \mathbb{A}^1$, that is: both y and the y_i can be canonically (thus simultaneously) seen inside $\mathrm{Gr}_0 \subset \mathrm{Gr}_{\mathbb{A}^1}$. Now, by [Kal05, Theorem 2] we know that there exists at least a point $y \in \overline{X} \cap Y$ such that, for any $(x_i) \to y$ as in the hypothesis of Whitney's Condition A, Whitney's Condition A is satisfied, and the same for Whitney's condition B. In other words, the space

$$\mathrm{Sing}(X,Y) = \{y \in \overline{X} \cap Y \mid y \text{ does not satisfy either Whitney's}$$

$$\text{Condition A or B for some choice of } (x_i), (y_i)\}$$

does not coincide with the whole $\overline{X} \cap Y$. Let $\pi : \mathcal{G}\mathrm{r}_{\mathcal{R}\mathrm{an}} \to \mathrm{Ran}(M)$ be the natural map. Note that X and Y are acted upon by $\mathcal{R}\mathrm{an}\mathcal{A}\mathrm{rc} \times_{\mathrm{Ran}(M)} \pi(X)$ and $\mathcal{R}\mathrm{an}\mathcal{A}\mathrm{rc} \times_{\mathrm{Ran}(M)} \pi(Y) \simeq G_{\mathcal{O}}$ respectively (recall that Y is a subset of $\mathrm{Gr} = \mathrm{Gr}_0 \subset \mathrm{Gr}_{\mathbb{A}^1}$), and this action is transitive on the fibers over any point of $\mathrm{Ran}(M)$. Now, the actions of $\mathcal{R}\mathrm{an}\mathcal{A}\mathrm{rc}$ and $G_{\mathcal{O}}$ take Whitney-regular points with respect to X to Whitney-regular points with respect to X, since it preserves all strata. Therefore, if y is a "regular" point as above, the whole Y is made of regular points, and we conclude. □

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[5] X and Y are said to satisfy Whitney's condition A if this is satisfied for any $(x_i) \subset X$ tending to $y \in Y$. The space is said to satisfy Whitney's condition A if every pair of strata satisfies it.

[6] Idem.

2.B.4 Details from the proof of [HY19, Proposition 3.17]

The following proof has been communicated to us by Jeremy Hahn and Allen Yuan, expanding the one contained in [HY19]. We report the details, as communicated by them, because we need them for our purposes.

- We adapt to the notations of *loc. cit.*, so that

$$\mathcal{F}\text{act}\mathcal{G}\text{r}_{1,x} = \text{Gr}(\{x\}), (\mathcal{F}\text{act}\mathcal{G}\text{r}_1)_{\text{Ran}^{\leq n}(D)} = \text{Gr}(D),$$

$$p : \text{Gr}(D) \to \text{Ran}^{\leq n}(D), i : \text{Gr}(\{x\}) \to \text{Gr}(D).$$

- First of all, note that every connected component of Gr$(\{x\})$ is simply connected. Indeed, there are at least two ways to see this. One is to note that there is an explicit Iwahori cell decomposition, by even cells. So as a CW complex Gr$(\{x\})$ is built only out of 0-cells, 2-cells, 4-cells, etc. and therefore has trivial fundamental group at any arbitrary basepoint. A second way is to note that the affine Grassmannian is the based loop space of a Lie group G^{an}, and therefore its fundamental group at any basepoint corresponds to the second homotopy group of the Lie group at a corresponding basepoint (well-defined up to homotopy). However, π_2 of any Lie group is trivial. The Iwahori cell decomposition is a refinement of the filtration we consider, so Gr$(\{x\})$ is just an explicit skeleton which again has only even cells.

- Let Ran$^n(D)$ be the open subset of Ran$^{\leq n}(D)$ consisting of those subsets od D having exactly n elements, and let Gr$^n(D)$ be its preimage along p. Now suppose $n = 1$. Then Gr$^1(D)$ is (the underlying complex topological space of) the so-called Beilinson-Drinfeld Grassmannian, restricted from M to D. Call Gr$^1(M)$ the Beilinson-Drinfeld Grassmannian over the whole M. Then Gr$^1(M) \to$ Ran$^1(M) = M$ is a fiber bundle, because following [Zhu16, Equation 3.1.10] Gr$^1(M)$ can be rewritten as the underlying complex topological space of $\hat{X} \times^{G_{\mathcal{O}}}$ Gr$(\{x\})$. Now $\hat{X}$ is a $G_{\mathcal{O}}$-torsor over X, hence the corresponding map of topological spaces $\hat{M} \to M$ is a locally trivial fibration (a fiber bundle). Therefore by [AP12, Proposition 2.6.4] Gr$(M) \to M$ is a locally trivial fibration as well, that is a fiber bundle with fiber Gr$(\{x\})$.

- The map Gr(Ran$(M)) \to$ Ran(M) admits the lifting property with respect to homotopies going from "higher cardinality" strata to more "lower cardinality" strata. This comes from the previous step and from some "folklore" fact (as communicated to us by Hahn and Yuan) which we believe is a consequence of [Zhu09, Proposition 1.2.4]. We plan to prove a formal statement in a forthcoming paper.

As for the map $\mathcal{C}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(\{E_i\})) \to \mathcal{C}\mathcal{A}\mathrm{rc}_k(\mathrm{Ran}(\{D_i\}))$, by similar arguments we can reduce ourselves to the case $k = 1$. But since the datum of $G_{\mathcal{O}}$ is completely local, the verification boils down to the fact that the map $\mathrm{Ran}(\{E_i\}) \to \mathrm{Ran}(\{D_i\})$ is a homotopy equivalence.

Chapter 3

Whitney stratifications and conically smooth structures

Contents

3.1 Background

In this section we briefly review the definition of Whitney stratified space and we recall the basic properties of conically stratified and conically smooth spaces. By doing this, we will also introduce the necessary notations to state our main result, conjectured in [AFT17, Conjecture 1.5.3].

3.1.1 Smooth stratifications of subsets of manifolds (Thom, Mather)

We take the following definitions from [Mat70], with minimal changes made in order to connect the classical terminology to the one used in [AFT17].

Definition 3.1.1. Let M be a smooth manifold. A **smooth stratification** of a subset $Z \subset M$ is a partition of Z into smooth submanifolds of M. More generally, if M is a C^μ-manifold, then a C^μ stratification of a subset Z of M is a partition of Z into C^μ submanifolds of M.

Remark 3.1.2. In particular, all strata of a smoothly stratified space $Z \subset M$ are locally closed subspaces of Z.

Definition 3.1.3 (Whitney's Condition B in $\mathbb{R}^n$)**.** Let X, Y be smooth submanifolds of $\mathbb{R}^n$, and let $y \in Y$ be a point. The pair (X, Y) is said to satisfy **Whitney's Condition B** at y if the following holds. Let $(x_i) \subset X$ be a sequence converging to y, and $(y_i) \subset Y$ be another sequence converging to y. Suppose that $T_{x_i}X$ converges to some vector space τ in the r-Grassmannian of $\mathbb{R}^n$ and that the lines x_iy_i converge to some line l in the 1-Grassmannian (projective space) of $\mathbb{R}^n$. Then $l \subset \tau$.

Definition 3.1.4 (Whitney's condition B)**.** Let X, Y be smooth submanifolds of a smooth n-dimensional manifold M, and $y \in Y$. The pair (X, Y) is said to satisfy Whitney's Condition B at y if there exist a chart of M $\phi : U \to \mathbb{R}^n$ around y such that $(\phi(U \cap X), \phi(U \cap Y))$ satisfies Whitney's Condition B at $\phi(y)$.

Definition 3.1.5 (Whitney stratification)**.** Let M be a smooth manifold of dimension n. A smooth stratification $(Z, \mathscr{S})$ on a subset Z of M is said to satisfy the Whitney conditions if

- (local finiteness) each point has a neighbourhood intersecting only a finite number of strata;
- (condition of the frontier) if Y is a stratum of $\mathscr{S}$, consider its closure $\bar{Y}$ in M. Then we require that $(\bar{Y} \setminus Y) \cap Z$ is a union of strata, or equivalently that $S \in \mathscr{S}, S \cap \bar{Y} \neq \varnothing \Rightarrow S \subset \bar{Y}$;
- (Whitney's condition B) Any pair of strata of $\mathscr{S}$ satisfies Whitney's condition B when seen as smooth submanifolds of M.

Given two strata of a Whitney stratification X and Y, we say that $X < Y$ if $X \subset \bar{Y}$. This is a partial order on $\mathscr{S}$.

3.1.2 Conical and conically smooth stratifications (Lurie, Ayala-Francis-Tanaka)

Definition 3.1.6. Let P be a partially ordered set. The Alexandrov topology on P is defined as follows. A subset $U \subset P$ is open if it is closed upwards: if $p \leq q$ and $p \in U$ then $q \in U$.

With this definition, closed subsets are downward closed subsets and locally closed subsets are "convex" subsets: $p \leq r \leq q, p, q \in U \Rightarrow r \in U$.

Definition 3.1.7 (Stratified space)**.** A stratification on a topological space X is a continuous map $s : X \to P$ where P is a poset endowed with the Alexandrov topology. The fibers of the points $p \in P$ are subspaces of X and are called the strata. We denote the fiber at p by X_p and by $\mathscr{S}$ the collection of these strata.

In this definition we do not assume any smooth structure, neither on the ambient space nor on the strata. Note that, by continuity of s, the strata are locally closed subsets of X.

Note also that the condition of the frontier in Definition 3.1.5 implies that any Whitney stratified space is stratified in the sense of Lurie's definition: indeed, one obtains a map towards the poset $\mathscr{S}$ defined by $S < T \iff S \subset \bar{T}$, which is easily seen to be continuous by the condition of the frontier.

Definition 3.1.8. A stratified map between stratified spaces (X, P, s) and (Y, Q, t) is the datum of a continuous map $f : X \to Y$ and an order-preserving map $\phi : P \to Q$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[d, "t"] \\
P \arrow[r, "\phi"] & Q
\end{tikzcd}$$

commute.

Definition 3.1.9. Let (Z, P, s) be a stratified topological space. We define $C(Z)$ (as a set) as

$$\frac{Z \times [0, 1)}{\{(z, 0) \sim (z', 0)\}}.$$

Its topology and stratified structure are defined in [Lur17, Definition A.5.3]. When Z is compact, then the topology is the quotient topology. Note that the stratification of $C(Z)$ is over $P^{\triangleleft}$, the poset obtained by adding a new initial element to P: the stratum over this new point is the vertex of the cone, and the other strata are of the form $X \times (0, 1)$, where X is a stratum of Z.

Note 3.1.10. A very useful notion relative to stratified spaces (see for example from [AFT17, Definition 2.4.4]) is the notion of **depth** of a stratified space at a point. For example, let Z be an unstratified space of Lebesgue covering dimension n. Then the depth of the cone $C(Z)$ at the cone point is $n + 1$.

Definition 3.1.11 ([Lur17, Definition A.5.5])**.** Let (X, P, s) be a stratified space, $p \in P$, and $x \in X_p$. Let $P_{>p} = \{q \in P \mid q > p\}$. A **conical chart** at x is the datum of a stratified space $(Z, P_{>p}, t)$, an unstratified space Y, and a P-stratified open embedding

$$\begin{tikzcd}
Y \times C(Z) \arrow[rr, hook] \arrow[dr] & & X \arrow[dl] \\
& P &
\end{tikzcd}$$

whose image hits x. Here the stratification of $Y \times C(Z)$ is induced by the stratification of $C(Z)$, namely by the maps $Y \times C(Z) \to C(Z) \to P_{\geq p} \to P$ (see Definition 3.1.9).

A stratified space is **conically stratified** if it admits a covering by conical charts.

More precisely, the conically stratified spaces we are interested in are the so-called C^0**-stratified spaces** defined in [AFT17, Definition 2.1.15]. Here we recall the two important properties of a C^0-stratified space $(X, s : X \to P)$:

- every stratum X_p is a *topological manifold*;
- there is a *basis* of the topology of X formed by conical charts

$$\mathbb{R}^i \times C(Z) \to X$$

  where Z is a *compact* C^0-stratified space over the relevant $P_{>p}$. Note that Z will have depth strictly less than X; this observation will be useful in order to make many inductive arguments work.

Hence the definition of [AFT17] may be interpreted as a possible analog of the notion of topological manifold in the stratified setting: charts are continuous maps which establish a stratified homeomorphism between a small open set of the stratified space and some "basic" stratified set.

Following this point of view, one may raise the question of finding an analog of "smooth manifold" (or, more precisely, "smoothly differentiable structure") in the stratified setting. We refer to [AFT17, Definition 3.2.21] for the definition of a **conically smooth structure** (and to the whole Section 3 there for a complete understanding of the notion), which is a very satisfactory answer to this question. A C^0-stratified space together with a conically smooth structure is called a **conically smooth stratified space**.

The definition of conically smooth structure is rather elaborate. As in the case of C^0-stratified spaces, here we just give a couple of important and enlightening properties of these conically smooth stratified spaces:

- any conically smooth stratified space is a C^0-stratified space;
- all strata have an induced structure of *smooth* manifold, like in the case of Whitney stratifications;
- there is a notion of atlas, in the sense of a system of charts whose domains are the so-called basics, i.e. stratified spaces of the form $\mathbb{R}^i \times C(Z)$ where Z is equipped with a conically smooth atlas: indeed, to make this definition rigorous, the authors of [AFT17] employ an inductive argument on the depth, where the case of depth equal to zero corresponds to the usual notion of an atlas for a smooth manifold, and to pass to a successive inductive step they observe that, whenever there is an open stratified embedding $\mathbb{R}^i \times C(Z) \hookrightarrow M$, then $\mathrm{depth} Z < \mathrm{depth} M$.

This system admits a notion of "smooth" change of charts, in the sense that charts centered at the same point admit a subchart which maps into both of them in a "rigid" way. We recommend to look at the proof of Theorem 3.2.7 for a more precise explanation of this property.

- the definition of conically smooth space is intrinsic, in the sense that it does not depend on a given embedding of the topological space into some smooth manifold, in contrast to the case of Whitney stratifications (see Definition 3.1.1 and Definition 3.1.5);
- in [AFT17] the authors also introduced a notion of conically smooth maps, which differs substantially from the "naive" requirement of being stratified and smooth along each stratum that one has in the case of Whitney stratifications, and hence define a category $\mathcal{S}$trat of conically smooth stratified spaces. In this setting, they are able to build up a very elegant theory and prove many desirable results such as a functorial resolution of singularities to smooth manifolds with corners and the existence of tubular neighbourhoods of conically smooth submanifolds. These results allow to equip $\mathcal{S}$trat with a Kan-enrichment (and hence, a structure of ∞-category); also, the hom-Kan complex of conically smooth maps between two conically smooth spaces has the "correct" homotopy type (we refer to the introduction to [AFT17] for a more detailed and precise discussion on this topic), allowing to define a notion of tangential structure naturally extending the one of a smooth manifold and to give a very simple description of the exit-path ∞-category of a conically smooth stratified space.

Up to now, the theory of conically smooth spaces has perhaps been in need of a good quantity of explicit examples, specially of topological nature. The following conjecture goes in the direction of providing a very broad class of examples coming from differential geometry and topology.

Conjecture 3.1.12 ([AFT17, Conjecture 1.5.3]). *Let $(M, \mathscr{S})$ be a Whitney stratified space. Then it admits a conically smooth structure in the sense of [AFT17].*

The rest of the chapter is devoted to the proof of this conjecture (Theorem 3.2.7).

3.2 Whitney stratifications are conically smooth

3.2.1 Whitney stratifications are conical

We will need the following lemma, whose proof (to our knowledge) has never been written down.

Lemma 3.2.1. *Let $(M, \mathscr{S})$ be a Whitney stratified space, T a smooth unstratified manifold, and let $f : M \to T$ be a proper map of topological spaces which is a smooth submersion on the strata. Then for every $p \in T$ the fiber of f at p has a natural Whitney stratification inherited from M.*

Proof. First of all, by definition of smoothly stratified space we may suppose that $M \subset S$ for some manifold S of dimension n. Again by definition the problem is local, and we may then suppose $M = \mathbb{R}^n$.

We want to prove Whitney's condition B for any pair of strata of the form $X = X' \cap f^{-1}(p)$ and $Y = Y' \cap f^{-1}(p)$, where X', Y' are strata of M and $p \in T$. To this end, we reformulate the problem in the following way: consider the product $M \times T$ with its structure maps $\pi_1 : M \times T \to M, \pi_2 : M \times T \to T$, and its naturally induced Whitney stratification. Consider also the following two stratified subspaces of $M \times T$: the graph Γ_f and the subspace $\pi_2^{-1}(p)$. Note that we can see Γ_f as a homeomorphic copy of M inside the product (diffeomorphically on the strata). Having said that, the intersection $\Gamma_f \cap \pi_2^{-1}(p)$ is exactly the fiber $f^{-1}(p)$. Consider now strata X, Y in $f^{-1}(p)$ as above, seen as strata of Γ_f. Consider sequences $x_i \subset X, y_i \subset Y$ both converging to some $y \in Y$. Let l_i be the line between x_i and y_i, and suppose that $l_i \to l, T_{x_i}X \to \tau$. By compactness of the Grassmannians $\mathrm{Gr}(n, 1)$ and $\mathrm{Gr}(n, \dim T_{x_i}X)$ (which is independent of i), there exists a subsequence (x_{i_j}) such that $T_{x_{i_j}}\Gamma_{f|_{X'}}$ converge to some vector space $V \supset \tau$. Since the stratification on M is Whitney, we obtain that $l \subset V$. On the other hand, applying the same argument to $\pi_2^{-1}(p)$ (which is again stratified diffeomorphic to M via the map π_1), we obtain that, up to extracting another subsequence, $T_{x_{i_j}}(\pi_2^{-1}(p) \cap \pi_1^{-1}(X))$ converges to some $W \supset \tau$. Again, since the stratification on M is Whitney, we obtain that $l \subset W$. Note that the lines l_i and l only depend on the points x_i, y_i and on the embedding of M into some real vector space, and not on the subspace we are working with.

Now we would like to show that $\tau = V \cap W$, and this will follow from a dimension argument that uses the fact that $f|_X$ is a smooth submersion onto T.

Note that $\dim \tau = \dim T_{x_i}X$ for every i. Moreover, by the submersion hypothesis, this equals $\dim X' - \dim T$. Also,

$$\dim V = \dim T_{x_{i_j}}\Gamma_{f|_{X'}} = \dim X'$$

and

$$\dim W = \dim T_{x_{i_j}}(\pi_2^{-1}(p) \cap \pi_1^{-1}(X')) = \dim X'.$$

To compute $\dim V \cap W$, it suffices to compute $\dim(V + W)$, which by convergence coincides with $\dim(T_{x_{i_j}}\Gamma_{f|_{X'}} + T_{x_{i_j}}(\pi_2^{-1}(p) \cap \pi_1^{-1}(X')))$. Let $V_j = T_{x_{i_j}}\Gamma_{f|_{X'}}, W_j = T_{x_{i_j}}(\pi_2^{-1}(p) \cap \pi_1^{-1}(X')))$. We have a map of vector spaces

$$V_j \oplus W_j \to T_{x_{i_j}}X' \oplus T_{f(x_{i_j})}T$$

sending $(v, w) \mapsto (w - v, \mathrm{d}f_{x_{i_j}}v)$. This map is surjective (since $\mathrm{d}f$ is) and is zero on the subspace $\{(v, v) \mid v \in V_j \cap W_j\}$; hence it induces a surjective map

$$V_j + W_j \to T_{x_{i_j}}X' \oplus T_{f(x_{i_j})}T.$$

It follows that

$$\dim(T_{x_{i_j}}\Gamma_{f|_{X'}} + T_{x_{i_j}}(\pi_2^{-1}(p) \cap \pi_1(X'))) \geq \dim X' + \dim T$$

and therefore

$$\dim V \cap W = \dim V + \dim W - \dim(V + W) \leq \dim X' - \dim T = \dim \tau.$$

Since $\tau \subset V \cap W$ the proof is complete. ☐

Lemma 3.2.2. *Any open subset of a Whitney stratified manifold inherits a natural Whitney stratification by restriction.*

Proof. Unlike the previous lemma, this is just a direct verification allowed by the fact that tangent spaces to open subsets of strata coincide with the tangent spaces to the original strata. One can also apply the more general and very useful argument appearing in [Gib+76, (1.3), (1.4) and discussion below]. ☐

Lemma 3.2.3 (Thom's first isotopy lemma, [Mat72, (8.1)]). *Let $f : X \to Y$ be a C^2 mapping, and let A be a closed subset of X which admits a C^2 Whitney stratification $\mathscr{S}$. Suppose $f|_A : A \to Y$ is proper and that for each stratum U of $\mathscr{S}$, $f|_U : U \to T$ is a submersion. Then $f|_A : A \to Y$ is a locally trivial fibration.*

We recommend the reading of Mather's two papers [Mat70] and [Mat72] to understand the behaviour of Whitney stratified spaces, specially in order to understand the notion of tubular neighbourhood around a stratum, which is the crucial one in order to prove our main result. We refer to [Mat70, Section 6] for a tractation of tubular neighbourhoods. Here we just recall the definition:

Definition 3.2.4. Let S be a manifold and $X \subset M$ be a submanifold. A **tubular neighborhood** T of X in M is a triple (E, ε, ϕ), where $\pi : E \to X$ is a vector bundle with an inner product $\langle , \rangle$, ε is a positive smooth function on X, and ϕ is a diffeomorphism of $B_\varepsilon = \{e \in E \mid \langle e, e \rangle < \varepsilon(\pi(e))\}$ onto an open subset of S, which commutes with the zero section ζ of E:

$$\begin{tikzcd}
B_\varepsilon \arrow[dr, "\phi"] & \\
X \arrow[u, "\zeta"] \arrow[r, hook] & S.
\end{tikzcd}$$

From [Mat72, Corollary 6.4] we obtain that any stratum W of a Whitney stratified space $(M, \mathscr{S})$ has a tubular neighbourhood, which we denote by (T_W, ε_W); the relationship with the previous notation is the following: T_W is $\phi(B_\varepsilon) \cap M$ (recall that a priori $\phi(B_\varepsilon) \subset S$, the ambient manifold)[1]. We also denote by ρ the tubular (or distance) function

$$T_W \to \mathbb{R}_{\geq 0}$$

$$v \mapsto \langle v, v \rangle$$

with the notation as in Definition 3.2.4. Note that $\rho(v) < \varepsilon(\pi(v))$.

A final important feature of the tubular neighbourhoods of strata constructed in Mather's proof is that they satisfy the so-called "control conditions" or "commutation relations". Namely, consider two strata $X < Y$ of a Whitney stratified space M. Then, if T_X and T_Y are the tubular neighbourhoods relative to X and Y as constructed by Mather, one has that

$$\pi_Y \pi_X = \pi_Y$$

$$\rho_X \pi_Y = \rho_X.$$

We explain the situation with an example.

Example 3.2.5. Let M be the real plane $\mathbb{R}^2$ and $\mathscr{S}$ the stratification given by

$$X = \{(0,0)\}$$

$$Y = \{x = 0\} \setminus \{(0,0)\}$$

$$Z = M \setminus \{x = 0\}.$$

We take $\mathbb{R}^2$ itself as the ambient manifold. Then Mather's construction of the tubular neghbourhoods associated to the strata gives a result like in Fig. 3.1. Here the circle is T_X, and the circular segment is a portion of T_Y around a point of Y. We can see here that T_Y is not a "rectangle" around the vertical line, as one could imagine at first thought, because the control conditions impose that the distance of a point in T_W from the origin of the plane is the same as the distance of its "projection" to Y from the origin.

Keeping this example in mind (together with its upper-dimensional variants) for the rest of the tractation may be a great help for the visualization of the arguments used in our proofs.

Now we closely review the proof of [Mat72, Theorem 8.3], which is essential for the next section. This review is also useful to fix some notations. Note that we will use euclidean disks D^n (and not euclidean spaces $\mathbb{R}^n$) as domains of charts for smooth manifolds, because this will turn out to be useful in Section 3.2.2 in order to define some "shrinking" maps in an explicit way.

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[1] Also, we usually identify this subspace of M with its preimage in the "abstract" tubular neighbourhood $B_\varepsilon \subset E$.

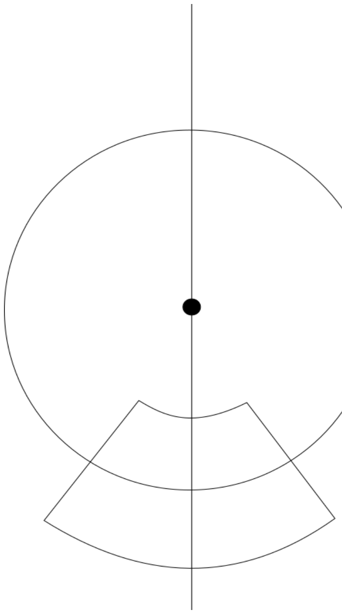

Figure 3.1: Tubular neighbourhoods in $(\mathbb{R}^2, (X, Y, Z))$.

Theorem 3.2.6 ([Mat72, Theorem 8.3]). *Let M be a space endowed with a Whitney stratification $\mathscr{S}$. Then M is conically stratified, and its conical charts are of the type $D^i \times C(Z)$, where $D^i \subset \mathbb{R}^i$ is the unit open disk, and Z is a compact topological space endowed with a natural Whitney stratification.*

Proof. Denote by $\pi_W : T_W \to W$ the projection, $\rho_W : T_W \to \mathbb{R}_{\geq 0}$ the tubular function and $\varepsilon_W : W \to \mathbb{R}_{>0}$ the "radius" function of T_W. We examine closely the proof of [Mat72, Theorem 8.3]. Choose a positive smooth function ε' on W such that $\varepsilon' < \varepsilon_W$. Let N be the set

$$\{x \in T_W \mid \rho_W(x) \leq \varepsilon'(\pi_W(x))\}.$$

Let also

$$A = \{x \in T_W \mid \rho_W(x) = \varepsilon'(\pi_W(x))\}$$

and $f = \pi_W|_A : A \to W$. Note that f is a proper stratified submersion, since π_W is a proper stratified submersion and for any stratum S of M the differential of $\pi_W|_S$ vanishes on the normal to $A \cap S$. Hence by Lemma 3.2.1 the restriction of the stratification of M to any fiber of f is again Whitney. Consider the mapping

$$g : N \setminus W \to W \times (0, 1]$$

defined by

$$g(x) = \left(\pi_W(x), \frac{\rho_W(x)}{\varepsilon'(\pi_W(x))}\right).$$

The space $N \setminus W$ inherits from M a Whitney stratification (see Lemma 3.2.2) and, by [Mat70, Lemma 7.3 and above], the map g is a proper stratified submersion. Thus, since $A = g^{-1}(W \times \{1\})$, by Lemma 3.2.3 one gets a stratified[2] homeomorphism h fitting in the commuting triangle

$$\begin{tikzcd} N \setminus W \arrow[rr, "h"] \arrow[dr, "g"'] & & A \times (0,1] \arrow[dl, "f \times \mathrm{id}"] \\ & W \times (0,1] & \end{tikzcd}.$$

Furthermore, since $W = \rho^{-1}(0) \subseteq N$, h extends to a homeomorphism of pairs

$$(N, W) \xrightarrow{(h,\mathrm{id})} (M(f), W),$$

where $M(f)$ is the mapping cylinder of f (we recall that $f : A \to W$ is the projection $(\pi_W|_A)$).

If $D^i \subset \mathbb{R}^i$ is the unit open disk, for any euclidean chart $j : D^i \hookrightarrow W$, the pullback of f along j becomes a projection $D^i \times Z \to D^i$. Note that Z is compact by properness of f, and has an induced Whitney stratification being a fiber of f, as we have noticed above. Finally,

$$M(f) \simeq M(D^i \times Z \xrightarrow{\mathrm{pr}_1} D^i) \simeq C(Z) \times D^i.$$

□

From now on the conical charts obtained through the procedure explained in the previous theorem will be referred to as the **Thom-Mather charts** associated to the Whitney stratified space M. The rest of this chapter will be devoted to prove that these charts constitute a conically smooth structure (atlas) for M, as conjectured in [AFT17, Conjecture 1.5.3 (3)].

3.2.2 Whitney stratifications are conically smooth

We take all the definitions and notations from [AFT17], specially from Section 3. In particular, we recall that the definition of conically smooth structure is given in [AFT17, Definition 3.2.21].

Let now $(M, \mathscr{S})$ be a Whitney stratified space. Given a chosen system of tubular neighbourhoods around the strata along with their distance and projection functions $\{\rho_X, \pi_X\}$, we have an induced collection of Thom-Mather charts associated to this choice. Call $\mathcal{A}$ this collection. We are now going to

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[2] With respect to the Whitney stratification induced on A, see Lemma 3.2.1.

prove that this is a conically smooth atlas in the sense of [AFT17, Definition 3.2.10]. We will then prove (Remark 3.2.9) that different choices of systems of tubular neghbourhoods induce equivalent conically smooth atlases, again in the sense of [AFT17, Definition 3.2.10].

Theorem 3.2.7 (Main Theorem). *If $(M, \mathscr{S})$ is a Whitney stratified space, then the Thom-Mather charts exhibit a conically smooth structure on $(M, \mathscr{S})$.*

Proof. The proof will proceed by induction on the depth of $(M, \mathscr{S})$ (see Note 3.1.10). The case of depth 0 is obvious, since any Whitney stratified space over a discrete poset is just a disconnected union of strata which are smooth manifolds. Thus, we may assume that for any Whitney stratified set $(M', \mathscr{S}')$ with $\mathrm{depth}(M', \mathscr{S}') < \mathrm{depth}(M, \mathscr{S})$, the Thom-Mather charts induce a conically smooth structure on $(M', \mathscr{S}')$.

Now we need to show that the Thom-Mather charts induce an atlas of $(M, \mathscr{S})$ in the sense of [AFT17, Definition 3.2.10]. We know that the charts cover the space M. By Theorem 3.2.6, a Thom-Mather chart is in particular an open embedding of the form $D^i \times C(Z) \hookrightarrow M$ where Z has a Whitney stratification $\mathscr{S}'$ and $\mathrm{depth}(Z, \mathscr{S}') < \mathrm{depth}(M, \mathscr{S})$; thus, by the inductive hypothesis, Z is conically smooth and this implies that the Thom-Mather chart is a basic in the sense of [AFT17, Definition 3.2.4].

Hence it remains to prove that the "atlas" axiom is satisfied: that is, if $m \in M$ is a point, $u : \mathbb{R}^i \times C(Z) \to M$ and $v : \mathbb{R}^j \times C(W) \to M$ are Thom-Mather charts with images U and V, such that $m \in U \cap V$, then there is a commuting diagram

$$\begin{tikzcd}
D^k \times C(T) \arrow[r, "f"] \arrow[d, "g"] & D^i \times C(Z) \arrow[d, "u"] \\
D^j \times C(W) \arrow[r, "v"] & M
\end{tikzcd} \tag{3.2.1}$$

such that $x \in \mathrm{Im}(uf) = \mathrm{Im}(vg)$ and that f and g are maps of basics in the sense of [AFT17, Definition 3.2.4].

It is sufficient to consider strata X, Y such that $X < Y$ (that is X is in the closure of Y) and $m \in Y$. In particular, X will have dimension strictly less than Y. [3]

In this setting, we may reduce to the case when u is a Thom-Mather chart for X which also contains $m \in Y$ and v is a Thom Mather chart for a neighbourhood of M in Y, such that $v^{-1}(m) = (0, *)$ ($*$ is the cone point). Consider $v^{-1}(U \cap Y)$ as an open subset of $D^j \times *$. This open subset contains some closed ball of radius δ and dimension j centered at 0; denote it by $\overline{B}_\delta \times *$. Also, let $\rho_Y : V \to \mathbb{R}_{>0}$ be the "distance from Y" function associated to the Thom-Mather chart v, and let γ be a positive continuous

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[3] One may use Example 3.2.5 as a guiding example, with m a point on $\{x = 0\} \setminus \{(0, 0)\}$.

function on Y (defined at least locally around m) such that there is an inclusion

$$\{n \in V \mid \rho_Y(n) < \gamma(\pi_Y(n)), \pi_Y(n) \in U \cap Y\} \subseteq V \cap U.$$

Let $\bar{\gamma}$ be defined as follows. Let $\varepsilon_Y : V \cap Y \to \mathbb{R}_{>0}$ be the radius function asociated to the Thom-Mather chart v. Note that ε_Y is equal to the function

$$y \mapsto \sup\{\rho_Y(n) \mid n \in V, \pi_Y(n) = y\}$$

("maximum radius function" for v). Then it makes sense to define

$$\bar{\gamma} = \min_{v(\overline{B}_\delta \times *)} (\gamma/\varepsilon_Y) > 0.$$

Now let us consider the self-embedding

$$D^j \overset{i}{\hookrightarrow} \overline{B}_\delta \subset D^j$$

where i is of the form $(t_1, \dots, t_j) \mapsto (\frac{t_1}{a_1}, \dots, \frac{t_j}{a_j})$ (in such a way that $m \in \mathrm{Im}(v \circ i)$). We call

$$\psi : D^j \times C(W) \xhookrightarrow{i \times (\cdot \bar{\gamma})} D^j \times C(W).$$

This construction is a way to "give conical parameters" for a sufficiently small open subspace of $v^{-1}(U \cap V)$: the multiplication by $\bar{\gamma}$ is the rescalation of the cone coordinate, while i is the rescalation of the "euclidean" coordinate (i.e. the one relative to the D^j component). By construction, $m \in \mathrm{Im}(v \circ \psi) \subseteq U \cap V$. In particular, the image is contained in $U \setminus X$.

Lemma 3.2.8. *The function ψ is a map of basics.*

Proof. We prove that:

- ψ is conically smooth along D^j. Indeed, the map on the bottom row of the diagram in [AFT17, Definition 3.1.4] takes the form

$$(0,1) \times \mathbb{R}^j \times D^j \times C(W) \to (0,1) \times \mathbb{R}^j \times D^j \times C(W)$$

$$(t, (v_1, \dots, v_j), (u_1, \dots, u_n), [s, z]) \mapsto$$
$$\left(t, \left(\frac{v_1}{a_1}, \dots \frac{v_j}{a_j}\right), \left(\frac{u_1}{a_1}, \dots, \frac{u_j}{a_j}\right), \left[\frac{s}{\bar{\gamma}}, z\right]\right). \qquad (3.2.2)$$

  As one can see from the formula, this indeed extends to $t = 0$, and the extension is called $\tilde{D}\psi$; the differential $D\psi$ of ψ is the restriction of $\tilde{D}\psi$ to $t = 0$.
  The same argument works for higher derivatives.

- $D\psi$ is injective on vectors. This is an immediate verification using the formula (3.2.2).
- We have that $\mathcal{A}_{\psi^{-1}((D^j \times C(W))\setminus D^j)} = \psi^*_{\psi^{-1}((D^j \times C(W))\setminus D^j)}\mathcal{A}_{(D^j \times C(W))\setminus D^j}$. This is proven by looking at the definition of ψ: charts are only rescaled along the cone coordinate, or rescaled and translated in the unstratified part.

□

Now consider the open subset $D^i \times Z \times (0,1) \subset D^i \times C(Z)$. By [AFT17, Lemma 3.2.9], basics form a basis for basics, and therefore we may find a map of basics $\phi : D^{i'} \times C(Z') \hookrightarrow D^i \times C(Z)$ whose image is contained in $D^i \times Z \times (0,1)$. Therefore we have a diagram

$$\begin{tikzcd}
 & D^{i'} \times Z' \arrow[d, hook, "u'"] \\
D^j \times C(W) \arrow[r, hook, "v\circ\psi"] & U \setminus X
\end{tikzcd}$$

But now, depth$(U \setminus X) <$ depth$(U) \leq$ depth(M), and $U \setminus X$ with its natural stratification as an open subset of M is Whitney by Lemma 3.2.2 (or also by definition of Thom-Mather chart). Therefore, by induction we may find maps of basics f', g' sitting in the diagram

$$\begin{tikzcd}
D^k \times C(T) \arrow[r, "f'"] \arrow[d, "g'"] & D^{i'} \times Z' \arrow[d, hook, "u'"] \\
D^j \times C(W) \arrow[r, hook, "v\circ\psi"] & U \setminus X
\end{tikzcd}$$

Let us define f as the composition

$$D^k \times C(T) \xrightarrow{f'} D^{i'} \times C(Z') \xrightarrow{\phi} D^i \times C(Z)$$

and g as the composition

$$D^k \times C(T) \xrightarrow{g'} D^j \times C(W) \xrightarrow{\psi} D^j \times C(W).$$

Now

$$u \circ f = u' \circ f' = v \circ \psi \circ g' = v \circ g.$$

Since ϕ, ψ, f', g' are maps of basics, then also f and g are, and this completes the proof. □

Remark 3.2.9. By [Mat70, Proposition 6.1], different choices of Thom-Mather charts induce equivalent conically smooth atlases in the sense of [AFT17, Definition 3.2.10]. Indeed, the construction of a Thom-Mather atlas $\mathcal{A}$ depends on the choice of a tubular neighbourhood for each stratum X, along with its distance and projection functions ρ_X, π_X. Thus, let $\mathcal{A}, \mathcal{A}'$ be two conically smooth atlases induced by different choices of a system of tubular neighbourhoods as above. We want to prove that $\mathcal{A} \cup \mathcal{A}'$ is again an atlas. The nontrivial part of the verification is the following. Let us fix two strata $X < Y$, and a point $y \in Y$; take ϕ_X a Thom-Mather chart associated to the $\mathcal{A}$-tubular neighbourhood T_X of X, and that ψ'_Y a Thom-Mather associated to the $\mathcal{A}'$-tubular neighbourhood T'_Y of Y. We want to verify the "atlas condition" (3.2.1); let T_Y be the $\mathcal{A}$-tubular neighbourhood of Y. Now by [Mat70, Proposition 6.1] there is an isotopy between T'_Y and T_Y fixing Y. By pulling back ψ'_Y to T_Y along this isotopy, we obtain an $\mathcal{A}$-Thom Mather chart ψ_Y around y; we are now left with two $\mathcal{A}$-charts ϕ_X and ψ_Y and we finally can apply the fact that $\mathcal{A}$ is an atlas.

Chapter 4

Derived Brauer map via twisted sheaves

Contents

4.1 Introduction and reminders

4.1.1 Reminders on gerbes and twisted sheaves

Construction 4.1.1. Let X be a qcqs scheme. For us, a $\mathbb{G}_{\mathrm{m}}$-gerbe will be what is defined for example in [Ols16, Definition 12.2.2] (take $\boldsymbol{\mu} = \mathbb{G}_{\mathrm{m}} \times X$ there). Let G be a $\mathbb{G}_{\mathrm{m}}$-gerbe over X. The derived ∞-category of quasicoherent sheaves on G is denoted by $\mathrm{QCoh}(G)$ and it is a presentable stable compactly generated $\mathcal{O}_X$-linear category. Inside this category, we will recall the definition of G-twisted sheaves on X. This notion dates back to Giraud [Gir71] and later to Max Lieblich's thesis [Lie08], and has been developed in the derived setting by Bergh and Schnürer [BS19] (using the language of triangulated categories) and Binda and Porta [BP21] (using the language of stable ∞-categories).

Let now $\mathcal{F} \in \mathrm{QCoh}(G)$. Let $\mathcal{I}_G$ be the inertia group stack of G over X. Thus $\mathcal{F}$ is endowed with a canonical right action by $\mathcal{I}_G$, called the *inertial action*. For an explicit definition, see [BS19, Section 3].

Proposition 4.1.2. *Let G be a $\mathbb{G}_{\mathrm{m}}$-gerbe over a qcqs scheme X. The pullback functor* $\mathrm{QCoh}(X) \to \mathrm{QCoh}(G)$ *establishes an equivalence between* $\mathrm{QCoh}(X)$ *and the full subcategory of* $\mathrm{QCoh}(G)$ *spanned by those sheaves on which the inertial action is trivial.*

Note that the banding $\mathbb{G}_{\mathrm{m}} \times G \to \mathcal{I}_G$ induces a right action ρ of $\mathbb{G}_{\mathrm{m}}$ on any sheaf $\mathcal{F} \in \mathrm{QCoh}(G)$, by composing the banding with the inertial action. On the other hand, for any character $\chi : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, $\mathbb{G}_{\mathrm{m}}$ acts on $\mathcal{F}$ on the left by scalar multiplication precomposed with χ. Let us call this latter action σ_χ.

Remark 4.1.3. Let 1 be the trivial character of $\mathbb{G}_{\mathrm{m}}$. Proposition 4.1.2 can be restated as: the pullback functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(G)$ induces an equivalence between $\mathrm{QCoh}(X)$ and the full subcategory of $\mathrm{QCoh}(G)$ where $\rho = \sigma_1$.

Let G be a $\mathbb{G}_{\mathrm{m}}$-gerbe over a qcqs scheme X, and χ a character of $\mathbb{G}_{\mathrm{m}}$. We define the category of χ-homogeneous sheaves over G, informally, as the full subcategory $\mathrm{QCoh}_\chi(G)$ of $\mathrm{QCoh}(G)$ spanned by those sheaves on which $\rho = \sigma_\chi$.

Remark 4.1.4. The above definition is a little imprecise, in that it does not specify the equivalences $\rho(\gamma, \mathcal{F}) \simeq \sigma_\chi(\gamma, \mathcal{F}), \gamma \in \mathbb{G}_{\mathrm{m}}, \mathcal{F} \in \mathrm{QCoh}(G)$. A formal definition is given in [BP21, Definition 5.14]. There, the Authors define an idempotent functor $(-)_\chi : \mathrm{QCoh}(G) \to \mathrm{QCoh}(G)$, taking the "$\chi$-homogeneous component". This functor is t-exact ([BP21, Proof of Lemma 5.17], and comes with canonical maps $i_{\chi,\mathcal{F}} : \mathcal{F}_\chi \to \mathcal{F}$. The category $\mathrm{QCoh}_\chi(G)$ is defined as the full subcategory of $\mathrm{QCoh}(G)$ spanned by those $\mathcal{F}$ such that $i_\chi(\mathcal{F})$ is an equivalence. To fix the notations, let us make these constructions explicit. If X is a scheme and G a $\mathbb{G}_{\mathrm{m}}$-gerbe, we have the following diagram

$$\begin{tikzcd}
G \arrow[r, "u"] & G \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "act_\alpha"] \arrow[dl, "p"] \arrow[d, "q"] & G \arrow[d, "\pi"] \\
\mathrm{B}\mathbb{G}_{\mathrm{m}} & G \arrow[r, "\pi"] & X
\end{tikzcd}$$

where p, q are the projections, u is the atlas of the trivial gerbe and act_α is the morphism induced by the bending α of G. See [BP21, Section 5] for more specific description of these maps. Given $\mathcal{F} \in \mathrm{QCoh}(G)$ and χ a character of $\mathbb{G}_{\mathrm{m}}$, the Authors define

$$(\mathcal{F})_\chi := u^*(q^* q_*(\mathrm{act}_\alpha^*(\mathcal{F}) \otimes \mathcal{L}_\chi^\vee) \otimes \mathcal{L}_\chi)$$

and the morphism $i_\chi(\mathcal{F}) : (\mathcal{F})_\chi \to \mathcal{F}$ to be the pullback through u of the counit of the adjunction $q^* \dashv q_*$. We will denote by L_χ the line bundle over $\mathrm{B}\mathbb{G}_{\mathrm{m}}$ associated to the character χ, while $\mathcal{L}_\chi$ is the pullback of L_χ to the trivial gerbe $G \times \mathbb{G}_{\mathrm{m}}$, i.e. $\mathcal{L}_\chi := p^* L_\chi$.

[BP21] also prove that there is a decomposition

$$\mathrm{QCoh}(G) \simeq \prod_{\chi \in \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})} \mathrm{QCoh}_{\chi}(G)$$

building on the fact that $\mathcal{F} \simeq \bigoplus_{\chi \in \mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})} \mathcal{F}_{\chi}$. The same result was previously obtained by Lieblich in the setting of abelian categories and by Bergh-Schnürer in the setting of triangulated categories.

Definition 4.1.5. In the case when χ is the identity character $\mathrm{id} : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, $\mathrm{QCoh}_{\chi}(G)$ is usually called the *category of G-twisted sheaves on X*.

The relationship between categories of twisted sheaves and Azumaya algebras has been intensively studied, see [De 04], [De], [Lie08], [HR17], [BS19], [BP21]. Given a $\mathbb{G}_{\mathrm{m}}$-gerbe G over X, its category of twisted sheaves $\mathrm{QCoh}_{\mathrm{id}}(G)$ admits a compact generator, whose algebra of endomorphisms is a derived Azumaya algebra A_G. In contrast, if we restrict ourselves to the setting of abelian categories and consider *abelian* categories of twisted sheaves, this reconstruction mechanism does not work anymore. This is one of the reasons of the success of Toën's derived approach.

The construction of the category of twisted sheaves gives rise to a functor

$$\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$$

$$G \mapsto \mathrm{QCoh}_{id}(G)$$

taking values in the 2-groupoid $\mathcal{B}\mathrm{r}^{\dagger}(X)$ of compactly generated invertible categories (see Theorem 4.2.6). To be precise, this was already known, see for instance [HR17, Example 9.3]. This functor corresponds to a section of the map $\mathcal{B}\mathrm{r}^{\dagger}(X) \xrightarrow{\sim} \mathrm{Map}(X, \mathrm{B}^2\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{Z}) \to \mathrm{Map}(X, \mathrm{B}^2\mathbb{G}_{\mathrm{m}})$. This section is not fully faithful nor essentially surjective. However, one can observe that $\mathrm{QCoh}_{\mathrm{id}}(G)$ is not just an ∞-category, but also carries a t-structure which is compatible with filtered colimits, since as recalled in Remark 4.1.4 the functor $(-)_{\mathrm{id}}$ is t-exact. This additional datum allows to "correct" the fact that $\mathrm{QCoh}_{\mathrm{id}}(-)$ is not an equivalence. Indeed, we can change the target from $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ to the ∞-category of stable $\mathcal{O}_X$-linear presentable categories *with a t-structure* compatible with filtered colimits. Under the association $(\mathcal{C}, \mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}) \mapsto \mathcal{C}_{\geq 0}$, this ∞-category is equivalent to the category $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ of so-called *Grothendieck prestable* $\mathcal{O}_X$-linear ∞-categories (see Definition 4.1.11). Therefore, we have a functor

$$\Psi : \mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X) \tag{4.1.1}$$

$$G \mapsto \mathrm{QCoh}_{\mathrm{id}}(G)_{\geq 0}.$$

4.1.2 Reminders on stable and prestable linear categories

Definition 4.1.6. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is *prestable* if the following conditions are satisfied:

- The ∞-category $\mathcal{C}$ is pointed and admits finite colimits.
- The suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful.
- For every morphism $f : Y \to \Sigma Z$ in $\mathcal{C}$, there exists a pullback and pushout square

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d] & Y \arrow[d, "f"] \\ 0 \arrow[r] & \Sigma Z. \end{tikzcd}$$

Proposition 4.1.7 ([Lur18, Proposition C.1.2.9])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then the following conditions are equivalent:*

- *$\mathcal{C}$ is prestable and has finite limits.*
- *There exists a stable ∞-category $\mathcal{D}$ equipped with a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$.*

Proposition 4.1.8 ([Lur18, Proposition C.1.4.1])**.** *Let $\mathcal{C}$ be a presentable ∞-category. Then the following conditions are equivalent:*

- *$\mathcal{C}$ is prestable and filtered colimits in $\mathcal{C}$ are left exact.*
- *There exists a stable ∞-category $\mathcal{D}$ equipped with a t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ compatible with filtered colimits, and an equivalence $\mathcal{C} \simeq \mathcal{D}_{\geq 0}$.*

Definition 4.1.9 ([Lur18, Definition C.1.4.2])**.** Let $\mathcal{C}$ be a presentable ∞-category. We will say that $\mathcal{C}$ is *Grothendieck* if it satisfies the equivalent conditions of Proposition 4.1.8. Following [Lur18, Definition C.3.0.5], we denote the ∞-category of Grothendieck presentable ∞-categories (and colimit-preserving functors between them) by $\mathcal{G}\mathrm{roth}_\infty$. We also denote the category of presentable stable ∞-categories (and colimit-preserving functors between them) by $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$.

Remark 4.1.10. By [Lur17] and [Lur18, Theorem C.4.2.1], both $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ and $\mathcal{G}\mathrm{roth}_\infty$ inherit a symmetric monoidal structure from $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ which we denote again by $\otimes$.

Definition 4.1.11. Let X be a qcqs scheme. An $\mathcal{O}_X$-linear prestable ∞-category is an object of

$$\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X) := \mathrm{Mod}_{\mathrm{QCoh}(X)_{\geq 0}}(\mathcal{G}\mathrm{roth}_{\infty}^{\otimes}).$$

A stable presentable $\mathcal{O}_X$-linear ∞-category is an object of

$$\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X) := \mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{L},\otimes}).$$

Remark 4.1.12. There exists a stabilization functor $\mathrm{st}_X : \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$, induced by the usual stabilization procedure.

Remark 4.1.13. The category $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ has a tensor product $- \otimes_{\mathrm{QCoh}(X)_{\geq 0}} -$ (which we will abbreviate by $\otimes$) induced by the Lurie tensor product of presentable ∞-categories. See [Lur18, Theorem C.4.2.1] and [Lur18, Section 10.1.6] for more details. The same is true for $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$. The stabilization functor $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ is symmetric monoidal with respect to these structures.

$\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ and $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ satisfy a very important "descent" property, which is what Gaitsgory [Gai15] calls 1-affineness.

Construction 4.1.14. The functors

$$\mathrm{CAlg}_k \to \mathcal{C}\mathrm{at}_{\infty}$$
$$R \mapsto \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(\mathrm{Spec}\, R)$$
$$R \mapsto \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(\mathrm{Spec}\, R)$$

can be right Kan extended to functors

$$\mathcal{Q}\mathrm{Stk}^{\mathrm{St}}, \mathcal{Q}\mathrm{Stk}^{\mathrm{PSt}} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_{\infty}.$$

This gives a meaning to the expressions $\mathcal{Q}\mathrm{Stk}^{\mathrm{St}}(X)$, $\mathcal{Q}\mathrm{Stk}^{\mathrm{PSt}}(X)$, which can be thought of as "the category of sheaves of QCoh-linear (resp. $\mathrm{QCoh}_{\geq 0}$-linear) (pre)stable categories on X". For any $X \in \mathrm{Sch}_k$, there are well-defined "global sections functors"

$$\mathcal{Q}\mathrm{Stk}^{\mathrm{St}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X), \mathcal{Q}\mathrm{Stk}^{\mathrm{PSt}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$$

constructed in [Lur18, discussion before Theorem 10.2.0.1].

Theorem 4.1.15. *Let X be a qcqs scheme over k. Then the global sections functors*

$$\mathcal{Q}\mathrm{Stk}^{\mathrm{St}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$$
$$\mathcal{Q}\mathrm{Stk}^{\mathrm{PSt}}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$$

are symmetric monoidal equivalences.

Proof. For the stable part, this is [Lur11, Proposition 6.5]. For the prestable part, this is the combination of [Lur18, Theorem 10.2.0.2] and [Lur18, Theorem D.5.3.1]. Symmetric monoidality follows from the fact that the inverse of the global sections functor (the "localization functor", see [BP21, Section 2.3]) is strong monoidal. ☐

This theorem means that, if X is a qcqs scheme over k, every (Grothendieck pre)stable presentable $\mathcal{O}_X$-linear ∞-category $\mathcal{C}$ has an associated sheaf of ∞-categories on X having $\mathcal{C}$ as category of global sections. We will make substantial use of this fact in the present work. The main reason why we will always assume our base scheme X to be qcqs is because it makes this theorem hold.

Definition 4.1.16. We denote by $\mathcal{B}\mathrm{r}(X)$ the maximal ∞-groupoid contained in $\mathcal{Q}\mathrm{Stk}^{\mathrm{PSt}}(X)$ and generated by $\otimes_{\mathrm{QCoh}(X)_{\geq 0}}$-invertible objects which are compactly generated categories, and equivalences between them. We denote by $\mathcal{B}\mathrm{r}^{\dagger}(X)$ the maximal ∞-groupoid contained in $\mathcal{Q}\mathrm{Stk}^{\mathrm{St}}(X)$ generated by $\otimes_{\mathrm{QCoh}(X)}$-invertible objects which are compactly generated categories, and equivalences between them.

We call $\mathrm{dBr}(X) := \pi_0 \mathcal{B}\mathrm{r}(X)$ the *derived Brauer group* of X and $\mathrm{dBr}^{\dagger}(X) := \pi_0(\mathcal{B}\mathrm{r}^{\dagger}(X))$ the *extended derived Brauer group* of X.

By [Lur18, Theorem 10.3.2.1], to be compactly generated is a property which satisfies descent. The same is true for invertibility, since the global sections functor is a symmetric monoidal equivalence by Theorem 4.1.15. Therefore, $\mathcal{B}\mathrm{r}^{\dagger}(X)$ is equivalent to the ∞-category of invertible and compactly generated categories in $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}$, and an analogous statement is true for $\mathcal{B}\mathrm{r}(X)$.

Remark 4.1.17. The stabilization functor mentioned in Remark 4.1.12 restricts to a functor $\mathcal{B}\mathrm{r}(X) \to \mathcal{B}\mathrm{r}^{\dagger}(X)$, whose homotopy fiber at any object $\mathcal{C} \in \mathcal{B}\mathrm{r}^{\dagger}(X)$ is discrete and can be identified with the collection of all t-structures $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ satisfying the following conditions:

- The t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is right complete and compatible with filtered colimits.
- The Grothendieck prestable ∞-category $\mathcal{C}_{\geq 0}$ is an $\otimes$-invertible object of $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$.
- The Grothendieck prestable ∞-category $\mathcal{C}_{\geq 0}$ and its $\otimes$-inverse are compactly generated.

See [Lur18, Remark 11.5.7.3] for further comments.

Consider now the functor Ψ introduced in Eq. (4.1.1). In analogy to the stable setting, we will prove that it factors through $\mathcal{B}\mathrm{r}(X) \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ (see Theorem 4.2.6).

The inverse will be described as follows. By Theorem 4.1.15, if X is a quasicompact quasiseparated scheme, and M a prestable presentable ∞-category equipped with an action of $\mathrm{QCoh}(X)_{\geq 0}$, then M is the category of global sections over X of a unique sheaf of prestable $\mathrm{QCoh}(X)_{\geq 0}$-linear categories $\mathcal{M}$.

Definition 4.1.18. Let X be a quasicompact quasiseparated scheme, and M be an element of $\mathcal{B}\mathrm{r}(X)$. Then we define the $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ as the sheaf of categories

$$S \mapsto \mathcal{E}\mathrm{quiv}\Big(\mathrm{QCoh}(S)_{\geq 0}, \mathcal{M}(S)\Big).$$

Finally, observe that both $\mathcal{B}\mathrm{r}(X)$ and $\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)$ have symmetric monoidal structures: on the first one, we have the tensor product $\otimes$ inherited by $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ (Remark 4.1.13), and on the second one we have the rigidified product $\star$ of $\mathbb{G}_{\mathrm{m}}$-gerbes. Although the two structures are well-known to experts, we will recall them in Section 4.2.1.

We are now ready to formulate our main theorem.

Theorem 4.1.19. *Let X be a qcqs scheme over a field k. Then there is a symmetric monoidal equivalence of 2-groupoids*

$$\Phi : \mathcal{B}\mathrm{r}(X)^{\otimes} \longleftrightarrow \mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)^{\star} : \Psi$$

where

- $\Phi(M) = \mathcal{T}\mathrm{riv}_{\geq 0}(M)$
- $\Psi(G) = \mathrm{QCoh}_{\mathrm{id}}(G)_{\geq 0}$.

Remark 4.1.20. By taking the π_0, one obtains an isomorphism of abelian groups

$$\mathrm{dBr}(X) \simeq \mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}}).$$

This isomorphism appears also in [Lur18, Example 11.5.7.15], but as mentioned before, it is a consequence of the equivalence of ∞-categories

$$\mathcal{B}\mathrm{r}(X) \simeq \mathrm{Map}(X, \mathrm{B}^2\mathbb{G}_{\mathrm{m}})$$

whose proof however does never refer to the notion of $\mathbb{G}_{\mathrm{m}}$-gerbe.

4.2 Study of the derived Brauer map

4.2.1 Symmetric monoidal structures

We begin by describing the symmetric monoidal structure on the category of $\mathbb{G}_{\mathrm{m}}$-gerbes.

Let X be a scheme and G_1 and G_2 be two $\mathbb{G}_{\mathrm{m}}$-gerbes on X. One can costruct the product $G_1 \star G_2$, which is a $\mathbb{G}_{\mathrm{m}}$-gerbe such that its class in cohomology is the product of the classes of G_1 and G_2 (see [BP21, Conjecture 5.23]). Clearly, this is not enough to define a symmetric monoidal structure on the category of

$\mathbb{G}_{\mathrm{m}}$-gerbes. The idea is to prove that this $\star$ product has a universal property in the ∞-categorical setting, which allows us to define the symmetric monoidal structure on the category of $\mathbb{G}_{\mathrm{m}}$-gerbes using the theory of simplicial colored operads and ∞-operads (see Chapter 2 of [Lur17]).

Construction 4.2.1. Let AbGer(X) be the $(2,1)$-category of abelian gerbes over X and AbGr(X) the $(1,1)$-category of sheaves of abelian groups over X. We have the so-called banding functor

$$\mathrm{Band} : \mathrm{AbGer}(X) \longrightarrow \mathrm{AbGr}(X),$$

see for example Chapter 3 of [BS19]. It is easy to prove that Band is symmetric monoidal with respect to the two Cartesian symmetric monoidal structures of the source and target, that is it extends to a symmetric monoidal functor Band : $\mathrm{AbGer}(X)^{\times} \to \mathrm{AbGr}(X)^{\times}$ of colored simplicial operads (and therefore also of ∞-operads).

Recall that, given a morphism of sheaf of groups $\phi : \mu \to \mu'$ and a μ-gerbe G, we can construct a μ'-gerbe, denoted by $\phi_* G$, and a morphism $\rho_\phi : G \to \phi_* G$ whose image through the banding functor is exactly ϕ. This pushforward construction is essentially unique and verifies weak funtoriality. This follows from the following result: if G is a gerbe banded by μ, then the induced banding functor

$$\mathrm{Band}_{G/} : \mathrm{AbGer}(X)_{G/} \longrightarrow \mathrm{AbGr}(X)_{\mu/}$$

is an equivalence and the pushforward construction is an inverse (see [BS19, Proposition 3.9]). This also implies that Band is a coCartesian fibration.

Let $\mathcal{F}\mathrm{in}_*$ be the Segal category of pointed finite set. Consider now the morphism $\mathcal{F}\mathrm{in}_* \to \mathrm{AbGr}(X)^{\times}$ induced by the algebra object $\mathbb{G}_{\mathrm{m},X}$ in $\mathrm{AbGr}(X)^{\times}$. We can consider the following pullback diagram

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "B"] & \mathrm{AbGer}(X)^{\times} \arrow[d, "\mathrm{Band}"] \\
\mathcal{F}\mathrm{in}_* \arrow[r, "\mathbb{G}_{m,X}"] & \mathrm{AbGr}(X)^{\times};
\end{tikzcd}$$

thus B is a coCartesian fibration. This implies that $\mathcal{G}$ is a symmetric monoidal structure over the fiber category $\mathcal{G}_{<1>} := B^{-1}(<1>)$ which is exactly the category of $\mathbb{G}_{\mathrm{m}}$-gerbes. The operadic nerve of $\mathcal{G}$ will be the symmetric monoidal ∞-category of $\mathbb{G}_{\mathrm{m}}$-gerbes (see [Lur17, Proposition 2.1.1.27]).

We claim that this symmetric monoidal structure coincides with the $\star$ product of gerbes defined in [BS19]. In fact, following the rigidification procedure in [Ols16, Exercise 12.F] and [AOV08, Appendix A], we can define the $\star$ product as the pushforward of the multiplication map $m : \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$, i.e. if G_1 and G_2 are two $\mathbb{G}_{\mathrm{m}}$-gerbes over X, then

$$G_1 \star G_2 := m_*(G_1 \times_X G_2).$$

Remark 4.2.2. Let us describe $\mathcal{G}$ as a simplicial colored operad. The objects (or colors) of $\mathcal{G}$ are $\mathbb{G}_\mathrm{m}$-gerbes over X. Let $\{G_i\}_{i\in I}$ be a sequence of objects indexed by a finite set I and $\mathcal{H}$ another object; we denote by $\prod_{i\in I} G_i$ the fiber product of G_i over X. The simplicial set of multilinear maps $\mathrm{Mul}(\{G_i\}_{i\in I}, \mathcal{H})$ is the full subcategory of the 1-groupoid of morphisms $\mathrm{Map}_X(\prod_{i\in I} G_i, \mathcal{H})$ of gerbes over X such that its image through the banding functor is the n- fold multiplication map of $\mathbb{G}_\mathrm{m}$, where n is the cardinality of I. Note that because the simplicial sets of multilinear maps are Kan complexes by definition, then they are fibrant simplicial sets. This is why the operadic nerve gives us a symmetric monoidal structure, see [Lur17, Proposition 2.1.1.27].

Definition 4.2.3. We denote by $\mathrm{Ger}_{\mathbb{G}_\mathrm{m}}(X)^\star$ the symmetric monoidal ∞-groupoid of $\mathbb{G}_\mathrm{m}$-gerbes over X, with the symmetric monoidal structure given by Construction 4.2.1.

Remark 4.2.4. Note that $\mathrm{Ger}_{\mathbb{G}_\mathrm{m}}(X)$ is just a $(2,1)$-category, since gerbes are by definition 1-truncated stacks. Also, it is 0-truncated (i.e. a 2-groupoid) since morphisms of $\mathbb{G}_\mathrm{m}$-gerbes are always equivalences.

Next, we will describe the symmetric monoidal structure on $\mathcal{B}\mathrm{r}(X)$. We know that it is the restriction of the symmetric monoidal structure of $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$, see Remark 4.1.13.

Remark 4.2.5. We now describe explicitly the symmetric monoidal structure of $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ using the language of simplicial colored operads. The same description will apply to the prestable case. This will come out useful in the rest of the chapter.

The simplicial colored operad $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ can be described as follows:

1. the objects (or colors) are stable presentable ∞-categories which are modules over $\mathrm{QCoh}(X)$ (see [Lur17, Section 4.5] for the precise definition of module over an algebra object);
2. given $\{M_i\}_{i\in I}$ a sequence of objects indexed by a finite set I and N another object, the simplicial set of multilinear maps $\mathrm{Mul}(\{M_i\}_{i\in I}, N)$ is the mapping space $\mathrm{Fun}^\mathrm{L}(\prod_{i\in I} M_i, N)$ of functors of stable presentable categories which are $\mathrm{QCoh}(X)$-linear and preserve small colimits separately in each variable.

4.2.2 Derived categories of twisted sheaves

Recall from Section 4.1.1 the definition of $\mathrm{QCoh}_{\mathrm{id}}(G)$ and $\mathrm{QCoh}_{\mathrm{id}}(G)_{\geq 0}$. Our aim in this subsection is to prove the following statement:

Theorem 4.2.6. *Let X be a qcqs scheme. The functors*

$$\mathrm{QCoh}_{\mathrm{id}}(-) : \mathrm{Ger}_{\mathbb{G}_\mathrm{m}}(X)^\star \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)^\otimes$$

and

$$\mathrm{QCoh}_{\mathrm{id}}(-)_{\geq 0} : \mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)^{\star} \to \mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)^{\otimes}$$

carry a symmetric monoidal structure with respect to the $\star$-symmetric monoidal structure on the left hand side and to the $\otimes$-symmetric monoidal structures on the right hand sides. In particular, since every $\mathbb{G}_{\mathrm{m}}$-gerbe is $\star$-invertible and $\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)$ *is a 2-groupoid,* $\mathrm{QCoh}_{\mathrm{id}}(-)$ *takes values in* $\mathcal{B}\mathrm{r}^{\dagger}(X)$ *and* $\mathrm{QCoh}_{\mathrm{id}}(-)_{\geq 0}$ *takes values in* $\mathcal{B}\mathrm{r}(X)$*.*

Remark 4.2.7. Notice that we need to explicit how $\mathrm{QCoh}_{\mathrm{id}}(-)$ (or $\mathrm{QCoh}_{\mathrm{id}}(-)_{\geq 0}$) acts on the mapping spaces as well. We decided to stick with the covariant notation, i.e we define the image of a morphism of $\mathbb{G}_{\mathrm{m}}$-gerbes $f : G_1 \to G_2$ to be f_*. We omit the proof of the fact that f_* sends twisted sheaves to twisted sheaves, which follows from a straightforward computation.

Although there are no real issues with the contravariant notation (considering pullbacks instead of pushforwards), there are some technical complications when one wants to prove that a contravariant functor is symmetric monoidal.[1] Note that, if f is any morphism of gerbes, then, $f_* = (f^*)^{-1}$.

We will prove Theorem 4.2.6 in two different steps.

Remark 4.2.8. Let C be a presentable ∞-category, $p : C \to C$ be an endofunctor and $\eta : p \Rightarrow \mathrm{id}_C$ be a natural transformation. Let C^0 be the equalizer of the following diagram

$$\begin{tikzcd} C \arrow[r, bend left, "p"{name=U}] \arrow[r, bend right, "\mathrm{id}_C"'{name=D}] & C \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

which can be described as the full subcategory of C spanned by the elements X of C such that $\eta(X)$ is an equivalence. Now consider (C_1, p_1, η_1) and (C_2, p_2, η_2) triples as the above one, and let $\rho : C_1 \to C_2$ be a functor and $\alpha : p_2 \circ \rho \Rightarrow \rho \circ p_1$ be an equivalence of functors. If $(\mathrm{id}_\rho * \eta_1) \circ \alpha = (\eta_2 \circ \mathrm{id}_\rho)$ then (ρ, α) is a morphism between the two diagrams and therefore there exists a unique (up to homotopy) morphism $C_1^0 \to C_2^0$ compatible with all the data.

If G is a $\mathbb{G}_{\mathrm{m}}$-gerbe and χ is a character of $\mathbb{G}_{\mathrm{m}}$, the endofunctor $(-)_\chi$ of $\mathrm{QCoh}(G)$ and the natural transformation i_χ introduced in Remark 4.1.4 form a diagram as above. In this situation, $\mathrm{QCoh}_\chi(G)$ is exactly the equalizer.

Proposition 4.2.9. *Let X be a quasicompact quasiseparated scheme over a field k. Let $G, G' \to X$ be two $\mathbb{G}_{\mathrm{m}}$-gerbes over X, and χ, χ' two characters of $\mathbb{G}_{\mathrm{m}}$. The external tensor product establishes a t-exact equivalence*

$$\boxtimes : \mathrm{QCoh}_\chi(G) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}_{\chi'}(G') \xrightarrow{\sim} \mathrm{QCoh}_{(\chi,\chi')}(G \times_X G')$$

---

[1] In fact, one would need to use the construction of the "opposite of an ∞-operad", see [Bea], [Lur17, Remark 2.4.2.7].

where $G \times_X G'$ *is seen as a* $\mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$*-gerbe on* X*.*

Proof. First we prove that the external tensor product induces a t-exact equivalence

$$\boxtimes : \mathrm{QCoh}(G) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(G') \simeq \mathrm{QCoh}(G \times_X G'). \tag{4.2.1}$$

By Theorem 4.1.15, every side of the sought equivalence is the ∞-category of global sections of a sheaf in categories over X, which means that the equivalence is étale-local on X. Therefore, by choosing a covering $U \to X$ which trivializes both G and G', we can reduce to the case $G = U \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \to U, G' = U \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \to U$, both maps being the projection to U. But by [BP21, Corollary 5.6], $\mathrm{QCoh}(U \times \mathrm{B}\mathbb{G}_{\mathrm{m}}) \simeq \mathrm{QCoh}(U) \times \mathrm{QCoh}(\mathrm{B}\mathbb{G}_{\mathrm{m}})$, and this proves our claim. Finally, t-exactness follows from the Künneth formula.

Now we prove that the restriction of the functor (4.2.1) to $\mathrm{QCoh}_{\chi}(G) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}_{\chi'}(G')$ lands in $\mathrm{QCoh}_{(\chi,\chi')}(G \times_X G')$. It is enough to construct an equivalence

$$\alpha(\mathcal{F} \boxtimes \mathcal{F}') : (\mathcal{F} \boxtimes \mathcal{F}')_{(\chi,\chi')} \simeq (\mathcal{F})_{\chi} \boxtimes (\mathcal{F}')_{\chi'}$$

for every $\mathcal{F} \boxtimes \mathcal{F}'$. With the notation of Remark 4.1.4, we can do this using the following chain of equivalence:

$$\begin{aligned}(\mathcal{F} \boxtimes \mathcal{F}')_{(\chi,\chi')} &= (p \times p')_*(\mathrm{act}^*_{(\alpha,\alpha')}(\mathcal{F} \boxtimes \mathcal{F}') \otimes \mathcal{L}^{\vee}_{(\chi,\chi')}) \\ &\simeq (p \times p')_*((\mathrm{act}_{\alpha} \times \mathrm{act}_{\alpha'})^*(\mathcal{F} \boxtimes \mathcal{F}') \otimes (\mathcal{L}^{\vee}_{\chi} \boxtimes \mathcal{L}^{\vee}_{\chi'})) \\ &\simeq (p \times p')_*((\mathrm{act}^*_{\alpha}\mathcal{F} \otimes \mathcal{L}^{\vee}_{\chi}) \boxtimes (\mathrm{act}^*_{\alpha'}\mathcal{F}' \otimes \mathcal{L}^{\vee}_{\chi'})) \simeq (\mathcal{F})_{\chi} \boxtimes (\mathcal{F}')_{\chi'}.\end{aligned}$$

A straightforward computation shows that α is in fact a natural transformation and verifies the condition described in Remark 4.2.8.

It remains to prove that the restricted functor induces an equivalence, which will automatically be t-exact. But again, it suffices to show this locally, and in the local case this just reduces to the fact that

$$\mathrm{QCoh}_{\chi}(U \times \mathrm{B}\mathbb{G}_{\mathrm{m}}) \otimes_{\mathrm{QCoh}(U)} \mathrm{QCoh}_{\chi'}(U \times \mathrm{B}\mathbb{G}_{\mathrm{m}}) \simeq \mathrm{QCoh}(U) \otimes_{\mathrm{QCoh}(U)} \mathrm{QCoh}(U) \to \mathrm{QCoh}(U)$$

is an equivalence. ☐

Remark 4.2.10. Using the associativity of the box product $\boxtimes$, the same proof works in the case of a finite number of $\mathbb{G}_{\mathrm{m}}$-gerbes.

Proposition 4.2.11. *Let* $G \star G'$ *be the* $\mathbb{G}_{\mathrm{m}}$*-gerbe defined in Construction 4.2.1. Then the pullback along* $\rho : G \times_X G' \to G \star G'$ *establishes a t-exact equivalence*

$$\mathrm{QCoh}_{\mathrm{id}}(G \star G') \xrightarrow{\sim} \mathrm{QCoh}_{(\mathrm{id},\mathrm{id})}(G \times_X G').$$

Proof. Note that the character $m : \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ given by multiplication induces a line bundle $L_{\mathrm{id},\mathrm{id}}$ on $\mathrm{B}\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$. One can prove easily that this line bundle coincides with the external product of L_{id}, the universal line bundle on $\mathrm{B}\mathbb{G}_{\mathrm{m}}$, with itself. This means that what we denote by $\mathcal{L}_{(\mathrm{id},\mathrm{id})} \in \mathrm{QCoh}(G \times_X G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{G}_{\mathrm{m}})$ has the form $\mathcal{L}_{\mathrm{id}} \boxtimes \mathcal{L}_{\mathrm{id}}$ (again with the usual notations of Remark 4.1.4).

We need to prove that ρ^* sends id-twisted sheaves in $(\mathrm{id}, \mathrm{id})$-twisted sheaves. To do this, we will construct an equivalence

$$\alpha(\mathcal{F}) : (\rho^*\mathcal{F})_{(\mathrm{id},\mathrm{id})} \simeq \rho^*(\mathcal{F}_{\mathrm{id}})$$

for every $\mathcal{F}$ in $\mathrm{QCoh}(G \star G')$ and apply Remark 4.2.8. An easy computation shows that α is in fact a natural transformation and it verifies the condition described in Remark 4.2.8.

By construction of the $\star$ product, one can prove that the following diagram

$$\begin{tikzcd}
G \times_X G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "\mathrm{act}_{(\alpha,\alpha')}"] \arrow[d, "{(\rho,\mathrm{B}m)}"] & G \times_X G' \arrow[d, "\rho"] \\
G \star G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "\mathrm{act}_{\alpha\alpha'}"] & G \star G'
\end{tikzcd}$$

is commutative, where $\mathrm{act}_{(\alpha,\alpha')}$ is the action map of $G \times_X G'$ defined by the product bending, $act_{\alpha\alpha'}$ is the action map of $G \star G'$ and $\mathrm{B}m$ is the multiplication map $m : \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ at the level of classfying stacks. This implies that

$$act^*_{(\alpha,\alpha')}(\rho^*\mathcal{F}) = (\rho, \mathrm{B}m)^* act^*_{\alpha\alpha'}(\mathcal{F}).$$

Consider now the following diagram:

$$\begin{tikzcd}
G \times_X G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[dr, "{(\mathrm{id},\mathrm{B}m)}"] \arrow[drr, bend left, "\tilde{q}_{(\alpha,\alpha')}"] \arrow[ddr, bend right, "{(\rho,\mathrm{B}m)}"'] & & \\
& G \times_X G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "q_{(\alpha,\alpha')}"] \arrow[d, "{(\rho,\mathrm{id})}"] & G \times_X G' \arrow[d, "\rho"] \\
& G \star G' \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "q_{\alpha\alpha'}"] & G \star G'
\end{tikzcd}$$

where $\tilde{q}_{(\alpha,\alpha')}$, $q_{(\alpha,\alpha')}$ and $q_{\alpha\alpha'}$ are the projections. This is a commutative diagram and the square is a pullback.

Finally, we can compute the (id, id)-twisted part of $\rho^*\mathcal{F}$:

$$\begin{aligned}(\rho^*\mathcal{F})_{\mathrm{id,id}} =&\tilde{q}_{(\alpha,\alpha'),*}(act^*_{(\alpha,\alpha')}(\rho^*\mathcal{F}) \otimes \mathcal{L}^\vee_{\mathrm{id,id}})\\ =&q_{(\alpha,\alpha'),*}(\mathrm{id},\mathrm{B}m)_*\Big((\mathrm{id},\mathrm{B}m)^*(\rho,\mathrm{id})^*(act^*_{\alpha\alpha'}(\mathcal{F}))\otimes\mathcal{L}^\vee_{\mathrm{id,id}}\Big);\end{aligned}$$

notice that since $\mathrm{B}m^*L_{\mathrm{id}} = L_{\mathrm{id,id}}$ we have $(\rho,\mathrm{B}m)^*\mathcal{L}_{\mathrm{id}} = \mathcal{L}_{\mathrm{id,id}}$. Furthermore, an easy computation shows that the unit $\mathrm{id} \to (\mathrm{id},\mathrm{B}m)_*(\mathrm{id},\mathrm{B}m)^*$ is in fact an isomorphism, because of the explicit description of the decomposition of the stable ∞-category of quasi-coherent sheaves over $\mathrm{B}\mathbb{G}_{\mathrm{m}}$. These two facts together give us that

$$\begin{aligned}(\rho^*\mathcal{F})_{\mathrm{id,id}} =&q_{(\alpha,\alpha'),*}(\mathrm{id},\mathrm{B}m)_*\Big((\mathrm{id},\mathrm{B}m)^*(\rho,\mathrm{id})^*(act^*_{\alpha\alpha'}(\mathcal{F}))\otimes\mathcal{L}^\vee_{\mathrm{id,id}}\Big)\\ =&q_{(\alpha,\alpha'),*}(\mathrm{id},\mathrm{B}m)_*(\mathrm{id},\mathrm{B}m)^*(\rho,\mathrm{id})^*\Big(act^*_{\alpha\alpha'}(\mathcal{F})\otimes\mathcal{L}^\vee_{\mathrm{id}}\Big)\\ =&q_{(\alpha,\alpha'),*}(\rho,\mathrm{id})^*\Big(act^*_{\alpha\alpha'}(\mathcal{F})\otimes\mathcal{L}^\vee_{\mathrm{id}}\Big)\\ =&\rho^*q_{\alpha\alpha',*}\Big(act^*_{\alpha\alpha'}(\mathcal{F})\otimes\mathcal{L}^\vee_{\mathrm{id}}\Big)\\ =&\rho^*((\mathcal{F})_{\mathrm{id}}).\end{aligned}$$

To finish the proof, we need to verify that ρ^* restricted to id-twisted sheaves is an equivalence with the category of (id, id)-twisted sheaves. Again, this can be checked étale locally, therefore we can reduce to the case $G \simeq G' \simeq X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$ where the morphism ρ can be identified with $(\mathrm{id}_X, \mathrm{B}m)$. We know that for the trivial gerbe we have that $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}_{\mathrm{id}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}})$ where the map is described by $\mathcal{F} \mapsto \pi^*\mathcal{F}\otimes\mathcal{L}_{\mathrm{id}}$, π being is the structural morphism of the (trivial) gerbe. A straightforward computation shows that, using the identification above, the morphism ρ^* is the identity of $\mathrm{QCoh}(X)$.

Finally, t-exactness follows from the fact that ρ is flat (see [AOV08, Theorem A.1]). ☐

Remark 4.2.12. One can prove easily that ρ_* is an inverse of the morphism ρ^* which we have just described. It is still true that send twisted sheaves to twisted sheaves and it has the same functorial property of the pullback, due to the natural adjunction. Furthermore, ρ_* is t-exact because ρ is a morphism of gerbes whose image through the banding functor $\mathrm{Band}(\rho)$ is a surjective morphism of groups with a linearly reductive group as kernel. This implies that it is the structure morphism of a gerbe banded by a linearly reductive group, therefore ρ_* is exact.

Proof of Theorem 4.2.6. First of all, we start with the stable case. We have to lift $\mathrm{QCoh}_{\mathrm{id}}(-)$ from a morphism of ∞-groupoid to a symmetric monoidal functor, i.e. to extend the action of $\mathrm{QCoh}_{\mathrm{id}}(-)$ to multilinear maps. Let $\{G_i\}_{i\in I}$ be a sequence of $\mathbb{G}_{\mathrm{m}}$-gerbes indexed by a finite set I and H be a $\mathbb{G}_{\mathrm{m}}$-gerbe, then we can define a morphism of simplicial set

$$\mathrm{QCoh}_{\mathrm{id}}(-) : \mathrm{Mul}(\{G_i\}_{i\in I}, H) \longrightarrow \mathrm{Mul}(\{\mathrm{QCoh}_{\mathrm{id}}(G_i)\}_{i\in I}, \mathrm{QCoh}_{\mathrm{id}}(H))$$

by the following rule: if $f : \prod_{i\in I} G_i \to H$ is a multilinear map, we define $\mathrm{QCoh}_{\mathrm{id}}(f)$ to be the composition $f_* \circ \boxtimes^n : \prod_{i\in I} \mathrm{QCoh}_{\mathrm{id}}(G_i) \to \mathrm{QCoh}_{\mathrm{id}}(H)$ where $\boxtimes^n : \prod_{i\in I} \mathrm{QCoh}_{\mathrm{id}}(G_i) \to \mathrm{QCoh}_{\mathrm{id}}(\prod_{i\in I} G_i)$ is just the n-fold box product. The fact that this association is well-defined follows from Proposition 4.2.9 and Proposition 4.2.11. A priori, $\mathrm{QCoh}_{\mathrm{id}}(-)$ is a lax-monoidal functor from $\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)^{\star}$ to $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)^{\times}$, where $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)^{\times}$ is the symmetric monoidal structure on $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{St}}(X)$ induced by the product of ∞-categories. However, f_* and $\boxtimes^n$ are both $\mathrm{QCoh}(X)$-linear and preserves small colimits. Notice that f_* preserves small colimits because it is exact, due to the fact that it is a μ-gerbe, with μ a linearly reductive group (see Remark 4.2.12). This implies that $\mathrm{QCoh}_{\mathrm{id}}(-)$ can be upgraded to a morphism of ∞-operad if we take the operadic nerve. It remains to prove that it is symmetric monoidal, i.e. to prove that it sends coCartesian morphism to coCartesian morphism. This follows again from the isomorphisms described in Proposition 4.2.11 and Proposition 4.2.9.

The prestable case can be dealt with in the exact same way. Because the equivalences in Proposition 4.2.9 and Proposition 4.2.11 are t-exact, they restrict to the prestable connective part of the ∞-categories. The fact that they remain equivalences can be checked étale locally. □

4.2.3 Gerbes of positive trivializations

Definition 4.2.13. Let $M \in \mathcal{B}\mathrm{r}(X)$. We define the functor

$$\mathcal{T}\mathrm{riv}_{\geq 0}(M) : \mathrm{Sch}_{/X} \to \mathcal{S}$$

$$(S \to X) \mapsto \mathcal{E}\mathrm{quiv}_{\mathrm{QCoh}(S)_{\geq 0}}(\mathrm{QCoh}(S)_{\geq 0}, \mathcal{M}(S))$$

where $\mathcal{M}$ is the stack of categories associated to M (see Theorem 4.1.15).

Our aim in this subsection is to prove that for every $M \in \mathcal{B}\mathrm{r}(X)$, the functor $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ has a natural structure of a gerbe over X, and also that the functor $\mathcal{T}\mathrm{riv}_{\geq 0}(-) : \mathcal{B}\mathrm{r}(X) \to \mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)$ can be promoted to a symmetric monoidal functor $\mathcal{B}\mathrm{r}(X)^{\otimes} \to \mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)^{\star}$. Let us recall the main result about the Brauer space proven in [Lur18] (specialized to the case of qcqs schemes).

Theorem 4.2.14 ([Lur18, Theorem 11.5.7.11]). *Let X be a qcqs scheme. Then for every $u \in \mathrm{dBr}(X) = \pi_0(\mathcal{B}\mathrm{r}(X))$, there exists an étale covering $f : U \to X$ such that $f^*u = 0$ in $\mathrm{dBr}(U)$; that is, any representative of f^*u is equivalent to $\mathrm{QCoh}(U)_{\geq 0}$ as an ∞-category.*

Lemma 4.2.15. *Let X be a qcqs scheme. Then the stack* $\mathcal{E}\mathrm{quiv}_{\mathcal{Q}\mathcal{C}\mathrm{oh}(X)_{\geq 0}}(\mathcal{Q}\mathcal{C}\mathrm{oh}(X)_{\geq 0}, \mathcal{Q}\mathcal{C}\mathrm{oh}(X)_{\geq 0})$ *is equivalent to* $\mathrm{B}\mathbb{G}_{\mathrm{m}} \times X$.

Proof. Let $S \to X$ be a morphism of schemes. A $\mathrm{QCoh}(S)$-linear autoequivalence of $\mathrm{QCoh}(S)$ is determined by the image of $\mathcal{O}_S$, and therefore amounts to the datum of a line bundle $\mathcal{L}$ concentrated in degree 0 (the degree must be nonnegative because we are in the connective setting, and if it were positive then the inverse functor would be given by tensoring by a negatively-graded line bundle, which is impossible).

This implies that the desired moduli space is $\mathcal{P}\mathrm{ic} \times X$, which is the same as $\mathrm{B}\mathbb{G}_{\mathrm{m}} \times X$. □

Proposition 4.2.16. *Let X be a qcqs scheme, and $M \in \mathcal{B}\mathrm{r}(X)$. Then $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ has a natural structure of $\mathbb{G}_{\mathrm{m}}$-gerbe over X. Furthermore, given an isomorphism $f : M \to N$ of prestable invertible categories, then the morphism $\mathcal{T}\mathrm{riv}_{geq0}(f)$ defined by the association $\phi \mapsto f \circ \phi$ is a morphism of $\mathbb{G}_{\mathrm{m}}$-gerbes.*

Proof. To prove that $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ is a $\mathbb{G}_{\mathrm{m}}$-gerbe we need to verify that both the structure map and the diagonal of $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ are epimorphisms and to provide a $\mathbb{G}_{\mathrm{m}}$-banding. The first two assertions are evident from the fact that $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$ is locally of the form $X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$. Now we provide the banding in the following way. First of all, notice that $\mathbb{G}_{\mathrm{m},X}$ can be identified with the automorphism group of the identity endofunctor of $\mathrm{QCoh}(X)_{\geq 0}$, namely $\mathcal{O}_X$-linear invertible natural transformations of $\mathrm{id}_{\mathrm{QCoh}(X)_{\geq 0}}$. Let $\mathcal{I}_{\mathcal{T}\mathrm{riv}_{\geq 0}(M)}$ be the inertia stack of $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$. We define a functor

$$\alpha_M : \mathcal{T}\mathrm{riv}_{\geq 0}(M) \times \mathbb{G}_{\mathrm{m}} \longrightarrow \mathcal{I}_{\mathcal{T}\mathrm{riv}_{\geq 0}(M)}$$

as $\alpha_M(\phi, \lambda) := (\phi, \mathrm{id}_\phi * \lambda) \in \mathcal{I}_{\mathcal{T}\mathrm{riv}_{\geq 0}}$ for every (ϕ, λ) object of $\mathcal{T}\mathrm{riv}_{\geq 0}(M)$, $*$ being the horizontal composition of natural transformations. Since being an isomorphism is an étale-local property, we can reduce to the trivial case, for which it is a straightforward computation. The second part of the statement follows from a straightforward computation. □

Lemma 4.2.17. *The functor $\mathcal{T}\mathrm{riv}_{\geq 0}(-)$ is symmetric monoidal.*

Proof. To upgrade $\mathcal{T}\mathrm{riv}_{\geq 0}$ to a symmetric monoidal functor, we need to define its action on multilinear maps. Let $\{M_i\}_{i \in I}$ be a sequence of invertible prestable $\mathcal{O}_X$-linear ∞-categories indexed by a finite set I and N be another invertible prestable $\mathcal{O}_X$-linear category. Then we define

$$\mathcal{T}\mathrm{riv}_{\geq 0}(-) : \mathrm{Mul}\Big(\{M_i\}_{i \in I}, N\Big) \longrightarrow \mathrm{Mul}\Big(\{\mathcal{T}\mathrm{riv}_{\geq 0}(M_i)\}_{i \in I}, \mathcal{T}\mathrm{riv}_{\geq 0}(N)\Big)$$

in the following way: if $f : \prod_{i \in I} M_i \to N$ is a morphism in $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$ which preserves small colimits separately in each variable, we have to define the image $\mathcal{T}\mathrm{riv}_{\geq 0}(f)$ as a functor

$$\mathcal{T}\mathrm{riv}_{\geq 0}(f) : \prod_{i \in I} \mathcal{T}\mathrm{riv}_{\geq 0}(M_i) \longrightarrow \mathcal{T}\mathrm{riv}_{\geq 0}(N)$$

such that Band($\mathcal{T}\mathrm{riv}_{\geq 0}(f)$) is the n-fold multiplication of $\mathbb{G}_{\mathrm{m}}$, where n is the cardinality of I. We define it on objects in the following way: if $\{\phi_i\}$ is an object of $\prod_{i\in I}(\mathcal{T}\mathrm{riv}_{\geq 0}(M_i))$, then we set

$$\mathcal{T}\mathrm{riv}_{\geq 0}(f)\Big(\{\phi_i\}\Big) := f \circ \prod \phi_i$$

where $\prod \phi_i$ is just the morphism induced by the universal property of the product.

First of all we need to prove that $f \circ \prod \phi_i$ is still an equivalence. Let $S \in \mathrm{Sch}_X$ and $f : \prod_{i\in I} M_i \to N$ morphism in $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(X)$, we can consider the following diagram:

$$\begin{tikzcd}
\mathrm{QCoh}(S) \arrow[r, "\prod \phi_i"] \arrow[d, "\otimes\phi_i"'] & \prod_{i\in I} \mathcal{M}_i(S) \arrow[dl, "u"'] \arrow[d, "f(S)"] \\
\bigotimes_{i\in I} \mathcal{M}_i(S) \arrow[r, "\tilde{f}(S)"] & \mathcal{N}(S)
\end{tikzcd} \tag{4.2.2}$$

where the tensors are in fact relative tensors over $\mathrm{QCoh}(S)$. The diagonal map u is the morphism universal between all the $\mathrm{QCoh}(X)$-linear morphisms from $\prod_{i\in I} \mathcal{M}_i(S)$ which preserve small colimits separately in each variable.Equivalently, one can say that it is a coCartesian morphism in $\mathcal{L}\mathrm{in}\mathcal{C}\mathrm{at}^{\mathrm{PSt}}(S)^{\otimes}$.

Thus, it is enough to prove that both $\otimes\phi_i$ and $\tilde{f}$ are equivalences. Because the source of the functor $\mathcal{T}\mathrm{riv}_{\geq 0}(-)$ is the ∞-groupoid dBr(S), the morphism $\tilde{f}(S)$ is an equivalence. Furthermore, the morphisms ϕ_i are equivalences, therefore it follows from the functoriality of the relative tensor product that $\otimes\phi_i$ is an equivalence. A straightforward but tedious computation shows that it is defined also at the level of 1-morphism and it is in fact a functor.

It remains to prove that Band$\Big(\mathcal{T}\mathrm{riv}_{\geq 0}(f)\Big)$ is the n-fold multiplication of $\mathbb{G}_{\mathrm{m}}$, where n is the cardinality of I. It is equivalent to prove the commutativity of the following diagram:

$$\begin{tikzcd}
\prod_{i\in I} \mathcal{T}\mathrm{riv}_{\geq 0}(M_i) \times {\mathbb{G}_{\mathrm{m}}}^n \arrow[r, "(\mathcal{T}\mathrm{riv}_{\geq 0}(f){,}m^n)"] \arrow[d, "\prod \alpha_{M_i}"] & \mathcal{T}\mathrm{riv}_{\geq 0}(N) \times \mathbb{G}_{\mathrm{m}} \arrow[d, "\alpha_N"] \\
\prod_{i\in I} \mathcal{J}_{\mathcal{T}\mathrm{riv}_{\geq 0}(M_i)} \arrow[r, "\mathcal{J}_{\mathcal{T}\mathrm{riv}_{\geq 0}(f)}"] & \mathcal{J}_{\mathcal{T}\mathrm{riv}_{\geq 0}(N)}
\end{tikzcd}$$

where m^n is the n-fold multiplicatoon of $\mathbb{G}_{\mathrm{m}}$. The notation follows the one in the proof of Lemma 4.2.16. Using diagram 4.2.2 again, we can reduce to the following straightforward statement: let $\lambda_1, \ldots, \lambda_n$ be $\mathrm{QCoh}(S)$-linear automorphisms of $\mathrm{id}_{\mathrm{QCoh}(S)}$, which can be identified with elements of $\mathbb{G}_{\mathrm{m}}(S)$; then the tensor of the natural transformations coincide with the product as elements of $\mathbb{G}_{\mathrm{m}}$, i.e $\lambda_1 \otimes \cdots \otimes \lambda_n = m^n(\lambda_1 \ldots \lambda_n)$.

Finally, because we are considering ∞-groupoid, the condition of being stricly monoidal is automotically satisfied.

□

4.2.4 Proof of the equivalence

The goal of this subsection is to prove that the constructions

$$M \mapsto \mathcal{T}\mathrm{riv}_{\geq 0}(M)$$

and

$$G \mapsto \mathrm{QCoh}_{\mathrm{id}}(G)_{\geq 0}$$

establish a (symmetric monoidal) categorical equivalence between the 2-groupoids $\mathcal{B}\mathrm{r}(X)$ and $\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)$, thus proving Theorem 4.1.19.

Proposition 4.2.18. *The functor* $\mathcal{T}\mathrm{riv}_{\geq 0}(-)$ *is fully faithful.*

Proof. We want to prove that for any $M, M' \in \mathcal{B}\mathrm{r}(X)$ the map

$$\mathrm{Map}_{\mathcal{B}\mathrm{r}(X)}(M, M') \to \mathrm{Map}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)}(\mathcal{T}\mathrm{riv}_{\geq 0}(M), \mathcal{T}\mathrm{riv}_{\geq 0}(M'))$$

is a homotopy equivalence of spaces (more precisely, an equivalence of 1-groupoids). First of all, by using the grouplike monoid structure of $\mathcal{B}\mathrm{r}(X)$ together with Lemma 4.2.17, we can reduce to the case $M' = \mathrm{QCoh}(X)_{\geq 0} = \mathbf{1}$.

If M does not lie in the connected component of $\mathbf{1}$, then what we have to check is that

$$\mathrm{Map}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)}(\mathcal{T}\mathrm{riv}_{\geq 0}(M), \mathcal{T}\mathrm{riv}_{\geq 0}(\mathbf{1})) = \varnothing.$$

But if this space contained an object, then in particular we would have an equivalence at the level of global sections between $\mathrm{Equiv}_{\mathrm{QCoh}(X)_{\geq 0}}(M, \mathbf{1})$ and $\mathrm{Equiv}_{\mathrm{QCoh}(X)_{\geq 0}}(\mathbf{1}, \mathbf{1})$. But the first space is empty by hypothesis, while the second is not.

On the other hand, if M lies in the connected component of $\mathbf{1}$, by functoriality of $\mathcal{T}\mathrm{riv}_{\geq 0}(-)$ we can suppose that $M = \mathbf{1}$. In this case, we have to prove that the map

$$\mathrm{Map}_{\mathcal{B}\mathrm{r}(X)}(\mathbf{1}, \mathbf{1}) \to \mathrm{Map}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)}(\mathcal{T}\mathrm{riv}_{\geq 0}(\mathbf{1}), \mathcal{T}\mathrm{riv}_{\geq 0}(\mathbf{1})) \simeq \mathrm{Map}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}, X \times \mathrm{B}\mathbb{G}_{\mathrm{m}})$$

is an equivalence of groupoids. But the first space is the groupoid $\mathrm{Pic}(X)$, the latter is the space of maps $X \to \mathrm{B}\mathbb{G}_{\mathrm{m}}$, and the composite map is the one sending a line bundle over X to the map $X \to \mathrm{B}\mathbb{G}_{\mathrm{m}}$ that classifies it. ☐

Proposition 4.2.19. *Let* X *be a qcqs scheme. Then there is a natural equivalence of functors*

$$\mathrm{Id}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(X)} \Rightarrow \mathcal{T}\mathrm{riv}_{\geq 0} \circ \mathrm{QCoh}_{\mathrm{id}}(-)_{\geq 0}.$$

Proof. Let G be a $\mathbb{G}_{\mathrm{m}}$-gerbe over X. Let us observe that for any $S \to X$ we have

$$G(S) \simeq \mathcal{E}\mathrm{quiv}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}(S)}(S \times \mathrm{B}\mathbb{G}_{\mathrm{m}}, G_S).$$

Indeed, there is an evident map of stacks over X

$$F : \mathcal{E}\mathrm{quiv}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}, G) \to G$$

and up to passing to a suitable étale covering of X, the map becomes an equivalence: in fact the choice of an equivalence $\phi : S \times \mathrm{B}\mathbb{G}_{\mathrm{m}} \to S \times \mathrm{B}\mathbb{G}_{\mathrm{m}}$ *of gerbes over* S amounts to the choice of a map $S \to \mathrm{B}\mathbb{G}_{\mathrm{m}}$, because ϕ must be a map over S (hence $\mathrm{pr}_{\mathrm{B}\mathbb{G}_{\mathrm{m}}} \circ \phi = \mathrm{pr}_{\mathrm{B}\mathbb{G}_{\mathrm{m}}}$) and it must respect the banding . Therefore, F is an equivalence over X, and this endows $\mathcal{E}\mathrm{quiv}_{\mathrm{Ger}_{\mathbb{G}_{\mathrm{m}}}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}, G)$ with a natural structure of a $\mathbb{G}_{\mathrm{m}}$-gerbe over X.

The construction $\phi \mapsto \phi_*$ thus provides a morphism of stacks

$$G \to \mathcal{E}\mathrm{quiv}_{\mathcal{QC}\mathrm{oh}(X)}\Big(\mathrm{QCoh}_{\mathrm{id}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}), \mathrm{QCoh}_{\mathrm{id}}(G)\Big).$$

Now since the pushforward ϕ_* of an equivalence ϕ of stacks is t-exact, the construction $\phi \mapsto \phi_*$ factors through the stabilization map

$$\mathcal{E}\mathrm{quiv}_{\mathcal{QC}\mathrm{oh}_{\mathrm{id}}}\Big(\mathcal{QC}\mathrm{oh}_{\mathrm{id}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}})_{\geq 0}, \mathcal{QC}\mathrm{oh}_{\mathrm{id}}(G)\Big) \to \mathcal{E}\mathrm{quiv}_{\mathcal{QC}\mathrm{oh}(X)}\Big(\mathcal{QC}\mathrm{oh}_{\mathrm{id}}(X \times \mathrm{B}\mathbb{G}_{\mathrm{m}}), \mathcal{QC}\mathrm{oh}_{\mathrm{id}}(G)\Big).$$

Note that the stabilization map makes sense in the id-twisted context since the stabilization functor has descent (it is the limit of iterated loops), and thus we can reduce to the case when G is trivial.

We have therefore obtained a map of stacks over X

$$G \to \mathcal{E}\mathrm{quiv}_{\mathcal{QC}\mathrm{oh}(X)_{\geq 0}}\Big(\mathcal{QC}\mathrm{oh}(X)_{\geq 0}, \mathcal{QC}\mathrm{oh}_{\mathrm{id}}(G)_{\geq 0}\Big) = \mathcal{T}\mathrm{riv}_{\geq 0}(\mathrm{QCoh}_{\mathrm{id}}(G)_{\geq 0}).$$

But now, $\mathcal{T}\mathrm{riv}_{\geq 0}(\mathrm{QCoh}_{\mathrm{id}}(G))$ is a $\mathbb{G}_{\mathrm{m}}$-gerbe over X (note that this was not true before passing to the connective setting), and therefore to prove that the map is an equivalence it suffices to prove that it agrees with the bandings, that is, that it is a map of $\mathbb{G}_{\mathrm{m}}$-gerbes. This follows from unwinding the definitions.

□

We are now ready to prove our main result.

Proof of Theorem 4.1.19: Section 4.2.2 and Section 4.2.3 tell us that the two functors are symmetric monoidal and take values in the sought ∞-categories. The fact that they form an equivalence follows from Proposition 4.2.18 and Proposition 4.2.19. □

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