

Fine corrections in the effective string describing SU(2) Yang-Mills theory in three dimensions

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ABSTRACT: We present a study of the effective string that describes the infrared dynamics of SU(2) Yang-Mills theory in three dimensions. By combining high-precision lattice simulation results for Polyakov-loop correlators at finite temperatures close to (and less than) the deconfinement one with the analytical constraints from renormalization-group arguments, from the exact integrability of the two-dimensional Ising model that describes the universality class of the critical point of the theory, from conformal perturbation theory, and from Lorentz invariance, we derive tight quantitative bounds on the corrections to the effective string action beyond the Nambu-Gotō approximation. We show that these corrections are compatible with the predictions derived from a bootstrap analysis of the effective string theory, and have a value which does not allow to prove the Axionic String *Ansatz* for this model.

KEYWORDS: Bosonic Strings, Conformal Field Theory, Lattice Quantum Field Theory, Wilson, 't Hooft and Polyakov loops

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1 Introduction

One of the most promising approaches to understand and model the non-perturbative behavior of confining Yang-Mills theories is the “effective string theory” (EST) description, in which the flux tube joining together a quark-antiquark pair is modeled as a thin vibrating string [1–5]. Recently, there has been a lot of progress in this context. In particular, it has been realized that the EST enjoys “low-energy universality” [6–12]: due to the peculiar features of the string action and to the symmetry constraints imposed by Poincaré invariance in the target space, the first few terms of the long-distance expansion of the string action are fixed and hence universal. This implies that the EST is much more predictive than typical effective theories, and in fact during the past few years its predictions have been confirmed by many simulations in lattice gauge theories (for recent reviews, see for instance refs. [12–14]).

At the same time, it was also realized that the simplest Lorentz-invariant EST, which is the well-known Nambu-Gotō model [1, 2], is an exactly integrable, irrelevant, perturbation of the two-dimensional free Gaussian model [15], driven by the $T\bar{T}$ operator of the $D-2$ free

bosons¹ that represent the transverse degrees of freedom of the string [16]. This observation stimulated much work and led to interesting novel results, whose relevance extends even beyond the original application to Yang-Mills theory [17–24]. In particular, these findings are at the basis of an S-matrix bootstrap approach that can be used to constrain the EST action beyond the Nambu-Gotō approximation [25, 26].

Indeed, it is by now clear that the Nambu-Gotō action should be considered only as a leading-order approximation of the actual EST describing the infrared dynamics of confining gauge theories. Going beyond this approximation is one of the most interesting open problems in this context. The terms beyond the Nambu-Gotō action encode important physical information and their study could be of great importance to understand the mechanism underlying confinement or the physical degrees of freedoms from which the confining string arises.

A natural way to study these corrections would be to perform high-precision simulations of the interquark potential in different lattice gauge theories. This approach, however, is hampered by the existence of boundary terms [27–30] related to the finiteness of the physical flux tubes studied in lattice simulations. These boundary terms provide a dominant contribution to the corrections beyond the Nambu-Gotō action and make the detection of other terms challenging (if possible at all: see below for a detailed discussion of this issue).

However, it can be shown that these corrections become subleading and can be neglected if one looks at the interquark potential at finite temperature, in the neighborhood of the deconfinement transition, but still in the confining phase. With this motivation, in this work we address a study of the simplest non-trivial non-Abelian lattice gauge theory: the SU(2) Yang-Mills theory in $(2 + 1)$ dimensions, in the range of temperatures $0.8T_c \leq T \leq T_c$, where T_c denotes the deconfinement temperature.

This model has been the subject of several lattice studies in the past, including e.g. refs. [27–36], because it is a particularly simple lattice gauge theory based on a non-Abelian Lie group and allows one to study the non-perturbative features of Yang-Mills theories to much better numerical precision than one could obtain in lattice simulations of quantum chromodynamics.

There is another important reason why the choice of the SU(2) lattice gauge theory in $(2 + 1)$ dimensions is helpful. Since the deconfinement transition for this model is of the second order [37, 38], renormalization group arguments show that in the neighborhood of the deconfinement transition the model is in the same universality class of the bidimensional Ising model: this is the celebrated Svetitsky-Yaffe conjecture [39]. According to this correspondence, the Polyakov loop correlator is mapped to the spin-spin correlator of the two-dimensional Ising model which, thanks to the exact integrability of the model, is exactly known. As will be shown in this work, the quantitative accuracy of this mapping is confirmed by our new sets of high-precision non-perturbative results, obtained by Monte Carlo simulations; moreover, the knowledge of the exact form of the spin-spin correlator in the spin model provides us with a tool to extract the temperature-dependence of

¹We will denote in the following with D the number of spacetime dimensions of the target lattice theory and with $d \equiv D - 1$ the number of spacelike directions.

the ground-state energy of the theory with high accuracy, and to compare these results with effective-string predictions. In particular, this will allow us to precisely quantify the corrections with respect to the Nambu-Gotō action, which is one of the main goals of this work.

This article is organized as follows. The next four sections contain introductory material: section 2 is devoted to a description of the lattice setup, in section 3 we summarize the Svetitsky-Yaffe conjecture, section 4 reviews known results on the spin-spin correlator of the two-dimensional Ising model, while section 5 presents a brief introduction to the effective string theory. Our results and a description of lattice simulations are presented in section 6, while in the last section 7 we summarize our findings and list some concluding remarks.

2 Definitions and lattice setup

As mentioned above, in this work we focus on the three-dimensional SU(2) Yang-Mills theory at finite temperature. We regularize the theory on a finite cubic lattice of spacing a and sizes aN_t in the $\hat{0}$ (“Euclidean-time”) direction and aN_s in the two other (“spatial”) directions, labelled as $\hat{1}$ and $\hat{2}$. To simplify notations, in the following we will set $a = 1$. Periodic boundary conditions are assumed in all directions and we always take $N_s \gg N_t$. We use the standard Wilson action [40]

$$S_w = -\frac{2}{g^2} \sum_x \sum_{0 \leq \mu < \nu \leq 2} \text{Tr} U_{\mu\nu}(x) \tag{2.1}$$

where the plaquette $U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x)$ is defined as the path-ordered product of link variables $U_\mu(x)$ (taking values in the fundamental representation of the SU(2) group) along the elementary square having the site x as a corner and lying in the oriented (μ, ν) plane. g^2 is the squared bare coupling, which has energy dimension one. In the following, we will often use the parameter β , defined as $\beta = 4/g^2$.

This lattice model has been studied in various works in the past. These include, in particular, ref. [32], in which the following scale setting was presented:

$$\sqrt{\sigma_0(\beta)} = \frac{1.324(12)}{\beta} + \frac{1.20(11)}{\beta^2} + \mathcal{O}(\beta^{-3}), \tag{2.2}$$

where σ_0 denotes the zero-temperature string tension.² In the following, we will also discuss the relationship between σ_0 and the finite-temperature string tension $\sigma(T)$. For some of our simulations at $\beta = 9.0$, we used the high-precision scale setting that was recently reported in ref. [36]. In addition, we will also analyze the results of a set of simulations at $\beta = 16.0$, that were presented in ref. [41].

The temperature T is related to the extent of the shortest compact size of the lattice as $N_t = 1/T$: as a consequence, T can be varied by changing N_t , or the lattice spacing (which can be varied continuously by tuning β), or both. We will study the system in the confining phase, just below the deconfinement temperature T_c , in the temperature range

²Note that eq. (2.2) expresses the square root of the zero-temperature string tension as a dimensionless quantity, in lattice units. Reinstating the lattice spacing a , the left-hand side of eq. (2.2) would be replaced by $a\sqrt{\sigma_0}$.

$0.8 \leq T/T_c \leq 1$. Very accurate estimates of T_c for various values of N_t , that we will use in the following, can be found in ref. [42].

The Polyakov loop through a point of spatial coordinates \vec{x} is defined as the normalized trace of the closed Wilson line in the $\hat{0}$ direction:

$$P(\vec{x}) = \frac{1}{2} \text{Tr} \prod_{t=0}^{N_t} U_0(t, \vec{x}). \quad (2.3)$$

The two-point correlation function of Polyakov loops is then defined as

$$G(R) = \left\langle \sum_{\vec{x}} P(\vec{x}) P(\vec{x} + R\hat{k}) \right\rangle, \quad (2.4)$$

where \hat{k} denotes one of the two spatial directions, the sum is over all spatial coordinates \vec{x} , while the $\langle \dots \rangle$ average is taken over all values of all of the $U_\mu(x)$ variables, with a measure that is proportional to the product of the Haar measures of all $U_\mu(x)$ matrices and to $\exp(-S_w)$, and normalized in such a way that the expectation value of the identity operator is 1.

2.1 Finite-temperature interquark potential

In a finite-temperature setting, one can define the “interquark potential” (or, more precisely, the potential energy associated with a pair of static fundamental color sources) V from the free energy associated with the Polyakov-loop two-point correlation function:

$$G(R) \equiv \exp \left[-\frac{V(R, N_t)}{T} \right] = \exp [-N_t V(R, N_t)]. \quad (2.5)$$

For sufficiently large values of the spatial separation R between the color sources, in the confining phase one expects $V(R, N_t)$ to tend to a linearly rising function of R :

$$G(R) \simeq \exp [-\sigma(T)N_t R], \quad (2.6)$$

where $\sigma(T)$ denotes a temperature-dependent string tension. As we will see below, $\sigma(T)$ is a decreasing function of T and vanishes exactly at the deconfinement point [43, 44]. From $V(R, N_t)$ and $\sigma(T)$ it is also possible to derive the zero-temperature potential $V(R)$ as the $T \rightarrow 0$ limit of $V(R, N_t)$, and, accordingly, the zero-temperature string tension σ_0 as the $T \rightarrow 0$ limit of $\sigma(T)$.

It is interesting to note that the correlator defined in eq. (2.5) has an analogy with the expectation value of an ordinary Wilson loop, except for the boundary conditions, which in this case are fixed in the spatial directions and periodic in the compact-time direction. Accordingly, the resulting geometry for the world-sheet associated with the fluctuating string between the color sources is that of a cylinder, which is topologically different from the rectangular geometry associated with a Wilson loop.

The existence of periodic boundary conditions for the gauge fields along the compactified Euclidean-time direction is the basis for the interpretation of the thermal deconfinement transition in terms of dynamical breaking of a global symmetry described by a group that

is the center C of the gauge group (i.e. \mathbb{Z}_N if the gauge group is $SU(N)$) [45–47]. This symmetry, which can be thought of as the action of multiplying all timelike links at a given Euclidean time by the same element of C , has the Polyakov loop as an order parameter. In the low-temperature phase ($T < T_c$), the center symmetry is realized, and the expectation value of a Polyakov loop vanishes: this means that the free energy associated with a static, isolated color source is finite, i.e. color confinement. Conversely, in the high-temperature phase ($T > T_c$), center symmetry is spontaneously broken and the Polyakov loop has a non-zero expectation value: this signals that the free energy associated with a static color source is finite, i.e. that the theory is in its deconfined phase.

3 Svetitsky-Yaffe mapping

The peculiar role played by the Polyakov loops in the above discussion suggests to study the behavior of the theory in the vicinity of the deconfinement transition using an effective action that can be constructed by integrating out the spacelike link variables and projecting each Polyakov loop to the closest element of the center of the gauge group. Starting from a $(d + 1)$ -dimensional lattice gauge theory, we end up in this way with an effective action for the Polyakov loops, which will be a d -dimensional spin model, having the center of the original gauge group as a global symmetry.

This integration cannot be performed exactly, and one usually resorts to some kind of strong-coupling expansion. Notwithstanding this, some general insight on the behavior of the model can be deduced by simple renormalization group arguments [39]. Indeed, if the phase transition is continuous, in the vicinity of the critical point the fine details of the Hamiltonian describing the effective spin model can be neglected, and the latter can be shown to belong to the same universality class of the simplest spin model, with only nearest-neighbor interactions, sharing the same symmetry-breaking pattern. This means, in our case, that the deconfinement transition of the $SU(2)$ lattice gauge theory in three dimensions, which is continuous, belongs to the same universality class of the symmetry-breaking phase transition of the two-dimensional Ising model. As is well known, this model is exactly integrable [48] and in particular, as we will see in the next section, an exact expression for the spin-spin correlator is known. This fact will play an important role in the following.

Let us list a few important features of the gauge-spin mapping suggested by the Svetitsky-Yaffe conjecture.

- The ordered (low-temperature) phase of the spin model corresponds to the deconfined (high-temperature) phase of the original gauge theory. These are the phases in which the order parameters of the two theories (namely, the Polyakov loop for the gauge theory and the spin for the spin model) have non-zero expectation values.
- The Polyakov loop is mapped to the spin operator, while the plaquette is mapped to the energy operator of the effective spin model. Accordingly, the Polyakov-loop correlator in the confining phase, in which we are interested in this work, is mapped to the spin-spin correlator in the disordered, high-temperature phase of the spin model.

- Thermal perturbations from the critical point in the original gauge theory, which are driven by the plaquette operator, are mapped to thermal perturbations of the effective spin model, driven by the energy operator. Notice however the change in sign: an *increase* in temperature of the original gauge theory corresponds to a *decrease* of the temperature of the effective spin model.

An important consequence of this mapping is that, in the vicinity of the deconfinement point, we can use the spin-spin correlator of the Ising model to model the behavior of the Polyakov loop correlator of the lattice theory: this poses very tight constraints on the EST that describes this correlator. In this respect, the exact integrability of the two-dimensional Ising model gives us a unique opportunity to study the dynamics of the SU(2) lattice gauge theory in $(2 + 1)$ dimensions.

4 The spin-spin correlator of the two-dimensional Ising model

The spin-spin correlator of the two-dimensional Ising model can be written as a finite determinant with a size that depends on the separation of the spins [49]. In the scaling limit, these determinants can be rewritten as suitable solutions of an equation of the Painlevé type [50]. These solutions have a very different form depending on the phase of the model. In particular, denoting the correlation length by ξ and the R/ξ ratio as t , in the disordered phase which is the focus of our interest in this work they can be expanded in the short-distance ($R \ll \xi$) and in the long-distance ($R \gg \xi$) limits as follows.

- In the $R \ll \xi$ regime, the two-point spin correlator can be expanded as

$$\langle \sigma(0)\sigma(R) \rangle = \frac{k_s}{R^{\frac{1}{4}}} \left[1 + \frac{t}{2} \ln \left(\frac{e^{\gamma_E} t}{8} \right) + \frac{1}{16} t^2 + \frac{1}{32} t^3 \ln \left(\frac{e^{\gamma_E} t}{8} \right) + O(t^4 \ln^2 t) \right], \quad (4.1)$$

where $\gamma_E = 0.57721\dots$ denotes the Euler-Mascheroni constant, while k_s is a non-universal constant, which can be evaluated exactly in the case of the two-dimensional Ising model on a square lattice [50] (a results which, due to its non-universal nature, is not of relevance for our present purposes).

- Conversely, in the $R \gg \xi$ regime, the two-point spin correlator admits the expansion

$$\langle \sigma(0)\sigma(R) \rangle = k_l K_0(t) \quad (4.2)$$

where, again, k_l is a non-universal constant which can be evaluated exactly in the case of the two-dimensional Ising model on a square lattice (but which is not relevant for our discussion), while K_0 is the modified Bessel function of order zero, whose long-distance expansion is

$$K_0(t) \simeq \sqrt{\frac{\pi}{2t}} e^{-t} \left[1 + O\left(\frac{1}{t}\right) \right]. \quad (4.3)$$

We conclude this section with some important observations.

1. The short-distance expansion of eq. (4.1) can also be obtained using conformal perturbation theory (CPT): an approach proposed in 1987 by Zamolodchikov [51], which for the past thirty years has proved to be a powerful analytical tool to describe statistical-mechanics models and quantum field theories in the vicinity of a critical point. As a matter of fact, the full agreement of the CPT result in the Ising case with the exact expansion of eq. (4.1) represents one of the most stringent and successful tests of CPT [52–54]. While original applications of CPT were limited to two-dimensional models (see, for example, refs. [54–59]), recently it has also been extended to three-dimensional models [60, 61]. The reason why the CPT approach is important for our present discussion is that, thanks to it, the results that we discuss in this article do not necessarily require the exact integrability of the underlying spin model but can be extended to any pair of lattice gauge theory and spin model with a second-order deconfinement/symmetry-breaking transition. Indeed this approach was already followed in ref. [62] for the mapping between the (3 + 1)-dimensional SU(2) Yang-Mills theory and the Ising model in three dimensions.
2. The K_0 function describing the long-distance behavior is the typical expression that one obtains for the correlator of a two-dimensional Euclidean quantum field theory with an isolated, massive excitation. In the Ising case, this is the Majorana fermion which describes the model in the continuum limit. As for the short-distance expansion, also this result can be extended to models that are not exactly integrable. Indeed the symmetric phase of a generic spin model is always described, in the continuum limit, by an appropriate set of (possibly interacting) massive particles. In the long-distance limit, the spin-spin correlator for any of these theories will be dominated by the lowest mass in the spectrum, whose correlation function in d dimensions is given by a $K_{(d-2)/2}$ modified Bessel function.
3. It is interesting to observe the shift in the exponent of the $1/R$ term in eq. (4.1) and in eq. (4.2). At short distance (where we expect deviations with respect to the Nambu-Gotō action) the power is fixed by the universality class of the model (in our case $1/4$), and, in general, will depend on the gauge group of the underlying gauge theory. At large distances, instead, the asymptotic behavior of the Bessel function in eq. (4.3) implies that the power is always $1/2$: this is in agreement with the Nambu-Gotō result, as we will discuss in section 5.

5 Effective string theory predictions

There is by now a rich literature on the many properties and features of the effective string theory description of the confining flux tube. Here we will limit ourselves to a brief discussion of the features that are most relevant for our problem. For a general introduction to EST, we refer the reader to the recent reviews [12–14].

The main idea behind EST is that confinement of color charges can be associated with the formation of a thin string-like flux tube [1–5], which leads, for large separations between the color sources, to a linearly confining potential.

In a finite-temperature setting it is possible to show, with very mild assumptions, that the EST description implies the following form for the Polyakov-loop correlator [6, 7]:

$$\langle P(0)P^\dagger(R) \rangle = \sum_n |v_n(N_t)|^2 2R \left(\frac{E_n}{2\pi R} \right)^{\frac{D-1}{2}} K_{(D-3)/2}(E_n R), \quad (5.1)$$

where D denotes the number of spacetime dimensions (in our case $D = 3$), while E_n are the energy levels of the string and $v_n(N_t)$ their amplitudes, which in general depend on the inverse temperature N_t . The physical meaning of eq. (5.1) is that the Polyakov-loop correlation function can be modelled in terms of an infinite series of modified Bessel functions of the second kind: this is expected to hold independently from the type of string that is considered, as long as the energy spectrum is characterized by the existence of isolated states, and these states are stable against decay by glueball radiation.

At large R , the right-hand side of eq. (5.1) is dominated by the lowest energy level E_0 and, setting $D = 3$, we end up with exactly the same expression that we found for the long-distance behavior of the spin-spin correlator in the two-dimensional Ising model, eq. (4.2). It is interesting to note that this equality is not simply a consequence of the $SU(2)$ /Ising correspondence that we are studying here: instead, it is much more general and holds also for $D > 3$ [14]. As we mentioned above, any spin model with an isolated ground state in the spectrum is described by a modified Bessel function of the same type that appears in the EST description. This is an important consistency check of both the EST picture³ and of the Svetitsky-Yaffe mapping in this high temperature limit.

The Nambu-Gotō string model [1, 2] is the simplest Poincaré invariant EST. It has a simple geometric interpretation, since it associates each possible configuration that the string can span in the target space with a quantum weight proportional to the area of the world-sheet surface. As such, the Nambu-Gotō string action can be thought of as a straightforward generalization of the relativistic action for a pointlike particle to a bosonic string. The Nambu-Gotō action can be written as follows:

$$S_{\text{NG}} = \sigma_0 \int_{\Sigma} d^2\xi \sqrt{g}, \quad (5.2)$$

where $g \equiv \det g_{\alpha\beta}$ and $g_{\alpha\beta} = \partial_\alpha X_\mu \partial_\beta X^\mu$ is the induced metric on the reference world-sheet surface Σ , where we denote the world-sheet coordinates as $\xi \equiv (\xi^0, \xi^1)$. This action has only one free parameter: the string tension σ_0 , which has dimension two.

In order to perform calculations with the Nambu-Gotō action one has first to fix its invariance under reparametrizations of the string world-sheet coordinates. A common choice is the so-called “physical gauge”, in which the two world-sheet coordinates are identified with the longitudinal degrees of freedom of the string: $\xi^0 = X^0$ and $\xi^1 = X^1$, so that the string action can be expressed as a function only of the $(D - 2)$ degrees of freedom describing transverse displacements, X^i , with $i = 2, \dots, (D-1)$, which are assumed to be single-valued functions of the world-sheet coordinates. In the physical gauge, the

³In particular, as we will see below, for the Nambu-Gotō action, both $v(N_t)$ and E_n can be evaluated exactly thanks to the exact integrability of the model.

determinant of the metric has the form

$$g = 1 + \partial_0 X_i \partial_0 X^i + \partial_1 X_i \partial_1 X^i + \partial_0 X_i \partial_0 X^i \partial_1 X_j \partial_1 X^j - (\partial_0 X_i \partial_1 X^i)^2 \quad (5.3)$$

and the Nambu-Gotō action can then be written as a low-energy expansion in the number of derivatives of the transverse degrees of freedom of the string which, by a suitable redefinition of the fields, can be rephrased as an expansion around the limit of an infinitely long string. The first few terms in this expansion are

$$S = \sigma_0 R N_t + \frac{\sigma_0}{2} \int d^2 \xi \left[\partial_\alpha X_i \cdot \partial^\alpha X^i + \frac{1}{8} (\partial_\alpha X_i \cdot \partial^\alpha X^i)^2 - \frac{1}{4} (\partial_\alpha X_i \cdot \partial_\beta X^i)^2 + \dots \right]. \quad (5.4)$$

It is important to stress, however, that the physical gauge discussed above is anomalous in $D \neq 26$. Hence, in the three-dimensional case we are interested in, eq. (5.4) only describes an *effective* version of the original Nambu-Gotō action. However, thanks to the low-energy universality theorem discussed below, it is known that the corrections to eq. (5.4) due to the anomaly only appear at high orders in the low-energy expansion. Finding the leading corrections with respect to the physical-gauge limit is one of the goals of this paper.

Despite its apparent complexity, it can be shown that all the additional terms in the expansion of eq. (5.4) beyond the Gaussian one conspire to yield an exactly integrable, irrelevant perturbation of the Gaussian term [15], driven by the $T\bar{T}$ operator of $D - 2$ free bosons [16].

Thanks to this exact integrability, the partition function of the model can be written explicitly.⁴ For the two-point Polyakov-loop correlation function that we are considering here (i.e. for the partition function with Dirichlet boundary conditions in the R direction and periodic boundary conditions in the N_t direction), the expression in D spacetime dimensions is

$$G(R) = \sum_{n=0}^{\infty} w_n \frac{2R\sigma_0 N_t}{E_n} \left(\frac{\pi}{\sigma_0}\right)^{\frac{D-2}{2}} \left(\frac{E_n}{2\pi R}\right)^{\frac{D-1}{2}} K_{(D-3)/2}(E_n R), \quad (5.5)$$

where the energy levels E_n are given by

$$E_n = \sigma_0 N_t \sqrt{1 + \frac{8\pi}{\sigma_0 N_t^2} \left(n - \frac{D-2}{24}\right)}, \quad (5.6)$$

and the weights w_n can be obtained from the expansion in powers of q of the Dedekind function that describes the large- R limit of eq. (5.5) (for a detailed derivation, see ref. [63]):

$$\left(\prod_{r=1}^{\infty} \frac{1}{1 - q^r}\right)^{D-2} = \sum_{k=0}^{\infty} w_k q^k. \quad (5.7)$$

⁴The explicit expression for the partition function was actually found even before this $T\bar{T}$ study, first by using the constraints imposed by the open-closed string duality [7] and then using a Dp -brane formalism [63]. In the context of the $T\bar{T}$ formalism it was recently described in refs. [64, 65] for periodic boundary conditions, and in ref. [66], for the Dirichlet boundary conditions of relevance for our study. In addition, the latter reference also studied the extension to compactified transverse dimensions.

For $D = 3$ we have simply $w_k = p_k$, the number of partitions of the integer k . Similar expressions can be obtained also for the other geometries, e.g. for the rectangle (relevant for the description of the Wilson loop) [67] and for the torus (which can model an interface) [68].

From the discussion above we see that, as anticipated, for the effective Nambu-Gotō string we have an exact expression both for the $v(N_t)$ amplitudes⁵ and for the energy levels E_n . In particular, for the $D = 3$ case the lowest state is

$$E_0 = \sigma_0 N_t \sqrt{1 - \frac{\pi}{3\sigma_0 N_t^2}} = \sigma(T) N_t, \tag{5.8}$$

where we defined the “temperature-dependent string tension” as

$$\sigma(T) \equiv \sigma_0 \sqrt{1 - \frac{\pi}{3\sigma_0 N_t^2}}. \tag{5.9}$$

As we have seen in the previous section, E_0 is the inverse of the correlation length, thus the Nambu-Gotō EST predicts a critical temperature [69, 70]

$$\frac{T_{c,NG}}{\sqrt{\sigma_0}} = \sqrt{\frac{3}{\pi(D-2)}} \tag{5.10}$$

and a critical index $\nu = 1/2$. This prediction, however, is inconsistent with the Svetitsky-Yaffe mapping, which for our model predicts the two-dimensional Ising value $\nu = 1$. Moreover, the prediction for the critical temperature is quantitatively wrong (albeit close to the correct one). These observations suggest that, in order to obtain the correct EST describing the gauge theory, one should necessary go beyond the Nambu-Gotō approximation.

5.1 Effective string action beyond the Nambu-Gotō approximation

The discussion above shows that the pure Nambu-Gotō action cannot be the actual effective string action. Discovering the correct (subleading) terms of the effective string action beyond the Nambu-Gotō approximation is one of the major open challenges in present studies of the EST, and is, in fact, the main goal of this article.

There are essentially two classes of terms which one should address: “bulk terms” and “boundary terms”. Let us look at them in more detail.

5.1.1 Beyond the Nambu-Gotō approximation: bulk terms and low-energy universality

From an effective-action point of view, there is no reason to constrain the coefficients of the higher-order terms in eq. (5.4) to the values they take in the derivative expansion of the Nambu-Gotō action. One should instead assume the most general form for such an effective action:

$$S = S_{cl} + \frac{\sigma_0}{2} \int d^2\xi \left[\partial_\alpha X_i \cdot \partial^\alpha X^i + c_2 (\partial_\alpha X_i \cdot \partial^\alpha X^i)^2 + c_3 (\partial_\alpha X_i \cdot \partial_\beta X^i)^2 + \dots \right], \tag{5.11}$$

⁵We remark that this expression for the amplitudes is correct only in the effective-string limit (i.e. assuming the physical gauge), as also the amplitudes are affected by the anomaly. As we only study the ground-state energy E_0 , we do not consider this issue further in the following.

and then fix the coefficients (which, in this context, would play the role of low-energy constants of the effective theory) order by order, either using results from Monte Carlo simulations or (in the case of quantum chromodynamics) from experiments.

However, one of the most interesting results of the last few years, known as “low-energy universality of the EST” is that the c_i coefficients are not arbitrary, but must satisfy a set of constraints to enforce the Poincaré invariance of the gauge theory in the target space [6–11]. This same result can also be obtained in an independent way, using a bootstrap type of analysis: this was done in refs. [25, 26] in the framework of the S-matrix approach pioneered in ref. [15]. These constraints are particularly restrictive for a three-dimensional theory: the first few terms of the expansion exactly coincide with those that are obtained from the expansion of the Nambu-Gotō action, while the first correction with respect to the Nambu-Gotō action, in the high-temperature regime which we are studying here, appears only at order $1/N_t^7$. Moreover, even the coefficient of this correction can be constrained: using the notations of refs. [25, 26], this additional term can be written as

$$-\frac{32\pi^6}{225} \frac{\gamma_3}{\sigma^3 N_t^7} \tag{5.12}$$

where γ_3 is a new parameter which together with σ_0 defines the EST. By using a bootstrap analysis it is possible to show that γ_3 is constrained to be $\gamma_3 > -\frac{1}{768}$.

The γ_3 parameter encodes some important information on the effective string theory. For example, it can be shown that only if $\gamma_3 \geq 0$ then the Axionic String *Ansatz* (ASA) discussed in refs. [18, 19] is certainly correct, while nothing can be said on it if $\gamma_3 < 0$.

5.1.2 Beyond the Nambu-Gotō approximation: boundary corrections

Boundary corrections to the EST encode the effect of possible interactions of the flux tube with the static color sources at its ends. As we will see below, at zero and very low temperature the boundary correction behaves as $1/R^4$, and hence, is the dominant contribution beyond the Nambu-Gotō approximation. Its presence makes it almost impossible to detect the much weaker (and more interesting, being related to the nature of the confining flux tubes) effects due to bulk correction terms discussed above. However, as will be shown below, in the high-temperature regime which we studied in this work, the boundary term actually becomes *subleading*, making it possible to access the bulk corrections using Monte Carlo simulations.

Like the bulk terms, also the boundary terms are strongly constrained by Lorentz invariance: the first boundary correction compatible with the spacetime symmetries of the underlying gauge theory is [71]

$$b_2 \int d\xi_0 \left[\frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right], \tag{5.13}$$

with an arbitrary, non-universal coefficient b_2 . This coefficient has been estimated in some recent lattice studies [27–30, 71]: in particular, for the SU(2) Yang-Mills theory in three dimensions it was found to be $b_2 \simeq -0.025/(\sqrt{\sigma})^3$. The lowest-order term of the expansion

of eq. (5.13) is

$$S_{b,2}^{(1)} = b_2 \int d\xi_0 (\partial_0 \partial_1 X)^2. \quad (5.14)$$

The contribution of this term to the interquark potential was evaluated in ref. [72] using a ζ -function regularization:

$$\langle S_{b,2}^{(1)} \rangle = -b_2 \frac{\pi^3 N_t}{60R^4} E_4 \left(e^{-\frac{\pi N_t}{R}} \right) \quad (5.15)$$

where E_4 denotes the fourth-order Eisenstein series:

$$E_4(q) \equiv 1 + \frac{2}{\zeta(-3)} \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \quad (5.16)$$

and $\zeta(s)$ is the Riemann ζ function.

In the low-temperature ($N_t \gg R$) regime (which is the one that is most often studied in lattice calculations) eq. (5.15) amounts to a $1/R^4$ contribution to the interquark potential. As it scales with a larger power of R , this term obfuscates the evidence of bulk corrections in numerical results. However, using the modular properties of the Eisenstein function,

$$E_4 \left(e^{-\frac{\pi N_t}{R}} \right) = \left(\frac{2R}{N_t} \right)^4 E_4 \left(e^{-\frac{4\pi R}{N_t}} \right), \quad (5.17)$$

it is easy to see that in the $R \gg N_t$ (“high-temperature”) regime, the boundary correction becomes

$$\langle S_{b,2}^{(1)} \rangle = -b_2 \frac{4\pi^3}{15N_t^3} E_4 \left(e^{-\frac{4\pi R}{N_t}} \right), \quad (5.18)$$

which does not contain terms linear in R and thus it does not contribute to the temperature-dependent string tension. We will make use of this property in the analysis of our numerical results.

6 Simulation setting and results

In this section we present the results of a new set of Monte Carlo simulations. The calculations were run with the parallel C++ code developed for the studies presented in refs. [73, 74]. The elements of the SU(2) group are stored as four complex numbers in double precision, and are updated using a combination of local heat-bath [75, 76] and overrelaxation [77] steps.

We performed two sets of simulations. In the first, which was mainly devoted to testing the Svetitsky-Yaffe mapping and to a general study of the deviations with respect to the Nambu-Gotō EST predictions, we fixed a few values of N_t and varied the temperature by changing β . In the second set of simulations, which was aimed to a high-precision study of the corrections to the Nambu-Gotō action, we chose the opposite strategy and fixed three values of β and varied the temperature by changing N_t . Let us discuss these simulations in detail.

$N_t \times N_s^2$	β	T/T_c	n_{conf}
9×96^2	11.3048	0.80	2.5×10^5
	11.72873	0.83	2.5×10^5
9×160^2	12.15266	0.86	2.5×10^5
7×96^2	9.228023	0.83	2.5×10^5
	9.561566	0.86	2.5×10^5

$N_t \times N_s^2$	β	T/T_c	n_{conf}
8×96^2	10.10736	0.80	2.5×10^5
	10.486386	0.83	2.5×10^5
	10.865412	0.86	2.5×10^5
6×96^2	8.258494	0.86	2.5×10^5
	8.546581	0.89	2.5×10^5

Table 1. Information on the first set of lattice simulations.

6.1 Test of the Svetitsky-Yaffe mapping

We chose four values of N_t , namely $N_t = 6, 7, 8,$ and 9 , for which a very precise determination of β_c is known from ref. [42], and we performed a large set of simulations for different values of β , see table 1. We chose the values of β so as to keep the correlation length ξ in the range $10 \lesssim \xi \lesssim 25$ lattice spacings, in order to control both lattice artifacts and finite-size effects.

In addition, in figure 1 we present a data sample corresponding to $T = 0.62T_c$ (a temperature lower than the rest of those considered in this work, that corresponds to a shorter correlation length and allows a better visualization of data collapse), again from simulations with $N_t = 6, 7, 8,$ and 9 : the figure, showing the Polyakov loop correlator as a function of the distance (in units of the inverse temperature) reveals a clear collapse of data. The decay of the correlator can be described very well in terms of a single exponential (which, in this semilogarithmic plot, manifests itself in the approximately linear behavior of the data) over a wide range of distances. Leaving aside the points at values of R of the order of a few lattice spacings, which are affected by non-negligible discretization effects, the slight bending of the data from short to intermediate distances, before the onset of the purely exponential decay, is a signature of the effective string corrections that will be discussed in detail below.

In the analysis of these correlators, one must pay particular attention to finite-size effects, which could become particularly severe when the correlation length grows as the critical temperature is approached. Fortunately, the knowledge of the exact solution of the Ising model in two dimensions allows us to test the Svetitsky-Yaffe mapping in the short-distance regime, using eq. (4.1), which is less sensitive to finite-size corrections. For the subset of our data sets corresponding to correlation lengths shorter than $N_s/6$, we found that finite-size effects could be accounted for by including the contribution from the first periodic copy of the system in the fit, enabling us to cross-check our results using *also* the long-distance behavior of the Ising correlator. The role of finite-size corrections will be analyzed more in detail below, in the discussion of our determination of the γ_3 coefficient.

In the following, we will discuss in detail only one of our simulations, the one at $N_t = 9$ and $\beta = 12.15266$, corresponding to $T/T_c = 0.86$ and $\xi \simeq 22$, for which we could test the Svetitsky-Yaffe mapping both at short and at long distances. To control finite-size effects we simulated the model on a lattice with $N_s = 160$. We found analogous results for all the other cases when we could perform both fits, as reported in table 3.

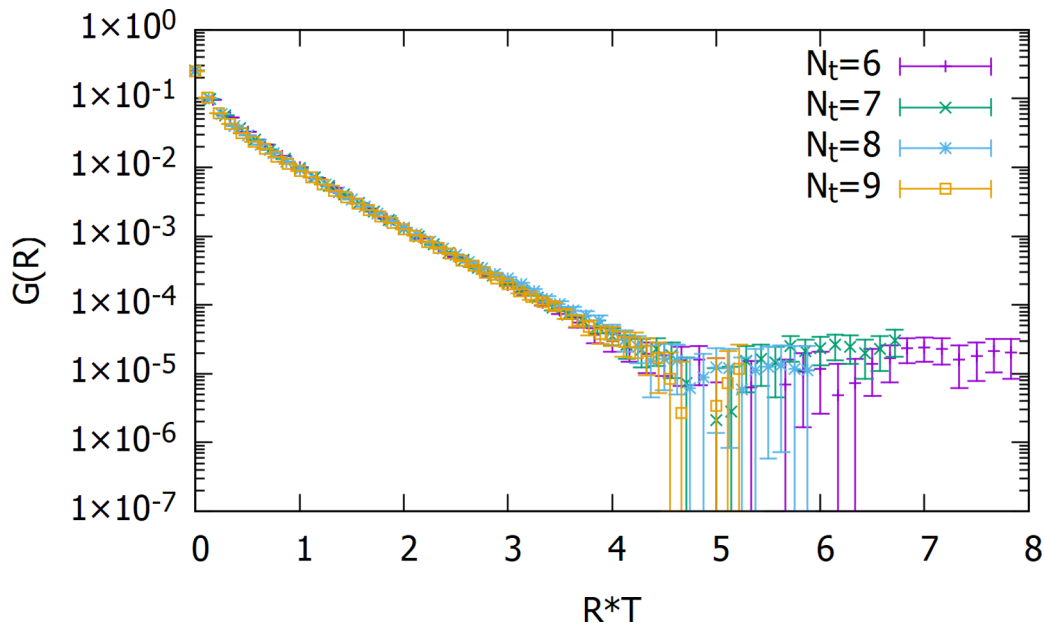


Figure 1. Collapse of data in the Polyakov-loop two-point correlation function $G(R)$ obtained for different values of N_t , from 6 to 9, at the same temperature, $T = 0.62T_c$. The data, shown using a logarithmic scale for the vertical axis, are plotted as a function of the distance between the loops, R , in units of the inverse temperature.

We first tested the Svetitsky-Yaffe mapping, fitting our data to the Ising expressions for the spin-spin correlator reported in eq. (4.1) and in eq. (4.2). To account for the first periodic copy of the lattice, the long-distance fits were performed with the following function

$$G(R) = k_l \left[K_0 \left(\frac{R}{\xi} \right) + K_0 \left(\frac{N_s - R}{\xi} \right) \right] \quad (6.1)$$

which, like eq. (4.2), has only two free parameters: k_l and ξ . The results of the fits are listed in table 2 and shown in figure 2 and in figure 3.

From table 2 it is clear that both at short and at long distances we could fit a wide range of data with a good reduced χ^2 . As expected, the short-distance fit inspired by the Ising-model correlation function works well only up to distances of the order of the correlation length ($R \in [5 - 26]$). What is instead surprising is that the long-distance fit works all the way down to $R = 6$, exhibiting remarkable agreement with the simulation results. This agreement holds also for the other values of β that we tested. What is more important, the two estimates of the correlation length agree with each other within their uncertainties. This means that the two-point Polyakov-loop correlator is described well by the Ising spin-spin correlation function in the whole range of distances that we studied (down to $R > 4$). It is tempting to guess that the good quality of the long-distance fit is due to the fact that, as mentioned above, the fitted function coincides with the EST prediction if one neglects all higher-order states of the spectrum except for E_0 . In this respect it is worth noting that the minimum value of R that can be described by the fit almost coincides with the critical radius R_c at which the Nambu-Gotō action is expected to break

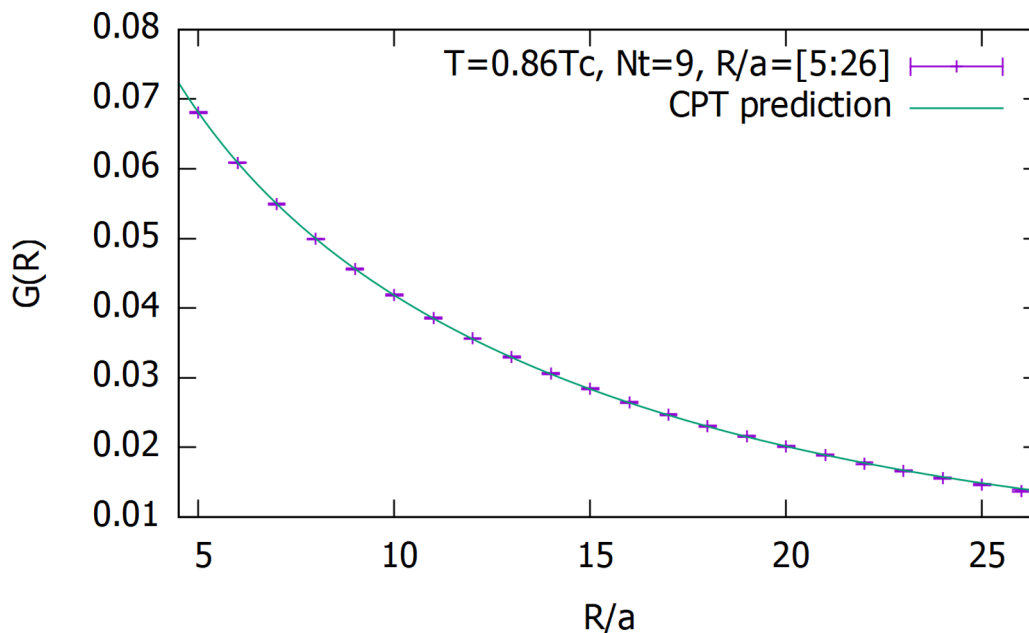


Figure 2. Fit of the Monte Carlo results obtained at $N_t = 9$, $\beta = 12.15266$, $T/T_c = 0.86$ according to the short-distance approximation of the spin-spin Ising correlator of eq. (4.1).

	R_{\min}	R_{\max}	amplitude	ξ	χ_{red}^2
eq. (4.1)	5	26	$k_s = 0.1534(2)$	22.05(6)	0.92
eq. (6.1)	6	47	$k_l = 0.0415(4)$	22.13(17)	0.33

Table 2. Results of the fits to the short- and long-distance behaviors, according to the Svetitsky-Yaffe mapping, of the Polyakov loop correlator for $N_t = 9$, $\beta = 12.15266$ which corresponds to $T/T_c = 0.86$.

down due to the tachyonic singularity [12], which at $\beta = 12.15266$ is $R_c = \sqrt{\frac{\pi}{12\sigma}} \simeq 4.4$. Since higher-order states in the EST expression cannot be detected within the precision of our simulations, we cannot use them to extract information on the corrections beyond Nambu-Gotō. To obtain information on these corrections, it is more convenient to use a different strategy, that will be discussed in the next subsection.

In order to appreciate the agreement, in figure 3 we show the results of both fits.

6.2 Identification of EST corrections beyond the Nambu-Gotō approximation

The most efficient way to identify EST corrections beyond the Nambu-Gotō approximation is to study the behavior of the ground-state energy E_0 as a function of the temperature, as the deconfinement transition is approached from below, at a fixed value of β (i.e. at fixed lattice spacing a) and varying the temperature by changing the value of N_t . For this purpose, we performed additional sets of simulations, whose details are reported in table 4, in table 5, and in table 6. For each of the configuration ensembles produced in these runs, we extracted the value of E_0 (defined as the inverse of the correlation length ξ). The results of these fits are reported in table 7, in table 8, and in table 9, respectively.

N_t		T/T_c	R_{\min}	R_{\max}	amplitude	ξ	χ_{red}^2
6	eq. (4.1)	0.86	5	16	$k_s = 0.1630(4)$	14.13(5)	1.10
	eq. (6.1)	0.86	6	47	$k_l = 0.0491(5)$	14.25(15)	0.09
	eq. (4.1)	0.89	5	26	$k_s = 0.1660(4)$	18.65(7)	1.30
	eq. (6.1)	0.89	6	47	$k_l = 0.0473(6)$	18.40(24)	0.33
7	eq. (4.1)	0.83	5	15	$k_s = 0.1562(4)$	13.37(5)	1.15
	eq. (6.1)	0.83	6	47	$k_l = 0.04716(24)$	13.62(7)	0.33
	eq. (4.1)	0.86	5	22	$k_s = 0.1587(3)$	17.03(5)	0.83
	eq. (6.1)	0.86	6	47	$k_l = 0.0458(3)$	17.03(10)	0.19
8	eq. (4.1)	0.80	5	15	$k_s = 0.1497(4)$	12.75(5)	1.55
	eq. (6.1)	0.80	6	47	$k_l = 0.0454(7)$	13.07(17)	0.13
	eq. (4.1)	0.83	5	20	$k_s = 0.1526(4)$	15.65(6)	1.35
	eq. (6.1)	0.83	6	47	$k_l = 0.0445(4)$	15.86(13)	0.09
	eq. (4.1)	0.86	6	32	$k_s = 0.1543(3)$	20.36(7)	0.96
	eq. (6.1)	0.86	6	47	$k_l = 0.0429(4)$	20.05(20)	0.05
9	eq. (4.1)	0.80	5	17	$k_s = 0.1462(3)$	14.47(4)	0.75
	eq. (6.1)	0.80	6	47	$k_l = 0.0432(4)$	14.74(13)	0.16
	eq. (4.1)	0.83	5	25	$k_s = 0.1492(3)$	17.81(7)	1.01
	eq. (6.1)	0.83	6	47	$k_l = 0.0422(4)$	17.96(18)	0.23
	eq. (4.1)	0.86	5	26	$k_s = 0.1534(2)$	22.05(6)	0.92
	eq. (6.1)	0.86	6	47	$k_l = 0.0415(4)$	22.13(17)	0.33

Table 3. Results of the fits to the short- and long-distance behavior, according to the Svetitsky-Yaffe mapping, of the Polyakov-loop correlator for different values of β and N_t .

At temperatures close to the deconfinement transition, the large values of the correlation length required a careful treatment of effects due to the finiteness of the spatial extent of the system. We addressed this issue in two ways.

1. For some N_t values, we repeated our simulations on lattices of larger spatial sizes, $N_s = 160$ and $N_s = 240$, as shown in table 4, in table 5, and in table 6.
2. We generalized eq. (6.1) by including the contributions to the correlator not only from the first periodic copy of the system, but all of the copies whose contribution was larger than the statistical error of the simulations.

For the remaining sets of simulations, i.e. those corresponding to $N_t \geq 7$ for $\beta = 9$, to $N_t \geq 9$ for $\beta = 12.15266$, and to $N_t \geq 10$ for $\beta = 13.42445$, we fitted our data using eq. (6.1).

Studying the EST corrections through the analysis of the ground-state energy has two main advantages. Firstly, in this setting the Nambu-Gotō expectation for E_0 is exactly

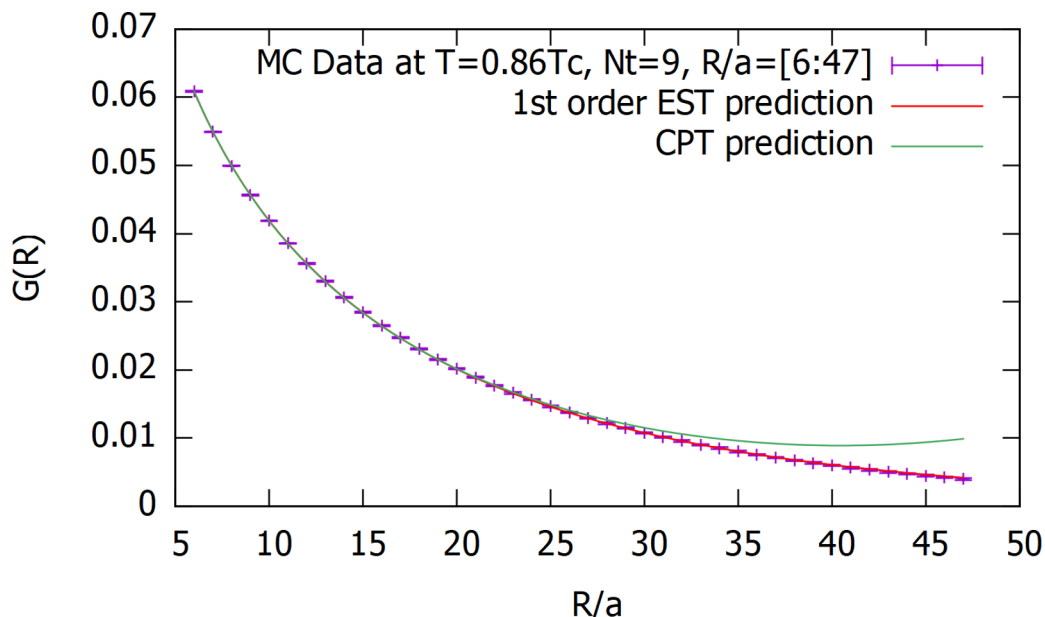


Figure 3. Fit of the data at $N_t = 9$, $\beta = 12.15266$, $T/T_c = 0.86$ combining both the short- and long-distance approximations of the spin-spin Ising correlator, which are respectively given by eq. (4.1) and by eq. (6.1).

known and is given by eq. (5.8), which we recall here

$$E_0 = N_t \sigma(N_t) = N_t \sigma_0 \sqrt{1 - \frac{\pi}{3N_t^2 \sigma_0}} = N_t \sigma_0 \sqrt{1 - \frac{T^2}{T_{c,NG}^2}}. \quad (6.2)$$

Note that eq. (6.2) predicts a mean-field critical index $\nu = 1/2$ for the correlation length $\xi = 1/E_0$: this is obviously incompatible with the prediction from the Svetitsky-Yaffe mapping, from which one would expect the two-dimensional Ising critical index $\nu = 1$, i.e. a linear scaling as $T \rightarrow T_c$ from below:

$$\xi \sim \left(1 - \frac{T}{T_c}\right)^{-1}, \quad (6.3)$$

or, equivalently:

$$\xi \sim \left(1 - \frac{N_{t,c}}{N_t}\right)^{-1}, \quad (6.4)$$

which translates into the following *Ansatz* for the form of the ground-state energy in the vicinity of the deconfinement point:

$$E_0 \sim 1 - \frac{N_{t,c}}{N_t}. \quad (6.5)$$

Thus, the dependence of E_0 on the temperature is an ideal probe for corrections with respect to the Nambu-Gotō approximation.

Moreover, this approach follows very closely the one used in the bootstrap analysis of refs. [25, 26]. From those works, using the low-energy universality we can search for

β	N_t	N_s	T/T_c	n_{conf}
9	6	160	0.935	2.0×10^5
	7	96	0.801	2.0×10^5
	8	96	0.701	2.0×10^5
	9	96	0.623	2.0×10^5
	10	96	0.561	2.0×10^5
	11	96	0.510	2.0×10^5
	12	96	0.468	2.0×10^5

Table 4. Information on the simulations at $\beta = 9$.

β	N_t	N_s	T/T_c	n_{conf}
12.15266	8	240	0.960	2.0×10^5
	9	160	0.853	2.0×10^5
	10	96	0.768	2.0×10^5
	11	96	0.698	2.0×10^5
	12	96	0.640	2.0×10^5
	13	96	0.591	2.0×10^5
	14	96	0.549	2.0×10^5

Table 5. Information on the simulations at $\beta = 12.15266$.

β	N_t	N_s	T/T_c	n_{conf}
13.42445	9	240	0.947	2.0×10^5
	10	160	0.852	2.0×10^5
	11	160	0.775	2.0×10^5
	12	96	0.710	2.0×10^5
	13	96	0.655	2.0×10^5
	14	96	0.609	2.0×10^5
	15	96	0.568	2.0×10^5

Table 6. Information on the simulations at $\beta = 13.42445$.

numerical evidence of the first correction to the Nambu-Gotō approximation, which is expected to appear at the order $1/N_t^7$ and can be parametrized as

$$E_0(N_t) = N_t \sigma_0 \sqrt{1 - \frac{\pi}{3N_t^2 \sigma_0}} - \frac{32\pi^6 \gamma_3}{225\sigma_0^3 N_t^7}. \tag{6.6}$$

Following these observations, we first tried to fit the data using a functional form motivated by the Nambu-Gotō model ($\nu = 1/2$) and one based on the Ising model ($\nu = 1$). Unsurprisingly, we found that neither choice describes the data accurately: this is clearly visible in figure 4, in figure 5, and in figure 6. For all the three β values the Nambu-Gotō

N_t	R_{\min}	R_{\max}	k_l	E_0	χ_{red}^2
6	7	47	0.0417(6)	0.0294(5)	0.07
7	7	47	0.0495(9)	0.0871(13)	0.58
8	8	47	0.0475(15)	0.1250(25)	0.56
9	8	47	0.0501(15)	0.1691(36)	0.30
10	9	47	0.0432(33)	0.1941(63)	2.00
11	9	47	0.0469(41)	0.2381(80)	0.72
12	9	47	0.0392(32)	0.2581(69)	0.09

Table 7. Best-fit estimates for E_0 for different values of N_t at $\beta = 9$.

N_t	R_{\min}	R_{\max}	k_l	E_0	χ_{red}^2
8	8	47	0.0338(10)	0.0135(5)	0.36
9	10	47	0.0416(4)	0.0452(3)	0.33
10	9	47	0.0416(13)	0.0695(17)	0.45
11	10	47	0.0423(16)	0.0922(22)	0.35
12	10	47	0.0414(20)	0.1132(32)	0.13
13	11	47	0.0408(26)	0.1334(42)	0.25
14	11	47	0.0355(16)	0.1440(30)	0.18

Table 8. Best-fit estimates for E_0 for different values of N_t at $\beta = 12.15266$.

N_t	R_{\min}	R_{\max}	k_l	E_0	χ_{red}^2
9	9	47	0.0338(13)	0.0151(7)	0.32
10	11	47	0.0391(6)	0.0401(5)	0.06
11	12	47	0.0407(8)	0.0619(8)	0.05
12	11	47	0.0387(7)	0.0777(9)	0.52
13	12	47	0.0399(12)	0.0979(16)	0.06
14	12	47	0.0390(13)	0.1135(20)	0.19
15	12	47	0.0359(17)	0.1254(27)	0.25

Table 9. Best-fit estimates for E_0 for different values of N_t at $\beta = 13.42445$.

curve fits the data well at large N_t (i.e. at low temperature), but it misses the approach to the critical point, when the deconfinement transition is approached. On the contrary, the linear fit agrees with the data near the critical point, as expected from the Svetitsky-Yaffe correspondence, but this agreement holds only for the first few values of N_t . For larger values, a linear fit is not consistent with the Monte Carlo data.

β	$N_{t,\min}$	$N_{t,\max}$	k_4	σ_0	χ_{red}^2	literature
9	6	12	0.040(8)	0.02603(19)	1.60	0.02583(3)
12.15266	8	14	0.054(5)	0.01366(5)	0.89	0.01371(29)
13.42445	9	15	0.053(8)	0.01104(5)	1.33	0.01108(23)

Table 10. Results of the fits of our numerical data to eq. (6.7). In the last column we report the values of σ_0 quoted in ref. [36] for $\beta = 9$, and in ref. [32] for $\beta = 12.15266$ and for $\beta = 13.42445$.

This failure is indeed in agreement with the low-energy universality: the latter suggests that the correct behavior at short distances should not be modelled by assuming an expression like the one in eq. (6.5), but rather by adding a suitable $1/N_t^7$ correction to the Nambu-Gotō approximation.

Moreover, since low-energy universality suggests that, starting from $O(N_t^{-7})$, there may also be any possible higher-order corrections, we truncate for consistency the Nambu-Gotō expression to this order. Based on this reasoning, we assume the following form for the N_t dependence of the ground state energy:

$$E_0(N_t) = \text{Taylor}_4(E_0) + \frac{k_4}{(\sigma_0)^3 N_t^7}, \tag{6.7}$$

with

$$\text{Taylor}_4(E_0) \equiv \sigma_0 N_t - \frac{\pi}{6N_t} - \frac{\pi^2}{72(\sigma_0)N_t^3} - \frac{\pi^3}{432(\sigma_0)^2 N_t^5} - \frac{5\pi^4}{10368(\sigma_0)^3 N_t^7}, \tag{6.8}$$

where the Taylor expansion is completely known, and the only free parameters of the fit are the zero-temperature string tension σ_0 and the k_4 coefficient.

Remarkably, these fits yield very good reduced χ^2 values for the data at all the three β values. The detailed results are reported in table 10, and shown in figure 7, in figure 8 and in figure 9. In particular, the quality of the fits improves as one goes toward the continuum limit. Moreover, the best-fit values obtained for σ_0 are fully consistent with those that were independently obtained in ref. [36] (for $\beta = 9$) and in ref. [32] (for $\beta = 12.15266$ and for $\beta = 13.42445$). In fact, the precision of our results for σ_0 is even better than the one from the latter reference, so that, in principle, the approach that we used for the determination of σ_0 could even be used for scale setting (although much more precise scale-setting methods exist today [78, 79]).

Another non-trivial consistency check of our analysis is that the three values of k_4 that we found should agree within their uncertainties, since the scale dependence of this coefficient is already accounted for by the $1/\sigma_0^3$ normalization in eq. (6.7). As table 10 shows, this expectation is indeed confirmed in our fit results, and the three values are compatible within their errors. This agreement is also confirmed by the analysis reported in table 11, where we performed the same fits as in table 10 but fixing σ_0 to the values reported in the literature. The agreement between the three values of β that we tested is clear. We remark that all of these data could be fitted with only one free parameter, k_4 .

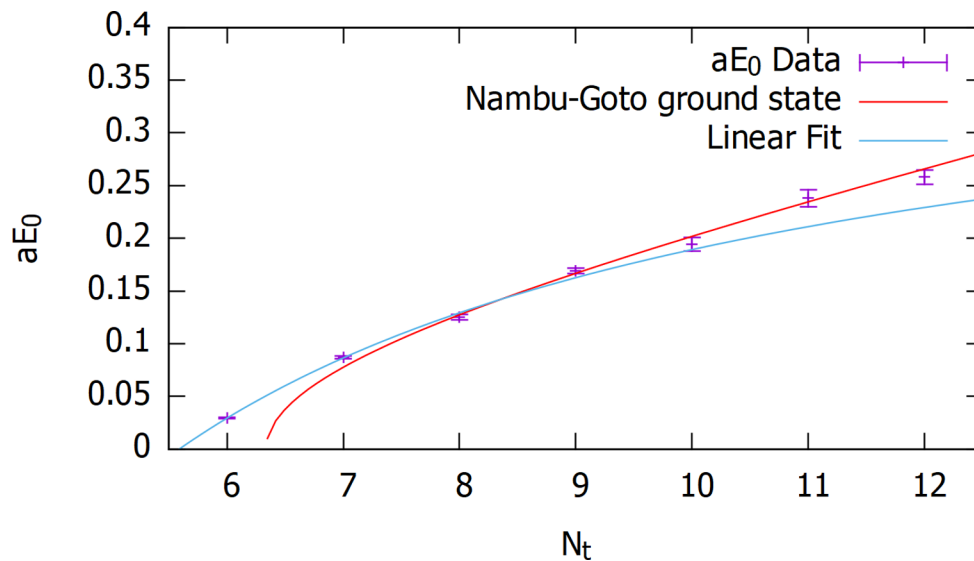


Figure 4. Fits of our data for the ground-state energy E_0 at $\beta = 9$ according to the prediction from the Nambu-Gotō model, eq. (6.2) (red line) and according to the *Ansatz* of a linear dependence of E_0 on the temperature, according to eq. (6.5) (blue curve). Note that the quantity on the horizontal axis of this plot is the inverse of the temperature, in units of the lattice spacing.

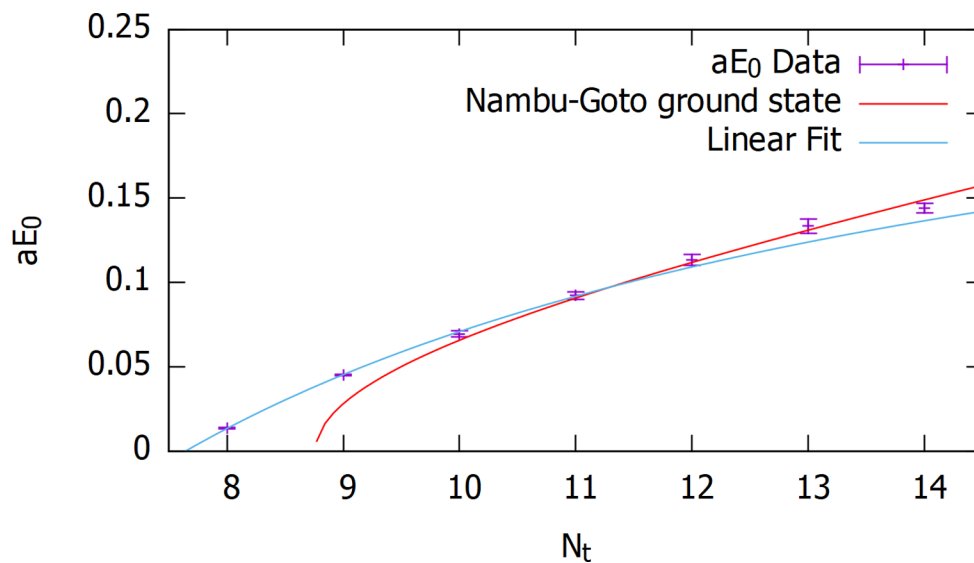


Figure 5. Same as in figure 4, but for the data at $\beta = 12.15266$.

β	$N_{t,\min}$	$N_{t,\max}$	k_4	χ_{red}^2
9	6	12	0.048(3)	1.62
12.15266	8	14	0.049(2)	0.89
13.42445	9	15	0.048(4)	1.24

Table 11. Results of the fits of our numerical data to eq. (6.7) using as input the values for σ_0 obtained from the literature.

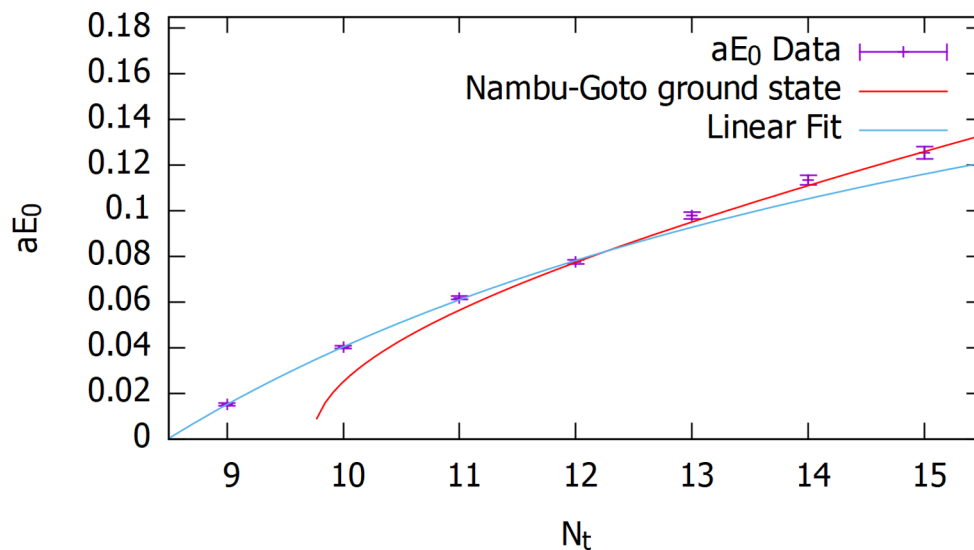


Figure 6. Same as in figure 4, but for $\beta = 13.42445$.

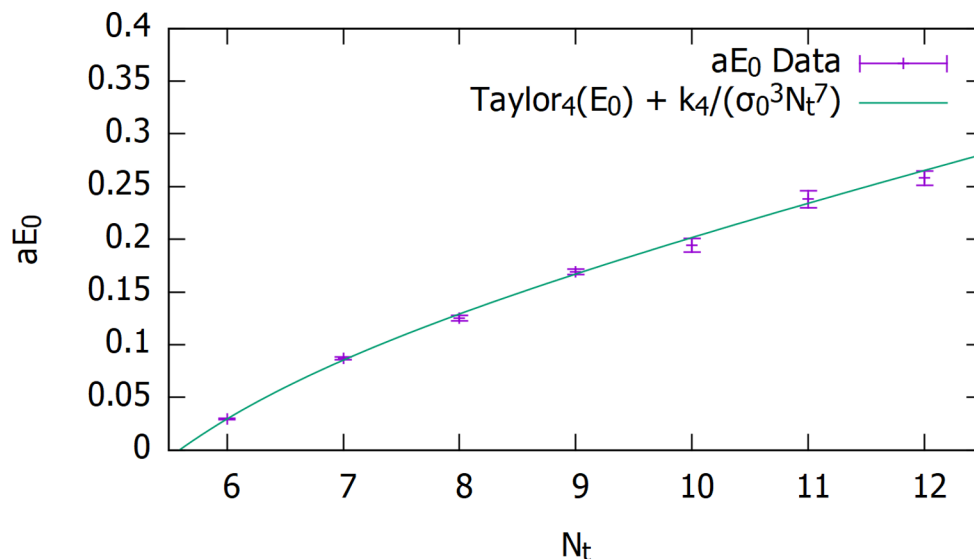


Figure 7. Fit of our numerical results for the ground-state energy at $\beta = 9$ to eq. (6.7).

As our final result for k_4 , we quote a weighted average of the three values from table 10: $k_4 = 0.050(8)$, with a somewhat conservative estimate for the uncertainty. From this result, using eq. (5.12), we obtain

$$\gamma_3 = -\frac{225}{32\pi^6} k_4 = -0.00037(6), \quad (6.9)$$

which is well inside the bound $\gamma_3 \geq -\frac{1}{768} \simeq -0.0013$ derived in refs. [25, 26].

The fact that γ_3 is negative was already noted in ref. [41] and is non-trivial: in particular, as shown in ref. [25], it does not allow to prove the Axionic String *Ansatz* for the EST describing this gauge theory.

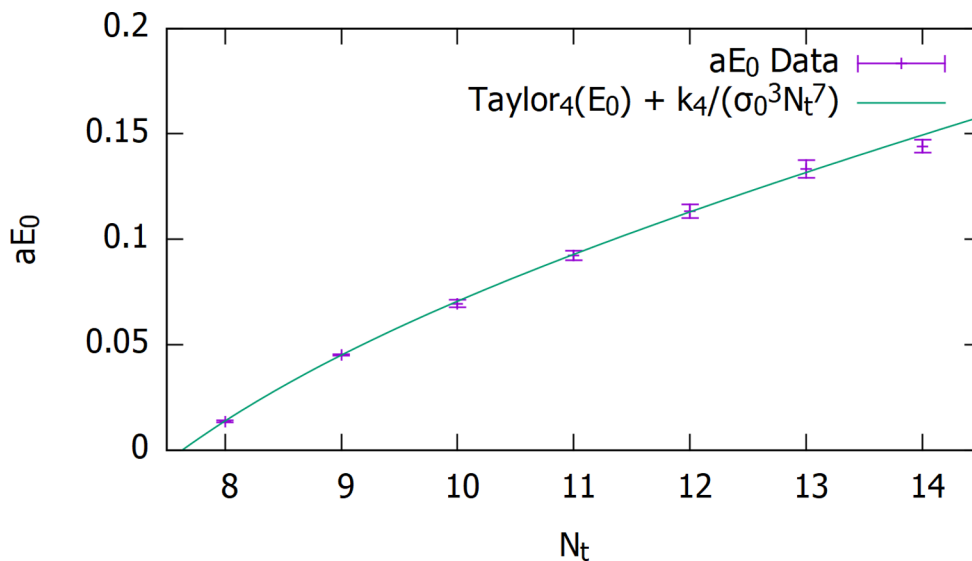


Figure 8. Same as in figure 7, but for $\beta = 12.15266$.

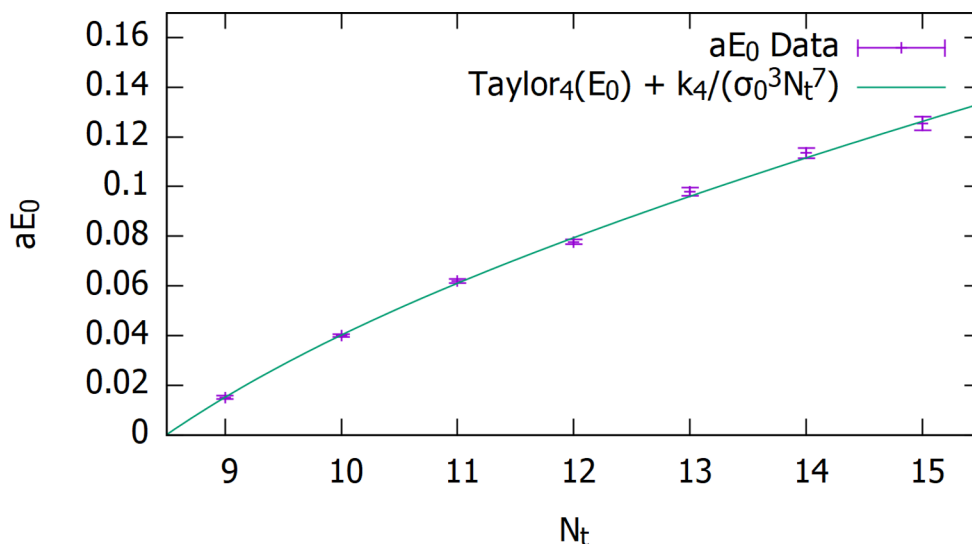


Figure 9. Same as in figure 7, but for $\beta = 13.42445$.

The contribution associated with the γ_3 term is only the first term of an infinite series of higher-order corrections to the Nambu-Gotō action. Unfortunately, our data do not allow us to extract information about the subleading terms, due to the fact that discretization effects on N_t are too large. All our attempts to fit the data adding a further term in the Taylor expansion, i.e. fitting the data with:

$$E_0(N_t) = \text{Taylor}_5(E_0) + \frac{k_4}{(\sigma_0)^3 N_t^7} + \frac{k_5}{(\sigma_0)^4 N_t^9}, \quad (6.10)$$

led to values for k_4 and k_5 compatible with zero within their uncertainties (the two terms tend to compensate each other in the fit).

We can compare our results with those obtained in ref. [41] at $\beta = 16.0$. Fitting the values for E_0 reported in ref. [41] with eq. (6.7), one finds $k_4 \simeq 0.80$, but using instead eq. (6.10) one finds $k_4 = 0.069(25)$ and $k_5 = 0.045(22)$, with a string tension $\sigma_0 = 0.007644(4)$, in full agreement with the value quoted in ref. [41], which is $\sigma_0 = 0.0076416(46)$, and a reduced $\chi^2 = 1.33$. This k_4 value is compatible with the one that we found at smaller values of β , suggesting that simulations at even larger β may lead to reliable estimates also for k_5 .

Finally, it is interesting to compare our result for γ_3 with the one obtained in refs. [19, 22, 80] for the SU(6) Yang-Mills theory in three dimensions, using the lattice data from ref. [81], which was of similar magnitude but opposite in sign. Moreover, estimates for k_4 can be extracted for the SU(4), SU(6), and SU(8) theories using the results for the ground-state energy from ref. [41]: this leads to results that are different for the different gauge groups, showing explicitly that, at this level of resolution, the EST is not universal anymore: instead it encodes, as it should, the specific properties of the underlying Yang-Mills theory.

7 Concluding remarks

In this work we discussed the results of a set of high-precision simulations of the Polyakov-loop correlator in the SU(2) lattice gauge theory in three dimensions. All our simulations were run at finite temperature, in the vicinity of the deconfinement transition, in the range $0.8T_c \leq T \leq T_c$, where it is easier to compare the simulations with the EST predictions. Moreover, invoking the Svetitsky-Yaffe conjecture, in this regime one can compare the analytical solution of the two-dimensional Ising model with the gauge-theory data. The results of this comparison revealed remarkable agreement between our numerical results for the Polyakov-loop correlator and the exact expression of spin-spin correlator of the two-dimensional Ising model. We could extract very precise values for the ground-state energy E_0 of the effective string describing this gauge theory, and quantify the deviations from the predictions that can be derived approximating the effective string action with the Nambu-Gotō action.

We conclude with some comments on these results.

1. The type of comparison that we carried out is not limited to theories having a critical point in the universality class of the two-dimensional Ising model (or to another exactly integrable model). Indeed, in principle conformal perturbation theory allows one to work out the form of the spin-spin correlator for any model characterized by a continuous phase transition. As a consequence, the Svetitsky-Yaffe mapping can be used also for spin models that are not exactly integrable. An example of this approach was discussed in ref. [62], where results for the SU(2) Yang-Mills theory in four dimensions were compared with the Ising model in three dimensions.
2. As we discussed in section 5, an interesting consequence of the effective string description is that the long-distance behavior of the correlator of a Yang-Mills theory in $(d+1)$ dimensions is dominated by a $K_{(d-2)/2}(E_0)$ Bessel function, in exact agreement with the long-distance behavior predicted by the Svetitsky-Yaffe conjecture for the spin-spin correlator of the underlying d -dimensional spin model. This holds for any

EST (under very mild conditions) and for any spin model, and can be regarded as a non-trivial check of mutual consistency for the effective string and the Svetitsky-Yaffe conjecture.

3. The value that we found for the γ_3 coefficient for the EST describing the SU(2) Yang-Mills theory in three dimensions does not allow to prove the Axionic String *Ansatz*. It would be interesting to extend our analysis to the SU(4) gauge theory, for which results compatible with the Axionic String *Ansatz* were recently obtained in refs. [82, 83]. Similarly, it would also be interesting to explore the same type of contribution to the EST describing gauge theories in four dimensions, as the axionic string is expected to play an important role in the description of the low-energy dynamics of quantum chromodynamics [21].
4. Historically, one of the problems of the EST description of Yang-Mills theories was its universality, i.e. the fact that it predicted essentially the same behavior (with only a mild dependence on the number of spacetime dimensions), for models as different as the three-dimensional \mathbb{Z}_2 gauge model as the four-dimensional SU(3) Yang-Mills theory. This feature is now understood as a universality that manifests itself only at low energy (or, equivalently, a side-effect of the high accuracy of the Nambu-Gotō approximation of EST), while the details related to the gauge group (and, possibly, to the confining mechanism into play) may be encoded in the higher-order EST corrections, which are not expected to be universal. In view of the fact that our results indicate that the three-dimensional SU(2) gauge theory is not necessarily described by an axionic effective string, while the opposite conclusion was recently obtained for the SU(6) Yang-Mills theory [19, 22, 80], quantifying these corrections and understanding how they depend on the gauge group would be of great importance. The numerical precision of current Monte Carlo studies of lattice gauge theory is sufficient to probe the fine details of the effective string theory at very high orders, and to test the accurate theoretical predictions that have been formulated during the past few years [13]. This is possible thanks not only to the increase in computing power, but also to the deployment of increasingly sophisticated simulation algorithms, among which we would like to mention non-equilibrium Monte Carlo calculations [84, 85] based on Jarzynski's theorem [86, 87]: as an example of their use for the problems discussed in the present paper, we refer the readers to the recent study of terms $O(L^{-7})$ in the effective-string description of a fluctuating interface of linear size L [88].

Acknowledgments

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