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Stable Weakly Shadowable Volume-preserving Systems Are Volume-hyperbolic

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Abstract We prove that any C^1-stable weakly shadowable volume-preserving diffeomorphism defined on a compact manifold displays a dominated splitting $E \oplus F$. Moreover, both E and F are volume-hyperbolic. Finally, we prove the version of this result for divergence-free vector fields. As a consequence, in low dimensions, we obtain global hyperbolicity.

Keywords Weak shadowing, dominated splitting, hyperbolicity

MR(2010) Subject Classification 37C50, 37D30, 37C10

1 Introduction

1.1 Shadowing in Dynamical Systems

It is a very rich field in smooth dynamics the relation between the stability of a certain property (with respect to a given topology) and some hyperbolic behavior on the tangent action of the dynamical system. Structural stability, shadowing-type properties, robust transitivity, stable ergodicity, topological stability, expansiveness, specification, L^p-shadowing, inverse shadowing, weak shadowing, are some successful examples of that (see e.g. [1, 3, 6, 16–21, 35], and the references therein). Here, we are interested in the weak shadowing property.

The shadowing in dynamics aims, in brief terms, to obtain shadowing of approximate trajectories in a given dynamical system by true orbits of the system.

The weak shadowing property is a relaxed form of shadowing and, informally speaking, allows the pseudo-orbits to be approximated by true orbits if one forgets the time parameterization and consider only the distance between the orbit and the pseudo-orbit as two sets in

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the phase space. We intend, in this paper, to study the weak shadowing property for volume-preserving diffeomorphisms and also for volume-preserving flows.

There are limitations about the information we can capture from a fixed dynamical system that displays some shadowing-type property, since another system arbitrarily close to it may be absent of that property. Thence, it is of great utility and natural to consider that a selected model can be slightly perturbed in order to obtain the same property — the stable weakly shadowable systems. However, it is worth to mention that stability in the volume-preserving setting only allows us to consider perturbations which preserves the volume-form and not evolves in the broader space of dissipative diffeomorphisms/flows. So, the results already proved for dissipative diffeomorphisms/flows are not applicable to our conservative context.

In [31], it is proved that if a diffeomorphism defined in a surface has the C^1-stable weak shadowing property, then it satisfies the axiom A and the no-cycle condition. However, the converse does not hold (see [23]). In [24], we can find more details on the relation between C^1-stability of weakly shadowing systems and structural stability in surfaces. In [14], the weak shadowing property is proved to be generic (in the C^1-sense) for diffeomorphisms in closed manifolds (this results is also valid in the volume-preserving context cf. [14, §2.5]). In the two-dimensional volume-preserving case (thus symplectic because of the low dimension assumption), C^1-weakly shadowing implies hyperbolicity (see [20]), and C^1-weakly shadowable symplectomorphisms and Hamiltonians are partially hyperbolic (see [6, 7]). In this paper, we generalize the results in [6, 7, 19, 20, 34] proving that any C^1-stable weakly shadowable volume-preserving diffeomorphism displays a dominated splitting $E \oplus F$ (Theorem 1). Moreover, both E and F are volume-hyperbolic. With respect to the flow setting, the literature is absent on exploring this type of shadowing. We begin to develop this concept proving similar results (Theorem 2).

1.2 Basic Definitions for the Discrete-time Case

Let M be a d-dimensional ($d \geq 2$) Riemannian closed and connected manifold and let $d(\cdot,\cdot)$ denote the distance on M inherited by the Riemannian structure. We endow M with a volume-form and let μ denote the Lebesgue measure related to it. Actually, in [22] we find an atlas formed by a finite collection of smooth volume-preserving charts $\{\varphi_j\colon \mathbb{R}^d \to U_j \subset M\}_{i=1}^k$, where U_j are open sets and each φ_j pullbacks the volume on $\mathbb{R}^d$ into the volume-form. Let $\mathrm{Diff}^{\,1}_{\mu}(M)$ denote the set of volume-preserving diffeomorphisms defined on M, i.e., those diffeomorphisms such that $\mu(B) = \mu(f(B))$ for any μ-measurable subset B. Consider this space endowed with the C^1 Whitney topology. The Riemannian inner-product induces a norm $\|\cdot\|$ on the tangent bundle T_xM. We will use the usual uniform norm of a bounded linear map A given by $\|A\| = \sup_{\|v\|=1} \|A\cdot v\|$.

Fix some diffeomorphism $f \in \mathrm{Diff}^{\,1}_{\mu}(M)$. Given $\delta > 0$, we say that a sequence of points $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is a *δ-pseudo-orbit of f* if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that a sequence of points $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is *weakly ϵ-shadowed by the f-orbit of x* if $\{x_i\}_{i\in\mathbb{Z}} \subset B_\epsilon(\{\bigcup_{i\in\mathbb{Z}} f^i(x)\})$ where, for $A \subset M$, we have $B_\epsilon(A) := \{y \in M\colon d(y, A) < \epsilon\}$. The diffeomorphism $f \in \mathrm{Diff}^{\,1}_{\mu}(M)$ has the *weak shadowing property* if for any $\epsilon > 0$, there exists $\delta > 0$ such that, any δ-pseudo orbit is weakly ϵ-shadowed by the f-orbit of some $x \in M$. We say that f is *C^1-stable weakly* *shadowing* if there is a C^1-neighborhood $U(f)\cap \mathrm{Diff}_{\mu}^{\,1}(M)$ of f, such that any $g \in U(f)$ has the weak shadowing property.

A diffeomorphism f is said to be *transitive* if there is an f-dense orbit in M. We observe that a transitive diffeomorphism has the weakly shadowing property. Thus, C^1-stable transitivity implies the C^1-stable shadowing property.

Given an f-invariant set $\Lambda \subseteq M$, we say that Λ is *uniformly hyperbolic* if the tangent vector bundle over Λ splits into two Df-invariant subbundles $T_\Lambda M = E^u \oplus E^s$ such that $\|Df|_{E^s}\| \le \frac{1}{2}$ and $\|Df^{-1}|_{E^u}\| \le \frac{1}{2}$. When $\Lambda = M$, we say that f is *Anosov*. Clearly, there are lots of Anosov diffeomorphisms which are not volume-preserving. We say that an f-invariant set $\Lambda \subseteq M$ admits an *ℓ-dominated splitting* if there exists a continuous decomposition of the tangent bundle $T_\Lambda M$ into Df-invariant subbundles E and F such that

$$\|Df^{\ell}(x)|_F\|\,\|(Df^{\ell}(x)|_E)^{-1}\| \le \frac{1}{2},$$

in this case we say $E \succ_\ell F$ (i.e. E ℓ-dominates F).

Finally, we say that an f-invariant set $\Lambda \subseteq M$ is *uniformly partially hyperbolic*, if we have a splitting $E^s \oplus E^c \oplus E^u$ of $T_\Lambda M$ such that E^s is uniformly contracting, E^u is uniformly expanding, $E^c \succ E^s$ and $E^u \succ E^c$. When M is partially hyperbolic for f we say that f is a *partially hyperbolic diffeomorphism.*

We say that the Df-invariant splitting $E \oplus C \oplus F$ of $T_\Lambda M$ where $C \succ E$ and $F \succ E$ is *volume partially hyperbolic* if the volume is uniformly contracted on the bundle E and expanded on the bundle F, i.e., there exists $\ell \in \mathbb{N}$ such that $|\det Df^{\ell}(x)_{|E}| < \frac{1}{2}$ and $|\det Df^{-\ell}(x)_{|F}| < \frac{1}{2}$. When C is trivial, we say that $E \oplus F$ is *volume-hyperbolic.*

It is proved in [8, Proposition 0.5] that, in the volume-preserving context, the existence of a dominated splitting implies volume-hyperbolicity.

A *periodic orbit* for a diffeomorphism f is a point $p \in M$ such that $f^{\pi}(p) = p$, where π is the least positive integer satisfying the equality. Given a periodic orbit p of period π of a diffeomorphism f we say that p:

- is *hyperbolic* if $Df^{\pi}(p)$ has no norm one eigenvalues;
- has *trivial real spectrum* if $Df^{\pi}(p)$, has only real eigenvalues of norm one (thus equal to -1 or 1) and there exists $0 \le k \le d$ such that 1 has multiplicity k and -1 has multiplicity $d-k$.

Clearly, if x is an f-periodic point with period π and with trivial real spectrum, we can split $T_xM = E_x^+ \oplus E_x^-$ such that $Df^{\pi}(x)\colon E_x^+ \to E_x^+$ is the identity map and $Df^{\pi}(x)\colon E_x^- \to E_x^-$ is the minus identity map.

1.3 Basic Definitions for the Continuous-time Case

We present the basic set up for the context of flows. In this setting we assume that the dimension of M is greater than two. Given a C^r $(r \ge 1)$ vector field $X\colon M \to TM$, the solution of the equation $x' = X(x)$ gives rise to a C^r flow, X^t; by the other side given a C^r flow, we can define a C^{r-1} vector field by considering $X(x) = \frac{d}{dt}X^t(x)|_{t=0}$. We say that X is *divergence-free* if its divergence is equal to zero. Note that, by Liouville formula, a flow X^t is volume-preserving if and only if the corresponding vector field, X, is divergence-free.

Let $\mathfrak{X}^r_\mu(M)$ denote the space of C^r divergence-free vector fields and we consider the usual C^r Whitney topology on this space.

For $\delta > 0$, we say that

$$\{(x_i, t_i) : x_i \in M, t_i \geq 1\}_{i \in \mathbb{Z}}$$

is a $(\delta, 1)$-pseudo orbit of X if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$.

We say that X has the weak shadowing property if, for every $\epsilon > 0$, there is $\delta > 0$ such that for any $(\delta, 1)$-pseudo orbit $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$, there is a point $x \in M$ such that $\{x_i\}_{i \in \mathbb{Z}} \subset B_\epsilon(\{\bigcup_{t \in \mathbb{R}} X^t(x)\})$.

We say that $X \in \mathfrak{X}^1_\mu(M)$ is *C^1-stable weakly shadowable* if any $Y \in \mathfrak{X}^1_\mu(M)$ sufficiently C^1-close to X is also weakly shadowable.

Given a vector field X, we denote by $\mathrm{Sing}(X)$ the set of *singularities* of X, i.e. those points $x \in M$ such that $X(x) = \vec{0}$. Let $R := M \setminus \mathrm{Sing}(X)$ be the set of *regular* points. Given $x \in R$, we consider its normal bundle $N_x = X(x)^\perp \subset T_xM$ and define the associated *linear Poincaré flow* by $P^t_X(x) := \Pi_{X^t(x)} \circ DX^t(x)$, where $\Pi_{X^t(x)} \colon T_{X^t(x)}M \to N_{X^t(x)}$ is the projection along the direction of $X(X^t(x))$. In the same way as we did in the discrete-time case, we define uniform hyperbolicity, dominated splitting, partial hyperbolicity and volume-hyperbolicity for the linear Poincaré flow P^t_X, in subsets of R and related to subbundles of the normal bundle N. We also observe that, when $\Lambda \subseteq M$ is compact, the (partial) hyperbolicity of the tangent map DX^t on Λ implies the (partial) hyperbolicity for the linear Poincaré flow of X on Λ (see [15, Proposition 1.1]).

Given a closed orbit p of period π of a flow X^t, we say that p:

• is *hyperbolic* if $P^\pi_X(p)$ has no norm one eigenvalues;

• has *trivial real spectrum* if $P^\pi_X(p)$, has only real eigenvalues of norm one (thus equal to -1 or 1) and there exists $0 \leq k \leq d-1$ such that 1 has multiplicity k and -1 has multiplicity $d-1-k$.

If x is an X^t-periodic point with period π and with trivial real spectrum, we can split $N_x = N^+_x \oplus N^-_x$ such that $P^\pi_X(x) \colon N^+_x \to N^+_x$ is the identity map and $P^\pi_X(x) \colon N^-_x \to N^-_x$ is the minus identity map.

1.4 Statement of the Results and Some Applications

As we already said, here we begin by developing the generalized versions of the results in [6, 7, 19, 20, 34]. Our first result is the following.

Theorem 1 *Let $f \in \mathrm{Diff}^1_\mu(M)$ be a C^1-stable weakly shadowing diffeomorphism. Then, M admits a volume-hyperbolic dominated splitting.*

Notice that f is not necessarily C^1-robustly transitive and this is exactly the interesting case because otherwise we could use the arguments on robust transitivity developed in [8, §7]. We observe that Bonatti and Viana [11, §6.2] build an open subset of partially hyperbolic (but not Anosov) transitive diffeomorphisms on 3-dimensional manifolds. Since their construction can be made conservative, we observe that Theorem 1 is optimal for dimension $d \geq 3$.

We observe that a quite complete construction of this type of behavior, in the volume-preserving context, was done in [27]. Actually, Rodriguez-Hertz et al. [27] build *volume-preserving blenders* which are a prototype example of robust transitive dynamics. We also observe that these results were used to prove recent important results in the volume-preserving setting; namely, Catalan [13] proved that in the complement of Anosov (volume-preserving) diffeomorphisms we have densely robust heterodimensional cycles and Rodriguez-Hertz [28] proved a new criterium for ergodicity among partial hyperbolic diffeomorphisms with central direction with dimension two, that is, stable ergodic diffeomorphisms are C^1-dense among volume-preserving partially hyperbolic diffeomorphisms with two-dimensional center bundle.

As we already said, by [8, Proposition 0.5], we obtain that a dominated splitting $TM = E \oplus F$ with $F \succ E$ implies that E is uniformly volume hyperbolic (contracting) and F is uniformly volume hyperbolic (expanding). Thus, Theorem 1 implies the result in [20].

With respect to the three-dimensional case, Theorem 1 implies that, in the presence of C^1-stability of the weak shadowing property we get that $TM = E \oplus F$ has a domination $F \succ E$. Since the dimension of the splitting is constant (see [9]), one of the subbundles E or F are one-dimensional and thus uniformly hyperbolic. In fact, the diffeomorphism is *coarsely* partially hyperbolic (cf. [29, p. 122]). Rodriguez-Hertz [29, Corollary 1.7] presented sufficient conditions to have a (proper) partial hyperbolic volume-preserving diffeomorphism, i.e., a splitting into three one-dimensional sub bundles $E^u \oplus E^c \oplus E^s$ with E^u and E^s uniform hyperbolic expanding and contracting, respectively.

Finally, we present the corresponding versions for the flow context.

Theorem 2 *Let $X \in \mathfrak{X}^1_\mu(M)$ be a C^1-stable weakly shadowing vector field. Then, M admits a volume-hyperbolic dominated splitting for the linear Poincaré flow.*

As an immediate consequence of previous result and [15, Proposition 1.1], we obtain

Corollary 1.1 *Let $X \in \mathfrak{X}^1_\mu(M)$ be a C^1-stable weakly shadowing vector field and M is three-dimensional. Then X is an Anosov flow.*

We observe that, as we already mention C^1-stability of transitivity implies C^1-stability of weak shadowing. Thence, Theorem 2 gives a different way of proving the main result in [3]. We notice that Ferreira (see [18]) using the C^1-stability of shadowing was able to obtain global hyperbolicity for divergence-free vector fields.

2 Discrete-time Case

2.1 Linear Conservative Cocycles over Large Periodic Systems

To prove that any volume-preserving diffeomorphism f which is C^1-stable weakly shadowing does not contains trivial real spectrum periodic orbits, we will use the following very useful volume-preserving version of Franks' lemma (see the dissipative version in [19, Lemma 3.1]) that can be established following the volume-preserving arguments in [8, Proposition 7.4].

Lemma 2.1 *Let $f \in \mathrm{Diff}^{\,1}_\mu(M)$ and a C^1-neighborhood of f, $U(f)$ be given. Then there exist $\delta_0 > 0$ and $U_0(f)$ such that, for any $g \in U_0(f)$, a finite set $\{x_i\}_{i=1}^l$, a neighborhood U of $\{x_i\}_{i=1}^l$ and volume-preserving linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ $(1 \le i \le l)$ satisfying the inequality $||L_i - Dg(x_i)|| < \delta_0$ for all $1 \le i \le l$, there exist $\epsilon_0 > 0$ and $\tilde{g} \in U(f)$ such that*

a) *$\tilde{g}(x) = g(x)$ if $x \in M \backslash U$ and*

b) *for all $1 \le i \le l$ we have $\tilde{g}(x) = \varphi_{g(x_i)} \circ L_i \circ \varphi^{-1}_{x_i}(x)$ if $x \in B(x_i, \epsilon_0)$.*

Item b) above implies that $\tilde{g}(x) = g(x)$ if $x \in \{x_i\}_{i=1}^{l}$ and $D\tilde{g}(x_i)$ is conjugated to L_i via the tangent maps of the local volume-preserving charts.

We start this section by recalling some basic definitions introduced in the paper [10, §2.3]. Let $f \in \mathrm{Diff}_{\mu}^{\,1}(M)$ and consider a set $\Sigma \subseteq M$ which is a countable union of periodic orbits of f. Moreover, we assume that the number of orbits of Σ with period equal to τ is finite for any $\tau > 0$. A *large periods system* (LPS) is a four-tuple $\mathcal{A} = (\Sigma, f, T\Sigma, A)$, where $T\Sigma$ is the restriction to Σ of the tangent bundle over M and $A\colon \Sigma \to \mathrm{SL}(n, \mathbb{R})$ is a continuous map, where $\mathrm{SL}(n, \mathbb{R})$ stands for the special linear $(n^2 - 1)$-dimensional Lie group of $n \times n$ matrices with real entries. In fact, for $x \in \Sigma$, A_x is a linear map from T_xM to $T_{f(x)}M$ and we identify these spaces with $\mathbb{R}^n$. The quintessential example of an LPS, and associated with the dynamics of the volume-preserving diffeomorphism, is obtained by taking the so-called *dynamical cocycle* given by $A_x = Df(x)$. Given an LPS $\mathcal{A} = (\Sigma, f, T_\Sigma, A)$, the *cocycle identity* associated with it is given by

$$A_x^{m+n} = A_{f^n(x)}^m \cdot A_x^n, \tag{2.1}$$

where $x \in \Sigma$ and $m, n \in \mathbb{N}$. The LPS $\mathcal{A} = (\Sigma, f, T_\Sigma, A)$ is *bounded* if there exists $K > 0$ such that $\|A_x\| \leq K$ for all $x \in \Sigma$. Since the LPS $\mathcal{A}$ evolves in $\mathrm{SL}(n, \mathbb{R})$, we say that it is *conservative*, in fact, $|\det A_x| = 1, \forall\, x \in \Sigma$.

An LPS $\mathcal{B} = (\Sigma, f, T\Sigma, B)$ is a *conservative perturbation* of a bounded LPS $\mathcal{A}$ if, for every $\epsilon > 0$, $\|A_x - B_x\| < \epsilon$, up to points x belonging to a finite number of orbits, and $\mathcal{B}$ is conservative. A bounded LPS $\mathcal{A}$ is *strictly without dominated decomposition* if the only invariant subsets of Σ that admit a dominated splitting for A are finite sets.

Let us now present a key result about LPS which is the conservative version of [10, Theorem 2.2].

Theorem 2.2 *Let $\mathcal{A}$ be a conservative and bounded* LPS*. If $\mathcal{A}$ is strictly without dominated decomposition, then there exist a conservative perturbation $\mathcal{B}$ of $\mathcal{A}$ and an infinite set $\Sigma' \subset \Sigma$ which is f-invariant such that for every $x \in \Sigma'$ the linear map $B_x^{\pi(x)}$ has all the eigenvalues real and with the same modulus* (*thus equal to* 1 *or to* -1)*.*

As in [10], we can also obtain the following more user friendly result:

Corollary 2.3 *Given any $K > 0$ and $\epsilon > 0$, there exist $\pi_0, \ell \in \mathbb{N}$ such that, for any conservative and K-bounded* LPS *$\mathcal{A}$, over a periodic orbit x with period $\pi(x) > \pi_0$, we have either*

(i) *that $\mathcal{A}$ has an ℓ-dominated splitting along the orbit of x or else*

(ii) *there exists an ϵ-C^0-perturbation $\mathcal{B}$ of $\mathcal{A}$, such that $B_x^{\pi(x)}$ has all eigenvalues equal to* 1 *and* -1.

All the perturbations which are used in the proof of [10, Theorem 2.2] are rotations and directional homotheties, i.e., diagonal linear maps for a fixed basis. They are made in the linear cocycle setting which evolves in the general linear group $\mathrm{GL}(n, \mathbb{R})$, and clearly can also be done in $\mathrm{SL}(n, \mathbb{R})$ with some additional care. Then, Lemma 2.1 allows to realize them as perturbations of a fixed volume-preserving diffeomorphism. Therefore, the proof given by Bonatti et al. [10] can be carried on to our volume-preserving setting without additional obstructions. As a conclusion, we obtain the following crucial result.

Lemma 2.4 *Let $f \in \mathrm{Diff}_\mu^{\,1}(M)$ and fix $\epsilon_0 > 0$. There exist $\pi_0, \ell \in \mathbb{N}$ such that, for any periodic orbit x with period $\pi(x) > \pi_0$, we have either*

(i) *that f has an ℓ-dominated splitting along the orbit of x or else*

(ii) *for any neighborhood U of $\bigcup_n f^n(x)$, there exists an ϵ-C^1-perturbation g of f, coinciding with f outside U and on $\bigcup_n f^n(x)$, and such that $Dg^{\pi(x)}(x)$ has all eigenvalues equal to* 1 *and* -1.

Since the fact that previous result is very useful, we can improve it in order to be used in the sequel.

Remark 2.5 Let $Dg^{\pi(x)}(x)\colon T_xM \to T_xM$ be the linear map given in Lemma 2.4 (ii). We can assume that x has trivial real spectrum. Actually, we can obtain a proof of this by considering [10, Proposition 3.7] where we can obtain $h \in \mathrm{Diff}_\mu^{\,1}(M)$ arbitrarily C^1-close to g and such that $Dh^{\pi(x)}(x)$ has all eigenvalues real of multiplicity 1, and with different modulus. Moreover, the Lyapunov exponents of $Dh^{\pi(x)}(x)$ can be chosen arbitrarily close to those of $Dg^{\pi(x)}(x)$. Finally, another small and volume-preserving perturbation can be done in order to preserve the simplicity of the spectrum and with all eigenvalues equal to 1 or -1.

2.2 Proof of Theorem 1

The following result is almost the volume-preserving counterpart of [19, Lemma 3.2]. Another fundamental results which guarantee the absent of elliptic behavior among C^1-stable weakly shadowable maps can be found in [7, Main Lemma 1] and in [20, Proposition 3.5]. In brief terms next lemma says that fixed/periodic local trivial dynamics is an ingredient against the stability of weak shadowing.

Lemma 2.6 *Fix some C^1-weakly shadowable volume-preserving diffeomorphism f and $\delta_0 > 0$ such that any $g \in \mathrm{Diff}_\mu^{\,1}(M)$ δ_0-C^1-close to f is also weakly shadowable. Let $U_0(f)$ be given by* Lemma 2.1 *with respect to $U(f)$. Then, for any $g \in U_0(f)$, g does not contains periodic points with trivial real spectrum.*

Proof Let $\dim M = d$ and let us suppose that there is a volume-preserving $g \in U_0(f)$ that have a period orbit p with all eigenvalues equal to 1 and -1. Assume that p is such that $g(p) = p$. Then $Dg(p)$ has k eigenvalues equal to 1 and $n-k$ eigenvalues equal to -1 and $T_pM = E_p^+ \oplus E_p^-$ where E_p^+ corresponds to the subspace of the eigenvalue 1 and E_p^- corresponds to the subspace of eigenvalue -1.

By Lemma 2.1, there is $\epsilon_0 > 0$ and a stable weakly shadowable $\tilde{g} \in U(f)$, such that $\tilde{g}(p) = g(p) = p$ and $\tilde{g}(x) = \varphi_{g(p)} \circ Dg(p) \circ \varphi_p^{-1}(x)$ if $x \in B(p, \epsilon_0)$, reducing ϵ_0 if necessary.

The next computations will be yield in $E_p^+(\epsilon_0)$ (the other case is similar using that $\tilde{g}^2$ has all eigenvalues equal to 1 and that, if g has the weak shadowing property, then g^2 also has the weak shadowing property).

Since $Dg(p)|_{E_p^+} = \mathrm{id}$, where $\mathrm{id}\colon E_p^+ \to E_p^+$ is the identity map on E_p^+, there is a small arc $\mathcal{I}_p \subset B(p, \epsilon_0) \cap \varphi_p(E_p^+(\epsilon_0))$ with center p such that $\tilde{g}(\mathcal{I}_p) = \mathcal{I}_p$ and $\tilde{g}|_{\mathcal{I}_p}$ is id (i.e., the *identity map*).

Take $\epsilon_1 < \epsilon_0$, $v_1 \in E_p^+(\epsilon_1)$ with $\|v_1\| = \epsilon_2 = \frac{\epsilon_1}{2}$ and set

$$\mathcal{I}_p \supset \varphi_p(\{tv_1 : t \in [-1, 1]\}) \cap B(p, \epsilon_1).$$

Put $\epsilon = \frac{\epsilon_1}{5}$ and let $0 < \delta < \epsilon$ be the number of the weak shadowing property of $\tilde{g}$.

Now, we are going to construct a δ-pseudo-orbit $\{x_k\}_{k\in\mathbb{Z}}$ of $\tilde{g}$ in $\mathcal{I}_p$ which cannot be weakly ϵ-shadowed by any $\tilde{g}$-true orbit of a point in M.

We take a finite sequence $\{w_k\}_{k=0}^{T}$ in $E_p^+(\epsilon_1)$ for some $T > 0$, such that $w_0 = O_p$, $w_T = v_1$ and $|w_k - w_{k+1}| < \delta$ for $0 \le k \le T-1$. Here the w_k are chosen such that if $w_k = t_k v_1$, then $|t_k| < |t_{k+1}|$ for $0 \le k \le (T-1)$. Finally, define

- $x_k = \varphi_p(w_0)$ for $k < 0$;
- $x_k = \varphi_p(w_k)$ for $0 \le k \le T-1$;
- $x_k = \tilde{g}^{k-T}(\varphi_p(w_T))$ for $k \ge T$.

Then $\{x_k\}_{k\in\mathbb{Z}}$ is a δ-pseudo-orbit of $\tilde{g}$ in $B(p,\epsilon_2)$ and since $\tilde{g}$ is stable weakly shadowable, there is $y \in M$ weakly ϵ-shadowing $\{x_k\}_{k\in\mathbb{Z}}$.

The local structure of $\tilde{g}$ in a neighborhood of $\mathcal{I}_p$ in M is the direct product of the identity map, $\tilde{g}|_{\varphi_p(E_p^+(\epsilon_2))\cap B(p,\epsilon_2)}$ by the minus identity map $\tilde{g}|_{\varphi_p(E_p^-(\epsilon_2))\cap B(p,\epsilon_2)}$.

We may assume that $y \in B(x_0, \epsilon)$.

If $y \in \mathcal{I}_p$, then since $\tilde{g}^i(y) = y$ for $i \in \mathbb{Z}$, $d(\tilde{g}^i(y), x_T) > \epsilon$ by the choice of ϵ.

If $y \notin \mathcal{I}_p$, then, $\tilde{g}(y) = y$ or else $\tilde{g}^2(y) = y$, for all $i \in \mathbb{Z}$, and $d(\tilde{g}^i(y), x_T) > \epsilon$ by the choice of ϵ. This proves the lemma. □

Proof of Theorem 1 Take the Pugh–Robinson residual (general density theorem, see [25]), intersected with Bonatti–Crovisier residual [12] and call it $\mathcal{R}$. As $\mathrm{Diff}_\mu^1(M)$ endowed with the C^1-topology is a Baire space, we can take a sequence of $f_n \in \mathcal{R}$ with $f_n \to f$ (in the C^1-topology). There exist periodic points p_n (for f_n and with period π_n) such that $\limsup_n \bigcup_i f^i(p_n) = M$ (in the Hausdorff metric sense[1)]). Clearly, $\pi_n \to +\infty$. Define

$$\Sigma = \bigcup_{n\in\mathbb{N}} \{p_n, f_n(p_n), \ldots, f_n^{\pi_n - 1}(p_n)\}. \tag{2.2}$$

Notice that $\overline{\Sigma} = M$.

We now define an $\mathcal{A} = (\Sigma, g, T\Sigma, A)$; Σ is defined in (2.2), $T\Sigma = T_x M$ where $x \in \Sigma$, $g(f_n^i(p_n)) = f_n^{i+1}(p_n)$ for any $i \in \{0, 1, \ldots, \pi_n - 1\}$ and $A(g_n^i(p_n)) = Df_n(f_n^i(p_n))$.

Of course that we are in the presence of an LPS. By Corollary 2.3, there exists a uniform dominated splitting over Σ. Since $\overline{\Sigma} = M$ and the dominated splitting extends to the closure (cf. [9]), we obtain that M has a dominated splitting with respect to the LPS $\mathcal{A} = (\Sigma, g, T\Sigma, A)$.

Realizing dynamically the cocycle, by Theorem 2.2 and Lemma 2.6, we obtain that f, C^1-stable weakly shadowable, has a dominated splitting over M. Finally, [8, Proposition 0.5] guarantees the volume-hyperbolic statement and the theorem is proved. □

3 Continuous-time Case

3.1 Linear Traceless Differential Systems over Large Periods Systems

We begin by recalling some definitions introduced in [10], in Subsection 2.1 and first developed for flows in [3]. Let $X \in \mathfrak{X}_\mu^1(M)$ and consider a set $\Sigma \subset M$ which is a countable union of closed orbits of X^t. A *Linear Differential System* (LDS) is a four-tuple $\mathscr{A} = (\Sigma, X^t, N_\Sigma, A)$, where N_Σ

1) We recall that the Hausdorff distance between two compact subsets $A, B \subseteq M$ is given by $d_H(A,B) = \max\{\sup_{y\in B} d(y,A), \sup_{x\in A} d(x,B)\}$.

is the restriction to Σ of the normal bundle of X over $M\backslash\mathrm{Sing}(X)$ and $A\colon \Sigma \to \mathrm{GL}(n-1,\mathbb{R})$ is a continuous map. In fact, for $x \in \Sigma$, A_x is a linear map of N_x and we identify this space with $\mathbb{R}^{n-1}$. The natural LDS associated with the dynamics of the vector field is obtained by taking $A_x = \Pi \circ DX_x$.

Given $\mathcal{A} = (\Sigma, X^t, N_\Sigma, A)$, the *linear variational equation* (or *equation of first variations*) associated to $\mathcal{A}$ is given by

$$\frac{d}{dt}u(t,x) = A(X^t(x)) \cdot u(t,x), \tag{3.1}$$

where $x \in \Sigma$ and $u(t,x)$ is a map between N_x and $N_{X^t(x)}$.

The matriciant (or solution) of the system (3.1) with initial condition $u(0,x) = \mathrm{id}$ is, for each t and x, a linear map $\Phi_A^t(x)\colon N_x \to N_{X^t(x)}$. We call the map A the *infinitesimal generator* of Φ_A. It is easy to see that $\Phi_A^t(x) = P_X^t(x)$ when the infinitesimal generator is $\Pi \circ DX$. We say that $\mathcal{A} = (\Sigma, X^t, N_\Sigma, A)$ is *bounded*, if there exists $K > 0$ such that $\|A_x\| \le K$ for all $x \in \Sigma$. The LDS $\mathcal{A}$ is said to be a *large period system* if the number of orbits of Σ with period less or equal to τ is finite for any $\tau > 0$. We say that the LDS $\mathcal{A}$ is *traceless* (or *conservative*) if, for all $x \in \Sigma$, we have

$$|\det \Phi_A^t(x)| \|X(X^t(x))\| = \|X(x)\|. \tag{3.2}$$

In fact we observe that, if for a given $X \in \mathfrak{X}_\mu^1(M)$, we have $\mathrm{Sing}(X) = \emptyset$, then there exists $\rho\colon M \to \mathbb{R}$ with $\rho(x) = \|X(x)\|^{-1}$ such that $\rho X(x)$, in (3.2), satisfies $|\det \Phi_A^t(x)| = 1$. Now, by (3.3), we get $\int_0^t \mathrm{tr}(A(X^s(x)))ds = 0$. It follows from Liouville's formula that

$$\det \Phi_A^t(x) = \exp\left(\int_0^t \mathrm{tr}(A(X^s(x)))ds\right). \tag{3.3}$$

An LDS $\mathcal{B} = (\Sigma, X^t, N_\Sigma, B)$ is a *traceless perturbation* of a bounded LDS $\mathcal{A}$ if, for every $\epsilon > 0$, $\|A_x - B_x\| < \epsilon$, up to points x belonging to a finite number of orbits, and $\mathcal{B}$ is conservative. From (3.3) it follows that $\mathcal{B}$ is traceless if and only if $\mathrm{tr}(B) = \mathrm{tr}(A)$.

Gronwall's inequality gives that

$$\|\Phi_A^t(x) - \Phi_B^t(x)\| \le \exp(K|t|)\|A_x - B_x\|.$$

In particular, Φ_B^1 is a perturbation of Φ_A^1 in the sense introduced in [10] and in Subsection 2.1 for the discrete case. A bounded LDS $\mathcal{A}$ is *strictly without dominated decomposition* if the only invariant subsets of Σ that admit a dominated splitting for Φ_A^t are finite sets.

Now, we present a result about LDS which is the flow version of Theorem 2.2.

Theorem 3.1 *Let $\mathcal{A}$ be a traceless, large period and bounded* LDS. *If $\mathcal{A}$ is strictly without dominated decomposition, then there exist a traceless perturbation $\mathcal{B}$ of $\mathcal{A}$ and an infinite set $\Sigma' \subset \Sigma$ which is X^t-invariant such that for every $x \in \Sigma'$ the linear map $\Phi_B^{\pi(x)}(x)$ has all the eigenvalues real and with the same modulus* (*thus equal to* 1 *or to* -1).

Like in Theorem 2.2, once again the perturbations are made in the linear traceless differential systems and Franks' lemma for volume-preserving vector fields (Lemma 3.2) allows us to realize them as perturbations of a fixed volume-preserving flow.

Let us introduce now a useful concept. Fix $X \in \mathfrak{X}_\mu^1(M)$, $\tau > 0$ and $p \in M$ non-periodic (or with period larger than τ). A *one-parameter area-preserving linear family* $\{A_t\}_{t\in\mathbb{R}}$ associated with $\{X^t(p);\ t \in [0,\tau]\}$ is defined as follows:

- $A_t\colon N_p \to N_p$ is a linear map for all $t \in \mathbb{R}$,
- $A_t = \mathrm{id}$ for all $t \leq 0$, and $A_t = A_\tau$ for all $t \geq \tau$,
- $A_t \in \mathrm{SL}(n, \mathbb{R})$, and
- the family A_t is C^∞ on the parameter t.

The following result, proved in [3, Lemma 3.2] is now stated for $X \in \mathfrak{X}^1_\mu(M)$ instead of $X \in \mathfrak{X}^4_\mu(M)$ because of the improved *smooth C^1 pasting lemma* proved in [5, Lemma 5.2].

Lemma 3.2 *Given $\epsilon > 0$ and a vector field $X \in \mathfrak{X}^1_\mu(M)$, there exists $\xi_0 = \xi_0(\epsilon, X)$ such that $\forall \tau \in [1, 2]$, for any periodic point p of period greater than 2, for any sufficient small flowbox $\mathcal{T}$ of $\{X^t(p);\ t \in [0, \tau]\}$ and for any one-parameter linear family $\{A_t\}_{t\in[0,\tau]}$ such that $\|A'_t A_t^{-1}\| < \xi_0$, $\forall t \in [0, \tau]$, there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying the following properties:*

1. *Y is ϵ-C^1-close to X;*
2. *$Y^t(p) = X^t(p)$ for all $t \in \mathbb{R}$;*
3. *$P^\tau_Y(p) = P^\tau_X(p) \circ A_\tau$, and*
4. *$Y|_{\mathcal{T}^c} \equiv X|_{\mathcal{T}^c}$.*

In overall, we obtain the following result:

Lemma 3.3 *Let $X \in \mathfrak{X}^1_\mu(M)$ and fix a small $\epsilon_0 > 0$. There exist $\pi_0, \ell \in \mathbb{N}$ such that, for any closed orbit x with period $\pi(x) > \pi_0$, we have either*

(i) *that P^t_X has an ℓ-dominated splitting along the orbit of x or else*

(ii) *for any neighborhood U of $\bigcup_t X^t(x)$, there exists an ϵ-C^1-perturbation Y of X, coinciding with X outside U and on $\bigcup_t X^t(x)$, and such that $P^{\pi(x)}_Y(x)$ has all eigenvalues equal to 1 and -1.*

3.2 Set Us Free of Singularities

In order to rule out singularities in the context of C^1-stable weak shadowable volume-preserving flows, we will recall some useful results. The first one was proved in [3, Lemma 3.3].

Lemma 3.4 *Let σ be a singularity of $X \in \mathfrak{X}^1_\mu(M)$. For any $\epsilon > 0$, there exists $Y \in \mathfrak{X}^\infty_\mu(M)$, such that Y is ϵ-C^1-close to X and σ is a linear hyperbolic singularity of Y.*

The second one was proved in [33, Proposition 4.1] generalizing Doering's theorem in [15]. Observe that, in our context, the singularities of hyperbolic type are all saddles.

Proposition 3.5 *If $Y \in \mathfrak{X}^1(M)$ admits a linear hyperbolic singularity of saddle-type, then the linear Poincaré flow of Y does not admit any dominated splitting over $M\backslash\mathrm{Sing}(Y)$.*

Finally, since by Poincaré recurrence, any $X \in \mathfrak{X}^1_\mu(M)$ is chain transitive, the following result is a direct consequence of [2].

Proposition 3.6 *In $\mathfrak{X}^1_\mu(M)$ chain transitive flows equal topologically mixing flows in a C^1-residual subset.*

The following theorem is proved borrowing some arguments in [1, Theorem 15].

Theorem 3.7 *If $X \in \mathfrak{X}^1_\mu(M)$ is C^1-stable weak shadowable vector field, then X has no singularities.*

Proof Let $X \in \mathfrak{X}^1_\mu(M)$ be a C^1-stable weak shadowable vector field and fix a small C^1 neighborhood $\mathcal{U} \subset \mathfrak{X}^1_\mu(M)$ of X. The proof is by contradiction. Assume that $\mathrm{Sing}(X) \neq \emptyset$.

Using Lemma 3.4, there exists $Y \in \mathcal{U}$ with a linear saddle-type singularity $\sigma \in \mathrm{Sing}(Y)$. By Proposition 3.6, there exist $Z_n \in \mathfrak{X}^1_\mu(M)$ C^1-close to Y which is topologically mixing. We can find $W_n \in \mathfrak{X}^1_\mu(M)$ C^1-close to Z_n having a W_n-closed orbit p_n such that the Hausdorff distance between M and $\bigcup_t W^t_n(p_n)$ is less than $\frac{1}{n}$.

Now we consider jointly Lemma 3.8 and Lemma 3.3 obtaining that $P^t_{W_n}$ is ℓ-dominated over the W_n-orbit of p_n where ℓ is uniform on n. Since W_n converges in the C^1-sense to Y and $\limsup_n \bigcup_t W^t_n(p_n) = M$, we obtain that $M \backslash \mathrm{Sing}(Y)$ has an ℓ-dominated splitting which contradicts Proposition 3.5. □

3.3 Proof of Theorem 2

The following result is the continuous-time counterpart of Lemma 2.6.

Lemma 3.8 *Fix some C^1-weakly shadowable volume-preserving vector field $X \in \mathfrak{X}^1_\mu(M)$. Then, any $Y \in \mathfrak{X}^1_\mu(M)$ sufficiently C^1-close to X does not contains closed orbits with trivial real spectrum.*

Proof Let $X \in \mathfrak{X}^1_\mu(M)$ be a volume preserving vector field and $U(X)$ a C^1-neighbourhood of X in $\mathfrak{X}^1_\mu(M)$ where the weak shadowing property holds. Let p be a closed orbit of X and U_p a small neighbourhood of p in M. Let us also assume that all eigenvalues of $P^\pi_X(p)$, the linear Poincaré flow in p, are -1 and 1.

Now, we transpose our objects to the euclidian space using the volume-preserving charts given by Moser's theorem (see [22]). Then, there exists a smooth conservative change of coordinates $\varphi_p\colon T_pM \to U_p$ such that $\varphi_p(\vec{0}) = p$. Let $f_X : \varphi_p(N_p) \to \Sigma$ stand for the Poincaré map associated with X^t, where Σ denotes the Poincaré section through p, and take $\mathcal{V}$ a C^1-neighbourhood of f_X.

By the careful construction of a *local linearized* divergence-free vector field done in [4], we take $\mathcal{T}$ a small flowbox of $\{X^t(p)\colon t \in [0,\tau]\}$, $\tau < \pi$ where we have that there is a (local linear) divergence-free vector field $Z \in U(X)$, $f_Z \in \mathcal{V}$ and a small $\epsilon_0 > 0$ such that

$$f_Z(x) = \begin{cases} \varphi_p \circ P^\pi_X \circ \varphi_p^{-1}(x), & x \in B_{\epsilon_0}(p) \cap \varphi_p(N_p); \\ f_X(x), & x \notin B_{4\epsilon_0}(p) \cap \varphi_p(N_p). \end{cases}$$

The next computations will be yield in $N^+_p(\epsilon_0)$. Take $v \in N^+_p(\epsilon_1)$, $\epsilon_1 < \epsilon_0$ with $||v|| = \epsilon_2 = \frac{\epsilon_1}{2}$ and set $\mathcal{I}_p = \{sv\colon 0 \le s \le 1\}$.

Fix $0 < \epsilon < \frac{\epsilon_2}{2}$ and let $0 < \delta < \epsilon$ be the number of the weak shadowing property of Z^t. Now, we are going to construct a $(\delta, 1)$-pseudo-orbit of Z^t belonging to $\varphi_p(\mathcal{I}_p)$ which cannot be weakly ϵ-shadowed by any true orbit $y \in M$.

We take a finite sequence $\{w_k\}_{k=0}^T \in N^+_p(\epsilon_1)$ for some $T > 0$, such that $w_0 = 0_p$, $w_T = v$ and $|w_k - w_{k+1}| < \delta$ for $0 \le k \le T-1$. Here w_k are chosen such that if $w_k = s_k v$ then $s_k < s_{k+1}$ for $0 \le k \le T-1$. Finally, we define

- $x_k = \varphi_p(w_0)$, $t_k = \pi$ for $k < 0$;
- $x_k = \varphi_p(w_k)$, $t_k = \pi$ for[2)] $0 \le k \le T-1$;
- $x_k = f_Z^{k-T}(\varphi_p(w_T))$, $t_k = \pi$ for $k \ge T$.

2) Observe that we are considering that the return time at the transversal section is the same and equal to π. Clearly, it is not exactly equal to π, however it is as close to π as we want by just squeezing the flowbox.

Then $\{(x_i, t_i)\}_{i\in\mathbb{Z}}$ is a $(\delta, 1)$-pseudo orbit of Z^t in $B(p, \epsilon_2)$ and since Z is weakly shadowable, there is $y \in M$ such that $\{x_i\}_{i\in\mathbb{Z}} \subset B(Z^t(y), \epsilon)$, $\forall t \in \mathbb{R}$.

We may assume that $y \in B(x_0, \epsilon) \cap \varphi_p(N_{p,\epsilon})$. Note that, in $N_p^+(\epsilon_1)$ we have

$$d(x_0, x_T) = d(\varphi_p(w_0), \varphi_p(w_T)) = d(\varphi_p(\vec{0}), \varphi_p(v)) = d(p, \varphi_p(v)) \approx \|v\| = \epsilon_2 > 2\epsilon,$$

where $\approx$ says that $d(p, \varphi_p(v))$ is very close to $d(\vec{0}, v) = \|v\|$, because φ_p is very close to be an isometry when ϵ is close to zero.

On the other hand, since Z is weakly shadowable, we have that, for some $\iota = n\pi$,

$$d(x_0, x_T) \le d(x_0, y) + d(y, x_T) = d(x_0, y) + d(Z^\iota(y), x_T) < 2\epsilon,$$

which is a contradiction and the lemma follows. □

Proof of Theorem 2 The proof goes as the one did in Theorem 1. Take the Pugh–Robinson residual (general density theorem, see [25]) intersected with the residual in [2] and call it $\mathcal{R}$. As $\mathfrak{X}^1_\mu(M)$ endowed with the C^1-topology is a Baire space, we can take a sequence of $X_n \in \mathcal{R}$ with $X_n \to X$ (in the C^1-topology). There exist closed orbits p_n (for X_n and with period π_n) such that $\limsup_n \bigcup_t X^t(p_n) = M$ in the Hausdorff metric sense. Clearly, $\pi_n \to +\infty$. Define

$$\Sigma = \bigcup_{n\in\mathbb{N}} \{X_n^t(p_n) \colon 0 \le t \le \pi_n\}. \tag{3.4}$$

Notice that $\overline{\Sigma} = M$.

We now define an LDS $\mathscr{A} = (\Sigma, Y^t, N_\Sigma, A)$; Σ is defined in (3.4), $N_\Sigma = N_x$ where $x \in \Sigma$. We define a one-parameter map by $Y^t(X_n^r(p_n)) = X_n^{t+r}(p_n)$ for any $r \in [0, \pi_n]$. Observe that Y^t is a flow; clearly,

- $Y^0(X_n^r(p_n)) = X_n^r(p_n)$ and
- $Y^{t+s}(X_n^r(p_n)) = X_n^{t+s+r}(p_n) = Y^t(X_n^{s+r}(p_n)) = Y^t(Y^s(X_n^r(p_n)))$.

Finally, we define the linear action on N_Σ by $\Phi_A^t(Y_n^r(p_n)) = P_{X_n}^t(X_n^r(p_n))$. Since X_n are divergence-free, the map A is traceless.

This time we are in the presence of an LDS. Reasoning analogously with the discrete-time case, using Theorem 3.1 and Lemma 3.8, we obtain that there exists a uniform dominated splitting over Σ. Since $\overline{\Sigma} = M$ and, by Theorem 3.7, we have $\mathrm{Sing}(Y) = \emptyset$, the dominated splitting extends to the closure and we obtain that M has a dominated splitting with respect to the LDS $\mathcal{A} = (\Sigma, Y^t, N_\Sigma, A)$.

Finally, we realize dynamically the traceless LDS, by Theorem 3.1 and Lemma 3.2, we obtain that X^t, C^1-stable weakly shadowable, has a dominated splitting over M for P_X^t. The volume-hyperbolicity can be obtained as in [8, Proposition 0.5]. □

4 Conclusions and Possible Generalizations

In overall, as it was expected, shadowing is a property that allows us to go further than weak shadowing. However, surprisingly, shadowing does not take us as far as we would expect when compared to weak shadowing. In fact, in low dimensions, the C^1-stability of the shadowing property implies global hyperbolicity exactly as the C^1-stability of the weak shadowing property. Moreover, even in the multidimensional case, both properties share several properties, because both avoid the presence of singularities (for flows), both display a dominated splitting in the whole manifold and both have a hyperbolic behavior; the shadowing has hyperbolicity and the weak shadowing has volume-hyperbolicity.

We believe that the techniques developed in the present paper should be useful to prove that volume-preserving dynamical systems which exhibits C^1-shadowing-like properties display a weak form of hyperbolicity. Actually, it is our guess that the results in [1] can be generalized not only for our class but also for dissipative flows in higher dimensions. Thus, systems with C^1-stability of the average shadowing property and also the asymptotic average shadowing property should also have a dominated splitting. Furthermore, the C^1-stability of weak shadowing for dissipative flows is not studied yet, and our results should enlighten the solution for that problem.

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